Stochastic Calculus and Applications to Finance Homework

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Questions marked with a star are longer or more difficult.
The clarity of argumentation will play a great role in the mark.

1 Multidimensional Black-Scholes model

A time horizon $0 < T < \infty$ is fixed. On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ one defines $W = (W(t))_{0\leq t\leq T}$ a d-dimensional standard Brownian motion (in particular its component are independent and $W(0) = 0$). One notes $(\mathcal{F}_t)_{0\geq t\geq T}$ the natural filtration of W (defined by $\mathcal{F}_t = \mathcal{F}_t^W$). One has $r > 0$, $\mu \in \mathbb{R}^m$, $\sigma \in \mathbb{R}^{m\times d}$.

One has m risky assets, whose prices at time $0 \leq t \leq T$ are denoted $S_i(t)$ for $1 \leq i \leq m$ (note that m is possibly different from d). We will sometimes consider the vector valued process defined by $S(t) = (S_1(t), ..., S_m(t))^\top$.

The model is the following:

$$\forall 1 \leq i \leq m, \qquad dS_i(t) = \mu_i S_i(t)dt + S_i(t)\sum_{j=1}^{d} \sigma_{ij} dW_j(t). \tag{1}$$

The constant r represents the short interest rate. One notes

$$\forall 0 \leq t \leq T, \qquad D(t) = \exp(-rt) \text{ and } S_0(t) = \exp(rt),$$

respectively the Discount factor and the value of the non-risky asset.

1.1 Exploration of some properties of the model

1. **Let $X(t) = (X_1(t), ..., X_m(t))^\top$ a m-dimensional process defined by its starting point $(X_1(0), ..., X_m(0))^\top \in (R_+^*)^m$ and**

$$\forall 1 \leq i \leq m,\ \forall 0 \leq t \leq T, \qquad X_i(t) = X_i(0)\exp\left[\left(\mu_i - \frac{1}{2}\sum_{j=1}^{d}\sigma_{ij}^2\right)t + \sum_{j=1}^{d}\sigma_{ij}W_j(t)\right].$$

   Show that X solves (1) (to be seen as a multidimensional SDE).

   We will use Ito's rule on the function $f(x) = X_i(0)\cdot\exp(x)$. We can see that $f'' = f' = f$. We can introduce a new d-dimensional random process Y given by

$$Y_i(t) = \left(\mu_i - \frac{1}{2}\sum_{j=1}^{d}\sigma_{ij}^2\right)t + \sum_{j=1}^{d}\sigma_{ij}W_j(t).$$

   Now we have

$$dX_i(t) = df(Y_i(t)).$$

We will use some properties of the bracket, such as Levy's theorem, i.e. $\langle B_i, B_j\rangle_t = \mathbb{I}\{i=j\}t$, for a multidimensional Brownian motion B, and $\langle cX\rangle_t = c^2\langle X\rangle_t$, because we can think of bracket as of a quadratic variation. And finally, the bracket of continuous process of FV is zero. So, we have

$$\begin{aligned}
dX_i(t) &= f'(Y_i(t))dY_i(t) + \frac{1}{2}f''(Y_i(t))d\langle Y_i\rangle_t \\
&= X_i(t)\left(\mu_i - \frac{1}{2}\sum_{j=1}^{d}\sigma_{ij}^2\right)dt + X_i(t)\sum_{j=1}^{d}\sigma_{ij}dW_j(t) + \frac{1}{2}X_i(t)\sum_{j=1}^{d}\sigma_{ij}^2 dt \\
&= \mu_i X_i(t)dt + X_i(t)\sum_{j=1}^{d}\sigma_{ij}dW_j(t).
\end{aligned}$$

So, X indeed solves equation (1).

2. **Is the solution to (1) unique? Show that each $S_i(t)$ remains strictly positive for S solution to (1).**

We can rewrite equation (1) as a matrix equation

$$[dS_i(t)]_{i=1}^m = [\mu_i S_i(t)]_{i=1}^m dt + [S_i(t)\sigma_{ij}]_{i,j=1}^{m\times d}[dW_i]_{i=1}^d,$$

or

$$dS(t) = b(S(t))dt + \sigma(S(t))dW(t),$$

where

$$b(x) = \begin{bmatrix}\mu_1 x_1\\ \mu_2 x_2\\ \vdots\\ \mu_m x_m\end{bmatrix} \text{ and } \sigma(x) = \begin{bmatrix} x_1\sigma_{11} & x_1\sigma_{12} & \dots & x_1\sigma_{1d}\\ x_2\sigma_{21} & x_2\sigma_{22} & \dots & x_2\sigma_{2d}\\ \vdots & \vdots & \ddots & \vdots\\ x_m\sigma_{11} & x_m\sigma_{12} & \dots & x_m\sigma_{md}\end{bmatrix}$$

In order to use Theorem 5.2.1. for the uniqueness of the solution, we need to prove next properties of b and σ. Firstly, we need to prove that they are globally Lipschitz, which means that there exist $K>0$, such that for all $x, y \in \mathbb{R}^m$ we have

$$||b(x)-b(y)|| + \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{d}(\sigma_{ij}(x)-\sigma_{ij}(y))^2} \le K\cdot||x-y||.$$

We can see that

$$||b(x)-b(y)||^2 = \sqrt{\sum_{i=1}^{m}\mu_i^2(x_i-y_i)^2} \le \max_{1\le 1\le m}\mu_i\cdot||x-y||,$$

and, similarly,

$$\sqrt{\sum_{i=1}^{m}\sum_{j=1}^{d}(\sigma_{ij}(x)-\sigma_{ij}(y))^2} = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{d}\sigma_{ij}^2(x_i-y_i)^2} \le \max_{\substack{1\le i\le m\\ 1\le j\le d}}\sigma_{ij}\cdot\sqrt{d}\cdot||x-y||^2.$$

So we have that our K is equal to $\mu_{\max} + \sigma_{\max}\cdot\sqrt{d}$. We also need a linear growth condition, or for the same K to have

$$||b(x)||^2 + \sum_{i=1}^{m}\sum_{j=1}^{d}\sigma_{ij}^2(x) \le K^2(1+||x||^2),$$

for all x. Using the same principle as before, we can find such K. At the end it is enough to take the grater of K's in order to have the theorem. Thus, we have a unique solution to equation (1). From the Question 1, we know the unique solution S_i and we can see that all S_i are strictly positive, thanks to exponential function and the strict positiveness of $X(0)$.

3. **For S solution to (1) we set $Y_i(t) = \log(S_i(t))$ for any $1 \leq i \leq m$. Write down $dY_i(t)$ for any $1 \leq i \leq m$.**

We can see that

$$Y_i(t) = \log S_i(t) = \left(\mu_i - \frac{1}{2} \sum_{j=1}^{d} \sigma_{ij}^2 \right) t + \sum_{j=1}^{d} \sigma_{ij} W_j(t) + \log S_i(0).$$

We can see that we have already computed dY_t in the first question, so we got

$$dY_i(t) = \left(\mu_i - \frac{1}{2} \sum_{j=1}^{d} \sigma_{ij}^2 \right) dt + \sum_{j=1}^{d} \sigma_{ij} dW_j(t).$$

4*. **Show then that for any $1 \leq i, j \leq m$ and any $0 \leq t \leq T$,**

$$\operatorname{cov}(Y_i(t), Y_j(t)) = \left[\sigma\sigma^\top\right]_{ij} \cdot t.$$

Interpret the matrix valued parameter σ in the model.

We should use the bilinear property of covariance, as well as the fact that $\operatorname{cov}(c, X) = 0$. So we have,

$$\begin{aligned}
\operatorname{cov}(Y_i(t), Y_j(t)) &= \operatorname{cov}\left(\sum_{k=1}^{d} \sigma_{ik} W_k(t), \sum_{l=1}^{d} \sigma_{jl} W_l(t) \right) \\
&= \sum_{k=1}^{d} \sum_{l=1}^{d} \sigma_{ik}\sigma_{jl} \operatorname{cov}(W_k(t), W_l(t)).
\end{aligned}$$

Having in mind that W is standard Brownian motion, we know that

$$\operatorname{cov}(W_k(t), W_l(t)) = \mathbb{I}\{k = l\} \operatorname{var}(W_k(t)) = \mathbb{I}\{k = l\} \cdot t,$$

i.e. it is t, when $k = l$, and zero otherwise. Thus, we have

$$\operatorname{cov}(Y_i(t), Y_j(t)) = \sum_{k=1}^{d} \sigma_{ik}\sigma_{jk} \cdot t = \left[\sigma\sigma^\top\right]_{ij} \cdot t$$

We can see from the Question 3 that Y is multidimensional Brownian motion with the drift and starting point in $\log S_i(0)$. This means that $Y(t)$ is multidimensional normal distribution with some mean and covariance matrix Σ. We have seen that the covariance matrix of $Y(t)$ is given by

$$t\sigma\sigma^\top.$$

1.2 Exhibition of a risk-neutral probability measure

1. **We consider a portfolio containing at time $0 \leq t \leq T$, $H_i(t)$ shares of S_i for $0 \leq i \leq m$. We note $V(t)$ the value of this portfolio at time $0 \leq t \leq T$ that is to say**

$$V(t) = \sum_{i=0}^{m} H_i(t) S_i(t).$$

We assume that this portfolio is self-financing that is to say

$$dV(t) = \sum_{i=0}^{m} H_i(t) dS_i(t). \tag{2}$$

Show that

$$dV(t) = rV(t)dt + \sum_{i=1}^{m} \frac{H_i(t)}{D(t)} d(D(t)S_i(t)). \tag{3}$$

We will use that $S_0(t) = \exp(rt)$, which means that $dS_0(t) = rS_0(t)dt$. Also, we have

$$V(t) - \sum_{i=1}^{m} H_i(t) S_i(t) = H_0(t) S_0(t).$$

Now we have

$$\begin{aligned} dV(t) &= H_0(t) dS_0(t) + \sum_{i=1}^{m} H_i(t) dS_i(t) \\ &= rH_0(t)S_0(t)dt + \sum_{i=1}^{m} H_i(t) dS_i(t) \\ &= r\left(V(t) - \sum_{i=1}^{m} H_i(t)S_i(t)\right) dt + \sum_{i=1}^{m} H_i(t) dS_i(t) \\ &= rV(t)dt + \sum_{i=1}^{m} H_i(t) \left(dS_i(t) - rS_i(t)dt\right). \end{aligned}$$

We know that $D(t) = \exp(-rt)$, which means that

$$\begin{aligned} d(D(t)S_i(t)) &= dD(t)S_i(t) + dS_i(t)D(t) \\ &= -rD(t)S_i(t)dt + D(t)dS_i(t) \\ &= D(t) \left(dS_i(t) - rS_i(t)dt\right), \end{aligned}$$

or

$$dS_i(t) - rS_i(t)dt = \frac{d(D(t)S_i(t))}{D(t)},$$

which concludes the proof.

2. **In the context of Question 1. write down $d(D(t)V(t))$. Under which condition on the processes $D(t)S_i(t)$ (for $1 \leq i \leq m$) can we say that the process $D(t)V(t)$ is a martingale?**

By Ito's formula, and the fact that $D(t)$ is of FV, which means that bracket $\langle D, V\rangle_t$ is zero, we get

$$\begin{aligned} d(D(t)V(t)) &= -rD(t)V(t)dt + D(t)dV(t) \\ &= -rD(t)V(t)dt + D(t)\left(rV(t)dt + \sum_{i=1}^{m} \frac{H_i(t)}{D(t)} d(D(t)S_i(t))\right) \\ &= \sum_{i=1}^{m} H_i(t)d(D(t)S_i(t)). \end{aligned}$$

In order to $D(t)V(t)$ be a martingale, all the processes $D(t)S_i(t)$ must also be martingales.

3*. **One says that $\lambda \in \mathbb{R}^d$ solves the market price of risk equations if**

$$\forall 1 \le i \le m, \quad \mu_i - r = \sum_{j=1}^{d} \sigma_{ij}\lambda_j.$$

In the sequel one notes $\widetilde{W} = (\widetilde{W}(t))_{0\le t\le T}$ the process defined by

$$\widetilde{W}(t) = W(t) + \lambda t, \quad \forall 0 \le t \le T,$$

with λ solution of the market price of risk equations (we assume such a solution exists and is unique, for all μ_i, $1 \le i \le m$, $r > 0$). Construct a probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ (locally equivalent to $\mathbb{P}$) such that, for any $1 \le i \le m$, the process $D(t)S_i(t)$ is a $(\mathcal{F}_t)$-martingale under $\mathbb{Q}$. What can you say about $\widetilde{W}$ under $\mathbb{Q}$? In the sequel $\mathbb{Q}$ is called the *risk-neutral probability measure.*

We have seen that

$$d(D(t)S_i(t)) = D(t)(dS_i(t) - rS_i(t)dt).$$

If we now use the assumption of the model

$$dS_i(t) = \mu_i S_i(t)dt + S_i(t)\sum_{j=1}^{d} \sigma_{ij}dW_j(t),$$

we will obtain

$$\begin{aligned} d(D(t)S_i(t)) &= D(t)\left(\mu_i S_i(t)dt + S_i(t)\sum_{j=1}^{d} \sigma_{ij}dW_j(t) - rS_i(t)dt\right) \\ &= D(t)S_i(t)\left((\mu_i - r)dt + \sum_{j=1}^{d} \sigma_{ij}dW_j(t)\right). \end{aligned}$$

If we now use an assumption of λ being the solution of the market price, we will have

$$\begin{aligned} d(D(t)S_i(t)) &= D(t)S_i(t)\left(\sum_{j=1}^{d} \sigma_{ij}\lambda_j dt + \sum_{j=1}^{d} \sigma_{ij}dW_j(t)\right) \\ &= D(t)S_i(t)\sum_{j=1}^{d} \sigma_{ij}(\lambda_j dt + dW_j(t)) \\ &= D(t)S_i(t)\sum_{j=1}^{d} \sigma_{ij}d\widetilde{W}_j(t). \end{aligned}$$

So, in oreder to $D(t)S_i(t)$ to be a martingale, it is sufficient to construct $\mathbb{Q}$ such that $\widetilde{W}$ is a Brownian motion.

If we put $X \equiv -\lambda$, we will have that $\mathbb{E}_{\mathbb{P}}\left[\frac{1}{2}\int_0^T ||X_s||^2 ds\right] < +\infty$, we can use the Girsanove theorem on W and X. It says that for $\mathbb{Q}$ locally equivalent to $\mathbb{P}$, by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} |_{\mathcal{F}_t} = Z_t,$$

where

$$Z_t = \exp\left(\sum_{i=1}^{d}\int_0^t X_i(s)dW_i(s) - \frac{1}{2}\int_0^t \|X(s)\|^2 ds\right),$$

we have that $\widetilde{W}$ is d-dimensional $(\mathcal{F}_t)$-standard Brownian motion under $\mathbb{Q}$.

4. **Show that**

$$\forall 1 \le i \le m, \quad dS_i(t) = rS_i(t)dt + S_i(t)\sum_{j=1}^{d}\sigma_{ij}d\widetilde{W}_j(t).$$

What can you say about the dynamic of the $S_i(t)$'s under $\mathbb{Q}$? Comment.

We can start from the right hand side, and have

$$\begin{aligned}
rS_i(t)dt + S_i(t)\sum_{j=1}^{d}\sigma_{ij}d\widetilde{W}_j(t) &= rS_i(t)dt + S_i(t)\sum_{j=1}^{d}\sigma_{ij}(\lambda_j dt + dW_j(t)) \\
&= \left(\mu_i - \sum_{j=1}^{d}\sigma_{ij}\lambda_j\right)S_i(t)dt + S_i(t)\sum_{j=1}^{d}\sigma_{ij}\lambda_j dt \\
&+ S_i(t)\sum_{j=1}^{d}\sigma_{ij}dW_j(t) \\
&= \mu_i S_i(t)dt + S_i(t)\sum_{j=1}^{d}\sigma_{ij}dW_j(t).
\end{aligned}$$

In the previous calculation we used that λ is a solution of the market price of risk equations, and the definition of $\widetilde{W}$. We can see that under $\mathbb{Q}$, we again have Multidimensional Black-Scholes model for risky asset of price S. This time, the trend of the model (which was μ under $\mathbb{P}$) becomes constant vector of short interest rates r.

1.3 Risk-neutral pricing formula

In this part we are under the assumptions of Part 1.2, and follow the same notations.

Let $h \in L^2(\Omega, \mathcal{F}, \mathbb{Q})$. We will show that the price at time $0 \le t \le T$ of a derivative product paying h at time T is given by

$$\mathbb{E}_{\mathbb{Q}}\left[\frac{D(T)}{D(t)}h \mid \mathcal{F}_t\right].$$

Note that h can be *path-dependent*, that is to say h is a function possibly not only of the values of the risky assets at maturity (for example we could have $h = S_1(T) + \int_0^T S_2(s)ds$, ...). In these conditions it is not clear that we could solve the problem using a PDE, as we have done in the course (the final condition on the PDE has played a crucial role in Chapter 5). This can motivate that this time we choose a different way to solve the problem.

Besides, of course, like in the one-dimensional context, we define the price of the derivative at time t as the value $V(t)$ of some self-financing replicating portfolio. Our task is now to construct this portfolio.

One notes

$$M(t) = \mathbb{E}_{\mathbb{Q}}[D(T)h \mid \mathcal{F}_t], \quad \forall 0 \leq t \leq T.$$

1. **What can you say about the process $M(t)$?**

   We will use the Theorem 1 from the Appendix on the process $M(t)$, so we need to prove that $M(t)$ is a square-integrable $(\mathcal{F}_t)$-martingale. From the assumption that $h \in L^2(\Omega, \mathcal{F}, \mathbb{Q})$ we can check that $M(t)$ is also in $L^2(\Omega, \mathcal{F}, \mathbb{Q})$. We have

   $$\mathbb{E}_{\mathbb{Q}}\left[M(t)^2\right] = \mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}[D(T)h \mid \mathcal{F}_t]^2\right] = \mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}[e^{-rT}h \mid \mathcal{F}_t]^2\right].$$

   We want to get an upper bound by $\mathbb{E}_{\mathbb{Q}}[h^2]$, because we know this is the finite. In order to have h^2, we can use that $(\mathbb{E}[X])^2 \leq \mathbb{E}\left[X^2\right]$, which is a special case of Jensen's inequality. So we have

   $$\begin{aligned} \mathbb{E}_{\mathbb{Q}}\left[M(t)^2\right] &\leq \mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}[e^{-2rT}h^2 \mid \mathcal{F}_t]\right] \\ &= e^{-2rT}\mathbb{E}_{\mathbb{Q}}\left[h^2\right] < +\infty. \end{aligned}$$

   We used the fact that $\mathcal{F}_t$ is sub-σ-algebra of $\mathcal{F}$, so we can apply the

   $$\mathbb{E}_{\mathbb{Q}}[\mathbb{E}_{\mathbb{Q}}[h \mid \mathcal{F}_t]] = \mathbb{E}_{\mathbb{Q}}[h].$$

   We can also see that the process $(M(t))_{0 \leq t \leq T}$ is obviously $(\mathcal{F}_t)$-adapted, since for any $0 \leq t \leq T$ the random variable

   $$M(t) = \mathbb{E}_{\mathbb{Q}}[D(T)h \mid \mathcal{F}_t]$$

   is $\mathcal{F}_t$-measurable by definition of the conditional expectation with respect to $\mathcal{F}_t$. Lastly, we can prove that $M(t)$ is $(\mathcal{F}_t)$-martingale. We will use the same property for the conditional expectation as we have already used in proving square-integrability. So, we have

   $$\mathbb{E}_{\mathbb{Q}}[M(t) \mid \mathcal{F}_s] = \mathbb{E}_{\mathbb{Q}}[\mathbb{E}_{\mathbb{Q}}[D(T)h \mid \mathcal{F}_t] \mid \mathcal{F}_s] = \mathbb{E}_{\mathbb{Q}}[D(T)h \mid \mathcal{F}_s] = M(s).$$

2. **Justify that $(\mathcal{F}_t)$ is also the natural filtration of $\widetilde{W}$ (i.e. $\mathcal{F}_t = \mathcal{F}_t^W = \mathcal{F}_t^{\widetilde{W}}$).**

   We know that $\mathcal{F}_t = \mathcal{F}_t^W$, and we want to show that $\mathcal{F}_t = \mathcal{F}_t^{\widetilde{W}}$. We are going to do it by proving both inclusions.

   ($\supseteq$) We used in the Question 3 in the Part 1.2 Girsanov theorem, which gave us that $\widetilde{W}$ is $(\mathcal{F}_t)$-adapted. On the other hand, $(\mathcal{F}_t^{\widetilde{W}})$ is a natural filtration of $\widetilde{W}$, which is the smallest sub-σ-algebra of $\mathcal{F}$ containing all the X_s, for $s \leq t$. So we have for any $0 \leq t \leq T$ that $\mathcal{F}_t^{\widetilde{W}} \subseteq \mathcal{F}_t$.

   ($\subseteq$) We know that for any $0 \leq t \leq T$ we have $W_t = \widetilde{W}_t - \lambda t$, which means that knowing the paths of $\widetilde{W}$, we can be able to recover the paths of W. This implies that $\mathcal{F}_t^W \subseteq \mathcal{F}_t^{\widetilde{W}}$.

3*. **Show that there exists adapted processes $H_i(t)$, $1 \le i \le m$, such that**

$$dM(t) = \sum_{i=1}^{m} H_i(t)D(t)S_i(t) \sum_{j=1}^{d} \sigma_{ij} d\widetilde{W}_j(t). \tag{4}$$

Hint: Of course you may use the results provided in the Appendix...

Obviously, we will use the Theorem 1 from the Appendix. So, we need to recall what we have. We know that $\widetilde{W}$ is d-dimensional Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{Q})$ with its natural filtration $(\mathcal{F}_t)$. We have seen that M is square-integrable $(\mathcal{F}_t)$-martingale under $\mathbb{Q}$. Now, according to Theorem 1, we know that there exist unique adapted processes $\Gamma_j(t)$, for $1 \le j \le d$, such that $\Gamma_j \in \Pi_2(W_j)$, for all $1 \le j \le d$, and for all t it is

$$M(t) = M(0) + \sum_{j=1}^{d} \int_0^t \Gamma_j(s) d\widetilde{W}_j(s).$$

Now we can rewrite this equation as

$$dM(t) = \sum_{j=1}^{d} \Gamma_j(t) d\widetilde{W}_j(t).$$

We want to prove that there exist processes H_i such that

$$\Gamma_j(t) = \sum_{i=1}^{m} H_i(t)D(t)S_i(t)\sigma_{ij}. \tag{5}$$

We can rewrite that as

$$\sum_{i=1}^{m} \sigma_{ij} Y_i(t) = \frac{\Gamma_j(t)}{D(t)},$$

where $Y_i(t) = H_i(t)S_i(t)$. Now, we have a matrix equation

$$\sigma^\top \cdot Y(t) = \frac{\Gamma(t)}{D(t)}. \tag{6}$$

So, we would like to apply the Lemma 1 from the Appendix on the matrix $A = \sigma$, and vector $c = \frac{\Gamma(t)}{D(t)}$, so we can find one solution $y_0 = Y(t)$. It is said in the Question 3 of Part 1.2 that the market price of risk equations have a unique solution λ, for any value of μ_i, $1 \le i \le m$, $r > 0$. In other words, the equation

$$\sigma\lambda = b$$

has a unique solution $\lambda \in \mathbb{R}^d$ for any $b \in \mathbb{R}^m$. Using now Lemma 1 of the homework, the equation

$$\sigma^\top y = c$$

has a solution $y \in \mathbb{R}^m$ for any $c \in \mathbb{R}^d$. Finally, if $Y(t)$ is solution of the equation (6), then we can define $H_i(t)$ as

$$H_i(t) = \frac{Y_i(t)}{S_i(t)},$$

and we will obtain (5).

4. **Define now $H_0(t)$ (with the help of h, $D(t)$ the $H_i(t)$'s for $1 \leq i \leq m$ and some conditional expectation) in order to have**

$$\forall 0 \leq t \leq T, \quad V(t) = \sum_{i=0}^{m} H_i(t) S_i(t) = \mathbb{E}_{\mathbb{Q}} \left[\frac{D(T)}{D(t)} h \mid \mathcal{F}_t \right].$$

We know that $D(t) = \exp(-rt)$ is $\mathcal{F}_t$ measurable, because $D(t)$ is deterministic, so if we know $\mathcal{F}_t$, we know t, and we also know $D(t)$, so we can put it outside the expectation to obtain

$$V(t) = \sum_{i=0}^{m} H_i(t) S_i(t) = \frac{1}{D(t)} \mathbb{E}_{\mathbb{Q}} \left[D(T) h \mid \mathcal{F}_t \right] = \frac{M(t)}{D(t)}.$$

Now we can see that

$$H_0(t) = \frac{\frac{M(t)}{D(t)} - \sum\limits_{i=1}^{m} H_i(t) S_i(t)}{S_0(t)}$$

5*. **Verify that the strategy defined by $H_i(t)$, $0 \leq i \leq m$, is self-financing and replicates h at maturity T (in order to be convincing do not hesitate do again some previous computations in "the reverse sense"). Conclude.**

We have defined strategy $(H, H_0)^\top$. To show that the strategy is replicating we have

$$V_T = D(T) M(T) = \mathbb{E}_{\mathbb{Q}} \left[\frac{D(T)}{D(T)} h \mid \mathcal{F}_T \right] = h,$$

because h is $\mathcal{F}_T$-measurable. Now, let us show that the strategy is self-financing. We have

$$\begin{aligned}
dV(t) &= d(\exp(rt) M(t)) = r \exp(rt) M(t) dt + \exp(rt) dM(t) \\
&= rV(t) + \exp(rt) \sum_{i=1}^{m} H_i(t) D(t) S_i(t) \sum_{j=1}^{d} \sigma_{ij} d\widetilde{W}_j(t) \\
&= rV(t) + \sum_{i=1}^{m} H_i(t) S_i(t) \sum_{j=1}^{d} \sigma_{ij} (dW_j(t) + \lambda_j dt) \\
&= rV(t) + \sum_{i=1}^{m} H_i(t) S_i(t) \left(\sum_{j=1}^{d} \sigma_{ij} dW_j(t) + \sum_{j=1}^{d} \sigma_{ij} \lambda_j dt \right) \\
&= rV(t) + \sum_{i=1}^{m} H_i(t) S_i(t) \left(\sum_{j=1}^{d} \sigma_{ij} dW_j(t) + (\mu_i - r) dt \right) \\
&= rV(t) + \sum_{i=1}^{m} H_i(t) \left(dS_i(t) - rS_i(t) dt \right) \\
&= rV(t) + \sum_{i=1}^{m} \frac{H_i(t)}{D(t)} d(D(t) S_i(t)).
\end{aligned}$$

Using the Question 1 in the Part 1.2 we can conclude that the strategy is self-financing. So, we showed that there exists a self-financing strategy that replicates h. Its value at time $0 \leq t \leq T$ is

$$\frac{M(t)}{D(t)} = \mathbb{E}_{\mathbb{Q}} \left[\frac{D(T)}{D(t)} h \mid \mathcal{F}_t \right].$$

1.4 Appendix (helpful admitted results)

Theorem 1 (**Brownian martingale representation theorem**). *Let $B(t)$ a d-dimensional Brownian motion defined on some probability space $(E, \mathcal{E}, \mathbb{P})$ and $(\mathcal{G}_t)$ its natural filtration. If $(N(t))$ is a $(\mathcal{G}_t)$-martingale under $\mathbb{P}$, with $\mathbb{E}_{\mathbb{P}}|N(t)|^2 < \infty$ for any t, there exist unique adapted processes $\Gamma_j(t)$, $1 \le j \le d$, satisfying $\Gamma_j \in \Pi_2(B_j)$ for all $1 \le j \le d$, s.t.,*

$$\forall t, \quad N(t) = N(0) + \sum_{j=1}^{r} \int_0^t \Gamma_j(s) dB_j(s).$$

Lemma 1. *Let $A \in \mathbb{R}^{m \times d}$ and $b \in \mathbb{R}^m$, $c \in \mathbb{R}^d$. If the equation*

$$Ax = b$$

has a unique solution $x_0 \in \mathbb{R}^d$, then the equation

$$A^\top y = c$$

has at least one solution $y_0 \in \mathbb{R}^m$ (If $m = d$ such a solution is unique).

2 Feynman-Kac formula

In this problem a time horizon $0 < T < \infty$ is fixed. We are given some bounded coefficients $b : [0,T] \times \mathbb{R} \to \mathbb{R}$ and $\sigma : [0,T] \times \mathbb{R} \to \mathbb{R}$ satisfying the conditions

$$\begin{aligned} |b(t,y) - b(t,x)| + |\sigma(t,y) - \sigma(t,x)| &\le K|y - x|, \quad \forall (t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}, \\ 0 < m &\le \sigma^2(t,x), \quad \forall (t,x) \in [0,T] \times \mathbb{R} \end{aligned}$$

for some constants $0 < K, m < \infty$. We are also given continuous bounded functions $f : \mathbb{R} \to \mathbb{R}$, $k : [0,T] \times \mathbb{R} \to \mathbb{R}_+$, $g : [0,T] \times \mathbb{R} \to \mathbb{R}$, with

$$\lim_{|x| \to \infty} |f(x)| = 0,$$

and $f \in L^2(R)$.

A one-dimensional standard $(\mathcal{F}_t)$-Brownian motion W is defined on some probability space $(\Omega, \mathcal{F}, \mathbb{B})$. We consider the Stochastic Differential Equation (SDE)

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t. \tag{7}$$

Note that we do not specify any starting point for this SDE because this starting point will change. More precisely for any $(t,x) \in [0,T] \times \mathbb{R}$ we note $\mathbb{P}^{t,x} = \mathbb{P}(\cdot \mid X_t = x)$. Consider then the Partial Differential Equation (PDE)

$$(\mathcal{P}) \left\{ \begin{array}{rcll} -\partial_t v(t,x) + k(t,x)v(t,x) &=& \mathcal{A}_t v(t,x) + g(t,x), & \forall (t,v) \in [0,T) \times \mathbb{R} \\ v(T,x) &=& f(x), & \forall x \in \mathbb{R} \\ \lim_{|x| \to \infty} |v(t,x)| &=& 0, & \forall t \in [0,T] \end{array} \right.$$

with for any $0 \le t < T$ the operator $\mathcal{A}_t$ acting of functions in φ in $C^2(\mathbb{R})$ by

$$(\mathcal{A}_t \varphi)(x) = \frac{1}{2} a(t,x) \varphi''(x) + b(t,x) \varphi'(x), \quad \forall x \in \mathbb{R},$$

where $a(t,x) = \sigma^2(t,x)$. The classic theory of parabolic PDEs says that a solution $v(t,x)$ of class $C([0,T]\times\mathbb{R};\mathbb{R})\cap C^{1,2}([0,T)\times\mathbb{R};\mathbb{R})$ exists to $(\mathcal{P})$. Moreover it is bounded, and its first space derivative satisfies

$$\sup_{(u,x)\in[0,s]\times\mathbb{R}} |\partial_x v(u,x)| \le M_s,$$

for any $s < T$.

We wish to show the stochastic representation

$$v(t,x) = \mathbb{E}^{t,x}\left[f(X_T)\exp\left(-\int_t^T k(\theta,X_\theta)d\theta\right) + \int_t^T g(u,X_u)\exp\left(-\int_t^u k(\theta,X_\theta)d\theta\right)du\right] \quad (8)$$

1. **First justify that for any $(t,x)\in[0,T]\times\mathbb{R}$ a solution X to (5) starting from x at time t exists.**

   From the first assumption we can see that coefficients b and σ are globally Lipschitz functions. To apply the Theorem 5.2.1 on one dimensional case we need one more condition, i.e.

   $$|b(t,x)|^2 + \sigma^2(t,x) \le K^2(1+|x|^2).$$

   But we assumed that b and σ are bounded, which is stronger assumption then the previous condition, thus we can apply given Theorem, which says that we have unique solution to (5), given that $X_0 = x$.

From questions 2. to 4. a time $t\in[0,T)$ is fixed.

2. **We set $Z_s = \exp\left(-\int_t^s k(\theta,X_\theta)d\theta\right)$ for any $t\le s<T$. Is the process Z of finite variation? Justify that**

   $$dZ_s = -k(s,X_s)Z_s ds.$$

   To prove that Z is of FV we can notice that k is positive, which means that $Z_\cdot(\omega)$ is decreasing, and thus Z is of FV. We know that X has almost surly continuous paths, so we can choose $\omega\in\Omega$ such that $X_\cdot(\omega)$ is continuous. We can observe a mapping $\Phi : [t,T) \to R_+^*$ given by

   $$\Phi(s) = Z_s(\omega) = \exp\left(-\int_t^s k(\theta,X_\theta(\omega))d\omega\right).$$

   Then, Φ is continuous due to continuity of exponential function, integral up to time s, function k, and $X_\cdot(\omega)$. Mapping Φ is also differentiable, so we can differentiate Φ and obtain

   $$\Phi'(s) = -k(s,X_s(\omega))\exp\left(-\int_t^s k(\theta,X_t\theta(\omega))d\theta\right) = -k(s,X_s(\omega))Z_s(\omega).$$

   We proved that

   $$dZ_s = -k(s,X_s)Z_s ds.$$

3*. **Let $t\le s<T$. Show that**

$$v(s,X_s)Z_s = v(t,X_t) - \int_t^s Z_u g(u,X_u)du + \int_t^s Z_u\partial_x v(u,X_u)\sigma(u,X_u)dW_u.$$

Advice: Here computations may seem a bit long. Take the time to treat this question, it is the heart of the proof.

We can see from the question that we should use Ito's formula on semimartingale (s, X_s, Z_s) by function $(s, x, z) \mapsto v(s, x)z$ for $s \in [t, T)$. We will use the previous question and the fact that Z is of FV, so the bracket of Z with any other process will always be zero. The same applies for (s). We will also need $\langle X \rangle_u$. In order to compute it we will use the fact that $\langle B \rangle_t = t$, for some Brownian motion B, and the fact that bracket will square constant multiplier. Thus

$$d\langle X \rangle_t = \sigma^2(t, X_t) d\langle W \rangle_t = \sigma^2(t, X_t)dt.$$

Also, we see that $Z_t = 1$, and we have

$$\begin{aligned}
v(s, X_s)Z_s &= v(t, X_t)Z_t + \int_t^s \partial d_t v(u, X_u)Z_u du + \int_t^s \partial_x v(u, X_u)Z_u dX_u \\
&+ \int_t^s v(u, X_u)dZ_u + \frac{1}{2}\int_t^s \partial^2_{xx} v(u, X_u)Z_u d\langle X \rangle_u \\
&= v(t, X_t) + \int_t^s Z_u \partial_t v(u, X_u)du + \int_t^s Z_u b(u, X_u)\partial_x v(u, X_u)du \\
&+ \int_t^s Z_u \partial_x v(u, X_u)\sigma(u, X_u)dW_u - \int_t^s Z_u v(u, X_u)k(u, X_u)du \\
&+ \frac{1}{2}\int_t^s \partial^2_{xx} v(u, X_u)Z_u \sigma^2(u, X_u)du.
\end{aligned}$$

We are now going to group integrals with du, because everything else is where it should be. We get

$$\int_t^s Z_u \left(\partial_t v(u, X_u) + \underbrace{b(u, X_u)\partial_x v(u, X_u) + \frac{1}{2}\partial^2_{xx} v(u, X_u)\sigma^2(u, X_u)}_{\mathcal{A}_u v(u, X_u)} - v(u, X_u)k(u, X_u) \right) du.$$

Finally, using first equation in $(\mathcal{P})$ we have

$$v(s, X_s)Z_s = v(t, X_t) - \int_t^s Z_u g(u, X_u)du + \int_t^s Z_u \partial_x v(u, X_u)\sigma(u, X_u)dW_u.$$

4. **Show then that for any** $t \leq s < T$

$$v(t, x) = \mathbb{E}^{t,x}\left[v(s, X_s)Z_s + \int_t^s Z_u g(u, X_u)du \right].$$

We will start from what we got in the previous question. We would like to take the expectation on that equation. Firstly, we can notice that process

$$\left(\int_t^s Z_u \partial_x v(u, X_u)\sigma(u, X_u)dW_u \right)_s$$

is a martingale with value zero at time t. This means that the expectation will be zero, i.e.

$$\mathbb{E}^{x,t}\left[\int_t^s Z_u \partial_x v(u, X_u)\sigma(u, X_u)dW_u \right] = 0.$$

We can also notice that

$$\mathbb{E}^{x,t}[v(t, X_t)] = \mathbb{E}[v(t, X_t) \mid X_t = x] = v(t, x).$$

Finally, we can take the expectation of the result of the Question 3 and we will get the result.

5. **Conclude that we have (6), for any** $(t, x) \in [0, T] \times \mathbb{R}$**.**

   Hint: Remember of the convergence theorems you have encountered in your life in Probability or Integration theory.

   We would like to change the place of limit and expectation, which suggests that we should use the dominated convergence theorem. After using it we have

$$\begin{aligned} v(t,x) = \lim_{s \to T} v(t,x) = \lim_{s \to T} \mathbb{E}^{t,x}\left[v(s, X_s)Z_s + \int_t^s Z_u g(u, X_u)du\right] \\ = \mathbb{E}^{t,x}\left[v(T, X_T)Z_T + \int_t^T Z_u g(u, X_u)du\right]. \end{aligned}$$

   We used the fact that v is continuous. If we put $f(X_T) = v(T, X_T)$ we will get the result (6). The only thing left to be done is the justification that we can use D.C.T. Since v and g are bounded functions, and Z is bounded by 1, we can find constant C such that

$$v(s, X_s)Z_s + \int_t^s Z_u g(u, X_u)du \leq C,$$

   and thus we can apply D.C.T.