# NEGATIVE EIGENVALUES OF THE SCHRÖDINGER EQUATION: AN APPROACH THROUGH FRACTIONAL INTEGRATION AND MORREY SPACES

by

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# PREFACE

This dissertation consists of three chapters. The first chapter contains an introduction. The second chapter contains quite technical proofs of inequalities for functions in certain function spaces bearing the name of Morrey. The second chapter also includes direct estimates on solutions of a certain time–independent Schrödinger equation. The third chapter deals with estimates of negative eigenvalues of the Schrödinger equation. These problems are quite different in nature. The first problem requires uniqueness, whereas the other implies existence of multiple solutions: If $u_0$ is an eigenfunction satisfying $(-\triangle + V)u_0 = \lambda u_0$ for $x \in \Omega$ and $u(x) = 0$ for $x \in \partial\Omega$, then $cu_0$ is a solution of $(-\triangle + (V - \lambda))u_0 = 0$ for $x \in \Omega$ and $u(x) = 0$ for $x \in \partial\Omega$ for all values of $c \in \mathbf{R}^3$. One should therefore expect the analysis of the two problems to require quite different techniques. This turns out not to be so. The key tool for solving both problems is the Morrey space inequalities developed in chapter II. The Morrey space inequalities reduces the theory to the $\mathcal{L}^p$–theory, although a bit more work is necessary.

The content of the dissertation is presented in the same chronological order by which the results where discovered. When read in this order, all the results appear naturally and are merely "simple" deductions of the previous results. For the patient reader with no urge for "action" this may be a good way to read this dissertation. The third chapter is however more "action–packed". Chapter II and III have therefore been written such that they can be read independently. A certain amount of redundant information has been included to achieve this. The reader must not feel guilty about skipping familiar material. The author recommends that the reader starts reading chapter II and skip to chapter III for a rest whenever the technicalities of chapter II become too overwhelming.

Chapter II has previously been published in [22]. Only minor changes have been made from [22] to chapter II.

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# CHAPTER I

# Introduction

We shall present four types of mathematical results in this dissertation. The first kind is one of pure mathematical interest; fractional integration. Fractional integration is the study of convolution of generic functions against fractions, hence the name. We shall consider fractions of the form $|x|^{-\alpha}$ where $0 < \alpha < 3$ and $x \in \mathbf{R}^3$.

The second kind of result is a direct estimate of the $\mathcal{L}^p$–norm of the solution of the equation $(-\triangle + V)u = 0$ for $|x| < R$ and $u(x) = 1$ if $|x| = R$. We have not attempted to make these results as general as possible: The domain $|x| < R$ can be replaced by any bounded region in $\mathbf{R}^3$, and the boundary condition by more general functions. The results can also be extended to estimate $u(x)$ in terms of the potential and the distance from $x$ to the boundary. The tools for this have already been developed in [6].

The third kind of results describes the distribution of eigenvalues for the Schrödinger equation. Only minor improvements have been made from the existing theory. But hopefully the proofs presented may shed some new light on the existing theory. Several of the results in [15] have been given new proofs in the framework of Morrey spaces.

The fourth kind of results is that which concerns the theory of Morrey spaces.

Most of the results concerning Morrey spaces may belong to one of the above categories. But some of these results are not too interesting, except as preliminary results, in the above categories. They do however yield information about the behavior of Morrey spaces.

None of the above categories were first studied here. Each of the topics (fractional integration, $\mathcal{L}^p$–estimates of solutions of elliptic partial differential equations, eigenvalue estimates of the Schrödinger operator and Morrey spaces) contains a rich mathematical tradition. We shall attempt to mention some of the people that have provided insight into these areas. Unfortunately many names must be left out as the author does not know the full list, and the full list would take up a lot of space as well.

## 1.1 Morrey Spaces

There are several kinds of Morrey or Morrey–Campanato spaces out there. Almost all the different definitions of Morrey spaces gives some sort of constraint on how fast some $\mathcal{L}^p$–norm on a ball can grow in terms of the size of the balls. Often one subtracts the average of the function on the ball before integrating, [28].

The Morrey spaces we shall consider were defined by Gilbarg and Trudinger in [18]. We say that a function $f$ is in the Morrey space $M_q^r$ if

$$\left( \int_Q |f(x)|^q \, dx \right)^{1/q} \leq C|Q|^{1/q-1/r} , \tag{1.1}$$

holds for all cubes $Q \subset \mathbf{R}^3$. In the harmonic analysis litterature, [3], $M_q^r$ is frequently denoted $\mathcal{L}^{q,\lambda}(\mathbf{R}^n)$, where $\lambda = n(1-q/r)$ is a dimensionality measure of the regularity of $f$. The smallest constant $C$ such that (1.1) holds is the Morrey norm of $f$ with respect to $q$ and $r$, and it is denoted by $\|f\|_{q,r}$. Other authors also consider $Q \subset \mathbf{R}^3$, and they may also write $M^r$ for $M_1^r$ (cf. [18, 26, 27]). Federbush considers in [13] a space that is almost equivalent to $M_2^p$ for $p > 3$. Weighted Morrey spaces has been considered as well. See for example [6].

On a first encounter with the Morrey space $M_q^r$, one may want to think of $M_q^r$ as a space consisting of functions that look like $\mathcal{L}^q$ locally and $\mathcal{L}^r$ globally. This is not the best way to think about $M_q^r$. One should rather view it as an extended $\mathcal{L}^r$ space where the functions are allowed their "worst $\mathcal{L}^r$ point behavior" on a set of dimension $3(1 - q/r)$ (we shall return to this point later). It is this dimensionality property of Morrey spaces that make them so useful. They have been frequently used in studying elliptic partial differential equations arising from Brownian motion in the work of Conlon et al. as seen in [5, 6, 7, 8]. The importance of dimensionality in these elliptic partial differential equations is related to recurrence properties of Brownian motion. It is well known that 3–dimensional Brownian motion recurs infinitely often to a set of dimension 1. The elliptic partial differential equations studied by Conlon is related to Brownian motion through expectation values of expressions involving a potential evaluated at the site of the Brownian motion at a given time. If the potential is at it's worst behavior on a set of dimension 1 then the solution will still blow up. The worst behavior for the Schrödinger equation is roughly given by a point singularity in $\mathcal{L}^{3/2}$, or $r = 3/2$ for the Morrey spaces. This means that we must require $3(1 - q/r) < 1$ or $q > 1$. So the potential $V$ should be in $M_{3/2}^q$ for $q > 1$. For the equation $(-\triangle + \vec{b} \cdot \nabla)u = 0$ the worst behavior is $\mathcal{L}^3$, or $r = 3$ for the Morrey spaces. It follows that we must require $q > 2$. This equation was studied in [6, 7].

## 1.2 Morrey Spaces and Partial Differential Equations

We have already seen that Morrey spaces are natural to use in elliptic partial differential equations related to Brownian motion. This is the case for the Schrödinger equation related to Brownian motion through the Feynman–Kac formula and the equation $(-\triangle + \vec{b} \cdot \nabla)u = 0$ related to Brownian motion with drift through the Cameron–Martin formula. Similarly parabolic equations can be studied by applying the corresponding time–dependent Feynman–Kac or Cameron–Martin formula. More general elliptic partial differential equations are currently being analyzed by Conlon and Song, [9].

An equally important partial differential equation, the Navier–Stokes equation, turns out to find Morrey spaces useful. The Navier–Stokes equation is a nonlinear parabolic partial differential equation. Although there is no known stochastic process generating the solution to the Navier–Stokes equation this should not come as a surprise, as there are stochastic processes similar to Brownian motion known to generate other nonlinear elliptic and parabolic partial differential equations. Flow in porous media [12] is such an example. Works on Navier–Stokes equation features Morrey spaces on several occasions. Examples are [13, 14, 17, 19, 26, 27]. Federbush showed in [13] that for initial values in $M_2^p(\mathbf{R}^3)$ with $p > 3$ a solution to the Navier–Stokes equation exists. Analogous results for flow in $\mathbf{R}^n$) for $n \geq 3$ was done in [26]. The appearance of Morrey spaces in many applications is no doubt due to their dimensionality feature.

## 1.3 Fractional Integration and Convolution Inequalities

Young's inequality tells us that the operator $(T_g f)(x) = \int dy\ f(x-y)g(y)$ satisfies $||T_g f||_r \leq ||f||_q ||g||_p$ if $1 + 1/r = 1/p + 1/q$ and $1 \leq p, q, r \leq \infty$. If we try to apply this to the operator $(Tf)(x) = \int dy\ f(y)|x-y|^{-3/p}$ we find that $|x|^{-3/p}$ is almost in $\mathcal{L}^p$, but not quite. The operator $T$ is nevertheless a bounded operator satisfying $\|Tf\|_r \leq C_p\|f\|_q$ for $1 + 1/r = 1/p + 1/q$, $1 < p, q, r < \infty$ and $C_p$ a sufficient large constant depending on $p$. This inequality was first discovered by Hardy, Littlewood and Sobolev and is appropriately named after it's discoverers. A simple proof of the Hardy–Littlewood–Sobolev inequality may be found in [25]. The Hardy–Littlewood–Sobolev inequality extends Young's inequality for a very specialized, although important, class of choices of $g$. There is also a generalization of Young's inequality that is more general and is referred to as the weak Young's inequality, [18]. It says that $\|T_g f\|_r \leq C\|f\|_q\|g\|_{w,p}$ for $g \in \mathcal{L}^p_w$, $1 < p, q, r < \infty$ and $\|g\|_{w,p}$ denoting the usual norm for weak $\mathcal{L}^p$–spaces. Since $|x|^{-3/p} \in \mathcal{L}^p_w$ for $1 < p < \infty$ it follows that the weak Young's inequality is stronger than the Hardy–Littlewood–Sobolev inequality.

The Hardy–Littlewood–Sobolev inequality is frequently featured in the theory of Sobolev inequalities. As $u$ can be related to $\nabla u$ via convolution by $|x|^{-1}$ in $\mathbf{R}^3$ it follows that one can relate $\|u\|_r$ to $\|\nabla u\|_q$ for $1 + 1/r = 1/3 + 1/q$, $1 < q, r < \infty$ by the Hardy–Littlewood–Sobolev inequality. This makes the Hardy–Littlewood–Sobolev inequality a powerful tool in the study of Sobolev inequalities. Morrey spaces are also used in the theory of Sobolev inequalities. The proof of the John Nirenberg inequality is one such important example.

A key result of this dissertation is perhaps an extension of the Hardy–Littlewood–Sobolev inequality to Morrey spaces. It says that $T$ is a bounded operator from $M^r_q$ to $M^t_s$ where $q, r, s$ and $t$ satisfy $1 + 1/t = 1/p + 1/r$, $1 < q < r < \infty$, $1 < s < t < \infty$, $1 < p < \infty$ and $s/t = q/r$. $T$ satisfies the inequality $\|Tf\|_{s,t} \leq C\|f\|_{q,r}$. This result was first proved in [3], but a different proof has been provided in Theorem 11. This inequality allows us to extend Sobolev inequalities and study the time–independent

Schrödinger equation in Morrey spaces. The reason the Schrödinger equation comes into play is that the Green's function for the Laplacian contains a term of the form $|x-y|^{-1}$.

This extension of the Hardy–Littlewood–Sobolev inequality is not merely made so that we can introduce Morrey spaces. As we shall see in chapter 2, Morrey spaces will simplify the proof of the Fefferman–Phong estimate for the number of negative eigenvalues, [15].

## 1.4 Eigenvalue Problems

For a lot of applications in physics as well as in mathematics it is useful to know the distribution of eigenvalues of the Schrödinger equation. For one thing this would tell us the energy levels or spectrum of a given atom or molecule for which we know the potential. This is however wishful thinking. The problem is too hard for exact or even efficient numerical solutions. On the other hand, even non–exact solutions, some even containing unspecified constants, have lead to dazzling results! The stability of matter, [11, 21], and the atomic structure of matter, [15], being two such examples.

There are two types of estimates of eigenvalues we shall pursue here: upper and lower estimates of the lowest eigenvalue and upper estimates of $N(V,\lambda)$, where $N(V,\lambda)$ is the number of negative eigenvalues below $\lambda$. The function $N(V,\lambda)$ contains all information about the eigenvalue distribution, so an exact estimate of $N(V,\lambda)$ would tell us where all the negative eigenvalues are located.

There appears to be two schools of thought on the eigenvalue problem. The first school of thought strive for estimates that are better for the typical "nice" function. This is the "best constant" type of estimates. Perhaps the most famous such result is the following result by Cwickel, Lieb and Rosenblum, [23]:

**Theorem 1**

$$N(V,\lambda) \le c_n \left|\{(x,\xi):\ |\xi|^2 + V(x) < \lambda\}\right| \ ,$$

*for some constants $c_n$ depending only on the dimension $n$.* □

If we let $\lambda = 0$, $n = 3$ the result reduces to the following corollary, [23]:

**Corollary 2** *The number of bound states of the Schrödinger equation is bounded by*

$$N(V,0) \le 0.116 \int_{\mathbf{R}^3} |V(x)|^{3/2}\, dx \ . \tag{1.2}$$

□

The second school of thought is not overly concerned with finding best constants, but attempts rather to find estimates that are valid for larger classes of functions. We shall pursue such estimates in chapter III. It is typical for these estimates to contain unspecified constants. One such result is the Fefferman–Phong estimate, [15]. To state this result we need to explain what a minimal cube is.

Make a dyadic decomposition of $\mathbf{R}^3$ into cubes. Define a dyadic cube $Q$ to be minimal with respect to $\epsilon > 0, p > 1$ if $\int_Q |V(x)|^p\, dx \ge \epsilon^p |Q|^{1-2p/3}$ and $\int_{Q'} |V(x)|^p\, dx < \epsilon^p |Q'|^{1-2p/3}$ for all dyadic cubes $Q' \subset Q$. The number of minimal dyadic cubes with respect to $\epsilon$ is denoted by $N_\epsilon(V)$. The Fefferman–Phong estimate says that

$$N(V,0) \le C_\epsilon N_\epsilon(V) \ , \tag{1.3}$$

where $C_\epsilon$ is finite for sufficiently small $\epsilon > 0$.

One example where (1.2) does not work, whereas (1.3) does work, is the potential $V(x) = |x|^{-2}$. We shall later present a proof of (1.3) (Theorem 28) different from that presented in [15], but more along the lines of the proof of Kerman and Sawyer in [20]. In [15], Fefferman asked whether there were similar "$\mathcal{L}\log\mathcal{L}$" type results. Chang, Wilson and Wolff,[1], proved that "$\mathcal{L}\log\mathcal{L}$" was not possible, and proceeded to find the largest possible "$\mathcal{L}^{\phi}$"–type extensions of (1.3). Other related results can also be found in [20, 2] .

## 1.5 What's New?

This dissertation contains several results. Some of these results were already well known, but they fit well into our framework. The others are new results. We have already mentioned the classical theory, so let us talk about what's new:

1. The first kind of results relates to the behavior of the operator $(T_g f)(x) = \int g(x)f(y)|x-y|^{3/p}\, dy$ in Morrey spaces. The main result is Theorem 4, which is optimal only for the case when $g \in M_u^v$ and $1/v + 1/p = 1$. Optimal results can however be extracted from this theorem by a kind of bootstrapping technique that we develop to extend the Hardy–Littlewood–Sobolev inequality to Morrey spaces. The extension of the Hardy–Littlewood–Sobolev inequality to Morrey spaces is a new result that shall prove useful in several application.

2. The problem that led to this dissertation in the first place was that of extending the classical theory to the "drift case", i.e. potentials of the form $V = -\nabla\cdot \vec{b}$. We have basically two kinds of results for the drift case. Firstly there is the problem of estimating the $\mathcal{L}^p$–norm of the solution of the equation $(-\triangle - \nabla\cdot \vec{b})$ $u(x) = 0$ for $|x| < R$ and $u(x) = 1$ for $|x| = R$. Secondly there is the problem of estimating the number of negative eigenvalues of the operator $-\triangle - \nabla\cdot \vec{b}$. It is interesting to note that the first problem required much more insight beyond the corresponding $V \in M_p^{3/2}$ problem than did the eigenvalue problem. It was this

insight that led to the tools required for the analysis of the eigenvalue problem in the first place. It is well worth noting that the estimates for eigenvalues for $V \in M_p^{3/2}$ involves $b \in M_{2p}^3$, where $b(x) = |V(x)|^{1/2}$, whereas when analyzing the drift case this intermediate $b$–function is completely unnecessary!

3. Finally we have proved old results about the function $N(V, \lambda)$ with new techniques. The techniques involved use the extended Hardy–Littlewood–Sobolev inequality. There seem to be no reason why these techniques shouldn't be applicable when analyzing other elliptic partial differential equations.

## 1.6 Suggested Problems for the Future

The author would like to suggest some problems worth investigating in the future:

1. Weak Young's inequality extends the Hardy–Littlewood–Sobolev inequality to weak $\mathcal{L}^p$–spaces. In other words, we can replace $|x|^{-3/p}$ by functions in $\mathcal{L}^p_w$. Theorem 11 is an extension of the Hardy–Littlewood–Sobolev inequality to Morrey spaces. Is there a generalization of this result where $|x|^{-3/p}$ can be replaced by functions $g$ where $g$ is taken from some "weak Morrey space"?

2. The Fefferman–Phong estimate appeared before wavelets became fashionable. The proof presented in [15] has however a clear wavelet flavor about it. The author is convinced that the proof in [15] could be repeated using orthonormal wavelet bases with compact support. Maybe one could take it even further and use this wavelet approach to find numerical estimates for the smallest eigenvalue, and possibly also to the other eigenvalues? It would be interesting to see if such a scheme is feasible and if it could produce useful results.

# CHAPTER II

# Fractional Integration and Morrey Spaces

## 2.1 Introduction

Let $V : \mathbf{R}^3 \to \mathbf{R}$ be the potential for the 3-dimensional Schrödinger operator $-\triangle + V$. Consider the problem of estimating the solution of the problem

$$\triangle u(x) + V(x)u(x) = 0 \tag{2.1}$$
$$u(x)|_{\partial\Omega} \;=\; 1\,,$$

where $\Omega = \{x : x \in \mathbf{R}^3 \text{ and } |x| < R\}$. We wish to estimate the $\mathcal{L}^p$ norm of $u$. A first attempt will give bounds in terms of $\|V\|_{3/2}$. This is analogous to (1.2). Is there an analogue to (1.3) as well. The answer is "yes". Conlon and Redondo showed in [8] that $\|u\|_1$ is bounded by a certain Morrey norm of $V$. We will "generalize" this to estimate $\|u\|_p$, when $1 \le p < \infty$. Furthermore we will investigate analogous estimates for potentials of the form $V = -\nabla\cdot \vec{b}$, $\vec{b}$: $\mathbf{R}^3 \to \mathbf{R}^3$.

## 2.2 Morrey Spaces

We will say that a function $f$ is in the Morrey space $M_q^r(\Omega)$, $1 \le q \le r$ if we have

$$\left(\int_{Q\cap\Omega} |f(x)|^q \, dx\right)^{1/q} \le C|Q|^{1/q-1/r}\,, \tag{2.2}$$

for all cubes $Q$. The smallest constant $C$ such that (2.2) holds is the Morrey norm of $f$ with respect to $q$ and $r$, and it is denoted by $\|f\|_{q,r}$. Applying Hölder's inequality to (2.2) we also see that

$$\left(\int_{Q\cap\Omega} |f(x)|^{q'}\,dx\right)^{1/q'} \le \|f\|_{q,r}|Q|^{1/q'-1/r}\ ,$$

for all $1 \le q' \le q$. This tells us that $M^r_{q'}(\Omega) \subset M^r_q(\Omega)$ for $1 \le q' \le q \le r$. Note also that $M^r_r(\Omega) = \mathcal{L}^r(\Omega)$.

We will simply write $M^r_q$ for $M^r_q(\mathbf{R}^3)$.

### 2.2.1 Maximal Cubes

By definition of $M^r_q$, the largest value $\int_Q |f|^q$ can have for a cube $Q$ is

$$\int_Q |f|^q = \|f\|^q_{q,r}|Q|^{1-q/r}\ . \tag{2.3}$$

We will call any dyadic cube $Q$ with the property (2.3) a maximal cube. A natural question to ask is how big $f$ can be on a maximal cube. Let $f(x) = C$ be constant. We then get

$$C = \|f\|_{q,r}|Q|^{-1/r}\ . \tag{2.4}$$

One interesting thing about (2.4) is that $C$ does not depend on $q$ if $\|f\|_{q,r}$ is fixed. This means that if $f(x)$ is constant on a maximal cube $Q$ then the constant depends only on $r$ and the size of the cube.

### 2.2.2 Dimensionality of Small Maximal Cubes

We now know how big $f$ can be on a maximal cube. But we don't know how many maximal cubes we can have. To see this we let $Q_0$ be a cube of unit size and $N_n$ be the number of maximal dyadic cubes $Q_n$ contained in $Q_0$ of size $2^{-3n}$. We then have the inequality

$$N_n\|f\|^q_{q,r}|Q_n|^{1-q/r} \le \|f\|^q_{q,r}|Q_0|^{1-q/r}$$

or equivalently

$$N_n \leq 2^{3n(1-q/r)} . \tag{2.5}$$

(2.5) tells us that the maximal dimension of small maximal cubes is $3(1-q/r)$. This gives us a nice intuitive feel for what a typical function in $M_q^r$ looks like.

## 2.3 The Schrödinger Equation for Potentials in $\mathcal{L}^p$

To understand (2.1) better we will analyze the solution of (2.1) for some simple potentials $V$, but first we need to do some preparatory work.

### 2.3.1 The Perturbation Series

Substituting $v(x) = u(x) - 1$ into (2.1) we get

$$\begin{aligned} \triangle v(x) + v(x)V(x) &= -V(x) \\ v(x)|_{\partial\Omega} &= 0 . \end{aligned} \tag{2.6}$$

Applying the inverse Dirichlet Laplacian to both sides of (2.6) gives us

$$\begin{aligned} (\triangle + V)v &= -V \\ (1 + \triangle^{-1}V)v &= -\triangle^{-1}V \\ v &= (1 - (-\triangle^{-1}V))^{-1}(-\triangle^{-1}V) \\ v &= \sum_{n=1}^{\infty}(-\triangle^{-1}V)^n . \end{aligned}$$

Using $u(x) = v(x) + 1$ this gives us

$$u(x) = 1 + \sum_{n=1}^{\infty}(-\triangle^{-1}V)^n . \tag{2.7}$$

The Green's function, $G_D(x,y)$, for the Dirichlet Laplacian, $-\triangle$, is given by

$$G_D(x,y) = \frac{1}{4\pi}\left\{\frac{1}{|x-y|} - \frac{R}{|y|}\frac{1}{|x-\overline{y}|}\right\} , \tag{2.8}$$

where $\overline{y} = R^2 y/|y|^2$. Using (2.7) and (2.8) we may now write an explicit formula for $u(x)$

$$
\begin{aligned}
u(x) \;&=\; 1 + \int_\Omega G_D(x,x_1)V(x_1)\,dx_1 \qquad (2.9)\\
&+ \int_\Omega G_D(x,x_1)V(x_1)\,\int_\Omega G_D(x_1,x_2)V(x_2)\,dx_2\,dx_1 + \cdots .
\end{aligned}
$$

### 2.3.2 The Green's Function

We recall that the Green's function $G_D(x,y)$ satisfies the properties $G_D(x,y) = G_D(y,x)$ and $G_D(x,y) \geq 0$ for all $x, y \in \Omega$. From the positivity of $G_D(x,y)$ it follows that

$$|G_D(x,y)| \leq \frac{1}{4\pi}\frac{1}{|x-y|} \qquad (2.10)$$

and

$$\frac{R}{|y|}\frac{1}{|x-\overline{y}|} \leq \frac{1}{|x-y|}\,. \qquad (2.11)$$

Using the symmetry $G_D(x,y) = G_D(y,x)$ when differentiating with respect to $y$ we transform (2.8) into

$$\vec{\nabla}_y\, G_D(x,y) = \frac{1}{4\pi}\left\{-\frac{y-x}{|y-x|^3} + \frac{R}{|x|}\frac{y-\overline{x}}{|y-\overline{x}|^3}\right\}\,. \qquad (2.12)$$

Application of (2.11) to (2.12) gives us an upper bound for $|\,\vec{\nabla}_y\, G_D(x,y)|$

$$|\,\vec{\nabla}_y\, G_D(x,y)| \leq \frac{1}{2\pi}\frac{1}{|x-y|^2}\,.$$

### 2.3.3 $\mathcal{L}^\infty$ estimates for $p > 3/2$

Because of (2.10) we may bound $u$, using (2.9), by

$$
\begin{aligned}
u(x) \;&\leq\; 1 + C\int_\Omega \frac{V(x_1)}{|x-x_1|}\,dx_1 \qquad (2.13)\\
&+C^2\int_\Omega \frac{V(x_1)}{|x-x_1|}\,\int_\Omega \frac{V(x_2)}{|x_1-x_2|}\,dx_2\,dx_1 + \cdots .
\end{aligned}
$$

(2.13) is now just a bunch of convolutions nested inside one another, so we may estimate $u$ using Hölder's inequality. Since $1/|x| \in \mathcal{L}^q$ for $q < 3$, we get $\|V * 1/|\cdot|\,\|_\infty \leq \|V\|_p \|1/|\cdot|\,\|_q = C_p \|V\|_p$ for $1 = 1/p + 1/q$, i.e. $p > 3/2$. Using this convolution inequality repeatedly we get

$$\|u\|_\infty \leq \sum_{n=0}^{\infty} (C_p \|V\|_p)^n \,. \tag{2.14}$$

If $\|V\|_p \leq (2C_p)^{-1}$ we have $\|u\|_\infty \leq 1 + c_p \|V\|_p$.

#### 2.3.4 $\mathcal{L}^q$ estimates for $p = 3/2$

Since $1/|x|$ 'almost' is in $\mathcal{L}^3$ we wish we could choose $q = 3$. The Hardy–Littlewood–Sobolev inequality allows us to do just that. It states ([25]) that $\|V * 1/|\cdot|^{3/q}\|_r \leq C_{p,q} \|V\|_p$ for $1/r = 1/p + 1/q - 1$ and $1 < p, q, r < \infty$. For $q = 3$ we have $1/r = 1/p - 1/3$. The problem is that we cannot set $p = 3/2$ in this formula since $r = \infty$ is not allowed in the Hardy–Littlewood–Sobolev inequality. We get around this by using $1 < p < 3/2$ and $\|V\|_p \leq |\Omega|^{1/p - 2/3} \|V\|_{3/2}$. We then have $\|V * 1/|\cdot|\,\|_r \leq C_r |\Omega|^{1/p-2/3} \|V\|_{3/2}$ for $1/r = 1/p - 2/3$, $1 < p < 3/2$. Using Hölder's inequality we get

$$\|V(V * 1/|\cdot|)\|_p \leq C_r |\Omega|^{1/r} \|V\|_{3/2} \,.$$

Using the Hölder and the Hardy–Littlewood–Sobolev inequality every other time we get

$$(\mathrm{Av}_{x\in\Omega} |u(x)|^r)^{1/r} \leq \sum_{n=0}^{\infty} (C_r \|V\|_{3/2})^n \,, \tag{2.15}$$

for $1 < r < \infty$. Here $\mathrm{Av}_{x\in\Omega}|u(x)|^r$ is shorthand notation for the average of $|u(x)|^r$ over $\Omega$, i.e. $(\mathrm{Av}_{x\in\Omega}|u(x)|^r)^{1/r} := (\int_\Omega |u(x)|^r \, dx/|\Omega|)^{1/r}$. Equation (2.15) is analogous to (2.14). If $\|V\|_{3/2} \leq (2C_r)^{-1}$ we have $(\mathrm{Av}_{x\in\Omega}|u(x)|^r)^{1/r} \leq c_r \|V\|_{3/2}$.

### 2.3.5 A Counterexample for $p < 3/2$

For $\mathcal{L}^p$, $p < 3/2$ we cannot expect to bound the $\mathcal{L}^r$ norm of $u$ for any $r$. To see this we let

$$V_\epsilon(x) = -\frac{2\epsilon}{|x|^2}\left(1 - \log(1/|x|) + 2\epsilon[\log(1/|x|)]^2\right) . \tag{2.16}$$

We may then verify that $u(x) = \exp\{\epsilon[\log(1/|x|)]^2\}$ is a solution of (2.1) for $R = 1$. Since $u(x)$ has a singularity that grows faster than $|x|^{-n}$ for any n in the origin, $u \notin \mathcal{L}^r$ for any $r > 0$, regardless of how small $\epsilon > 0$ is. In other words we cannot perturb the Laplacian with general potentials from $\mathcal{L}^p$ when $p < 3/2$.

### 2.3.6 What about $V(x) = \epsilon|x|^{-2}$

If we examine the potential $V(x) = \epsilon|x|^{-2}$ we see that the solution of (2.1) for $R = 1$ is $u(x) = |x|^\alpha$, where $\alpha(\alpha+1) = -\epsilon$. This means that $V \in \mathcal{L}^r$, $r < \infty$ if $\epsilon > 0$ is sufficiently small. We cannot take $r = \infty$ since $u(x)$ has a singularity at $x = 0$. On the other hand the solution behaves like the general case for potentials in $\mathcal{L}^{3/2}$, but $V \notin \mathcal{L}^{3/2}$ and is not covered by the result of section 3.4. We must therefore seek to find a finer partition than the $\mathcal{L}^p$ spaces in order to include $\epsilon/|x|^2$ into our theory. This finer partition we will consider is the Morrey spaces we introduced in section 2. We will attempt to develop the analogue theory for Morrey spaces by generalizing the Hardy–Littlewood–Sobolev inequality to Morrey spaces.

## 2.4 Generalizing the Hardy–Littlewood–Sobolev Inequality

Since we will be dealing with a lot of cubes, we will find it convenient to introduce the notation $Q(x,l)$ to denote a cube with center at $x$ and side length $l$. Also $CQ$ will denote a cube with the same center as $Q$, but with side length $Cl$.

### 2.4.1 The Naive Approach

Define an operator $T$ by

$$(Tf)(x) = \int_{\mathbf{R}^3} \frac{f(y)}{|x-y|^{3/p}}\, dy \ . \tag{2.17}$$

Naively, since $f \in M^r_q$ implies $f \in \mathcal{L}^q_{\text{loc}}$, we may try to use the Hardy–Littlewood–Sobolev inequality locally and take care of the larger scales by cruder estimates. This gives the following estimate

**Theorem 3** *The operator $T : M^r_q \to M^t_s$ defined by (2.17) is a bounded operator, satisfying $\|Tf\|_{s,t} \leq C\|f\|_{q,r}$ for some constant $C = C(p,q,r)$ if $1 < p < \infty$, $1 < s \leq t < \infty$, $1 < q \leq r < \infty$, $1/t = 1/r + 1/p - 1$ and $1/s = 1/q + 1/p - 1$.* □

Since by (2.4) $t$ is essentially the only variable controlling the size of $Tf$ on maximal cubes, we would expect $s$ to be such that the dimension of small maximal cubes is preserved, i.e. $s/t = q/r$, since $|x|^{-3/p}$ has only one singularity. $s$ and $t$ of Theorem 3 does however not possess this property, and we will see later that Theorem 3 is indeed not optimal.

**Proof of Theorem 3.** We need to prove that

$$\left(\int_Q |Tf(x)|^s\, dx\right)^{1/s} \leq C\|f\|_{q,r}|Q|^{1/s-1/t}$$

holds for all cubes $Q$. Decompose $f$ according to $Q$, that is $f = f_1 + f_2$ and $f_1(x) = \chi_{4Q}(x)f(x)$. We then have by Minkowski's inequality that

$$\left(\int_Q |Tf|^s\right)^{1/s} \leq \left(\int_Q |Tf_1|^s\right)^{1/s} + \left(\int_Q |Tf_2|^s\right)^{1/s} \ . \tag{2.18}$$

By use of the Hardy–Littlewood–Sobolev inequality we have

$$\left(\int_Q |Tf_1|^s\right)^{1/s} \ \leq \ \left(\int_{\mathbf{R}^3} |Tf_1|^s\right)^{1/s}$$

$$
\begin{aligned}
&= \|Tf_1\|_s \\
&\leq C\|f_1\|_q \\
&= \left(\int_{4Q} |f|^q\right)^{1/q} \\
&\leq C|Q|^{1/q-1/r}\|f\|_{q,r} \\
&= C|Q|^{1/s-1/t}\|f\|_{q,r} \ , \qquad (2.19)
\end{aligned}
$$

since $1/s = 1/q + 1/p - 1$. Whereas for $f_2$ we have

$$
\begin{aligned}
\left(\int_Q |Tf_2|^s\right)^{1/s} &\leq \left(\int_Q \left(\int_{\mathbf{R}^3\setminus 4Q} \frac{|f(y)|}{|x-y|^{3/p}}\, dy\right)^s dx\right)^{1/s} \\
&\leq \sum_{n=1}^{\infty} \left(\int_Q \left(\int_{2^nQ} 2^{-3n/p}|Q|^{-1/p}|f(y)|\, dy\right)^s dx\right)^{1/s} \\
&\leq \sum_{n=1}^{\infty} 2^{3n(1-1/r-1/p)}|Q|^{1/s+(1-1/r-1/p)}\|f\|_{q,r} \\
&= \|f\|_{q,r}|Q|^{1/s-1/t}\sum_{n=1}^{\infty} 2^{-3n/t} \\
&= C\|f\|_{q,r}|Q|^{1/s-1/t} . \qquad (2.20)
\end{aligned}
$$

Combining (2.19), (2.20) and (2.18) we deduce that

$$
\left(\int_Q |Tf|^s\right)^{1/s} \leq C\|f\|_{q,r}|Q|^{1/s-1/t}
$$

for all cubes $Q$. This proves Theorem 3. ■

### 2.4.2 The operator $T_g$

To extend Theorem 3 we define a slightly more general operator $T_g f$ by

$$
(T_g f)(x) = \int_{\mathbf{R}^3} \frac{g(x)f(y)}{|x-y|^{3/p}}\, dy \ . \qquad (2.21)
$$

Note that $T_1 f(x) = Tf(x)$.

**Theorem 4** *If $g \in M_u^v$, $f \in M_q^r$ and* $\mathrm{supp}(f)$ *is contained in a bounded domain, then $T_g : M_q^r \to M_s^t$ is a bounded operator satisfying*

$$
\|T_g f\|_{s,t} \leq C\|f\|_{q,r}\|g\|_{u,v} \ ,
$$

*where $C = C(p, q, r, s, t, u, v)$ is a constant and $p, q, r, s, t, u, v$ satisfies the conditions*

1. $1 < p < \infty$, $1 < q \leq r < \infty$, $1 < s \leq t < \infty$ *and* $1 < u \leq v \leq \infty$.

2. $s < u$.

3. $1/t = 1/v + 1/r + 1/p - 1$ *and* $1/r + 1/p > 1$

4. $s \geq q$ *and* $1/s = 1/v + 1/q + 1/p - 1$. □

**Corollary 5** *If* $1/v + 1/p = 1$ *then* $T_g f : M_q^r \to M_q^r$ *is a bounded operator with* $\|T_g f\|_{q,r} \leq C\|g\|_{u,v}\|f\|_{q,r}$, *provided* $1 < q < u$, $q \leq r$, $u \leq v$, $1 < p, q, r, u, v < \infty$ *and* $1/r + 1/p > 1$. □

To prove Corollary 5 we merely check all the conditions of Theorem 4 when $1/p + 1/v = 1$.

**Remarks.**

1. $p = 3/2$, $v = 3$ in Corollary 5 gives Theorem 2.1 of [7].

2. Letting $u = v = \infty$ and $g(x) = 1$ in Theorem 4 we recover Theorem 3.

3. $u = v = \infty$, $q = r$ and $s = t$ recovers the Hardy–Littlewood–Sobolev inequality.

4. The proof of Theorem 4 was found by applying techniques developed in [7] and [20].

Although Theorem 4 does not seem to generalize Theorem 3 by remark 2, we shall see later that we can in fact prove a stronger version of Theorem 3 using Theorem 4 and a different choice of $g$. Specifically we will see that Theorem 3 holds for larger values of $s$ as has previously been shown in Theorem 2 of [3] (Theorem 11, section 4.3).

The proof of Theorem 4 is rather long. We will therefore find it convenient to prove several lemmas before concluding the proof.

**Proof of Theorem 2**

We will first verify that

$$\left(\int_Q |T_g f|^s\right)^{1/s} \leq C\|f\|_{q,r}\|g\|_{u,v}|Q|^{1/s-1/t}$$

holds for large cubes. Since $\mathrm{supp}(f)$ is finite, we may define a cube $Q_0$ with $Q_0 = Q(0, 2^{n_0})$ such that $\mathrm{supp}(f) \subset (1/3)Q_0$. Let

$$(T_{g,0}f)(x) = \chi_{\mathbf{R}^3\setminus Q_0}(x)(T_g f)(x) \tag{2.22}$$

and

$$(S_{n,B}f)(x) = 2^{-3n/p}\int_{B(x,2^n)} |f(y)|\, dy\ . \tag{2.23}$$

We immediately see from (2.23) that

$$|T_{g,0}f(x)| \leq \sum_{n=n_0}^{\infty} |g(x)| S_{n,B}f(x)\ . \tag{2.24}$$

**Lemma 6** $T_{g,0} : M_q^r \to M_s^t$ *defined by (2.22) is a bounded operator satisfying*

$$\|T_{g,0}f\|_{s,t} \leq C\|f\|_{q,r}\|g\|_{u,v}\ . \tag{2.25}$$

□

**Proof of Lemma 6.** If $Q$ is a small cube satisfying $|Q| \leq |Q_0|$ we have from (2.24)

$$\begin{aligned}
\left(\int_Q |T_{g,0}f|^s\right)^{1/s} &\leq \sum_{n=n_0}^{\infty}\left(\int_Q |g(x)|^s |S_{n,B}f(x)|^s\, dx\right)^{1/s} \\
&\leq \sum_{n=n_0}^{\infty} 2^{-3n/p} C\|f\|_{q,r} 2^{3n(1-1/r)}\left(\int_Q |g(x)|^s\right)^{1/s} \\
&\leq C\|f\|_{q,r}\|g\|_{u,v}|Q|^{1/s-1/v}\sum_{n=n_0}^{\infty} 2^{3n(1-1/r-1/p)} \\
&\leq C\|f\|_{q,r}\|g\|_{u,v}|Q|^{1/s-1/t}\ ,
\end{aligned} \tag{2.26}$$

since $1-1/r-1/p<0$. For large cubes $|Q|\geq|Q_0|$ we have

$$|T_{g,0}f(x)| \leq \sum_{n=n_0}^{\infty} 2^{-3n/p} \int_{B(0,2^n)} |f(y)|\, dy \chi_{B(0,2^{n-1})}(x)|g(x)| \ ,$$

so that

$$\begin{aligned} \|T_{g,0}f\|_s &\leq \sum_{n=n_0}^{\infty} \|S_{n,B}f(0)|g|\chi_{B(0,2^{n-1})}\|_s \\ &\leq C\sum_{n=n_0}^{\infty} 2^{-3n/p}\|f\|_1\|g\|_{u,v}|B(0,2^{n-1})|^{1/s-1/v} \\ &\leq \|f\|_{q,r}\|g\|_{u,v}|Q_0|^{1-1/r}\sum_{n=n_0}^{\infty} 2^{3n(-1/p+1/s-1/v)} \\ &\leq \|f\|_{q,r}\|g\|_{u,v}|Q|^{1/s-1/t} \ , \end{aligned} \tag{2.27}$$

since $1/s-1/p-1/v<0$, $s<u$ and $|Q|\geq|Q_0|$. (2.26) and (2.27) now proves Lemma 6. ■

To examine $Tf(x)$ on large scales, we define a dyadic version of $Tf(x)$. Let $K = Q(\cdot,2^{n_K})$ be the 'mother cube'. Make a dyadic subdivision of $K$ by successively dividing each cube into 8 smaller cubes of equal size. We define dyadic versions of $S_{n,B}$ and $T_{g,0}$.

$$S_nf(x) \stackrel{def}{=} 2^{-3n/p}\int_{Q_n} |f(y)|\, dy \ ,$$

where $Q_n$ is the unique dyadic sub cube of the form $Q_n = Q(\cdot,2^n)$ that contains $x$. Similarly

$$T_Kf(x) \stackrel{def}{=} \sum_{n=-\infty}^{n_K} |g(x)|S_nf(x) \ .$$

If we can prove that $\|T_Kf\|_{s,t}\leq C\|f\|_{q,r}\|g\|_{u,v}$ for all $K=Q(\cdot,2^{2+n_0})$ then we also have $\|Tf\|_{s,t}\leq C\|f\|_{q,r}\|g\|_{u,v}$. To see this we need the following lemma.

**Lemma 7** *There exists a constant $C$ such that for all $s\geq 1$*

$$\int_{Q\cap Q_0} |Tf(x)|^s\, dx \leq \int_{Q_0}\int_{Q\cap Q_0} |T_{\tilde{Q}_0(z)}f(x)|^s\, dx\frac{dz}{|Q_0|} \ , \tag{2.28}$$

*where $Q_0 = Q(0,2^{n_0})$ and $\tilde{Q}_0(z) = Q(z,2^{2+n_0})$.* □

**Proof of Lemma 7.** Applying Jensen's inequality to the right hand side of equation (2.28) gives

$$\int_{Q_0}\int_{Q\cap Q_0}|T_{\tilde{Q}_0(z)}f(x)|^s\,dx\frac{dz}{|Q_0|}\geq\int_{Q\cap Q_0}\left\{\int_{Q_0}T_{\tilde{Q}_0(z)}f(x)\frac{dz}{|Q_0|}\right\}^s\,dx\,. \tag{2.29}$$

But because $\mathrm{supp}(f)\subset(1/3)Q_0$ we have

$$\int_{Q_0}T_{\tilde{Q}_0(z)}f(x)\frac{dz}{|Q_0|}\geq C|Tf(x)|\,, \tag{2.30}$$

for all $x\in Q_0$. (2.30) and (2.29) now proves (2.28). ∎

**Proof of Theorem 4.** Assume $\|T_Kf\|_{s,t}\leq C\|f\|_{q,r}\|g\|_{u,v}$ is true. We have

$$\left(\int_Q|Tf|^s\right)^{1/s}\leq\left(\int_{Q\cap Q_0}|Tf|^s\right)^{1/s}+\left(\int_{Q\setminus Q_0}|Tf|^s\right)^{1/s}\,. \tag{2.31}$$

The second term is bounded by Lemma 6, whereas the first term can be bounded by Lemma 7. For if $K=Q(z,2^{2+n_0})$ then $Q\cap Q_0$ can be covered by 8 or less dyadic sub cubes, $Q'$, of $K$ such that $|Q'|\leq|Q|$. Therefore

$$\begin{aligned}\left(\int_{Q\cap Q_0}|T_Kf|^s\right)^{1/s} &\leq \sum_{Q'}\left(\int_{Q'}|T_Kf|^s\right)^{1/s} \\ &\leq 8C\|f\|_{q,r}\|g\|_{u,v}|Q|^{1/s-1/t}\,,\end{aligned} \tag{2.32}$$

and $(\int_{Q\cap Q_0}|Tf|^s)^{1/s}\leq C\|f\|_{q,r}\|g\|_{u,v}|Q|^{1/s-1/t}$ by Lemma 7. This proves Theorem 4 by (2.31). ∎

Lemmas 6 and 7 now allow us to reduce the proof of Theorem 4 to the same theorem, but with $T_g$ replaced by $T_K$, $K=4Q_0$ and the cube $Q$ replaced by a dyadic cube $Q'\subset K$. We first prove the inequality $(\int_{Q'}|T_Kf|^s)^{1/s}\leq C\|f\|_{q,r}\|g\|_{u,v}|Q'|^{1/s-1/t}$ for a subset of all the dyadic cubes $Q'\subset K$.

**Lemma 8** *If a dyadic cube $Q'$ of $K$ satisfies*

$$|Q|^{\epsilon-1/p}\int_Q|f(y)|\,dy\leq|Q'|^{\epsilon-1/p}\int_{Q'}|f(y)|\,dy \tag{2.33}$$

*for all dyadic cubes $Q \subset Q'$, where $Q'$ is a dyadic sub cube of $K$ and $\epsilon > 0$ is sufficiently small, then*

$$\left(\int_{Q'} |T_K f|^s\right)^{1/s} \leq C\|f\|_{q,r}\|g\|_{u,v}|Q'|^{1/s-1/t}\,, \tag{2.34}$$

*if $s < u$.* □

**Proof of Lemma 8.** Let the side length of the dyadic sub cube $Q'$ be $2^N$, i.e. $Q' = Q(\cdot, 2^N)$. By use of Minkowski's inequality we have

$$\left(\int_{Q'} |T_K f|^s\right)^{1/s} \leq \Sigma_1 + \Sigma_2\,,$$

where

$$\begin{aligned}
\Sigma_1 &\stackrel{\text{def}}{=} \left(\int_{Q'} |g(x)|^s \left(\sum_{n=N}^{n_K} S_n f(x)\right)^s dx\right)^{1/s} \\
\Sigma_2 &\stackrel{\text{def}}{=} \left(\int_{Q'} |g(x)|^s \left(\sum_{n=-\infty}^{N} S_n f(x)\right)^s dx\right)^{1/s} .
\end{aligned}$$

We need now only show that equation (2.34) holds for both $\Sigma_1$ and $\Sigma_2$.

**Part I: $\Sigma_1$.** Since we have

$$\sum_{n=N}^{n_K} S_n f(x) \leq C\|f\|_{q,r}|Q'|^{1-1/r-1/p}\,,$$

for $1/r + 1/p > 1$, we see that

$$\begin{aligned}
\Sigma_1 &\leq C\|f\|_{q,r}|Q'|^{1-1/r-1/p}\left(\int_{Q'} |g(x)|^s\right)^{1/s} \\
&\leq C\|f\|_{q,r}\|g\|_{u,v}|Q'|^{1/s-1/t}\,,
\end{aligned} \tag{2.35}$$

if $s \leq u$.

**Part II: $\Sigma_2$.** We use the formula

$$\left(\sum_{k=-\infty}^{N} a_k\right)^s \leq \sum_{k=-\infty}^{N} s a_k \left(\sum_{n=k}^{N} a_n\right)^{s-1}\,, \tag{2.36}$$

which holds for $a_k \geq 0$, $k \in \mathbf{Z}$. Choose $a_n = |S_n f(x)|$ to see that

$$\Sigma_2^s \leq \int_{Q'} s \sum_{k=-\infty}^{N} |S_k f(x)|\, |g(x)|^s \left( \sum_{n=k}^{N} |S_n f(x)| \right)^{s-1} dx \, .$$

By use of (2.33) we have

$$\sum_{n=k}^{N} |S_n f(x)| \leq C 2^{3\epsilon(N-k)} |Q'|^{-1/p} \int_{Q'} |f(y)|\, dy \, ,$$

when $\epsilon > 0$. Therefore

$$\begin{aligned} \Sigma_2^s \quad \leq \quad & \left( |Q'|^{-1/p} \int_{Q'} |f(y)|\, dy \right)^{s-1} \times \\ & \sum_{k=-\infty}^{N} s\; 2^{3\epsilon(s-1)(N-k)} \int_{Q'} |g(x)|^s |S_k f(x)|\, dx \, . \end{aligned} \tag{2.37}$$

The size of the integral in equation (2.37) depends in some subtle way on the size of $g$ on the dyadic sub cubes of size $2^{3k}$. To study this effect we decompose $g$ according to its size, i.e.

$$|g(x)| \leq \sum_{m=-\infty}^{\infty} 2^m \chi_{E_m}(x) \, ,$$

where

$$E_m \stackrel{\text{def}}{=} \left\{ x : 2^{m-1} \leq |g(x)| < 2^m \right\} \, .$$

Since $g \in M_u^v$ we may estimate the size of $E_m$ intersected with an arbitrary cube. Clearly by the definition of $E_m$ we have

$$\int_Q 2^{(m-1)u} \chi_{E_m}(x)\, dx \leq \int_Q |g(x)|^u\, dx \leq \left( \|g\|_{u,v} |Q|^{1/u-1/v} \right)^u \, .$$

From this inequality it follows that

$$\frac{|E_m \cap Q|}{|Q|} \leq \left( \frac{\|g\|_{u,v}}{2^{m-1} |Q|^{1/v}} \right)^u \, . \tag{2.38}$$

Using this decomposition together with the definition of $S_n f(x)$ we have

$$\int_{Q'} |g(x)|^s |S_k f(x)|\, dx \leq \sum_{\substack{Q_k \subset Q' \\ Q_k \text{ dyadic}}} |Q_k|^{-1/p} \sum_{m=-\infty}^{\infty} 2^{ms} |Q_k \cap E_m| \int_{Q_k} |f(y)|\, dy \, .$$

Defining

$$I_m \stackrel{\text{def}}{=} \sum_{\substack{Q_k \subset Q' \\ Q_k \text{ dyadic}}} |Q_k|^{-1/p} |Q_k \cap E_m| \int_{Q_k} |f(y)|\, dy$$

we may write equation (2.37)

$$\Sigma_2^s \le C \left( |Q'|^{-1/p} \int_{Q'} |f(y)|\, dy \right)^{s-1} \times \sum_{k=-\infty}^{N} \sum_{m=-\infty}^{\infty} 2^{3\epsilon(s-1)(N-k)} 2^{ms} I_m \, . \tag{2.39}$$

Let us estimate $I_m$ by using the growth condition (2.33)

$$I_m \le |Q'|^{-1/p} \int_{Q'} |f(y)|\, dy 2^{3\epsilon(N-k)} \left( \sum_{\substack{Q_k \subset Q' \\ Q_k \text{ dyadic}}} |Q_k \cap E_m| \right) ,$$

where it is implicitly assumed that $Q_k$ are dyadic sub cubes of $Q'$ with side length $2^k$. Since $\cup_{Q_k \subset Q'} Q_k = Q'$, and the $Q_k$ are disjoint

$$I_m \le |Q'|^{-1/p} \int_{Q'} |f(y)|\, dy \; 2^{3\epsilon(N-k)} |Q' \cap E_m| \, . \tag{2.40}$$

If we do not use equation (2.33) in estimating $I_m$, but instead use $Q_k \cap E_m \subset Q_k \Rightarrow |Q_k \cap E_m| < |Q_k|$, we can bound $I_m$ in a second way

$$\begin{aligned} I_m &\le 2^{-3(N-k)(1-1/p)} |Q'|^{1-1/p} \sum_{\substack{Q_k \subset Q' \\ Q_k \text{ dyadic}}} \frac{|Q_k \cap E_m|}{|Q_k|} \int_{Q_k} |f| \\ &\le 2^{-3(N-k)(1-1/p)} |Q'|^{1-1/p} \int_{Q'} |f(y)|\, dy \, . \end{aligned} \tag{2.41}$$

Equations (2.40) and (2.41) now enables us to estimate $S_2$. Inserting (2.40) and (2.41) into (2.39) we get

$$\begin{aligned} \Sigma_2^s &\le C|Q'| \left\{ |Q'|^{-1/p} \int_{Q'} |f| \right\}^s \times \sum_{k=-\infty}^{N} \sum_{m=-\infty}^{\infty} 2^{3\epsilon s(N-k)} 2^{ms} \\ &\quad \min \left\{ \frac{|Q' \cap E_m|}{|Q'|} , 2^{-3(N-k)(1-1/p+\epsilon)} \right\} . \end{aligned} \tag{2.42}$$

Using equation (2.38) we see that

$$\frac{|Q' \cap E_m|}{|Q'|} \le 2^{-3(N-k)(1-1/p+\epsilon)} ,$$

when $m \geq m_0$, if

$$m_0 = \log_2 \left( 2\|g\|_{u,v} |Q'|^{-1/v} 2^{3(N-k)(1-1/p+\epsilon)/u} \right) .$$

Split the right hand side of equation (2.42) up into two parts according to whether $m < m_0$ or not. This gives us

$$\Sigma_2^s \leq C|Q'|^{1-s/p} \left( \int_{Q'} |f| \right)^s (\mathcal{S}_1 + \mathcal{S}_2) , \tag{2.43}$$

where

$$\begin{aligned} \mathcal{S}_1 &= \sum_{k=-\infty}^{N} \sum_{m<m_0} 2^{3\epsilon s(N-k)} 2^{ms} 2^{-3(N-k)(1-1/p+\epsilon)} \\ &\leq C \sum_{k=-\infty}^{N} 2^{m_0 s} 2^{-3(N-k)((1-1/p)-\epsilon(s-1))} \\ &= C \left( \|g\|_{u,v} |Q'|^{-1/v} \right)^s \sum_{k=-\infty}^{N} 2^{-3(N-k)((1-1/p)(1-s/u)-\epsilon(s-1+s/u))} \\ &\leq C \|g\|_{u,v}^s |Q'|^{-s/v} , \end{aligned} \tag{2.44}$$

if $1 - s/u > 0$ or equivalently $s < u$ and $\epsilon$ is sufficiently small. For $\mathcal{S}_2$ we have similarly

$$\mathcal{S}_2 \leq \sum_{k=-\infty}^{N} \sum_{m>m_0-1} 2^{3\epsilon s(N-k)} 2^{ms} \left( \frac{\|g\|_{u,v}}{2^{m-1}|Q'|^{1/v}} \right)^u ,$$

by use of equation (2.38). Continuing this calculation we get an estimate similar to that of (2.44).

$$\begin{aligned} \mathcal{S}_2 &\leq C\|g\|_{u,v}^u |Q'|^{-u/v} \sum_{k=-\infty}^{N} 2^{3\epsilon s(N-k)} 2^{m_0(s-u)} \\ &\leq C\|g\|_{u,v}^s |Q'|^{-s/v} \sum_{k=-\infty}^{N} 2^{-3(N-k)((1-1/p)(1-s/u)-\epsilon(s-1+s/u))} \\ &\leq C\|g\|_{u,v}^s |Q'|^{-s/v} , \end{aligned} \tag{2.45}$$

if $\epsilon$ is sufficiently small and $s < u$. Equations (2.44) and (2.45) may be substituted into equation (2.43). Keeping track of exponents we find that

$$\Sigma_2 \leq C\|g\|_{u,v} |Q'|^{1/s-1/p-1/v} \int_{Q'} |f|$$

$$\leq \; C\|g\|_{u,v}\|f\|_{q,r}|Q'|^{1/s-1/t} \; .$$

This estimate together with (2.35) proves Lemma 8. ∎

It remains to prove the inequality (2.34) for cubes that doesn't satisfy (2.33). We will assume $f \in \mathcal{L}^{\infty}$. Since none of our estimates will depend on $\|f\|_{\infty}$ we can do this without loss of generality. Let $N_0(x) := N$ and $\mathcal{F}_0 = \{Q'\}$, where as before the side length of $Q'$ is $2^N$. Define a function $N_1 : Q' \to \{-\infty, \ldots, N-1, N\}$ by the following rules

$\mathcal{A}_1$ $N_1(x) = -\infty$ if

$$|Q|^{-1/p+\epsilon} \int_Q |f| \leq |Q'|^{-1/p+\epsilon} \int_{Q'} |f|$$

for all dyadic sub cubes $Q$ of $Q'$ such that $x \in Q \subset Q'$.

$\mathcal{B}_1$ Otherwise $N_1(x)$ is defined by letting $2^{N_1(x)}$ be the length of a side of the largest dyadic sub cube of $Q$ such that

$$|Q|^{-1/p+\epsilon} \int_Q |f| > |Q'|^{-1/p+\epsilon} \int_{Q'} |f| \, ,$$

and $x \in Q \subset Q'$.

Corresponding to the function $N_1$ we define the sets $\mathcal{G}_1$ and $\mathcal{F}_1$ as follows

$$\mathcal{G}_1 \stackrel{def}{=} \{x \in Q' : N_1(x) = -\infty\}$$

and

$$\mathcal{F}_1 \stackrel{def}{=} \{Q : Q \text{ occured as a largest subcube in step } \mathcal{B}_1\} \; .$$

Similarly we define $N_2$, $\mathcal{G}_2$ and $\mathcal{F}_2$ as follows. Define $N_2 : Q' \to \{-\infty, \ldots, N-1, N\}$ by the following rules

$\mathcal{A}_2$ $N_2(x) = -\infty$ if $x \in \mathcal{G}_1$ or

$$|Q|^{-1/p+\epsilon}\int_Q |f| \le |\overline{Q}|^{-1/p+\epsilon}\int_{\overline{Q}} |f|$$

for all dyadic sub cubes $Q$ of $Q'$ such that $x \in Q \subset \overline{Q} \in \mathcal{F}_1$.

$\mathcal{B}_2$ Otherwise $N_2(x)$ is defined by letting $2^{N_2(x)}$ be the length of a side of the largest dyadic sub cube of $Q$ such that

$$|Q|^{-1/p+\epsilon}\int_Q |f| > |\overline{Q}|^{-1/p+\epsilon}\int_{\overline{Q}} |f| \, ,$$

and $x \in Q \subset \overline{Q} \in \mathcal{F}_1$.

Corresponding to the function $N_2$ we define the sets $\mathcal{G}_2$ and $\mathcal{F}_2$ as follows

$$\mathcal{G}_2 \stackrel{def}{=} \{x \in Q' : N_2(x) = -\infty\} \setminus \mathcal{G}_1$$

and

$$\mathcal{F}_2 \stackrel{def}{=} \{Q : Q \text{ occured as a largest subcube in step } \mathcal{B}_2\} \, .$$

In a similar fashion we define recursively $N_j(x)$, $\mathcal{G}_j$ and $\mathcal{F}_j$ for $j = 3, 4, \ldots$. These will obey the following properties

1. $\cup_{j=1}^{\infty}\mathcal{G}_j = Q'$

2. $\cup_{Q\in\mathcal{F}_k} Q = Q' \setminus \cup_{j=1}^{k}\mathcal{G}_j$

3. $N_{k+1}(x) \le N_k(x) - 1$

4. For all cubes $Q \in \mathcal{F}_k$ and $\overline{Q} \in \mathcal{F}_{k-1}$ such that $Q \subset \overline{Q}$ we have the inequality

$$|Q|^{-1/p+\epsilon}\int_Q |f| > |\overline{Q}|^{-1/p+\epsilon}\int_{\overline{Q}} |f|$$

5. $N_k(x) = -\infty$ if $x \in \cup_{j=1}^{k}\mathcal{G}_j$.

*2,3,4* and *5* follows immediately from the construction. *1* follows from *2* and the fact that if $f \in \mathcal{L}^\infty$ then $\mathcal{F}_k$ is empty for some finite value of $k$, since by $\mathcal{A}_k$ we have $\|f\|_\infty > \mathrm{Av}_Q|f| > (|Q'|/|Q|)^{1-1/p+\epsilon}\mathrm{Av}_{Q'}|f| = 2^{(k+i)(1-1/p+\epsilon)}\mathrm{Av}_{Q'}|f| \to \infty$ as $k \to \infty$, where $i \geq 0$ and $Q \in \mathcal{F}_k$.

Since the upper bound of $\Sigma_1$, (2.35), didn't depend on the condition (2.33) we need only analyze $\Sigma_2$. We have the following lemma

**Lemma 9**

$$\Sigma_2^s \leq C\|g\|_{u,v} \sum_{j=0}^{\infty} \sum_{Q\in\mathcal{F}_j} |Q|^{1-s/p-s/v} \left( \int_Q |f(x)|\, dx \right)^s . \tag{2.46}$$

□

**Proof of Lemma 9:** Since $f(x) \in \mathcal{L}^\infty(Q')$, $N_j(x) = -\infty$ for some finite value of $j$ for almost all $x \in Q'$. Using (2.36) we have the following inequality for $\Sigma_2^s$

$$\begin{aligned}
\Sigma_2^s &= \int_{Q'} |g(x)|^s \,|\sum_{n=-\infty}^{N} S_n f(x)|^s \, dx \\
&\leq s \int_{Q'} \sum_{k=-\infty}^{N} |S_k f(x)|\, |g(x)|^s (\sum_{n=k}^{N} |S_n f(x)|)^{s-1} \, dx \\
&= \sum_{j=1}^{\infty} s \int_{Q'} \sum_{k=N_j(x)+1}^{N_{j-1}(x)} |S_k f(x)|\, |g(x)|^s (\sum_{n=k}^{N} |S_n f(x)|)^{s-1} \, dx \\
&= C \sum_{j=1}^{\infty} a_j \ ,
\end{aligned}$$

where

$$a_j \stackrel{def}{=} \int_{Q'} \sum_{k=N_j(x)+1}^{N_{j-1}(x)} |S_k f(x)|\, |g(x)|^s (\sum_{n=k}^{N} |S_n f(x)|)^{s-1} \, dx \ . \tag{2.47}$$

Since all the cubes considered in the expression for $a_1$ satisfies (2.33) we conclude that

$$a_1 \leq C\|g\|_{u,v} |Q'|^{1-s/p-s/v} (\int_{Q'} |f(x)|\, dx)^s \ .$$

When treating $a_j$ we observe that since

$$a_j = \int_{N_{j-1}(x) > -\infty} [\text{Interior of equation (2.47)}]\, dx$$

$$\leq \sum_{Q\in\mathcal{F}_{j-1}} \int_Q [\text{Interior of equation (2.47)}]\, dx\ ,$$

we may treat the integral over the individual cubes $Q \in \mathcal{F}_{j-1}$. But we need as before an $\mathcal{L}^\infty$ type of estimate on $\sum_{n=k}^N |S_n f(x)|$. Fix a cube $\overline{Q} \in \mathcal{F}_{j-1}$ with $\overline{Q} = Q(\cdot, 2^M)$. Using the conditions $\mathcal{A}_j$ and $\mathcal{B}_j$ given in the Calderon–Zygmund decomposition we can bound $\sum_{n=M}^N |S_n f(x)|$ and $\sum_{n=k}^{M-1} |S_n f(x)|$. For the first sum we have

$$\begin{aligned}\sum_{n=M}^{N} |S_n f(x)| &\leq \sum_{n=M}^{N} 2^{3\epsilon(M-n)}|\overline{Q}|^{-1/p}\int_{\overline{Q}} |f(x)|\,dx \\ &\leq C|\overline{Q}|^{-1/p}\int_{\overline{Q}} |f(x)|\,dx\ ,\end{aligned}$$

whereas for the second sum we have

$$\begin{aligned}\sum_{n=k}^{M-1} |S_n f(x)| &\leq \sum_{n=k}^{M-1} 2^{3\epsilon(M-n)}|\overline{Q}|^{-1/p}\int_{\overline{Q}} |f(x)|\,dx \\ &\leq C2^{3\epsilon(M-k)}|\overline{Q}|^{-1/p}\int_{\overline{Q}} |f(x)|\,dx\ .\end{aligned}$$

Putting these two estimates together with the argument of Lemma 8 shows that

$$\int_Q [\text{Interior of equation (2.47)}]\, dx \leq |Q|^{1-s/p-s/v}(\int_Q |f(x)|\,dx)^s\ ,$$

which concludes the proof of Lemma 9. ■

We now need to estimate the individual terms of (2.46).

**Lemma 10**

$$\sum_{j=0}^{\infty}\sum_{Q\in\mathcal{F}_j} |Q|^{1-s/p-s/v}\left(\int_Q |f(x)|\,dx\right)^s \leq C\left(\int_{Q'} |f(x)|^q\,dx\right)^{s/q} \tag{2.48}$$

*if* $s \geq q$ *and* $1/s = 1/v + 1/q + 1/p - 1$. □

**Proof of Lemma 10.** Since $\|f\|_\infty \leq \infty$ we may assume that there exist some $\tau \geq 1$ such that $\mathcal{F}_\tau$ is empty. It follows that $Q' = \cup_{j=1}^{\tau}\mathcal{G}_j$. Consider for the rest of this proof the cube $Q$ to be a fixed element of $\mathcal{F}_j$ unless anything else is indicated. It is then evident that $Q \subset \cup_{m=j+1}^{\tau}\mathcal{G}_m$.

**Part I; estimating $|Q \cap \mathcal{G}_m|$.** We want to estimate $|Q \cap \mathcal{G}_m|$ for $m \geq j+1$. To do this we consider $\int_Q |f|$.

$$\begin{aligned}
\int_Q |f| &\geq \sum_{i=m}^{\tau} \int_{Q \cap \mathcal{G}_i} |f| \\
&= \sum_{\overline{Q} \in \mathcal{F}_{m-1};\, \overline{Q} \subset Q} \int_{\overline{Q}} |f| \\
&\geq \sum_{\overline{Q} \in \mathcal{F}_{m-1};\, \overline{Q} \subset Q} \left( \frac{|Q|}{|\overline{Q}|} \right)^{\epsilon - 1/p} \int_Q |f| \qquad (2.49) \\
&\geq \sum_{\overline{Q} \in \mathcal{F}_{m-1};\, \overline{Q} \subset Q} \left( \frac{|Q|}{|\overline{Q}|} \right)^{-1/p} \int_Q |f| \qquad (2.50) \\
&= \sum_{\overline{Q} \in \mathcal{F}_{m-1};\, \overline{Q} \subset Q} \left( \frac{|Q|}{|\overline{Q}|} \right)^{1-1/p} \frac{|\overline{Q}|}{|Q|} \int_Q |f| \\
&\geq \frac{2^{3(1-1/p)(m-j-1)}}{|Q|} \int_Q |f| \sum_{\overline{Q} \in \mathcal{F}_{m-1};\, \overline{Q} \subset Q} |\overline{Q}| \\
&\geq \frac{2^{3(1-1/p)(m-j-1)}}{|Q|} \int_Q |f| \, |\mathcal{G}_m \cap Q| \, .
\end{aligned}$$

Here we have used *2* and *3* of the Calderon–Zygmund decomposition in obtaining (2.49) and (2.50). Dividing both sides by $\int_Q |f|$ (we may assume $\int_Q |f| \neq 0$) we formally get the inequality

$$\frac{|\mathcal{G}_m \cap Q|}{|Q|} \leq 2^{-3(1-1/p)(m-j-1)} \, , \qquad (2.51)$$

which we will use to prove equation (2.48).

**Part II; inequality (2.48).** By use of Hölder's inequality we have

$$\begin{aligned}
\left( \int_Q |f| \right)^s &\leq \left( \sum_{m=j+1}^{\tau} \int_{Q \cap \mathcal{G}_m} |f| \right)^s \\
&\leq \left( \sum_{m=j+1}^{\tau} a_m^{q'} \right)^{s/q'} \times \left\{ \sum_{m=j+1}^{\tau} a_m^{-q} \left( \int_{Q \cap \mathcal{G}_m} |f| \right)^q \right\}^{s/q} ,
\end{aligned}$$

where $1/q + 1/q' = 1$, and

$$a_m = \left\{ \left( \frac{3}{2} \right)^{3(1-1/p)(m-j-1)} \frac{|Q \cap \mathcal{G}_m|}{|Q|} \right\}^{1/q'} .$$

In view of (2.51) we have

$$\sum_{m=j+1}^{\tau} a_m^{q'} \leq \sum_{m=j+1}^{\tau} \left(\frac{3}{4}\right)^{3(1-1/p)(m-j-1)} \leq C\,, \tag{2.52}$$

where $C$ depends on $p$ only. By Jensen's inequality we have

$$\left(\frac{1}{|Q\cap\mathcal{G}_m|}\int_{Q\cap\mathcal{G}_m}|f(x)|\,dx\right)^q \leq \frac{1}{|Q\cap\mathcal{G}_m|}\int_{Q\cap\mathcal{G}_m}|f(x)|^q\,dx\,, \tag{2.53}$$

since $dx/|Q\cap\mathcal{G}_m|$ is a probability measure on $Q\cap\mathcal{G}_m$. Using (2.52) and (2.53) we get

$$\begin{aligned}\left(\int_Q |f|\right)^s &\leq C\left\{\sum_{m=j+1}^{\tau}\left(\frac{2}{3}\right)^{3(1-1/p)(q-1)(m-j-1)}\right.\\ &\qquad\left.\left(\frac{|Q|}{|Q\cap\mathcal{G}_m|}\right)^{q-1}\left(\int_{Q\cap\mathcal{G}_m}|f|\right)^q\right\}^{s/q}\\ &\leq C|Q|^{s(1-1/q)}\left\{\sum_{m=j+1}^{\tau}\left(\frac{2}{3}\right)^{3(1-1/p)(q-1)(m-j-1)}\int_{Q\cap\mathcal{G}_m}|f|^q\right\}^{s/q}.\end{aligned}$$

Multiply by $|Q|^{1-s/p-s/v}$ and sum over cubes $Q\in\mathcal{F}_j$

$$\begin{aligned}\textstyle\sum_{Q\in\mathcal{F}_j} &|Q|^{1-s/p-s/v}\left(\textstyle\int_Q|f|\right)^s\\ &\leq C\textstyle\sum_{Q\in\mathcal{F}_j}|Q|^{1-s(1/p+1/v+1/q-1)}\left\{\sum_{m=j+1}^{\tau}\left(\frac{2}{3}\right)^{3(1-1/p)(q-1)(m-j-1)}\right.\\ &\qquad\left.\textstyle\int_{Q\cap\mathcal{G}_m}|f|^q\right\}^{s/q}\\ &\leq C\textstyle\sum_{Q\in\mathcal{F}_j}\left\{\sum_{m=j+1}^{\tau}\left(\frac{2}{3}\right)^{3(1-1/p)(q-1)(m-j-1)}\int_{Q\cap\mathcal{G}_m}|f|^q\right\}^{s/q}\\ &\leq C\left\{\textstyle\sum_{m=j+1}^{\tau}\left(\frac{2}{3}\right)^{3(1-1/p)(q-1)(m-j-1)}\int_{\mathcal{G}_m}|f|^q\right\}^{s/q},\end{aligned}$$

since $s/q\geq 1$. We now sum over $j$, interchange the order of summation and arrive at (2.48) in the end.

$$\begin{aligned}\textstyle\sum_{j=1}^{\tau} &\textstyle\sum_{Q\in\mathcal{F}_j}|Q|^{1-s/p-s/v}\left(\int_Q|f|\right)^s\\ &\leq C\left\{\textstyle\sum_{j=1}^{\tau}\sum_{m=j+1}^{\tau}\left(\frac{2}{3}\right)^{3(1-1/p)(q-1)(m-j-1)}\int_{\mathcal{G}_m}|f|^q\right\}^{s/q}\\ &\leq C\left\{\textstyle\sum_{m=1}^{\tau}\sum_{j=1}^{m-1}\left(\frac{2}{3}\right)^{3(1-1/p)(q-1)(m-j-1)}\int_{\mathcal{G}_m}|f|^q\right\}^{s/q}\\ &\leq C\left(\textstyle\int_{Q'}|f|^q\right)^{s/q}.\end{aligned}$$

This concludes the proof of Lemma 10. ■

Lemmas 9 and 10 proves that

$$\begin{aligned}\left(\int_{Q'} |T_K f|^s \, dx\right)^{1/s} &\leq C\|g\|_{u,v}\left(\int_{Q'} |f|^q\right)^{1/q} \\ &\leq C\|g\|_{u,v}\|f\|_{q,r}|Q|^{1/q-1/r} \\ &= C\|g\|_{u,v}\|f\|_{q,r}|Q|^{1/s-1/t} .\end{aligned}$$

This is exactly what we had left of the proof of Theorem 4.

### 2.4.3 Bootstrapping Our Way to the Hardy–Littlewood–Sobolev Inequality

Our intuition about Morrey spaces told us that $Tf$ defined by (2.17) should satisfy $s/t = q/r$ to preserve the dimensionality of the small maximal cubes (See the comment to Theorem 3). This turns out to be right.

**Theorem 11** *If $f \in M_q^r$ have support contained in a bounded domain then $\|Tf\|_{s,t} \leq C\|f\|_{q,r}$ for some constant $C = C(p,q,r)$ if $s \leq t$, $q \leq r$, $1 < p,q,r,s,t < \infty$, $1/t = 1/r + 1/p - 1$ and $s/t = q/r$.* □

Theorem 11 was pervously proved in [3], but we shall nevertheless provide a different proof.

**Proof of Theorem 11.** Instead of choosing $g(x) = 1$ in Theorem 4, we choose $g = (Tf)^{\alpha-1}$ for some $\alpha > 1$. This turns out to work better. Assume we already know that $Tf \in M_{s_0}^t$ and $\|Tf\|_{s_0,t} \leq \|f\|_{q,r}$ (e.g. $s_0$ given by Theorem 3). If $s_0 < s$ we wish to improve the index $s_0$. Ultimately we would like to show $Tf \in M_s^t$. Note that

$$Tf \in M_{s_0}^t \Leftrightarrow g \in M_{s_0/(\alpha-1)}^{t/(\alpha-1)}$$

and

$$\|g\|_{s_0/(\alpha-1),t/(\alpha-1)} = \|Tf\|_{s_0,t}^{\alpha-1} .$$

We also have $(T_g f) = (Tf)^\alpha$, so that

$$Tf \in M_s^t \Leftrightarrow T_g f \in M_{s/\alpha}^{t/\alpha} .$$

If we can show that $T_g f \in M_{s/\alpha}^{t/\alpha}$ and $\|T_g f\|_{s/\alpha,t/\alpha} \leq C\|g\|_{s_0/(\alpha-1),t/(\alpha-1)}\|f\|_{q,r}$ we will be done since

$$\|Tf\|_{s,t} = \|T_g f\|_{s/\alpha,t/\alpha}^{1/\alpha} \leq (C\|g\|_{s_0/(\alpha-1),t/(\alpha-1)}\|f\|_{q,r})^{1/\alpha} \leq C\|f\|_{q,r} .$$

Verifying the conditions of Theorem 4 we get

$$\alpha/t = (\alpha-1)/t + 1/p + 1/r - 1 = \alpha/t ,$$

which certainly is consistent. Also $s/\alpha \leq s_0/(\alpha-1)$, $s/\alpha \geq q$ and $\alpha/s = (\alpha - 1)/t + 1/p + 1/q - 1$ must be satisfied. The last equation can be simplified to $\alpha(1/s - 1/t) = 1/q - 1/r$ or equivalently $1/s = 1/t + 1/\alpha(1/q - 1/r)$. If $\alpha = s/q$ is a valid choice then $1/s = 1/t + q/s(1/q - 1/r)$ or simply $s/t = q/r$. This would prove the theorem. If $\alpha = s/q$ is not a valid choice, we may at least improve the value of $s_0$ by some factor less than $\alpha/(\alpha-1)$. Choosing $\alpha = s_0/q$ we get $\alpha/(\alpha-1) = s_0/(s_0-q) > 1$ and the improved value of $s_0$ is given by $1/s_0' = 1/t + q/s_0(1/q - 1/r) = 1/s_0 + 1/t - q/rs_0 = 1/s_0 + q/r(1/s - 1/s_0) < 1/s_0$ if $s_0 < s$, so that we can always improve $s_0$ if $s_0 < s$. If on the other hand $s_0 > (s-q)$ then $\alpha = s/q$ is a valid choice, so that $s = qt/r$ is proved in one step. Since $s, s_0$ are assumed to be finite it follows that the proof must end in finitely many iterations.■

### 2.4.4 Optimality of Theorem 11

The intuitive notion of preserving the dimension of small maximal cubes should give us the best possible value for $s$. To see this we construct a counterexample for larger values of $s$.

**Theorem 12** *The operator* $T : M_q^r \to M_s^t$ *is an unbounded operator for* $1/t = 1/r + 1/p - 1$, $s > tq/r$. $\square$

**Proof of Theorem 12.** Let $f(x) = 2^{3n/r}$ for $x$ in the cubes $Q_1, Q_2, \cdots, Q_{N_n}$ and $f(x) = 0$ elsewhere. Here $N_n = [2^{3n(1-q/r)}]$ and $\{Q_i\}_{i=1}^{N_n}$ are some cubes with volume $|Q_i| = 2^{-3n}$. Assume for the moment that we can distribute the cubes $\{Q_i\}_{i=1}^{N_n}$ in such a way that they are pairwise disjoint, lie inside the cube $Q_0 = Q(0,1)$, $f \in M_q^r$ and $\|f\|_{q,r} \leq 4$. If this is the case then for $x \in Q_i$

$$\begin{aligned}(Tf)(x) &= \int_{\mathbf{R}^3} \frac{f(y)}{|x-y|^{3/p}}\, dy \\ &\geq \int_{Q_i} \frac{f(y)}{|x-y|^{3/p}}\, dy \\ &\geq 2^{-3n(1-1/r-1/p)} \\ &= 2^{3n/t}\,.\end{aligned}$$

This tells us that

$$\begin{aligned}\int_{Q_0} |Tf|^s\, dx &\geq N_n 2^{-3n} 2^{3ns/t} \\ &= 2^{3n(s/t-q/r)}\,.\end{aligned}$$

If $\|Tf\|_{s,t} \leq C\|f\|_{q,r}$ we must however have

$$\int_{Q_0} |Tf|^s\, dx \leq C\ ,$$

since $|Q_0| = 1$. But because $2^{3n(s/t-q/r)} \leq C$ for all $n$ implies $s/t \leq q/r$ we conclude that $T$ is unbounded. It remains only to show that we can place $Q_1, Q_2, \cdots, Q_{N_n}$ inside $Q_0$ in such a way that $f \in M_q^r$ and $\|f\|_{q,r} \leq 4$. Let $Q$ be any cube $Q(\cdot, 2^{-m})$. Since $(\int_Q |f|^q)^{1/q} \leq 4|Q|^{1/q-1/r}$ is automatically satisfied for $|Q| \leq 2^{-3n}$ we may assume $0 \leq m < n$. Let $N_{n,m}$ be the number of cubes among $Q_1, Q_2, \cdots, Q_{N_n}$ inside the cube $Q$ (Assume for simplicity that all of the cubes intersecting $Q$ lies inside the closure of $Q$). We must have

$$\int_Q |f(x)|^q \, dx = N_{n,m} 2^{-3n(1-q/r)} \le 4|Q|^{1-q/r} ,$$

i.e. $N_{n,m} \le 2^{3(n-m)(1-q/r)}$. If we can place $Q_1, Q_2, \cdots, Q_{N_n}$ inside $Q_0$ without violating this condition we have $f \in M_q^r$ and $\|f\|_{q,r} \le 4$. We do this recursively. Partition $Q_0$ into 8 distinct cubes of equal size. Place 1/8'th of the cubes $Q_1, Q_2, \cdots, Q_{N_n}$ in each of the half cubes. To place 1/8'th of them in one half cube $Q(\cdot, 1/2)$ we partition each of the half cubes into 8 new cubes. Into each of these we place 1/8'th of the 1/8'th of the cubes $Q_1, Q_2, \cdots, Q_{N_n}$ that we were given. We continue this process until we have to place less than 8 of the cubes into one of the partitioned cubes. This we do randomly. Since $N_n \le 2^{3n} = 8^n$ the process will work and it guarantees that $\|f\|_{q,r} \le 4$. This completes the proof of Theorem 12. ■

## 2.5 The Schrödinger Equation for $V \in M_p^{3/2}$

We will try to bound the $\mathcal{L}^q$ norms, $1 \le q < \infty$, of the solution $u$ of (2.1) in terms of $\|V\|_{p,3/2}$ using Theorem 11.

### 2.5.1 Why is $M_p^{3/2}$ the "Right" Space

It may seem like we are pulling the space $M_p^{3/2}$ out of a magic hat. This is not so. To understand why $M_p^{3/2}$ is the right choice, we scale the function $u(x)$ solving (2.1). Call the scaled function $u_R$; $u_R(x) = u(Rx)$. $u_R$ then satisfies the equation

$$\triangle u_R(x) + R^2 V(Rx) u_R(x) = 0$$

for $|x| \le 1$ with the boundary condition

$$u_R(x) = 1$$

for $|x| = 1$. Let $V_R(x) = R^2 V(Rx)$. A reasonable requirement for a norm on the potential $V$ is that the norm should be invariant under the transformation $V \to V_R$. The spaces $M_p^{3/2}$, $1 \le p \le 3/2$ have this invariance property. Next we would like to understand what values of $p$ we should expect. To do this we study $u$ via the Feynman–Kac formula, i.e.

$$u(x) = E_x \left[ e^{\int_0^\tau V(X(t))dt} \right] .$$

Here $E_x$ denotes the expectation taken with respect to a three dimensional Brownian motion $X(t)$ started at $x$ at time 0. $\tau$ is the first time the Brownian motion exits the region $\{x : |x| \le R\}$. Since two dimensional Brownian Motion recurs infinitely often to a point , we may expect three dimensional Brownian Motion to recur infinitely often to a set of dimension 1. Therefore it is necessary to avoid potentials that are big on sets of dimension $\ge 1$. By the analysis of section 2.2 we expect $V$ to be big on a set of dimension $3(1 - 2p/3)$. This means that we need to have $p > 1$. This turns out to be right.

### 2.5.2 Hölder's Inequality for Morrey Spaces

Before continuing, we will prove a convenient analogue of Hölder's inequality. We already have an extended version of the Hardy–Littlewood–Sobolev inequality. In obtaining (2.15) we used Hölder's inequality as well as the Hardy–Littlewood–Sobolev inequality. So we will have to generalize Hölder's inequality to Morrey spaces.

**Lemma 13** *If $f \in M_{q_1}^{r_1}$ and $g \in M_{q_2}^{r_2}$ then $fg \in M_{q_3}^{r_3}$ and $\|fg\|_{q_3,r_3} \le \|f\|_{q_1,r_1}\|g\|_{q_2,r_2}$ if $1 \le q_i \le r_i \le \infty$ for $i = 1, 2$ and 3, $1/r_3 = 1/r_1 + 1/r_2$ and $1/q_3 = 1/q_1 + 1/q_2$.* $\square$

**Proof of Lemma 13.** Using Hölder's inequality we get

$$\left( \int_Q |f(x)g(x)|^{q_3} \, dx \right)^{1/q_3} \le \left( \int_Q |f(x)|^{q_1} \, dx \right)^{1/q_1} \left( \int_Q |g(x)|^{q_2} \, dx \right)^{1/q_2}$$

$$\begin{aligned} &\leq \|f\|_{q_1,r_1}\|g\|_{q_2,r_2}|Q|^{(1/q_1+1/q_2)-(1/r_1+1/r_2)} \\ &= \|f\|_{q_1,r_1}\|g\|_{q_2,r_2}|Q|^{1/q_3-1/r_3} \ . \end{aligned} \tag{2.54}$$

Since (2.54) is independent of the choice of the cube $Q$ we conclude that $fg \in M_{q_3}^{r_3}$ and $\|fg\|_{q_3,r_3} \leq \|f\|_{q_1,r_1}\|g\|_{q_2,r_2}$. ■

### 2.5.3 $\mathcal{L}^q$ Estimates for $V \in M_p^{3/2}$

**Theorem 14** *If $V \in M_p^{3/2}$, $u$ is the solution of (2.1), $v(x) = u(x) - 1$, $1 < p \leq 3/2$, $1 < s \leq t < \infty$ and $s < 2pt/3$ then*

$$\|v\|_{s,t} \leq C_1\|V\|_{p,3/2}|\Omega|^{1/t} \tag{2.55}$$

*and*

$$(\mathrm{Av}|v|^s)^{1/s} \leq C_2\|V\|_{p,3/2} \ , \tag{2.56}$$

*if $\|V\|_{p,3/2} < \epsilon$ and $\epsilon = \epsilon(p, q, r, s, t, C_1, C_2)$ is sufficiently small.* □

If we choose $s = 1$, then equation (2.56) reduces to the statement of Theorem 1.1 (b) of [8].

**Proof of Theorem 14.** By use of Lemma 13 we have

$$\|V\|_{q,r} \leq \|V\|_{p,3}|\Omega|^{1/r-2/3}$$

for $1 \leq q < p$, $q \leq r < 3/2$. By use of Theorem 11 we have

$$\|\int_\Omega G_D(\cdot, y)V(y)\,dy\|_{s,t} \leq C\|V\|_{p,3/2}|\Omega|^{1/r-2/3}$$

for $1/t = 1/r - 2/3$ and $s/t = q/r$. Using Hölder's inequality we see that

$$\|V(\cdot)\int_\Omega G_D(\cdot, y)V(y)\,dy\|_{q',r} \leq C\|V\|_{p,3/2}^2|\Omega|^{1/t} \ ,$$

where $1/q' = 1/s + 1/p$. It is easy to see that $q' > q$ since

$$\frac{1}{q'} = \frac{r}{tq} + \frac{1}{p} = \frac{r}{q}(\frac{1}{r} - \frac{2}{3}) + \frac{1}{p} = \frac{1}{q} + (\frac{1}{p} - \frac{2r}{3q}) = \frac{1}{q} + (\frac{1}{p} - \frac{2t}{3s}) \leq \frac{1}{q}$$

Consequently, $\|\cdot\|_{q,r} \leq \|\cdot\|_{q',r}$, which implies that

$$V(\cdot)\int_\Omega G_D(\cdot,y)V(y)\,dy \in M_q^r\ .$$

We now repeat the process for

$$\int_\Omega G_D(\cdot,y)V(y)\int_\Omega G_D(y,z)V(z)\,dz\,dy\ ,$$

and find

$$\begin{aligned} \|v\|_{s,t} &\leq \sum_{n=1}^{\infty}\left(C\|V\|_{p,3/2}\right)^n|\Omega|^{1/t} \\ &\leq C_1\|V\|_{p,3/2}|\Omega|^{1/t} \end{aligned}$$

if $\|V\|_{p,3/2} \leq 1/(2C)$. Since $s = qt/r < 2pt/3$ if $q < 2pr/3$ or $3q/(2p) < r < 3/2$, which is possible for all $1 \leq q < p$ by choosing $r$ sufficiently close to $3/2$, we have proved (2.55) for all $s < 2pt/3$. To prove (2.56) we observe that

$$\begin{aligned} \left(\int_\Omega |v|^s\,dx\right)^{1/s} &\leq \|v\|_{s,t}|\Omega|^{1/s-1/t} \\ &\leq C_1\|V\|_{p,3/2}|\Omega|^{1/s}\ , \end{aligned}$$

by (2.55). Fixing $t$ and dividing by $|\Omega|^{1/s}$ we get (2.56). ■

### 2.5.4 A Counterexample for $p = 1$

To see that the condition $p > 1$ in Theorem 14 is optimal we construct a function $V \in M_1^{3/2}$ for which $\|u\|_1$ is unbounded.

**Theorem 15** *The inequality*

$$\mathrm{Av}_\Omega|v| \leq C\|V\|_{1,3/2} \tag{2.57}$$

*does not hold in general for* $V \in M_1^{3/2}$ *and* $\|V\|_{1,3/2} < \epsilon$, *regardless of how small* $\epsilon > 0$ *is.* □

**Proof of Theorem 15.** By the comments about scaling in section 5.1 we see that we can choose $R = 8$ without loss of generality. Define the cubes $Q_i = Q((i \cdot 2^{-m}, 0, 0), 2^{-m})$ for $i = 0, 1, 2, \cdots, 2^m - 1$, where $m$ is some preassigned integer greater than 2. Let

$$V(x) = \begin{cases} \epsilon 2^{2m} & \text{if } x \in Q_i \text{ for } i = 0, 1, 2, \cdots, 2^m - 1 \\ 0 & \text{otherwise} . \end{cases}$$

To see that $V(x)$ is a counterexample observe first that it is in the desired space

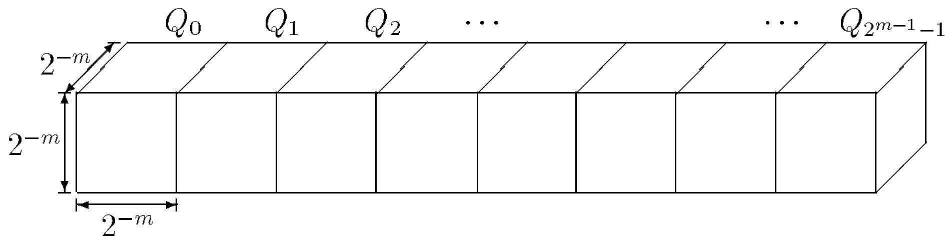


Figure 2.1: Placement of the cubes $\{Q_i\}_{i=0}^{2^m-1}$

$M_1^{3/2}$. Let $Q$ be any cube of side length $l$. Then $V(x) = 0$ everywhere in $Q$ except possibly on a strip of maximal width $2^{-m} \times 2^{-m}$ and maximal length $l$, i.e.

$$\int_Q |V(x)| \, dx \leq (l 2^{-2m}) \epsilon 2^{2m} = \epsilon l = \epsilon |Q|^{1/3} .$$

Therefore $V \in M_1^{3/2}$ and $||V||_{1,3/2} = \epsilon$.

To prove that $\mathrm{Av}_\Omega v$ is big we note that due to the positivity of $V$ we have

$$\begin{aligned} v(x) &= \int_\Omega G_D(x,y) V(y) \, dy \\ &\quad + \int_\Omega G_D(x,y) V(y) \int_\Omega G_D(y,z) V(z) \, dz \, dy + \cdots \\ &\geq \int_\Omega G_D(x,y) V(y) \int_\Omega G_D(y,z) V(z) \, dz \, dy . \end{aligned}$$

Taking the average on both sides we get

$$\mathrm{Av}_\Omega v(x) \geq \frac{C}{R} \int_{\Omega/4} \int_{\Omega/4} \frac{V(y)V(z)}{|y-z|} \, dy \, dz ,$$

because

$$G_D(x,y) = \frac{1}{4\pi}\left\{\frac{1}{|x-y|} - \frac{R}{|y|}\frac{1}{|x-\overline{y}|}\right\}$$

and

$$\frac{R}{|y|}\frac{1}{|x-\overline{y}|} \le \frac{R}{|y|(|\overline{y}|-|x|)} \le \frac{R}{R^2-|y|\,|x|} \le \frac{16}{15R} \le \frac{8}{15}\frac{1}{|x-y|}\ ,$$

so that $G_D(x,y) \ge 7/(60\pi|x-y|)$ if $x,y \in \Omega/4$. Since $\mathrm{supp}(V) \subset \Omega/4$ we have

$$\begin{aligned}
\int_\Omega \frac{V(z)}{|y-z|}\,dz \;&\ge\; C\sum_{i=1}^{2^m}(\epsilon 2^{2m})2^{-3m}\frac{1}{i2^{-m}} \\
&\ge\; C\epsilon\log(2^{m-1}) \\
&\ge\; C\epsilon m\ ,
\end{aligned}$$

for $m \ge 3$ and $y \in Q_i$ for some $i \in \{0,1,\cdots,2^m-1\}$. We therefore have

$$\begin{aligned}
\mathrm{Av}_\Omega v(x) \;&\ge\; C\int_{\Omega/4} V(y)\int_{\Omega/4}\frac{V(z)}{|y-z|}\,dz\,dy \\
&\ge\; C\epsilon m\sum_{i=0}^{2^m-1}\int_{Q_i} V(y)\,dy \\
&=\; C\epsilon^2 m\ . \qquad (2.58)
\end{aligned}$$

Letting $m \to \infty$ we see by (2.58) that (2.57) cannot hold regardless of how small $\epsilon > 0$ is. ■

## 2.6 Potentials of the Form $V = -\vec{\nabla}\cdot\vec{b}$

The solution $u(x)$ of (2.1) is given, as noted in section 5.1, by the Feynman–Kac formula

$$u(x) = E_x\left[e^{\int_0^\tau V(X(t))\,dt}\right]\ . \qquad (2.59)$$

Let $V(x) = V_+(x) - V_-(x)$, where $V_+(x) = \mathrm{Max}\{0, V(x)\}$. We see from (2.59) that

$$u(x) \le E_x\left[e^{\int_0^\tau V_+(X(t))\,dt}\right]\ ,$$

so that by Theorem 14

$$(\mathrm{Av}_\Omega |u|^s)^{1/s} \leq 1 + C \|V_+\|_{p,3/2} \;, \tag{2.60}$$

for all $1 < p \leq 3/2$ and $1 \leq s < \infty$. The estimate (2.60) is good if $V(x)$ is positive, but can be very bad for potentials with large fluctuations. We would ideally like to consider potentials with more fluctuations than allowed in $M_1^{3/2}$. One such potential takes values 1 or -1 with probability 1/2 in the cubes $Q(n,1)$ for $n \in \mathbf{Z}^3$. Letting $R \to \infty$ one can show that the quantity

$$\tilde{u} = \frac{E_x\left[e^{\int_0^\infty V(X(t))\,dt}\right]}{E_V\,E_x\left[e^{\int_0^\infty V(X(t))\,dt}\right]}$$

exists with probability one (see [5]). (Here $E_V$ denotes the expectation with respect to the value of $V$ on the cubes $Q(n,1)$.)

These problems require that one understand how the fluctuations affects the value of $\tilde{u}$. This is a difficult problem, so we shall be content to consider potentials $V = -\vec{\nabla} \cdot \vec{b}$, $\vec{b}$: $\mathbf{R}^3 \to \mathbf{R}^3$. This allows $V$ to change sign rapidly, and still we will be able to bound $\|u\|_s$ in terms of $\| \vec{b} \|_{p,3}$ for $2 < p \leq 3$.

### 2.6.1 Perturbation Series Revisited

Assume that $\vec{b}$: $\mathbf{R}^3 \to \mathbf{R}^3$ is continuously differentiable. This allows us to do integration by parts, and yet our final estimate will not depend on this, so we may dispose of the condition later.

Inserting $V = -\vec{\nabla} \cdot \vec{b}$ to the perturbation series (2.9) we get

$$\begin{aligned} u(x) &= 1 - \int_\Omega G_D(x,x_1)\,\vec{\nabla} \cdot \vec{b}\,(x_1)\,dx_1 \\ &\quad + \int_\Omega G_D(x,x_1)\,\vec{\nabla} \cdot \vec{b}\,(x_1) \int_\Omega G_D(x_1,x_2)\,\vec{\nabla} \cdot \vec{b}\,(x_2)\,dx_2\,dx_1 \\ &\quad - \int_\Omega G_D(x,x_1)\,\vec{\nabla} \cdot \vec{b}\,(x_1) \int_\Omega G_D(x_1,x_2)\,\vec{\nabla} \cdot \vec{b}\,(x_2) \end{aligned}$$

$$\int_\Omega G_D(x_2,x_3)\,\vec{\nabla}\cdot\vec{b}\,(x_3)\,dx_3\,dx_2\,dx_1$$
$$+\cdots\,. \tag{2.61}$$

Applying integration by parts to (2.61) and using $G_D(x,y)=0$ if $x$ or $y$ is in $\partial\Omega$, we obtain the formula

$$\begin{aligned}
u(x) \;=\;& 1+\int_\Omega \vec{\nabla}_{x_1}\,G_D(x,x_1)\cdot\vec{b}\,(x_1)\,dx_1 \\
&+\int_\Omega \vec{\nabla}_{x_1}\,G_D(x,x_1)\cdot\vec{b}\,(x_1)\int_\Omega \vec{\nabla}_{x_2}\,G_D(x_1,x_2)\cdot\vec{b}\,(x_2)\,dx_2\,dx_1 \\
&+\int_\Omega G_D(x,x_1)\,\vec{b}\,(x_1)\cdot\vec{\nabla}_{x_1}\int_\Omega \vec{\nabla}_{x_2}\,G_D(x_1,x_2)\cdot\vec{b}\,(x_2)\,dx_2\,dx_1 \\
&+\int_\Omega \vec{\nabla}_{x_1}\,G_D(x,x_1)\cdot\vec{b}\,(x_1)\int_\Omega \vec{\nabla}_{x_2}\,G_D(x_1,x_2)\cdot\vec{b}\,(x_2) \\
&\int_\Omega \vec{\nabla}_{x_3}\,G_D(x_2,x_3)\cdot\vec{b}\,(x_3)\,dx_3\,dx_2\,dx_1 \\
&+\int_\Omega \vec{\nabla}_{x_1}\,G_D(x,x_1)\cdot\vec{b}\,(x_1)\int_\Omega G_D(x_1,x_2)\,\vec{b}\,(x_2)\cdot\vec{\nabla}_{x_2} \\
&\int_\Omega \vec{\nabla}_{x_3}\,G_D(x_2,x_3)\cdot\vec{b}\,(x_3)\,dx_3\,dx_2\,dx_1 \\
&+\int_\Omega G_D(x,x_1)\,\vec{b}\,(x_1)\cdot\vec{\nabla}_{x_1}\int_\Omega \vec{\nabla}_{x_2}\,G_D(x_1,x_2)\cdot\vec{b}\,(x_2) \\
&\int_\Omega \vec{\nabla}_{x_3}\,G_D(x_2,x_3)\cdot\vec{b}\,(x_3)\,dx_3\,dx_2\,dx_1 \\
&+\int_\Omega G_D(x,x_1)\,\vec{b}\,(x_1)\cdot\vec{\nabla}_{x_1}\int_\Omega G_D(x_1,x_2)\,\vec{b}\,(x_2)\cdot\vec{\nabla}_{x_2} \\
&\int_\Omega \vec{\nabla}_{x_3}\,G_D(x_2,x_3)\cdot\vec{b}\,(x_3)\,dx_3\,dx_2\,dx_1 \\
&+\cdots\,.
\end{aligned} \tag{2.62}$$

(2.62) is a bit more complicated than (2.9). To organize (2.62) we introduce the operators

$$S_1 f(x) = \int_\Omega G_D(x,y)f(y)\,dy \tag{2.63}$$
$$S_2\,\vec{f}\,(x) = \int_\Omega \vec{\nabla}_y\,G_D(x,y)\cdot\vec{f}\,(y)\,dy \tag{2.64}$$
$$S_2^*\,\vec{f}\,(x) = \int_\Omega \vec{\nabla}_x\,G_D(x,y)\cdot\vec{f}\,(y)\,dy \tag{2.65}$$
$$\overrightarrow{S_3}\,\vec{f}\,(x) = \vec{\nabla}_x\int_\Omega \vec{\nabla}_y\,G_D(x,y)\cdot\vec{f}\,(y)\,dy\,. \tag{2.66}$$

From now on we will suppress the vector sign in $\vec{S}_3$ and merely write $S_3$. We may then rewrite (2.62) using the operators defined by (2.63), (2.64), (2.65) and (2.66)

$$\begin{aligned} u(x) &= 1 + S_2 \vec{b} + S_2 \vec{b} S_2 \vec{b} + S_1 \vec{b} \cdot S_3 \vec{b} + S_2 \vec{b} S_2 \vec{b} S_2 \vec{b} \\ &\quad + S_2 \vec{b} S_1 \vec{b} \cdot S_3 \vec{b} + S_1 \vec{b} S_2^* \vec{b} \cdot S_3 \vec{b} + S_1 \vec{b} \cdot S_3 \vec{b} S_2 \vec{b} \\ &\quad + \cdots . \end{aligned} \tag{2.67}$$

More formally we may write

$$u(x) = \prod_{k=0}^{\infty} \left( \sum_{i_k=0}^{\infty} \sum_{j_k=0}^{\infty} (S_2 \vec{b})^{i_k} S_1 \vec{b} (S_2^* \vec{b})^{j_k} \cdot S_3 \vec{b} \right) .$$

By controlling the Morrey norms of the operators $S_1, S_2, S_2^*$ and $S_3$ we get an analogous result to that of Theorem 14.

### 2.6.2 Generalizing the Calderon–Zygmund Inequality

Define the operator

$$H_{i,j} f(x) = \frac{\partial^2}{\partial x_i \partial x_j} \int_{\mathbf{R}^3} \frac{f(y)}{|x-y|} \, dy . \tag{2.68}$$

The Calderon–Zygmund inequality then states that

$$||H_{i,j} f||_p \leq C ||f||_p ,$$

for all $i, j = 1, 2$ and $3$, where $C$ depends only on $1 < p < \infty$ (See [18] or [24]). We would like to have a similar result for Morrey spaces.

**Theorem 16** *$H_{i,j} : M_q^r \to M_q^r$ defined by (2.68) is a bounded operator for all $i, j = 1, 2$ or $3$ if $1 < q \leq r < \infty$ and $||H_{i,j} f||_{q,r} \leq C ||f||_{q,r}$, where $C$ is a universal constant depending on $q$ and $r$ only.* □

**Proof of Theorem 16.** The Calderon–Zygmund inequality proves the theorem for $q = r$. To prove the theorem more generally, we decompose f according to the cube $Q$

we wish to integrate $H_{i,j}$ over; $f(x) = f_1(x) + f_2(x)$ and $f_1(x) = \chi_{4Q}(x) f(x)$, where $4Q$ is the cube with 4 times the side length of $Q$ and with the same center. Since $f(x) \in M_q^r$ implies that $f \in \mathcal{L}_{loc}^q$, it follows that $f_1 \in \mathcal{L}^q$. We may therefore apply the Calderon–Zygmund inequality to $f_1(x)$

$$\begin{aligned} \|H_{i,j} f_1\|_q &\leq C \|f_1\|_q \\ &= C \left( \int_{4Q} |f(x)|^q \, dx \right)^{1/q} \\ &\leq C \|f\|_{q,r} |4Q|^{1/q - 1/r} \\ &= C \|f\|_{q,r} |Q|^{1/q-1/r} \, . \end{aligned}$$

On the other hand, we also have

$$\begin{aligned} \left( \int_Q |H_{i,j} f_1(x)|^q \, dx \right)^{1/q} &\leq \left( \int_{\mathbf{R}^3} |H_{i,j} f_1(x)|^q \, dx \right)^{1/q} \\ &= \|H_{i,j} f_1\|_q \\ &\leq C \|f\|_{q,r} |Q|^{1/q-1/r} \, . \end{aligned}$$

We must prove a similar inequality for $H_{i,j} f_2(x)$. Since the partial derivatives of second order of $1/|x-y|$ is integrable over $x \in Q$ when $y \notin 4Q$, we may carry the partial derivatives inside the integral sign! For $x \in Q$ we have

$$\begin{aligned} |H_{i,j} f_2(x)| &= \left| \frac{\partial^2}{\partial x_i \partial x_j} \int_{\mathbf{R}^3 \setminus 4Q} \frac{f(y)}{|x-y|} \, dy \right| \\ &\leq C \int_{\mathbf{R}^3 \setminus 4Q} \frac{|f(y)|}{|x-y|^3} \, dy \\ &\leq C \sum_{n=0}^{\infty} \int_{2^{n+2} Q} \frac{|f(y)|}{2^{3(n+2)} |Q|} \, dy \\ &\leq C \sum_{n=0}^{\infty} 2^{-3n} |Q|^{-1} \|f\|_{q,r} |2^{n+2} Q|^{1-1/r} \\ &\leq C \sum_{n=0}^{\infty} 2^{-3n/r} |Q|^{-1/r} \|f\|_{q,r} \\ &\leq C \|f\|_{q,r} |Q|^{-1/r} \, . \end{aligned}$$

Using Minkowski's inequality we get

$$\begin{aligned}\left(\int_Q |H_{i,j}f(x)|^q\,dx\right)^{1/q} &= \left(\int_Q |H_{i,j}f_1(x)+H_{i,j}f_2(x)|^q\,dx\right)^{1/q}\\ &\le \left(\int_Q |H_{i,j}f_1(x)|^q\,dx\right)^{1/q} + \left(\int_Q |H_{i,j}f_2(x)|^q\,dx\right)^{1/q}\\ &\le C\|f\|_{q,r}|Q|^{1/q-1/r}\ ,\end{aligned}$$

from which it follows that $H_{i,j}f \in M_q^r$ if $f \in M_q^r$ and $\|H_{i,j}f\|_{q,r} \le C\|f\|_{q,r}$, where C is a universal constant depending only on $q, r$. ■

Now that we know the behavior of $H_{i,j}$ we can go on to study the behavior of $S_3$.

**Theorem 17** *If $f \in M_q^r(\Omega)$ and $1 < q \le r < \infty$ then $S_3 f \in M_q^r(\Omega)$ and $\|S_3 f\|_{q,r} \le C\|f\|_{q,r}$ for some constant $C$ depending only on $q$ and $r$.* □

When differentiating $G_D(x,y)$ with respect to $y_j$ we see that $\partial/\partial y_j|x-y|^{-1} = -\partial/\partial x_j|x-y|^{-1}$. Applying this formula we may write

$$4\pi S_3 f(x) = \sum_{i,j=1}^{3} -(H_{i,j}(f\chi_\Omega)(x) + K_{i,j}f(x))\ \vec{\mathbf{e}}_i\ , \tag{2.69}$$

where $\vec{\mathbf{e}}_i$ is the $i$'th unit vector, and

$$K_{i,j}f(x) = \frac{\partial}{\partial x_i}\int_\Omega f(y)\frac{\partial}{\partial y_j}\left\{\frac{R}{|y|}\frac{1}{|x-\overline{y}|}\right\}\,dy\ . \tag{2.70}$$

To prove Theorem 17 we need to show that $\|K_{i,j}f\|_{q,r} \le C\|f\|_{q,r}$.

**Lemma 18** *If $f \in M_q^r(\Omega)$ and $1 < q \le r < \infty$ then $K_{i,j}f \in M_q^r(\Omega)$ and $\|K_{i,j}f\|_{q,r} \le C\|f\|_{q,r}$.* □

**Proof of Lemma 18.** From (2.70) we see that the operator $K_{i,j}$ has a singular kernel only when $x$ is at the boundary. We will find it convenient to split the operator into two new operators; $L_{i,j}$ and $M_{i,j}$.

$$K_{i,j}f(x) = L_{i,j}f(x) + M_{i,j}f(x)\ , \tag{2.71}$$

where

$$L_{i,j}f(x) = \frac{\partial}{\partial x_i}\int_{B(0,R/2)} f(y)\frac{\partial}{\partial y_j}\left\{\frac{R}{|y|}\frac{1}{|x-\overline{y}|}\right\}\,dy \tag{2.72}$$

and

$$M_{i,j}f(x) = \frac{\partial}{\partial x_i}\int_{R/2\le|y|<R} f(y)\frac{\partial}{\partial y_j}\left\{\frac{R}{|y|}\frac{1}{|x-\overline{y}|}\right\}\,dy\ . \tag{2.73}$$

Upon differentiating $R/(|y|\,|x-\overline{y}|)$, we find

$$\begin{aligned}\frac{\partial^2}{\partial x_i\partial y_j}\left\{\frac{R}{|y|}\frac{1}{|x-\overline{y}|}\right\} &= \frac{Ry_j}{|y|^3}\frac{x_i-\overline{y}_i}{|x-\overline{y}|^3} - \frac{3R}{|y|^3}\frac{x_i-\overline{y_i}}{|x-\overline{y}|^5}\times\\ &\quad \left(2y_j\overline{y}\cdot(x-\overline{y}) - (x_j-\overline{y_j})R^2\right)\ ,\end{aligned} \tag{2.74}$$

for $i\neq j$, and

$$\begin{aligned}\frac{\partial^2}{\partial x_i\partial y_i}\left\{\frac{R}{|y|}\frac{1}{|x-\overline{y}|}\right\} &= \frac{Ry_i}{|y|^3}\frac{x_i-\overline{y}_i}{|x-\overline{y}|^3} + \left(\frac{R^2}{|y|^2}-\frac{2R^2y_i^2}{|y|^4}\right)\times\\ &\quad \frac{R}{|y|}\frac{1}{|x-\overline{y}|^3} - \frac{3R}{|y|^3}\frac{x_i-\overline{y_i}}{|x-\overline{y}|^5}\times\\ &\quad \left(2y_i\overline{y}\cdot(x-\overline{y}) - (x_i-\overline{y_i})R^2\right)\ ,\end{aligned} \tag{2.75}$$

for $i=j$. Since $|y|<R/2$ for the operator $L_{i,j}$, we have

$$|y|\,|x-\overline{y}| \ge |y|(|\overline{y}|-|x|) \ge R^2/2\ . \tag{2.76}$$

Using (2.76) together with (2.74) and (2.75) respectively, we see that

$$\begin{aligned}\left|\frac{\partial^2}{\partial x_i\partial y_j}\left\{\frac{R}{|y|}\frac{1}{|x-\overline{y}|}\right\}\right| &\le \frac{R}{(R^2/2)^2} + \frac{3R\times 3R^2}{(R^2/2)^3}\\ &= \frac{76}{R^3}\ ,\end{aligned} \tag{2.77}$$

for $i\neq j$, and

$$\begin{aligned}\left|\frac{\partial^2}{\partial x_i\partial y_i}\left\{\frac{R}{|y|}\frac{1}{|x-\overline{y}|}\right\}\right| &\le \frac{R}{(R^2/2)^2} + \frac{3R^3}{(R^2/2)^3} + \frac{3R\times 3R^2}{(R^2/2)^3}\\ &= \frac{100}{R^3}\ ,\end{aligned} \tag{2.78}$$

for $i = j$. From this we see by (2.77) and (2.78) that

$$\begin{aligned}
\left(\int_Q |L_{i,j}f(x)|^q\, dx\right)^{1/q} &\le \left(\frac{C}{R^3}(\|f\|_{q,r}|\Omega|^{1/q-1/r})^q|Q|\right)^{1/q} \\
&\le C\|f\|_{q,r}|\Omega|^{-1/r}|Q|^{1/q} \\
&\le C\|f\|_{q,r}|Q|^{1/q-1/r}\,, \qquad (2.79)
\end{aligned}$$

since $|Q| < |\Omega|$ may be assumed for $L_{i,j}f \in M_q^r(\Omega)$. This shows that $\|L_{i,j}f\|_{q,r} \le C\|f\|_{q,r}$. If we can get the same inequality for $M_{i,j}$ we will be done by (2.71). Let us try to reduce the operator $M_{i,j}$ to the form of the operator $H_{i,j}$. This can be done in two steps. First we make all partial $y$-derivatives in to partial $x$-derivatives. Secondly we substitute $y$ into $\overline{y}$ to get the denominator on the form $|x-y|$.

The actual calculations are as follows

$$\begin{aligned}
\frac{\partial}{\partial y_j}\left\{\frac{R}{|y|}\frac{1}{|x-\overline{y}|}\right\} &= -\frac{Ry_j}{|y|^3}\frac{1}{|x-\overline{y}|} + \frac{R^3}{|y|^3}\frac{x_j-\overline{y}_j}{|x-\overline{y}|^3} - \\
&\quad \frac{2R^3y_j\, y\cdot(x-\overline{y})}{|y|^5|x-\overline{y}|^3} \\
&= -\frac{Ry_j}{|y|^3}\frac{1}{|x-\overline{y}|} - \frac{\partial}{\partial x_j}\left(\frac{R^3}{|y|^3}\frac{1}{|x-\overline{y}|}\right) + \\
&\quad \vec{\nabla}_x\cdot\left(\frac{2R^3y_jy}{|y|^5}\frac{1}{|x-\overline{y}|}\right)\,. \qquad (2.80)
\end{aligned}$$

We decompose $M_{i,j}$ according to the terms in (2.80).

$$M_{i,j}f(x) = -M_{i,j}^1f(x) - M_{i,j}^2f(x) + M_{i,j}^3f(x)\,, \qquad (2.81)$$

where

$$M_{i,j}^1f(x) = \frac{\partial}{\partial x_i}\int_{R/2\le|y|<R}\frac{Ry_j}{|y|^3}\frac{f(y)}{|x-\overline{y}|}\,dy\,, \qquad (2.82)$$

$$M_{i,j}^2f(x) = \frac{\partial^2}{\partial x_i\partial x_j}\int_{R/2\le|y|<R}\frac{R^3}{|y|^3}\frac{f(y)}{|x-\overline{y}|}\,dy \qquad (2.83)$$

and

$$M_{i,j}^3f(x) = \sum_{k=1}^3\frac{\partial^2}{\partial x_i\partial x_k}\int_{R/2\le|y|<R}\frac{2R^3y_jy_k}{|y|^5}\frac{f(y)}{|x-\overline{y}|}\,dy\,. \qquad (2.84)$$

Substituting $\overline{y} = R^2 y/|y|^2$ we get

$$\frac{\partial \overline{y_i}}{\partial y_j} = \begin{cases} -\frac{2R^2 y_i y_j}{|y|^4} & \text{if } i \neq j \\ \frac{R^2}{|y|^2} - \frac{2R^2 y_i^2}{|y|^4} & \text{if } i = j\ . \end{cases} \tag{2.85}$$

From (2.85) we find that the Jacobian determinant is given by

$$\operatorname{Det}\left(\frac{\partial \overline{y_i}}{\partial y_j}\right)_{i,j=1}^{3} = -\left(\frac{R^2}{|y|^2}\right)^3 , \tag{2.86}$$

and we may write (2.82), (2.83) and (2.84) on the following form

$$M^1_{i,j} f(x) = \frac{\partial}{\partial x_i}\int_{R\leq |z|<2R} \frac{f^1(z)}{|x-z|}\, dz\ , \tag{2.87}$$

$$M^2_{i,j} f(x) = \frac{\partial^2}{\partial x_i \partial x_j}\int_{R\leq |z|<2R} \frac{f^2(z)}{|x-z|}\, dz \tag{2.88}$$

and

$$M^3_{i,j} f(x) = \sum_{k=1}^{3} \frac{\partial^2}{\partial x_i \partial x_k}\int_{R\leq |z|<2R} \frac{f^3_k(z)}{|x-z|}\, dz\ , \tag{2.89}$$

where the functions $f^1$, $f^2$ and $f^3_k$ are related to $f$ by the relations

$$f^1(y) = \frac{R\overline{y}_j}{|\overline{y}|^3}\left(\frac{|\overline{y}|^2}{R^2}\right)^3 f(\overline{y})\ , \tag{2.90}$$

$$f^2(y) = \frac{R^3}{|\overline{y}|^3}\left(\frac{|\overline{y}|^2}{R^2}\right)^3 f(\overline{y}) \tag{2.91}$$

and

$$f^3_k(y) = \frac{2R^3\overline{y}_j\overline{y}_k}{|\overline{y}|^5}\left(\frac{|\overline{y}|^2}{R^2}\right)^3 f(\overline{y})\ . \tag{2.92}$$

Using $R/2 \leq |\overline{y}| < R$ in (2.90), (2.91) and (2.92), we see that

$$|f^1(y)| \leq \frac{|f(\overline{y})|}{R}\ , \tag{2.93}$$

$$|f^2(y)| \leq |f(\overline{y})| \tag{2.94}$$

and

$$|f^3_k(y)| \leq 2|f(\overline{y})|\ . \tag{2.95}$$

Let $\overline{f}(y) = f(\overline{y})$. If $f \in M_q^r(B(0,R) \setminus B(0,R/2))$ then $\overline{f} \in M_q^r(B(0,2R) \setminus B(0,R))$ and $\|\overline{f}\|_{q,r} \leq C\|f\|_{q,r}$ since $|y_1 - y_2| \leq C|\overline{y_1} - \overline{y_2}|$ for all $y_1, y_2 \in B(0,R) \setminus B(0,R/2)$. Using this together with (2.93), (2.94) and (2.95) we see that $f^1$, $f^2$ and $f_k^3$ are all in $M_q^r$, with

$$\|f^1\|_{q,r} \leq \frac{C\|f\|_{q,r}}{R} , \tag{2.96}$$

$$\|f^2\|_{q,r} \leq C\|f\|_{q,r} \tag{2.97}$$

and

$$\|f_k^3\|_{q,r} \leq C\|f\|_{q,r} . \tag{2.98}$$

By application of Theorem 16 it follows immediately from (2.88), (2.89), (2.97) and (2.98) that $M_{i,j}^2 : M_q^r \to M_q^r$ and $M_{i,j}^3 : M_q^r \to M_q^r$ are bounded operators satisfying $\|M_{i,j}^k f\|_{q,r} \leq C\|f\|_{q,r}$ for $i,j = 1,2,3$ and $k = 2,3$.

For $k = 1$ we need a different approach. We note that $|\partial/\partial x_i |x-z|^{-1}| \leq |x-z|^{-2}$, and that $\|f\|_{q,r} \leq \|f\|_{q',r'}(2R)^{3(1/r'-1/r)}$ for $f \in M_q^r(2\Omega)$, $r' \leq r$, $q' \leq q$, $1 \leq q' \leq r' \leq \infty$ and $1 \leq q \leq r \leq \infty$. Using this we can apply Theorem 3 to the operator $M_{i,j}^1$. If $1 < q \leq r < \infty$ we get

$$\|M_{i,j}^1 f\|_{q,r} \leq CR^{1-\alpha}\|M_{i,j}^1 f\|_{u,v} \leq CR^{1-\alpha}\|f^1\|_{s,t} \leq CR\|f^1\|_{q,r} \leq C\|f\|_{q,r} ,$$

where

$$1/t = 1/r + \alpha/3 , \quad 1/v = 1/t - 1/3 , \quad 1/r = 1/v + (1-\alpha)/3 ,$$

$$1/s = 1/q + \beta/3 , \quad 1/u = 1/s - 1/3 , \quad 1/q = 1/u + (1-\beta)/3 ,$$

$1 < s \leq t < \infty$ and $1 < u \leq v < \infty$, for some appropriate choice of $0 \leq \alpha \leq 1$ and $0 \leq \beta \leq 1$. One possible choice of $\alpha$ and $\beta$ is

$$\beta = \begin{cases} 0 & \text{if } 1 < q < 2 \text{ and } q \leq r \\ 1 & \text{if } 2 \leq q \leq r \end{cases}$$

and

$$\beta = \begin{cases} 0 & \text{if } 1 < q \leq r < 2 \\ 3(1/2 - 1/r) & \text{if } 1 < q < 2 \text{ and } 2 \leq r \leq 6 \\ 1 & \text{if } r > 6 \text{ or } q \geq 2\,. \end{cases}$$

We leave it to the reader to show that this choice of $\alpha$ and $\beta$ works. Using (2.81) we now see by the triangle inequality for Morrey spaces that $\|M_{i,j}f\|_{q,r} \leq C\|f\|_{q,r}$, where C is again a universal constant depending only on $q$ and $r$. By use of (2.79) and (2.71) we get $\|K_{i,j}f\|_{q,r} \leq C\|f\|_{q,r}$, which proves Lemma 18. ■

**Proof of Theorem 17.** Theorem 17 follows directly from (2.69), Theorem 16 and Lemma 18. ■

### 2.6.3 $\mathcal{L}^q$ Estimates for $\vec{b} \in M_p^3$

Let $V \in M_q^r$ and assume $u \in M_s^t$. We will verify that the operators $S_1, S_2, S_2^*$ and $S_3$ map the spaces $M_q^r, M_s^t$ and $M_u^v$ around as in figure 2.2, for some values of $q, r, s, t, u$ and $v$. If all the arrows of figure 2.2 is correct and the norms of all the

$$\begin{tikzcd}
M_q^r \arrow[loop left, "S_3"] \arrow[rr, shift right, "S_2^*"'] \arrow[dr, shift left, "S_2"] & & M_u^v \arrow[ll, shift right, "\text{Dot product with } \vec{b}"'] \arrow[dl, "S_1"] \\
& M_s^t \arrow[ul, shift left, "\text{Multiplication by } \vec{b}"] &
\end{tikzcd}$$

Figure 2.2: Mapping diagram for the operators $S_1, S_2, S_2^*, S_3$, multiplication by $\vec{b}$ and dot product with $\vec{b}$.

operators $S_1, S_2, S_2^*$ and $S_3$ are bounded, then

**Theorem 19** *If $u$ is the solution of (2.1) with $V = -\vec{\nabla}_y \cdot \vec{b}$, $v(x) = u(x) - 1$, $b \in M_p^3$, $2 + 3/t < p \leq 3$, $1 < s \leq t < \infty$ and $s \leq pt/3$ then*

$$\|v\|_{s,t} \leq C_1 \|b\|_{p,3} |\Omega|^{1/t} \tag{2.99}$$

*and*

$$(\mathrm{Av}_\Omega |v|^s)^{1/s} \leq C_2 \|b\|_{p,3} \ , \tag{2.100}$$

*if $\|b\|_{p,3} < \epsilon$ and $\epsilon = \epsilon(p, s, t, C_1, C_2)$ is sufficiently small.* □

**Proof of Theorem 19.** Note that in the perturbation series (2.67) there are exactly $2^n$ terms with $n + 1$ of the operators $S_1, S_2, S_2^*$ and $S_3$ and $n + 1$ multiplications / dot products with $\vec{b}$. This means that if figure 2.2 is right and the operator norms of $S_1, S_2, S_2^*$ and $S_3$ are all bounded by a constant then

$$\|v\|_{s,t} \leq \sum_{n=1}^{\infty} 2^{n-1} (C \| \vec{b} \|_{p,3})^n \leq C_1 \| \vec{b} \|_{p,3}$$

if $\| \vec{b} \|_{p,3} \leq 1/(4C)$. This would prove Theorem 19. Let us list all the results we have for the operators of figure 2.2 in a table and then verify that we don't have any contradictions. We assume in the table that the conditions $1 < u \leq v < \infty$, $1 < s \leq t < \infty$ and $1 < q \leq r < \infty$ are already satisfied. we can easily check that the first condition of each of the entries of table 1 holds if $1 < v < 3/2 < r < 3 < t < \infty$ and $1/v = 1/r + 1/3$, $1/r = 1/t + 1/3$. In other words if we choose a value of $t$ in the region $3 < t < \infty$ then $v$ and $r$ are uniquely and consistently determined according to the first condition of each entry in table 2.1. We need only check that the second conditions give rise to no contradictory statements. Clearly $u < q < s$ by $1/u = 1/p + 1/q$ and $1/q = 1/p + 1/s$, but since $u > 1$ and $s < \infty$ is required this implies that $2/p < 1$ or $p > 2$. This is satisfied by the assumption $2 < p \leq 3$ in the statement of the theorem. Using $1/r = 1/t + 1/3$, $1/q = 1/p + 1/s$, $1/v = 1/t + 2/3$

| Operator | Norm Estimate | Conditions for validity | Proof |
|---|---|---|---|
| $S_1$ | $\|\|S_1 f\|\|_{s,t} \leq C\|\|f\|\|_{u,v}$ | $1/t = 1/v - 2/3$ | Theorem 11 |
| | | $s/t \leq u/v$ | |
| $S_2$ | $\|\|S_2 f\|\|_{s,t} \leq C\|\|f\|\|_{q,r}$ | $1/t = 1/r - 1/3$ | Theorem 11 |
| | | $s/t \leq q/r$ | |
| $S_2^*$ | $\|\|S_2^* f\|\|_{q,r} \leq C\|\|f\|\|_{u,v}$ | $1/r = 1/v - 1/3$ | Theorem 11 |
| | | $q/r \leq u/v$ | |
| $S_3$ | $\|\|S_3 f\|\|_{q,r} \leq C\|\|f\|\|_{q,r}$ | $1 < q \leq r < \infty$ | Theorem 17 |
| | | | |
| Multiplication | $\|\|\vec{b} f\|\|_{q,r} \leq \|\|\vec{b}\|\|_{p,3}\|\|f\|\|_{s,t}$ | $1/r = 1/3 + 1/t$ | Lemma 13 |
| by $\vec{b}$ | | $1/q = 1/p + 1/s$ | |
| Dot product | $\|\|\vec{b} \cdot \vec{f}\|\|_{q,r} \leq \|\|\vec{b}\|\|_{p,3}\|\|\vec{f}\|\|_{s,t}$ | $1/v = 1/3 + 1/r$ | Lemma 13 |
| with $\vec{b}$ | | $1/u = 1/p + 1/q$ | |

Table 2.1: The operators of figure 2 and boundedness of these.

and $1/u = 2/p + 1/s$ we see that the conditions $s/t \leq u/v$, $s/t \leq q/r$ and $q/r \leq u/v$

is equivalent to

$$\begin{array}{ccccc} \frac{s}{t} & \le & \frac{q}{r} & \le & \frac{u}{v} \\ & & \Updownarrow & & \\ \frac{s}{t} & \le & \frac{\frac{1}{3}+\frac{1}{t}}{\frac{1}{p}+\frac{1}{s}} & \le & \frac{\frac{2}{3}+\frac{1}{t}}{\frac{2}{p}+\frac{1}{s}} \\ & & \Updownarrow & & \\ 1 & \le & \frac{\frac{t}{3}+1}{\frac{s}{p}+1} & \le & \frac{\frac{2t}{3}+1}{\frac{2s}{p}+1} \\ & & \Updownarrow & & \\ & \frac{s}{p} & \le & \frac{t}{3} & \\ & & \Updownarrow & & \\ & s & \le & \frac{pt}{3}\,. & \end{array}$$

Also we must have $u > 1$ or $1/u = 2/p + 1/s < 1$, i.e. $p > 2s/(s-1)$. This combined with $s \le pt/3$ tells us that we must have $2(pt/3)/(pt/3 - 1) < p$ or equivalently $p > 2 + 3/t$. This proves that (2.99) holds for $s \le pt/3$ and $2 + 3/t < p \le 3$, but since we can choose $s, t$ arbitrary large, and since $\|v\|_{s',t'} \le |\Omega|^{1/t' - 1/t} \|v\|_{s,t}$ for $s' < s$ and $t' < t$ it follows that (2.99) must hold for all values $1 < s \le t < \infty$ of $s$ and $t$. (2.100) follows from (2.99) as it did in Theorem 14. ■

### 2.6.4 A Counterexample for $1 \le p < 2$.

Before we start looking for counterexamples, it may be healthy to ask why they should exist. One reasonable requirement for (2.1) to have a solution is that $V$ is integrable, so that we can evaluate $\int_\Omega G_D(x,y)V(y)\ dy$. On the other hand, $\vec{b} \in M_p^3$ typically means that $|\ \vec{b}\ |$ is of size $2^n$ on $2^{(3-p)n}$ dyadic cubes of size $2^{-3n}$. We may expect $\vec{\nabla}_y \cdot \vec{b}$ to be of size $2^{2n}$ on these cubes. This means that $\int_{Q_0} |\ \vec{\nabla}_y \cdot \vec{b}\ | \approx 2^{(3-p)n} 2^{2n} 2^{-3n} = 2^{(2-p)n}$. If $p < 2$ then $2^{(2-p)n} \to \infty$ as $n \to \infty$, so we may expect $V = -\vec{\nabla}_y \cdot \vec{b}$ not to be integrable.

Unfortunately this is not the case for $p = 2$, and we have not been able to decide whether $p = 2$ is possible or not in Theorem 19. We have however been able to construct a counterexample for $1 < p < 2$ according to the principle of nonintegrability of $V$.

Let us first describe how we construct our candidate for a counterexample. Start by defining a function $f : [-1/2, 1/2] \to \mathbf{R}$ by

$$f(x_3) = \begin{cases} 1/2 - x_3 & \text{if} \quad 0 \leq x_3 \leq 1/2 \\ 1/2 + x_3 & \text{if} \quad -1/2 \leq x_3 \leq 0 \,. \end{cases}$$

Using this function we define a 'mother wavelet', $\vec{b^0}$, from which we get $\vec{b}$ by translations and dilations of $\vec{b^0}$.

$$b_3^0 \stackrel{def}{=} 2^n f(2^n x_3) \chi_{[-1,1]\times[-2^{-n-1},2^{-n-1}]\times[-2^{-n-1},2^{-n-1}]}(x_1, x_2, x_3)$$

and $\vec{b^0} = (0, 0, b_3^0)$. Explicitly $\vec{b}$ is given by $\vec{b^0}$ by the formula

$$\vec{b}\,(x_1, x_2, x_3) = \sum_{i=-[2^{n\delta}]}^{i=[2^{n\delta}]} \epsilon\, \vec{b^0}\,(x_1, x_2, i \cdot 2^{-n\delta} + x_3)\,, \tag{2.101}$$

where $0 < \delta < 1$.

**Theorem 20** *For a given $1 \leq p < 2$ there exists some $\delta$; $0 < \delta < 1$ such that $\vec{b}$ given by (2.101) is a counter example to (2.100).* □

**Proof of Theorem 20.** Define some cubes $Q_i^j$ by

$$\begin{aligned} Q_i^j &= [-1 + (i-1)2^{-n}, -1 + i2^{-n}] \times [-2^{-n-1}, 2^{-n-1}] \\ &\quad \times [-j2^{-n\delta} - 2^{-n-1}, -j2^{-n\delta} + 2^{-n-1}]\,. \end{aligned}$$

With this definition we have

$$\mathrm{supp}(\vec{b}) = \cup_{i=1}^{2^{n+1}} \cup_{j=-[2^{n\delta}]}^{[2^{n\delta}]} Q_i^j$$

Differentiating $\vec{b^0}$ we find that

$$\vec{\nabla}\cdot\vec{b^0}=\begin{cases} -2^{2n} & \text{if } 0<x_3<2^{-n-1},\ -1\le x_1\le 1 \\ & \text{and } -2^{-n-1}\le x_2\le 2^{-n-1} \\ 2^{2n} & \text{if } -2^{-n-1}<x_3<0,\ -1\le x_1\le 1 \\ & \text{and } -2^{-n-1}\le x_2\le 2^{-n-1} \\ 0 & \text{otherwise}\ , \end{cases}$$

so that $\vec{\nabla}\cdot\vec{b}$ is $-2^{2n}$ in the upper half of $Q_i^j$,

$$\begin{aligned}(Q_i^j)^+ &= [-1+(i-1)2^{-n},-1+i2^{-n}]\times[-2^{-n-1},2^{-n-1}] \\ &\quad\times[-j2^{-n\delta},-j2^{-n\delta}+2^{-n-1}]\ ,\end{aligned}$$

and $2^{2n}$ in the lower part of $Q_i^j$,

$$\begin{aligned}(Q_i^j)^- &= [-1+(i-1)2^{-n},-1+i2^{-n}]\times[-2^{-n-1},2^{-n-1}] \\ &\quad\times[-j2^{-n\delta}-2^{-n-1},-j2^{-n\delta}]\ .\end{aligned}$$

We have summed up this information about $V=-\vec{\nabla}\cdot\vec{b}$ schematically in figure 2.3.

Because $|\ \vec{b}\ |\approx 2^n$ in all the $Q_i^j$'s and there are $\approx 2^{n(1+\delta)}$ $Q_i^j$'s, we suspect that $\vec{b}\in M^3_{2-\delta}$ by equating the dimensions $1+\delta$ and $3-p$. This is easily verified. Since $\text{supp}(\vec{b})\subset B(0,2)$ we may assume $|Q|\le 1$.

We let $Q$ be an arbitrary cube with side length $0\le l\le 1$. We then have the inequality

$$\begin{aligned}\int_Q|\ \vec{b}\ |^{2-\delta} &\le \epsilon^{2-\delta}2^{n(2-\delta)}(\ \#\text{ cubes } Q_i^j \text{ inside } Q)\cdot 2^{-3n} \\ &\le \epsilon^{2-\delta}2^{n(2-\delta)}\left(\frac{l}{2^{-n}}\right)\left(\frac{l}{2^{-n\delta}}\right)2^{-3n}\end{aligned}$$

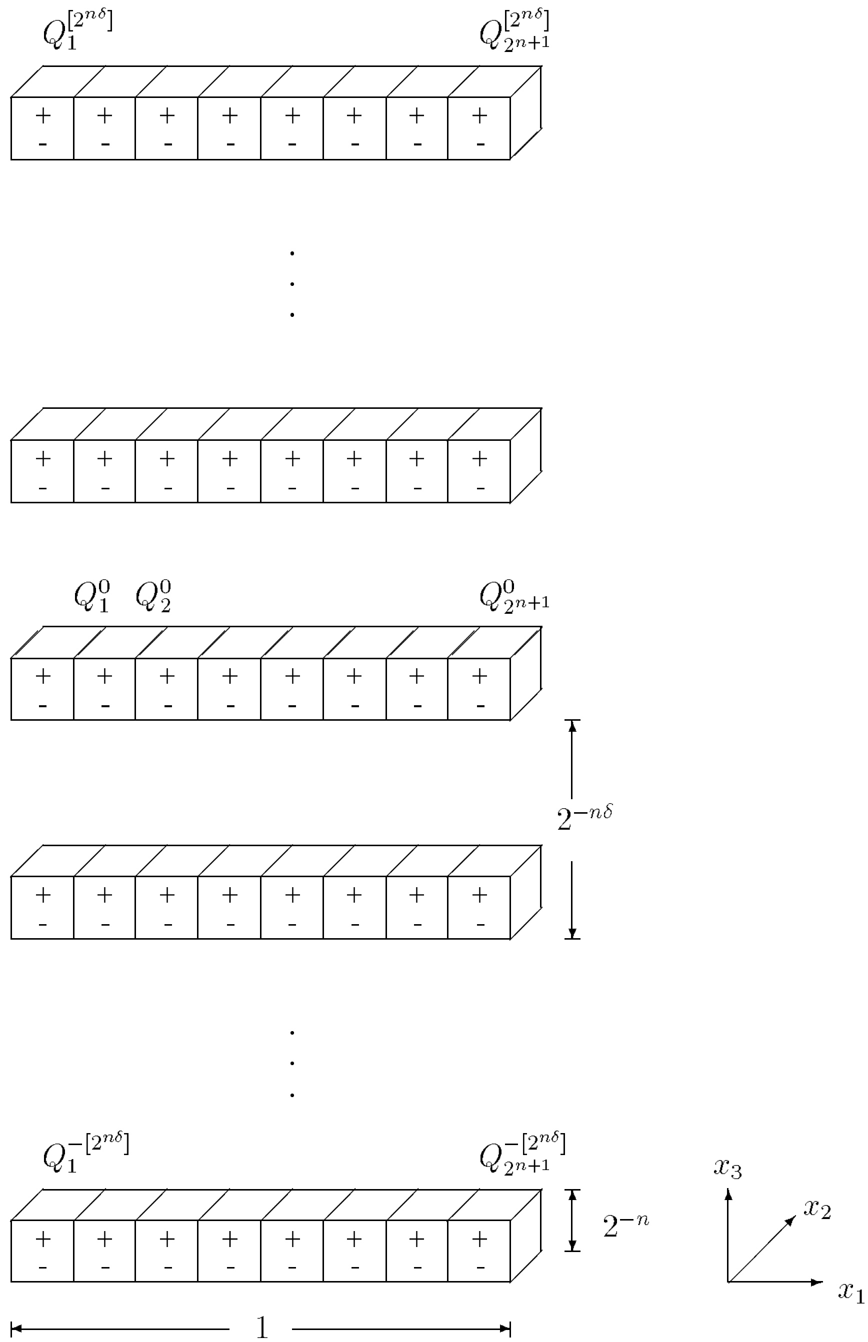


Figure 2.3: $|V(x)| = \epsilon 2^{2n}$ in supp$(V)$. The sign of $V = -\nabla \cdot \vec{b}$ is indicated by $+$ or $-$ in the appropriate box halves.

$$
\begin{aligned}
&= \epsilon^{2-\delta}|Q|^{2/3} \\
&\leq \epsilon^{2-\delta}|Q|^{1-(2-\delta)/3},
\end{aligned}
$$

since $|Q| \leq 1$ and $(2-\delta)/3 > 1/3$. This proves $\vec{b} \in M^3_{2-\delta}$ and $\|b\|_{2-\delta,3} \leq \epsilon$.

To prove that $\mathrm{Av}_\Omega u \to \infty$ as $n \to \infty$ is harder. It will be important to use the symmetries of $V = -\vec{\nabla} \cdot \vec{b}$ extensively. Note that $V$ is an odd function in $x_3$. It follows that $\int_\Omega G_D(x,y)V(y)\, dy$ is an odd function in $x_3$ as well, so that $\mathrm{Av}_\Omega \int_\Omega G_D(x,y)V(y)\, dy = 0$, or equivalently $Av_\Omega E_x\left[\int_0^\tau V(X(t))\, dt\right] = 0$. Iterating this principle we see that

$$
Av_\Omega E_x\left[\left(\int_0^\tau V(X(t))\, dt\right)^{2k+1}\right] = 0 ,
$$

for $k = 0, 1, 2, \cdots$. We may therefore write

$$
\mathrm{Av}_\Omega u(x) = 1 + \mathrm{Av}_\Omega u_2(x) + \mathrm{Av}_\Omega u_4(x) + \cdots ,
$$

where $u_k(x) = \frac{1}{k!} E_x\left[\left(\int_0^\tau V(X(t))\, dt\right)^k\right]$.

We will now estimate the behavior of $u_k(x)$ on the cubes $Q_i^j$ away from the middle, $\overline{(Q_i^j)^+} \cap \overline{(Q_i^j)^-}$. Let us divide $Q_i^j$ into four parts (see figure 2.4) $(Q_i^j)_0^+$, $(Q_i^j)_1^+$, $(Q_i^j)_0^-$ and $(Q_i^j)_1^-$. $(Q_i^j)_0^+$ and $(Q_i^j)_1^+$ are respectively the upper and lower half of $(Q_i^j)^+$. Similarly $(Q_i^j)_0^-$ and $(Q_i^j)_1^-$ are the respectively lower and upper half of $(Q_i^j)^-$. Note that $V(x)$ is an odd function in $Q_i^j$ with respect to the dividing plane between $(Q_i^j)^-$ and $(Q_i^j)^+$. We note that the same will be true modulo some small error for $u_1(x), u_2(x), \cdots$. By the alternating series theorem, we may write for $x \in (Q_i^j)_0^+$

$$
\begin{aligned}
u_1(x) &\geq C \int_{Q_i^j} \frac{V(y)}{|x-y|}\, dy \\
&\geq C \left( \int_{(Q_i^j)_0^+} \frac{V(y)}{|x-y|}\, dy - \int_{(Q_i^j)_0^-} \frac{V(y)}{|x-y|}\, dy \right) \\
&\geq C' 2^n \int_{(Q_i^j)_0^+} V(y)\, dy \\
&\geq C'' \epsilon .
\end{aligned}
$$

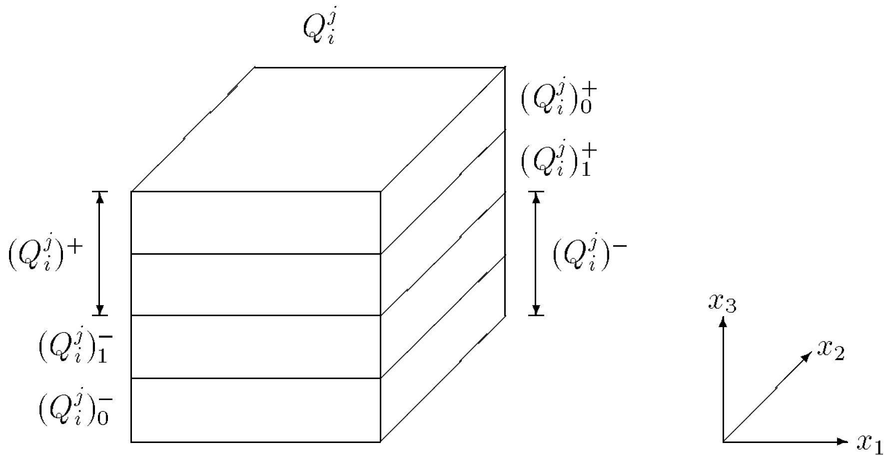


Figure 2.4: Partitioning $Q_i^j$.

Similarly we have $u_1(x) \leq -C''\epsilon$ for $x \in (Q_i^j)_0^-$. Also note that for $x \in (Q_i^j)_1^+$ $u_1(x)$ is either positive or negligibly small and negative. In fact

$$\int_\Omega \frac{V(y)}{|x-y|}\, dy \geq -C\epsilon 2^{-n(1-\delta)}$$

for $x \in (Q_i^j)_1^+$, which is negligibly small in comparison with $C''\epsilon$. It follows that

$$\begin{aligned} u_2(x) &\approx \int_\Omega \frac{V(y)}{|x-y|} \int_\Omega \frac{V(z)}{|y-z|}\, dy\, dz \\ &\geq 2^{n(1+\delta)} 2^{2n} \frac{C\epsilon^2}{R} 2^{-3n} - 2^{n(\delta-1)} 2^{2n} 2^{-3n} \frac{C\epsilon^2}{R} 2^{n(\delta+1)} \\ &\geq C\epsilon^2 2^{n\delta}\ , \end{aligned}$$

where $C = C(R)$ for $x \in (Q_i^j)_0^+$. We note that $\|u_2\|_q$ is unbounded as $n \to \infty$ for $q > \frac{2}{\delta} - 1$, since

$$\int_\Omega |u_2(x)|^q\, dx \geq \epsilon^{2q} 2^{n(-2+\delta(q+1))}\ .$$

We continue by estimating $u_3(x), u_4(x), \cdots$ by the same method and find that

$$u_{2k}(x) \geq C_k \epsilon^{2k} 2^{nk\delta}\ ,$$

so that

$$\int_\Omega u_{2k}(x)\, dx \geq C_k \epsilon^{2k} 2^{n(-2+\delta(k+1))}\ .$$

It is now evident that given $\delta$ we can find a $k$ such that $\int_\Omega u_{2k}(x)\, dx \to \infty$ as $n \to \infty$. It follows that $\int_\Omega u(x)\, dx \to \infty$ as well, so $b$ is indeed a counterexample for $1 \leq p < 2$.

■

# CHAPTER III

# Eigenvalue Estimates

## 3.1 Preliminaries

We are concerned with estimating the negative eigenvalues of the Schrödinger operator $-\triangle + V$. We shall assume that there are only finitely many negative eigenvalues, and we will resort to the following notation:

- $L \stackrel{def}{=} -\triangle + V$, where $\triangle = \partial^2/\partial x_1 + \partial^2/\partial x_2 + \partial^2/\partial x_3$.
- Eigenvalues: $\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N < 0$.
- $N(V, \lambda) = \#\{i : \lambda_i \leq \lambda\}$ = the number of eigenvalues of $L$ below $\lambda$.

The general strategy for bounding $N(V, \lambda)$ from above will be to find a subspace of $\mathcal{L}^2(\mathbf{R})$, $H_K$, such that $\dim(\mathcal{L}^2(\mathbf{R})/H_K) = K$ and

$$< Lu, u > \geq \lambda \|u\|_2^2 \tag{3.1}$$

for all $u \in H_K$. We then have $N(V, \lambda) \leq K$.

In order to get the estimate (3.1) we apply some classical Sobolev inequalities (see [18]) extended to some Morrey spaces.

## 3.2 Morrey Spaces and Sobolev Inequalities

We shall define Morrey spaces according to [7], [22] and [6]. We will say that $f$ is in the Morrey space $M_q^r(\Omega)$, $1 \le q \le r$ if for all cubes $Q$ we have

$$\left(\int_{Q\cap\Omega} |f(x)|^q\right)^{1/q} \le C|Q|^{1/q-1/r} \, . \tag{3.2}$$

The smallest constant such that (3.2) holds is the Morrey norm of $f$ with respect to $q$ and $r$ and it is denoted $||f||_{q,r}$. From [22] we have the following theorem that we shall find convenient to use:

**Theorem 21** *If*

$$(Tf)(x) \stackrel{def}{=} \int_{\mathbf{R}^3} \frac{b(x)f(y)}{|x-y|^{3/p}}\, dy \; ,$$

$f \in M_u^v(\mathbf{R}^3)$ *and* $1/v + 1/p = 1$ *then* $||Tf||_{q,r} \le C||b||_{u,v}||f||_{q,r}$ *when* $1 < q < u$, $q \le r$, $u \le v$, $1 < p, q, r, u, v < \infty$ *and* $1/r + 1/p > 1$. □

The theorem is stated and proved as Corollary 3 of [22]. Similar fractional integral theorems are also proved in [3],[7] and [20]. We shall be mostly concerned with the $p = 3/2$ case of the theorem. This special case is Theorem 1.2 of [7]. In order to use Theorem 21 to estimate eigenvalues we also need some statements from the theory of Sobolev spaces. Although these can be found in many references discussing partial differential equations and Sobolev spaces we shall state and prove the two statements we need.

**Lemma 22** *If $u$ has compact support in $B(0, R/2) = \{x : |x| \le R/2\}$ and is differentiable then*

$$u(x) = \frac{1}{4\pi} \int_{|x-y|\le R} \frac{(x-z)\cdot\nabla u(z)}{|x-z|^3}\, dz \tag{3.3}$$

*and if $u$ is differentiable, but not necessarily with compact support then*

$$|u(x) - \mathrm{Av}_Q u| \le C \int_Q \frac{|\nabla u(z)|}{|x-z|^2}\, dz \ , \tag{3.4}$$

*where* $\mathrm{Av}_Q u = \int_Q u(y)\, dy/|Q|$, *and $Q$ denotes any cube in* $\mathbf{R}^3$ *containing $x$.* $\square$

**Proof:** If $u$ is differentiable we have

$$\begin{aligned} u(x) - u(y) &= \int_0^1 \frac{d}{dt} u(y + t(x-y))\, dt \\ &= \int_0^1 (x-y) \cdot \nabla u(y + t(x-y))\, dt \ , \end{aligned} \tag{3.5}$$

for $x \in B(0, R/2)$. We define $S_R$ to be $S_R = \{y : |x-y| = R\}$. Since $u$ has compact support in $B(0, R/2)$ it follows that $u(y) = 0$ if $y \in S_R$ and $x \in B(0, R/2)$. We average (3.5) over $S_R$ to get

$$u(x) = \frac{1}{|S_R|} \int_{y \in S_R} dy \ \int_0^1 dt\ (x-y) \cdot \nabla u(y + t(x-y))\ .$$

We now use the surface variable $y$ together with $t$ to make a volume variable $z = y + t(x-y) = x + (1-t)(y-x)$. The change of variables gives us

$$u(x) = \frac{R^2}{|S_R|} \int_{|z-x| \le R} \frac{(x-z) \cdot \nabla u(z)}{|x-z|^3}\, dz \ ,$$

which upon using $|S_R| = 4\pi R^2$ gives us (3.3). To prove (3.4) we average (3.5) over the cube $Q$

$$u(x) - \mathrm{Av}_Q u = \int_Q dy \ \int_0^1 dt\ (x-y) \cdot \nabla u(y + t(x-y))\ .$$

If we extend $\nabla u$ to be zero outside $Q$ and let $B$ be a ball centered at $x$ with radius $\mathrm{diam}(Q)$ we get

$$\begin{aligned} |u(x) - \mathrm{Av}_Q u| &\le \frac{1}{|Q|} \int_Q dy \ \int_0^1 dt\ |x-y| \cdot |\nabla u(y + t(x-y))| \\ &= \frac{1}{|Q|} \int_0^{\mathrm{diam}(Q)} dR\ R^2 \int_{\partial B(x,R)} \int_0^1 dt\ |x-y| \cdot |\nabla u(y + t(x-y))|\ . \end{aligned}$$

By the previous variable change, $(dy, dt) \to dz$, the integral becomes

$$
\begin{aligned}
|u(x) - \mathrm{Av}_Q u| &\leq \frac{1}{|Q|} \int_0^{\mathrm{diam}(Q)} dR\, R^2 \int_{B(x,R)} dz\, \frac{|\nabla u(z)|}{|x-z|^2} \\
&\leq \frac{1}{|Q|} \int_0^{\mathrm{diam}(Q)} dR\, R^2 \int_{B} dz\, \frac{|\nabla u(z)|}{|x-z|^2} \\
&= \frac{1}{3} \int_Q dz\, \frac{|\nabla u(z)|}{|x-z|^2} \,,
\end{aligned}
$$

which proves (3.4). ■

**Lemma 23** *If $b \in M_p^3$ and $2 < p \leq 3$ and if $u$ is differentiable and has compact support then*

$$\|bu\|_2 \leq C\|b\|_{p,3}\|\nabla u\|_2 \,. \tag{3.6}$$

□

**Proof:** Use (3.3) of Lemma 22. Noting that $M_2^2 = \mathcal{L}^2$ and $\|\cdot\|_{2,2} = \|\cdot\|_2$, (3.6) is merely a consequence of Theorem 21. ■

## 3.3 Upper and Lower Bounds for the First Eigenvalue

We shall try to estimate the first eigenvalue, $\lambda_0$, of $L$ when the potential $V$ is in the Morrey space $M_p^{3/2}$, $1 < p \leq 3/2$. Also we shall find a lower estimate of the first eigenvalue when the potential is of divergence form $V = -\nabla \cdot b$ with $|b| \in M_p^3$, $2 < p \leq 3/2$. For the first case we have

**Theorem 24** *If $V \in M_p^{3/2}$, $1 < p \leq 3/2$ then*

$$E_{\mathrm{small}} \leq \lambda_0 \leq E_{\mathrm{big}} \,, \tag{3.7}$$

*where*

$$E_{\mathrm{small}} = C_{\mathrm{sm}} \mathrm{inf}_Q \{c(\mathrm{diam}Q)^{-2} - (\mathrm{Av}_Q |V|^p)^{1/p}\} \tag{3.8}$$

*and*

$$E_{\text{big}} = C_{\text{big}} \inf_Q \{c(\text{diam}Q)^{-2} + (8\text{Av}_{2Q}V_+ - \text{Av}_Q V_-)\} \ , \tag{3.9}$$

*which simplifies to*

$$E_{\text{big}} = C_{\text{big}} \inf_Q \{c(\text{diam}Q)^{-2} - \text{Av}_Q V_-\} \ , \tag{3.10}$$

*when* $V(x) \leq 0$. □

Here $V(x) = V_+(x) - V_-(x)$, $V_+(x) = V(x)\chi_{\{x:V(x)\geq 0\}}(x)$, $Q$ is any cube and $2Q$ is the cube centered at the same center as $Q$, but with twice the side length of $Q$.

Theorem 24 is merely Theorem 5 of [15], but we shall nevertheless present a proof different from that of [15] as we believe this new proof may provide some new insight. Other estimates of the first eigenvalue can be found in [20] and [1]. We shall divide the proof in two parts; the lower and the upper estimate. The upper estimate is proved in exactly the same manner as in [15].

### 3.3.1 Proof of the Upper Estimate

Since $\lambda_0$ is the smallest eigenvalue we have

$$\lambda_0 = \inf_{u\neq 0} \frac{< Lu, u >}{< u, u >} \leq \frac{< Lu_0, u_0 >}{< u_0, u_0 >} \ ,$$

for $u_0$ a fixed function not identically zero. We can simplify $< Lu_0, u_0 >$ as follows

$$\begin{aligned} < Lu_0, u_0 > &= \int (-\triangle u_0 + V u_0) u_0 \\ &= \int |\nabla u_0|^2 + \int V |u_0|^2 \ , \end{aligned}$$

if $u_0$ has compact support. We shall let $u_0 = \phi((x-c)/d)$, where $\phi$ is a $C^2$–function with the properties

$$\phi(x) = \begin{cases} 1 & \text{for} \quad x \in Q(0,1) \\ 0 & \text{for} \quad x \in \mathbf{R}^3 \setminus Q(0,2) \end{cases}$$

and $0 \le \phi(x) \le 1$ for $x \in Q(0,2) \setminus Q(0,1)$. Here $Q(c,d)$ denotes a cube of side length $d$ centered at $c \in \mathbf{R}^3$.

By changing the variables we get

$$\lambda_0 \le \frac{d\|\nabla\phi\|_2^2 + \int_{Q(c,2d)} V_+(x)\, dx - \int_{Q(c,d)} V_-(x)\, dx}{d^3\|\phi\|_2^2}\,, \tag{3.11}$$

since $u_0(x) \equiv 1$ on $Q(c,d)$, $0 \le u_0(x) \le 1$ and $u_0(x) \equiv 0$ on $\mathbf{R}^3 \setminus Q(c,2d)$. $\phi$ is merely a fixed $C^2$–function, so $\|\nabla\phi\|_2^2$ and $\|\phi\|_2^2$ are merely constants with respect to $d$ and $c$. We therefore get

$$\lambda_0 \le C' d^{-2} + C'' d^{-3} \left( \int_{Q(c,2d)} V_+(x) - \int_{Q(c,d)} V_-(x) \right) .$$

Writing $d = (\mathrm{diam}Q)/\sqrt{3}$, $Q = Q(c,d)$ and $2Q = Q(c,2d)$ we get

$$\begin{aligned} E_{\mathrm{big}} &= \inf_Q \left\{ C'(\mathrm{diam}Q)^{-2} + C''(8\mathrm{Av}_{2Q}V_+ - \mathrm{Av}_Q V_-) \right\} \\ &= C_{\mathrm{big}} \inf_Q \left\{ c(\mathrm{diam}Q)^{-2} + (8\mathrm{Av}_{2Q}V_+ - \mathrm{Av}_Q V_-) \right\} , \end{aligned}$$

which proves (3.9). ■

### 3.3.2 A Partition of Unity

To find a lower estimate we define a particular kind of partition of unity. Define

$$\lambda = \inf_Q \left\{ c(\mathrm{diam}Q)^{-2} - (\mathrm{Av}_Q |V|^p)^{1/p} \right\} \tag{3.12}$$

so that $E_{\mathrm{small}} = C_{\mathrm{sm}}\lambda$ (We may here assume $\lambda < 0$).

**Lemma 25** *We claim that there exists a partition of unity $\{\Phi_n\}_{n\in\mathbf{Z}^3}$ such that*

1. *Each $\Phi_n$ has compact support in a cube $Q_n$, with $\mathrm{diam}(Q_n) = (-\lambda/c)^{-1/2}$.*

2. *The cubes $\{(1/3)Q_n\}_{n\in\mathbf{Z}^3}$ are disjoint and cover $\mathbf{R}^3$.*

3. *$0 \le \Phi_n(x) \le 1$ for all $n \in \mathbf{Z}^3$, $x \in \mathbf{R}^3$.*

*4.* $\sum_{n\in\mathbf{Z}^3}\Phi_n(x)\equiv 1$*.*

*5.* $|\nabla\cdot\sqrt{\Phi_n(x)}|\le C_\Phi(\mathrm{diam}Q)^{-1}$ *for all* $n\in\mathbf{Z}^3$*,* $x\in\mathbf{R}^3$ *for some universal constant* $C_\Phi$*.* □

**Proof:** We may without loss of generality assume $\lambda=-c$, since $\lambda\neq -c$, $\lambda<0$ can be obtained by scaling of the partition of unity for $\lambda=-c$. For $\lambda=-c$ we can make a partition of unity from the quadratic B–spline associated with the Battle–Lemarié family (see [10]). For another construction of $\Phi_n$ see [4]. We define the piecewise quadratic B–spline $\phi$ by

$$\phi(x)=\begin{cases}\frac{1}{2}(x+1)^2 & \text{for} \quad -1\le x\le 0\\ \frac{3}{4}-(x-\frac{1}{2})^2 & \text{for} \quad 0\le x\le 1\\ \frac{1}{2}(x-2)^2 & \text{for} \quad 1\le x\le 2\,.\end{cases}$$

It is evident that $0\le\phi\le 1$, and one can easily check that $\sum_{n=-\infty}^{\infty}\phi(x-n)\equiv 1$ for all $x\in\mathbf{R}^3$. (Since $\sum_{n=-\infty}^{\infty}\phi(x-n)$ is periodic with period 1, it is enough to verify that $\phi(x)+\phi(x+1)+\phi(x+2)=1$ for $-1\le x\le 0$). Note also that

$$\sqrt{\phi(x)}=\begin{cases}\frac{1}{\sqrt{2}}(x+1) & \text{for} \quad -1\le x\le 0\\ \sqrt{\frac{3}{4}-(x-\frac{1}{2})^2} & \text{for} \quad 0\le x\le 1\\ \frac{1}{\sqrt{2}}(x-2) & \text{for} \quad 1\le x\le 2\,,\end{cases}$$

so that $|\partial/\partial x\sqrt{\phi(x)}|\le 1$ for $-1\le x\le 2$. If we now define $\Phi_n(x)\equiv\phi(x_1-n_1)$ $\phi(x_2-n_2)\;\phi(x_3-n_3)$, where $x=(x_1,x_2,x_3)\in\mathbf{R}^3$, $n=(n_1,n_2,n_3)\in\mathbf{Z}^3$. With this definition it is immediately seen that $\Phi_n$ satisfies *(1)*–*(5)* with $C_\Phi=3$. ■

Using the tools of Lemma 23 and Lemma 25 we are ready to prove the lower estimate for $\lambda_0$.

### 3.3.3 Proof of the Lower Estimate

We want to find a lower estimate for $\|\nabla u\|_2^2 + \int V|u|^2$. We treat each term separately in the beginning. Let $\phi_n(x)$ to be such that $\phi_n^2(x)$ is a partition of unity satisfying the requirements (*1*)–(*5*) of Lemma 25. Using $\sum_{n\in\mathbf{Z}^3}\phi_n^2 \equiv 1$ we get

$$\begin{aligned}\int_{\mathbf{R}^3} V|u|^2 &= \int_{\mathbf{R}^3} V|u|^2 \sum_{n\in\mathbf{Z}^3}\phi_n^2\\ &= \sum_{n\in\mathbf{Z}^3}\int_{Q_n} V|\phi_n u|^2\\ &= \sum_{n\in\mathbf{Z}^3}\int V_n|u_n|^2\,,\end{aligned}$$

where $u_n = \phi_n u$ and $V_n = V\chi_{Q_n}$. Note that since

$$\lambda = \inf_Q\{c(\mathrm{diam}Q)^{-2} - (\mathrm{Av}_Q|V|^p)^{1/p}\}$$

we have

$$(\mathrm{Av}_Q|V_n|^p)^{1/p} \le 2c(\mathrm{diam}Q)^{-2}\,, \tag{3.13}$$

or equivalently $V_n \in M_p^{3/2}$ with $\|V_n\|_{p,3/2} \le 2c$. To see (3.13) assume that $\mathrm{diam}(Q) \le \mathrm{diam}(Q_n) = (-\lambda/c)^{-1/2}$ (since $V_n \equiv 0$ outside $Q_n$) which implies $-\lambda \le c(\mathrm{diam}Q)^{-2}$. (3.13) then follows by fixing $Q$ and isolating $(\mathrm{Av}_Q|V|^p)^{1/p}$ in (3.12). Using Lemma 23 we now get

$$\begin{aligned}\int_{\mathbf{R}^3} V|u|^2 &= \sum_{n\in\mathbf{Z}^3}\int V_n|u_n|^2\\ &\ge -\sum_{n\in\mathbf{Z}^3} C^2\|V_n\|_{p,3/2}\|\nabla u_n\|_2^2\\ &\ge -2cC^2\sum_{n\in\mathbf{Z}^3}\|\nabla u_n\|_2^2\,.\end{aligned}$$

$C$ is here the constant from Lemma 23. We shall assume $c$ is small enough that $2cC^2 < 1$, so that

$$\int_{\mathbf{R}^3} V|u|^2 \ge -\sum_{n\in\mathbf{Z}^3}\|\nabla u_n\|_2^2\,. \tag{3.14}$$

For the $\|\nabla u\|_2^2$ term we have $\|\nabla u\|_2^2 = \int |\nabla u|^2 = \int \sum_{n\in\mathbf{Z}^3} \phi_n^2 |\nabla u|^2$. Using the identity $|\nabla(\phi_n u)|^2 = |\nabla\phi_n|^2|u|^2 + 2u\phi_n\nabla u\cdot\nabla\phi_n + |\phi_n|^2|\nabla u|^2$ we get

$$\begin{aligned} \|\nabla u\|_2^2 &= \sum_{n\in\mathbf{Z}^3}\int_{Q_n}|\nabla(\phi_n u)|^2 - \int_{Q_n}|\nabla\phi_n|^2|u|^2 \\ &\quad -2\int_{Q_n}(\phi_n\nabla\phi_n)\cdot(u\nabla u)\,. \end{aligned} \tag{3.15}$$

The last term may be left out as we have

$$\begin{aligned} 2\sum_{n\in\mathbf{Z}^3}\int_{Q_n}(\phi_n\nabla\phi_n)\cdot(u\nabla u) &= \sum_{n\in\mathbf{Z}^3}\frac{1}{2}\int_{Q_n}\nabla(\phi_n^2)\cdot\nabla(u^2) \\ &= \frac{1}{2}\int_{\mathbf{R}^3}\nabla(\sum_{n\in\mathbf{Z}^3}\phi_n^2)\cdot\nabla(u^2) \\ &= \frac{1}{2}\int_{\mathbf{R}^3}\nabla(1)\cdot\nabla(u^2) \\ &= 0\,. \end{aligned} \tag{3.16}$$

Putting together (3.15) and (3.16) we see that

$$\|\nabla u\|_2^2 = \sum_{n\in\mathbf{Z}^3}\int_{Q_n}|\nabla(\phi_n u)|^2 - \int_{Q_n}|\nabla\phi_n|^2|u|^2\,. \tag{3.17}$$

(3.14) and (3.17) give us the inequality

$$\begin{aligned} < Lu, u > &= \|\nabla u\|_2^2 + \int_{\mathbf{R}^3} V|u|^2 \\ &\geq -\sum_{n\in\mathbf{Z}^3}\int_{Q_n}|\nabla\phi_n|^2|u|^2\,. \end{aligned} \tag{3.18}$$

Using (*1*),(*2*) and (*5*) of Lemma 25 we get

$$\begin{aligned} < Lu, u > &\geq -C_\Phi^2\sum_{n\in\mathbf{Z}^3}(\mathrm{diam}Q_n)^{-2}\int_{Q_n}|u|^2 \\ &= C\lambda\sum_{n\in\mathbf{Z}^3}\int_{Q_n}|u|^2 \\ &\geq C_{\mathrm{sm}}\lambda\int|u|^2 \\ &= E_{\mathrm{sm}}\int|u|^2\,. \end{aligned}$$

From this inequality it follows immediately that $\lambda_0 > E_{\mathrm{sm}}$. ■

## 3.4 The Drift Case

Next we shall be concerned with the case $V = -\nabla \cdot b$. We will ignore questions regarding estimates of $E_{\text{big}}$. In a case where $b$ is given this can probably be done using test functions as previously demonstrated in Theorem 24.

To estimate $E_{\text{small}}$ we shall reduce the problem to that of Theorem 24. We need to find a number $\lambda$ such that $< Lu, u > \geq \lambda \|u\|_2^2$. By integration by parts the left hand side can be rewritten

$$< Lu, u > = \int |\nabla u|^2 + 2 \int (b \cdot \nabla u) u \,.$$

Using the inequality $yz \geq -(y^2/2 + z^2/2)$ with $y = |\nabla u|$ and $z = 2|b|u$ we get

$$< Lu, u > \geq \frac{1}{2} \int |\nabla u|^2 - 2 \int |b|^2 |u|^2 \,.$$

Therefore if $\int |\nabla u|^2 - 4 \int |b|^2 |u|^2 \geq 2\lambda \|u\|_2^2$ then $< Lu, u > \geq \lambda \|u\|_2^2$. Defining $\tilde{V} = -4|b|^2$, $\tilde{\lambda} = 2\lambda$ the problem reduces to the more familiar

$$\int |\nabla u|^2 + \int \tilde{V} |u|^2 \geq \tilde{\lambda} \|u\|_2^2 \,. \tag{3.19}$$

The argument of Theorem 24 applied to (3.19) now yields

**Corollary 26** *If $V = -\nabla \cdot b$, $|b| \in M_p^3$ and $2 < p \leq 3$ then*

$$\lambda_0 \geq \frac{C_{\text{sm}}}{2} \inf_Q \{c(\text{diam} Q)^{-2} - 4(\text{Av}_Q |b|^p)^{2/p}\} \,.$$

$\square$

## 3.5 Estimates for $N(V, \lambda)$

We want to estimate the total number of negative eigenvalues below $\lambda$. This quantity is denoted $N(V, \lambda)$. We know that $N(V, \lambda) = 0$ if $< Lu, u > \geq \lambda < u, u >$

for all $u \in \mathcal{L}^2(\mathbf{R}^3)$. Similarly we know that if $< Lu, u > \geq \lambda < u, u >$ for all $u$ in some space $H \subset \mathcal{L}^2(\mathbf{R}^3)$ with index $N$ in $\mathcal{L}^2(\mathbf{R}^3)$ then $N(V, \lambda) \leq N$.

As the case $\lambda = 0$ is slightly different from the case $\lambda < 0$, we shall consider these separately. We first consider estimates for $N(V, 0)$. In order to state our theorems we need however some new terminology.

### 3.5.1 Non–perturbative Cubes

All cubes from here on are dyadic unless otherwise stated. A dyadic cube have center coordinates $(k_1 2^{-n}c, k_2 2^{-n}c, k_3 2^{-n}c)$ and side length $2^{-n}$ for some fixed constant $c > 0$ and some integers $k_1$, $k_2$, $k_3$ and $n$.

The strategy for constructing $H$, will be to "cut out" cubes where $(\mathrm{Av}_Q|V|^p)^{1/p}$ is particularly large compared to $\epsilon(\mathrm{diam}Q)^{-2}$, where $\epsilon > 0$ is a sufficiently small constant ($\epsilon$ is less than the constant $c$ of Theorem 24, equation (3.8)). Letting $H$ consist of functions that are orthogonal to the characteristic functions on the cubes we choose to "cut out", we can lower our estimated value of $< Lu, u >$ by using (3.4) of Lemma 22.

From here on we shall consider the cubes for which $|Q|^{2/3}(\mathrm{Av}|V|^p)^{1/p}$ is particularly large. We shall refer to a dyadic cube, $Q$, as non–perturbative (with respect to $\gamma$) if $(\mathrm{Av}|V|^p)^{1/p} > \gamma|Q|^{-2/3}$. We denote the set of all non–perturbative cubes by $\mathcal{Q}^{\mathrm{pert}}$.

If $V$ has compact support then $|Q|^{2/3}(\mathrm{Av}|V|^p)^{1/p}$ must become small as $|Q| \to \infty$ (at least for $p < 3/2$). Therefore there exist for each perturbative cube $Q$ a maximal cube $Q_{\max} \in \mathcal{Q}^{\mathrm{pert}}$ such that $Q \subset Q_{\max}$. The set of all maximal cubes will be denoted $\mathcal{Q}^{\max} = \{Q : Q \in \mathcal{Q}^{\mathrm{pert}} \text{ and } Q \text{ is maximal } \}$.

Similarly, since we are assuming that $|Q|^{2/3}(\mathrm{Av}|V|^p)^{1/p} \leq \epsilon$ for sufficiently small cubes (here $\epsilon > 0$ is a fixed sufficiently small number satisfying $\epsilon \leq \gamma$), we can find a minimal cube $Q_{\min} \in \mathcal{Q}^{\text{pert}}$ for each $Q \in \mathcal{Q}^{\text{pert}}$ such that $Q_{\min} \subset Q$. These cubes we will denote by $\mathcal{Q}^{\min} = \{Q : Q \in \mathcal{Q}^{\text{pert}}$ and $Q$ is minimal$\}$.

Next we denote the first generation descendants of $Q \in \mathcal{Q}^{\text{pert}}$ by $\mathcal{D}(Q)$; i.e. $\mathcal{D}(Q) = \{Q' : Q' \in \mathcal{Q}^{\text{pert}}$, $Q' \subset Q$ and there is no $Q'' \in \mathcal{Q}^{\text{pert}}$ such that $Q' \subset Q'' \subset Q\}$. The cubes $Q \in \mathcal{Q}^{\text{pert}}$ for which $\mathcal{D}(Q)$ contains more than one element will be referred to as branching cubes, and the set of such cubes by $\mathcal{Q}^{\text{branch}} = \{Q : Q \in \mathcal{Q}^{\text{pert}}$ and $\mathcal{D}(Q)$ contains at least two elements$\}$.

Finally we shall refer to the first generation descendants of a branching cube as siblings. The set of such cubes will be denoted $\mathcal{Q}^{\text{sibling}} = \{Q : Q \in \mathcal{D}(Q')$, $Q' \in \mathcal{Q}^{\text{branch}}\}$.

To summarize these concept we say that maximal cubes are contained in no other non–perturbative cubes; minimal cubes contain no other non–perturbative cubes; branching cubes contain at least two distinct non–perturbative cubes that are first generation descendants or maximal relative to itself; sibling cubes are simply the maximal non–perturbative cubes of a branching cube, or if one wish the first generation descendants of the branching cubes.

The concepts of minimal, maximal, branching and sibling cubes were introduced in [15]. We shall essentially follow the strategy of [15] to estimate $N(V, 0)$, but with some interesting modifications. Non–perturbative cubes were also studied in [6]. The following results (Lemma 27, Theorem 28 and Lemma 29) are due to C. Fefferman [15].

**Lemma 27** *If* $|\mathcal{Q}^{\text{pert}}| < \infty$ *and* $N = |\mathcal{Q}^{\min}|$ *then* $|\mathcal{Q}^{\min} \cup \mathcal{Q}^{\max} \cup \mathcal{Q}^{\text{branch}} \cup \mathcal{Q}^{\text{sibling}}| < 4N$.

$\square$

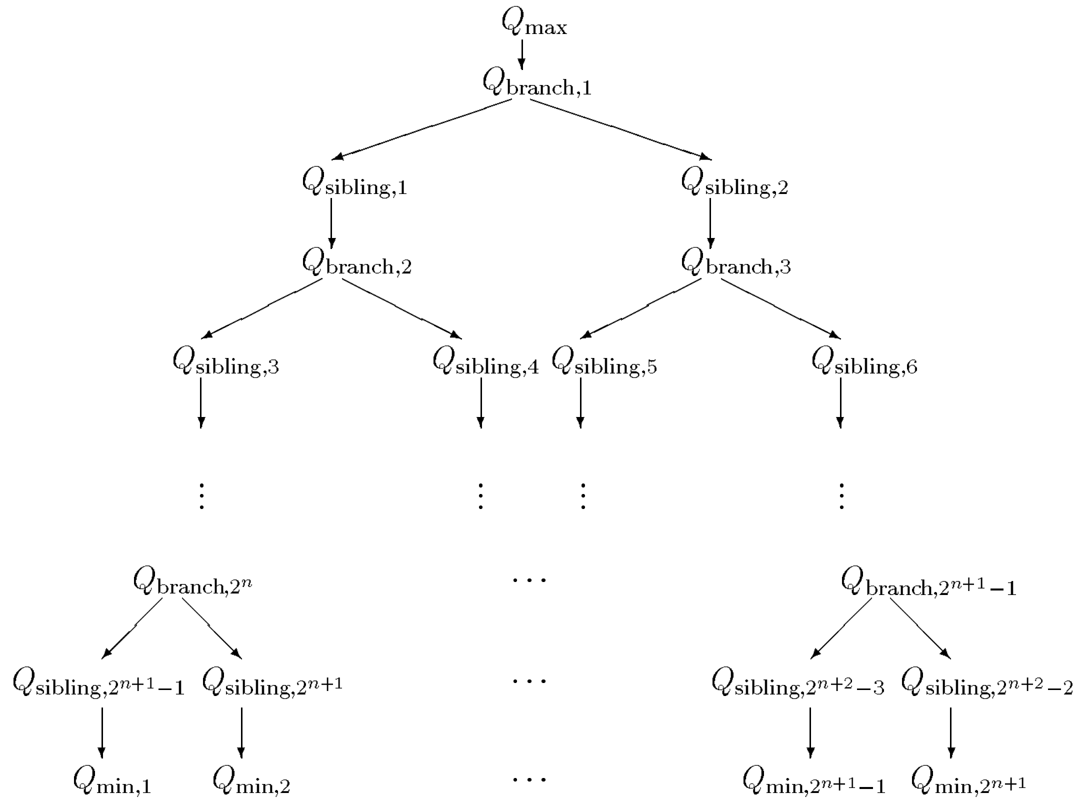


Figure 3.1: Worst Case Containment Diagram for the Non–perturbative Cubes

**Proof:** From Figure 3.5.1 it is easily realized that if $\mathcal{Q}^{\rm pert}$ consists of one single tree then

$$|\mathcal{Q}^{\rm sibling}| \le |\mathcal{Q}^{\rm min}| + 2^{-1}|\mathcal{Q}^{\rm min}| + 2^{-2}|\mathcal{Q}^{\rm min}| + \cdots + 2^{1-k}|\mathcal{Q}^{\rm min}| \;,$$

$$|\mathcal{Q}^{\rm branch}| \le 2^{-1}|\mathcal{Q}^{\rm min}| + 2^{-2}|\mathcal{Q}^{\rm min}| + \cdots + 2^{-k}|\mathcal{Q}^{\rm min}|$$

and

$$|\mathcal{Q}^{\rm max}| \le 2^{-k}|\mathcal{Q}^{\rm min}| \;,$$

where $2^{k-1} \le |\mathcal{Q}^{\rm min}| \le 2^k$. It is easily seen that when $|\mathcal{Q}^{\rm min}| = 2^k$, $k = 0, 1, 2, \ldots$ the ratio between $|\mathcal{Q}^{\rm min} \cup \mathcal{Q}^{\rm max} \cup \mathcal{Q}^{\rm branch} \cup \mathcal{Q}^{\rm sibling}|$ and $|\mathcal{Q}^{\rm min}|$ is maximized on the intervals $(2^{k-1}, 2^k]$ and in fact

$$|\mathcal{Q}^{\rm min} \cup \mathcal{Q}^{\rm max} \cup \mathcal{Q}^{\rm branch} \cup \mathcal{Q}^{\rm sibling}| = (4 - 2^{1-k})|\mathcal{Q}^{\rm min}| < 4N - 1 \;, \qquad (3.20)$$

when $2^{k-1} < |\mathcal{Q}^{\min}| = N \leq 2^k$. If $\mathcal{Q}^{\text{pert}}$ consists of several trees this inequality will still hold and

$$|\mathcal{Q}^{\min} \cup \mathcal{Q}^{\max} \cup \mathcal{Q}^{\text{branch}} \cup \mathcal{Q}^{\text{sibling}}| < 4N \ .$$

∎

The proof of Lemma 27 can be made more precise. For the enhanced mathematical rigor we refer the interested reader to Lemma 5 of [15]. The interested reader may also observe that the inequality can be sharpened to $|\mathcal{Q}^{\min} \cup \mathcal{Q}^{\max} \cup \mathcal{Q}^{\text{branch}} \cup \mathcal{Q}^{\text{sibling}}| \leq 4N - 2$ as shown in (3.20).

### 3.5.2 Construction of $H$ and Estimates of $N(V,0)$

The subspace $H$ of $\mathcal{L}^2(\mathbf{R}^3)$ that we shall use to estimate $N(V,0)$ will consist of $u \in \mathcal{L}^2(\mathbf{R}^3)$ satisfying $< u, \chi_Q >= 0$ for all $Q \in \mathcal{Q}^{\min} \cup \mathcal{Q}^{\max} \cup \mathcal{Q}^{\text{branch}} \cup \mathcal{Q}^{\text{sibling}}$ and $< u, \chi_P >$ for some polygonal regions $P$ corresponding to the cubes $Q$. For simplicity we shall let $\mathcal{Q}^H = \mathcal{Q}^{\min} \cup \mathcal{Q}^{\max} \cup \mathcal{Q}^{\text{branch}} \cup \mathcal{Q}^{\text{sibling}}$. In particular we have that the index of $H$ in $\mathcal{L}^2(\mathbf{R}^3)$ is $|\mathcal{Q}^H|$ plus the number of polygonal regions that was added. By Lemma 27 $|\mathcal{Q}^H|$ is less than $4N$ and the number of polygonal regions is similarly bounded by $CN$. If $< Lu, u > \geq 0$ for all $u \in H$ then we have the following theorem:

**Theorem 28** *If $V$ has compact support and $(\text{Av}_Q|V|^p)^{1/p} \leq \epsilon |Q|^{-2/3}$ $(1 < p < 3/2)$ for all sufficiently small dyadic cubes $Q$ and $N = |\mathcal{Q}^{\min}|$ then $N(V,0) < CN$. $C$ can be set to 1 if all the non–perturbative cubes are minimal, 7 if all the non–perturbative cubes are non–branching and 49 for the general case. $C$ does not depend on the region of support of $V$.* □

The assumptions on the potential in Theorem 28 insure that $|\mathcal{Q}^{\text{pert}}| < \infty$ and that $V \in M_p^{3/2}$. This is so that we can use Lemma 27 as well as Lemma 23 with $b = |V|^{1/2}$.

Theorem 28 was previously stated and proved in [15] and [20]. The proof presented here resembles that of [20], but the geometric method for dealing with overlapping cubes simplifies that of [20].

To prove Theorem 28 we shall divide $V$ into potentials $V_i$ that live on the perturbative cubes of $\mathcal{Q}^H$. The potentials $V_i$ will all have Morrey norms sufficiently small to insure that $< L_i u, u > \geq 0$ for all $u \in H$. Here $L_i$ is the Laplace operator with potential $V_i$.

### 3.5.3 Chopping Up the Potential

We apply a "divide and conquer" technique to our potential. Each time we encounter a cube in $\mathcal{Q}^H$ we cut out a part of the potential. When finished with this subdivision procedure we are left with the potentials

$$V_i(x) \stackrel{def}{=} V(x)\chi_{\{x:x\in E_i\}}(x)\,, \quad i = 0, 1, 2, \ldots, M\,,$$

where

$$E_0 = \mathbf{R}^3 \setminus \cup_{Q\in\mathcal{Q}^H} Q$$

and

$$E_i = Q_i \setminus \cup_{Q\in\mathcal{Q}^H, Q\neq Q_i} Q$$

for $i = 1, 2, \ldots, M$ and $Q_i \in \mathcal{Q}^H$. The main point with this construction is that all the potentials $V_i$ now have small Morrey norms, $||V_i||_{p,3/2} \leq \epsilon$.

**Lemma 29** *The inequalities* $(\mathrm{Av}_Q |V_i|^p)^{1/p} \leq c|Q|^{-2/3}$ $(1 < p < 3/2)$ *for* $c > 0$ *holds for* $i = 0, 1, 2, \ldots, M$ *and all dyadic cubes* $Q$*, provided* $V$ *has compact support and* $V \in \mathcal{L}^\infty$ *or* $(\mathrm{Av}_Q |V_i|^p)^{1/p} \leq \epsilon|Q|^{-2/3}$ *holds for all sufficiently small cubes* $Q$*, and* $\epsilon$ *a small number depending on* $c$*.* □

The constant $c$ is related to the constant $\gamma$ by $c = k\gamma$, where $k \leq 8^{2/p}$.

**Proof:** Firstly we note that we may restrict ourselves to dyadic cubes, as we can divide any given cube into 8 disjoint parts that are contained in dyadic cubes of approximately the same size as the original cube. This means that Lemma 29 will hold for a larger constant $c$ if it holds for some constant $c'$ for all dyadic cubes.

We shall prove Lemma 29 according to what kind of cube $V_i$ corresponds to. We shall divide the potential into 4 types: $V_0$, $V_i$'s corresponding to minimal cubes, $V_i$'s corresponding to branching cubes and the rest of the $V_i$'s. For simplicity we shall refer to potentials corresponding to minimal cubes as minimal potentials, potentials corresponding to maximal cubes as maximal potentials and so forth.

**Case I: Perturbative Potentials**

The only perturbative potential is $V_0$. For $Q \in \mathcal{Q}^{\text{pert}}$ we have $Q \cap E_0 = \emptyset$ since all non–perturbative cubes are contained in a maximal cube. $(\text{Av}_Q|V_0|^p)^{1/p} \leq c|Q|^{-2/3}$ therefore holds trivially for all $c > \gamma$.

**Case II: Minimal and Branching Potentials**

If our non–perturbative potential corresponds to a cube $Q_i$ we have

$$\int_Q |V_i|^p \, dx = \int_{Q_i} |V_i|^p \, dx \leq c|Q_i|^{1-2p/3} \leq c|Q|^{1-2p/3}$$

for all dyadic cubes $Q \supset Q_i$ if $1 < p \leq 3/2$. We may therefore assume that $Q \subseteq Q_i$ for the non–perturbative potentials as the inequality $(\text{Av}_Q|V_i|^p)^{1/p} \leq c|Q|^{-2/3}$ holds whenever $(\text{Av}_{Q_i}|V_i|^p)^{1/p} \leq c|Q_i|^{-2/3}$ holds. If the potential is minimal then there are no non–perturbative cubes inside the minimal cube, and so $(\text{Av}_Q|V_i|^p)^{1/p} \leq c|Q|^{-2/3}$ holds for $c > 8^{1/p}\gamma$ since it holds for all dyadic cubes inside the minimal cube.

For the branching potentials we have that the descendants of the branching cubes are sibling cubes, and so any cube properly contained in $E_i$ for $V_i$ a branching potential cannot be non–perturbative. This implies that $(\mathrm{Av}_Q|V_i|^p)^{1/p} \leq c|Q|^{-2/3}$ holds for $c > 8^{1/p}\gamma$ as for the minimal potentials.

**Case III: Non–minimal, Non–branching Non–perturbative Potentials**

If our non–minimal, non–branching non–perturbative cube is denoted $Q_i$ and we focus on the subgraph of the containment diagram of $\mathcal{Q}^{\text{pert}}$ we see that there is a *unique* first generation "$\mathcal{Q}^H$–descendant" of $Q_i$ that is either a minimal or a branching cube. We shall call this cube $Q^{\#}$. It is clear that $E_i = Q_i \setminus Q^{\#}$ by the uniqueness of $Q^{\#}$. To see that $(\mathrm{Av}_Q|V_i|^p)^{1/p} \leq c|Q|^{-2/3}$ we now perform a Calderon–Zygmund decomposition of our non–perturbative cube $Q \subset Q_i$. Note that $Q^{\#} \subset Q$ since otherwise $Q_i$ would be a branching cube. We bisect $Q$ and continue to bisect each of the sub cubes $Q_\alpha$ if $Q^{\#} \subset Q_\alpha$. Note that if $Q^{\#} \cap Q_\alpha = \emptyset$ then $Q_\alpha \notin \mathcal{Q}^{\text{pert}}$ because if $Q_\alpha \in \mathcal{Q}^{\text{pert}}$ then $Q_i \in \mathcal{Q}^{\text{branch}}$. The only non–perturbative cube in the decomposition is therefore $Q^{\#}$. All the cubes in the Calderon–Zygmund decomposition satisfies $(\mathrm{Av}_Q|V_i|^p)^{1/p} \leq \gamma|Q|^{-2/3}$ by virtue of the fact that $Q^{\#}$ is the only non–perturbative cube as non–branching cubes can only have one non–perturbative sub cube. But $Q^{\#}$ is a minimal or branching cube and satisfies $(\mathrm{Av}_Q|V_i|^p)^{1/p} \leq c|Q|^{-2/3}$ by case II for $c = 8^{1/p}\gamma$. Assuming $|Q| = 2^{3n}|Q^{\#}|$ we see that there are exactly $(2^3 - 1) = 7$ cubes $Q_\alpha$ with a certain volume $2^{-3k}|Q|$, for each value of $k = 1, 2, \ldots, n-1$. I.e. there are 7 cubes on each volume level between $Q_i$ and $Q^{\#}$. We therefore have

$$\begin{aligned}
\int_Q |V_i|^p &\leq \sum_{k=1}^{n}\sum_{i=1}^{7}\int_{Q_{k,i}} |V_i|^p + \int_{Q^{\#}} |V_i|^p \\
&\leq \sum_{k=1}^{n}\sum_{i=1}^{7} c|Q_{k,i}|^{1-2p/3} + c|Q^{\#}|^{1-2p/3} \\
&= \sum_{k=1}^{n} 7c2^{-3k(1-2p/3)}|Q|^{1-2p/3} + c2^{-3n(1-2p/3)}|Q|^{1-2p/3} \\
&\leq 8c\sum_{k=1}^{n} 2^{-3k(1-2p/3)}|Q|^{1-2p/3}
\end{aligned}$$

$$\leq \; c'|Q|^{1-2p/3}\,,$$

when $p < 3/2$. It follows that since all other dyadic cubes of $E_i$ are perturbative $(\mathrm{Av}_Q|V_i|^p)^{1/p} \leq c|Q|^{-2/3}$ holds for all dyadic cubes $Q \subset E_i$ if $V_i$ is a non–minimal, non–branching potential.

Since Case I–III cover all the different potentials we have finished the proof of Lemma 29. Note that the constant $c$ of Lemma 29 is related to $\gamma$ by $c \leq k\gamma$ for a universal constant $k$. Since $\gamma$ may be any number $\gamma > \epsilon$ it is evident that $c \leq k'\epsilon$ for some universal constant $k'$. If $\epsilon$ is chosen to be small enough we can make $c$ correspondingly small. Note that $\mathcal{L}^\infty$–potentials satisfy $(\mathrm{Av}_Q|V|^p)^{1/p} \leq \epsilon|Q|^{-2/3}$ for sufficiently small cubes $Q$ (whose size depends only on $\epsilon$ and $\|V\|_\infty$). ■

### 3.5.4 Theorem 28 for Minimal Cubes

If all the non–perturbative cubes are minimal, then $\mathcal{Q}^{\mathrm{H}}$ will consist of *disjoint* minimal cubes. Denote these cubes $Q_1, Q_2, \cdots, Q_N$. We then have

$$\begin{aligned}\int_{\mathbf{R}^3} V|u|^2 \;&=\; \sum_{i=1}^{N}\int_{Q_i} V_i(x)|u(x)|^2\,dx + \int_{\mathbf{R}^3} V_0(x)|u(x)|^2\,dx \\ &\geq\; -C'\|\nabla u\|_2^2 - \sum_{i=1}^{N} C\int_{Q_i}\left(\int_{Q_i}\frac{|\nabla u(y)|b_i(x)}{|x-y|^2}\,dy\right)^2 dx\,, \qquad (3.21)\end{aligned}$$

where $b_i(x) = |V_i(x)|^{1/2}$. Here we have used that $\int_{\mathbf{R}^3} V_0(x)|u(x)|^2\,dx \geq -C'\|\nabla u\|_2^2$ by Lemma 23 if $\|V_0\|_{p,3/2} \leq \epsilon$, where $\epsilon$ is sufficiently small. We also used that if $u \in H$ then $< u, \chi_{Q_i} >= 0$ whence $\mathrm{Av}_{Q_i}u = 0$ for $i = 1, 2, \ldots, N$, so by (3.4) of Lemma 22 we have that

$$|u(x)| \leq C\int_{Q_i}\frac{|\nabla u(z)|}{|x-z|^2}\,dz\,.$$

To proceed we use Theorem 21 with $b(x) = b_i(x)$, $f(y) = \chi_{Q_i}(y)|\nabla u(y)|$. The statement of Theorem 21 then tells us that

$$\int_{Q_i}\left(\int_{Q_i}\frac{|\nabla u(y)|b_i(x)}{|x-y|^2}\,dy\right)^2 dx \;\leq\; C\|b_i\|_{2p,3}^2\|\chi_{Q_i}\nabla u\|_2^2$$

$$= C\|V_i\|_{p,3/2}\|\chi_{Q_i}\nabla u\|_2^2 .$$

Summing over $i = 1, 2, \ldots, N$ as in (3.21) we get

$$\begin{aligned}\int V|u|^2 &\geq -C'\|\nabla u\|_2^2 - C\sum_{i=1}^{N}\|\chi_{Q_i}\nabla u\|_2^2 \\ &\geq -C''\|\nabla u\|_2^2 ,\end{aligned}$$

since $\sum_{i=1}^N \|\chi_{Q_i}\nabla u\|_2^2 = \sum_{i=1}^N \int_{Q_i} |\nabla u|^2 \leq \int_{\mathbf{R}^3} |\nabla u|^2$, when $Q_i$, $i = 1, 2, \ldots, N$, are disjoint. Here $C''$ depends on the number $\gamma$ which we use to decide whether a cube is perturbative with respect to. If $\gamma$ is sufficiently small, then $C''$ is less than 1, and it follows that $\int |\nabla u|^2 + \int V|u|^2 \geq 0$ or $< Lu, u >\geq 0$ for $u \in H$. Thus the number of eigenvalues is given by the index of $H$ in $\mathcal{L}^2(\mathbf{R}^3)$. In other words $N(V, 0) \leq N$ which is the statement of Theorem 26 for the case when all non–perturbative cubes are minimal. ■

### 3.5.5 Theorem 28 for Non–Branching Cubes

If all the non–perturbative cubes are non–branching then all the cubes of $\mathcal{Q}^{\mathrm{H}}$ are either minimal or maximal. If a cube is maximal, but not minimal, it must contain a unique minimal cube. So not all of the cubes in $\mathcal{Q}^{\mathrm{H}}$ need to be disjoint, and we cannot use the same strategy as for the minimal cube case of the previous section. There are two ways we can resolve the problem with the overlapping cubes. We can either use a smaller value of $\gamma$, or we can subdivide the regions where the cubes overlap. We shall follow the last strategy. To do this we have to extend (3.4) of Lemma 22. If we attempt to make (3.4) hold for as general domains as possible we get the following result:

**Lemma 30** *If $\Omega$ is any bounded convex domain in $\mathbf{R}^3$ then*

$$|u(x) - \mathrm{Av}_\Omega u| \leq \frac{(\mathrm{diam}(\Omega))^3}{3|\Omega|}\int_\Omega \frac{|\nabla u(y)|}{|x-y|^2}\, dz ,$$

*where* $\operatorname{diam}(\Omega) = \sup\{d(x,y) : x, y \in \Omega\}$. □

**Proof:** The proof of Lemma 30 is almost the same as that of Lemma 22, except that extra care has been taken. First we note that we need $\Omega$ to be convex to write

$$u(x) - u(y) = \int_0^1 \frac{d}{dt} u(y + t(x-y))\, dt$$

for all $x, y \in \Omega$ as in (3.5). Secondly, we follow the rest of the proof of Lemma 22 and see that the constant $C$ in (3.4) is bounded by $(\operatorname{diam}(\Omega))^3/(3|\Omega|)$. ■

If we do not allow branching cubes, each maximal cube can contain at most one minimal cube. We now want to avoid interaction between the maximal and minimal cubes. To do this we divide up the maximal cubes into convex regions and apply Lemma 30. We must beware, however, that $(\operatorname{diam}(\Omega))^3/(3|\Omega|)$ does not become to large. This is the case if our domain $\Omega$ is "skinny". So we wish to divide up the region in between a maximal cube and the minimal cube inside into convex sets that aren't too "skinny". We can do this by extending diagonals from the sides of the sides and corners of the minimal cubes on the inside. We leave it to the reader as a simple exercise in geometry to verify that $(\operatorname{diam}(\Omega))^3/(3|\Omega|)$ is bounded for these regions. Some example figures have been drawn in Figure 3.5.5. Figure 3.5.5 is a two dimensional simplification, as the procedure is slightly harder to visualize in 3 dimensional space.

We shall call the "non–skinny", convex polygonal regions $P_1, P_2, \ldots$. Note that for each maximal cube we get at most 6 such regions. If we also include the minimal cubes among the $P_i$'s we may write:

$$\int V|u|^2 = \sum_{i=1}^{M_P} \int_{P_i} V_{j(i)}(x)|u(x)|^2\, dx + \int_{\mathbf{R}^3} V_0(x)|u(x)|^2\, dx\ , \tag{3.22}$$

where $M_P$ is the number of polygonal regions and $j(i)$ is the $j$ for which $P_i \subset E_j$. Since we have at most 6 polygonal regions for each pair of minimal and maximal

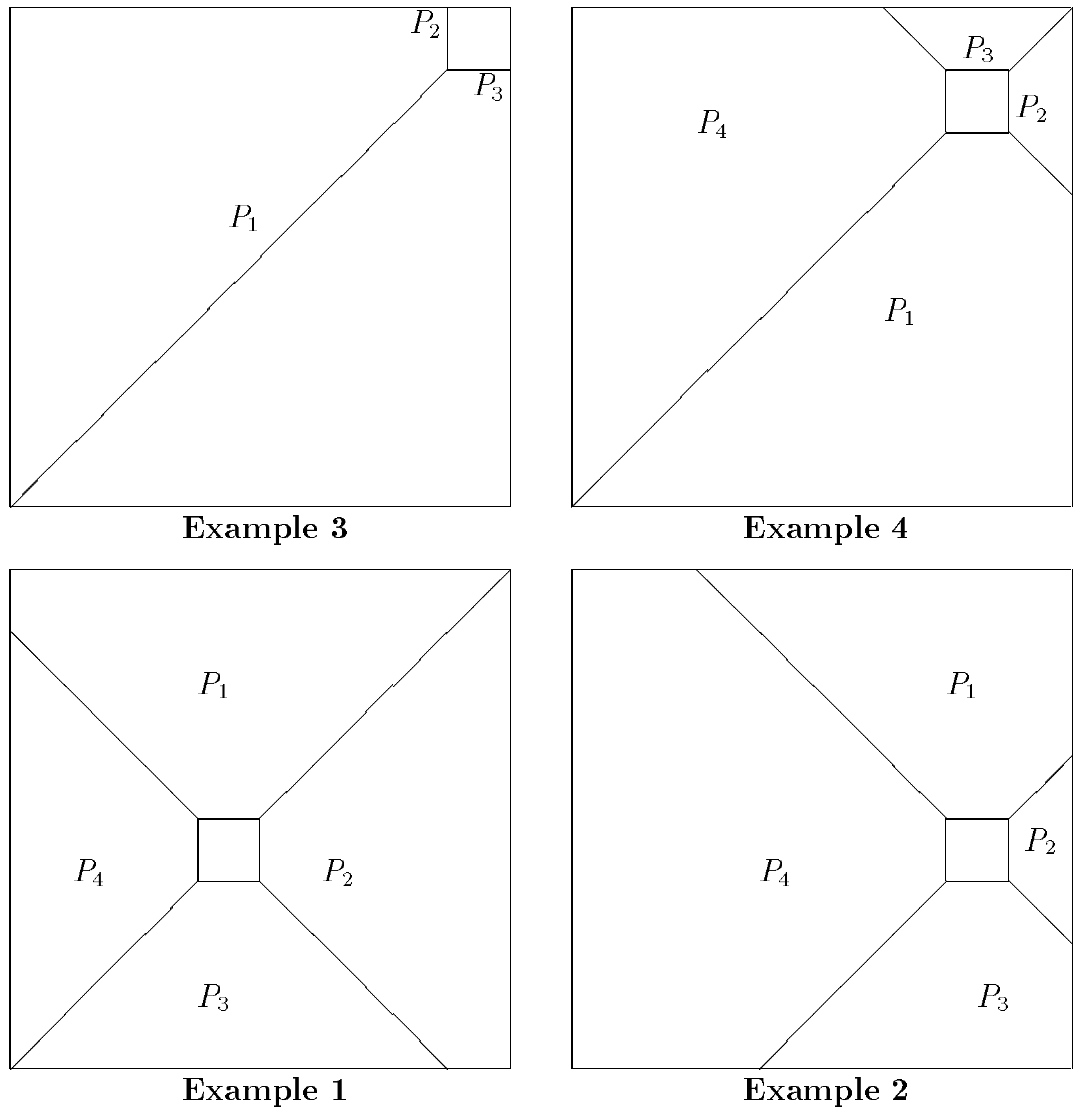


Figure 3.2: Subdividing the region between a maximal and a minimal cube into "non–skinny", convex, polygonal regions $P_1, P_2, \ldots$.

cubes we see that $M_P < 6N + N = 7N$. As before we may concentrate on the $\sum_{i=1}^{M_P} \int_{P_i} V_{j(i)}|u|^2$ term since $\int_{\mathbf{R}^3} V_0|u|^2 \geq -C\|\nabla u\|_2^2$. To bound the polygonal regions we use Lemma 30.

$$\sum_{i=1}^{M_P} \int_{P_i} V_{j(i)}|u|^2 \geq -C \sum_{i=1}^{M_P} \int_{P_i} |V_{j(i)}| \left( \int_{P_i} \frac{|\nabla u(y)|}{|x-y|^2}\, dy \right)^2 dx\ , \tag{3.23}$$

seeing as we have chosen the $P_i$'s to be not too "skinny", i.e. the bound $(\mathrm{diam}(\Omega))^3/(3|\Omega|)$ is uniformly bounded. We now write (3.23) in terms of the cubes $Q_i$. We clearly have

$$\int_{P_i} |V_{j(i)}| \left( \int_{P_i} \frac{|\nabla u(y)|}{|x-y|^2}\, dy \right)^2 dx \leq \int_{Q_{j(i)}} \left( \int_{Q_{j(i)}} \frac{|b_{j(i)} v_i|}{|x-y|^2}\, dy \right)^2 dx \, ,$$

where $b_j(x) = |V_j(x)|^{1/2}$ and $v_i(x) = \chi_{P_i} |\nabla u(x)|$. Using Theorem 28 we now get

$$\begin{aligned} \int_{Q_{j(i)}} |V_{j(i)}| \left( \int_{P_i} \frac{|\nabla u(y)|}{|x-y|^2}\, dy \right)^2 dx \ &\leq\ C \|b_{j(i)}\|_{2p,3}^2 \|v_i\|_2^2 \\ &\leq\ C' \|\chi_{P_i} |\nabla u(x)|\|_2^2 \, . \end{aligned}$$

If we use this estimate in (3.23) we may bound (3.22) as follows:

$$\begin{aligned} \int V|u|^2 \ &\geq\ -C\|\nabla u\|_2^2 - C' \sum_{i=1}^{M_P} \|\chi_{P_i} \nabla u\|_2^2 \\ &\geq\ -C\|\nabla u\|_2^2 - C'\|\nabla u\|_2^2 \\ &\geq\ -C``\|\nabla u\|_2^2 \, , \end{aligned}$$

if $\|V_{j(i)}\|_{p,3/2}$ is sufficiently small (i.e. $\gamma$ chosen sufficiently small). We have in conclusion that if $< u, \chi_{P_i} >= 0$, $i = 1, 2, \ldots, M_P$, $M_P < 7N$, then $< Lu, u > \geq 0$. In other words: If there are no non–branching cubes then there are at most 7N negative eigenvalues. ■

We have so far learned how to deal with minimal and non–branching cubes. It only remains to tackle the hardest case: branching cubes.

### 3.5.6 Theorem 28 for Branching Cubes

This is the real thing! If we can avoid the interaction between the branching cubes and the other cubes we can finish the proof of Theorem 28. Our strategy will be the same as that of the previous section; namely to subdivide the branching cubes into "not too skinny", disjoint, convex, polygonal regions. If $Q_i$ is our branching cube and $E_i$ is as before $E_i = Q_i \setminus \cup_{Q \neq Q_i, Q \in \mathcal{Q}^{\mathrm{H}}} Q$, then we are in fact interested in splitting up $E_i$ into "non–skinny", convex, polygons. If $Q$ is a sibling cube of $Q_i$, then $Q$ will correspond to a cubic hole in $E_i$. In fact $E_i$ is merely a cube $Q_i$ with some cubic holes inside. All of the cubic holes correspond to a unique sibling cube of $Q_i$. When constructing the subdivision of $E_i$ we must take care that the number of "non–skinny", convex polygons making up $E_i$ does not exceed a constant times the number of sibling cubes. If we can do so then by the argument of the previous section the number of negative eigenvalues is bounded by the number of convex polygons plus the number of minimal cubes. So all we need to do is to devise an algorithm to construct the "non–skinny", convex polygons making up $E_i$. Well, here it is:

Let $\{P_i\}_{i=1}^{M_P}$ be the set of polygons dividing up the $E_i$'s.

**Step 1** *Find the smallest super sibling of $Q_i$.*

Consider the sibling cubes of $Q_i$ and search for the smallest dyadic cube containing all the sibling cubes of $Q_i$, i.e. find the smallest dyadic cube $Q$ such that $Q_i \setminus Q \subset E_i$. This cube will be referred to as the super sibling of $Q_i$ and we shall denote it by $Q_{\text{super},i}$. Note that in the dyadic bisection of the super sibling at least two of the 8 sub cubes have cubic holes in them, i.e. $Q \not\subset E_i$ for at least two of the sub cubes of the super sibling.

**Step 2** *If $Q_i = Q_{\text{super},i}$ goto step 4.*

If the super sibling is $Q_i$ itself then skip step 3. Note that this will be the case most of the time.

**Step 3** *Subdivide $Q_i \setminus Q_{\text{super},i}$.*

Use the division method devised in section 3.5.5 for subdividing the region between a maximal and a minimal cube to subdivide the region between $Q_i$ and $Q_{\text{super},i}$ into "non–skinny", convex polygons. Add each of these polygons to the list $\{P_i\}_{i=1}^{M_P}$.

**Step 4** *Bisect* $Q_{\text{super},i}$.

When bisecting the super sibling we will divide up the sibling cubes of $Q_i$. At least two of the sub cubes from the bisection will contain at least one sibling cubes of $Q_i$. We will denote the cubes arising from the bisection of $Q_i$, $Q_{\text{sub},i}$.

**Step 5** *Categorize and subdivide* $Q_{\text{sub},i}$.

For each of the sub cubes, $Q_{\text{sub},i}$, of $Q_{\text{super},i}$ we do the following:

1. If $Q_{\text{sub},i} \subset E_i$, i.e $Q_{\text{sub},i}$ contains none of $Q_i$'s sibling cubes, then we add $Q_{\text{sub},i}$ to the set of "non–skinny", convex polygons, $\{P_i\}_{i=1}^{M_P}$.
2. If $\#\{Q : Q \text{ is a sibling cube of } Q_i, Q \subset Q_{\text{sub},i}\} = 1$, i.e. there is only one sibling cube contained in $Q_{\text{sub},i}$, then subdivide $Q_{\text{sub},i} \setminus Q_{\text{sibling},i}$ according to the procedure of section 3.5.5. Here $Q_{\text{sibling},i}$ refers to the sibling cube of $Q_i$ that is contained in $Q_{\text{sub},i}$. The regions arising from the subdivision of $Q_{\text{sub},i} \setminus Q_{\text{sibling},i}$ are then added to the list $\{P_i\}_{i=1}^{M_P}$.
3. If $\#\{Q : Q \text{ is a sibling cube of } Q_i, Q \subset Q_{\text{sub},i}\} > 1$ then repeat step 1,2,3,4 and 5 with the cube $Q_{\text{sub},i}$ in place of $Q_i$.

Our space $H$ will consist of all $\mathcal{L}^2(\mathbf{R}^3)$ functions, $u$, such that $< u, \chi_{P_i} >= 0$, $i = 1, 2, \ldots, M_P$. If the number of $P_i$'s, $M_P$, is not too large compared to the number of minimal cubes, $N$, Theorem 28 follows from the argument of section 3.5.5. To show that $M_P$ is bounded by $N$, we note that there are less super siblings than there are sibling cubes. Each super sibling gives rise to at most 6 exterior $P_i$'s and at most 6 interior $P_i$'s (the interior regions are all cubes). For each sibling cube we also get 6 exterior $P_i$ regions. Also we get 6 interior $P_i$ regions for each non–minimal non–branching cube, and 1 $P_i$ region for each minimal cube. So we have at most (18 $\cdot$ #sibling cubes + 6 $\cdot$ #non–branching cubes + #minimal cubes) $P_i$ regions.

Using that the number of sibling cubes and the number of non–branching cubes are bounded by $2N$, we see that $M_P$ is less than $(18 \cdot 2 + 6 \cdot 2 + 1)N = 49N$. Thus the proof of Theorem 28 follows. The estimate $49N$ can be improved by devising more clever subdivision schemes. Note for example that $Q_{\text{super},i} = Q_i$ almost always, so that one should expect $37N$ to be a sufficiently large bound. ■

### 3.5.7 Estimates of $N(V, \lambda)$

If $\lambda > 0$ we can get estimates of $N(V, \lambda)$ merely by considering the function $V_\lambda(x) = V(x)\chi_{\{x:V(x)\leq\lambda\}}(x)$. This is however not the best we can do.

**Corollary 31** *Let $\lambda < 0$, $V$ be such that $(\mathrm{Av}_Q|V|^p)^{1/p} \leq \epsilon|Q|^{-2/3}$ for all sufficiently small dyadic cubes $Q$ and let $N_\lambda$ be the number of minimal cubes with side length less than $c_1\lambda^{-1/2}$. With these assumptions we have*

$$N(V, \lambda) < CN_\lambda \, , \tag{3.24}$$

*for some universal positive constant $C$.* □

**Proof:** We recall from the proof of the lower estimate of Theorem 24 that by defining a partition of unity $\sum_{n\in\mathbf{Z}^3} \phi_n^2(x) \equiv 1$ such that $\Phi_n(x) = \phi_n^2(x)$ is given by Lemma 25 then

$$\int_{\mathbf{R}^3} V|u|^2 = \sum_{n\in\mathbf{Z}^3} \int_{\mathbf{R}^3} V_n|u_n|^2$$

and

$$\int_{\mathbf{R}^3} |\nabla u|^2 \geq \left( \sum_{n\in\mathbf{Z}^3} \int_{\mathbf{R}^3} |\nabla u_n|^2 \right) - C_\phi^2 c_1^{-1} \lambda \int_{\mathbf{R}^3} |u|^2 \, ,$$

where $u_n(x) = \phi_n(x)u(x)$. Since $\phi_n$ has compact support in a cube of side length $3(c/\lambda)^{-1/2} = 3c_1\lambda^{-1/2}$ ($c$ is the constant of Lemma 25), it follows that if $C_\phi^2 c_1^{-1}/3 < 1$ then we only need to prove that

$$\int_{\mathbf{R}^3} |\nabla u|^2 + \int_{\mathbf{R}^3} V_n|u_n|^2 \geq 0 \, . \tag{3.25}$$

By the proof of Theorem 28 it follows that this holds for all functions $u_n \in H$, where $H$ consists of all functions orthogonal to characteristic functions on minimal, maximal, branching and sibling cubes with side length less than $c_1\lambda^{-1/2}$. If $\phi_n$ is supported in cubes with side length $3c_1\lambda^{-1/2}$ it follows that at most $3^3$ of the functions $\phi_n$ are nonzero on a given dyadic cube of side length less than $c_1\lambda^{-1/2}$. The identity $u_n(x) = \phi_n(x)u(x)$ tells us that (3.25) holds if $< u_n, \chi_{P_k} >= < u\phi_n, \chi_{P_k} >= < u, \phi_n\chi_{P_k} >= 0$, where $P_k$ are the polygonal regions defining $H$. Since the number of polygonal regions $P_k$ is at most $49N_\lambda$ it follows that only $27(49N_\lambda) = 1323N_\lambda$ values of $n, k$ gives $\phi_n\chi_{P_k} \not\equiv 0$. If $u \in H'$, where $H'$ consists of all functions orthogonal to $\phi_n\chi_{P_k}$ then $< Lu, u >\geq -\lambda||u||_2^2$. The index of $H'$ is less than $1323N_\lambda$, so it follows that $N(V, \lambda) \leq 1323N_\lambda$. This inequality can be improved to $N(V, \lambda) \leq 392N_\lambda$ by letting $\phi_n$ be supported in cubes of side length $2c_1\lambda^{-1/2}$, and only overlapping 8 times. The inequality can however not be improved to $N(V, \lambda) \leq 49N_\lambda$ as this would force $\phi_n$ to be characteristic functions and therefore unsuitable for our purposes as they are non–differentiable. ■

The result of corollary 31 tells us that the number of eigenvalues of size approximately $\lambda$ ought to correspond to minimal cubes with side length approximately $c_1\lambda^{-1/2}$.

### 3.5.8 Estimates of $N(V, \lambda)$ in the Drift Case

For the case $V = -\nabla \cdot b$ we proceed in the same manner as in section 3.4. By (3.19) we have that $< Lu, u >\geq \lambda||u||_2^2$ holds if

$$\int |\nabla u|^2 + \int \tilde{V}|u|^2 \geq \tilde{\lambda}||u||_2^2$$

is true. Here $\tilde{V} = -4|b|^2$, $\tilde{\lambda} = 2\lambda$. If $u \in H$, where $H$ is the space discussed in Theorem 28 or Corollary 31 corresponding to the potential $\tilde{V}$ then the argument of

Theorem 28 and Corollary 31 are valid, so we get the following result:

**Corollary 32** *If* $V = -\nabla \cdot b$, $|b| \in M^3_{2p}$, $1 < p \le 3/2$, $b$ *has compact support (only needed for estimates of* $N(V,0)$*) and* $(\mathrm{Av}_Q |b|^{2p})^{1/(2p)} \le \epsilon' |Q|^{-1/3}$ *for all sufficiently small cubes* $Q$ *and* $\epsilon'$ *fixed, but sufficiently small, then*

$$N(V,0) \le CN$$

*and*

$$N(V,\lambda) \le CN_\lambda \, , \tag{3.26}$$

*where* $N$ *is the number of minimal dyadic cubes of* $\tilde{V} = -4|b|^2$ *and* $N_\lambda$ *is the number of such cubes with side length less than* $c_1 \lambda^{-1/2}$. $\square$

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# BIBLIOGRAPHY


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# ABSTRACT

Negative Eigenvalues of the Schrödinger Equation: An Approach Through Fractional Integration and Morrey Spaces

by

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The time independent Schrödinger equation has been analyzed for singular potentials. Direct estimates of the solution of the Schrödinger equation and estimates of the negative eigenvalues of the Schrödinger operator are obtained for potentials in certain Morrey spaces and for potentials of divergence form.

The method of solution relies heavily on an extension of the Hardy–Littlewood–Sobolev inequality. This fractional integral inequality enables one to reproduce classical results for the negative eigenvalues by Fefferman–Phong, and to find direct estimates for potentials of divergence form.