

Basic Notions — Spectral Sequences

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- ▶ **Example:** let k be a field and consider a cochain complex C^\bullet of finite dimensional k -vector spaces

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- ▶ **Goal:** compute the cohomology H^\bullet of the complex C^\bullet

$$H^\bullet := H^0(C^\bullet) \oplus H^1(C^\bullet) \oplus H^2(C^\bullet).$$

Starting point: a filtration on our complex

- If we do not have any extra information about C^\bullet , there is not much more we can do besides applying the definition

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- ▶ But C^\bullet has often some useful filtration $F^\bullet C^\bullet$, say

$$\begin{array}{ccccccc} C^\bullet = F^0 C^\bullet & \cdots 0 & \longrightarrow & C^0 & \xrightarrow{d^0} & C^1 & \xrightarrow{d^1} & C^2 & \longrightarrow & 0 \cdots \\ \cup & & & \cup & & \cup & & \cup & & \\ F^1 C^\bullet & \cdots 0 & \longrightarrow & F^1 C^0 & \longrightarrow & F^1 C^1 & \longrightarrow & F^1 C^2 & \longrightarrow & 0 \cdots \\ \cup & & & \cup & & \cup & & \cup & & \\ 0 = F^2 C^\bullet & \cdots 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \cdots \end{array}$$

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2. We can then recover H^i as the associated graded vector space

$$\operatorname{Gr}(F^\bullet H^i) = F^1 H^i \oplus (H^i / F^1 H^i) \cong H^i \text{ for all } i \in \{0, 1, 2\}.$$

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- ▶ **Q1:** Can we compute the cohomologies of $F^1 C^\bullet$ and $C^\bullet / F^1 C^\bullet$?
- ▶ **Q2:** Are these cohomologies related to $F^1 H^\bullet$ and $H^\bullet / F^1 H^\bullet$?

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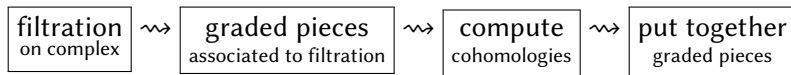
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So in this case we can carry out our naive strategy of computing each graded piece of the induced filtration in cohomology separately:



Possible scenario II: the bad

- Suppose we know nothing about C^\bullet . The *stupid filtration* yields

$$\begin{cases} F^1 C^\bullet := & \cdots \rightarrow 0 \rightarrow 0 \rightarrow C^1 \xrightarrow{d^1} C^2 \rightarrow 0 \rightarrow \cdots \\ C^\bullet / F^1 C^\bullet := & \cdots \rightarrow 0 \rightarrow C^0 \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow \cdots \end{cases}$$

Possible scenario III: the ugly

References



Timothy Y. Chow.

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Notices Amer. Math. Soc., 53(1):15–19, 2006.

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