Basic Notions — Spectral Sequences

Pedro Núñez

University of Freiburg

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“You could have invented spectral sequences”[1]

[1] Title of the expository article [Cho06]

Notation gets ugly very soon

Notation and conventions [Sta]

- Only finite dimensional vector spaces over $\mathbb{Q}$ for simplicity.
- (Cochain) complexes denoted with capital letters C and not $C^\bullet$.
- Cohomology again denoted just $H(C)$ and not $H^\bullet(C)$.
- Cohomology is regarded as a copmlex with 0 differentials

$$H(C) = \bigoplus_{i\in\mathbb{Z}} H^i(C)[-i] := \left(\cdots H^{-1}(C) \xrightarrow{0} H^0(C) \xrightarrow{0} H^1(C) \cdots\right).$$

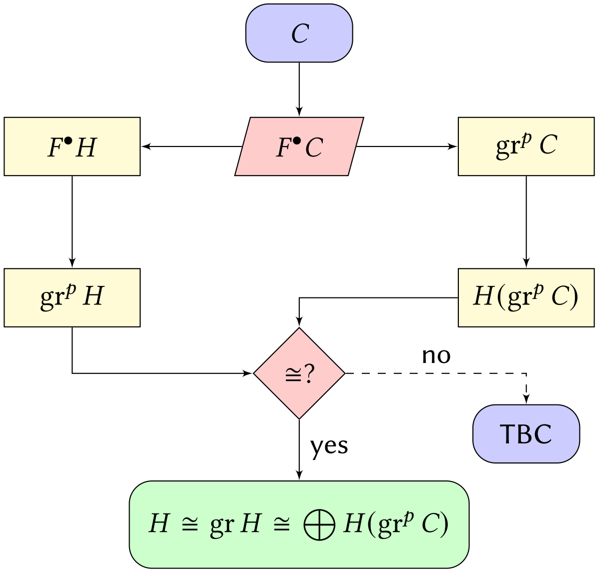
- If it is clear to which C we refer, denote $H(C)$ simply by H.
- (Decreasing) filtrations are always finite for simplicity:

$$F^\bullet A\colon \quad 0 = F^n A \subseteq \ldots \subseteq F^0 A = A.$$

- The graded object associated to a filtration $F^\bullet A$ is denoted

$$\operatorname{gr} A = \bigoplus_{p\in\mathbb{N}} \operatorname{gr}^p A := \bigoplus_{p\in\mathbb{N}} \left(F^p A / F^{p+1} A\right).$$

GOAL: compute cohomology with the help of a filtration



Filtration induced in cohomology

A filtration on the complex $F^\bullet C$ induces a filtration in cohomology

$$F^p H := \mathrm{im}(H(F^p C) \to H(C)) \subseteq H(C) = H,$$

where $H(F^p C) \to H(C)$ is induced by the inclusion map $F^p C \hookrightarrow C$.

Baby Example: 2 step filtration on a length 1 complex

$$
\begin{tikzcd}
C \arrow[d, phantom, "\shortparallel"] \\
F^0C & \cdots \arrow[r] & 0 \arrow[r] & C^0 \arrow[r, "d^0"] \arrow[d, phantom, "\cup"] & C^1 \arrow[r] \arrow[d, phantom, "\cup"] & 0 \arrow[r] & \cdots \\
F^1C \arrow[u, phantom, "\cup"] & \cdots \arrow[r] & 0 \arrow[r] & F^1C^0 \arrow[r, "d^0|_{F^1C^0}"] \arrow[d, phantom, "\cup"] & F^1C^1 \arrow[r] \arrow[d, phantom, "\cup"] & 0 \arrow[r] & \cdots \\
F^2C \arrow[u, phantom, "\cup"] \arrow[d, phantom, "\shortparallel"] & \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \\
0
\end{tikzcd}
$$

- Some $c \in C^0 \setminus F^1C^0$ may end up in F^1C^1 under d^0.
- The $\mathrm{gr}^p\, C$'s will not be able to detect this.
- Don't panic: spectral sequences will account for that.

Baby Example: filtration induced in cohomology

We first look at the map induced by $F^1C \hookrightarrow C$ in cohomology:

$$\begin{array}{ccc} H(F^1C) & = & \ker\left(d^0|_{F^1C}\right)[0] \oplus \left(\frac{F^1C^1}{\mathrm{im}\left(d^0|_{F^1C}\right)}\right)[-1] \\ \downarrow & & \downarrow \\ H(C) & = & \ker(d^0)[0] \oplus \left(\frac{C^1}{\mathrm{im}(d^0)}\right)[-1] \end{array}$$

The filtration on cohomology was given by its image, hence

$$F^1H = \ker\left(d^0|_{F^1C}\right)[0] \oplus \left(\frac{F^1C^1 + \mathrm{im}(d^0)}{\mathrm{im}(d^0)}\right)[-1]$$

Baby Example: graded cohomology pieces

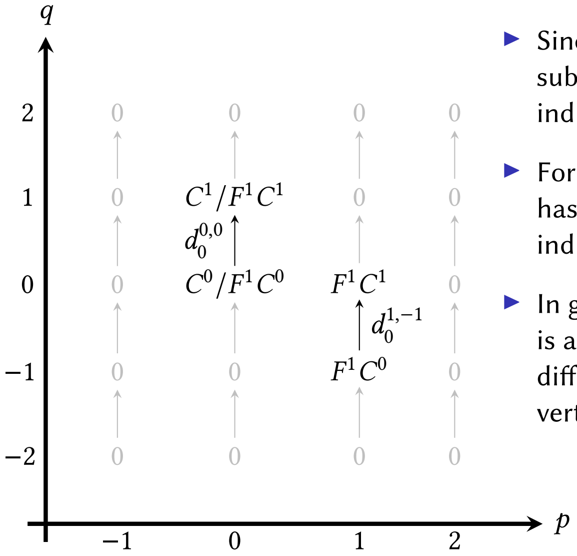
Recall that $\mathrm{gr}^0 H := F^0H/F^1H = H/F^1H$, hence (after $\cong$-theorem)

$$\mathrm{gr}^0 H = \left(\frac{\ker(d^0)}{\ker\left(d^0|_{F^1C}\right)}\right)[0] \oplus \left(\frac{C^1}{F^1C^1 + \mathrm{im}(d^0)}\right)[-1]$$

Similarly, $\mathrm{gr}^1 H = F^1H/0 = F^1H$, thus

$$\mathrm{gr}^1 H = \ker\left(d^0|_{F^1C}\right)[0] \oplus \left(\frac{F^1C^1 + \mathrm{im}(d^0)}{\mathrm{im}(d^0)}\right)[-1]$$

Baby Example: define $E_0^{p,q} := (\mathrm{gr}^p\, C)^{p+q}$ and visualize in $\mathbb{Z}^2$



- Since $F^1C \subseteq C$ is a subcomplex, $d^0\colon C^0 \to C^1$ induces the differential $d_0^{0,0}$.
- For the same reason, $d^0|_{F^1C}$ has image in F^1C^1, so it induces the differential $d_0^{1,-1}$.
- In general, since $F^{p+1}C \subseteq F^pC$ is a subcomplex, the original differentials induce the vertical differentials.

Baby Example: compute cohomology of the columns $\mathrm{gr}^p C$

From the $p = 0$ column in the "E_0-page" we compute

$$H(\mathrm{gr}^0 C) = \left(\frac{\ker(d^0) + (d^0)^{-1}(F^1C^1)}{F^1C^0} \right)[0] \oplus \left(\frac{C^1}{F^1C^1 + \mathrm{im}(d^0)} \right)[-1]$$

And from the $p = 1$ column in the "E_0-page" we compute

$$H(\mathrm{gr}^1 C) = \ker\left(d^0|_{F^1C}\right)[0] \oplus \left(\frac{F^1C^1}{\mathrm{im}(d^0|_{F^1C})} \right)[-1]$$

Baby Example: 1st approximation to the $\operatorname{gr}^0 H$ part

We compare $\operatorname{gr}^0 H$ (left column) to $H(\operatorname{gr}^0 C)$ (right column):

$$\begin{array}{cc}
\left(\dfrac{\ker(d^0)}{\ker(d^0|_{F^1C})}\right)[0] & \left(\dfrac{\ker(d^0)+(d^0)^{-1}(F^1C^1)}{F^1C^0}\right)[0] \\
\oplus & \oplus \\
\left(\dfrac{C^1}{F^1C^1+\operatorname{im}(d^0)}\right)[-1] & \left(\dfrac{C^1}{F^1C^1+\operatorname{im}(d^0)}\right)[-1]
\end{array}$$

- $H^1(\operatorname{gr}^0 C)$ does compute $\operatorname{gr}^0 H^1$ (bottom row),
- but $H^0(\operatorname{gr}^0 C)$ is not isomorphic to $\operatorname{gr}^0 H^0$ (top row).

Baby Example: 1^{st} approximation to the $\operatorname{gr}^1 H$ part

We compare now $\operatorname{gr}^1 H$ (left column) to $H(\operatorname{gr}^1 C)$ (right column):

$$\begin{array}{cc}
\ker\left(d^0|_{F^1C}\right)[0] & \ker\left(d^0|_{F^1C}\right)[0] \\
\oplus & \oplus \\
\left(\frac{F^1C^1+\operatorname{im}(d^0)}{\operatorname{im}(d^0)}\right)[-1] & \left(\frac{F^1C^1}{\operatorname{im}\left(d^0|_{F^1C}\right)}\right)[-1]
\end{array}$$

- In this case $H^0(\operatorname{gr}^1 C)$ does compute $\operatorname{gr}^1 H^0$ (top row),
- but $H^1(\operatorname{gr}^1 C)$ is not isomorphic to $\operatorname{gr}^1 H^1$ (bottom row).

Baby Example: define $E_1^{p,q} := H^{p+q}(\mathrm{gr}^p\, C)$ and plot in $\mathbb{Z}^2$

$$\begin{tikzcd}
q \\
1 & \frac{C^1}{F^1C^1+\mathrm{im}(d^0)} \arrow[r] & 0 \\
0 & \frac{\ker(d^0)+(d^0)^{-1}(F^1C^1)}{F^1C^0} \arrow[r, "d_1^{0,0}"] & \frac{F^1C^1}{\mathrm{im}\left(d^0|_{F^1C}\right)} \\
-1 & 0 \arrow[r] & \ker\left(d^0|_{F^1C}\right) \\
 & 0 & 1 & p
\end{tikzcd}$$

- The original differential d^0 induces the differential $d_1^{0,0}$.
- More generally, if we had started from a longer complex, the original differentials would induce the horizontal differentials.

Baby Example: turn the page by taking cohomology again

The “E_2-page” reads:

$$
\begin{array}{c|cc}
q & & \\
1 & \dfrac{C^1}{F^1C^1+\mathrm{im}(d^0)} & 0 \\[2ex]
0 & \dfrac{\ker(d^0)}{\ker\left(d^0|_{F^1C}\right)} & \dfrac{F^1C^1+\mathrm{im}(d^0)}{\mathrm{im}(d^0)} \\[2ex]
-1 & 0 & \ker\left(d^0|_{F^1C}\right) \\
\hline
 & 0 & 1 \quad p
\end{array}
$$

- If we had started from a longer complex, the original differentials would induce arrows of type $(2, -1)$.

Baby Example: 2^{nd} approximation to the $\mathrm{gr}^0 H$ part

We compare then $\mathrm{gr}^0 H$ (left column) to $E_2^{0,\bullet}$ (right column):

$$\left(\frac{\ker(d^0)}{\ker(d^0|_{F^1C})}\right)[0] \qquad\qquad \left(\frac{\ker(d^0)+F^1C^0}{F^1C^0}\right)[0]$$

$$\oplus \qquad\qquad\qquad\qquad \oplus$$

$$\left(\frac{C^1}{F^1C^1+\mathrm{im}(d^0)}\right)[-1] \qquad\qquad \left(\frac{C^1}{F^1C^1+\mathrm{im}(d^0)}\right)[-1]$$

After applying an $\cong$-theorem we see that the two agree!

Baby Example: 2^{nd} approximation to the $\operatorname{gr}^1 H$ part

We compare now $\operatorname{gr}^1 H$ (left column) to $E_2^{1,\bullet-1}$ (right column):

$$\begin{array}{cc}
\ker\left(d^0|_{F^1C}\right)[0] & \ker\left(d^0|_{F^1C}\right)[0] \\
\oplus & \oplus \\
\left(\frac{F^1C^1+\operatorname{im}(d^0)}{\operatorname{im}(d^0)}\right)[-1] & \left(\frac{F^1C^1}{\operatorname{im}(d^0)\cap F^1C^1}\right)[-1]
\end{array}$$

After applying an $\cong$-theorem we see that the two sides agree again!

Baby Example: reading off the result from the "E_2-page"

- $\operatorname{gr}^0 H \cong E_2^{0,0}[0] \oplus E_2^{0,1}[-1]$;
- $\operatorname{gr}^1 H \cong E_2^{1,-1}[0] \oplus E_2^{1,0}[-1]$.

$$\begin{cases} H^0 \cong \operatorname{gr}^0 H^0 \oplus \operatorname{gr}^1 H^0 \cong E_2^{0,0} \oplus E_2^{1,-1} \\ H^1 \cong \operatorname{gr}^1 H^0 \oplus \operatorname{gr}^1 H^1 \cong E_2^{0,1} \oplus E_2^{1,0} \end{cases}$$

$$\begin{array}{cccc}
p+q=1 & & & \\
p+q=0 & \dfrac{C^1}{F^1C^1+\operatorname{im}(d^0)} & 0 & \\
 & \dfrac{\ker(d^0)}{\ker\left(d^0|_{F^1C}\right)} & \dfrac{F^1C^1+\operatorname{im}(d^0)}{\operatorname{im}(d^0)} & \\
 & 0 & \ker\left(d^0|_{F^1C}\right) & H^1 \\
 & & & H^0
\end{array}$$

DEFINITION [Sta, Tag 011N]

A *spectral sequence* in an abelian category $\mathcal{A}$ is given by a system $(E_r, d_r)_{r\in\mathbb{N}}$ such that for all $r \in \mathbb{N}$ we have:

- E_r is an object of $\mathcal{A}$;
- $d_r\colon E_r \longrightarrow E_r$ is a morphism such that $d_r \circ d_r = 0$;
- and $E_{r+1} = \ker(d_r)/\operatorname{im}(d_r)$.

Existence [Sta, Tag 012M]

Let $F^\bullet C$ be a filtered complex in an abelian category $\mathcal{A}$. Then there is a spectral sequence $(E_r, d_r)_{r \in \mathbb{N}}$ of bigraded objects in $\mathcal{A}$ such that d_r has bidegree $(r, -r+1)$. More explicitly, the "E_0-page" is given by

- $E_0^{p,q} = \operatorname{gr}^p C^{p+q}$,
- $d_0^{p,q} = \operatorname{gr}^p(d^{p+q}) : E_0^{p,q} \longrightarrow E_0^{p,q+1}$;

and the "E_r-page" is given by

- $E_r^{p,q} = Z_r^{p,q} / B_r^{p,q}$, where
- $Z_r^{p,q} = (F^p C^{p+q} \cap d^{-1}(F^{p+r} C^{p+q+1}) + F^{p+1} C^{p+q}) / F^{p+1} C^{p+q}$, and
- $B_r^{p.q} = (F^p C^{p+q} \cap d(F^{p-r+1} C^{p+q-1}) + F^{p+1} C^{p+q}) / F^{p+1} C^{p+q}$;
- $d_r^{p,q} : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}$ induced by $d^{p+q} : F^p C^{p+q} \longrightarrow F^{p+r} C^{p+r+q-r+1}$.

If the filtration induced on each C^n was finite, then letting $r \longrightarrow \infty$ we would obtain the *limit* E_∞ of the spectral sequence, given by

$$E_\infty^{p,q} = (F^p C^{p+q} \cap \ker(d)) / (F^p C^{p+q} \cap \operatorname{im}(d)).$$

What's up with those crazy formulas for $E_r^{p,q} = Z_r^{p,q}/B_r^{p,q}$?

- Remember the falling rocks?
- These were elements in our complex that fell down a number of levels when we applied the differential d.

$Z_r^{p,q} = (F^pC^{p+q} \cap d^{-1}(F^{p+r}C^{p+q+1}) + F^{p+1}C^{p+q})/F^{p+1}C^{p+q}$ says:

"Elements in F^pC^{p+q} which will fall down at least r levels and we don't care what happens in $F^{p+1}C^{p+q}$."

And $B_r^{p,q} = (F^pC^{p+q} \cap d(F^{p-r+1}C^{p+q-1}) + F^{p+1}C^{p+q})/F^{p+1}C^{p+q}$ says:

"Elements in F^pC^{p+q} which came falling down at most $r-1$ levels and we don't care about the ones in $F^{p+1}C^{p+q}$."

CONVERGENCE: $E_r^{p,q} \Rightarrow H^{p+q}(C)$ [McC01, Theorem 2.6]

Let $F^\bullet C$ be a filtered complex in an abelian category $\mathcal{A}$ and assume that the filtration induced on each C^n is finite. Then:

- The induced filtration $F^\bullet H$ is finite.
- For each $p, q \in \mathbb{Z}$ there exists $r_0 \in \mathbb{N}$ such that

$$E_\infty^{p,q} = E_r^{p,q} \text{ for all } r \geqslant r_0.$$

- The spectral sequence *converges* to $F^\bullet H$, i.e.

$$E_\infty^{p,q} \cong \operatorname{gr}^p H^{p+q} \text{ for all } p, q \in \mathbb{Z}.$$

In particular, modulo extension problems (e.g. if we are working with vector spaces), we can recover H as

$$H^n(C) \cong \bigoplus_{p \in \mathbb{Z}} E_\infty^{p,n-p} \text{ for all } n \in \mathbb{Z}.$$

Applications of spectral sequences from filtrations

- Cellular homology and singular homology of a CW complex agree [McC01, Theorem 4.13].
- "Cheap Hodge decomposition" [Voi02, Remark 8.29].
- Two different ways to compute the cohomology of the total complex of a double complex.

To avoid getting too much into specific topics, we will focus on the third application. We will do this again with an example.

Snake Lemma: statement

Consider a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r] & 0 \\
0 \arrow[r] & A \arrow[r, "f"] \arrow[u, "a"] & B \arrow[r, "g"] \arrow[u, "b"] & C \arrow[r] \arrow[u, "c"] & 0
\end{tikzcd}$$

in which both rows are exact. Then there is an exact sequence

$$0 \to \ker a \to \ker b \to \ker c \to \operatorname{coker} a \to \operatorname{coker} b \to \operatorname{coker} b \to 0.$$

Snake Lemma: double complex and its total complex

We relabel the diagram to see it as a double complex

$$\begin{tikzcd}
0 \arrow[r] & C^{0,1} \arrow[r, "d_h^{0,1}"] & C^{1,1} \arrow[r, "d_h^{1,1}"] & C^{2,1} \arrow[r] & 0 \\
0 \arrow[r] & C^{0,0} \arrow[r, "d_h^{0,0}"] \arrow[u, "d_v^{0,0}"] & C^{1,0} \arrow[r, "d_h^{1,0}"] \arrow[u, "d_v^{1,0}"] & C^{2,0} \arrow[r] \arrow[u, "d_v^{2,0}"] & 0
\end{tikzcd}$$

whose total complex sC is then

$$\cdots \to 0 \to C^{0,0} \to C^{0,1} \oplus C^{1,0} \to C^{1,1} \oplus C^{2,0} \to C^{2,1} \to 0 \to \cdots$$

with differentials

$$d_{sC}^n := \begin{pmatrix} d_h^{n-1,1} & (-1)^n d_v^{n,0} \\ 0 & d_h^{n,0} \end{pmatrix}.$$

Snake Lemma: first filtration

The first filtration $F_I^\bullet sC$ is defined by

$$F_I^p sC^n := \bigoplus_{i+j=n, i\geqslant p} C^{i,j}.$$

Therefore $\mathrm{gr}_I^p sC = C^{p,\bullet}$ with the differential as before up to a sign. Associated to this filtration we get the spectral sequence

$$E_1^{p,q} = H^q(C^{p,\bullet}) \Rightarrow H^{p+q}(sC).$$

Right now we cannot immediately say much more, but we will come back to this soon.

Snake Lemma: second filtration

The second filtration $F_{II}^{\bullet} sC$ is defined by

$$F_{II}^{p} sC^{n} := \bigoplus_{i+j=n, j \geqslant p} C^{i,j}.$$

Therefore $\mathrm{gr}_{II}^{p} sC = C^{\bullet,p}$ with the differential as before.
In this case we get the spectral sequence

$$E_1^{p,q} = H^q(C^{\bullet,p}) \Rightarrow H^{p+q}(sC).$$

Since the rows $C^{\bullet,p}$ were exact, $E_1^{p,q} = 0$ for all $p, q \in \mathbb{Z}$.
Therefore $H(sC)$ must be zero.

Snake Lemma: back to the first spectral sequence

Using what we deduced from the second spectral sequence, we have now

$$E_1^{p,q} = H^q(C^{p,\bullet}) \Rightarrow 0.$$

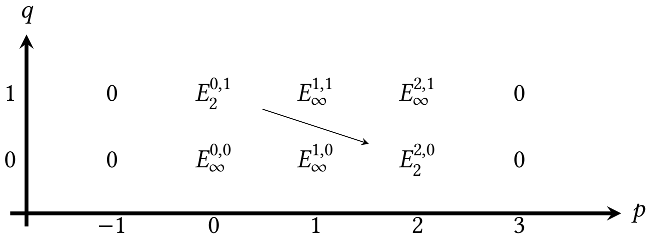
Let's draw the "E_1-page":

$$\begin{array}{c|ccccccccc}
q & & & & & & & & & \\
1 & 0 & \longrightarrow & \operatorname{coker}(d_v^{0,0}) & \longrightarrow & \operatorname{coker}(d_v^{1,0}) & \longrightarrow & \operatorname{coker}(d_v^{2,0}) & \longrightarrow & 0 \\
0 & 0 & \longrightarrow & \ker(d_v^{0,0}) & \longrightarrow & \ker(d_v^{1,0}) & \longrightarrow & \ker(d_v^{2,0}) & \longrightarrow & 0 \\
\hline
 & -1 & & 0 & & 1 & & 2 & & 3 \quad p
\end{array}$$

To show that the top and bottom rows are right and left exact respectively, we would like to have 0's at the corresopnding E_2 spots.

Snake Lemma: $E_2^{p,q}$ of the first spectral sequence

The "E_2-page" contains only one non-trivial differential:



- Since $E_r^{p,q} \Rightarrow 0$, all limit objects must be zero.
- So the top and bottom row in the previous page were right and left exact respectively, as we wanted.
- $E_3 = E_\infty = 0$, so $E_2^{0,1} \longrightarrow E_2^{2,0}$ is an isomorphism and we can glue

$$\ker(d_v^{1,0}) \longrightarrow \ker(d_v^{2,0}) \twoheadrightarrow E_2^{2,0} \cong E_2^{0,1} \hookrightarrow \operatorname{coker}(d_v^{0,0}) \longrightarrow \operatorname{coker}(d_v^{1,0}).$$

Thanks for your attention! Here are the references:

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