Basic Notions — Spectral Sequences

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- **Example:** let k be a field and consider a cochain complex $C^\bullet$ of finite dimensional k-vector spaces

$$\cdots \to 0 \to C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} C^2 \to 0 \to \cdots$$

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- **Goal:** compute the cohomology $H^\bullet$ of the complex $C^\bullet$

$$H^\bullet := H^0(C^\bullet) \oplus H^1(C^\bullet) \oplus H^2(C^\bullet).$$

Starting point

- If we do not have any extra information about $C^\bullet$, there is not much more we can do besides applying the definition

$$H^\bullet = \ker(d^0) \oplus \frac{\ker(d^1)}{\mathrm{im}(d^0)} \oplus \frac{C^2}{\mathrm{im}(d^1)}.$$

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- Luckily, $C^\bullet$ has often a natural filtration, for example

$$\begin{array}{ccccccc}
C^\bullet = F^0 C^\bullet & \qquad & C^0 & \xrightarrow{d^0} & C^1 & \xrightarrow{d^1} & C^2 \\
\cup\!\text{l} & & \cup\!\text{l} & & \cup\!\text{l} & & \cup\!\text{l} \\
F^1 C^\bullet & & F^1 C^0 & \longrightarrow & F^1 C^1 & \longrightarrow & F^1 C^2 \\
\cup\!\text{l} & & \cup\!\text{l} & & \cup\!\text{l} & & \cup\!\text{l} \\
F^2 C^\bullet & & F^2 C^0 & \longrightarrow & F^2 C^1 & \longrightarrow & F^2 C^2 \\
\cup\!\text{l} & & \cup\!\text{l} & & \cup\!\text{l} & & \cup\!\text{l} \\
0 = F^3 C^\bullet & & 0 & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

References

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