

# Basic Notions — Spectral Sequences

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“You could have invented spectral sequences”<sup>1</sup>



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<sup>1</sup>Title of the expository article [Cho06]

Notation gets ugly very soon



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- ▶ Cohomology is regarded as a complex with 0 differentials

$$H(C) = \bigoplus_{i \in \mathbb{Z}} H^i(C)[-i] := \left( \cdots H^{-1}(C) \xrightarrow{0} H^0(C) \xrightarrow{0} H^1(C) \cdots \right).$$

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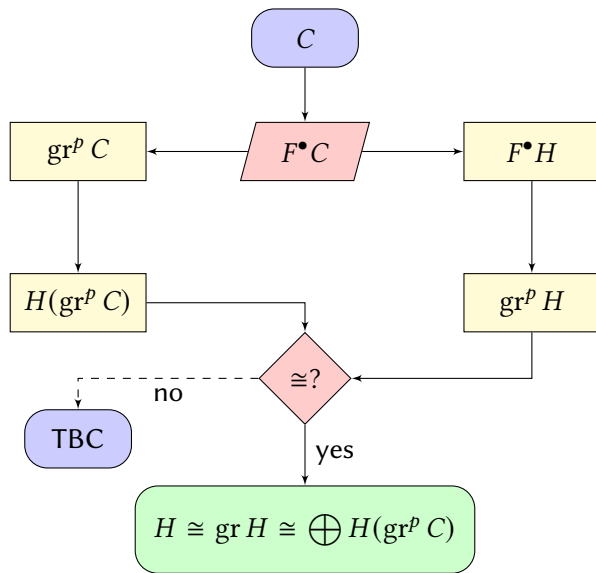
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- ▶ The graded object associated to a filtration  $F^\bullet A$  is denoted

$$\mathrm{gr} A = \bigoplus_{p \in \mathbb{N}} \mathrm{gr}^p A := \bigoplus_{p \in \mathbb{N}} (F^p A / F^{p+1} A).$$

# GOAL: compute cohomology with the help of a filtration



# Filtration induced in cohomology

A filtration on the complex  $F^\bullet C$  induces a filtration in cohomology

$$F^p H := \operatorname{im}(H(F^p C) \rightarrow H(C)) \subseteq H(C) = H,$$

where  $H(F^p C) \rightarrow H(C)$  is induced by the inclusion map  $F^p C \hookrightarrow C$ .

## BABY EXAMPLE: 2 step filtration on a length 1 complex

$$\begin{array}{ccccccc}
 C & & & & & & \\
 \parallel & & & & & & \\
 F^0 C & \cdots \rightarrow 0 \rightarrow C^0 \xrightarrow{d^0} C^1 \rightarrow 0 \rightarrow \cdots \\
 \cup \downarrow & & & \cup \downarrow & & & \cup \downarrow \\
 F^1 C & \cdots \rightarrow 0 \rightarrow F^1 C^0 \xrightarrow{d^0|_{F^1 C^0}} F^1 C^1 \rightarrow 0 \rightarrow \cdots \\
 \cup \downarrow & & & \cup \downarrow & & & \cup \downarrow \\
 F^2 C & \cdots \rightarrow 0 \rightarrow 0 \longrightarrow 0 \longrightarrow 0 \rightarrow \cdots \\
 \parallel & & & & & & \\
 0 & & & & & & 
 \end{array}$$

## BABY EXAMPLE: filtration induced in cohomology

We first look at the map induced by  $F^1 C \hookrightarrow C$  in cohomology:

$$\begin{array}{ccc} H(F^1 C) & = & \ker(d^0|_{F^1 C}) [0] \oplus \left( \frac{F^1 C^1}{\operatorname{im}(d^0|_{F^1 C})} \right) [-1] \\ \downarrow & & \downarrow \\ H(C) & = & \ker(d^0) [0] \oplus \left( \frac{C^1}{\operatorname{im}(d^0)} \right) [-1] \end{array}$$

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The filtration on cohomology was given by its image, hence

$$F^1 H = \ker(d^0|_{F^1 C}) [0] \oplus \left( \frac{F^1 C^1 + \operatorname{im}(d^0)}{\operatorname{im}(d^0)} \right) [-1]$$

## BABY EXAMPLE: graded cohomology pieces

Recall that  $\mathrm{gr}^0 H := F^0 H / F^1 H = H / F^1 H$ , hence (after  $\cong$ -theorem)

$$\mathrm{gr}^0 H = \left( \frac{\ker(d^0)}{\ker(d^0|_{F^1 C})} \right) [0] \oplus \left( \frac{C^1}{F^1 C^1 + \mathrm{im}(d^0)} \right) [-1]$$



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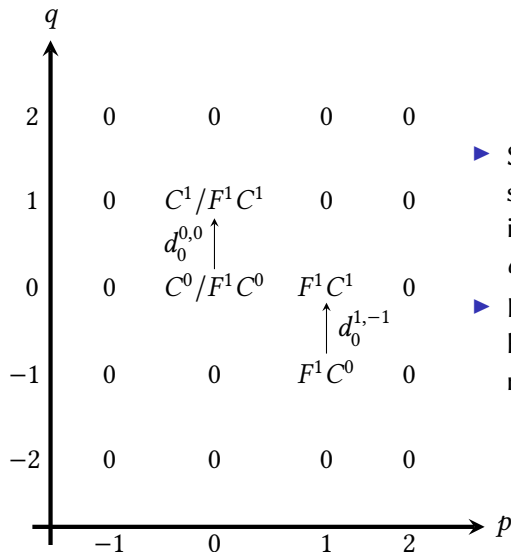
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Similarly,  $\mathrm{gr}^1 H = F^1 H / 0 = F^1 H$ , thus

$$\mathrm{gr}^1 H = \ker(d^0|_{F^1 C}) [0] \oplus \left( \frac{F^1 C^1 + \mathrm{im}(d^0)}{\mathrm{im}(d^0)} \right) [-1]$$

BABY EXAMPLE: define  $E_0^{p,q} := (\operatorname{gr}^p C)^{p+q}$  and visualize in  $\mathbb{Z}^2$



- ▶ Since  $F^1C \subseteq C$  is a subcomplex,  $d^0: C^0 \rightarrow C^1$  induces a morphism  $d_0^{0,0}: C^0/F^1C^0 \rightarrow C^1/F^1C^1$ .
- ▶ For the same reason,  $d^0|_{F^1C}$  has image in  $F^1C^1$ , so we may see it as a morphism

$$d_0^{1,-1}: F^1C^0 \rightarrow F^1C^1.$$

# References



Timothy Y. Chow.

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*Notices Amer. Math. Soc.*, 53(1):15–19, 2006.

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The Stacks project authors.

The stacks project.

<https://stacks.math.columbia.edu>, 2020.

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