Basics of Monodromy

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Basic Notions — University of Freiburg

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Motivating example — A multi-valued function

Consider $z = re^{\theta i} \mapsto z^2 = r^2 e^{2\theta i}$ on $\mathbb{C}$. Local inverse on $\mathbb{C}^\times$:

$$z = re^{\theta i} \mapsto \sqrt{z} = \sqrt{r} e^{\frac{\theta}{2} i}.$$

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Ambiguity: the previous expression is not well defined, as

$$re^{\theta i} = re^{(\theta + 2\pi) i} \mapsto \sqrt{r} e^{\frac{\theta}{2} i} \neq \sqrt{r} e^{\left(\frac{\theta}{2} + \pi\right) i}.$$

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Let $z_0 \in \mathbb{C}^\times$ and pick one value for $\sqrt{z_0}$. Let $\gamma : [0, 1] \to \mathbb{C}^\times$ be a path with $\gamma(0) = z_0$. Then the chosen $\sqrt{z_0}$ determines uniquely a value of $\sqrt{\gamma(t)}$ for all $t \in [0, 1]$, because we want $z \mapsto \sqrt{z}$ to be continuous.

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Theorem (Weierstraß)

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In particular, if we walk around a simply connected space, then the analytic continuation is single-valued everywhere. Hence:

“**monodromy**”, *mónos* (alone, only, single) and *drómos* (running).

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Pick a value of $\sqrt{1}$, say 1 itself, and let $\gamma\colon [0,1] \to \mathbb{C}^\times$ be a loop at 1 around 0. Extend $\sqrt{z}$ along γ as before.

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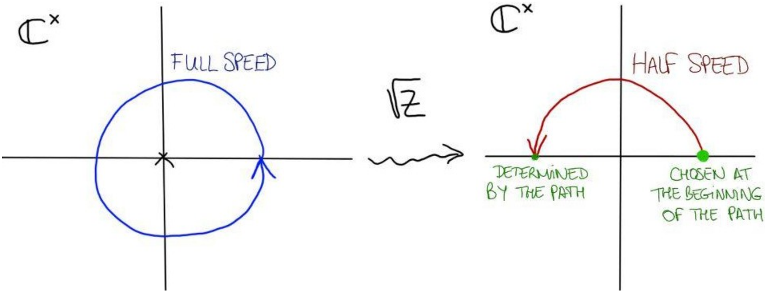

Figure: After the loop we arrive at -1, the other possible value of $\sqrt{1}$.

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The Monodromy Theorem became so famous that people kept using the word “monodromy” to talk about polydromy[1].

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Figure: This is an example of mathematical red herring principle.

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Exercise 0

This is the second red herring that appeared in this talk so far.
Can you spot the first one?

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- As we move z in $\mathbb{C}^\times$, the possible values of $\sqrt{z}$ form a nice **covering space** of $\mathbb{C}^\times$.
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- As we move z in $\mathbb{C}^\times$, the possible values of $\sqrt{z}$ form a nice **covering space** of $\mathbb{C}^\times$.
- If $p\colon Y \to X$ is a nice covering space, then $\pi_1(X, x)$ acts naturally on $p^{-1}(x)$. This is the **monodromy action**.
- If the fibres of p carry a natural vector space structure, we will be able to use the tools of **representation theory** to study polydromy.

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Covering spaces

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- A map $p\colon Y \to X$ has property $\mathbf{P}$ *locally on* X if every point $x \in X$ has an open neighbourhood $x \in U \subseteq X$ such that $\mathbf{P}$ is true for $p|_{p^{-1}(U)}\colon p^{-1}(U) \to U$.

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- $p\colon Y \to X$ is a *covering space* if locally on X it is isomorphic to a projection $X \times F \to X$ for some discrete space F.

Maps into covering spaces

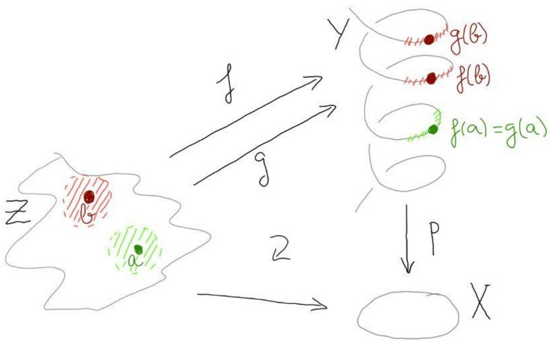

Figure: The set $\{z \in Z \mid f(z) = g(z)\}$ is open and closed, so if Z is connected and f and g agree on a single point, then they agree in all of Z.

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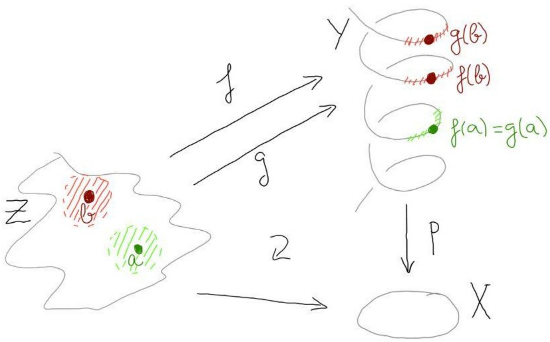

Figure: The set $\{z \in Z \mid f(z) = g(z)\}$ is open and closed, so if Z is connected and f and g agree on a single point, then they agree in all of Z.

In particular, if $p\colon Y \to X$ is a connected cover and $\phi \in \mathrm{Aut}(Y \mid X)$ fixes a point, then $\phi = \mathrm{id}_Y$.

Galois correspondence [X locally connected]

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Theorem ([?, Theorem 2.2.10])

Let $p\colon Y \to X$ be a Galois cover. Then there is a bijection

$$\left\{\begin{matrix}\text{Subgroups}\\ 0 \subseteq H \subseteq \mathrm{Aut}(Y \mid X)\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{Connected}\\ \text{intermediate}\\ \text{covers}\end{matrix} \quad \begin{tikzcd} Z \arrow[d, "q"'] & Y \arrow[l, dashed, "\exists"'] \arrow[dl, "p"] \\ X \end{tikzcd}\right\}$$

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$$\begin{aligned} H \quad &\mapsto \quad (H\backslash Y \to X) \\ \mathrm{Aut}(Y \mid Z) \quad &\mapsfrom \quad (Z \to X) \end{aligned}$$

Moreover, $q\colon Z \to X$ is Galois if and only if $H \subseteq \mathrm{Aut}(Y \mid X)$ is a normal subgroup, in which case we have

$$\mathrm{Aut}(Z \mid X) \cong G/H.$$

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1) $H \backslash Y \to X$ is a cover (local on X, hence may assume $Y = X \times F$).

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Exercise 1

A continuous action $G \circlearrowright Y$ is called even if each $y \in Y$ has an open nhood $y \in V \subseteq Y$ such that $\{gV\}_{g \in G}$ are pairwise disjoint. Show that $Y \longrightarrow G\backslash Y$ is then a cover and deduce that $Y \longrightarrow H\backslash Y$ is a cover.

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2) Define $\varphi\colon H \to \mathrm{Aut}(Y \mid H\backslash Y)$ by $\varphi(h)(y) := h \cdot y$. Since $H \circlearrowright Y$ is even, φ is injective. By "Maps into covering spaces" it is also surjective. Hence $\boxed{H \mapsto (H\backslash Y \to X) \mapsto \mathrm{Aut}(Y \mid H\backslash Y) \cong H}$.

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3) If $q\colon Z \longrightarrow X$ is an intermediate connected cover, then the map $Y \longrightarrow Z$ is a cover as well (local on Z, hence on X, hence we may assume that this map has the form $X \times F_Y \longrightarrow X \times F_Z$).

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4) Since $\mathrm{Aut}(Y \mid X) \circlearrowright p^{-1}(q(z))$ is transitive, $\mathrm{Aut}(Y \mid Z) \circlearrowright f^{-1}(z)$ is transitive as well by "Maps into covering spaces". Hence $Y \to Z$ is Galois and $\boxed{Z \mapsto \mathrm{Aut}(Y \mid Z) \mapsto \mathrm{Aut}(Y \mid Z)\backslash Y = Z}$.

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1) Let $z_0 \in W \subseteq Z$ be an open neighbourhood of a point in Z for which there exists a subdivision $0 = t_0 < t_1 < \cdots < t_m = 1$ such that $p \colon Y \longrightarrow X$ is trivial over $F(W \times [t_i, t_{i+1}])$ for all i.

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3) Iterate this process to obtain a local lifting $\tilde{F}\colon W \times [0,1] \to Y$.
4) Do the same for each point $z \in Z$. On the overlaps the extensions agree by “Maps into covering spaces” applied to each $\{z\} \times [0,1]$, because $\tilde{F}(z,0)$ has to be $\tilde{f}_0(z)$.

The monodromy action

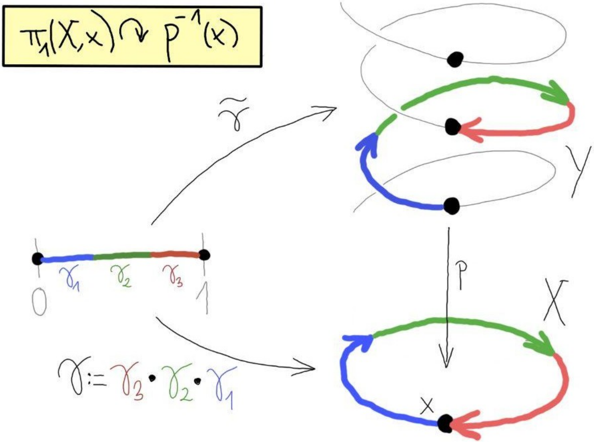

Figure: Given the class of a path $[\gamma] \in \pi_1(X, x)$ and a point $y \in p^{-1}(x)$, set $[\gamma] \cdot y := \tilde{\gamma}(1)$, where $\tilde{\gamma}$ is the unique lift of γ to the cover. Only defining concatenation the unconventional way we obtain a *left* action!

Cover↔Monodromy [X connected+locally 1-connected]

Theorem ([?, Theorem 2.3.4])

The functor

$$\begin{aligned} \mathrm{Fib}_x \colon \mathbf{Cov}(X) &\longrightarrow \pi_1(X,x) - \mathbf{Set} \\ (p\colon\, Y \to X) &\longmapsto \pi_1(X,x) \circlearrowright p^{-1}(x) \end{aligned}$$

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Exercise 2

Check that Fib_x *is a functor. [Hint: "Maps into covering spaces".]*

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5) $\pi : \tilde{X}_x \longrightarrow X$ is Galois, because $\pi_{[\gamma]}$ is an automorphism and $\pi_{[\gamma]}([x]) = [\gamma]$ (suffices to check transitivity on a single fibre).
6) $\boxed{\pi_1(X, x) \cong \mathrm{Aut}(\tilde{X}_x \mid X)^{\mathrm{op}}}$ via $[\gamma] \longmapsto ([\alpha] \mapsto [\alpha \cdot \gamma])$.
7) Let $\phi \in \mathrm{Aut}(\tilde{X}_x \mid X)$ and $y \in p^{-1}(x)$. Define $\phi \cdot y := \pi_y \circ \phi([x])$, i.e. the point in $\mathrm{Fib}_x(Y)$ corresponding to $\pi_y \circ \phi \in \mathrm{Hom}_X(\tilde{X}_x, Y)$. Then $\psi \cdot (\phi \cdot y)$ corresponds to $\pi_y \circ \phi \circ \psi = \pi_y \circ (\psi \circ^{\mathrm{op}} \phi)$. We get $\mathrm{Aut}(\tilde{X}_x \mid X)^{\mathrm{op}} \circlearrowright p^{-1}(x)$, which agrees with $\pi_1(X, x) \circlearrowright p^{-1}(x)$.

Sketch of proof — Part 2: applying Galois correspondence

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