

Basic Notions — Spectral Sequences

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“You could have invented spectral sequences”¹



¹Title of the expository article [Cho06]

Notation gets ugly very soon



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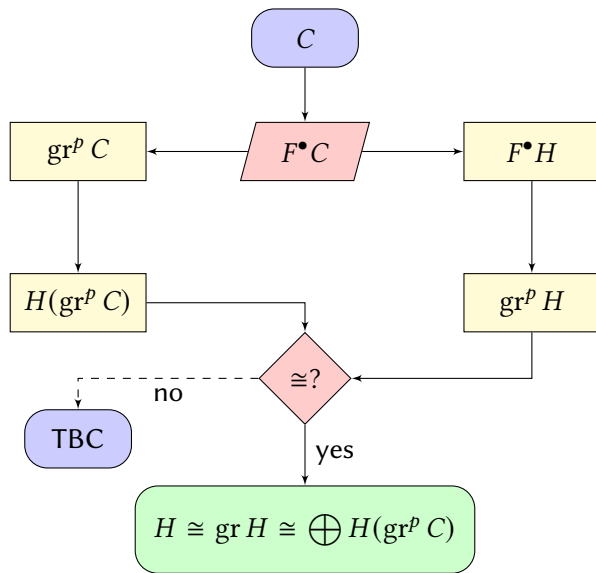
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- ▶ The graded object associated to a filtration $F^\bullet A$ is denoted

$$\mathrm{gr} A = \bigoplus_{p \in \mathbb{N}} \mathrm{gr}^p A := \bigoplus_{p \in \mathbb{N}} (F^p A / F^{p+1} A).$$

GOAL: compute cohomology with the help of a filtration



Filtration induced in cohomology

A filtration on the complex $F^\bullet C$ induces a filtration in cohomology

$$F^p H := \operatorname{im}(H(F^p C) \rightarrow H(C)) \subseteq H(C) = H,$$

where $H(F^p C) \rightarrow H(C)$ is induced by the inclusion map $F^p C \hookrightarrow C$.

BABY EXAMPLE: 2 step filtration on a length 1 complex

$$\begin{array}{ccccccc}
 C & & & & & & \\
 \parallel & & & & & & \\
 F^0 C & \cdots \rightarrow 0 \rightarrow C^0 \xrightarrow{d^0} C^1 \rightarrow 0 \rightarrow \cdots \\
 \cup \downarrow & & & \cup \downarrow & & & \cup \downarrow \\
 F^1 C & \cdots \rightarrow 0 \rightarrow F^1 C^0 \xrightarrow{d^0|_{F^1 C^0}} F^1 C^1 \rightarrow 0 \rightarrow \cdots \\
 \cup \downarrow & & & \cup \downarrow & & & \cup \downarrow \\
 F^2 C & \cdots \rightarrow 0 \rightarrow 0 \longrightarrow 0 \longrightarrow 0 \rightarrow \cdots \\
 \parallel & & & & & & \\
 0 & & & & & &
 \end{array}$$

BABY EXAMPLE: filtration induced in cohomology

We first look at the map induced by $F^1 C \hookrightarrow C$ in cohomology:

$$\begin{array}{ccc} H(F^1 C) & = & \ker(d^0|_{F^1 C}) [0] \oplus \left(\frac{F^1 C^1}{\operatorname{im}(d^0|_{F^1 C})} \right) [-1] \\ \downarrow & & \downarrow \\ H(C) & = & \ker(d^0) [0] \oplus \left(\frac{C^1}{\operatorname{im}(d^0)} \right) [-1] \end{array}$$

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The filtration on cohomology was given by its image, hence

$$F^1 H = \ker(d^0|_{F^1 C}) [0] \oplus \left(\frac{F^1 C^1 + \operatorname{im}(d^0)}{\operatorname{im}(d^0)} \right) [-1]$$

BABY EXAMPLE: graded cohomology pieces

Recall that $\mathrm{gr}^0 H := F^0 H / F^1 H = H / F^1 H$, hence (after \cong -theorem)

$$\mathrm{gr}^0 H = \left(\frac{\ker(d^0)}{\ker(d^0|_{F^1 C})} \right) [0] \oplus \left(\frac{C^1}{F^1 C^1 + \mathrm{im}(d^0)} \right) [-1]$$

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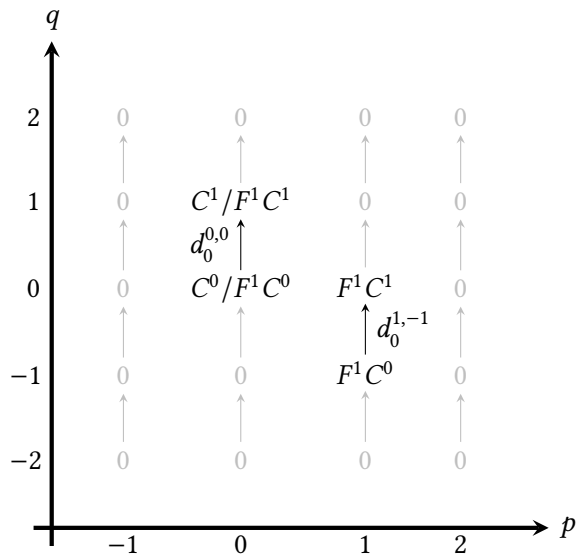
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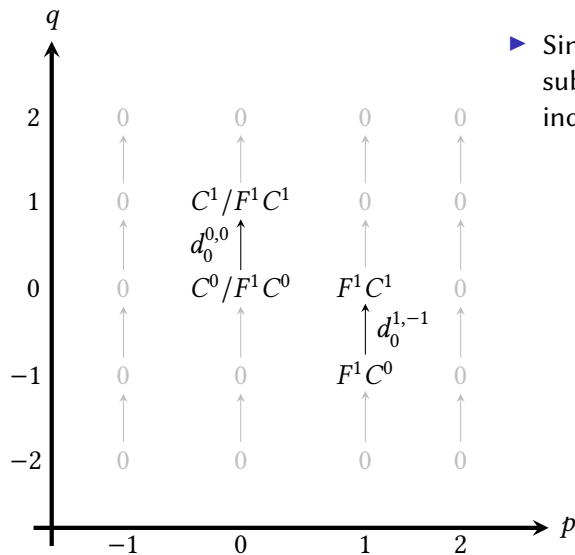
Similarly, $\mathrm{gr}^1 H = F^1 H / 0 = F^1 H$, thus

$$\mathrm{gr}^1 H = \ker(d^0|_{F^1 C}) [0] \oplus \left(\frac{F^1 C^1 + \mathrm{im}(d^0)}{\mathrm{im}(d^0)} \right) [-1]$$

BABY EXAMPLE: define $E_0^{p,q} := (\text{gr}^p C)^{p+q}$ and visualize in \mathbb{Z}^2

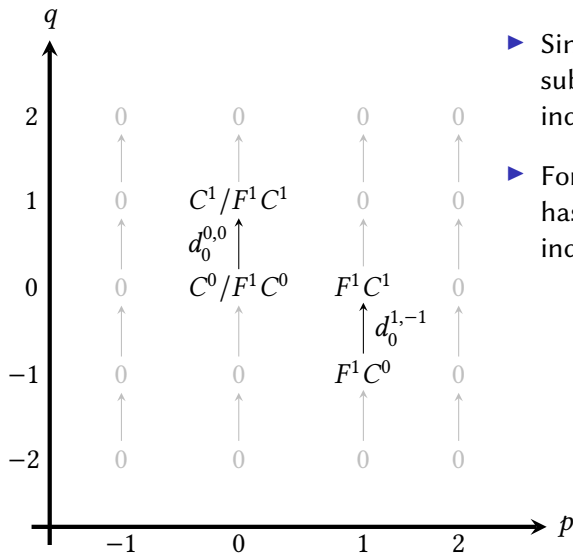


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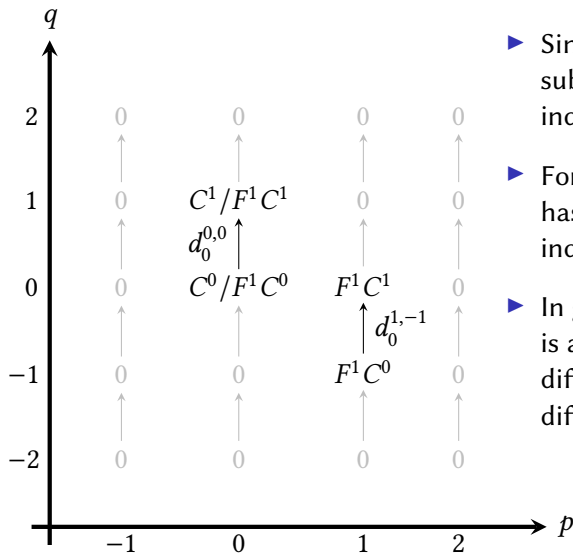
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- ▶ For the same reason, $d^0|_{F^1 C}$ has image in $F^1 C^1$, so it induces the differential $d_0^{1,-1}$.

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- ▶ For the same reason, $d^0|_{F^1 C}$ has image in $F^1 C^1$, so it induces the differential $d_0^{1,-1}$.
- ▶ In general, since $F^{p+1} C \subseteq F^p C$ is a subcomplex, the original differentials induce all vertical differentials.

References



Timothy Y. Chow.

You could have invented spectral sequences.

Notices Amer. Math. Soc., 53(1):15–19, 2006.

1



The Stacks project authors.

The stacks project.

<https://stacks.math.columbia.edu>, 2020.

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