Basics of Monodromy

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Basic Notions — University of Freiburg

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Motivating example — A multi-valued function

Consider $z = re^{\theta i} \mapsto z^2 = r^2 e^{2\theta i}$ on $\mathbb{C}$. Local inverse on $\mathbb{C}^{\times}$:

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Let $z_0 \in \mathbb{C}^\times$ and pick one value for $\sqrt{z_0}$. Let $\gamma : [0, 1] \to \mathbb{C}^\times$ be a path with $\gamma(0) = z_0$. Then the chosen $\sqrt{z_0}$ determines uniquely a value of $\sqrt{\gamma(t)}$ for all $t \in [0, 1]$, because we want $z \mapsto \sqrt{z}$ to be continuous.

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Theorem (Weierstraß)

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In particular, if we walk around a simply connected space, then the analytic continuation is single-valued everywhere. Hence:

“**monodromy**”, *mónos* (alone, only, single) and *drómos* (running).

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Pick a value of $\sqrt{1}$, say 1 itself, and let $\gamma \colon [0,1] \to \mathbb{C}^\times$ be a loop at 1 around 0. Extend $\sqrt{z}$ along γ as before.

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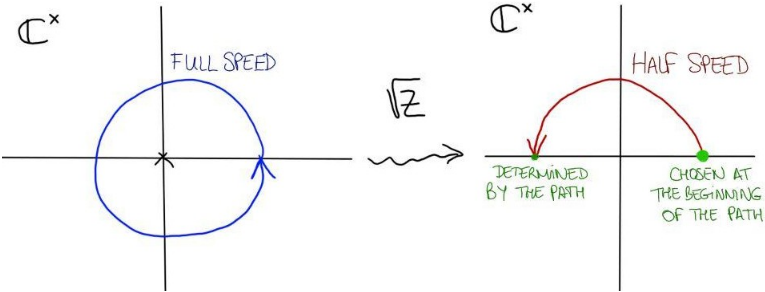

Figure: After the loop we arrive at -1, the other possible value of $\sqrt{1}$.

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Exercise 0

This is the second red herring that appeared in this talk so far. Can you spot the first one?

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- We can recover the covering space from the monodromy action!
- If the fibres of p carry a natural vector space structure, then one can use the tools of **representation theory** to study polydromy. This happens both naturally (e.g. when solving differential equations on a complex domain) and artificially (e.g. replacing the fibres by their cohomology groups).

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Covering spaces

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- $p\colon Y \to X$ is a *covering space* if locally on X it is isomorphic to a projection $X \times F \to X$ for some discrete space F.

Maps into covering spaces

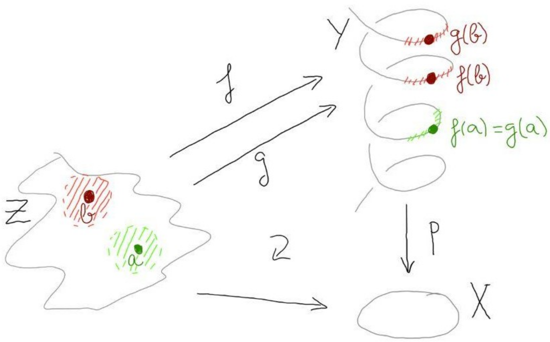

Figure: The set $\{z \in Z \mid f(z) = g(z)\}$ is open and closed, so if Z is connected and f and g agree on a single point, then they agree in all of Z.

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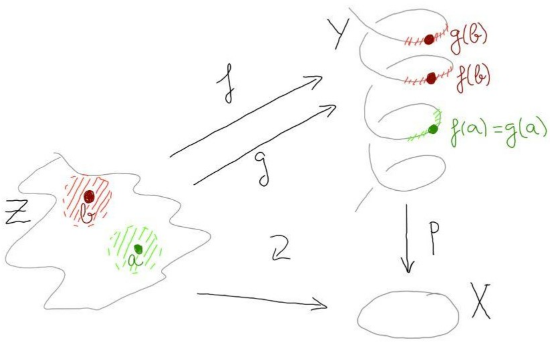

Figure: The set $\{z \in Z \mid f(z) = g(z)\}$ is open and closed, so if Z is connected and f and g agree on a single point, then they agree in all of Z.

In particular, if $p\colon Y \to X$ is a connected cover and $\phi \in \mathrm{Aut}(Y \mid X)$ fixes a point, then $\phi = \mathrm{id}_Y$.

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Theorem ([?, Theorem 2.2.10])

Let $p\colon Y \to X$ be a Galois cover. Then there is a bijection

$$\left\{\begin{matrix}\text{Subgroups}\\ 0 \subseteq H \subseteq \mathrm{Aut}(Y \mid X)\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{Connected}\\ \text{intermediate}\\ \text{covers}\end{matrix} \quad \begin{tikzcd} Z \arrow[d, "q"'] & Y \arrow[l, dashed, "\exists"'] \arrow[dl, "p"] \\ X & \end{tikzcd}\right\}$$

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Moreover, $q\colon Z \to X$ is Galois if and only if $H \subseteq \mathrm{Aut}(Y \mid X)$ is a normal subgroup, in which case we have

$$\mathrm{Aut}(Z \mid X) \cong G/H.$$

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Exercise 1

A continuous action $G \circlearrowright Y$ *is called* even if each $y \in Y$ has an open nhood $y \in V \subseteq Y$ such that $\{gV\}_{g \in G}$ are pairwise disjoint. Show that $Y \to G \backslash Y$ is then a cover and deduce that $Y \to H \backslash Y$ is a cover.

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2) Define $\varphi\colon H \to \mathrm{Aut}(Y \mid H\backslash Y)$ by $\varphi(h)(y) := h \cdot y$. Since $H \circlearrowright Y$ is even, φ is injective. By "Maps into covering spaces" it is also surjective. Hence $\boxed{H \mapsto (H\backslash Y \to X) \mapsto \mathrm{Aut}(Y \mid H\backslash Y) \cong H}$.

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3) If $q\colon Z \to X$ is an intermediate connected cover, then the map $Y \to Z$ is a cover as well (local on Z, hence on X, hence we may assume that this map has the form $X \times F_Y \to X \times F_Z$).

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4) Since $\mathrm{Aut}(Y \mid X) \circlearrowright p^{-1}(q(z))$ is transitive, $\mathrm{Aut}(Y \mid Z) \circlearrowright f^{-1}(z)$ is transitive as well by "Maps into covering spaces". Hence $Y \longrightarrow Z$ is Galois and $\boxed{Z \mapsto \mathrm{Aut}(Y \mid Z) \mapsto \mathrm{Aut}(Y \mid Z)\backslash Y = Z}$.

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1) Let $z_0 \in W \subseteq Z$ be an open neighbourhood of a point in Z for which there exists a subdivision $0 = t_0 < t_1 < \cdots < t_m = 1$ such that $p\colon Y \longrightarrow X$ is trivial over $F(W \times [t_i, t_{i+1}])$ for all i.

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3) Iterate this process to obtain a local lifting $\tilde{F}\colon W \times [0,1] \longrightarrow Y$.
4) Do the same for each point $z \in Z$. On the overlaps the extensions agree by “Maps into covering spaces” applied to each $\{z\} \times [0,1]$, because $\tilde{F}(z,0)$ has to be $\tilde{f}_0(z)$.

The monodromy action

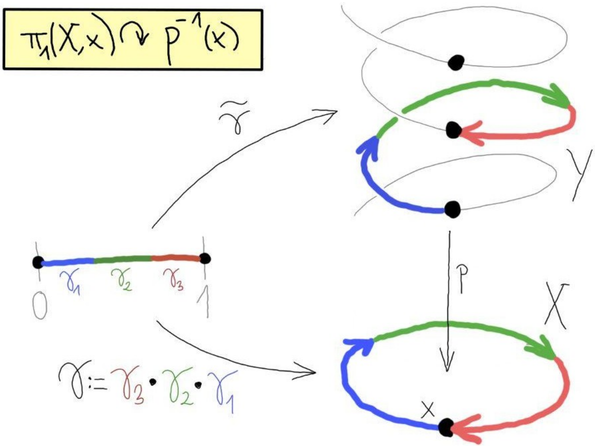

Figure: Given the class of a path $[\gamma] \in \pi_1(X, x)$ and a point $y \in p^{-1}(x)$, set $[\gamma] \cdot y := \tilde{\gamma}(1)$, where $\tilde{\gamma}$ is the unique lift of γ to the cover. Only defining concatenation the unconventional way we obtain a *left* action!

Cover↔Monodromy [X connected+locally 1-connected]

Theorem ([?, Theorem 2.3.4])

The functor

$$\begin{aligned}\mathrm{Fib}_x\colon \mathbf{Cov}(X) &\longrightarrow \pi_1(X,x) - \mathbf{Set}\\ (p\colon\ Y \to X) &\longmapsto \pi_1(X,x) \circlearrowright p^{-1}(x)\end{aligned}$$

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Exercise 2

Check that Fib_x is a functor. [Hint: "Maps into covering spaces".]

Sketch of proof — Part 1: the universal cover

1) The *universal cover* $\tilde{X}_x$ consists of homotopy classes of paths in X starting at X, and the projection $\pi\colon \tilde{X}_x \to X$ is $\pi([\alpha]) := \alpha(1)$.

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6) $\boxed{\pi_1(X, x) \cong \mathrm{Aut}(\tilde{X}_x \mid X)^{\mathrm{op}}}$ via $[\gamma] \longmapsto ([\alpha] \mapsto [\alpha \cdot \gamma])$.

Sketch of proof — Part 1: the universal cover

1) The *universal cover* $\tilde{X}_x$ consists of homotopy classes of paths in X starting at X, and the projection $\pi\colon \tilde{X}_x \longrightarrow X$ is $\pi([\alpha]) := \alpha(1)$.
2) Let $y \in p^{-1}(x)$. Define $\pi_y \in \mathrm{Hom}_X(\tilde{X}_x, Y)$ by $\pi_y([\alpha]) := \tilde{\alpha}(1)$.
3) Let $\phi \in \mathrm{Hom}_X(\tilde{X}_x, Y)$. Define $y \in p^{-1}(x)$ as $y := \phi([x])$.
4) These two maps are mutually inverse, so $\boxed{\mathrm{Fib}_x \cong \mathrm{Hom}_X(\tilde{X}_x, -)}$.
5) $\pi\colon \tilde{X}_x \longrightarrow X$ is Galois, because $\pi_{[\gamma]}$ is an automorphism and $\pi_{[\gamma]}([x]) = [\gamma]$ (suffices to check transitivity on a single fibre).
6) $\boxed{\pi_1(X, x) \cong \mathrm{Aut}(\tilde{X}_x \mid X)^{\mathrm{op}}}$ via $[\gamma] \longmapsto ([\alpha] \mapsto [\alpha \cdot \gamma])$.
7) Let $\phi \in \mathrm{Aut}(\tilde{X}_x \mid X)$ and $y \in p^{-1}(x)$. Define $\phi \cdot y := \pi_y \circ \phi([x])$, i.e. the point in $\mathrm{Fib}_x(Y)$ corresponding to $\pi_y \circ \phi \in \mathrm{Hom}_X(\tilde{X}_x, Y)$. Then $\psi \cdot (\phi \cdot y)$ corresponds to $\pi_y \circ \phi \circ \psi = \pi_y \circ (\psi \circ^{\mathrm{op}} \phi)$. We get $\mathrm{Aut}(\tilde{X}_x \mid X)^{\mathrm{op}} \circlearrowright p^{-1}(x)$, which agrees with $\pi_1(X, x) \circlearrowright p^{-1}(x)$.

Sketch of proof — Part 2: fully faithfulness

1) Let $\psi\colon \mathrm{Fib}_x(Y) \longrightarrow \mathrm{Fib}_x(Z)$ be $G := \pi_1(X, x)$-equivariant. Want some $f\colon Y \longrightarrow Z$ over X such that $\psi(y) = f(y)$ for all $y \in p^{-1}(x)$.

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6) $G_y \subseteq G_{\psi(y)}$ by G-equivar. $\rightsquigarrow f\colon Y = G_y\backslash\tilde{X}_x \longrightarrow G_{\psi(y)}\backslash\tilde{X}_x = Z$.
7) $f(y) = f(\pi_y([x])) = \pi_{\psi(y)}([x]) = \psi(y)$. For $y' \in p^{-1}(y)$, let γ s.t. $[\gamma]\cdot y = y'$, so that $\psi(y') = [\gamma]\cdot\psi(y)$. Then we have

$$f(y') = f \circ \tilde{\gamma}^Y(1) = \tilde{\gamma}^Z(1) = [\gamma]\cdot\psi(y) = \psi(y').$$

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7) If $g \cdot s = g' \cdot s$, then again $g'g^{-1} \in G_s = G_y$, so $\varphi(y) =$