

ex 1:

1)

(a) $X_i \sim \mathcal{U}([0; a])$

$$\underline{E[X_i]} = \int_{-\infty}^{+\infty} x f_{X_i}(x) dx = \int_{-\infty}^{+\infty} x \frac{1}{a} \mathbb{1}_{[0; a]}(x) dx = \int_0^a \frac{x}{a} dx = \left[\frac{x^2}{2a} \right]_0^a = \frac{a^2}{2a} = \underline{\frac{a}{2}}$$

$$V[X_i] = E[X_i^2] - (E[X_i])^2 = E[X_i^2] - \frac{a^2}{4}$$

$$E[X_i^2] = \int_{-\infty}^{+\infty} x^2 f_{X_i}(x) dx = \int_{-\infty}^{+\infty} x^2 \frac{1}{a} \mathbb{1}_{[0; a]}(x) dx = \int_0^a \frac{x^2}{a} dx = \left[\frac{x^3}{3a} \right]_0^a = \frac{a^3}{3a} = \frac{a^2}{3}$$

$$\text{donc: } \underline{V[X_i]} = \frac{a^2}{3} - \frac{a^2}{4} = \underline{\frac{a^2}{12}}$$

$$(b) \underline{E[A_n]} = \frac{2}{n} \sum_{i=1}^n E[X_i] = \frac{2}{n} n E[X_i] = \underline{a}$$

$$\underline{V[A_n]} = V\left[\frac{2}{n} \sum_{i=1}^n X_i\right] = \frac{4}{n^2} V\left[\sum_{i=1}^n X_i\right]$$

$$= \frac{4}{n^2} V[X_i] \text{ car les } (X_i)_{i=1, \dots, n} \text{ sont ind.}$$

$$= \frac{4}{n^2} \cdot n V[X_1] \text{ car les } (X_i)_{i=1, \dots, n} \text{ sont de même loi}$$

$$= \frac{4}{n} \cdot \frac{a^2}{12} = \underline{\frac{a^2}{3n}}$$

$$\text{Biais } [A_n] = E[A_n] - a = a - a = 0 \quad \underline{\text{c'est 1 estimateur sans biais de } a.}$$

$$2) B_n = \max(X_1, \dots, X_n)$$

(a) soit $t \in \mathbb{R}$,

$$F_{X_i}(t) = P(X_i \leq t) = \int_{-\infty}^t f_{X_i}(x) dx = \int_{-\infty}^t \frac{1}{a} \mathbb{1}_{[0; a]}(x) dx = \frac{1}{a} \int_{-\infty}^t \mathbb{1}_{[0; a]}(x) dx$$

$$\textcircled{1} \underline{\text{si } t \leq 0, F_{X_i}(t) = 0}$$

$$\textcircled{2} \underline{\text{si } t > 0, F_{X_i}(t) = \frac{1}{a} \int_0^t \mathbb{1}_{[0; a]}(x) dx}$$

$$\textcircled{2-1} \underline{\text{si } t \geq a, F_{X_i}(t) = \int_0^a \frac{1}{a} dx = \left[\frac{x}{a} \right]_0^a = \underline{1}}$$

$$\textcircled{2-2} \underline{\text{si } t \in]0; a[, F_{X_i}(t) = \frac{1}{a} \int_0^t dx = \frac{1}{a} \left[x \right]_0^t = \underline{\frac{t}{a}}}$$

$$\begin{aligned}
 * P(B_n \leq t) &= P(\max(X_1, \dots, X_n) \leq t) = P(X_1 \leq t, X_2 \leq t, \dots, X_n \leq t) \\
 &= \prod_{i=1}^n P(X_i \leq t) \text{ car } X_i \text{ sont indépendantes} \\
 &= [P(X_1 \leq t)]^n \text{ car } X_i \text{ ont la même loi} \\
 &= F_{X_1}(t)^n
 \end{aligned}$$

$$\forall t \in]0, a[, P(B_n \leq t) = \frac{t^n}{a^n}$$

$$\forall t \geq a, P(B_n \leq t) = 1^n = 1$$

$$(b) \forall t \in]0, a[, F_{B_n}(t) = P(B_n \leq t) = \frac{t^n}{a^n}$$

$$\forall t \in]0, a[, f_{B_n}(t) = (F_{B_n}(t))' = n \cdot \frac{t^{n-1}}{a^n}$$

On admet que: la densité de B_n est: $f_{B_n}(t) = n \frac{t^{n-1}}{a^n} \mathbb{1}_{]0, a[}(t)$

$$\begin{aligned}
 (c) * E[B_n] &= \int_{-\infty}^{+\infty} x \cdot f_{B_n}(x) dx = \int_{-\infty}^{+\infty} x \cdot n \frac{x^{n-1}}{a^n} \mathbb{1}_{]0, a[}(x) dx = \int_0^a x \cdot n \frac{x^{n-1}}{a^n} dx \\
 &= \int_0^a n \cdot \frac{x^n}{a^n} dx = \frac{n}{a^n} \cdot \left[\frac{x^{n+1}}{n+1} \right]_0^a = \frac{n}{a^n} \cdot \frac{a^{n+1}}{n+1} = \frac{n}{n+1} \cdot a \\
 \text{Biais}(B_n) &= E[B_n] - a = \frac{n}{n+1} \cdot a - a = \frac{-a}{n+1}
 \end{aligned}$$

$$* \text{On pose: } B_n^* = \frac{n+1}{n} \cdot B_n$$

$$\text{on a alors: } E[B_n^*] = \frac{n+1}{n} \cdot E[B_n] = \frac{n+1}{n} \cdot \frac{n}{n+1} \cdot a = a$$

$$\text{et: Biais}[B_n^*] = E[B_n^*] - a = a - a = 0$$

$$\begin{aligned}
 (d) V[B_n^*] &= \left(\frac{n+1}{n}\right)^2 V[B_n] = \left(\frac{n+1}{n}\right)^2 (E[B_n^2] - E[B_n]^2) = \left(\frac{n+1}{n}\right)^2 (E[B_n^2] - \left(\frac{na}{n+1}\right)^2) \\
 &= \left(\frac{n+1}{n}\right)^2 E[B_n^2] - a^2
 \end{aligned}$$

$$\begin{aligned}
 E[B_n^2] &= \int_{-\infty}^{+\infty} x^2 f_{B_n}(x) dx = \int_{-\infty}^{+\infty} x^2 \cdot n \frac{x^{n-1}}{a^n} \mathbb{1}_{]0, a[}(x) dx = \int_0^a \frac{n x^{n+1}}{a^n} dx \\
 &= \frac{n}{a^n} \left[\frac{x^{n+2}}{n+2} \right]_0^a \\
 &= \frac{n}{a^n} \frac{a^{n+2}}{n+2} = \frac{a^2 n}{n+2}
 \end{aligned}$$

$$\begin{aligned} \underline{V[B_n^*]} &= \left(\frac{n+1}{n}\right)^2 \frac{a^2 n}{n+2} - a^2 = \frac{(n+1)^2}{n(n+2)} a^2 - 1 = a^2 \left[\frac{n^2+2n+1}{n^2+2n} - \frac{n^2+2n}{n^2+2n} \right] \\ &= a^2 \cdot \frac{1}{n^2+2n} = \underline{\underline{\frac{a^2}{n(n+2)}}} \end{aligned}$$

$$3) \mathbb{E}[B_n^*] = \mathbb{E}[A_n] = a$$

résumés,

$$n^2+2n-3n = n^2-n = n(n-1) > 0 \text{ pour } n \text{ suffisamment grand}$$

$$\text{donc: } n^2+2n > 3n \Rightarrow \frac{1}{n^2+2n} < \frac{1}{3n} \Rightarrow \frac{a^2}{n^2+2n} < \frac{a^2}{3n} \Rightarrow \underline{V[B_n^*]} < \underline{V[A_n]}$$

On conseille d'utiliser B_n^* car ses valeurs sont moins dispersées que celles de A_n .

4)

* pour A_n : $\hat{a} = \frac{2}{n} (x_2 + \dots + x_n)$ [les $(x_i)_{i=2, \dots, n}$ sont les valeurs réellement observées]

$$= \frac{2}{20} \cdot 120 = \underline{12}$$

$$\hat{V}[A_n] = \frac{(\hat{a})^2}{3n} = \frac{12 \times 12}{3 \cdot 20} = \frac{4 \times 6}{10} = \frac{12}{5} = \underline{2,4}$$

* pour B_n^* : $\hat{a} = \frac{n+1}{n} \cdot \max(x_2, \dots, x_n) = \frac{21}{20} \cdot 11 = 11,55 \approx \underline{12}$

$$\hat{V}[B_n^*] = \frac{(\hat{a})^2}{n \cdot (n+2)} = \frac{21 \cdot 21 \cdot 11 \cdot 11}{20 \cdot 20 \cdot 20 \cdot 22} = \frac{21^2 \cdot 11}{20^3 \cdot 2} = \underline{0,30}$$

Les 2 estimateurs donnent la même estimation des temps d'attente (12 minutes) et la variance du premier est plus grande que celle du second.

ex 2: $Y = 2X+1$

1) soit $y \in \mathbb{R}$, $F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(2X+1 \leq y) = \mathbb{P}\left(X \leq \frac{y-1}{2}\right) = F_X\left(\frac{y-1}{2}\right)$

2) soit $z \in \mathbb{R}$, $F_Z(z) = \mathbb{P}(Z \leq z) = \mathbb{P}(Z \leq z, X < -2) + \mathbb{P}(Z \leq z, X \in [-2, 2]) + \mathbb{P}(Z \leq z, X > 2)$
 $= \mathbb{P}(X+2 \leq z, X < -2) + \mathbb{P}(0 \leq z, X \in [-2, 2]) + \mathbb{P}(X-2 \leq z, X > 2)$

① si $z < 0$, $F_Z(z) = \mathbb{P}(X+2 \leq z, X+2 < 0) + \underbrace{\mathbb{P}(0 \leq z, X \in [-2, 2])}_0 + \underbrace{\mathbb{P}(X-2 \leq z, X-2 > 0)}_0$
 $= \mathbb{P}(X+2 \leq z) + 0 + 0$
 $= \mathbb{P}(X \leq z-2) = \underline{F_X(z-2)}$

(4)

$$\begin{aligned}
 \textcircled{2} \text{ si } z \geq 0; F_Z(z) &= P(X+z \leq z, X+z < 0) + P(X \in [-2, 2]) + P(X-z \leq z, X-z > 0) \\
 &= P(X+z < 0) + P(X \in [-2, 2]) + P(0 < X-z \leq z) \\
 &= P(X < -z) + P(X \leq z) - P(X \leq -z) + P(2 < X \leq z+2) \\
 &= P(X \leq -z) + P(X \leq z) - P(X \leq -z) + P(X \leq z+2) - P(X \leq z) \\
 &= F_X(-z) + F_X(z) - F_X(-z) + F_X(z+2) - F_X(z) \\
 &= F_X(z+2)
 \end{aligned}$$

ex 3: $X \sim \mathcal{U}([-1, 1])$ [pour plus de détails voir dans quel intervalle intégrer, voir le cours/integration.pdf]

$$1) Y = X^2$$

$$\text{soit } y \in \mathbb{R}, F_Y(y) = P(Y \leq y) = P(X^2 \leq y)$$

$$\textcircled{1} \text{ si } y \leq 0, F_Y(y) = 0 \text{ [la carré est toujours positif]}$$

$$\textcircled{2} \text{ si } y > 0, F_Y(y) = P(-\sqrt{y} \leq X \leq \sqrt{y}) = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{2} \cdot \mathbb{1}_{[-1, 1]}(x) dx$$

$$\textcircled{2-1} \text{ si } y \geq 1 \text{ alors: } F_Y(y) = \int_{-1}^1 \frac{1}{2} dx = \left[\frac{x}{2} \right]_{-1}^1 = 1$$

$$\textcircled{2-2} \text{ si } y \in]0; 1[\text{ alors: } F_Y(y) = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{2} \cdot dx = \left[\frac{x}{2} \right]_{-\sqrt{y}}^{\sqrt{y}} = \frac{2\sqrt{y}}{2} = \sqrt{y}$$

$$2) Y = 2 - X$$

$$\text{soit } y \in \mathbb{R}, F_Y(y) = P(Y \leq y) = P(2 - X \leq y) = P(X \geq 2 - y)$$

$$= \int_{2-y}^{+\infty} \frac{1}{2} \mathbb{1}_{[-1, 1]}(x) dx$$

$$\textcircled{1} \text{ si } 2 - y \geq 1 \Leftrightarrow y \leq 1$$

$$\text{alors: } F_Y(y) = 0$$

$$\textcircled{2} \text{ si } 2 - y \leq -1 \Leftrightarrow y \geq 3$$

$$\text{alors: } F_Y(y) = \int_{-1}^1 \frac{1}{2} dx = \left[\frac{x}{2} \right]_{-1}^1 = 1$$

$$\textcircled{3} \text{ si } y \in]1, 3[\Leftrightarrow -1 < 2 - y < 1$$

$$\text{alors: } F_Y(y) = \int_{2-y}^{+\infty} \frac{1}{2} \mathbb{1}_{[-1, 1]}(x) dx = \int_{2-y}^1 \frac{1}{2} \mathbb{1}_{[-1, 1]}(x) dx = \left[\frac{x}{2} \right]_{2-y}^1 = \frac{1 - 2 + y}{2} = \frac{y - 1}{2}$$

3) $Y = e^X$
 soit $y \in \mathbb{R}$, $F_Y(y) = P(Y \leq y) = P(e^X \leq y)$

① si $y \leq 0$ alors $F_Y(y) = 0$
 ② si $y > 0$ alors: $F_Y(y) = P(X \leq \ln(y)) = \int_{-\infty}^{\ln y} \frac{1}{2} \mathbb{1}_{[-1,1]}(x) dx$

2.1 si $\ln y \leq -1 \Leftrightarrow y \leq \frac{1}{e}$

$F_Y(y) = 0$

2.2 si $\ln y \geq 1 \Leftrightarrow y \geq e$

alors: $F_Y(y) = \int_{-1}^1 \frac{1}{2} dx = \left[\frac{x}{2} \right]_{-1}^1 = 1$

2.3 si $y \in]\frac{1}{e}; e[\Leftrightarrow -1 < \ln y < 1$

alors: $F_Y(y) = \int_{-\infty}^{\ln y} \frac{1}{2} \mathbb{1}_{[-1,1]}(x) dx = \int_{-1}^{\ln y} \frac{1}{2} dx = \left[\frac{x}{2} \right]_{-1}^{\ln y} = \frac{\ln y + 1}{2}$

4) $Y = \ln |X|$

soit $y \in \mathbb{R}$, $F_Y(y) = P(\ln |X| \leq y) = P(|X| \leq e^y) = P(-e^y \leq X \leq e^y)$
 $= \int_{-e^y}^{e^y} \frac{1}{2} \mathbb{1}_{[-1,1]}(x) dx$

① si $e^y \geq 1 \Leftrightarrow y \geq 0$

alors: $F_Y(y) = \int_{-1}^1 \frac{1}{2} dx = \left[\frac{x}{2} \right]_{-1}^1 = 1$

② si $e^y < 1 \Leftrightarrow y < 0$

alors: $F_Y(y) = \int_{-e^y}^{e^y} \frac{1}{2} dx = \left[\frac{x}{2} \right]_{-e^y}^{e^y} = e^y$

ex 4:

$\xi: \Omega \rightarrow \{-1, 1\}$ avec $P(\xi = -1) = P(\xi = 1) = \frac{1}{2}$
 X et ξ sont indépendantes [général: $P(X \in [a, b], \xi = -1) = P(X \in [a, b]) \cdot P(\xi = -1)$] et $Y = X + \xi$

1) soit $y \in \mathbb{R}$,

$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P(X + \xi \leq y) = P(X + \xi \leq y, \xi = -1) + P(X + \xi \leq y, \xi = 1) \\ &= P(X - 1 \leq y, \xi = -1) + P(X + 1 \leq y, \xi = 1) \\ &= [P(X \leq y+1) \cdot P(\xi = -1) + P(X \leq y-1) \cdot P(\xi = 1)] \text{ car } X \text{ et } \xi \text{ sont indépendantes} \\ &= P(X \leq y+1) \cdot \frac{1}{2} + P(X \leq y-1) \cdot \frac{1}{2} \\ &= (F_X(y+1) + F_X(y-1)) \cdot \frac{1}{2} \end{aligned}$$

2) $f_Y(y) = (F_Y(y))' = \frac{1}{2} [f_X(y+1) + f_X(y-1)], \forall y \in \mathbb{R}$

Ex 5:

$X \sim \mathcal{E}(\lambda)$

$$\text{donc: } f_X: \mathbb{R} \rightarrow \mathbb{R}^+ \\ x \mapsto \begin{cases} \lambda e^{-\lambda x} & \text{si } x \geq 0 \\ 0 & \text{sinon} \end{cases}$$

$$\forall t \in \mathbb{R}, f_X(x) = \lambda e^{-\lambda x} \mathbb{1}_{[0, +\infty[}(x) dx$$

Soient t et $s \geq 0$,

$$\mathbb{P}(X > t+s | X > s) = \frac{\mathbb{P}(X > t+s, X > s)}{\mathbb{P}(X > s)} = \frac{\mathbb{P}(X > t+s)}{\mathbb{P}(X > s)}$$

Soit $u \in \mathbb{R}^+$

$$\mathbb{P}(X > u) = \int_u^{+\infty} f_X(x) dx = \int_u^{+\infty} \lambda e^{-\lambda x} \mathbb{1}_{[0, +\infty[}(x) dx = \int_u^{+\infty} \lambda e^{-\lambda x} dx = \left[-e^{-\lambda x} \right]_u^{+\infty}$$

$$\left[(e^{-\lambda x})' = -\lambda e^{-\lambda x} \right]$$

$$= \lim_{x \rightarrow +\infty} -\frac{1}{e^{-\lambda x}} + e^{-\lambda u}$$

$$= e^{-\lambda u}$$

$$\text{donc: } \underline{\mathbb{P}(X > t+s | X > s)} = \frac{e^{-\lambda(t+s)}}{e^{-\lambda s}} = e^{-\lambda t} = \underline{\mathbb{P}(X > t)}$$