Basic Notions — Spectral Sequences

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"You could have invented spectral sequences"[1]

[1] Title of the expository article [Cho06]

Notation gets ugly very soon

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- Cohomology is regarded as a copmlex with 0 differentials

$$H(C) = \bigoplus_{i\in\mathbb{Z}} H^i(C)[-i] := \left(\cdots H^{-1}(C) \xrightarrow{0} H^0(C) \xrightarrow{0} H^1(C) \cdots\right).$$

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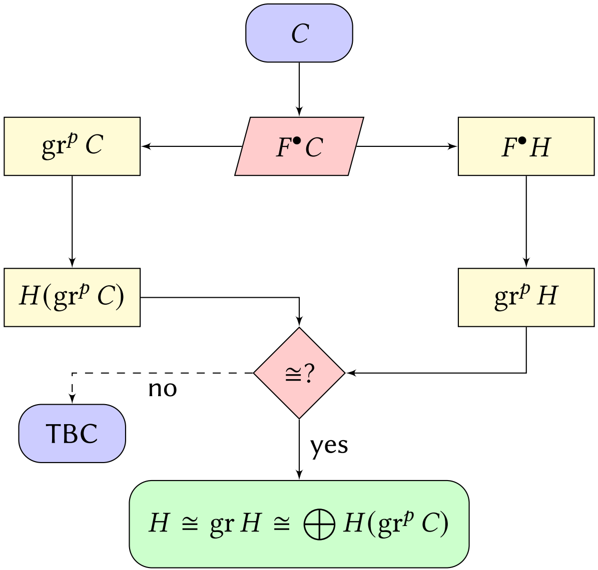
- The graded object associated to a filtration $F^\bullet A$ is denoted

$$\operatorname{gr} A = \bigoplus_{p\in\mathbb{N}} \operatorname{gr}^p A := \bigoplus_{p\in\mathbb{N}} \left(F^p A / F^{p+1} A\right).$$

Goal: compute cohomology with the help of a filtration



Filtration induced in cohomology

A filtration on the complx $F^\bullet C$ induces a filtration in cohomology

$$F^p H := \operatorname{im}(H(F^p C) \to H(C)) \subseteq H(C) = H,$$

where $H(F^p C) \to H(C)$ is induced by the inclusion map $F^p C \hookrightarrow C$.

Baby Example: 2 step filtration on a length 1 complex

$$
\begin{array}{ccccccccccc}
C \\
\parallel \\
F^0C & \cdots \longrightarrow & 0 & \longrightarrow & C^0 & \xrightarrow{\quad d^0 \quad} & C^1 & \longrightarrow & 0 & \longrightarrow & \cdots \\
\cup\!| & & & & \cup\!| & & \cup\!| \\
F^1C & \cdots \longrightarrow & 0 & \longrightarrow & F^1C^0 & \xrightarrow{d^0|_{F^1C^0}} & F^1C^1 & \longrightarrow & 0 & \longrightarrow & \cdots \\
\cup\!| & & & & \cup\!| & & \cup\!| \\
F^2C & \cdots \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
\parallel \\
0
\end{array}
$$

Baby Example: filtration induced in cohomology

We first look at the map induced by $F^1C \hookrightarrow C$ in cohomology:

$$\begin{array}{ccc} H(F^1C) & = & \ker\left(d^0|_{F^1C}\right)[0] \oplus \left(\frac{F^1C^1}{\mathrm{im}\left(d^0|_{F^1C}\right)}\right)[-1] \\ \downarrow & & \downarrow \\ H(C) & = & \ker(d^0)[0] \oplus \left(\frac{C^1}{\mathrm{im}(d^0)}\right)[-1] \end{array}$$

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\end{array}$$

The filtration on cohomology was given by its image, hence

$$F^1H = \ker\left(d^0|_{F^1C}\right)[0] \oplus \left(\frac{F^1C^1 + \mathrm{im}(d^0)}{\mathrm{im}(d^0)}\right)[-1]$$

Baby Example: graded cohomology pieces

Recall that $\mathrm{gr}^0 H := F^0 H / F^1 H = H / F^1 H$, hence (after $\cong$-theorem)

$$\mathrm{gr}^0 H = \left(\frac{\ker(d^0)}{\ker\left(d^0|_{F^1 C}\right)} \right) [0] \oplus \left(\frac{C^1}{F^1 C^1 + \mathrm{im}(d^0)} \right) [-1]$$

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Similarly, $\mathrm{gr}^1 H = F^1 H / 0 = F^1 H$, thus

$$\mathrm{gr}^1 H = \ker\left(d^0|_{F^1 C}\right) [0] \oplus \left(\frac{F^1 C^1 + \mathrm{im}(d^0)}{\mathrm{im}(d^0)} \right) [-1]$$

Baby Example: define $E_0^{p,q} := (\mathrm{gr}^p\, C)^{p+q}$ and visualize in $\mathbb{Z}^2$

$$
\begin{tikzcd}
q & & & & \\
2 & 0 & 0 & 0 & 0 \\
1 & 0 \arrow[u] & C^1/F^1C^1 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] \\
0 & 0 \arrow[u] & C^0/F^1C^0 \arrow[u, "d_0^{0,0}"] & F^1C^1 \arrow[u] & 0 \arrow[u] \\
-1 & 0 \arrow[u] & 0 \arrow[u] & F^1C^0 \arrow[u, "d_0^{1,-1}"'] & 0 \arrow[u] \\
-2 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] \\
 & -1 & 0 & 1 & 2 & p
\end{tikzcd}
$$

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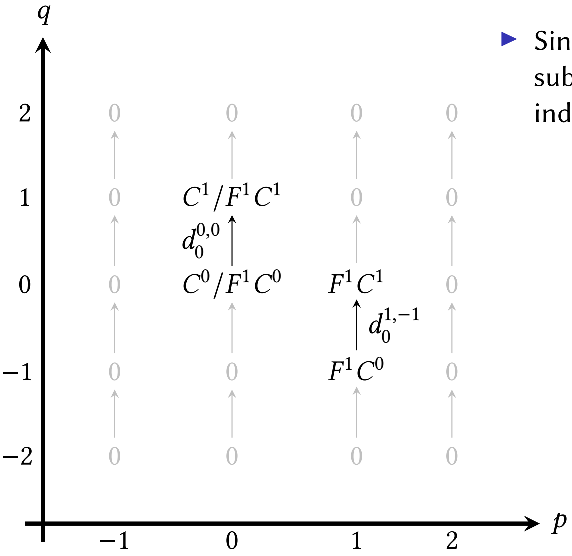

- Since $F^1C \subseteq C$ is a subcomplex, $d^0 : C^0 \to C^1$ induces the differential $d_0^{0,0}$.

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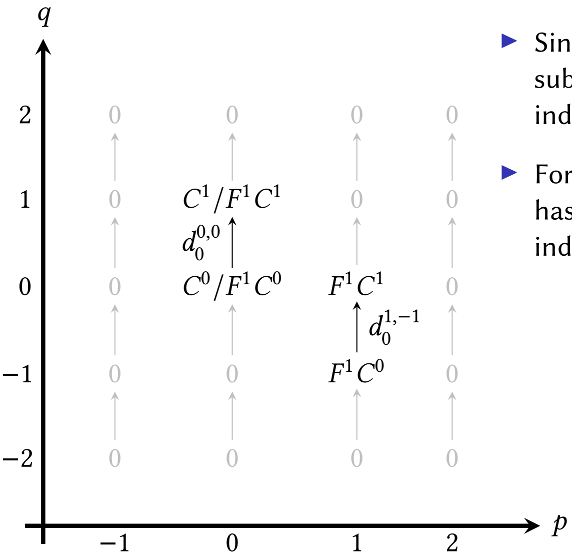

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- For the same reason, $d^0|_{F^1 C}$ has image in $F^1 C^1$, so it induces the differential $d_0^{1,-1}$.

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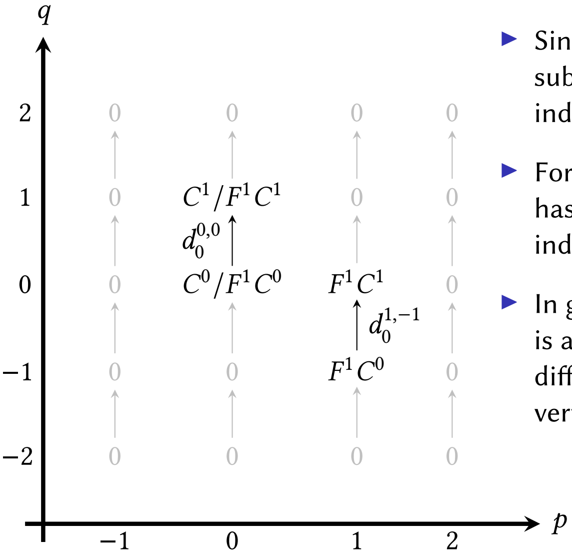

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- For the same reason, $d^0|_{F^1C}$ has image in F^1C^1, so it induces the differential $d_0^{1,-1}$.
- In general, since $F^{p+1}C \subseteq F^pC$ is a subcomplex, the original differentials induce the vertical differentials.

Baby Example: compute cohomology of the columns $\mathrm{gr}^p C$

From the $p = 0$ column in the "E_0-page" we compute

$$H(\mathrm{gr}^0 C) = \left(\frac{\ker(d^0) + (d^0)^{-1}(F^1 C)}{F^1 C^0}\right)[0] \oplus \left(\frac{C^1}{F^1 C^1 + \mathrm{im}(d^0)}\right)[-1]$$

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And from the $p = 1$ column in the "E_0-page" we compute

$$H(\mathrm{gr}^1 C) = \ker\left(d^0|_{F^1 C}\right)[0] \oplus \left(\frac{F^1 C^1}{\mathrm{im}(d^0|_{F^1 C})}\right)[-1]$$

References

Timothy Y. Chow.
You could have invented spectral sequences.
Notices Amer. Math. Soc., 53(1):15–19, 2006.
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The Stacks project authors.
The stacks project.
https://stacks.math.columbia.edu, 2020.
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