

Basics of Monodromy

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Basic Notions — University of Freiburg

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Motivating example — A multi-valued function

Consider $z = re^{\theta i} \mapsto z^2 = r^2 e^{2\theta i}$ on \mathbb{C} . Local inverse on \mathbb{C}^\times :

$$z = re^{\theta i} \mapsto \sqrt{z} = \sqrt{r} e^{\frac{\theta}{2} i}.$$

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Let $z_0 \in \mathbb{C}^\times$ and pick one value for $\sqrt{z_0}$. Let $\gamma: [0, 1] \rightarrow \mathbb{C}^\times$ be a path with $\gamma(0) = z_0$. Then the chosen $\sqrt{z_0}$ determines uniquely a value of $\sqrt{\gamma(t)}$ for all $t \in [0, 1]$, because we want $z \mapsto \sqrt{z}$ to be continuous.

The Monodromy Theorem

Theorem (Weierstraß)

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In particular, if we walk around a simply connected space, then the analytic continuation is single-valued everywhere. Hence:

“**monodromy**”, *mónos* (alone, only, single) and *drómos* (running).

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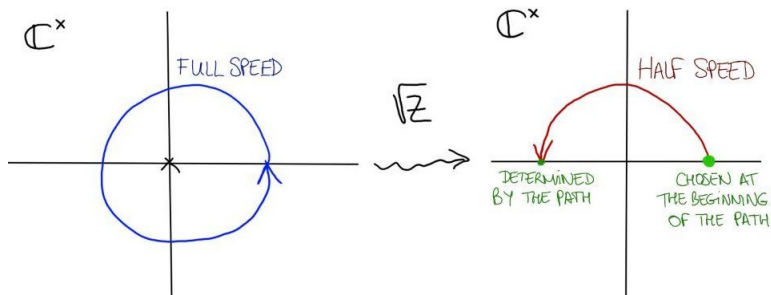



Figure: After the loop we arrive at -1 , the other possible value of $\sqrt{1}$.

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Exercise 0

*This is the second red herring that appeared in this talk so far.
Can you spot the first one?*

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- We can recover the covering space from the monodromy action.
- If the fibres of p carry a natural vector space structure, then one can use the tools of **representation theory** to study polydromy. This happens both naturally (e.g. when solving differential equations on a complex domain) and artificially (e.g. replacing the fibres by their cohomology groups).

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- $p: Y \rightarrow X$ is a *covering space* if locally on X it is isomorphic to a projection $X \times F \rightarrow X$ for some discrete space F .

Maps into covering spaces

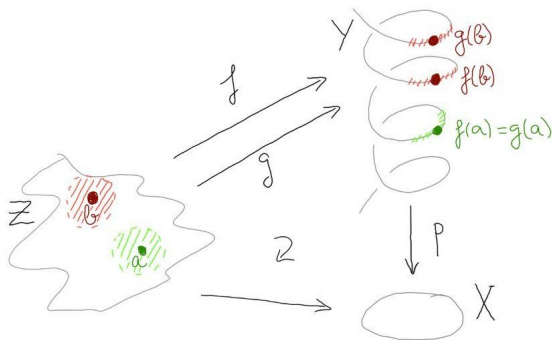


Figure: The set $\{z \in Z \mid f(z) = g(z)\}$ is open and closed, so if Z is connected and f and g agree on a single point, then they agree in all of Z .

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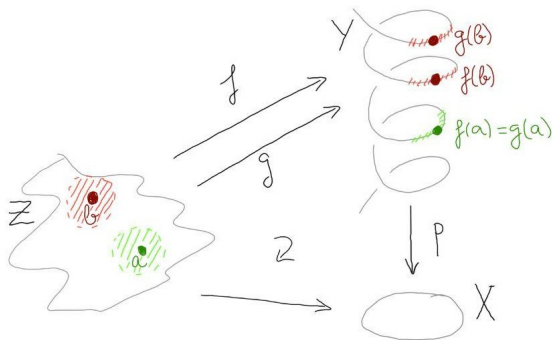


Figure: The set $\{z \in Z \mid f(z) = g(z)\}$ is open and closed, so if Z is connected and f and g agree on a single point, then they agree in all of Z .

In particular, if $p: Y \rightarrow X$ is a connected cover and $\phi \in \text{Aut}(Y \mid X)$ fixes a point, then $\phi = \text{id}_Y$.

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Theorem ([Sza09, Theorem 2.2.10])

Let $p: Y \rightarrow X$ be a Galois cover. Then there is a bijection

$$\left\{ \begin{array}{c} \text{Subgroups} \\ 0 \subseteq H \subseteq \text{Aut}(Y | X) \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{Connected} \\ \text{intermediate} \\ \text{covers} \end{array} \quad \begin{array}{ccc} Z & \xleftarrow{\exists} & Y \\ q \downarrow & \swarrow p & \\ X & & \end{array} \right\}$$

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Moreover, $q: Z \rightarrow X$ is Galois if and only if $H \subseteq \text{Aut}(Y | X)$ is a normal subgroup, in which case we have

$$\text{Aut}(Z | X) \cong G/H.$$

Proof of the bijection in the previous theorem

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Exercise 1

A continuous action $G \curvearrowright Y$ is called even if each $y \in Y$ has an open neighborhood $y \in V \subseteq Y$ such that $\{gV\}_{g \in G}$ are pairwise disjoint. Show that $Y \rightarrow G \backslash Y$ is then a cover and deduce that $Y \rightarrow H \backslash Y$ is a cover.

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2) Define $\varphi: H \rightarrow \text{Aut}(Y \mid H \backslash Y)$ by $\varphi(h)(y) := h \cdot y$. Since $H \curvearrowright Y$ is even, φ is injective. By “Maps into covering spaces” it is also surjective. Hence $H \mapsto (H \backslash Y \rightarrow X) \mapsto \text{Aut}(Y \mid H \backslash Y) \cong H$.

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- 4) Since $\text{Aut}(Y \mid X) \curvearrowright p^{-1}(q(z))$ is transitive, $\text{Aut}(Y \mid Z) \curvearrowright f^{-1}(z)$ is transitive as well by “Maps into covering spaces”. Hence $Y \rightarrow Z$ is Galois and $Z \mapsto \text{Aut}(Y \mid Z) \mapsto \text{Aut}(Y \mid Z) \backslash Y = Z$.

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- 1) Let $z_0 \in W \subseteq Z$ be an open neighbourhood of a point in Z for which there exists a subdivision $0 = t_0 < t_1 < \dots < t_m = 1$ such that $p: Y \rightarrow X$ is trivial over $F(W \times [t_i, t_{i+1}])$ for all i .

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- 3) Iterate this process to obtain a local lifting $\tilde{F}: W \times [0, 1] \rightarrow Y$.
- 4) Do the same for each point $z \in Z$. On the overlaps the extensions agree by “Maps into covering spaces” applied to each $\{z\} \times [0, 1]$, because $\tilde{F}(z, 0)$ has to be $\tilde{f}_0(z)$.

The monodromy action

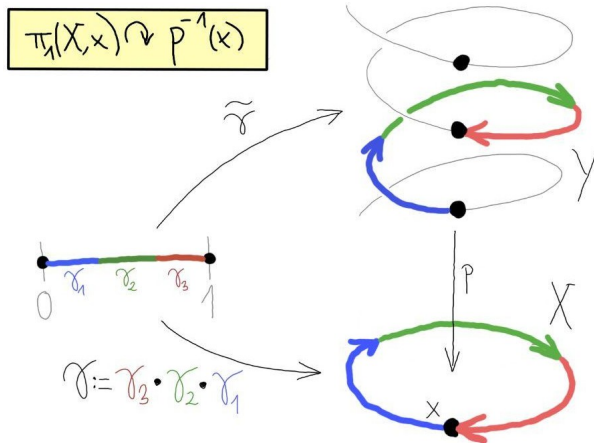


Figure: Given the class of a path $[y] \in \pi_1(X, x)$ and a point $y \in p^{-1}(x)$, set $[y] \cdot y := \tilde{\gamma}(1)$, where $\tilde{\gamma}$ is the unique lift of y to the cover. Only defining concatenation the unconventional way we obtain a *left* action!

Cover \leftrightarrow Monodromy [X connected+locally 1-connected]

Theorem ([Sza09, Theorem 2.3.4])

The functor

$$\mathrm{Fib}_x: \mathbf{Cov}(X) \longrightarrow \pi_1(X, x) - \mathbf{Set}$$

$$(p: Y \rightarrow X) \longmapsto \pi_1(X, x) \curvearrowright p^{-1}(x)$$

is an equivalence of categories from the category of covers of X to the category of sets endowed with a $\pi_1(X, x)$ -action.

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Exercise 2

Check that Fib_x is a functor. [Hint: “Maps into covering spaces”.]

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- 1) The *universal cover* \tilde{X}_x consists of homotopy classes of paths in X starting at x , and the projection $\pi: \tilde{X}_x \rightarrow X$ is $\pi([\alpha]) := \alpha(1)$.

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- 6) $\boxed{\pi_1(X, x) \cong \text{Aut}(\tilde{X}_x \mid X)^{\text{op}}}$ via $[y] \mapsto ([\alpha] \mapsto [\alpha \cdot \gamma])$.

Sketch of proof — Part 1: the universal cover

- 1) The *universal cover* \tilde{X}_x consists of homotopy classes of paths in X starting at x , and the projection $\pi: \tilde{X}_x \rightarrow X$ is $\pi([\alpha]) := \alpha(1)$.
- 2) Let $y \in p^{-1}(x)$. Define $\pi_y \in \text{Hom}_X(\tilde{X}_x, Y)$ by $\pi_y([\alpha]) := \tilde{\alpha}(1)$.
- 3) Let $\phi \in \text{Hom}_X(\tilde{X}_x, Y)$. Define $y \in p^{-1}(x)$ as $y := \phi([x])$.
- 4) These two maps are mutually inverse, so $\boxed{\text{Fib}_x \cong \text{Hom}_X(\tilde{X}_x, -)}$.
- 5) $\pi: \tilde{X}_x \rightarrow X$ is Galois, because $\pi_{[y]}$ is an automorphism and $\pi_{[y]}([x]) = [y]$ (suffices to check transitivity on a single fibre).
- 6) $\boxed{\pi_1(X, x) \cong \text{Aut}(\tilde{X}_x | X)^{\text{op}}}$ via $[y] \mapsto ([\alpha] \mapsto [\alpha \cdot y])$.
- 7) Let $\phi \in \text{Aut}(\tilde{X}_x | X)$ and $y \in p^{-1}(x)$. Define $\phi \cdot y := \pi_y \circ \phi([x])$, i.e. the point in $\text{Fib}_x(Y)$ corresponding to $\pi_y \circ \phi \in \text{Hom}_X(\tilde{X}_x, Y)$. Then $\psi \cdot (\phi \cdot y)$ corresponds to $\pi_y \circ \phi \circ \psi = \pi_y \circ (\psi \circ^{\text{op}} \phi)$. We get $\text{Aut}(\tilde{X}_x | X)^{\text{op}} \curvearrowright p^{-1}(x)$, which agrees with $\pi_1(X, x) \curvearrowright p^{-1}(x)$.

Sketch of proof — Part 2: fully faithfulness

- 1) Let $\psi: \text{Fib}_x(Y) \rightarrow \text{Fib}_x(Z)$ be $G := \pi_1(X, x)$ -equivariant. Want some $f: Y \rightarrow Z$ over X such that $\psi(y) = f(y)$ for all $y \in p^{-1}(x)$.

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- 4) Let $\pi_y: \tilde{X}_x \rightarrow Y$ corresponding to $y \in p^{-1}(x)$, so that

$Y = \text{Aut}(\tilde{X}_x \mid Y) \backslash \tilde{X}_x$

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- 5) Via $G \cong \text{Aut}(\tilde{X}_x | X)$ we can identify $G_y := \{g \in G \mid g \cdot y = y\}$ and $\{\phi \in \text{Aut}(\tilde{X}_x | X) \mid \pi_y \circ \phi = \pi_y\}$. Hence $\text{Aut}(\tilde{X}_x | Y) = G_y$.

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- 6) $G_y \subseteq G_{\psi(y)}$ by G -equivar. $\rightsquigarrow f: Y = G_y \backslash \tilde{X}_x \rightarrow G_{\psi(y)} \backslash \tilde{X}_x = Z$.
- 7) $f(y) = f(\pi_y([x])) = \pi_{\psi(y)}([x]) = \psi(y)$. For $y' \in p^{-1}(y)$, let γ s.t. $[\gamma] \cdot y = y'$, so that $\psi(y') = [\gamma] \cdot \psi(y)$. Then we have

$$f(y') = f \circ \tilde{\gamma}^Y(1) = \tilde{\gamma}^Z(1) = [\gamma] \cdot \psi(y) = \psi(y').$$

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- 7) If $g \cdot s = g' \cdot s$, then again $g'g^{-1} \in G_s = G_{q([x])}$, so $g \cdot q([x]) = g' \cdot q([x])$. And transitivity implies surjectivity.

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2. Galois correspondence

3. Monodromy action

4. Example

Back to our initial example

Let $\pi: \mathbb{C}^\times \rightarrow \mathbb{C}^\times$ be our map $z \mapsto z^2$, whose fibre over z_0 consists of the points $\pm\sqrt{z_0}$, and let $p: Y \rightarrow \mathbb{C}^\times$ be the cover in which the fibre over $z_0 \in \mathbb{C}^\times$ is the \mathbb{Q} -vector space spanned by $\{\sqrt{z_0}, -\sqrt{z_0}\}$ with the discrete topology².

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$$A := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

meaning that $\pi_1(\mathbb{C}^\times, 1) \rightarrow \mathrm{GL}_2(\mathbb{Q})$ maps $[\gamma]$ to A .

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Exercise 3

A subrepresentation of $\rho: G \rightarrow \mathrm{GL}(V)$ consists of a G -invariant subspace $W \subseteq V$ together with the induced $G \rightarrow \mathrm{GL}(W)$. Find all subrepresentations of this monodromy action. What are the fibres at a point $z_0 \in \mathbb{C}^\times$ of the corresponding covering spaces?

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Thanks for your attention!

Main reference:



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