Egyptian Palindromic Fractions?

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Who am I?

- Space Farmer: I use radar and multispectral images from satellites to help farmers.
- Erdös number 3:




- Olsen's inequality (Generalisation of the Hardy-Littlewood-Sobolev Inequality):

In studying a Schrödinger equation with perturbed potentials W on $\mathbb{R}^n$ (particularly for $n = 3$), P. A. Olsen [20] proved the following result.

Theorem 3.1 (Olsen). *For* $1 < p < \dfrac{n}{\alpha}$ *and* $0 \leq \lambda < n - \alpha p$, *we have*

$$\|W \cdot I_\alpha f\|_{p,\lambda} \leq C_{p,\lambda} \|W\|_{(n-\lambda)/\alpha,\lambda} \|f\|_{p,\lambda},$$

Are you smarter than a fifth grader?

This was a fifth grade extra credit problem:
Find distinct positive integers a, b, c, d, e such that

$$1 = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e}.$$

How many such solutions can you find?

Warm up

How many positive integer solutions a, b can you find to

$$\frac{1}{6} = \frac{1}{a} + \frac{1}{b}?$$

Solution:

1. $\frac{1}{6} = \frac{1}{} + \frac{1}{}$
2. $\frac{1}{6} = \frac{1}{} + \frac{1}{}$
3. $\frac{1}{6} = \frac{1}{} + \frac{1}{}$
4. $\frac{1}{6} = \frac{1}{} + \frac{1}{}$
5. $\frac{1}{6} = \frac{1}{} + \frac{1}{}$

Solve for a, b

$$\frac{1}{n} = \frac{1}{a} + \frac{1}{b}$$

Unit Fraction Expansion # 1

Solve for a, b

$$
\begin{aligned}
\frac{1}{n} &= \frac{1}{a} + \frac{1}{b} \\
\frac{ab\cancel{n}}{\cancel{n}} &= \frac{\cancel{a}bn}{\cancel{a}} + \frac{a\cancel{b}n}{\cancel{b}} \\
ab &= bn + an
\end{aligned}
$$

Unit Fraction Expansion # 1

Solve for a, b

$$\begin{aligned} \frac{1}{n} &= \frac{1}{a} + \frac{1}{b} \\ ab &= bn + an \\ ab - bn - an &= 0 \end{aligned}$$

Unit Fraction Expansion # 1

Solve for a, b

$$
\begin{aligned}
\frac{1}{n} &= \frac{1}{a} + \frac{1}{b} \\
ab &= bn + an \\
ab - bn - an &= 0 \\
n^2 - an - bn + ab &= n^2 \\
(a-n)(b-n) &= n^2
\end{aligned}
$$

Take any factorization of $n^2 = kl$ and solve $a - n = k$, $b - n = l$ giving $a = n + k$, $b = n + l$.

Example: $n = 6$

$6^2 = 1 \times 36$	$a = 6 + 1 = 7$	$b = 6 + 36 = 42$	$\frac{1}{6} = \frac{1}{7} + \frac{1}{42}$
$6^2 = 2 \times 18$	$a = 8$	$b = 24$	$\frac{1}{6} = \frac{1}{8} + \frac{1}{24}$
$6^2 = 3 \times 12$	$a = 9$	$b = 18$	$\frac{1}{6} = \frac{1}{9} + \frac{1}{18}$
$6^2 = 4 \times 9$	$a = 10$	$b = 15$	$\frac{1}{6} = \frac{1}{10} + \frac{1}{15}$
$6^2 = 6 \times 6$	$a = 12$	$b = 12$	$\frac{1}{6} = \frac{1}{12} + \frac{1}{12}$

Egyptian Fractions

According to Wikipedia The Free Encyclopedia "An Egyptian fraction is a sum of distinct unit fractions such as

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{10}."$$

Every fraction has at least one Egyptian fraction representation.

Why is this possible?

We will look at 5/6 (ignore that $5/6 = 1/2 + 1/3$). When we have clashes we can use the unit fraction expansion technique:

$$\begin{aligned}\frac{5}{6} &= \frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}\\ &= \frac{1}{6}+\left(\frac{1}{7}+\frac{1}{42}\right)+\left(\frac{1}{8}+\frac{1}{24}\right)+\left(\frac{1}{9}+\frac{1}{18}\right)+\left(\frac{1}{10}+\frac{1}{15}\right)\\ &= \frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{15}+\frac{1}{18}+\frac{1}{24}+\frac{1}{42}.\end{aligned}$$

I'll let you imagine how a proof could go...

Palindromes with 1,2 digits

A palindrome is a number that is the same when read forwards and backwards. For example

754838457

1 digit			2 digits		
1	2	3	11	22	33
4	5	6	44	55	66
7	8	9	77	88	99

9 palindromes   9 palindromes

Palindromes with 3 digits?

3 digits

101	202	303	404	505	606	707	808	909
111	212	313	414	515	616	717	818	919
121	222	323	424	525	626	727	828	929
131	232	333	434	535	636	737	838	939
141	242	343	444	545	646	747	848	949
151	252	353	454	555	656	757	858	959
161	262	363	464	565	666	767	868	969
171	272	373	474	575	676	777	878	979
181	282	383	484	585	686	787	888	989
191	292	393	494	595	696	797	898	999

90 palindromes

Palindromes with 4 digits?

4 digits

1001	2002	3003	4004	5005	6006	7007	8008	9009
1111	2112	3113	4114	5115	6116	7117	8118	9119
1221	2222	3223	4224	5225	6226	7227	8228	9229
1331	2332	3333	4334	5335	6336	7337	8338	9339
1441	2442	3443	4444	5445	6446	7447	8448	9449
1551	2552	3553	4554	5555	6556	7557	8558	9559
1661	2662	3663	4664	5665	6666	7667	8668	9669
1771	2772	3773	4774	5775	6776	7777	8778	9779
1881	2882	3883	4884	5885	6886	7887	8888	9889
1991	2992	3993	4994	5995	6996	7997	8998	9999

90 palindromes

Set of palindromes $\mathcal{A}$ with n digits: $\mathcal{A}_n$.

$$
\begin{aligned}
R = \sum_{a \in \mathcal{A}} \frac{1}{a} &\leq \sum_n \sum_{a \in \mathcal{A}_n} \frac{1}{10^{n-1}} \\
&= \sum_n \frac{|\mathcal{A}_n|}{10^{n-1}} \\
&= 9 + \frac{9}{10} + \frac{9\cancel{0}}{10\cancel{0}} + \frac{9\cancel{0}}{100\cancel{0}} + \frac{90\cancel{0}}{100\cancel{00}} + \frac{90\cancel{0}}{1000\cancel{00}} + \cdots \\
&= 9 + 2\left(\frac{9}{10} + \frac{9}{100} + \cdots\right) \\
&= 9 + \frac{18}{10}\left(1 + \frac{1}{10} + \frac{1}{100} + \cdots\right) \\
&= 9 + \frac{18}{10} \frac{1}{1 - \frac{1}{10}} \\
&= 9 + 2 \\
&= 11
\end{aligned}
$$

Reciprocal Sum of Palindromes #2

$R = 3.37028325949737332049215729 85\ldots$, therefore $\frac{7}{2} > R$ cannot be written as a sum of reciprocal palindromes.

Repeated palindromes

If we allow the palindromes to be repeated the sum of reciprocals can be arbitrarily large, so perhaps that will work?

From an representation perspective that would mean we can write

$$\frac{5}{6} = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6}$$

which isn't too Egyptian. Nevertheless, let's see where this all takes us.

Try it yourself

Let's take a break from slides and try it ourselves. Can we find $\frac{1}{n}$ as a sum of palindromic reciprocals for $10 \leq n \leq 100$.

See if you can find some reciprocal palindromic representations for $\frac{1}{n}$ for some integers $n > 10$. Here are some examples

$$
\begin{aligned}
\frac{1}{12} &= \frac{2}{33} + \frac{1}{44} \\
\frac{1}{13} &= \frac{1}{22} + \frac{1}{33} + \frac{1}{858} \\
\frac{1}{16} &= \frac{1}{22} + \frac{2}{121} + \frac{1}{2112} + \frac{1}{23232} \\
\frac{1}{81} &= \frac{1}{111} + \frac{1}{333} + \frac{1}{3003} + \frac{1}{2002002} + \frac{1}{6006006} + \frac{1}{3006006003} \\
&\quad + \frac{1}{4008008004} + \frac{1}{20222222202} + \frac{1}{27999999972} \\
\frac{1}{98} &= \frac{1}{99} + \frac{1}{9999} + \frac{1}{606606} + \frac{1}{707707}
\end{aligned}
$$

Palindromes in binary

For binary numbers the answer is still **no**. Why?

1. Palindromes can not end with 0, so all binary palindromes are odd.
2. A finite sum of reciprocal odd numbers can never equal $\frac{1}{2}$.

The sticking point is numbers ending with 0. Let's see if we can write $\frac{1}{10}$ as a sum of reciprocal palindromes?

Sum of reciprocal palindromes in base 10

The sticking point is numbers ending with 0. Let's see if we can write $\frac{1}{10}$ as a sum of reciprocal palindromes?

$$\begin{aligned}\frac{1}{10} &= \frac{1}{22}+\frac{1}{55}+\frac{1}{55}+\frac{1}{55}\\ &= \frac{1}{22}+\frac{3}{55}\end{aligned}$$

Since we are allowing repeats, we only need to check fractions of the form $\frac{1}{n}$.

Anti-Egyptian representations

One key to finding palindromic representations for a number n is to find a palindrome that is a multiple of n so that we can write

$$\frac{1}{n} = \frac{m}{\text{palindrome}}.$$

This is as far from an Egyptian representation as we can get. Essentially it's an *anti-Egyptian representation*.

Some clues?

$$\begin{aligned}\frac{1}{37} &= \frac{3}{111} \\ &= \frac{27}{999} \\ \frac{1}{81} &= \frac{12,345,679}{999,999,999}\end{aligned}$$

Fermat's little theorem

Fermat's Little Theorem (FLT) states that if p is a prime number then

$$a^{p-1} \equiv 1 (\mathrm{mod} p) \qquad \text{if } \gcd(a, p) = 1.$$

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Taking $a = 10$ and $p \neq 2, 5$ (since $\gcd(a, p) = 1$) that we see that p divides $10^{p-1} - 1 = 999 \cdots 999$ ($p - 1$ 9's).

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Example: $13|999, 999, 999, 999$ so we can conclude

$$\frac{1}{13} = \frac{76,923,076,923}{999,999,999,999}$$

Since 2,5 are already palindromes, the reciprocal of any prime can be written as a sum of reciprocal palindromes.

Prime Powers other than 2 and 5

Powers of primes also works. For primes $p \neq 2, 5$ we have $p^2 | 10^{p(p-1)} - 1$ since

$$10^{p(p-1)} - 1 = (10^{p-1} - 1)(1 + 10^{(p-1)} + 10^{2(p-1)} + \cdots + 10^{(p-1)(p-1)})$$

and $p | (10^{p-1} - 1)$ by FLT and

$$\begin{aligned} 1 + 10^{(p-1)} + 10^{2(p-1)} + \cdots + 10^{(p-1)(p-1)} &\equiv \\ 1 + 1^1 + 1^2 + \cdots + 1^{p-1} &\equiv p \bmod p \\ &\equiv 0 \bmod p. \end{aligned}$$

Since each of the factors are divisible by p, p^2 divides the product. Homework? Figure out cubes and higher prime powers.

Example with 49

We have $7 \mid (10^6 - 1)$ therefore $7^2 \mid (10^{42} - 1)$ and we get

$$\frac{1}{49} = \frac{20408163265306122448979591836734693877551}{999999999999999999999999999999999999999999}.$$

So, a mere 20 duodecillion unit palindrome reciprocals are needed.

Numbers not divisible by 2 or 5

If $\gcd(a, b) = 1$ and $a|10^k - 1$ and $b|10^l - 1$, then $ab|(10^{kl} - 1)$ since

$$10^{kl} - 1 = (10^k - 1)(1 + 10^k + \cdots + 10^{k(l-1)})$$

and

$$10^{kl} - 1 = (10^l - 1)(1 + 10^l + \cdots + 10^{l(k-1)})$$

Therefore if $gcd(n, 10) = 1$ we can build a repdigit that is a multiple of n from its representation as prime powers.

Example with 119

Take $a = 7$, $b = 17$, $n = ab = 7 \times 17 = 119$ then $7 \mid (10^6 - 1)$ and $17|(10^{16} - 1)$, so we have $119 \mid (10^{6\times16} - 1) = (10^{96} - 1)$. In fact it's also the case that $119 \mid (10^{48} - 1)$. Using this we get

$$\frac{1}{119} = \frac{8403361344537815126050420168067226890756302521}{999999999999999999999999999999999999999999999999}$$

An upper bound of 8.4 quattuordecillion unit palindrome reciprocals.

Remark

- We used repdigits of the form $999\cdots999$, but we could have also used repunits of the form $111\cdots111$.
- These representations requires a lot of repeats of some palindromes, and they are not efficient in terms of using the least number of palindromes.
- The hard cases still remains (multiples of 2,5 and 10).

This is a good point to see if we can find representations for $\frac{1}{13}$, $\frac{1}{17}$ and $\frac{1}{19}$ (all difficult numbers between 10 and 20 not dividing 2,5)

The trouble makers

So what to do about powers of 5 or 2?

- Can we multiply $999 \cdots 999$ by a number n such that $5^k | n$ and get a palindrome?
- Can we use $\frac{1}{n} = \frac{1}{a} + \frac{1}{b}$ to split the powers of 2 and 5 apart?

If all that works then maybe we can decompose any fraction into sums of reciprocal palindromes.

Multiplying by $999\cdots999$

Here's an example of multiplying a random number n ($n = 69,745,061$) by $10^k - 1$ for k=12. (It is important that $n < 10^k$)

$$69,745,061 \times (10^{12} - 1) = 69,745,060,999,930,254,939.$$

The first 8 red digits equals $n - 1$, while the blue digits are the *9's complement* of the red:

$$69,745,060 + 30,254,939 = 99,999,999$$

The middle green digits are the remaining $12 - 8$ nines from $10^k - 1$. We will use a *bar* to signify 9's complement, so

$$\overline{n-1} = \overline{69,745,060} = 30,254,939.$$

The digit reversal function

We need a function ρ that reverses the digits of a number:

$$\rho(n) = \rho(69, 745, 061) = 16, 054, 796.$$

For numbers that end with 0 we simply omit the leading 0's: $\rho(120) = 21$.

Palindromes are characterized by being fixed points of ρ:

$$\rho(n) = n.$$

Generating palindromes through multiplication

We saw that multiplying a number n by a repdigit $10^k - 1$ gives the result

$$n \times (10^k - 1) = (n-1)99\cdots99\overline{(n-1)}$$

This number would be a palindrome if the red reverse digits equal the blue, i.e. $\rho(n-1) = \overline{n-1}$. Is there a way to do this in general?

Constructing numbers whose 9 complement is its digit reversal

Here's how we can find n such that $\rho(n-1) = \overline{n-1}$:
Pick a number s not ending with 0 or 9 (why 9?), e.g. $s = 9375$ and compute $\overline{\rho(s)}$:

$$\rho(s) = 5739 \qquad \overline{\rho(s)} = 4260$$

Next we concatenate the two numbers into a bigger number:

$$\overline{n-1} = \{\overline{\rho(s)}, s\} = 42,609,375.$$

We find that

$$n-1 = 57,390,624$$

has the reverse digits of $\overline{n-1}$, so $\rho(n-1) = \overline{n-1}$. This works for *any* s not ending in 0 or 9. Thus $n = 57,390,625$ satisfies $\rho(n-1) = \overline{n-1}$.

An example

Here's another example: If we pick $s = 25$ we get $\rho(s) = 52$, $\overline{\rho(s)} = 47$ and $\overline{n-1} = 4725$. Consequently $n - 1 = 5274$ and $n = 5275$. Let's calculate $n \times (10^5 - 1)$:

$$5275 \times 99,999 = 527,494,725$$

and we can see this is a palindrome.

Making palindromes that are Powers of 5

We design an n such that $5^k | n$ and $n(10^l - 1)$ is a palindrome for **all** l such that $10^l > n$.

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Let $s = 10^k - 5^k$ and $\overline{n-1} = \{\overline{\rho(s)}, s\}$. By construction $\rho(n-1) = \overline{n-1}$ and

$$\overline{n-1} = s+10^k(\overline{\rho(s)}) = (2^k5^k-5^k)+2^k5^k(\overline{\rho(s)}) = 5^k\left(2^k + 2^k\overline{\rho(s)} - 1\right)$$

so 5^k divides $\overline{n-1}$. Since the last $2k$ digits of $n(10^l - 1)$ is $\overline{n-1}$ this means $5^k|n(10^l - 1)$ and therefore $5^k|n$.

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$$\overline{n-1} = s + 10^k(\overline{\rho(s)}) = (2^k 5^k - 5^k) + 2^k 5^k(\overline{\rho(s)}) = 5^k \left(2^k + 2^k \overline{\rho(s)} - 1\right)$$

so 5^k divides $\overline{n-1}$. Since the last $2k$ digits of $n(10^l - 1)$ is $\overline{n-1}$ this means $5^k | n(10^l - 1)$ and therefore $5^k | n$.

Explicitly the function that generates n from k is:

$$\kappa_5(k) \stackrel{\text{def}}{=} 1 + \overline{\{\overline{\rho(10^k - 5^k)}, 10^k - 5^k\}}.$$

Example palindromes that are Powers of 5

We take $k = 4$ as an example, so $s = 10^4 - 5^4 = 1000 - 625 = 9375$ and therefore $\overline{n-1} = 42609375$. Consequently $n = \kappa_5(4) = 57390625$ is a multiple of 5^4 (even a multiple of 5^6 actually!). Since n has 8 digits we must have $l \geq 8$. We have

$$
\begin{aligned}
n(10^8 - 1) &= 5,739,062,442,609,375 \\
n(10^9 - 1) &= 57,390,624,942,609,375 \\
n(10^{10} - 1) &= 573,906,249,942,609,375 \\
n(10^{11} - 1) &= 5,739,062,499,942,609,375
\end{aligned}
$$

Example palindromes that are Powers of 2

It works the same way for powers of 2. For $k = 3$ we have $s = 10^3 - 2^3 = 1000 - 8 = 992$ and $\overline{n-1} = 700992$. Consequently $n = \kappa_2(3) = 299008$ is a multiple of 2^3 (even a multiple of 2^{12} actually!). Since n has 6 digits we must have $l \geq 6$. We have

$$
\begin{aligned}
n(10^6 - 1) &= 299,007,700,992 \\
n(10^8 - 1) &= 29,900,799,700,992 \\
n(10^{10} - 1) &= 2,990,079,999,700,992
\end{aligned}
$$

Multiples of powers of 5: Case I

Case I: $n = 5^k m$ with $\gcd(m, 10) = 1$ and $m \mid (10^l - 1)$ and $10^l > 5^k$. In this case $\kappa_5(k)(10^l - 1)$ is a palindrome and $5^k m \mid \kappa_5(k)(10^l - 1)$.

Case I: $n = 5^k m$ with $\gcd(m, 10) = 1$ and $m \mid (10^l - 1)$ and $10^l > 5^k$. In this case $\kappa_5(k)(10^l - 1)$ is a palindrome and $5^k m \mid \kappa_5(k)(10^l - 1)$.

$k = 4$ and $m = 13$: We want to find a palindrome that is a multiple of $n = 13 \times 5^4 = 13 \times 625 = 8125$. We have already seen that $\kappa_5(4) = 57390625$ and by FLT $13 | (10^{12} - 1)$. We have

$$\kappa_5(4) \times (10^{12} - 1) = 57, 390, 624, 999, 942, 609, 375$$

and therefore we have the palindromic representation

$$\frac{1}{8125} = \frac{7, 063, 461, 538, 454, 475}{57, 390, 624, 999, 942, 609, 375}$$

Multiples of powers of 5: Case II

Case II: $10^l < 5^k$: If $m \mid (10^l - 1)$ and l is too small, then use $(10^l - 1) \mid (10^{jl} - 1)$ for $j = 2, 3, 4, \ldots$ and pick j so that $10^{jl} > 5^k$

Multiples of powers of 5: Case II

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As an example consider $n = 11 \times 5^4 = 6875$. We have $11|(10^2 - 1) = 99$, but $10^2 < \kappa_5(4) = 57390625$. We take $j = 4$ so that $11 \mid (10^8 - 1)$ and $10^8 > 57390625$. We have

$$\kappa_5(4) \times (10^8 - 1) = 5,739,062,442,609,375$$

and therefore we have the palindromic representation

$$\frac{1}{6875} = \frac{834,772,718,925}{5,739,062,442,609,375}$$

Numbers not ending in 0

Multiples of powers of 2 work in a similar way to multiples of powers of 5. Since numbers that are multiples of powers of 2 and powers of 5 will end in 0, we have covered all numbers not ending in 0!!

In fact we have shown for all n not ending in 0 that

$$\frac{1}{n} = \frac{b}{\text{palindrome}}$$

where b is an integer. For numbers ending in 0, this will no longer be possible. We will need at least 2 distinct palindromes in the denominator.

Possible $\neq$ practical

Let's understand how bad these decompositions are relative to what is possible. For $n = 65 = 5 \times 13$ the described methods gives

$$\frac{1}{65} = \frac{846,153,846,153}{54,999,999,999,945}$$

Everyone doing math competitions knows that $13 \mid 1001$, so clearly $1/65 = 77/5005$, but the smallest palindrome that is a multiple of 65 is 585

$$\frac{1}{65} = \frac{9}{585}$$

Instead of using 8.5×10^{11} palindromes we only need 9! We can do even better:

$$\frac{1}{65} = \frac{1}{66} + \frac{1}{5005} + \frac{1}{55055} + \frac{1}{66066}.$$

A truly Egyptian Palindromic Fraction needing only 4 palindromes!

Numbers divisible by 10

Let's consider all numbers, i.e. $n = 2^k 5^l m$ where $(m, 10) = 1$. By fraction decomposition we have

$$\frac{1}{n} = \frac{1}{2^k m(2^k + 5^l)} + \frac{1}{5^l m(2^k + 5^l)}$$

Consequently since the number $(2^k + 5^l)$ is not divisible by 2 or 5 we can use the previous result.

As an example take $n = 100$. We have

$$\frac{1}{100} = \frac{1}{4(4+25)} + \frac{1}{25(25+4)} = \frac{1}{116} + \frac{1}{725}$$

We calculate $\kappa_5(2) = 5725$, $\kappa_2(2) = 64$ and $29 \mid (10^{28} - 1)$ (but no lower exponents work) and using a calculator we see

$$\kappa_2(2)(10^{28} - 1) = 639999999999999999999999999936$$

and

$$\kappa_5(2)(10^{28} - 1) = 57249999999999999999999999994275$$

Example for final result #2

The resulting palindromic representation thus becomes

$$\frac{1}{100} = \frac{5517241379310344827586206896}{639999999999999999999999999936} + \frac{78965517241379310344827586199}{57249999999999999999999999994275}$$

A much more compact representation is possible with 7 palindromes instead of the 84 octillion ($8.4 * 10^{28}$) palindromes in the expression above.

$$\frac{1}{100} = \frac{1}{222} + \frac{1}{444} + \frac{1}{575} + \frac{1}{777} + \frac{2}{10101} + \frac{1}{52325}.$$

An Egyptian palindromic fraction with 7 palindromes:

$$\frac{1}{100} = \frac{1}{111} + \frac{1}{2002} + \frac{1}{3003} + \frac{1}{8778} + \frac{1}{40404} + \frac{1}{50505} + \frac{1}{52777725}$$

What questions to answer next?

- Can every unit fraction be written as an Egyptian palindromic fraction?
- What criteria guarantees a fraction can be written as an Egyptian palindromic fraction?
- How many reciprocal palindromes are needed? For $1/n$ is n reciprocal palindromes needed?
- Can you find better or cooler palindromic representations than I did?

Shortest palindromic representations I found

$$
\begin{aligned}
\frac{1}{10} &= \frac{1}{22}+\frac{3}{55}\\
\frac{1}{11} &= \frac{1}{11}\\
\frac{1}{12} &= \frac{2}{33}+\frac{1}{44}\\
\frac{1}{13} &= \frac{1}{22}+\frac{1}{33}+\frac{1}{858}\\
\frac{1}{14} &= \frac{1}{22}+\frac{2}{77}\\
\frac{1}{15} &= \frac{1}{33}+\frac{2}{55}\\
\frac{1}{16} &= \frac{1}{22}+\frac{2}{121}+\frac{1}{2112}+\frac{1}{23232}\\
\frac{1}{17} &= \frac{1}{55}+\frac{3}{77}+\frac{1}{595}\\
\frac{1}{18} &= \frac{1}{22}+\frac{1}{99}\\
\frac{1}{19} &= \frac{1}{33}+\frac{1}{99}+\frac{2}{171}+\frac{1}{1881}
\end{aligned}
$$

Can you improve on the "anti-Egyptian" palindromic fractions $\frac{1}{83} = \frac{9}{747}$ and $\frac{1}{85} = \frac{7}{595}$?

$$\begin{aligned}
\frac{1}{25} &= \frac{1}{33} + \frac{1}{252} + \frac{1}{404} + \frac{1}{505} + \frac{1}{909} + \frac{1}{5775} \\
\frac{1}{43} &= \frac{1}{44} + \frac{1}{2442} + \frac{1}{13431} + \frac{1}{26862} + \frac{1}{210012} + \frac{1}{420024} + \frac{1}{4620264} \\
\frac{1}{61} &= \frac{1}{101} + \frac{1}{171} + \frac{1}{1881} + \frac{1}{8888} + \frac{1}{4214124} + \frac{1}{8428248} \\
\frac{1}{81} &= \frac{1}{111} + \frac{1}{333} + \frac{1}{3003} + \frac{1}{2002002} + \frac{1}{6006006} + \frac{1}{3006006003} \\
&\quad + \frac{1}{4008008004} + \frac{1}{20222222202} + \frac{1}{27999999972} \\
\frac{1}{89} &= \frac{1}{99} + \frac{1}{909} + \frac{1}{33033} + \frac{1}{378873} + \frac{1}{979979} + \frac{1}{1112111} + \frac{1}{316565613} \\
\frac{1}{97} &= \frac{1}{111} + \frac{1}{777} + \frac{1}{111111} + \frac{1}{241142} + \frac{3}{26766762} + \frac{2}{221434122}
\end{aligned}$$

11, 13 and 18 were already Egyptian palindromic fractions. Here are the rest:

$$\begin{aligned}
\frac{1}{10} &= \frac{1}{11}+\frac{1}{121}+\frac{1}{2662}+\frac{1}{3993}+\frac{1}{5445}+\frac{1}{59895}\\
\frac{1}{12} &= \frac{1}{22}+\frac{1}{44}+\frac{1}{66}\\
\frac{1}{14} &= \frac{1}{22}+\frac{1}{77}+\frac{1}{88}+\frac{1}{616}\\
\frac{1}{15} &= \frac{1}{22}+\frac{1}{55}+\frac{1}{505}+\frac{1}{1111}+\frac{1}{6666}\\
\frac{1}{16} &= \frac{1}{22}+\frac{1}{99}+\frac{1}{242}+\frac{1}{363}+\frac{1}{23232}+\frac{1}{69696}\\
\frac{1}{17} &= \frac{1}{22}+\frac{1}{88}+\frac{1}{595}+\frac{1}{5005}+\frac{1}{8008}\\
\frac{1}{19} &= \frac{1}{22}+\frac{1}{171}+\frac{1}{969}+\frac{1}{3663}+\frac{1}{41514}
\end{aligned}$$

Egyptian palindromic fraction challenges

Ironically, we have the simple anti-Egyptian $\frac{1}{83} = \frac{9}{747}$ when allowing repeats. 9 different palindromes are also needed for an Egyptian representation.

$$
\begin{aligned}
\frac{1}{71} &= \frac{1}{77} + \frac{1}{2002} + \frac{1}{3003} + \frac{1}{4004} + \frac{1}{67876} + \frac{1}{3333333} + \frac{1}{6176716} + \frac{1}{24444442} \\
&\quad + \frac{1}{202222202} + \frac{1}{404444404} \\
\frac{1}{83} &= \frac{1}{88} + \frac{1}{3003} + \frac{1}{6776} + \frac{1}{9009} + \frac{1}{21912} + \frac{1}{39093} + \frac{1}{48984} + \frac{1}{747747} \\
&\quad + \frac{1}{143484341} \\
\frac{1}{85} &= \frac{1}{88} + \frac{1}{4114} + \frac{1}{8008} + \frac{1}{54145} + \frac{1}{77077} + \frac{1}{595595} \\
\frac{1}{95} &= \frac{1}{101} + \frac{1}{5005} + \frac{1}{5225} + \frac{1}{5775} + \frac{1}{17271} + \frac{1}{505505} + \frac{1}{909909} \\
\frac{1}{97} &= \frac{1}{101} + \frac{1}{4884} + \frac{1}{6666} + \frac{1}{22422} + \frac{1}{241142} + \frac{1}{246642} + \frac{1}{2214122} \\
&\quad + \frac{1}{4428244} + \frac{1}{24355342} + \frac{1}{245767542}
\end{aligned}
$$