

Basic Notions — Spectral Sequences

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“You could have invented spectral sequences”¹



¹Title of the expository article [Cho06]

Notation gets ugly very soon



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$$H(C) = \bigoplus_{i \in \mathbb{Z}} H^i(C)[-i] := \left(\cdots H^{-1}(C) \xrightarrow{0} H^0(C) \xrightarrow{0} H^1(C) \cdots \right).$$

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- ▶ If it is clear to which C we refer, denote $H(C)$ simply by H .
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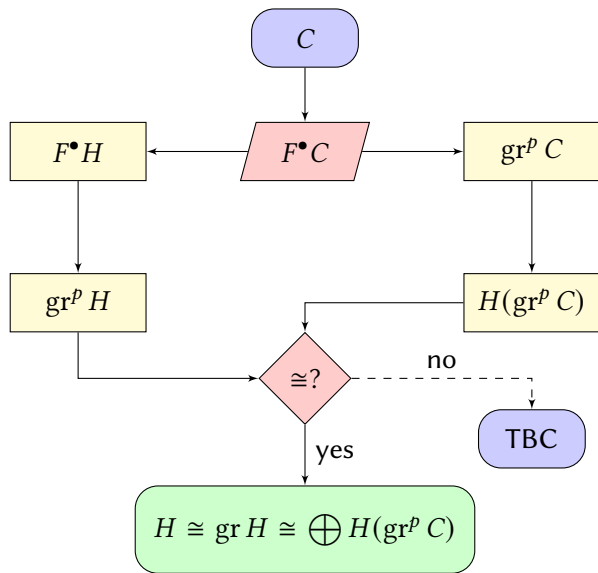
- ▶ If it is clear to which C we refer, denote $H(C)$ simply by H .
- ▶ (Decreasing) filtrations are always finite for simplicity:

$$F^\bullet A: \quad 0 = F^n A \subseteq \cdots \subseteq F^0 A = A.$$

- ▶ The graded object associated to a filtration $F^\bullet A$ is denoted

$$\mathrm{gr} A = \bigoplus_{p \in \mathbb{N}} \mathrm{gr}^p A := \bigoplus_{p \in \mathbb{N}} (F^p A / F^{p+1} A).$$

GOAL: compute cohomology with the help of a filtration



Filtration induced in cohomology

A filtration on the complex $F^\bullet C$ induces a filtration in cohomology

$$F^p H := \operatorname{im}(H(F^p C) \rightarrow H(C)) \subseteq H(C) = H,$$

where $H(F^p C) \rightarrow H(C)$ is induced by the inclusion map $F^p C \hookrightarrow C$.

BABY EXAMPLE: 2 step filtration on a length 1 complex

$$\begin{array}{ccccccc}
 C & & & & & & \\
 \parallel & & & & & & \\
 F^0 C & \cdots \rightarrow 0 \rightarrow C^0 \xrightarrow{d^0} C^1 \rightarrow 0 \rightarrow \cdots \\
 \cup \mid & & \cup \mid & & \cup \mid & & \\
 F^1 C & \cdots \rightarrow 0 \rightarrow F^1 C^0 \xrightarrow{d^0|_{F^1 C^0}} F^1 C^1 \rightarrow 0 \rightarrow \cdots \\
 \cup \mid & & \cup \mid & & \cup \mid & & \\
 F^2 C & \cdots \rightarrow 0 \rightarrow 0 \longrightarrow 0 \longrightarrow 0 \rightarrow \cdots \\
 \parallel & & & & & & \\
 0 & & & & & &
 \end{array}$$

BABY EXAMPLE: filtration induced in cohomology

We first look at the map induced by $F^1 C \hookrightarrow C$ in cohomology:

$$\begin{array}{ccc} H(F^1 C) & = & \ker(d^0|_{F^1 C}) [0] \oplus \left(\frac{F^1 C^1}{\operatorname{im}(d^0|_{F^1 C})} \right) [-1] \\ \downarrow & & \downarrow \\ H(C) & = & \ker(d^0) [0] \oplus \left(\frac{C^1}{\operatorname{im}(d^0)} \right) [-1] \end{array}$$

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The filtration on cohomology was given by its image, hence

$$F^1 H = \ker(d^0|_{F^1 C}) [0] \oplus \left(\frac{F^1 C^1 + \operatorname{im}(d^0)}{\operatorname{im}(d^0)} \right) [-1]$$

BABY EXAMPLE: graded cohomology pieces

Recall that $\mathrm{gr}^0 H := F^0 H / F^1 H = H / F^1 H$, hence (after \cong -theorem)

$$\mathrm{gr}^0 H = \left(\frac{\ker(d^0)}{\ker(d^0|_{F^1 C})} \right) [0] \oplus \left(\frac{C^1}{F^1 C^1 + \mathrm{im}(d^0)} \right) [-1]$$

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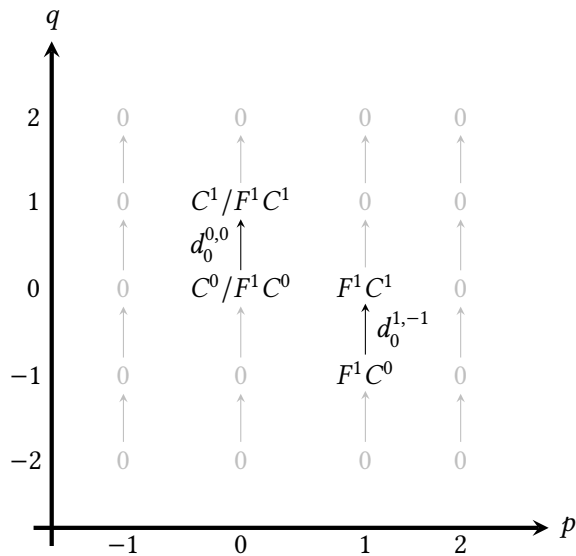
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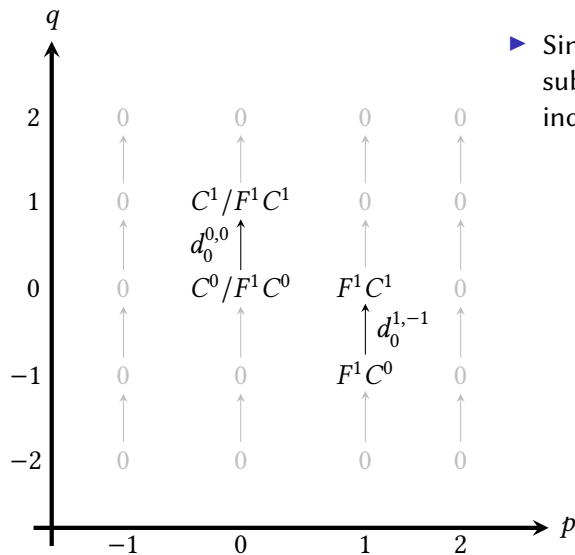
Similarly, $\mathrm{gr}^1 H = F^1 H / 0 = F^1 H$, thus

$$\mathrm{gr}^1 H = \ker(d^0|_{F^1 C}) [0] \oplus \left(\frac{F^1 C^1 + \mathrm{im}(d^0)}{\mathrm{im}(d^0)} \right) [-1]$$

BABY EXAMPLE: define $E_0^{p,q} := (\operatorname{gr}^p C)^{p+q}$ and visualize in \mathbb{Z}^2

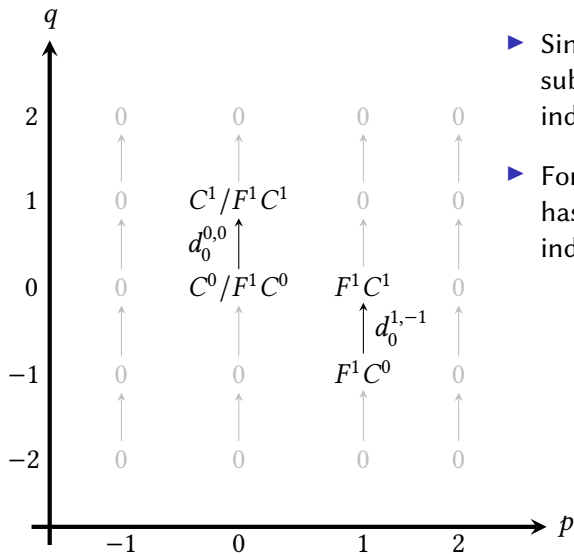


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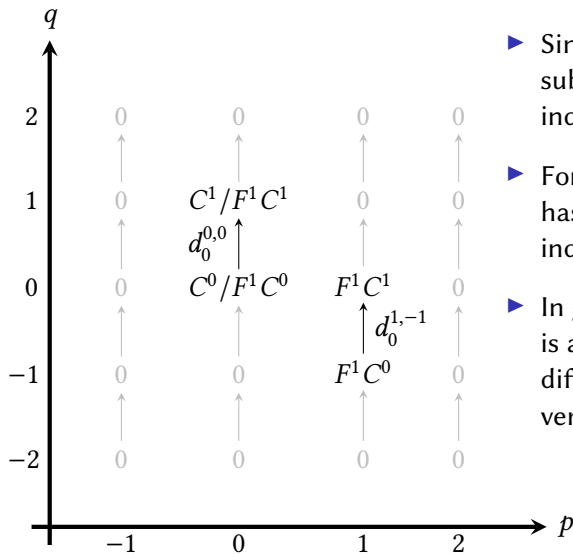
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- ▶ For the same reason, $d^0|_{F^1 C}$ has image in $F^1 C^1$, so it induces the differential $d_0^{1,-1}$.

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- ▶ For the same reason, $d^0|_{F^1 C}$ has image in $F^1 C^1$, so it induces the differential $d_0^{1,-1}$.
- ▶ In general, since $F^{p+1} C \subseteq F^p C$ is a subcomplex, the original differentials induce the vertical differentials.

BABY EXAMPLE: compute cohomology of the columns $\mathrm{gr}^p C$

From the $p = 0$ column in the “ E_0 -page” we compute

$$H(\mathrm{gr}^0 C) = \left(\frac{\ker(d^0) + (d^0)^{-1}(F^1 C^1)}{F^1 C^0} \right) [0] \oplus \left(\frac{C^1}{F^1 C^1 + \mathrm{im}(d^0)} \right) [-1]$$

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And from the $p = 1$ column in the “ E_0 -page” we compute

$$H(\text{gr}^1 C) = \ker(d^0|_{F^1 C}) [0] \oplus \left(\frac{F^1 C^1}{\text{im}(d^0|_{F^1 C})} \right) [-1]$$

BABY EXAMPLE: 1st approximation to the $\text{gr}^0 H$ part

We compare $\text{gr}^0 H$ (left column) to $H(\text{gr}^0 C)$ (right column):

$$\left(\frac{\ker(d^0)}{\ker(d^0|_{F^1 C})} \right) [0]$$

$$\oplus$$

$$\left(\frac{C^1}{F^1 C^1 + \text{im}(d^0)} \right) [-1]$$

$$\left(\frac{\ker(d^0) + (d^0)^{-1}(F^1 C^1)}{F^1 C^0} \right) [0]$$

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- $H^1(\mathrm{gr}^0 C)$ does compute $\mathrm{gr}^0 H^1$ (bottom row),

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- ▶ $H^1(\text{gr}^0 C)$ does compute $\text{gr}^0 H^1$ (bottom row),
- ▶ but $H^0(\text{gr}^0 C)$ is not isomorphic to $\text{gr}^0 H^0$ (top row).

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We compare now $\text{gr}^1 H$ (left column) to $H(\text{gr}^1 C)$ (right column):

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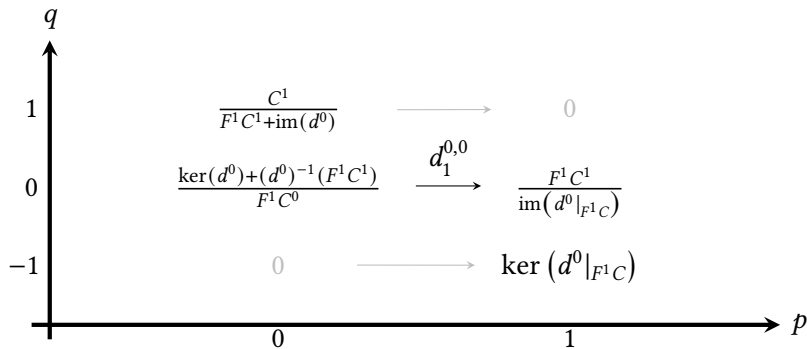
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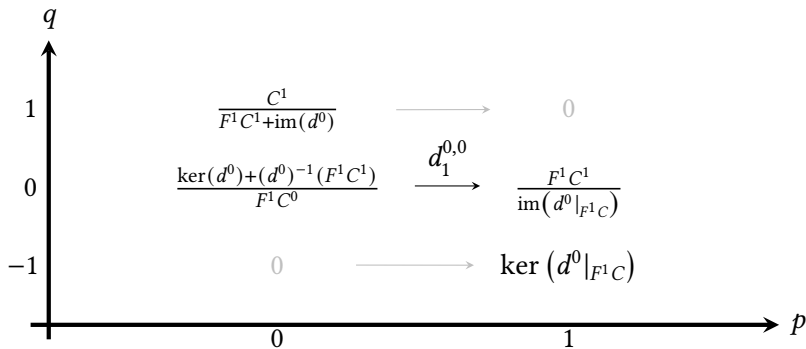
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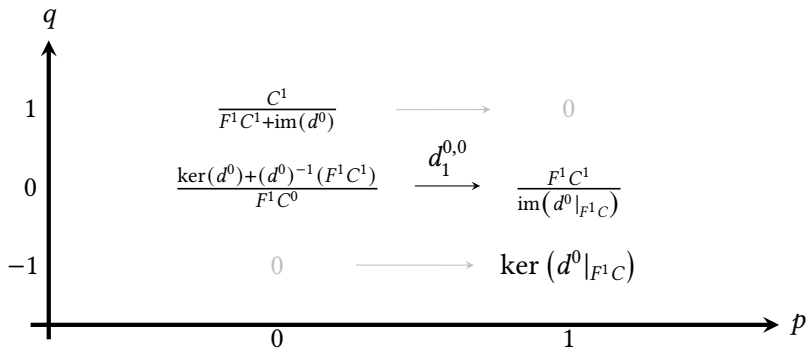


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- The original differential d^0 induces the differential $d_1^{0,0}$.

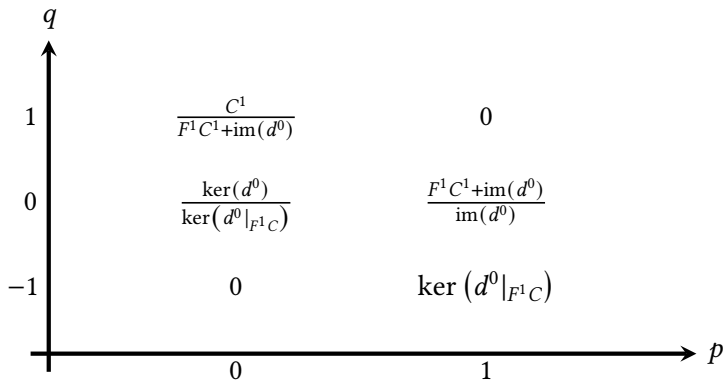
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- ▶ The original differential d^0 induces the differential $d_1^{0,0}$.
- ▶ More generally, if we had started from a longer complex, the original differentials would induce the horizontal differentials.

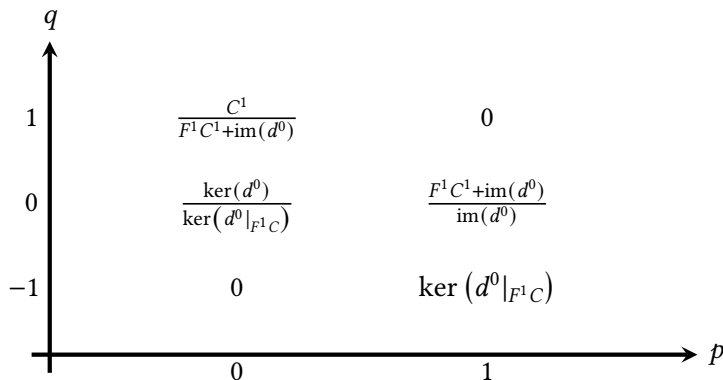
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We compare then $\text{gr}^0 H$ (left column) to $E_2^{0,\bullet}$ (right column):

$$\left(\frac{\ker(d^0)}{\ker(d^0|_{F^1 C})} \right) [0]$$

$$\oplus$$

$$\left(\frac{C^1}{F^1 C^1 + \text{im}(d^0)} \right) [-1]$$

$$\left(\frac{\ker(d^0) + F^1 C^0}{F^1 C^0} \right) [0]$$

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After applying an \cong -theorem we see that the two agree!

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We compare now $\text{gr}^1 H$ (left column) to $E_2^{1,\bullet-1}$ (right column):

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BABY EXAMPLE: reading off the result from the “ E_2 -page”

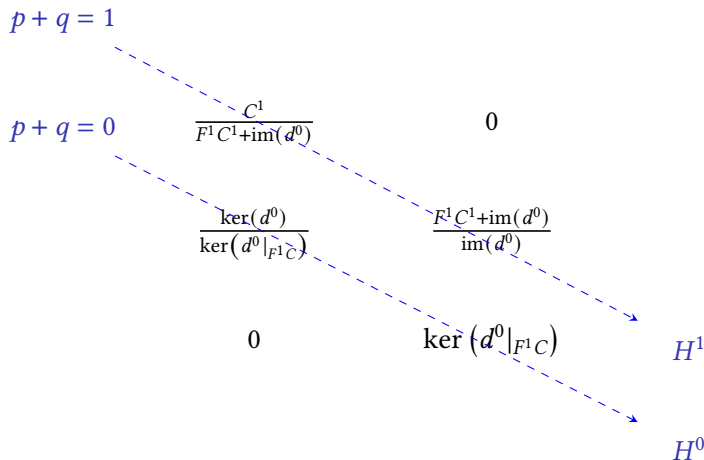
► $\mathrm{gr}^0 H \cong E_2^{0,0}[0] \oplus E_2^{0,1}[-1];$

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 - ▶ $\mathrm{gr}^1 H \cong E_2^{1,-1}[0] \oplus E_2^{1,0}[-1].$
- $$\begin{cases} H^0 \cong \mathrm{gr}^0 H^0 \oplus \mathrm{gr}^1 H^0 \cong E_2^{0,0} \oplus E_2^{1,-1} \\ H^1 \cong \mathrm{gr}^1 H^0 \oplus \mathrm{gr}^1 H^1 \cong E_2^{0,1} \oplus E_2^{1,0} \end{cases}$$



DEFINITION [Sta20, Tag 011N]

A *spectral sequence* in an abelian category \mathcal{A} is given by a system $(E_r, d_r)_{r \in \mathbb{N}}$ such that for all $r \in \mathbb{N}$ we have:

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- ▶ $d_r: E_r \rightarrow E_r$ is a morphism such that $d_r \circ d_r = 0$;
- ▶ and $E_{r+1} = \ker(d_r)/\operatorname{im}(d_r)$.

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- ▶ $E_0^{p,q} = \operatorname{gr}^p C^{p+q},$
- ▶ $d_0^{p,q} = \operatorname{gr}^p(d^{p+q}): E_0^{p,q} \rightarrow E_0^{p,q+1};$

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and the “ E_r -page” is given by

- ▶ $E_r^{p,q} = (F^p C^{p+q} \cap d^{-1}(F^{p+r} C^{p+q+1})) / (F^p C^{p+q} \cap d(F^{p-r} C^{p+q-1})),$
- ▶ $d_r^{p,q}: E_r^{p,q} \rightarrow E_r^{p+r, q-r+1}$ induced by $d^{p+q}: F^p C^{p+q} \rightarrow F^{p+r} C^{p+r+q-r+1}.$

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- ▶ $d_r^{p,q}: E_r^{p,q} \rightarrow E_r^{p+r, q-r+1}$ induced by
 $d^{p+q}: F^p C^{p+q} \rightarrow F^{p+r} C^{p+r+q-r+1}.$

If the filtration induced on each C^n was finite, then letting $r \rightarrow \infty$ we would obtain the *limit* E_∞ of the spectral sequence, given by

$$E_\infty^{p,q} = (F^p C^{p+q} \cap \ker(d)) / (F^p C^{p+q} \cap \operatorname{im}(d)).$$

CONVERGENCE: $E_r^{p,q} \Rightarrow H^{p+q}(C)$ [McC01, Theorem 2.6]

Let $F^\bullet C$ be a filtered complex in an abelian category \mathcal{A} and assume that the filtration induced on each C^n is finite. Then:

CONVERGENCE: $E_r^{p,q} \Rightarrow H^{p+q}(C)$ [McC01, Theorem 2.6]

Let $F^\bullet C$ be a filtered complex in an abelian category \mathcal{A} and assume that the filtration induced on each C^n is finite. Then:

- ▶ The induced filtration $F^\bullet H$ is finite.
- ▶ For each $p, q \in \mathbb{Z}$ there exists $r_0 \in \mathbb{N}$ such that

$$E_\infty^{p,q} = E_r^{p,q} \text{ for all } r \geq r_0.$$

- ▶ The spectral sequence *converges* to $F^\bullet H$, i.e.

$$E_\infty^{p,q} \cong \operatorname{gr}^p H^{p+q} \text{ for all } p, q \in \mathbb{Z}.$$

In particular, modulo extension problems (e.g. if we are working with vector spaces), we can recover H as

$$H^n(C) \cong \bigoplus_{p \in \mathbb{Z}} E_\infty^{p, n-p} \text{ for all } n \in \mathbb{Z}.$$

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- ▶ Cellular homology and singular homology of a CW complex agree [[McC01](#), Theorem 4.13].
- ▶ “Cheap Hodge decomposition” [[Voi02](#), Remark 8.29].
- ▶ Two different ways to compute the cohomology of the total complex of a double complex.

To avoid getting too much into specific topics, we will focus on the third application. We will do this again with an example.

SNAKE LEMMA: statement

Consider a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \longrightarrow & 0 \\ & & \uparrow a & & \uparrow b & & \uparrow c & & \\ 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \end{array}$$

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in which both rows are exact. Then there is an exact sequence

$$0 \rightarrow \ker a \rightarrow \ker b \rightarrow \ker c \rightarrow \operatorname{coker} a \rightarrow \operatorname{coker} b \rightarrow \operatorname{coker} c \rightarrow 0.$$

SNAKE LEMMA: double complex and its total complex

We relabel the diagram to see it as a double complex

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^{0,1} & \xrightarrow{d_h^{0,1}} & C^{1,1} & \xrightarrow{d_h^{1,1}} & C^{2,1} & \longrightarrow & 0 \\ & & \uparrow d_v^{0,0} & & \uparrow d_v^{1,0} & & \uparrow d_v^{2,0} & & \\ 0 & \longrightarrow & C^{0,0} & \xrightarrow{d_h^{0,0}} & C^{1,0} & \xrightarrow{d_h^{1,0}} & C^{2,0} & \longrightarrow & 0 \end{array}$$

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whose total complex sC is then

$$\cdots \rightarrow 0 \rightarrow C^{0,0} \rightarrow C^{0,1} \oplus C^{1,0} \rightarrow C^{1,1} \oplus C^{2,0} \rightarrow C^{2,1} \rightarrow 0 \rightarrow \cdots$$

with differentials

$$d_{sC}^n := \begin{pmatrix} d_h^{n-1,1} & (-1)^n d_v^{n,0} \\ 0 & d_h^{n,0} \end{pmatrix}.$$

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The first filtration $F_I^\bullet sC$ is defined by

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Right now we cannot immediately say much more, but we will come back to this soon!

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In this case we get the spectral sequence

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Since the rows $C^{\bullet, p}$ were exact, $E_1^{p,q} = 0$ for all $p, q \in \mathbb{Z}$.
Therefore $H(sC)$ must be zero!

SNAKE LEMMA: back to the first spectral sequence

Using what we deduced from the second spectral sequence, we have now

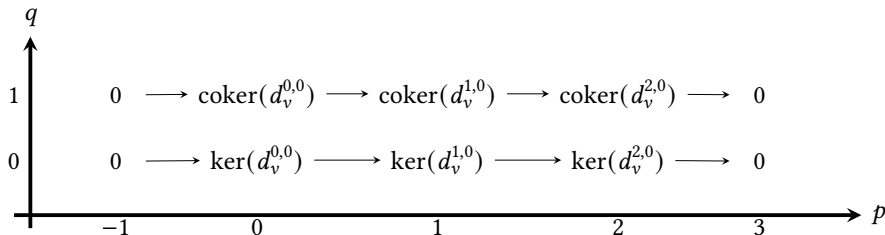
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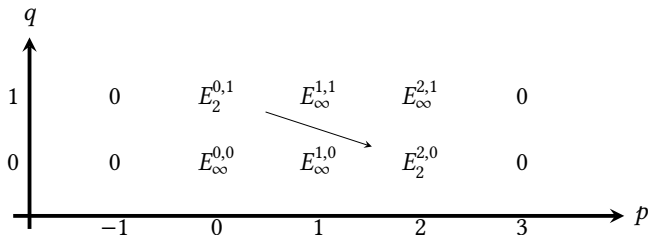
Let's draw the “ E_1 -page”:



To show that the top and bottom rows are right and left exact respectively, we would like to have 0's at the corresponding E_2 spots.

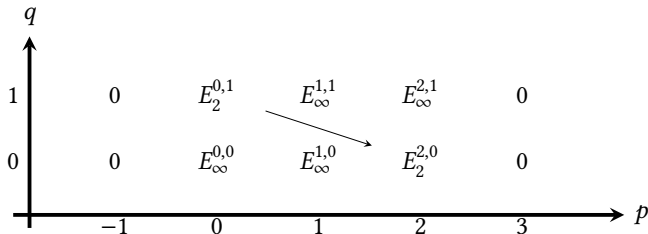
SNAKE LEMMA: $E_2^{p,q}$ of the first spectral sequence

The “ E_2 -page” contains only one non-trivial differential:



SNAKE LEMMA: $E_2^{p,q}$ of the first spectral sequence

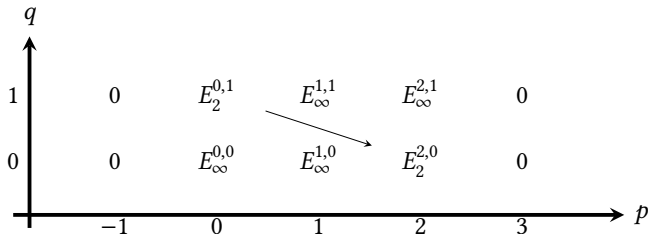
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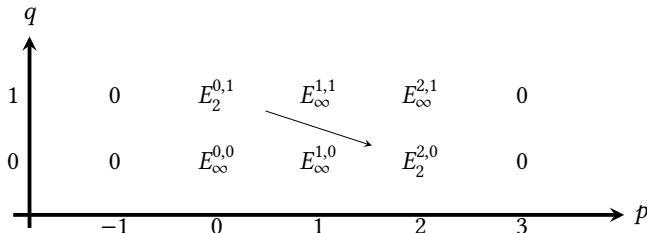
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SNAKE LEMMA: $E_2^{p,q}$ of the first spectral sequence

The “ E_2 -page” contains only one non-trivial differential:



- ▶ Since $E_r^{p,q} \Rightarrow 0$, all limit objects must be zero!
- ▶ So the top and bottom row in the previous page were right and left exact respectively, as we wanted.
- ▶ $E_3 = E_\infty = 0$, so $E_2^{0,1} \rightarrow E_2^{2,0}$ is an isomorphism and we can glue

$$\ker(d_v^{1,0}) \rightarrow \ker(d_v^{2,0}) \twoheadrightarrow E_2^{2,0} \cong E_2^{0,0} \hookrightarrow \operatorname{coker}(d_v^{0,0}) \rightarrow \operatorname{coker}(d_v^{1,0}).$$

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You could have invented spectral sequences.

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