Basics of Monodromy

Pedro Núñez

Basic Notions — University of Freiburg

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Motivating example — A multi-valued function

Consider $z = re^{\theta i} \mapsto z^2 = r^2 e^{2\theta i}$ on $\mathbb{C}$. Local inverse on $\mathbb{C}^\times$:

$$z = re^{\theta i} \mapsto \sqrt{z} = \sqrt{r} e^{\frac{\theta}{2} i}.$$

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Ambiguity: the previous expression is not well defined, as

$$re^{\theta i} = re^{(\theta + 2\pi) i} \mapsto \sqrt{r} e^{\frac{\theta}{2} i} \neq \sqrt{r} e^{\left(\frac{\theta}{2} + \pi\right) i}.$$

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Let $z_0 \in \mathbb{C}^\times$ and pick one value for $\sqrt{z_0}$. Let $\gamma : [0, 1] \longrightarrow \mathbb{C}^\times$ be a path with $\gamma(0) = z_0$. Then the chosen $\sqrt{z_0}$ determines uniquely a value of $\sqrt{\gamma(t)}$ for all $t \in [0, 1]$, because we want $z \mapsto \sqrt{z}$ to be continuous.

The Monodromy Theorem

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In particular, if we walk around a simply connected space, then the analytic continuation is single-valued everywhere. Hence:

"**monodromy**", *mónos* (alone, only, single) and *drómos* (running).

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Pick a value of $\sqrt{z}$ at z_0 as before and let $\gamma\colon [0,1] \to \mathbb{C}^\times$ be a loop at z_0, with $\gamma(0) = \gamma(1) = z_0$. Extend $\sqrt{z}$ along γ as before.

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Do we always arrive at the same value of $\sqrt{z_0}$ at the end of the loop?

Why are we then talking about monodromy?

The Monodromy Theorem became so famous that people kept using the word “monodromy” to talk about polydromy[1].

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Figure: This is an example of mathematical red herring principle.

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Exercise 0

This is the second red herring that appeared in this talk so far.
Can you spot the first one?

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- If $p\colon Y \to X$ is a nice covering space, then $\pi_1(X, x)$ acts naturally on $p^{-1}(x)$. This is the **monodromy action**.

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- As we move z in $\mathbb{C}^\times$, the possible values of $\sqrt{z}$ form a nice **covering space** of $\mathbb{C}^\times$.
- If $p\colon Y \to X$ is a nice covering space, then $\pi_1(X, x)$ acts naturally on $p^{-1}(x)$. This is the **monodromy action**.
- If the fibres of p carry a natural vector space structure, we will be able to use the tools of **representation theory** to study polydromy.

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Covering spaces

Let X be a topological space.

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- A map $p\colon Y \to X$ has property $\mathbf{P}$ *locally on* X if every point $x \in X$ has an open neighbourhood $x \in U \subseteq X$ such that $\mathbf{P}$ is true for $p|_{p^{-1}(U)}\colon p^{-1}(U) \to U$.

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- $p\colon Y \to X$ is a *covering space* if locally on X it is isomorphic to a projection $X \times F \to X$ for some discrete space F.

Maps into covering spaces

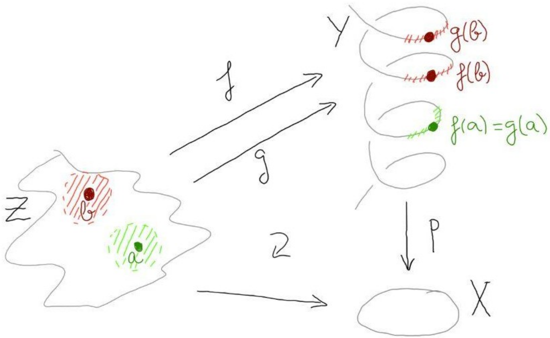

Figure: The set $\{z \in Z \mid f(z) = g(z)\}$ is open and closed, so if Z is connected and f and g agree on a single point, then they agree in all of Z.

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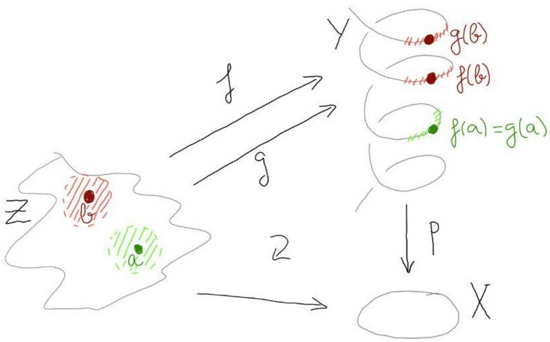

Figure: The set $\{z \in Z \mid f(z) = g(z)\}$ is open and closed, so if Z is connected and f and g agree on a single point, then they agree in all of Z.

In particular, if $p\colon Y \to X$ is a connected cover and $\phi \in \mathrm{Aut}(Y \mid X)$ fixes a point, then $\phi = \mathrm{id}_Y$.

Galois coverings

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Theorem 2

Let $p\colon Y \to X$ be a Galois cover. Then there is a bijection

$$\begin{aligned} \{\text{ Subgroups } H \subseteq \mathrm{Aut}(Y \mid X) \} &\leftrightarrow \{\text{ Intermediate covers } q\colon Z \to X \} \\ H \quad &\mapsto \quad (H\backslash Y \to X) \\ \mathrm{Aut}(Y \mid Z) \quad &\leftarrow\!\shortmid \quad (Z \to X) \end{aligned}$$

Moreover, $q\colon Z \to X$ is Galois if and only if $H \subseteq \mathrm{Aut}(Y \mid X)$ is a normal subgroup, in which case we have

$$\mathrm{Aut}(Z \mid X) = G/H.$$

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Local systems