Basic Notions — Spectral Sequences

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10th December 2020

"You could have invented spectral sequences"[1]

[1] Title of the expository article [Cho06]

Notation gets ugly very soon

Notation and conventions [Sta20]

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- Cohomology is regarded as a copmlex with 0 differentials

$$H(C) = \bigoplus_{i \in \mathbb{Z}} H^i(C)[-i] := \left(\cdots H^{-1}(C) \xrightarrow{0} H^0(C) \xrightarrow{0} H^1(C) \cdots \right).$$

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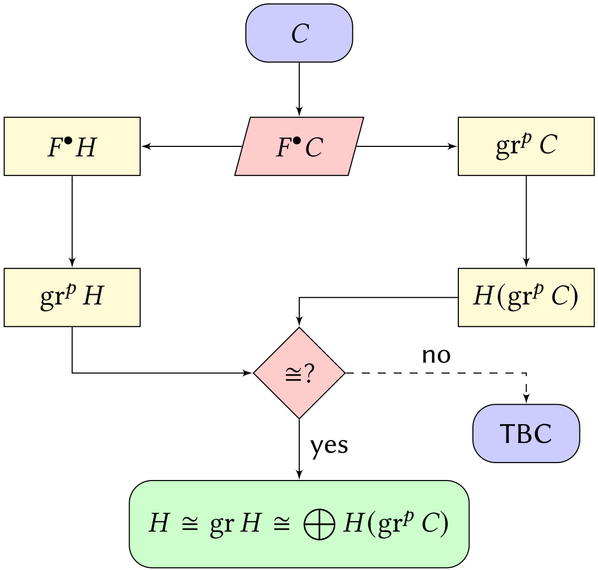
- (Decreasing) filtrations are always finite for simplicity:

$$F^\bullet A: \quad 0 = F^n A \subseteq \ldots \subseteq F^0 A = A.$$

- The graded object associated to a filtration $F^\bullet A$ is denoted

$$\operatorname{gr} A = \bigoplus_{p\in\mathbb{N}} \operatorname{gr}^p A := \bigoplus_{p\in\mathbb{N}} \left(F^p A / F^{p+1} A\right).$$

GOAL: compute cohomology with the help of a filtration



Filtration induced in cohomology

A filtration on the complx $F^\bullet C$ induces a filtration in cohomology

$$F^p H := \operatorname{im}(H(F^p C) \longrightarrow H(C)) \subseteq H(C) = H,$$

where $H(F^p C) \longrightarrow H(C)$ is induced by the inclusion map $F^p C \hookrightarrow C$.

Baby Example: 2 step filtration on a length 1 complex

$$
\begin{tikzcd}
C \arrow[d, phantom, "\shortparallel"] & & & & & & \\
F^0C \arrow[d, phantom, "\cup\!\!\mid"] & \cdots \arrow[r] & 0 \arrow[r] & C^0 \arrow[r, "d^0"] \arrow[d, phantom, "\cup\!\!\mid"] & C^1 \arrow[r] \arrow[d, phantom, "\cup\!\!\mid"] & 0 \arrow[r] & \cdots \\
F^1C \arrow[d, phantom, "\cup\!\!\mid"] & \cdots \arrow[r] & 0 \arrow[r] & F^1C^0 \arrow[r, "d^0|_{F^1C^0}"] \arrow[d, phantom, "\cup\!\!\mid"] & F^1C^1 \arrow[r] \arrow[d, phantom, "\cup\!\!\mid"] & 0 \arrow[r] & \cdots \\
F^2C \arrow[d, phantom, "\shortparallel"] & \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \\
0 & & & & & &
\end{tikzcd}
$$

Baby Example: filtration induced in cohomology

We first look at the map induced by $F^1C \hookrightarrow C$ in cohomology:

$$\begin{array}{ccc}
H(F^1C) & = & \ker\left(d^0|_{F^1C}\right)[0] \oplus \left(\frac{F^1C^1}{\mathrm{im}\left(d^0|_{F^1C}\right)}\right)[-1] \\
\downarrow & & \downarrow \\
H(C) & = & \ker(d^0)[0] \oplus \left(\frac{C^1}{\mathrm{im}(d^0)}\right)[-1]
\end{array}$$

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The filtration on cohomology was given by its image, hence

$$F^1H = \ker\left(d^0|_{F^1C}\right)[0] \oplus \left(\frac{F^1C^1 + \mathrm{im}(d^0)}{\mathrm{im}(d^0)}\right)[-1]$$

Baby Example: graded cohomology pieces

Recall that $\mathrm{gr}^0 H := F^0 H / F^1 H = H / F^1 H$, hence (after $\cong$-theorem)

$$\mathrm{gr}^0 H = \left(\frac{\ker(d^0)}{\ker\left(d^0|_{F^1 C}\right)} \right) [0] \oplus \left(\frac{C^1}{F^1 C^1 + \mathrm{im}(d^0)} \right) [-1]$$

Baby Example: graded cohomology pieces

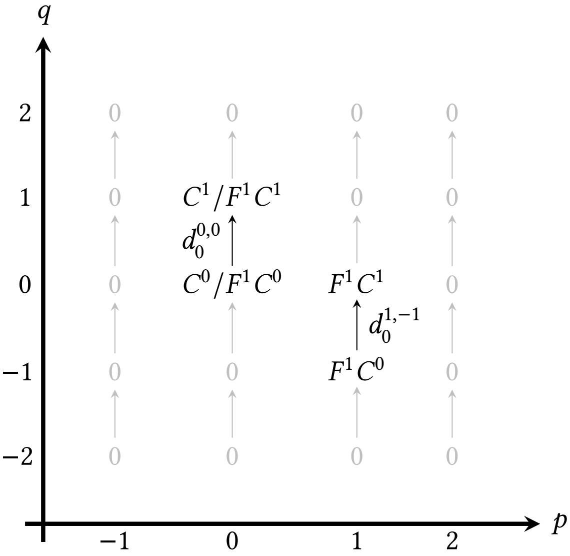
Recall that $\mathrm{gr}^0 H := F^0H/F^1H = H/F^1H$, hence (after $\cong$-theorem)

$$\mathrm{gr}^0 H = \left(\frac{\ker(d^0)}{\ker\left(d^0|_{F^1C}\right)}\right)[0] \oplus \left(\frac{C^1}{F^1C^1 + \mathrm{im}(d^0)}\right)[-1]$$

Similarly, $\mathrm{gr}^1 H = F^1H/0 = F^1H$, thus

$$\mathrm{gr}^1 H = \ker\left(d^0|_{F^1C}\right)[0] \oplus \left(\frac{F^1C^1 + \mathrm{im}(d^0)}{\mathrm{im}(d^0)}\right)[-1]$$

Baby Example: define $E_0^{p,q} := (\mathrm{gr}^p\, C)^{p+q}$ and visualize in $\mathbb{Z}^2$



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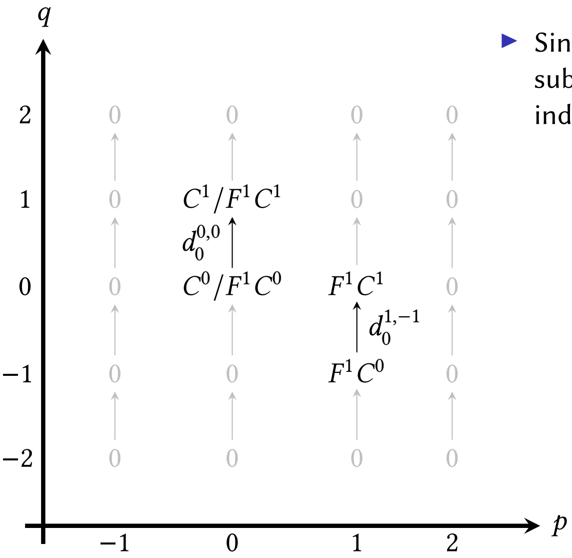

- Since $F^1C \subseteq C$ is a subcomplex, $d^0\colon C^0 \to C^1$ induces the differential $d_0^{0,0}$.

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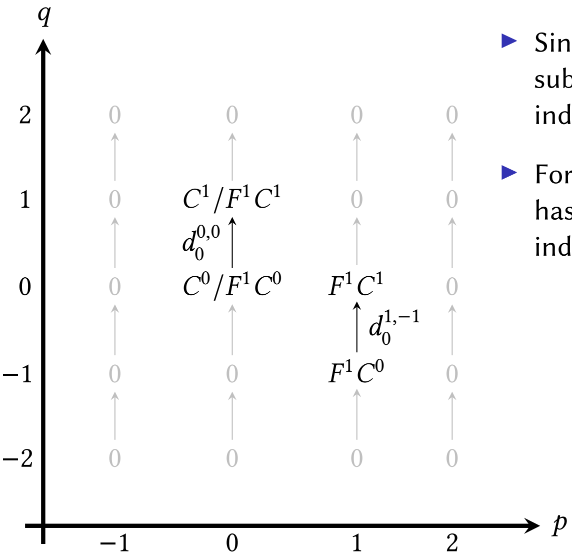

- Since $F^1C \subseteq C$ is a subcomplex, $d^0 : C^0 \to C^1$ induces the differential $d_0^{0,0}$.
- For the same reason, $d^0|_{F^1C}$ has image in F^1C^1, so it induces the differential $d_0^{1,-1}$.

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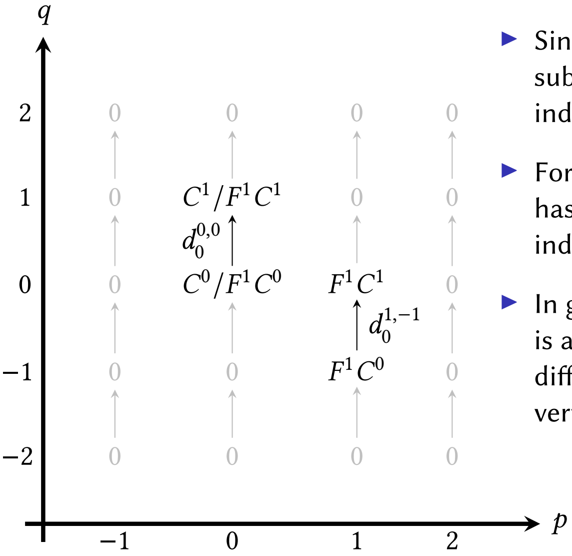

- Since $F^1C \subseteq C$ is a subcomplex, $d^0 : C^0 \to C^1$ induces the differential $d_0^{0,0}$.
- For the same reason, $d^0|_{F^1C}$ has image in F^1C^1, so it induces the differential $d_0^{1,-1}$.
- In general, since $F^{p+1}C \subseteq F^pC$ is a subcomplex, the original differentials induce the vertical differentials.

BABY EXAMPLE: compute cohomology of the columns $\mathrm{gr}^p\, C$

From the $p = 0$ column in the "E_0-page" we compute

$$H(\mathrm{gr}^0\, C) = \left(\frac{\ker(d^0) + (d^0)^{-1}(F^1C^1)}{F^1C^0}\right)[0] \oplus \left(\frac{C^1}{F^1C^1 + \mathrm{im}(d^0)}\right)[-1]$$

Baby Example: compute cohomology of the columns $\mathrm{gr}^p\, C$

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And from the $p = 1$ column in the "E_0-page" we compute

$$H(\mathrm{gr}^1\, C) = \ker\left(d^0|_{F^1C}\right)[0] \oplus \left(\frac{F^1C^1}{\mathrm{im}(d^0|_{F^1C})} \right)[-1]$$

Baby Example: 1^{st} approximation to the $\mathrm{gr}^0 H$ part

We compare $\mathrm{gr}^0 H$ (left column) to $H(\mathrm{gr}^0 C)$ (right column):

$$\left(\frac{\ker(d^0)}{\ker(d^0|_{F^1 C})}\right)[0] \qquad \left(\frac{\ker(d^0)+(d^0)^{-1}(F^1C^1)}{F^1C^0}\right)[0]$$

$$\oplus \qquad\qquad\qquad\qquad \oplus$$

$$\left(\frac{C^1}{F^1C^1+\mathrm{im}(d^0)}\right)[-1] \qquad \left(\frac{C^1}{F^1C^1+\mathrm{im}(d^0)}\right)[-1]$$

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- $H^1(\mathrm{gr}^0 C)$ does compute $\mathrm{gr}^0 H^1$ (bottom row),

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- $H^1(\operatorname{gr}^0 C)$ does compute $\operatorname{gr}^0 H^1$ (bottom row),
- but $H^0(\operatorname{gr}^0 C)$ is not isomorphic to $\operatorname{gr}^0 H^0$ (top row).

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We compare now $\mathrm{gr}^1 H$ (left column) to $H(\mathrm{gr}^1 C)$ (right column):

$$\ker\left(d^0|_{F^1C}\right)[0] \qquad\qquad \ker\left(d^0|_{F^1C}\right)[0]$$

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$$\left(\frac{F^1C^1+\mathrm{im}(d^0)}{\mathrm{im}(d^0)}\right)[-1] \qquad\qquad \left(\frac{F^1C^1}{\mathrm{im}\left(d^0|_{F^1C}\right)}\right)[-1]$$

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- In this case $H^0(\operatorname{gr}^1 C)$ does compute $\operatorname{gr}^1 H^0$ (top row),
- but $H^1(\operatorname{gr}^1 C)$ is not isomorphic to $\operatorname{gr}^1 H^1$ (bottom row).

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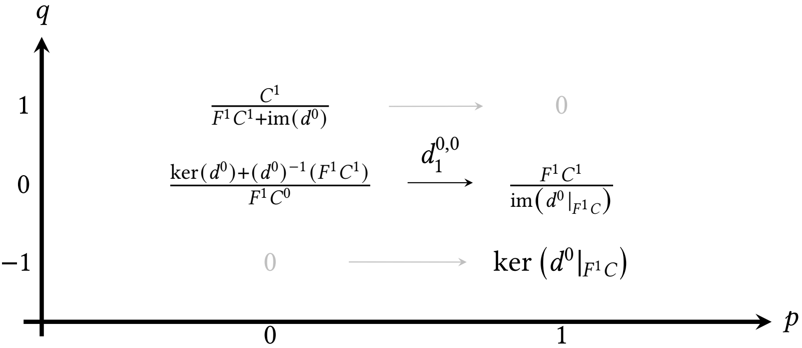

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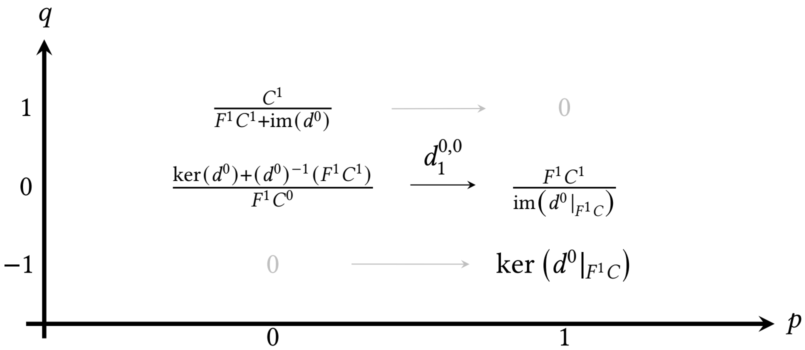

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q & & \\
1 & \frac{C^1}{F^1C^1+\mathrm{im}(d^0)} \arrow[r] & 0 \\
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-1 & 0 \arrow[r] & \ker\left(d^0|_{F^1C}\right) \\
 & 0 & 1 & p
\end{tikzcd}
$$

- The original differential d^0 induces the differential $d_1^{0,0}$.
- More generally, if we had started from a longer complex, the original differentials would induce the horizontal differentials.

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- We obtained the 1^{st} approximation by taking cohomologies on the "E_0-page".

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- We obtained the 1^{st} approximation by taking cohomologies on the "E_0-page".
- These 1^{st} approximations were then organized into the "E_1-page" in such a way that $\operatorname{gr}^p H^n$ would correspond to $E_1^{p,n-p}$.

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  - $\operatorname{gr}^1 H^0 = E_1^{1,-1}$, although $\operatorname{gr}^1 H^1 \neq E_1^{1,0}$.
- Continuing with this way to arrange things, we define the "E_2-page" by taking cohomologies at each point of the "E_1-page", that is

$$E_2^{p,q} := H^p(E_1^{\bullet,q}),$$

so that $\operatorname{gr}^p H^n$ would again correspond to $E_2^{p,n-p}$.

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We compare then $\mathrm{gr}^0 H$ (left column) to $E_2^{0,\bullet}$ (right column):

$$\left(\frac{\ker(d^0)}{\ker(d^0|_{F^1C})}\right)[0] \qquad\qquad \left(\frac{\ker(d^0)+F^1C^0}{F^1C^0}\right)[0]$$

$$\oplus \qquad\qquad\qquad\qquad \oplus$$

$$\left(\frac{C^1}{F^1C^1+\mathrm{im}(d^0)}\right)[-1] \qquad\qquad \left(\frac{C^1}{F^1C^1+\mathrm{im}(d^0)}\right)[-1]$$

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After applying an $\cong$-theorem we see that the two agree!

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We compare now $\operatorname{gr}^1 H$ (left column) to $E_2^{1,\bullet-1}$ (right column):

$\ker\left(d^0\vert_{F^1C}\right)[0]$	$\ker\left(d^0\vert_{F^1C}\right)[0]$
$\oplus$	$\oplus$
$\left(\frac{F^1C^1+\operatorname{im}(d^0)}{\operatorname{im}(d^0)}\right)[-1]$	$\left(\frac{F^1C^1}{\operatorname{im}(d^0)\cap F^1C^1}\right)[-1]$

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After applying an $\cong$-theorem we see that the two sides agree again!

Baby Example: reading off the result from the "E_2-page"

$$
\begin{array}{c|cc}
q & & \\
1 & \dfrac{C^1}{F^1C^1 + \mathrm{im}(d^0)} & 0 \\[2ex]
0 & \dfrac{\ker(d^0)}{\ker\left(d^0|_{F^1C}\right)} & \dfrac{F^1C^1 + \mathrm{im}(d^0)}{\mathrm{im}(d^0)} \\[2ex]
-1 & 0 & \ker\left(d^0|_{F^1C}\right) \\
\hline
 & 0 & 1 \quad p
\end{array}
$$

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q \ p	0	1
1	$\frac{C^1}{F^1C^1+\mathrm{im}(d^0)}$	0
0	$\frac{\ker(d^0)}{\ker\left(d^0\vert_{F^1C}\right)}$	$\frac{F^1C^1+\mathrm{im}(d^0)}{\mathrm{im}(d^0)}$
-1	0	$\ker\left(d^0\vert_{F^1C}\right)$

- $\mathrm{gr}^0 H \cong E_2^{0,0}[0] \oplus E_2^{0,1}[-1]$;

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q		
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-1	0	$\ker\left(d^0\vert_{F^1C}\right)$
	0	1 p

- $\mathrm{gr}^0 H \cong E_2^{0,0}[0] \oplus E_2^{0,1}[-1]$;
- $\mathrm{gr}^1 H \cong E_2^{1,-1}[0] \oplus E_2^{1,0}[-1]$.

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-1 & 0 & \ker\left(d^0|_{F^1C}\right) \\
\hline
 & 0 & 1 \quad p
\end{array}
$$

- $\mathrm{gr}^0 H \cong E_2^{0,0}[0] \oplus E_2^{0,1}[-1]$;
- $\mathrm{gr}^1 H \cong E_2^{1,-1}[0] \oplus E_2^{1,0}[-1]$.

$$
\begin{cases}
H^0 \cong \mathrm{gr}^0 H^0 \oplus \mathrm{gr}^1 H^0 \cong E_2^{0,0} \oplus E_2^{1,-1} \\
H^1 \cong \mathrm{gr}^1 H^0 \oplus \mathrm{gr}^1 H^1 \cong E_2^{0,1} \oplus E_2^{1,0}
\end{cases}
$$

Definition [Sta20, Tag 011N]

A *spectral sequence* in an abelian category $\mathcal{A}$ is given by a system $(E_r, d_r)_{r \in \mathbb{N}}$ such that for all $r \in \mathbb{N}$ we have:

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- E_r is an object of $\mathcal{A}$;
- $d_r \colon E_r \to E_r$ is a morphism such that $d_r \circ d_r = 0$;

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A *spectral sequence* in an abelian category $\mathcal{A}$ is given by a system $(E_r, d_r)_{r\in\mathbb{N}}$ such that for all $r \in \mathbb{N}$ we have:

- E_r is an object of $\mathcal{A}$;
- $d_r \colon E_r \longrightarrow E_r$ is a morphism such that $d_r \circ d_r = 0$;
- and $E_{r+1} = \ker(d_r)/\mathrm{im}(d_r)$.

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- $E_0^{p,q} = \operatorname{gr}^p C^{p+q}$,
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- $E_r^{p,q} = (F^p C^{p+q} \cap d^{-1}(F^{p+r}C^{p+q+1}))/(F^p C^{p+q} \cap d(F^{p-r}C^{p+q-1}))$,
- $d_r^{p,q}\colon E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}$ induced by $d^{p+q}\colon F^p C^{p+q} \longrightarrow F^{p+r} C^{p+r+q-r+1}$.

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If the filtration induced on each C^n was finite, then letting $r \longrightarrow \infty$ we would obtain the *limit* E_∞ of the spectral sequence, given by

$$E_\infty^{p,q} = (F^p C^{p+q} \cap \ker(d))/(F^p C^{p+q} \cap \mathrm{im}(d)).$$

Convergence: $E_r^{p,q} \Rightarrow H^{p+q}(C)$ [McC01, Theorem 2.6]

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Let $F^\bullet C$ be a filtered complex in an abelian category $\mathcal{A}$ and assume that the filtration induced on each C^n is finite. Then:

- The induced filtration $F^\bullet H$ is finite.
- For each $p, q \in \mathbb{Z}$ there exists $r_0 \in \mathbb{N}$ such that

$$E_\infty^{p,q} = E_r^{p,q} \text{ for all } r \geqslant r_0.$$

- The spectral sequence *converges* to $F^\bullet H$, i.e.

$$E_\infty^{p,q} \cong \operatorname{gr}^p H^{p+q} \text{ for all } p, q \in \mathbb{Z}.$$

In particular, modulo extension problems (e.g. if we are working with vector spaces), we can recover H as

$$H^n(C) \cong \bigoplus_{p \in \mathbb{Z}} E_\infty^{p,n-p} \text{ for all } n \in \mathbb{Z}.$$

Applications of spectral sequences from filtrations

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Applications of spectral sequences from filtrations

- Cellular homology and singular homology of a CW complex agree [McC01, Theorem 4.13].
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- Two different ways to compute the cohomology of the total complex of a double complex.

To avoid getting too much into specific topics, we will focus on the third application. We will do this again with an example.

Snake Lemma: statement

Consider a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r] & 0 \\ 0 \arrow[r] & A \arrow[r, "f"] \arrow[u, "a"] & B \arrow[r, "g"] \arrow[u, "b"] & C \arrow[r] \arrow[u, "c"] & 0 \end{tikzcd}$$

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in which both rows are exact. Then there is an exact sequence

$$0 \to \ker a \to \ker b \to \ker c \to \operatorname{coker} a \to \operatorname{coker} b \to \operatorname{coker} b \to 0.$$

Snake Lemma: double complex and its total complex

We relabel the diagram to see it as a double complex

$$\begin{tikzcd}
0 \arrow[r] & C^{0,1} \arrow[r, "d_h^{0,1}"] & C^{1,1} \arrow[r, "d_h^{1,1}"] & C^{2,1} \arrow[r] & 0 \\
0 \arrow[r] & C^{0,0} \arrow[r, "d_h^{0,0}"] \arrow[u, "d_v^{0,0}"] & C^{1,0} \arrow[r, "d_h^{1,0}"] \arrow[u, "d_v^{1,0}"] & C^{2,0} \arrow[r] \arrow[u, "d_v^{2,0}"] & 0
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\end{tikzcd}$$

whose total complex sC is then

$$\cdots \to 0 \to C^{0,0} \to C^{0,1} \oplus C^{1,0} \to C^{1,1} \oplus C^{2,0} \to C^{2,1} \to 0 \to \cdots$$

with differentials

$$d_{sC}^n := \begin{pmatrix} d_h^{d_h^{n-1,1}} & (-1)^n d_v^{n,0} \\ 0 & d_h^{n,0} \end{pmatrix}.$$

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The first filtration $F_I^\bullet sC$ is defined by

$$F_I^p sC^n := \bigoplus_{i+j=n, i \geqslant p} C^{i,j}.$$

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$$E_1^{p,q} = H^q(C^{p,\bullet}) \Rightarrow H^{p+q}(sC).$$

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Right now we cannot immediately say much more, but we will come back to this soon!

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The second filtration $F^{\bullet}_{II} sC$ is defined by

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Since the rows $C^{\bullet,p}$ were exact, $E_1^{p,q} = 0$ for all $p, q \in \mathbb{Z}$.
Therefore $H(sC)$ must be zero!

Snake Lemma: back to the first spectral sequence

Using what we deduced from the second spectral sequence, we have now

$$E_1^{p,q} = H^q(C^{p,\bullet}) \Rightarrow 0.$$

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Let's draw the "E_1-page":

$$\begin{array}{c|ccccccccc}
q & & & & & & & & & \\
1 & 0 & \longrightarrow & \operatorname{coker}(d_v^{0,0}) & \longrightarrow & \operatorname{coker}(d_v^{1,0}) & \longrightarrow & \operatorname{coker}(d_v^{2,0}) & \longrightarrow & 0 \\
0 & 0 & \longrightarrow & \ker(d_v^{0,0}) & \longrightarrow & \ker(d_v^{1,0}) & \longrightarrow & \ker(d_v^{2,0}) & \longrightarrow & 0 \\
\hline
 & -1 & & 0 & & 1 & & 2 & & 3 \quad p
\end{array}$$

To show that the top and bottom rows are right and left exact respectively, we would like to have 0's at the corresopnding E_2 spots.

Snake Lemma: $E_2^{p,q}$ of the first spectral sequence

The "E_2-page" contains only one non-trivial differential:

$$
\begin{tikzcd}
q \\
1 & 0 & E_2^{0,1} \arrow[drr] & E_\infty^{1,1} & E_\infty^{2,1} & 0 \\
0 & 0 & E_\infty^{0,0} & E_\infty^{1,0} & E_2^{2,0} & 0 \\
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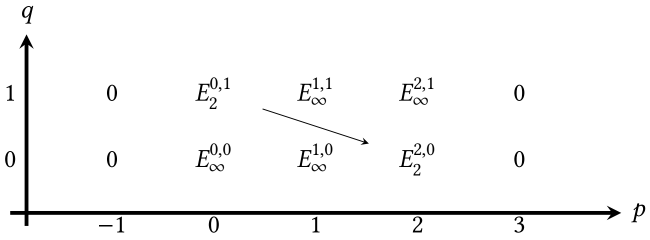

- Since $E_r^{p,q} \Rightarrow 0$, all limit objects must be zero!
- So the top and bottom row in the previous page were right and left exact respectively, as we wanted.
- $E_3 = E_\infty = 0$, so $E_2^{0,1} \longrightarrow E_2^{2,0}$ is an isomorphism and we can glue

$$\ker(d_v^{1,0}) \longrightarrow \ker(d_v^{2,0}) \twoheadrightarrow E_2^{2,0} \cong E_2^{0,0} \hookrightarrow \operatorname{coker}(d_v^{0,0}) \longrightarrow \operatorname{coker}(d_v^{1,0}).$$

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