

Basic Notions — Spectral Sequences

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- ▶ **Example:** let k be a field and consider a cochain complex C^\bullet of finite dimensional k -vector spaces

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- ▶ **Goal:** compute the cohomology H^\bullet of the complex C^\bullet

$$H^\bullet := H^0(C^\bullet)[0] \oplus H^1(C^\bullet)[-1] \oplus H^2(C^\bullet)[-2].$$

Starting point: a filtration on our complex

- ▶ If we do not have any extra information about C^\bullet , there is not much more we can do besides applying the definition

$$H^\bullet = \ker(d^0)[0] \oplus \frac{\ker(d^1)}{\operatorname{im}(d^0)}[-1] \oplus \frac{C^2}{\operatorname{im}(d^1)}[-2].$$

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- ▶ But C^\bullet has often some useful filtration $F^\bullet C^\bullet$, say

$$\begin{array}{ccccccc} C^\bullet = F^0 C^\bullet & \cdots 0 & \longrightarrow & C^0 & \xrightarrow{d^0} & C^1 & \xrightarrow{d^1} & C^2 & \longrightarrow & 0 \cdots \\ \cup & & & \cup & & \cup & & \cup & & \\ F^1 C^\bullet & \cdots 0 & \longrightarrow & F^1 C^0 & \longrightarrow & F^1 C^1 & \longrightarrow & F^1 C^2 & \longrightarrow & 0 \cdots \\ \cup & & & \cup & & \cup & & \cup & & \\ 0 = F^2 C^\bullet & \cdots 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \cdots \end{array}$$

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2. We can then recover H^i as the associated graded vector space

$$\operatorname{gr}(F^\bullet H^i) = F^1 H^i \oplus (H^i / F^1 H^i) \cong H^i \text{ for all } i \in \{0, 1, 2\}.$$

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- **Q1:** Can we compute the cohomologies of $F^1 C^\bullet$ and $C^\bullet / F^1 C^\bullet$?
- **Q2:** Are these cohomologies related to $F^1 H^\bullet$ and $H^\bullet / F^1 H^\bullet$?

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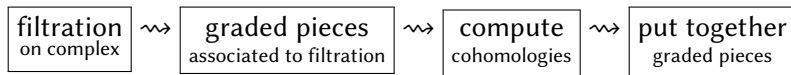
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So in this case we can carry out our naive strategy of computing each graded piece of the induced filtration in cohomology separately:



Possible scenario II: the bad

Suppose we know nothing about C^\bullet . The *stupid filtration*

$$\begin{cases} F^1 C^\bullet := \cdots \rightarrow 0 \rightarrow 0 \rightarrow C^1 \xrightarrow{d^1} C^2 \rightarrow 0 \rightarrow \cdots \\ F^2 C^\bullet := \cdots \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow C^2 \rightarrow 0 \rightarrow \cdots \end{cases}$$

induces the cohomology filtration

$$\begin{cases} F^1 H^\bullet := \operatorname{im}(H^\bullet(F^1 C^\bullet) \rightarrow H^\bullet(C^\bullet)) = H^1(C^\bullet)[-1] \oplus H^2(C^\bullet)[-2] \\ F^2 H^\bullet := \operatorname{im}(H^\bullet(F^2 C^\bullet) \rightarrow H^\bullet(C^\bullet)) = H^2(C^\bullet)[-2] \end{cases}$$

whose graded pieces are $H^0(C^\bullet)[0]$, $H^1(C^\bullet)[-1]$ and $H^2(C^\bullet)[-2]$.

So in this case computing $\operatorname{gr}^i H^\bullet(C^\bullet)$ is as hard as computing $H^i(C^\bullet)$ and we just wasted our time with this detour.

Possible scenario III: the ugly

Suppose now that we are somewhere in between the good and the bad scenarios. Namely, assume that we have some filtration $0 \subsetneq F^1 C^\bullet \subsetneq C^\bullet$ which does not correspond to a direct sum decomposition but such that we can compute the graded pieces $\text{gr}^i(H^\bullet(C^\bullet))$ of the filtration induced in cohomology.

References



Timothy Y. Chow.

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2