Basic Notions — Spectral Sequences

Pedro Núñez

University of Freiburg

10th December 2020

"You could have invented spectral sequences"[1]

[1]Title of the expository article [Cho06]

Notation gets ugly very soon

Notation and conventions [Sta20]

- Only finite dimensional vector spaces over $\mathbb{Q}$ for simplicity.

Notation and conventions [Sta20]

- Only finite dimensional vector spaces over $\mathbb{Q}$ for simplicity.
- (Cochain) complexes denoted with capital letters C and not $C^\bullet$.

Notation and conventions [Sta20]

- Only finite dimensional vector spaces over $\mathbb{Q}$ for simplicity.
- (Cochain) complexes denoted with capital letters C and not $C^\bullet$.
- Cohomology again denoted just $H(C)$ and not $H^\bullet(C)$.

Notation and conventions [Sta20]

- Only finite dimensional vector spaces over $\mathbb{Q}$ for simplicity.
- (Cochain) complexes denoted with capital letters C and not $C^\bullet$.
- Cohomology again denoted just $H(C)$ and not $H^\bullet(C)$.
- Cohomology is regarded as a copmlex with 0 differentials

$$H(C) = \bigoplus_{i\in\mathbb{Z}} H^i(C)[-i] := \left(\cdots H^{-1}(C) \xrightarrow{0} H^0(C) \xrightarrow{0} H^1(C) \cdots\right).$$

Notation and conventions [Sta20]

- Only finite dimensional vector spaces over $\mathbb{Q}$ for simplicity.
- (Cochain) complexes denoted with capital letters C and not $C^\bullet$.
- Cohomology again denoted just $H(C)$ and not $H^\bullet(C)$.
- Cohomology is regarded as a copmlex with 0 differentials

$$H(C) = \bigoplus_{i\in\mathbb{Z}} H^i(C)[-i] := \left(\cdots H^{-1}(C) \xrightarrow{0} H^0(C) \xrightarrow{0} H^1(C) \cdots\right).$$

- If it is clear to which C we refer, denote $H(C)$ simply by H.

Notation and conventions [Sta20]

- Only finite dimensional vector spaces over $\mathbb{Q}$ for simplicity.
- (Cochain) complexes denoted with capital letters C and not $C^\bullet$.
- Cohomology again denoted just $H(C)$ and not $H^\bullet(C)$.
- Cohomology is regarded as a copmlex with 0 differentials

$$H(C) = \bigoplus_{i\in\mathbb{Z}} H^i(C)[-i] := \left(\cdots H^{-1}(C) \xrightarrow{0} H^0(C) \xrightarrow{0} H^1(C) \cdots\right).$$

- If it is clear to which C we refer, denote $H(C)$ simply by H.
- (Decreasing) filtrations are always finite for simplicity:

$$F^\bullet A: \quad 0 = F^n A \subseteq \ldots \subseteq F^0 A = A.$$

Notation and conventions [Sta20]

- Only finite dimensional vector spaces over $\mathbb{Q}$ for simplicity.
- (Cochain) complexes denoted with capital letters C and not $C^\bullet$.
- Cohomology again denoted just $H(C)$ and not $H^\bullet(C)$.
- Cohomology is regarded as a copmlex with 0 differentials

$$H(C) = \bigoplus_{i\in\mathbb{Z}} H^i(C)[-i] := \left(\cdots H^{-1}(C) \xrightarrow{0} H^0(C) \xrightarrow{0} H^1(C) \cdots\right).$$

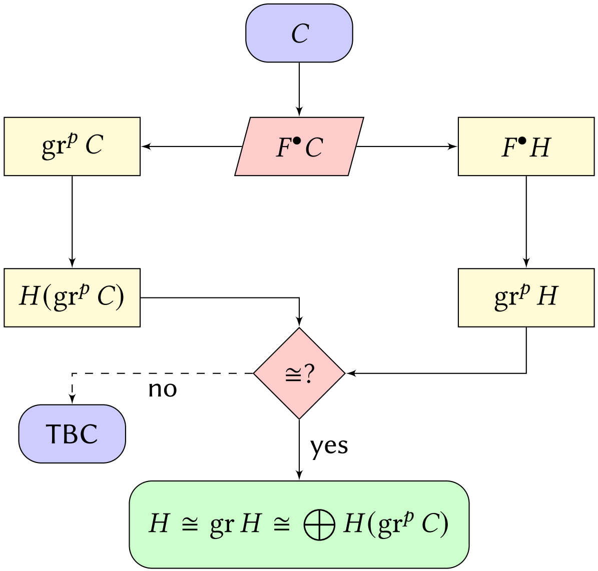
- If it is clear to which C we refer, denote $H(C)$ simply by H.
- (Decreasing) filtrations are always finite for simplicity:

$$F^\bullet A: \quad 0 = F^n A \subseteq \ldots \subseteq F^0 A = A.$$

- The graded object associated to a filtration $F^\bullet A$ is denoted

$$\operatorname{gr} A = \bigoplus_{p\in\mathbb{N}} \operatorname{gr}^p A := \bigoplus_{p\in\mathbb{N}} \left(F^p A / F^{p+1} A\right).$$

Goal: compute cohomology with the help of a filtration



Filtration induced in cohomology

A filtration on the complx $F^\bullet C$ induces a filtration in cohomology

$$F^p H := \operatorname{im}(H(F^p C) \to H(C)) \subseteq H(C) = H,$$

where $H(F^p C) \to H(C)$ is induced by the inclusion map $F^p C \hookrightarrow C$.

Baby Example: 2 step filtration on a length 1 complex

$$
\begin{array}{ccccccccccccc}
C & & & & & & & & & & & & \\
\parallel & & & & & & & & & & & & \\
F^0C & & \cdots & \to & 0 & \longrightarrow & C^0 & \xrightarrow{\quad d^0 \quad} & C^1 & \longrightarrow & 0 & \to & \cdots \\
\cup\!\mathsf{I} & & & & & & \cup\!\mathsf{I} & & \cup\!\mathsf{I} & & & & \\
F^1C & & \cdots & \to & 0 & \to & F^1C^0 & \xrightarrow{d^0|_{F^1C^0}} & F^1C^1 & \to & 0 & \to & \cdots \\
\cup\!\mathsf{I} & & & & & & \cup\!\mathsf{I} & & \cup\!\mathsf{I} & & & & \\
F^2C & & \cdots & \to & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \to & \cdots \\
\parallel & & & & & & & & & & & & \\
0 & & & & & & & & & & & &
\end{array}
$$

Baby Example: filtration induced in cohomology

We first look at the map induced by $F^1C \hookrightarrow C$ in cohomology:

$$\begin{array}{ccc} H(F^1C) & = & \ker\left(d^0|_{F^1C}\right)[0] \oplus \left(\frac{F^1C^1}{\mathrm{im}\left(d^0|_{F^1C}\right)}\right)[-1] \\ \downarrow & & \downarrow \\ H(C) & = & \ker(d^0)[0] \oplus \left(\frac{C^1}{\mathrm{im}(d^0)}\right)[-1] \end{array}$$

Baby Example: filtration induced in cohomology

We first look at the map induced by $F^1C \hookrightarrow C$ in cohomology:

$$
\begin{array}{ccc}
H(F^1C) & = & \ker\left(d^0|_{F^1C}\right)[0] \oplus \left(\dfrac{F^1C^1}{\mathrm{im}\left(d^0|_{F^1C}\right)}\right)[-1] \\
\downarrow & & \downarrow \\
H(C) & = & \ker(d^0)[0] \oplus \left(\dfrac{C^1}{\mathrm{im}(d^0)}\right)[-1]
\end{array}
$$

The filtration on cohomology was given by its image, hence

$$
F^1H = \ker\left(d^0|_{F^1C}\right)[0] \oplus \left(\frac{F^1C^1 + \mathrm{im}(d^0)}{\mathrm{im}(d^0)}\right)[-1]
$$

Baby Example: graded cohomology pieces

Recall that $\mathrm{gr}^0 H := F^0 H / F^1 H = H / F^1 H$, hence (after $\cong$-theorem)

$$\mathrm{gr}^0 H = \left(\frac{\ker(d^0)}{\ker\left(d^0|_{F^1 C}\right)} \right)[0] \oplus \left(\frac{C^1}{F^1 C^1 + \mathrm{im}(d^0)} \right)[-1]$$

Baby Example: graded cohomology pieces

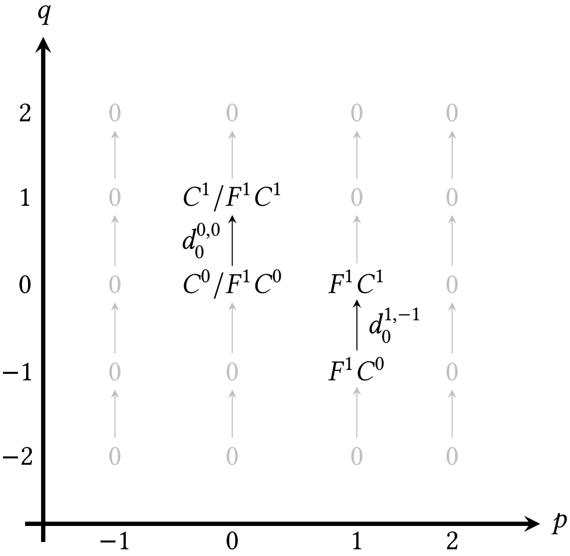
Recall that $\mathrm{gr}^0 H := F^0 H / F^1 H = H / F^1 H$, hence (after $\cong$-theorem)

$$\mathrm{gr}^0 H = \left(\frac{\ker(d^0)}{\ker\left(d^0|_{F^1 C}\right)} \right) [0] \oplus \left(\frac{C^1}{F^1 C^1 + \mathrm{im}(d^0)} \right) [-1]$$

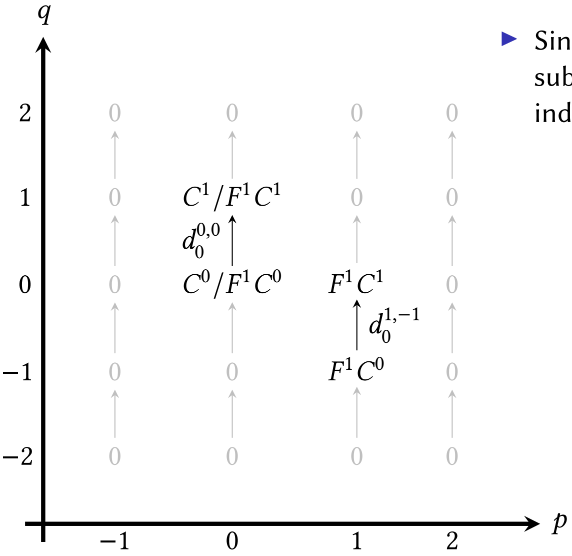
Similarly, $\mathrm{gr}^1 H = F^1 H / 0 = F^1 H$, thus

$$\mathrm{gr}^1 H = \ker\left(d^0|_{F^1 C}\right) [0] \oplus \left(\frac{F^1 C^1 + \mathrm{im}(d^0)}{\mathrm{im}(d^0)} \right) [-1]$$

Baby Example: define $E_0^{p,q} := (\mathrm{gr}^p\, C)^{p+q}$ and visualize in $\mathbb{Z}^2$

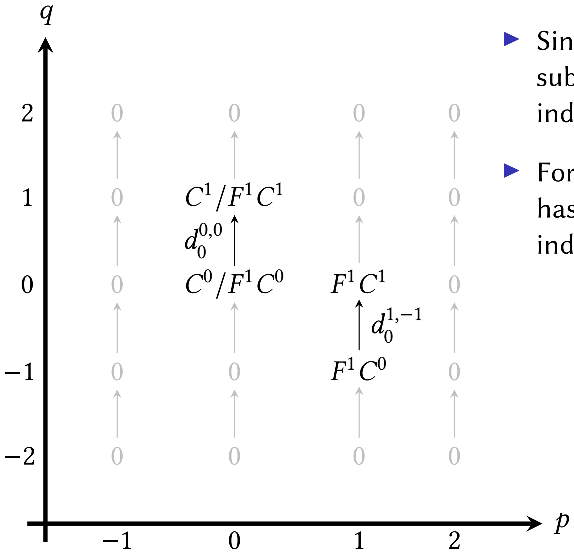


Baby Example: define $E_0^{p,q} := (\mathrm{gr}^p\, C)^{p+q}$ and visualize in $\mathbb{Z}^2$



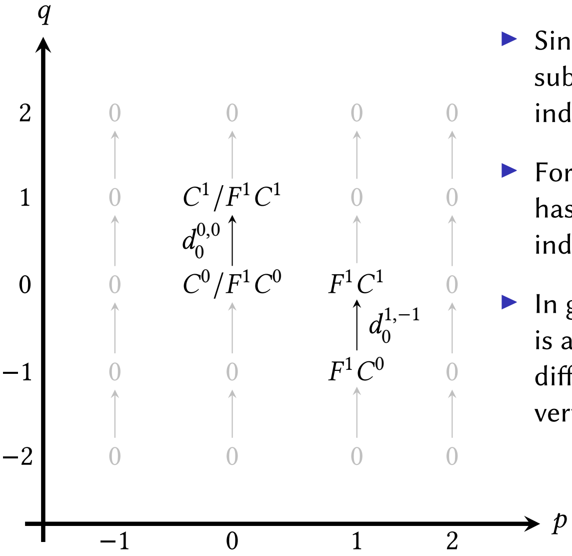
- Since $F^1C \subseteq C$ is a subcomplex, $d^0 : C^0 \to C^1$ induces the differential $d_0^{0,0}$.

Baby Example: define $E_0^{p,q} := (\mathrm{gr}^p\, C)^{p+q}$ and visualize in $\mathbb{Z}^2$



- Since $F^1C \subseteq C$ is a subcomplex, $d^0 : C^0 \to C^1$ induces the differential $d_0^{0,0}$.
- For the same reason, $d^0|_{F^1C}$ has image in F^1C^1, so it induces the differential $d_0^{1,-1}$.

BABY EXAMPLE: define $E_0^{p,q} := (\mathrm{gr}^p\, C)^{p+q}$ and visualize in $\mathbb{Z}^2$



- Since $F^1C \subseteq C$ is a subcomplex, $d^0 \colon C^0 \to C^1$ induces the differential $d_0^{0,0}$.
- For the same reason, $d^0|_{F^1C}$ has image in F^1C^1, so it induces the differential $d_0^{1,-1}$.
- In general, since $F^{p+1}C \subseteq F^pC$ is a subcomplex, the original differentials induce the vertical differentials.

Baby Example: compute cohomology of the columns $\mathrm{gr}^p\, C$

From the $p = 0$ column in the "E_0-page" we compute

$$H(\mathrm{gr}^0\, C) = \left(\frac{\ker(d^0) + (d^0)^{-1}(F^1 C^1)}{F^1 C^0} \right) [0] \oplus \left(\frac{C^1}{F^1 C^1 + \mathrm{im}(d^0)} \right) [-1]$$

Baby Example: compute cohomology of the columns $\mathrm{gr}^p C$

From the $p = 0$ column in the "E_0-page" we compute

$$H(\mathrm{gr}^0 C) = \left(\frac{\ker(d^0) + (d^0)^{-1}(F^1C^1)}{F^1C^0} \right)[0] \oplus \left(\frac{C^1}{F^1C^1 + \mathrm{im}(d^0)} \right)[-1]$$

And from the $p = 1$ column in the "E_0-page" we compute

$$H(\mathrm{gr}^1 C) = \ker\left(d^0|_{F^1C}\right)[0] \oplus \left(\frac{F^1C^1}{\mathrm{im}(d^0|_{F^1C})} \right)[-1]$$

Baby Example: 1^{st} approximation to the $\mathrm{gr}^0 H$ part

We compare $\mathrm{gr}^0 H$ (left column) to $H(\mathrm{gr}^0 C)$ (right column):

$$\left(\frac{\ker(d^0)}{\ker(d^0|_{F^1 C})}\right)[0] \qquad\qquad \left(\frac{\ker(d^0)+(d^0)^{-1}(F^1 C^1)}{F^1 C^0}\right)[0]$$

$$\oplus \qquad\qquad\qquad\qquad \oplus$$

$$\left(\frac{C^1}{F^1 C^1+\mathrm{im}(d^0)}\right)[-1] \qquad\qquad \left(\frac{C^1}{F^1 C^1+\mathrm{im}(d^0)}\right)[-1]$$

Baby Example: 1^{st} approximation to the $\operatorname{gr}^0 H$ part

We compare $\operatorname{gr}^0 H$ (left column) to $H(\operatorname{gr}^0 C)$ (right column):

$$\left(\frac{\ker(d^0)}{\ker(d^0|_{F^1C})}\right)[0] \qquad\qquad \left(\frac{\ker(d^0)+(d^0)^{-1}(F^1C^1)}{F^1C^0}\right)[0]$$

$$\oplus \qquad\qquad\qquad\qquad \oplus$$

$$\left(\frac{C^1}{F^1C^1+\operatorname{im}(d^0)}\right)[-1] \qquad\qquad \left(\frac{C^1}{F^1C^1+\operatorname{im}(d^0)}\right)[-1]$$

- $H^1(\operatorname{gr}^0 C)$ does compute $\operatorname{gr}^0 H^1$ (bottom row),

Baby Example: 1st approximation to the $\mathrm{gr}^0 H$ part

We compare $\mathrm{gr}^0 H$ (left column) to $H(\mathrm{gr}^0 C)$ (right column):

$$\left(\frac{\ker(d^0)}{\ker(d^0|_{F^1C})}\right)[0] \qquad \left(\frac{\ker(d^0)+(d^0)^{-1}(F^1C^1)}{F^1C^0}\right)[0]$$

$$\oplus \qquad\qquad \oplus$$

$$\left(\frac{C^1}{F^1C^1+\mathrm{im}(d^0)}\right)[-1] \qquad \left(\frac{C^1}{F^1C^1+\mathrm{im}(d^0)}\right)[-1]$$

- $H^1(\mathrm{gr}^0 C)$ does compute $\mathrm{gr}^0 H^1$ (bottom row),
- but $H^0(\mathrm{gr}^0 C)$ is not isomorphic to $\mathrm{gr}^0 H^0$ (top row).

Baby Example: 1^{st} approximation to the $\mathrm{gr}^1 H$ part

We compare now $\mathrm{gr}^1 H$ (left column) to $H(\mathrm{gr}^1 C)$ (right column):

$\ker\left(d^0\vert_{F^1 C}\right)[0]$	$\ker\left(d^0\vert_{F^1 C}\right)[0]$
$\oplus$	$\oplus$
$\left(\frac{F^1 C^1 + \mathrm{im}(d^0)}{\mathrm{im}(d^0)}\right)[-1]$	$\left(\frac{F^1 C^1}{\mathrm{im}\left(d^0\vert_{F^1 C}\right)}\right)[-1]$

Baby Example: 1st approximation to the $\mathrm{gr}^1 H$ part

We compare now $\mathrm{gr}^1 H$ (left column) to $H(\mathrm{gr}^1 C)$ (right column):

$$\begin{array}{cc}
\ker\left(d^0|_{F^1C}\right)[0] & \ker\left(d^0|_{F^1C}\right)[0] \\
\oplus & \oplus \\
\left(\frac{F^1C^1+\mathrm{im}(d^0)}{\mathrm{im}(d^0)}\right)[-1] & \left(\frac{F^1C^1}{\mathrm{im}\left(d^0|_{F^1C}\right)}\right)[-1]
\end{array}$$

- In this case $H^0(\mathrm{gr}^1 C)$ does compute $\mathrm{gr}^1 H^0$ (top row),

Baby Example: 1^{st} approximation to the $\operatorname{gr}^1 H$ part

We compare now $\operatorname{gr}^1 H$ (left column) to $H(\operatorname{gr}^1 C)$ (right column):

$$\begin{array}{cc}
\ker\left(d^0|_{F^1C}\right)[0] & \ker\left(d^0|_{F^1C}\right)[0] \\
\oplus & \oplus \\
\left(\dfrac{F^1C^1+\operatorname{im}(d^0)}{\operatorname{im}(d^0)}\right)[-1] & \left(\dfrac{F^1C^1}{\operatorname{im}\left(d^0|_{F^1C}\right)}\right)[-1]
\end{array}$$

- In this case $H^0(\operatorname{gr}^1 C)$ does compute $\operatorname{gr}^1 H^0$ (top row),
- but $H^1(\operatorname{gr}^1 C)$ is not isomorphic to $\operatorname{gr}^1 H^1$ (bottom row).

Baby Example: define $E_1^{p,q} := H^{p+q}(\mathrm{gr}^p\, C)$ and plot in $\mathbb{Z}^2$

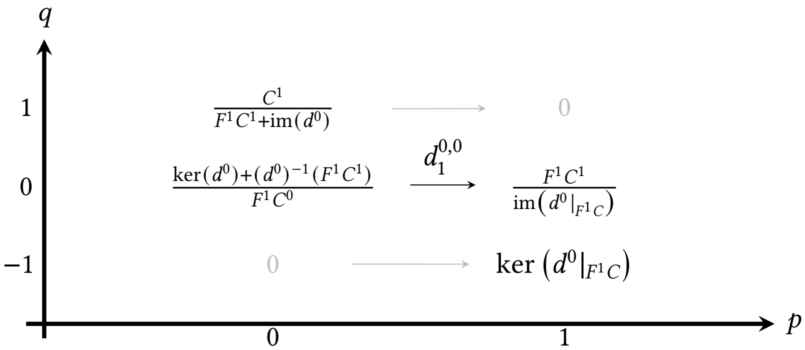
$$
\begin{array}{r|ccc}
q & & & \\
1 & \dfrac{C^1}{F^1C^1+\mathrm{im}(d^0)} & \longrightarrow & 0 \\[2ex]
0 & \dfrac{\ker(d^0)+(d^0)^{-1}(F^1C^1)}{F^1C^0} & \xrightarrow{d_1^{0,0}} & \dfrac{F^1C^1}{\mathrm{im}\left(d^0|_{F^1C}\right)} \\[2ex]
-1 & 0 & \longrightarrow & \ker\left(d^0|_{F^1C}\right) \\
\hline
 & 0 & & 1 \qquad p
\end{array}
$$

Baby Example: define $E_1^{p,q} := H^{p+q}(\mathrm{gr}^p\, C)$ and plot in $\mathbb{Z}^2$

$$
\begin{array}{c|ccc}
q & & & \\
1 & \dfrac{C^1}{F^1C^1+\mathrm{im}(d^0)} & \longrightarrow & 0 \\[2ex]
0 & \dfrac{\ker(d^0)+(d^0)^{-1}(F^1C^1)}{F^1C^0} & \xrightarrow{d_1^{0,0}} & \dfrac{F^1C^1}{\mathrm{im}\left(d^0|_{F^1C}\right)} \\[2ex]
-1 & 0 & \longrightarrow & \ker\left(d^0|_{F^1C}\right) \\
\hline
 & 0 & & 1 \quad p
\end{array}
$$

- The original differential d^0 induces the differential $d_1^{0,0}$.

Baby Example: define $E_1^{p,q} := H^{p+q}(\mathrm{gr}^p\, C)$ and plot in $\mathbb{Z}^2$



- The original differential d^0 induces the differential $d_1^{0,0}$.
- More generally, if we had started from a longer complex, the original differentials would induce the horizontal differentials.

Baby Example: 2^{nd} approximation to $\operatorname{gr} H$

- We obtained the 1^{st} approximation by taking cohomologies on the "E_0-page".

Baby Example: 2^{nd} approximation to $\operatorname{gr} H$

- We obtained the 1^{st} approximation by taking cohomologies on the "E_0-page".
- These 1^{st} approximations were then organized into the "E_1-page" in such a way that $\operatorname{gr}^p H^n$ would correspond to $E_1^{p,n-p}$.

Baby Example: 2^{nd} approximation to $\operatorname{gr} H$

- We obtained the 1^{st} approximation by taking cohomologies on the "E_0-page".
- These 1^{st} approximations were then organized into the "E_1-page" in such a way that $\operatorname{gr}^p H^n$ would correspond to $E_1^{p,n-p}$.
- For example, we have seen that
  - $\operatorname{gr}^0 H^1 = E_1^{0,1}$, altough $\operatorname{gr}^0 H^0 \neq E_1^{0,0}$;

Baby Example: 2^{nd} approximation to $\operatorname{gr} H$

- We obtained the 1^{st} approximation by taking cohomologies on the "E_0-page".
- These 1^{st} approximations were then organized into the "E_1-page" in such a way that $\operatorname{gr}^p H^n$ would correspond to $E_1^{p,n-p}$.
- For example, we have seen that
  - $\operatorname{gr}^0 H^1 = E_1^{0,1}$, altough $\operatorname{gr}^0 H^0 \neq E_1^{0,0}$;
  - $\operatorname{gr}^1 H^0 = E_1^{1,-1}$, although $\operatorname{gr}^1 H^1 \neq E_1^{1,0}$.

Baby Example: 2^{nd} approximation to $\operatorname{gr} H$

- We obtained the 1^{st} approximation by taking cohomologies on the "E_0-page".
- These 1^{st} approximations were then organized into the "E_1-page" in such a way that $\operatorname{gr}^p H^n$ would correspond to $E_1^{p,n-p}$.
- For example, we have seen that
  - $\operatorname{gr}^0 H^1 = E_1^{0,1}$, altough $\operatorname{gr}^0 H^0 \neq E_1^{0,0}$;
  - $\operatorname{gr}^1 H^0 = E_1^{1,-1}$, although $\operatorname{gr}^1 H^1 \neq E_1^{1,0}$.
- Continuing with this way to arrange things, we define the "E_2-page" by taking cohomologies at each point of the "E_1-page", that is

  $$E_2^{p,q} := H^p(E_1^{\bullet,q}),$$

  so that $\operatorname{gr}^p H^n$ would again correspond to $E_2^{p,n-p}$.

Baby Example: 2^{nd} approximation to the $\mathrm{gr}^0 H$ part

We compare then $\mathrm{gr}^0 H$ (left column) to $E_2^{0,\bullet}$ (right column):

$$\left(\frac{\ker(d^0)}{\ker(d^0|_{F^1 C})}\right)[0] \qquad \left(\frac{\ker(d^0)+F^1 C^0}{F^1 C^0}\right)[0]$$

$$\oplus \qquad \oplus$$

$$\left(\frac{C^1}{F^1 C^1+\mathrm{im}(d^0)}\right)[-1] \qquad \left(\frac{C^1}{F^1 C^1+\mathrm{im}(d^0)}\right)[-1]$$

Baby Example: 2^{nd} approximation to the $\mathrm{gr}^0 H$ part

We compare then $\mathrm{gr}^0 H$ (left column) to $E_2^{0,\bullet}$ (right column):

$$\left(\frac{\ker(d^0)}{\ker(d^0|_{F^1C})}\right)[0] \qquad \left(\frac{\ker(d^0)+F^1C^0}{F^1C^0}\right)[0]$$

$$\oplus \qquad\qquad \oplus$$

$$\left(\frac{C^1}{F^1C^1+\mathrm{im}(d^0)}\right)[-1] \qquad \left(\frac{C^1}{F^1C^1+\mathrm{im}(d^0)}\right)[-1]$$

After applying an $\cong$-theorem we see that the two agree!

Baby Example: 2^{nd} approximation to the $\mathrm{gr}^1 H$ part

We compare now $\mathrm{gr}^1 H$ (left column) to $E_2^{1,\bullet-1}$ (right column):

$\ker\left(d^0\vert_{F^1 C}\right)[0]$	$\ker\left(d^0\vert_{F^1 C}\right)[0]$
$\oplus$	$\oplus$
$\left(\frac{F^1 C^1 + \mathrm{im}(d^0)}{\mathrm{im}(d^0)}\right)[-1]$	$\left(\frac{F^1 C^1}{\mathrm{im}(d^0) \cap F^1 C^1}\right)[-1]$

Baby Example: 2^{nd} approximation to the $\operatorname{gr}^1 H$ part

We compare now $\operatorname{gr}^1 H$ (left column) to $E_2^{1,\bullet-1}$ (right column):

$\ker\left(d^0\vert_{F^1C}\right)[0]$	$\ker\left(d^0\vert_{F^1C}\right)[0]$
$\oplus$	$\oplus$
$\left(\frac{F^1C^1+\operatorname{im}(d^0)}{\operatorname{im}(d^0)}\right)[-1]$	$\left(\frac{F^1C^1}{\operatorname{im}(d^0)\cap F^1C^1}\right)[-1]$

After applying an $\cong$-theorem we see that the two sides agree again!

Baby Example: reading off the result from the "E_2-page"

$$
\begin{array}{c|cc}
q & & \\
1 & \frac{C^1}{F^1C^1+\mathrm{im}(d^0)} & 0 \\
0 & \frac{\ker(d^0)}{\ker\left(d^0|_{F^1C}\right)} & \frac{F^1C^1+\mathrm{im}(d^0)}{\mathrm{im}(d^0)} \\
-1 & 0 & \ker\left(d^0|_{F^1C}\right) \\
\hline
 & 0 & 1 \quad p
\end{array}
$$

Baby Example: reading off the result from the "E_2-page"

q		
1	$\frac{C^1}{F^1C^1+\mathrm{im}(d^0)}$	0
0	$\frac{\ker(d^0)}{\ker\left(d^0\vert_{F^1C}\right)}$	$\frac{F^1C^1+\mathrm{im}(d^0)}{\mathrm{im}(d^0)}$
-1	0	$\ker\left(d^0\vert_{F^1C}\right)$
	0	1 p

- $\mathrm{gr}^0 H \cong E_2^{0,0}[0] \oplus E_2^{0,1}[-1]$;

Baby Example: reading off the result from the “E_2-page”

$$
\begin{array}{c|cc}
q & & \\
1 & \dfrac{C^1}{F^1C^1+\mathrm{im}(d^0)} & 0 \\[2ex]
0 & \dfrac{\ker(d^0)}{\ker\left(d^0|_{F^1C}\right)} & \dfrac{F^1C^1+\mathrm{im}(d^0)}{\mathrm{im}(d^0)} \\[2ex]
-1 & 0 & \ker\left(d^0|_{F^1C}\right) \\
\hline
 & 0 & 1 \quad p
\end{array}
$$

- $\mathrm{gr}^0 H \cong E_2^{0,0}[0] \oplus E_2^{0,1}[-1]$;
- $\mathrm{gr}^1 H \cong E_2^{1,-1}[0] \oplus E_2^{1,0}[-1]$.

Baby Example: reading off the result from the "E_2-page"

$$
\begin{array}{c|cc}
q & & \\
1 & \dfrac{C^1}{F^1C^1+\mathrm{im}(d^0)} & 0 \\
0 & \dfrac{\ker(d^0)}{\ker\left(d^0|_{F^1C}\right)} & \dfrac{F^1C^1+\mathrm{im}(d^0)}{\mathrm{im}(d^0)} \\
-1 & 0 & \ker\left(d^0|_{F^1C}\right) \\
\hline
 & 0 & 1 \quad p
\end{array}
$$

- $\mathrm{gr}^0 H \cong E_2^{0,0}[0] \oplus E_2^{0,1}[-1]$;
- $\mathrm{gr}^1 H \cong E_2^{1,-1}[0] \oplus E_2^{1,0}[-1]$.

$$
\begin{cases}
H^0 \cong \mathrm{gr}^0 H^0 \oplus \mathrm{gr}^1 H^0 \cong E_2^{0,0} \oplus E_2^{1,-1} \\
H^1 \cong \mathrm{gr}^1 H^0 \oplus \mathrm{gr}^1 H^1 \cong E_2^{0,1} \oplus E_2^{1,0}
\end{cases}
$$

References

Timothy Y. Chow.
You could have invented spectral sequences.
Notices Amer. Math. Soc., 53(1):15–19, 2006.
1

The Stacks project authors.
The stacks project.
https://stacks.math.columbia.edu, 2020.
(document)