Basic Notions — Spectral Sequences

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10th December 2020

"You could have invented spectral sequences"[1]

[1]Title of the expository article [Cho06]

Notation gets ugly very soon

Notation and conventions [Sta20]

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- Cohomology is regarded as a copmlex with 0 differentials

$$H(C) = \bigoplus_{i \in \mathbb{Z}} H^i(C)[-i] := \left(\cdots H^{-1}(C) \xrightarrow{0} H^0(C) \xrightarrow{0} H^1(C) \cdots \right).$$

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- If it is clear to which C we refer, denote $H(C)$ simply by H.
- (Decreasing) filtrations are always finite for simplicity:

$$F^\bullet A: \quad 0 = F^n A \subseteq \ldots \subseteq F^0 A = A.$$

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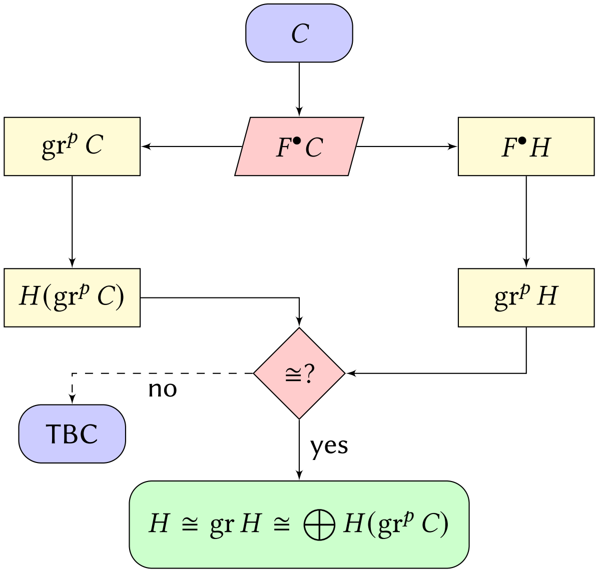
- (Decreasing) filtrations are always finite for simplicity:

$$F^\bullet A: \quad 0 = F^n A \subseteq \ldots \subseteq F^0 A = A.$$

- The graded object associated to a filtration $F^\bullet A$ is denoted

$$\operatorname{gr} A = \bigoplus_{p \in \mathbb{N}} \operatorname{gr}^p A := \bigoplus_{p \in \mathbb{N}} \left(F^p A / F^{p+1} A \right).$$

Goal: compute cohomology with the help of a filtration



Filtration induced in cohomology

A filtration on the complx $F^\bullet C$ induces a filtration in cohomology

$$F^p H := \operatorname{im}(H(F^p C) \to H(C)) \subseteq H(C) = H,$$

where $H(F^p C) \to H(C)$ is induced by the inclusion map $F^p C \hookrightarrow C$.

Baby Example: 2 step filtration on a length 1 complex

$$
\begin{tikzcd}
C \arrow[d, phantom, sloped, "="] \\
F^0C & \cdots \arrow[r] & 0 \arrow[r] & C^0 \arrow[r, "d^0"] & C^1 \arrow[r] & 0 \arrow[r] & \cdots \\
F^1C \arrow[u, phantom, sloped, "\subseteq"] & \cdots \arrow[r] & 0 \arrow[r] & F^1C^0 \arrow[u, phantom, sloped, "\subseteq"] \arrow[r, "d^0|_{F^1C^0}"] & F^1C^1 \arrow[u, phantom, sloped, "\subseteq"] \arrow[r] & 0 \arrow[r] & \cdots \\
F^2C \arrow[u, phantom, sloped, "\subseteq"] \arrow[d, phantom, sloped, "="] & \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[u, phantom, sloped, "\subseteq"] \arrow[r] & 0 \arrow[u, phantom, sloped, "\subseteq"] \arrow[r] & 0 \arrow[r] & \cdots \\
0
\end{tikzcd}
$$

Baby Example: filtration induced in cohomology

We first look at the map induced by $F^1C \hookrightarrow C$ in cohomology:

$$\begin{array}{ccc}
H(F^1C) & = & \ker\left(d^0|_{F^1C}\right)[0] \oplus \left(\frac{F^1C^1}{\mathrm{im}\left(d^0|_{F^1C}\right)}\right)[-1] \\
\downarrow & & \downarrow \\
H(C) & = & \ker(d^0)[0] \oplus \left(\frac{C^1}{\mathrm{im}(d^0)}\right)[-1]
\end{array}$$

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H(C) & = & \ker(d^0)[0] \oplus \left(\frac{C^1}{\mathrm{im}(d^0)}\right)[-1]
\end{tikzcd}$$

The filtration on cohomology was given by its image, hence

$$F^1H = \ker\left(d^0|_{F^1C}\right)[0] \oplus \left(\frac{F^1C^1 + \mathrm{im}(d^0)}{\mathrm{im}(d^0)}\right)[-1]$$

Baby Example: graded cohomology pieces

Recall that $\mathrm{gr}^0 H := F^0 H / F^1 H = H/F^1 H$, hence (after $\cong$-theorem)

$$\mathrm{gr}^0 H = \left(\frac{\ker(d^0)}{\ker\left(d^0|_{F^1 C}\right)} \right)[0] \oplus \left(\frac{C^1}{F^1 C^1 + \mathrm{im}(d^0)} \right)[-1]$$

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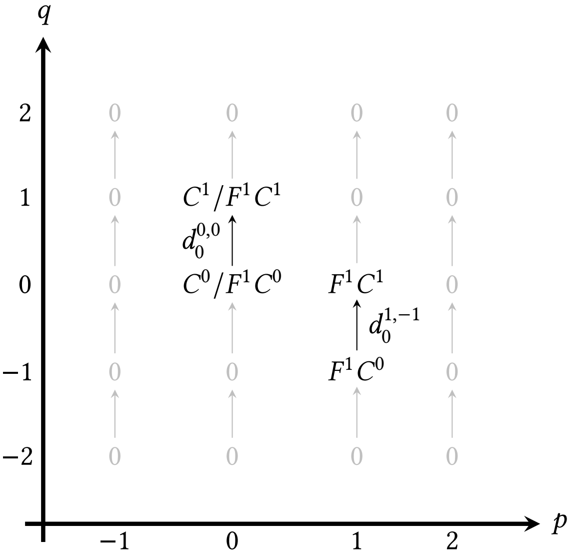
Similarly, $\mathrm{gr}^1 H = F^1 H/0 = F^1 H$, thus

$$\mathrm{gr}^1 H = \ker\left(d^0|_{F^1C}\right)[0] \oplus \left(\frac{F^1C^1 + \mathrm{im}(d^0)}{\mathrm{im}(d^0)} \right)[-1]$$

Baby Example: define $E_0^{p,q} := (\mathrm{gr}^p\, C)^{p+q}$ and visualize in $\mathbb{Z}^2$



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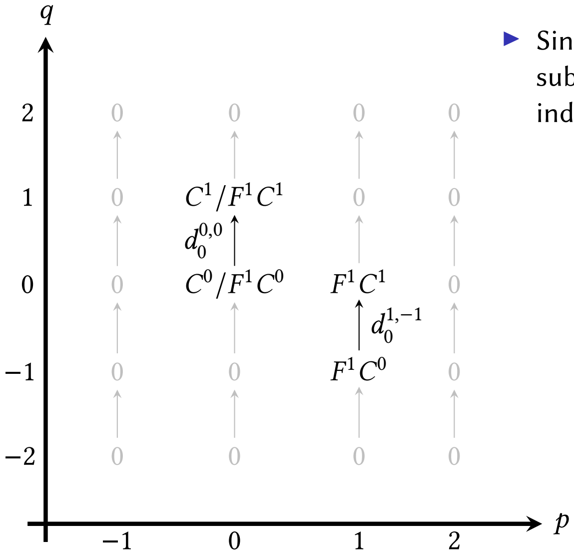

- Since $F^1C \subseteq C$ is a subcomplex, $d^0 \colon C^0 \to C^1$ induces the differential $d_0^{0,0}$.

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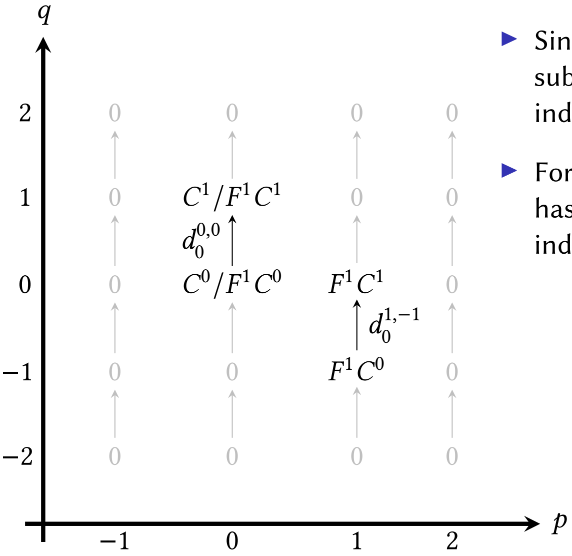

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- For the same reason, $d^0|_{F^1C}$ has image in F^1C^1, so it induces the differential $d_0^{1,-1}$.

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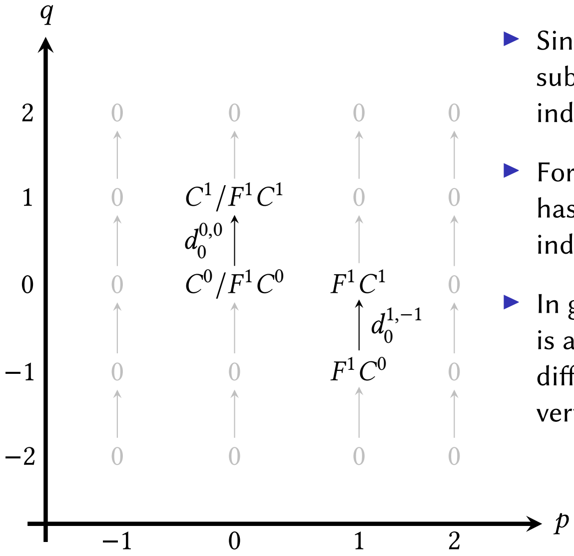

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- For the same reason, $d^0|_{F^1C}$ has image in F^1C^1, so it induces the differential $d_0^{1,-1}$.
- In general, since $F^{p+1}C \subseteq F^pC$ is a subcomplex, the original differentials induce the vertical differentials.

Baby Example: compute cohomology of the columns $\mathrm{gr}^p C$

From the $p = 0$ column in the "E_0-page" we compute

$$H(\mathrm{gr}^0 C) = \left(\frac{\ker(d^0) + (d^0)^{-1}(F^1 C^1)}{F^1 C^0} \right) [0] \oplus \left(\frac{C^1}{F^1 C^1 + \mathrm{im}(d^0)} \right) [-1]$$

Baby Example: compute cohomology of the columns $\mathrm{gr}^p\, C$

From the $p = 0$ column in the "E_0-page" we compute

$$H(\mathrm{gr}^0\, C) = \left(\frac{\ker(d^0) + (d^0)^{-1}(F^1C^1)}{F^1C^0}\right)[0] \oplus \left(\frac{C^1}{F^1C^1 + \mathrm{im}(d^0)}\right)[-1]$$

And from the $p = 1$ column in the "E_0-page" we compute

$$H(\mathrm{gr}^1\, C) = \ker\left(d^0|_{F^1C}\right)[0] \oplus \left(\frac{F^1C^1}{\mathrm{im}(d^0|_{F^1C})}\right)[-1]$$

Baby Example: 1^{st} approximation to the $\mathrm{gr}^0 H$ part

We compare $\mathrm{gr}^0 H$ (left column) to $H(\mathrm{gr}^0 C)$ (right column):

$$\left(\frac{\ker(d^0)}{\ker(d^0|_{F^1 C})}\right)[0] \qquad \left(\frac{\ker(d^0)+(d^0)^{-1}(F^1 C^1)}{F^1 C^0}\right)[0]$$

$$\oplus \qquad\qquad \oplus$$

$$\left(\frac{C^1}{F^1 C^1+\mathrm{im}(d^0)}\right)[-1] \qquad \left(\frac{C^1}{F^1 C^1+\mathrm{im}(d^0)}\right)[-1]$$

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$$\oplus \qquad\qquad\qquad\qquad \oplus$$

$$\left(\frac{C^1}{F^1 C^1+\operatorname{im}(d^0)}\right)[-1] \qquad\qquad \left(\frac{C^1}{F^1 C^1+\operatorname{im}(d^0)}\right)[-1]$$

- $H^1(\operatorname{gr}^0 C)$ does compute $\operatorname{gr}^0 H^1$ (bottom row),

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$$\oplus \qquad \oplus$$

$$\left(\frac{C^1}{F^1C^1+\mathrm{im}(d^0)}\right)[-1] \qquad \left(\frac{C^1}{F^1C^1+\mathrm{im}(d^0)}\right)[-1]$$

- $H^1(\mathrm{gr}^0 C)$ does compute $\mathrm{gr}^0 H^1$ (bottom row),
- but $H^0(\mathrm{gr}^0 C)$ is not isomorphic to $\mathrm{gr}^0 H^0$ (top row).

Baby Example: 1^{st} approximation to the $\mathrm{gr}^1 H$ part

We compare now $\mathrm{gr}^1 H$ (left column) to $H(\mathrm{gr}^1 C)$ (right column):

$$\ker\left(d^0|_{F^1 C}\right)[0] \qquad\qquad \ker\left(d^0|_{F^1 C}\right)[0]$$

$$\oplus \qquad\qquad \oplus$$

$$\left(\frac{F^1 C^1 + \mathrm{im}(d^0)}{\mathrm{im}(d^0)}\right)[-1] \qquad\qquad \left(\frac{F^1 C^1}{\mathrm{im}\left(d^0|_{F^1 C}\right)}\right)[-1]$$

Baby Example: 1$^{\text{st}}$ approximation to the $\operatorname{gr}^1 H$ part

We compare now $\operatorname{gr}^1 H$ (left column) to $H(\operatorname{gr}^1 C)$ (right column):

$\ker\left(d^0\vert_{F^1C}\right)[0]$	$\ker\left(d^0\vert_{F^1C}\right)[0]$
$\oplus$	$\oplus$
$\left(\dfrac{F^1C^1+\operatorname{im}(d^0)}{\operatorname{im}(d^0)}\right)[-1]$	$\left(\dfrac{F^1C^1}{\operatorname{im}\left(d^0\vert_{F^1C}\right)}\right)[-1]$

- In this case $H^0(\operatorname{gr}^0 C)$ does compute $\operatorname{gr}^0 H^0$ (top row),

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$\ker\left(d^0\vert_{F^1 C}\right)[0]$	$\ker\left(d^0\vert_{F^1 C}\right)[0]$
$\oplus$	$\oplus$
$\left(\frac{F^1 C^1 + \mathrm{im}(d^0)}{\mathrm{im}(d^0)}\right)[-1]$	$\left(\frac{F^1 C^1}{\mathrm{im}\left(d^0\vert_{F^1 C}\right)}\right)[-1]$

- In this case $H^0(\mathrm{gr}^0 C)$ does compute $\mathrm{gr}^0 H^0$ (top row),
- but $H^1(\mathrm{gr}^0 C)$ is not isomorphic to $\mathrm{gr}^0 H^1$ (bottom row).

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$$
\begin{tikzcd}
q & & \\
1 & \frac{C^1}{F^1C^1+\mathrm{im}(d^0)} \arrow[r] & 0 \\
0 & \frac{\ker(d^0)+(d^0)^{-1}(F^1C^1)}{F^1C^0} \arrow[r, "d_1^{0,0}"] & \frac{F^1C^1}{\mathrm{im}\left(d^0|_{F^1C}\right)} \\
-1 & 0 \arrow[r] & \ker\left(d^0|_{F^1C}\right) \\
 & 0 & 1 & p
\end{tikzcd}
$$

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$$\begin{array}{r|ccc}
q & & & \\
1 & \dfrac{C^1}{F^1C^1+\mathrm{im}(d^0)} & \longrightarrow & 0 \\
0 & \dfrac{\ker(d^0)+(d^0)^{-1}(F^1C^1)}{F^1C^0} & \xrightarrow{d_1^{0,0}} & \dfrac{F^1C^1}{\mathrm{im}\left(d^0|_{F^1C}\right)} \\
-1 & 0 & \longrightarrow & \ker\left(d^0|_{F^1C}\right) \\
\hline
 & 0 & & 1 \quad p
\end{array}$$

- The original differential d^0 induces the differential $d_1^{0,0}$.

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-1 & 0 \arrow[r] & \ker\left(d^0|_{F^1C}\right) \\
 & 0 & 1 & p
\end{tikzcd}$$

- The original differential d^0 induces the differential $d_1^{0,0}$.
- More generally, if we had started from a longer complex, the original differentials would induce the horizontal differentials.

References

Timothy Y. Chow.
You could have invented spectral sequences.
Notices Amer. Math. Soc., 53(1):15–19, 2006.
1

The Stacks project authors.
The stacks project.
https://stacks.math.columbia.edu, 2020.
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