VII. Conclusion

A method has been developed to improve the use of the Rayleigh-Ritz procedure. A criterion is established (flexible enough to take into account the desired accuracy) which is a measure of the cumulative improvement due to the addition of more and more terms in the series expansion. The exact roots of the determinantal equations are not calculated—it is shown that the exact calculation wastes time and does not improve the final accuracy. The convergence is accelerated by skipping unnecessary intermediate steps. The method does not use more terms than required by the terminating criterion; it ensures at the same time a large reduction of the computation time and an optimization of the accuracy. This is confirmed by comparison with other methods. The computation time is drastically reduced because the final results are obtained after the calculation of only a few (not more than 5 to 7) values of determinants of increasing order.

Acknowledgment

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Iterative Solutions of the Scalar Helmholtz Equation in Lossy Regions

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Abstract—**Iterative solutions to the finite difference equations derived from the scalar Helmholtz equation are found to diverge for domains greater than a certain size. A transform method is presented which produces convergence in larger domains. The method is illustrated by solutions for one- and two-dimensional cases involving lossy dielectric media.**

Introduction

THIS PAPER presents finite difference solutions to the scalar Helmholtz equation $(\nabla^2+k^2)\phi=0$ in domains for which the propagation constant k may be complex corresponding to a lossy dielectric medium. Finite difference methods have been used previously to obtain fields within lossless cavities [1] and in the cross sections of waveguides with arbitrarily shaped boundaries [2], [3].

It is found that in domains whose size exceeds a certain value, which is a function of k, the iterative solution using central differences diverges, irrespective of the net point spacing used [4]. This difficulty was not evident in previously published papers because of the smaller domain sizes in the treatment of dominant modes only.

In this paper, a transform is presented, the repeated application of which will produce convergence in domains of larger size. The technique is illustrated by computed results for a one-dimensional and a two-dimensional example, both involving lossy dielectric media.

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Difference Equations

The techniques to be described pertain to fields satisfying the scalar Helmholtz equation

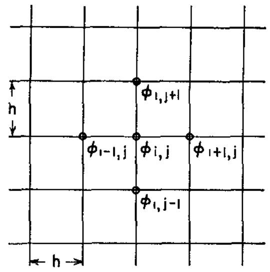

Fig. 1. Physical representation of the grid used in (3).

$$(\nabla^2 + k^2)\phi = 0 \tag{1}$$

where ϕ may represent either an electric or magnetic field quantity, and k^2, which is dependent on the characteristics of the medium, is given by

$$k^2 = \omega^2 \mu \epsilon. \tag{2}$$

For a lossy dielectric medium, ϵ is complex.

For the solution of (1) by finite differences, a mesh is drawn over the region with spacing h, as shown in Fig. 1 for the two-dimensional case.

Equation (1) can be written in finite difference form as

$$\phi_{i,j} = \frac{1}{\gamma}(\phi_{i-1,j} + \phi_{i+1,j} + \phi_{i,j-1} + \phi_{i,j+1}) \tag{3}$$

where $\gamma = 4 - h^2k^2$. The physical representation of this equation is shown in Fig. 1.

The set of equations (3) is commonly solved by iteration over the region using an overrelaxation factor to speed convergence. For most boundary value problems, central differences in the form of (3) must be used. It is possible to show that for a convergent solution to be obtained, the following conditions must be satisfied.

1) If the medium is lossless, the domain size may not exceed a certain value, which depends on the type of boundary conditions, and is typically $\frac{1}{2}$ wavelength.
2) If the medium is lossy, the domain size may be larger than $\frac{1}{2}$ wavelength, depending on the loss tangent. In the special case of a purely conductive region, there is no restriction on domain size.

The mathematical details of the restrictions are treated in [4]. They are related to the properties of the iteration matrix, which can be shown to have eigenvalues greater than unity when the domain exceeds a certain size, making the process unstable. Because of this difficulty, the range of boundary value problems which can be handled using conventional iteration of the central difference equations is restricted. A method is presented in the next section which allows problems to be solved even though the iterative process is unstable.

Transformation Method

Successive iterations over the network produce at each mesh point a sequence of numbers. In the study of functions which generate such numerical sequences, transformations have been developed which can accelerate convergence, and in some cases make a divergent sequence converge. Some of these are listed by Shanks [5] and Wynn [6].

For the application to boundary problems in which the number of eigenvalues of the iteration matrix with magnitudes greater than 1 is unknown, a useful transformation has been found to be

$$B_K = \frac{A_{K-1} \cdot A_{K+1} - A_K^2}{A_{K-1} + A_{K+1} - 2A_K} \tag{4}$$

where $A_1, A_2, \cdots A_K, \cdots$ are the values of ϕ produced by successive iterations at a single point.

This particular transformation is commonly known as Aitken's acceleration technique [7], and belongs to a general class of transformations which are discussed quite thoroughly by Shanks [5]. The transformation has previously been used to accelerate a convergent process; however, in this paper it is being applied to a divergent sequence.

The method used to achieve convergence is illustrated in the following example, given the one-dimensional problem

$$\frac{\partial^2 A}{\partial x^2} = -k^2 A \tag{5}$$

with

$$A(x = 0) = A(x = 5) = 1$$

and

$$k^2 = h^2 = 1.$$

In (5), the quantity A is analogous to the voltage along a TEM transmission line of length $5/2\pi$ wavelengths.

In order to simplify the example, a very coarse net spacing is used. For this reason, there is a slight discrepancy between the solutions of the difference equation and the differential equation. The difference equation corresponding to (5) is

$$A_i = A_{i-1} + A_{i+1} \tag{6}$$

with

$$A_0 = A_5 = 1.$$

Starting with initial values of $A_i = 0$ for $1 \leq i \leq 4$, the values for each iteration are listed in Table I. For each computation, the most recently computed value is used (method of successive displacements), with a relaxation factor of 1.

The results of this process are clearly divergent. However, the transformation (4) can be applied to the sequence for A_1 in the following manner:

$$B_{1,2} = \frac{1 \times 4 - 2^2}{1 + 4 - 2 \times 2} = 0$$

$$B_{1,3} = \frac{2 \times 10 - 4^2}{2 + 10 - 2 \times 4} = 1$$

$$B_{1,4} = \frac{4 \times 26 - 10^2}{4 + 26 - 2 \times 10} = 0.4$$

etc.

TABLE I

THE DIVERGENCE OF THE ITERATIVE APPROACH AS GIVEN BY (6)

Iteration (k)	A_0	A_1	A_2	A_3	A_4	A_5
	1	0	0	0	0	1
1		1	1	1	2	
2		2	3	5	6	
3		4	9	15	16	
4		10	25	41	42	
5		26	67	109	110	
6		68	177	287	288	
		.	.	.	.	
		.	.	.	.	
		.	.	.	.	

TABLE II

THE TRANSFORM DEFINED BY (4) IS APPLIED TO THE RESULTS LISTED IN TABLE I. THESE TRANSFORMED VALUES CONVERGE TO THE TRUE SOLUTION

Iteration (k)	B_0	B_1	B_2	B_3	B_4	B_5
2	1	0	0	−1.67	−0.67	1
3		1.0	−0.6	−1.25	−0.25	
4		0.4	−0.846	−1.095	−0.095	
5		0.154	−0.941	−1.036	−0.036	
6		0.059	−0.978	−1.014	−0.014	
7		0.0225	−0.991	−1.0053	−0.0053	
		.	.	.	.	
		.	.	.	.	
		.	.	.	.	
True Solution		0	−1	−1	0	

The transform can likewise be applied to the sequences for A_2, A_3, and A_4. The results are tabulated in Table II. These sequences converge to the true solution even though they were computed from the terms of divergent sequences.

DETERMINATION OF CONVERGENCE

In a given problem the number of eigenvalues with magnitude greater than 1 may not be known in advance. In this case, enough iterations should be performed and enough transformations applied so that:

1) convergent sequences are obtained;
2) applying the transformation one additional time produces a negligible change in the values;
3) if the difference equation is applied to the solution, the residuals are small.

MATHEMATICAL EXPLANATION

At any point in the grid, the error at the Kth iteration can be approximated in terms of the eigenvalues of the iteration matrix (λ_1, λ_2, etc.) and the initial errors (ϵ_1, ϵ_2, etc.) as [8]

$$\epsilon_K = \lambda_1{}^K\epsilon_1 + \lambda_2{}^K\epsilon_2 + \cdots \lambda_{N-1}{}^K\epsilon_{N-1}. \tag{7}$$

For all of the terms for which $|\lambda|<1$, as K become large $\lambda^K \to 0$, so that the only error terms left are those for which $|\lambda| \geq 1$. Since $A_K = A_{\text{EXACT}} + \epsilon_K$ for the case in which only one eigenvalue has a magnitude greater than 1, then

$$A_K = A_{\text{EXACT}} + \lambda^K\epsilon_1. \tag{8}$$

Substituting (8) in transform (4) yields

$$B_{iK} = \frac{(A_{i\text{EXACT}} + \lambda^{K-1}\epsilon_1)(A_{i\text{EXACT}} + \lambda^{K+1}\epsilon_1) - (A_{i\text{EXACT}} + \lambda^K\epsilon_1)^2}{(A_{i\text{EXACT}} + \lambda^{K-1}\epsilon_1) + (A_{i\text{EXACT}} + \lambda^{K+1}\epsilon_1) - 2(A_{i\text{EXACT}} + \lambda^K\epsilon_1)}. \tag{9}$$

Multiplying and collecting terms,

$$B_{iK} = \frac{(\lambda^{K+1} + \lambda^{K-1} - 2\lambda^K)\epsilon_1 A_{i\text{EXACT}}}{(\lambda^{K+1} + \lambda^{K-1} - 2\lambda^K)\epsilon_1}. \tag{10}$$

Canceling,

$$B_{iK} = A_{i\text{EXACT}}. \tag{11}$$

For the situation where two or more eigenvalues have a magnitude greater than 1, it is easily proved that the application of the transform (4) to the sequence A_i nullifies the effect of the largest eigenvalue. The application of the transformation to the B_i sequences nullifies the effect of the second largest eigenvalue, etc. For example,

$$C_K = \frac{B_{K-1} \cdot B_{K+1} - B_K{}^2}{B_{K-1} + B_{K+1} - 2B_K}. \tag{12}$$

The transformation can be applied as many times as there are eigenvalues with a magnitude greater than 1, and will also work equally well if the values of the numerical sequences are complex. This subject is elaborated in [4].

EXAMPLES

Example 1

A TEM transmission line is excited at one end by a voltage $V = 1\angle 0°$, and shorted at the far end, as shown in Fig. 2. The propagation constant is assumed to have a value given by $k^2 = 1 - j\,0.5$ which corresponds to a lossy dielectric. A net point spacing of $h = 0.5$ is chosen, resulting in 15 points along the line at which the voltage is to be evaluated.

The difference equation at the ith point is

$$V_i = \frac{1}{\gamma}(V_{i-1} + V_{i+1}) \tag{13}$$

where $V(i=0) = 1$, $V(i=15) = 0$, and $\gamma = 2 - h^2k^2$.

Since the domain size is greater than $\frac{1}{2}$ wavelength, the successive application of (13), starting from initial values of zero at the interior points, results in divergence. Moreover, since the total length is greater than two half-wavelengths, a single application of the transform given by (4) does not bring about convergence and a second application of the transform is necessary.

Computed results showing the amplitudes and phases of the voltages at the 15 mesh points are shown in Fig. 3. The values computed at each point following the first application of the transform are designated by B in the figure. The dashed line joins the B values which were obtained after 30 iterations. Further iterations caused the B values to diverge

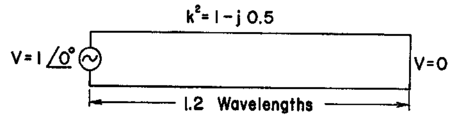

Fig. 2. Transmission line described by (13).

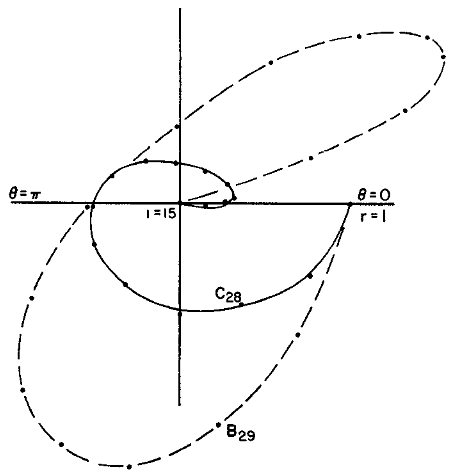

Fig. 3. Numerical results for Example 1 after 30 iterations of (13). Curve B is the result of a single application of the transform and C is the result of the application of the transform to B.

even more. Application of the transform to the B values yields the C values, shown by the solid line in Fig. 3, for 30 iterations. A third application of the transform yields values which are not discernably different from the C values shown.

This example was especially chosen to check the utility and accuracy of the transform method. Difference equation (13) may be solved analytically by alternative means as a check. This was done and the error was so small that the transform solution could not be discerned from the analytic solution when plotted on the graph.

Example 2

This example is chosen for a case in which no simple analytical solution of the difference equations exists. It is also intended to model the case of microwave diathermy applied to human tissue.

A two-dimensional waveguide with a sudden increase in dimension is filled with two different dielectric materials as shown in Fig. 4. The solution is given in terms of the magnetic field of a TM wave impressed in the waveguide. The only magnetic field component present is H_y, which is assumed to have unit amplitude at all points on the left-hand boundary. The remaining boundary conditions, as shown in the figure, correspond to perfectly conducting walls, except for the right-hand boundary, which is given an impedance condition corresponding to the wave impedance of plane waves in free space. The dielectric medium to the left is lossless, while that on the right has a loss tangent of unity, corresponding to certain human tissues.

A rather sparse network of points was used in order to conserve computer time, since the major motivation for this work was to verify the convergence of the solution, rather than to obtain accurate results. The netpoint spacing was

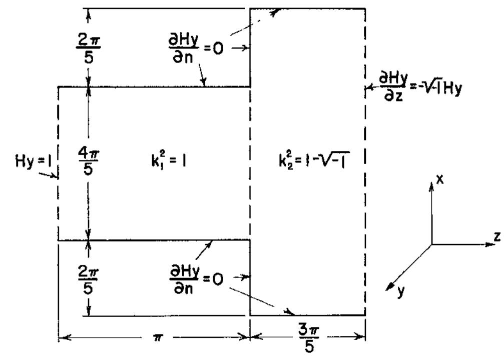

Fig. 4. Boundary conditions for Example 2.

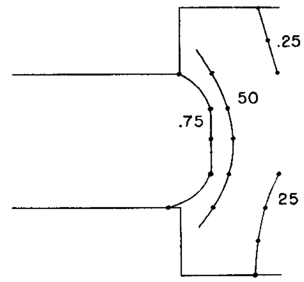

Fig. 5. Contours of equal amplitude for the numerical results of Example 2.

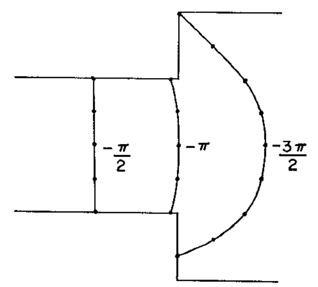

Fig. 6. Contours of equal phase for the numerical results of Example 2.

$h = \pi/5$. The transform was applied three times to obtain the field maps shown in Figs. 5 and 6. These are plotted, respectively, as contours of constant amplitude and contours of constant phase within the lossy region. The slight lack of symmetry is attributed to the ordering of the iteration process, and could be minimized by smoothing techniques. The calculations were performed on a Burroughs B5500 computer in an execution time of 45 seconds.

Conclusions

The transform technique presented in this paper allows central differences to be applied to a wider class of problems than have hitherto been presented.

The main difficulties encountered with the transformation are the following

1) If the iterative process applied to the original values by the difference equation (3) produces sequences that are extremely divergent, then it may not be possible to perform enough iterations to achieve a solution before the values diverge beyond the range of the computer.

2) The transformation (4) involves taking the difference between two large numbers which are close in value. This means that each time the transformation is applied, significant digits are lost.

In order to overcome these difficulties, it may be necessary to store the values and perform the arithmetic in multiple precision.

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The Finite Difference Solution of Microwave Circuit Problems

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Abstract—**Using finite difference methods this paper shows how solutions may be obtained with the aid of a digital machine to a wide range of microwave circuit problems. These problems include the parameters of TEM-mode transmission lines, the equivalent circuits of obstacles in these lines, the cutoff frequencies of the fundamental mode in a waveguide of very general cross section, and the equivalent circuits of obstacles in rectangular waveguide. Methods for deriving the appropriate finite difference equations are presented and optimum methods for their solution set out; singularities are also included in the treatment. The paper ends with a resumé of some typical results to problems of practical interest which have been obtained by these methods.**

I. Introduction

Many PROBLEMS in microwave engineering involve solution of the elliptic differential equation [1]–[4]

$$\nabla^2\phi + k^2\phi = 0 \tag{1}$$

subject to specified boundary conditions, where ϕ is a scalar potential function and $k=\omega/c$ is the phase constant. When $k=0$ (1) degenerates to Laplace's equation, which is of interest in connection with TEM-mode transmission lines. Otherwise it is referred to as the reduced scalar wave equation, which must be solved, for example, in considering cylindrical waveguides of any arbitrary cross section.

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Analytical solutions of this equation have been obtained, in general, only for boundary geometries belonging to coordinate systems in which it is separable, and then often in the form of a series not always convenient for engineering application. There is therefore a need for a simple means of obtaining approximate solutions to (1) which are sufficiently accurate for practical purposes. In this work a solution will be considered adequate when it involves a lesser uncertainty than is produced by manufacturing tolerances.

A common technique used to obtain solutions to (1) is to replace it by a large number of simultaneous linear algebraic equations. It is then assumed that by making the number of equations large, an adequate representation of the original problem may be obtained. The analytical complexities of solving the original partial differential equation are therefore traded off against the computational difficulties of solving a large system of simultaneous equations.

Prior to the advent of the modern digital computer, various ingenious techniques were devised for this purpose [5], [6]. In particular, the relaxation method of Southwell deserves mention [5], [7]. Nearly all these methods gave full scope to the computer for exercise of his intuition in speeding convergence and so are not suitable for direct translation to machine use. Further, all are so laborious that application to routine engineering problems is uninviting. The advent of the high speed digital computer has largely swept aside these difficulties and it is now possible to provide microwave engineers with a simple technique for solving (1) [1], [8]–[12]. This paper will be concerned with elaboration of this technique.