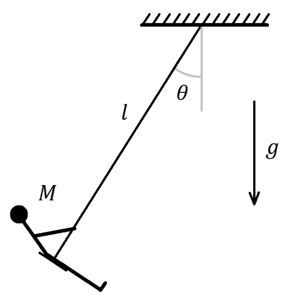
Problem 3 — Swinging on a Swing

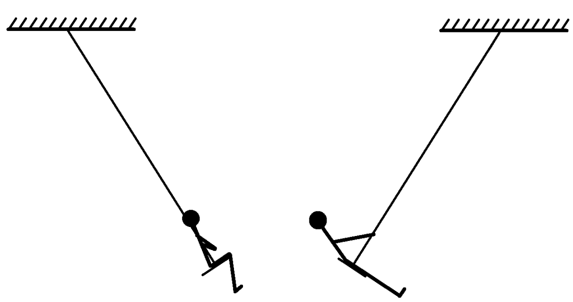
Swinging is a part of everyone's childhood memories. While as kids we learn how to pump a swing based on experience and through trial and error, the physics involved in this mechanism is also intriguing to study. In this problem, we will study two simplified mechanism for pumping a swing, acknowledging that much more complex and accurate models can be obtained with slight modifications.

To start, we will model the swing as a pendulum of length l and let the rider have a mass M and height h. Also let the amplitude of the pendulum, marked by the maximum deviation angle from the vertical, be indicated by θ. In the following two sub-problems, we will work towards finding the amplitude θ_n after n number of half-oscillations (the time-frame between two consequent full stops of the swing).

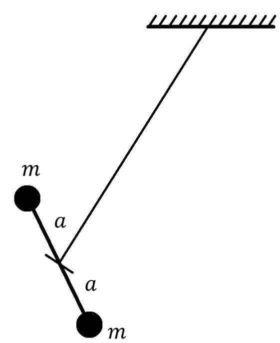


3.1 Pumping from the Sitting Position

This, perhaps more familiar method of pumping, involves the rider switching between the "sat" and the "leaned back" position, animated by the right and the left figure respectively.



This switch, which happens when the swing has come to a full stop at either extreme of its motion, entails a rotation of the rider's body. This rotation in turn causes a small rotation of the swing due to conservation of angular momentum, hence increasing its amplitude. The following is a simplified representation of the rider's body with a barbell of length $2a$ with two masses m attached on either side and attached to the end of the swing at its centre.



1. With reasonable estimates, write a and m in terms of the rider's mass M and height h. Also, what is a reasonable angle of rotation for the barbell to model the rider switching between the "sat" and "leaned back" positions?

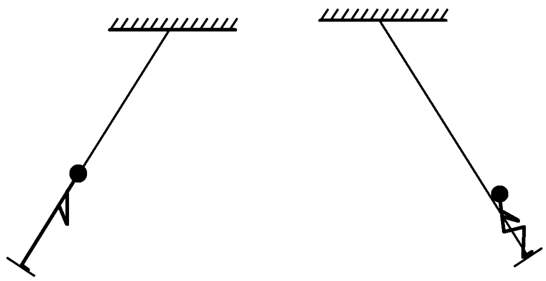
2. At the left-most extreme of swing motion, to increase the amplitude, does the rider switch from the "sat" position to the "leaned back" or vice versa? What about the right-most extreme? Therefore, when should the barbell rotate clockwise and when counter-clockwise?

3. Let us focus on only one switch for now. Write the conservation of angular momentum to obtain the increase in amplitude $\Delta\theta$ in terms of h and l, after the rider performs a switch (Hint: If you have not before, it will help to study the parallel axis theorem $I = I_{cm} + Md^2$).

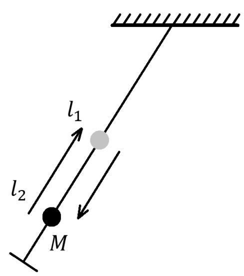
4. Assuming the rider starts pumping the swing from an initial amplitude of θ_0, find the amplitude θ_n after n half-oscillations.

3.2 Pumping from the Standing Position

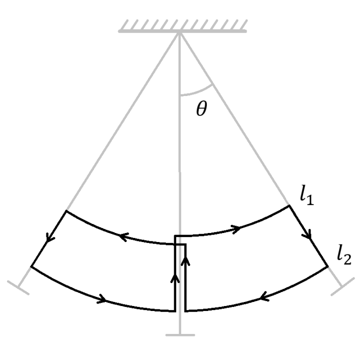
This mechanism of pumping involves the rider switching between the "stood" and "squatted" positions, animated by the left and right figure below respectively.



Switching between these two positions effectively lowers and heightens the rider's centre of mass. We can exploit this ability to pump the swing. First, let us construct a simple representation of the rider with only a mass M that is free to move between two lengths l_1 and l_2 along the length of the swing.



If the rider's centre of mass moves such that it follows the following path, they will be able to pump the swing.



1. With reasonable estimates, write l_1 and l_2 in terms of h and l.

2. Explain why this mechanism will increase the amplitude (Hint: Think about conservation of energy and work done by the rider).

3. Now, assume the swing has an amplitude of θ_n when the rider completes another half-oscillation. Find the new amplitude θ_{n+1} in terms of θ_n, h, and l (Note: You may assume that h/l is sufficiently small to ignore second order and above terms).

4. Assuming the rider starts pumping the swing from an initial amplitude of θ_0, find the amplitude θ_n after n half-oscillations.

5. Compare your answers to part (4) of the two sub-problems. Comment on differences and similarities. Which is a more suitable method of pumping for low amplitudes? What about high amplitudes?