Basic Notions — Spectral Sequences

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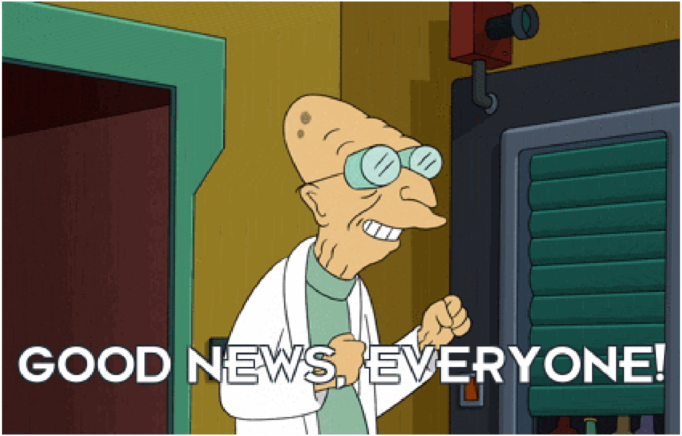
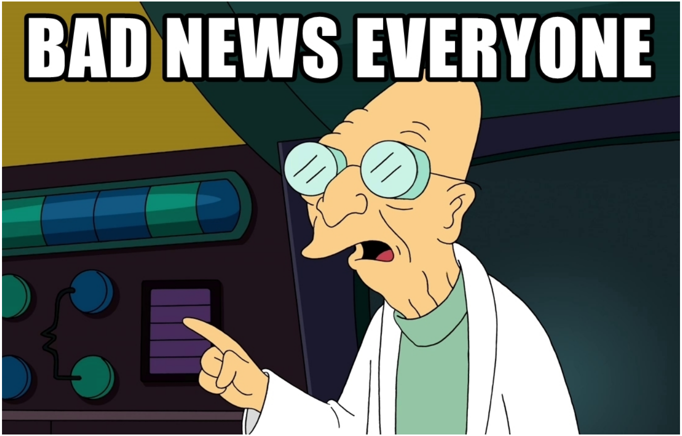
"You could have invented spectral sequences"[1]



[1] Title of the expository article [Cho06]

Notation gets ugly real quick



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$$H(C) = \bigoplus_{i\in\mathbb{Z}} H^i(C)[-i] := \left(\cdots H^{-1}(C) \xrightarrow{0} H^0(C) \xrightarrow{0} H^1(C) \cdots\right).$$

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- The graded object associated to a filtration $F^\bullet A$ is denoted

$$\operatorname{gr} A = \bigoplus_{p\in\mathbb{N}} \operatorname{gr}^p A := \bigoplus_{p\in\mathbb{N}} \left(F^p A / F^{p+1} A\right).$$

Goal: compute cohomology with the help of a filtration

C

$\operatorname{gr}^p C$ $\quad$ $F^\bullet C$ $\quad$ $F^\bullet H$

$H(\operatorname{gr}^p C)$ $\quad$ $\cong ?$ $\quad$ $\operatorname{gr}^p H$

$$H \cong \operatorname{gr} H \cong \bigoplus H(\operatorname{gr}^p C)$$

Starting point: a filtration on our complex

- If we do not have any extra information about $C^\bullet$, there is not much more we can do besides applying the definition

$$H^\bullet = \ker(d^0)[0] \oplus \frac{\ker(d^1)}{\operatorname{im}(d^0)}[-1] \oplus \frac{C^2}{\operatorname{im}(d^1)}[-2].$$

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- But $C^\bullet$ has often some <u>useful</u> filtration $F^\bullet C^\bullet$, say

$$\begin{array}{ccccccccccc}
C^\bullet = F^0 C^\bullet & & \cdots 0 & \longrightarrow & C^0 & \xrightarrow{d^0} & C^1 & \xrightarrow{d^1} & C^2 & \longrightarrow & 0 \cdots \\
\cup\!\shortmid & & & & \cup\!\shortmid & & \cup\!\shortmid & & \cup\!\shortmid & & \\
F^1 C^\bullet & & \cdots 0 & \rightarrow & F^1 C^0 & \longrightarrow & F^1 C^1 & \longrightarrow & F^1 C^2 & \rightarrow & 0 \cdots \\
\cup\!\shortmid & & & & \cup\!\shortmid & & \cup\!\shortmid & & \cup\!\shortmid & & \\
0 = F^2 C^\bullet & & \cdots 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \cdots
\end{array}$$

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$$\operatorname{gr}(F^\bullet H^i) = F^1 H^i \oplus \left(H^i / F^1 H^i\right) \cong H^i \text{ for all } i \in \{0, 1, 2\}.$$

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- **Q1:** Can we compute the cohomologies of $F^1C^\bullet$ and $C^\bullet/F^1C^\bullet$?
- **Q2:** Are these cohomologies related to $F^1H^\bullet$ and $H^\bullet/F^1H^\bullet$?

Possible scenario I: the good

- Suppose $C^\bullet = C_1^\bullet \oplus C_2^\bullet$ with $H^\bullet(C_1^\bullet)$ and $H^\bullet(C_2^\bullet)$ computable.

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and similarly $H^\bullet/F^1H^\bullet = H^\bullet(C_2^\bullet) = H^\bullet(C^\bullet/F^1C^\bullet)$.

So in this case we can carry out our naive strategy of computing each graded piece of the induced filtration in cohomology separately:

$$\boxed{\begin{array}{c}\text{filtration}\\ \text{on complex}\end{array}} \rightsquigarrow \boxed{\begin{array}{c}\text{graded pieces}\\ \text{associated to filtration}\end{array}} \rightsquigarrow \boxed{\begin{array}{c}\text{compute}\\ \text{cohomologies}\end{array}} \rightsquigarrow \boxed{\begin{array}{c}\text{put together}\\ \text{graded pieces}\end{array}}$$

Possible scenario II: the bad

Suppose we know nothing about $C^\bullet$. The *stupid filtration*

$$\begin{cases} F^1C^\bullet := & \cdots \to 0 \to 0 \to C^1 \xrightarrow{d^1} C^2 \to 0 \to \cdots \\ F^2C^\bullet := & \cdots \to 0 \to 0 \to 0 \to C^2 \to 0 \to \cdots \end{cases}$$

induces the cohomology filtration

$$\begin{cases} F^1H^\bullet := \mathrm{im}(H^\bullet(F^1C^\bullet) \to H^\bullet(C^\bullet)) = H^1(C^\bullet)[-1] \oplus H^2(C^\bullet)[-2] \\ F^2H^\bullet := \mathrm{im}(H^\bullet(F^2C^\bullet) \to H^\bullet(C^\bullet)) = H^2(C^\bullet)[-2] \end{cases}$$

whose graded pieces are $H^0(C^\bullet)[0]$, $H^1(C^\bullet)[-1]$ and $H^2(C^\bullet)[-2]$. So in this case computing $\mathrm{gr}^i H^\bullet(C^\bullet)$ is as hard as computing $H^i(C^\bullet)$ and we just wasted our time with this detour.

Possible scenario III: the ugly

Suppose now that we are somewhere in between the good and the bad scenarios. Namely, assume that we have some filtration $0 \subsetneq F^1C^\bullet \subsetneq C^\bullet$ which does not correspond to a direct sum decomposition but such that we can compute the graded pieces $\mathrm{gr}^i(H^\bullet(C^\bullet))$ of the filtration induced in cohomology.

References

Timothy Y. Chow.
You could have invented spectral sequences.
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The Stacks project authors.
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