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- Formulação forte:

Dadas as constantes reais $\alpha > 0$, $\beta \geq 0$ e $T > 0$, e as funções $f: [0,1] \times [0,T] \to \mathbb{R}$, $u_0 : [0,1] \to \mathbb{R}$ e $g: \mathbb{R} \to \mathbb{R}$ não linear, determine $u: [0,1] \times [0,T] \to \mathbb{R}$ tal que:

$$\begin{cases} u_t(x,t) - \alpha u_{xx}(x,t) + \beta u(x,t) + g(u(x,t)) = f(x,t) & x \in]0,1[\\ u(0,t) = u(1,t) = 0, \quad t \in [0,T] & t \in [0,T] \\ u(x,0) = u_0(x), \quad x \in [0,1] \end{cases}$$

- Formulação fraca:

Dadas as constantes reais $\alpha > 0$, $\beta \geq 0$ e $T > 0$, e as funções $f: [0,1] \times [0,T] \to \mathbb{R}$, $u_0 : [0,1] \to \mathbb{R}$ e $g: \mathbb{R} \to \mathbb{R}$ não linear, determine $u(t) \in V$, $t \in [0,T]$, tal que:

$$\begin{cases} (u'(t), v) + \kappa(u(t), v) + (g(u(t)), v) = (f(t), v) \quad \forall v \in V \\ u(0) = u_0 \end{cases}$$

Onde (w, v) e $\kappa(w, v)$ são os operadores:

$$(w, v) = \int_0^1 w(x)\, v(x)\, dx$$

$$\kappa(w, v) = \alpha \int_0^1 \frac{dw}{dx}(x) \frac{dv}{dx}(x) + \beta \int_0^1 w(x)\, v(x)\, dx$$

- Problema aproximado via Crank Nicolson Galerkin linearizado:

  - Solução aproximada em t_0: escolhemos $U^0 \in V_m$ como uma aproximação para u_0

  - Solução aproximada em t_1:

    - Parte 1: definimos $\tilde{U}^1 \in V_m$ como a relação de:

$$\left(\frac{\tilde{U}^1 - U^0}{\tau}, v\right) + \kappa\left(\frac{\tilde{U}^1 - U^0}{2}, v\right) + \left(g(U^0), v\right) = f(t_{1/2}, v) \quad \forall v \in V_m$$

    - Parte 2: definimos $U^1 \in V_m$ como a solução de:

$$\left(\frac{U^1 - U^0}{\tau} - v\right) + \kappa\left(\frac{U^1 + U^0}{2}, v\right) + \left(g\left(\frac{\tilde{U}^1 + U^0}{2}, v\right), v\right) = \left(f(t_{1/2}), v\right) \quad \forall v \in V_m$$

  - Solução aproximada em t_n, $n \geq 2$:

$$\left(\frac{U^n - U^{n-1}}{\tau}, v\right) + \kappa\left(\frac{U^n + U^{n-1}}{2}, v\right) + \left(g\left(\frac{3U^{n-1} - U^{n-2}}{2}\right), v\right) =$$

$$= \left(f(t_{n-1/2}), v\right) \quad \forall v \in V_m$$

- Formulação matricial:

• Para t_0: consideramos $U^0 = \sum_{j=1}^{m} c_j^0 \varphi_j$ como a interpolante de u_0.

• Para t_1:

- Parte 1: determinar $\tilde{U}^1 = \sum_{j=1}^{m} \tilde{c}_j^1 \varphi_j$ tal que

$$M \frac{\tilde{C}^1 - C^0}{t} + K \frac{\tilde{C}^1 + C^0}{2} + G(C^0) = F^{1/2}$$

ou

$$\left(M + \frac{t}{2} K\right) \tilde{C}^1 = \left(M - \frac{t}{2} K\right) C^0 - t\, G(C^0) + t F^{1/2}$$

- Parte 2: determinar $U^1 = \sum_{j=1}^{m} c_j^1 \varphi_j$ tal que

$$M \frac{C^1 - C^0}{t} + K \frac{C^1 + C^0}{2} + G\left(\frac{\tilde{C}^1 + C^0}{2}\right) = F^{1/2}$$

ou

$$\left(M + \frac{t}{2} K\right) C^1 = \left(M - \frac{t}{2} K\right) C^0 - t\, G\left(\frac{\tilde{C}^1 + C^0}{2}\right) + t F^{1/2}$$

• Para t_n, $n \geq 2$: determine $U^n = \sum_{j=1}^{m} c_j^n \varphi_j$ tal que

$$M \frac{C^n - C^{n-1}}{t} + K \frac{C^n + C^{n-1}}{2} + G\left(\frac{3C^{n-1} - C^{n-2}}{2}\right) = F^{n-1/2}$$

ou

$$\left(M + \frac{t}{2} K\right) C^n = \left(M - \frac{t}{2} K\right) C^{n-1} - t\, G\left(3\frac{C^{n-1} - C^{n-2}}{2}\right) - t F^{n-1/2}$$

Onde $G_i(c) = \int_0^1 g\left(\sum_{j=1}^{m} c_j \varphi_j\right) \varphi_i \, dx$, de maneira que

$$c = \begin{cases} C^0 & \text{quando } n=1 \text{ na parte } 1 \\[2ex] \dfrac{\bar{C}^1 + C^0}{2} & \text{quando } n=1 \text{ na parte } 2 \\[2ex] 3\dfrac{C^{n-1} - C^{n-2}}{2} & \text{quando } n \geq 2 \end{cases}$$

Para construir o vetor G, utilizamos um vetor G_a^e local:

$$G_a^e = \int_{x_1^e}^{x_2^e} g\left(c\varphi_1^e(x) + c\varphi_2^e(x)\right) \varphi_a^e(x) \, dx$$

$$= \frac{h}{2} \int_{-1}^{1} g\left(c\phi_1(\xi) + c\phi_2(\xi)\right) \phi_a(\xi) \, d\xi$$

$$\approx \frac{h}{2} \sum_{j=1}^{n} W_j \, g\left(c\phi_1(P_j) + c\phi_2(P_j)\right) \phi_a(P_j)$$

• Cálculo de U^0:

– Opção 1: como dito anteriormente, podemos tomar U^0 como a interpolante de u^0, fazendo

$$U^0(x_i) = \sum_{j=1}^{m} c_j^0 \varphi_j(x_i) = c_i^0$$

O que nos dá que $C^0 = [u_0(x_1), u_0(x_2), \dots, u_0(x_m)]^T$

– Opção 2: podemos tomar U^0 como a projeção L^2 de u_0. Sendo

• $U^0(x) = \sum_{j=1}^{m} c_j^0 \varphi_j(x)$

• $v_h = \varphi_i$ para $i = 1, 2, \dots, m$

Considere que

$$(U^0 - u_0, v_h) = 0 \qquad \forall v_h \in V_m$$

$$\downarrow$$

$$\left(\sum_{j=1}^{m} c_j^0 \varphi_j(x) - u_0, \varphi_i\right) = 0 \quad \text{para } i = 1, 2, \dots, m$$

$$\downarrow$$

$$\underbrace{M}_{\substack{K \\ \text{com} \\ \alpha = 0 \\ \beta = 1}} C^0 = \underbrace{[(u_0, \varphi_1), (u_0, \varphi_2), \dots, (u_0, \varphi_m)]^T}_{F \text{ com } f = u_0}$$

– Opção 3: tomando U^0 como a projeção H_0^1 de u_0.

Considere que

$$\left(\frac{d}{dx}(U^0 - u_0), \frac{d}{dx} v_h\right) = 0 \quad \forall v_h \in V_m$$

Tomando $U^0(x) = \sum_{j=1}^{m} c_j^0 \varphi_j(x)$ e $v_h = \varphi_i$ para $i = 1, 2, \dots, m$:

$$\left(\frac{d}{dx}\left(\sum_{j=1}^{m} c_j^0 \varphi_j(x), u_0\right), \frac{d}{dx}\varphi_i\right) = 0$$

$$\downarrow$$

$$\underbrace{\begin{bmatrix} (\partial_x \varphi_1, \partial_x \varphi_2) & \cdots & (\partial_x \varphi_m, \partial_x \varphi_1) \\ \vdots & \ddots & \vdots \\ (\partial_x \varphi_1, \partial_x \varphi_m) & \cdots & (\partial_x \varphi_m, \partial_x \varphi_m) \end{bmatrix}}_{\tilde{M}} C^0 = \underbrace{\begin{bmatrix} (\partial_x u_0, \partial_x \varphi_1) \\ \vdots \\ (\partial_x u_0, \partial_x \varphi_m) \end{bmatrix}}_{\tilde{F}}$$

Podemos montar $\tilde{M}$ e $\tilde{F}$ a partir dos vetores locais:

$$\tilde{M}_{a,b}^e = \alpha \int_{x_1^e}^{x_2^e} \partial_{xx} \varphi_b^e \, \partial_{xx} \varphi_a^e \, dx + \beta \int_{x_1^e}^{x_2^e} \partial_x \varphi_b^e \, \partial_x \varphi_a^e \, dx =$$

$$= \frac{2\alpha}{h} \int_{-1}^{1} \partial_{\xi\xi} \phi_b \, \partial_{\xi\xi} \phi_a \, d\xi + \frac{\beta h}{2} \int_{-1}^{1} \partial_{\xi\xi} \phi_a \, \partial_{\xi\xi} \phi_b \, dx$$

$$\approx \frac{2\alpha}{h} \sum_{j=1}^{n} W_j \, \partial_{\xi\xi} \phi_b(P_j) \, \partial_{\xi\xi} \phi_a(P_j) + \frac{\beta h}{2} \sum_{j=1}^{n} W_j \, \partial_\xi \phi_b(P_j) \, \partial_\xi \phi_a(P_j)$$

com $\alpha = 0$ e $\beta = 1$

$$\tilde{F}^e_{a,b} = \int_{x_1^e}^{x_2^e} \partial_x u_0 \, \partial_x \varphi_a^e \, dx =$$

$$= \frac{h}{2} \int_{-1}^{1} \partial_x (u_0(x(\xi, e)) \, \partial_\xi \phi_a(\xi) \, d\xi$$

$$\approx \frac{h}{2} \sum_{j=1}^{n} W_j \, \partial_x u_0(x(P_j, e)) \, \partial_\xi \phi_a(P_j)$$

– Opção 4: usando o operador κ para realizar a projeção de u_0. Considere que

$$\kappa(U^0 - u_0, v_h) = 0 \quad \forall v_h \in V_m$$

Tomando $U^0(x) = \sum_{j=1}^{m} C_j^0 \varphi_j(x)$ e $v_h = \varphi_i$ para $i = 1, 2, \ldots, m$:

$$\kappa\left(\sum_{j=1}^{m} C_j^0 \varphi_j - u_0, v_h\right) = 0$$

$$\downarrow$$

$$K C^0 = \underbrace{[\kappa(u_0, \varphi_1), \ldots, \kappa(u_0, \varphi_m)]^T}_{\tilde{F}}$$

Para calcular $\tilde{F}^e$:

$$\tilde{F}^e_a = \alpha \int_{x_1^e}^{x_2^e} \frac{d}{dx} u_0 \frac{d}{dx} \varphi_a^e \, dx + \beta \int_{x_1^e}^{x_2^e} u_0 \, \varphi_a^e \, dx =$$

$$= \frac{2\alpha}{h} \int_{-1}^{1} \frac{d}{dx} u_0(x(\xi, e)) \frac{d}{d\xi} \phi_a \, d\xi + \frac{\beta h}{2} \int_{-1}^{1} u_0(x(\xi, e)) \phi_a \, d\xi$$

$$\approx \frac{2\alpha}{h} \sum_{j=1}^{m} W_j \frac{d}{dx} u_0(x(P_j, e)) \frac{d}{d\xi} \phi_a(P_j) + \frac{\beta h}{2} \sum_{j=1}^{m} W_j u_0(x(P_j, e)) \phi_a(P_j)$$

Usando α e β do problema