Basic Notions — Spectral Sequences

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"You could have invented spectral sequences"[1]

[1] Title of the expository article [Cho06]

Notation gets ugly real quick

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- Cohomology is regarded as a copmlex with 0 differentials

$$H(C) = \bigoplus_{i\in\mathbb{Z}} H^i(C)[-i] := \left(\cdots H^{-1}(C) \xrightarrow{0} H^0(C) \xrightarrow{0} H^1(C) \cdots\right).$$

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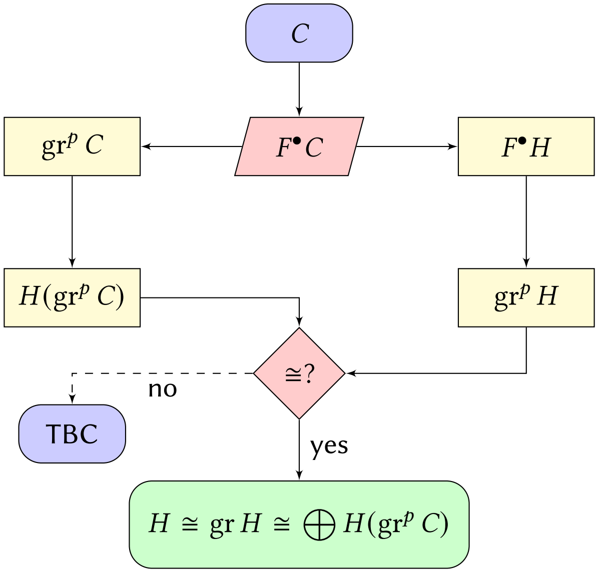
- The graded object associated to a filtration $F^\bullet A$ is denoted

$$\operatorname{gr} A = \bigoplus_{p\in\mathbb{N}} \operatorname{gr}^p A := \bigoplus_{p\in\mathbb{N}} \left(F^p A / F^{p+1} A\right).$$

Goal: compute cohomology with the help of a filtration



Filtration induced in cohomology

A filtration on the complx $F^\bullet C$ induces a filtration in cohomology

$$F^p H := \operatorname{im}(H(F^p C) \longrightarrow H(C)) \subseteq H(C) = H,$$

where $H(F^p C) \longrightarrow H(C)$ is induced by the inclusion map $F^p C \hookrightarrow C$.

Baby example: 2 step filtration on a length 1 complex

$$\begin{tikzcd}
C = F^0 C & \cdots \arrow[r] & 0 \arrow[r] & C^0 \arrow[r, "d^0"] & C^1 \arrow[r] & 0 \arrow[r] & \cdots \\
\rotatebox{90}{$\subseteq$} & & & \rotatebox{90}{$\subseteq$} & \rotatebox{90}{$\subseteq$} & & \\
F^1 C & \cdots \arrow[r] & 0 \arrow[r] & F^1 C^0 \arrow[r, "d^0|_{F^1 C^0}"] & F^1 C^1 \arrow[r] & 0 \arrow[r] & \cdots \\
\rotatebox{90}{$\subseteq$} & & & \rotatebox{90}{$\subseteq$} & \rotatebox{90}{$\subseteq$} & & \\
0 = F^2 C & \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

Baby example: filtration induced in cohomology

$$\begin{tikzcd}
H(F^1C) \arrow[d] & = & \ker\left(d^0|_{F^1C}\right)[0] \oplus \left(\frac{F^1C^1}{\mathrm{im}\left(d^0|_{F^1C}\right)}\right)[-1] \arrow[d] \\
H(C) & = & \ker(d^0)[0] \oplus \left(\frac{C^1}{\mathrm{im}(d^0)}\right)[-1]
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\end{tikzcd}$$

$$\Rightarrow F^1H = \ker\left(d^0|_{F^1C}\right)[0] \oplus \left(\frac{F^1C^1 + \mathrm{im}(d^0)}{\mathrm{im}(d^0)}\right)[-1]$$

References

Timothy Y. Chow.
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Notices Amer. Math. Soc., 53(1):15–19, 2006.
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The Stacks project authors.
The stacks project.
https://stacks.math.columbia.edu, 2020.
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