

Exercices sur les V.A.C.

ex 1:

1)

(a) $X_i \sim U([0; a])$

$$\mathbb{E}[X_i] = \int_{-\infty}^{+\infty} x f_{X_i}(x) dx = \int_{-\infty}^{+\infty} x \frac{1}{a} \mathbb{I}_{[0;a]}(x) dx = \int_0^a \frac{x}{a} dx = \left[\frac{x^2}{2a} \right]_0^a = \frac{a^2}{2a} = \frac{a}{2}$$

$$\mathbb{V}[X_i] = \mathbb{E}[X_i^2] - (\mathbb{E}[X_i])^2 = \mathbb{E}[X_i^2] - \frac{a^2}{4}$$

$$\mathbb{E}[X_i^2] = \int_{-\infty}^{+\infty} x^2 f_{X_i}(x) dx = \int_{-\infty}^{+\infty} x^2 \frac{1}{a} \mathbb{I}_{[0;a]}(x) dx = \int_0^a \frac{x^2}{a} dx = \left[\frac{x^3}{3a} \right]_0^a = \frac{a^3}{3a} = \frac{a^2}{3}$$

$$\text{donc: } \mathbb{V}[X_i] = \frac{a^2}{3} - \frac{a^2}{4} = \frac{a^2}{12}$$

$$(b) \mathbb{E}[A_h] = \frac{2}{n} \sum_{i=2}^h \mathbb{E}[X_i] = \frac{2}{n} h \mathbb{E}[X_i] = a$$

$$\begin{aligned} \mathbb{V}[A_h] &= \mathbb{V}\left[\frac{2}{n} \sum_{i=2}^h X_i\right] = \frac{4}{n^2} \mathbb{V}\left[\sum_{i=2}^h X_i\right] \\ &= \frac{4}{n^2} \mathbb{V}[X_i] \text{ car les } (X_i)_{i=2, \dots, n} \text{ sont ind.} \\ &= \frac{4}{n^2} \cdot n \mathbb{V}[X_2] \text{ car les } (X_i)_{i=2, \dots, n} \text{ sont de même loi} \\ &= \frac{4}{n} \cdot \frac{a^2}{12} = \frac{a^2}{3n} \end{aligned}$$

$$\text{Biais } [A_h] = \mathbb{E}[A_h] - a = a - a = 0 \quad \text{c'est l'estimateur sans biais de } a.$$

2) $B_h = \max(X_1, \dots, X_n)$

(a) $\forall t \in \mathbb{R}$,

$$F_{X_i}(t) = \mathbb{P}(X_i \leq t) = \int_{-\infty}^t f_{X_i}(x) dx = \int_{-\infty}^t \frac{1}{a} \mathbb{I}_{[0;a]}(x) dx = \frac{1}{a} \int_{-\infty}^t \mathbb{I}_{[0;a]}(x) dx$$

① si $t \leq 0$, $F_{X_i}(t) = 0$

② si $t > 0$, $F_{X_i}(t) = \frac{1}{a} \int_0^t \mathbb{I}_{[0;a]}(x) dx$

②-1 si $t \geq a$, $F_{X_i}(t) = \int_a^a \frac{1}{a} dx = \left[\frac{x}{a} \right]_0^a = 1$

②-2 si $t \in]0; a[$, $F_{X_i}(t) = \frac{1}{a} \int_0^t \frac{1}{a} dx = \frac{1}{a} \left[x \right]_0^t = \frac{t}{a}$

$$\begin{aligned}
 * \mathbb{P}(B_n \leq t) &= \mathbb{P}(\max(X_1, \dots, X_n) \leq t) = \mathbb{P}(X_1 \leq t, X_2 \leq t, \dots, X_n \leq t) \\
 &= \prod_{i=1}^n \mathbb{P}(X_i \leq t) \text{ car les } (X_i) \text{ sont indépendantes} \\
 &= [\mathbb{P}(X_1 \leq t)]^n \text{ car les } (X_i) \text{ sont de même loi} \\
 &= F_{X_1}(t)^n
 \end{aligned}$$

$$\forall t \in]0; a[, \mathbb{P}(B_n \leq t) = \frac{t^n}{a^n}$$

$$\forall t \geq a, \mathbb{P}(B_n \leq t) = 1^n = 1$$

$$(b) \forall t \in]0; a[, F_{B_n}(t) = \mathbb{P}(B_n \leq t) = \frac{t^n}{a^n}$$

$$\forall t \in]0; a[, f_{B_n}(t) = (F_{B_n}(t))' = n \cdot \frac{t^{n-1}}{a^n}$$

$$\text{On admet que : la densité de } B_n \text{ est : } f_{B_n}(t) = n \cdot \frac{t^{n-1}}{a^n} \quad]0; a[\quad (t)$$

$$\begin{aligned}
 (c) \mathbb{E}[B_n] &= \int_{-\infty}^{+\infty} x \cdot f_{B_n}(x) dx = \int_{-\infty}^{+\infty} x \cdot n \cdot \frac{x^{n-1}}{a^n} \mathbb{I}_{]0; a[}(x) dx = \int_0^a x \cdot n \cdot \frac{x^{n-1}}{a^n} dx \\
 &= \int_0^a n \cdot \frac{x^n}{a^n} dx = \frac{n}{a^n} \cdot \left[\frac{x^{n+1}}{n+1} \right]_0^a = \frac{n}{a^n} \cdot \frac{a^{n+2}}{n+2} = \frac{n}{n+2} \cdot a \\
 \text{Biais } (B_n) &= \mathbb{E}[B_n] - a = \frac{n}{n+2} \cdot a - \frac{n+2}{n+2} a = \frac{-a}{n+2}
 \end{aligned}$$

$$\text{On pose : } B_n^* = \frac{n+2}{n} \cdot B_n$$

$$\begin{aligned}
 \text{on a alors : } \mathbb{E}[B_n^*] &= \frac{n+2}{n} \cdot \mathbb{E}[B_n] = \frac{n+2}{n} \cdot \frac{n}{n+2} \cdot a = a \\
 \text{et : Biais } [B_n^*] &= \mathbb{E}[B_n^*] - a = a - a = 0
 \end{aligned}$$

$$\begin{aligned}
 (d) \mathbb{V}[B_n^*] &= \left(\frac{n+1}{n} \right)^2 \mathbb{V}[B_n] = \left(\frac{n+1}{n} \right)^2 \left(\mathbb{E}[B_n^2] - \mathbb{E}[B_n]^2 \right) = \left(\frac{n+1}{n} \right)^2 \left(\mathbb{E}[B_n^2] - \left(\frac{n}{n+2} \cdot a \right)^2 \right) \\
 &= \left(\frac{n+1}{n} \right)^2 \mathbb{E}[B_n^2] - a^2
 \end{aligned}$$

$$\begin{aligned}
 \mathbb{E}[B_n^2] &= \int_{-\infty}^{+\infty} x^2 f_{B_n}(x) dx = \int_{-\infty}^{+\infty} x^2 \cdot n \cdot \frac{x^{n-1}}{a^n} \mathbb{I}_{]0; a[}(x) dx = \int_0^a \frac{n x^{n+1}}{a^n} dx \\
 &= \frac{n}{a^n} \left[\frac{x^{n+2}}{n+2} \right]_0^a \\
 &= \frac{n}{a^n} \cdot \frac{a^{n+2}}{n+2} = \frac{a^{n+2} n}{n+2}
 \end{aligned}$$

$$\mathbb{V}[B_n^*] = \left(\frac{n+2}{n}\right)^2 \cdot \frac{a^2 n}{n+2} - a^2 = \frac{(n+2)^2}{n(n+2)} \cdot a^2 - 1 = a^2 \left[\frac{n^2 + 2n + 1}{n^2 + 2n} - \frac{n^2 + 2n}{n^2 + 2n} \right]$$

$$= a^2 \cdot \frac{2}{n^2 + 2n} = \frac{a^2}{n(n+2)}$$

3) $\mathbb{E}[B_n^*] = \mathbb{E}[A_n] = a$

résumé,

$$n^2 + 2n - 3n = n^2 - n = n(n-1) > 0 \text{ pour } n \text{ suffisamment grand}$$

$$\text{donc: } n^2 + 2n > 3n \Rightarrow \frac{1}{n^2 + 2n} < \frac{1}{3n} \Rightarrow \frac{a^2}{n^2 + 2n} < \frac{a^2}{3n} \Rightarrow \mathbb{V}[B_n^*] < \mathbb{V}[A_n]$$

On conseillera d'utiliser B_n^* car ses valeurs sont moins dispersées que celles de A_n .

4)

* pour A_n : $\hat{a} = \frac{1}{n} (x_1 + \dots + x_n)$ [les $(x_i)_{i=1, \dots, n}$ sont les valeurs réelles observées]

$$= \frac{1}{20} \cdot 120 = \underline{\underline{12}}$$

$$\mathbb{V}[A_n] = \frac{(\hat{a})^2}{3n} = \frac{12 \times 12}{3 \cdot 20} = \frac{4 \times 6}{10} = \frac{12}{5} = \underline{\underline{2,4}}$$

* pour B_n^* : $\hat{a} = \frac{n+1}{n} \cdot \max(x_1, \dots, x_n) = \frac{21}{20} \cdot 11 = 11,55 \approx \underline{\underline{12}}$

$$\mathbb{V}[B_n^*] = \frac{(\hat{a})^2}{n \cdot (n+2)} = \frac{21 \cdot 21 \cdot 11 \cdot 11}{20 \cdot 20 \cdot 20 \cdot 22} = \frac{21^2 \cdot 11}{20^3 \cdot 2} = \underline{\underline{0,30}}$$

Les 2 estimations donnent la même estimation du temps d'attente (12 minutes) et la variance du premier est plus grande que celle du second.

ex 2: $Y = 2X + 1$

1) soit $y \in \mathbb{R}$, $F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(2X+1 \leq y) = \mathbb{P}\left(X \leq \frac{y-1}{2}\right) = F_X\left(\frac{y-1}{2}\right)$

2) soit $z \in \mathbb{R}$, $F_Z(z) = \mathbb{P}(Z \leq z) = \mathbb{P}(Z \leq z, X < -2) + \mathbb{P}(Z \leq z, X \in [-2; 2]) + \mathbb{P}(Z \leq z, X > 2)$
 $= \mathbb{P}(X+2 \leq z, X < -2) + \mathbb{P}(0 \leq z, X \in [-2; 2]) + \mathbb{P}(X-2 \leq z, X > 2)$

① si $z < 0$, $F_Z(z) = \mathbb{P}(X+2 \leq z, X+2 < 0) + \underbrace{\mathbb{P}(0 \leq z, X \in [-2; 2])}_{0} + \underbrace{\mathbb{P}(X-2 \leq z, X-2 > 0)}_{0}$
 $= \mathbb{P}(X+2 \leq z) = F_X(z-2) = \underline{\underline{F_X(z-2)}}$

(4)

$$\begin{aligned}
 ② \text{ si } z > 0; F_Z(z) &= P(X+2 \leq z, X+2 < 0) + P(X \in [-2, z]) + P(X-2 \leq z, X-2 > 0) \\
 &= P(X+2 < 0) + P(X \in [-z, z]) + P(0 < X-2 \leq z) \\
 &= P(X < -2) + P(X \leq z) - P(X \leq -2) + P(X \leq z+2) - P(X \leq z) \\
 &= F_X(-2) + F_X(z) - F_X(-2) + F_X(z+2) - F_X(z) \\
 &= F_X(z+2)
 \end{aligned}$$

Ex 3: $X \sim U([-1; 1])$ [pour plus de précision pour savoir dans quel intervalle intégrer, voir le Cours/integration.pdf]

1) $Y = X^2$

$$\text{si } y \in \mathbb{R}, F_Y(y) = P(Y \leq y) = P(X^2 \leq y)$$

$$① \text{ si } y \leq 0, F_Y(y) = 0 \quad [\text{le carré est toujours positif}]$$

$$② \text{ si } y > 0, F_Y(y) = P(-\sqrt{y} \leq X \leq \sqrt{y}) = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{2} \cdot \mathbb{I}_{[-1; 1]}(x) dx$$

$$②-1 \text{ si } y \geq 1 \text{ alors: } F_Y(y) = \int_{-1}^1 \frac{1}{2} dx = \left[\frac{x}{2} \right]_{-1}^1 = \frac{1}{2}$$

$$②-2 \text{ si } y \in]0; 1[\text{ alors: } F_Y(y) = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{2} \cdot dx = \left[\frac{x}{2} \right]_{-\sqrt{y}}^{\sqrt{y}} = \frac{2\sqrt{y}}{2} = \sqrt{y}$$

2) $Y = 2-X$

$$\text{si } y \in \mathbb{R}, F_Y(y) = P(Y \leq y) = P(2-X \leq y) = P(X \geq 2-y)$$

$$= \int_{2-y}^{+\infty} \frac{1}{2} \mathbb{I}_{[-1; 1]}(x) dx$$

$$① \text{ si } 2-y \geq 1 \Leftrightarrow y \leq 1$$

$$\text{alors: } F_Y(y) = 0$$

$$② \text{ si } 2-y \leq -1 (\Rightarrow y \geq 3)$$

$$\text{alors: } F_Y(y) = \int_{-1}^1 \frac{1}{2} dx = \left[\frac{x}{2} \right]_{-1}^1 = \frac{1}{2}$$

$$③ \text{ si } y \in]1; 3[\Leftrightarrow -1 \leq 2-y \leq 1$$

$$\text{alors: } F_Y(y) = \int_{2-y}^1 \frac{1}{2} \mathbb{I}_{[-1; 1]}(x) dx = \int_{2-y}^1 \frac{1}{2} \mathbb{I}_{[-1; 1]}(x) dx = \left[\frac{x}{2} \right]_{2-y}^1 = \frac{1-2+y}{2} = \frac{y-1}{2}$$

$$3) Y = e^X$$

sit $y \in \mathbb{R}$, $F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(e^X \leq y)$

$$\textcircled{1} \text{ in } y \leq 0 \text{ also } F_Y(y) = 0$$

$$\textcircled{2} \text{ in } y > 0 \text{ also: } F_Y(y) = \mathbb{P}(X \leq \ln(y)) = \int_{-\infty}^{\ln y} \frac{1}{2} \mathbb{I}_{[-1,1]}(x) dx$$

$$\textcircled{2.1} \text{ in } \ln y \leq -1 \Leftrightarrow y \leq \frac{1}{e}$$

$$F_Y(y) = 0$$

$$\textcircled{2.2} \text{ in } \ln y \geq 1 \Leftrightarrow y \geq e$$

$$\text{also: } F_Y(y) = \int_{-1}^y \frac{1}{2} dx = \left[\frac{x}{2} \right]_{-1}^y = \frac{1}{2} (y + 1)$$

$$\textcircled{2.3} \text{ in } y \in \left[\frac{1}{e}; e \right] \Leftrightarrow -1 < \ln y < 1$$

$$\text{also: } F_Y(y) = \int_{-\infty}^{\ln y} \frac{1}{2} \mathbb{I}_{[-1,1]}(x) dx = \int_{-1}^{\ln y} \frac{1}{2} dx = \left[\frac{x}{2} \right]_{-1}^{\ln y} = \frac{\ln y + 1}{2}$$

$$4) Y = \ln |X|$$

sit $y \in \mathbb{R}$, $F_Y(y) = \mathbb{P}(\ln |X| \leq y) = \mathbb{P}(|X| \leq e^y) = \mathbb{P}(-e^y \leq X \leq e^y)$

$$= \int_{-e^y}^{e^y} \frac{1}{2} \mathbb{I}_{[-1,1]}(x) dx$$

$$\textcircled{1} \text{ in } e^y \geq 1 \Leftrightarrow y \geq 0$$

$$\text{also: } F_Y(y) = \int_{-1}^y \frac{1}{2} dx = \left[\frac{x}{2} \right]_{-1}^y = \frac{1}{2} (y + 1)$$

$$\textcircled{2} \text{ in } e^y < 1 \Leftrightarrow y < 0$$

$$\text{also: } F_Y(y) = \int_{-e^y}^{e^y} \frac{1}{2} dx = \left[\frac{x}{2} \right]_{-e^y}^{e^y} = \frac{e^y}{2}$$

ex 4:

$$\xi: \Omega \rightarrow \{-1,1\} \text{ and } \mathbb{P}(\xi = -1) = \mathbb{P}(\xi = 1) = \frac{1}{2}$$

X et ξ sont indépendantes [parce que: $\mathbb{P}(X \in [a,b], \xi = -1) = \mathbb{P}(X \in [a,b]) \cdot \mathbb{P}(\xi = -1)$] et $Y = X + \xi$

1) sit $y \in \mathbb{R}$,

$$\begin{aligned} F_Y(y) &= \mathbb{P}(Y \leq y) = \mathbb{P}(X + \xi \leq y) = \mathbb{P}(X + \xi \leq y, \xi = -1) + \mathbb{P}(X + \xi \leq y, \xi = 1) \\ &= \mathbb{P}(X - 1 \leq y, \xi = -1) + \mathbb{P}(X + 1 \leq y, \xi = 1) \\ &= [\mathbb{P}(X \leq y - 1) \cdot \mathbb{P}(\xi = -1) + \mathbb{P}(X \leq y + 1) \cdot \mathbb{P}(\xi = 1)] \text{ car } X \text{ et } \xi \text{ sont indépendantes} \\ &= [\mathbb{P}(X \leq y - 1) \cdot \frac{1}{2} + \mathbb{P}(X \leq y + 1) \cdot \frac{1}{2}] \\ &= (F_X(y+1) + F_X(y-1)) \cdot \frac{1}{2} \end{aligned}$$

$$2) f_Y(y) = (F_Y(y))' = \frac{1}{2} [f_X(y+1) + f_X(y-1)], \forall y \in \mathbb{R}$$

Ex 5: $X \sim \mathcal{E}(\lambda)$

dac: $f_X: \mathbb{R} \rightarrow \mathbb{R}^+$
 $x \mapsto \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{else} \end{cases}$

V.a.s, $f_X(x) = \lambda e^{-\lambda x} \mathbb{I}_{[0, +\infty]}(x)$

V.a.t t et s ≥ 0 ,

$$\mathbb{P}(X > t+s | X > s) = \frac{\mathbb{P}(X > t+s, X > s)}{\mathbb{P}(X > s)} = \frac{\mathbb{P}(X > t+s)}{\mathbb{P}(X > s)}$$

V.a.t u $\in \mathbb{R}^+$

$$\mathbb{P}(X > u) = \int_u^{+\infty} f_X(x) dx = \int_u^{+\infty} \lambda e^{-\lambda x} \mathbb{I}_{[0, +\infty]}(x) dx = \int_u^{+\infty} \lambda e^{-\lambda x} dx = \left[-e^{-\lambda x} \right]_u^{+\infty}$$

$$\left[(e^{-\lambda x})' \right] = -\lambda e^{-\lambda x}$$

$$= \lim_{x \rightarrow +\infty} -\frac{1}{e^{-\lambda x}} + e^{-\lambda u}$$

$$= e^{-\lambda u}$$

dac: $\mathbb{P}(X > t+s | X > s) = \frac{e^{-\lambda(t+s)}}{e^{-\lambda s}} = e^{-\lambda t} = \underline{\mathbb{P}(X > t)}$