Basic Notions — Spectral Sequences

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10th December 2020

"You could have invented spectral sequences"[1]

[1] Title of the expository article [Cho06]

Notation gets ugly very soon

Notation and conventions [Sta20]

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- Cohomology is regarded as a copmlex with 0 differentials

$$H(C) = \bigoplus_{i \in \mathbb{Z}} H^i(C)[-i] := \left(\cdots H^{-1}(C) \xrightarrow{0} H^0(C) \xrightarrow{0} H^1(C) \cdots \right).$$

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$$F^\bullet A: \quad 0 = F^n A \subseteq \ldots \subseteq F^0 A = A.$$

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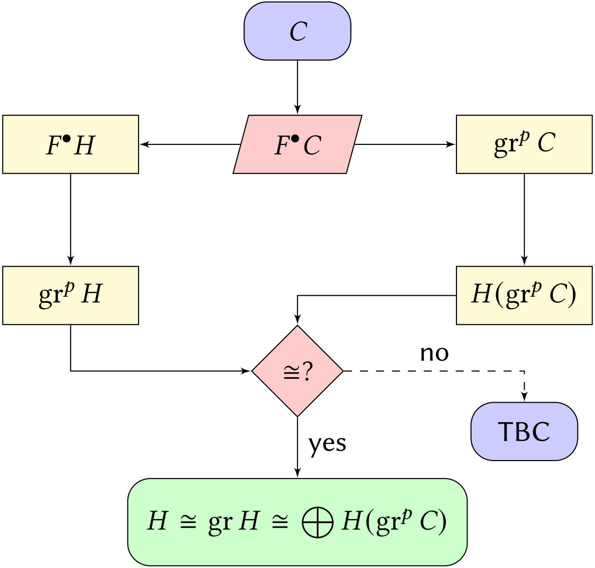
- (Decreasing) filtrations are always finite for simplicity:

$$F^\bullet A: \quad 0 = F^n A \subseteq \ldots \subseteq F^0 A = A.$$

- The graded object associated to a filtration $F^\bullet A$ is denoted

$$\operatorname{gr} A = \bigoplus_{p \in \mathbb{N}} \operatorname{gr}^p A := \bigoplus_{p \in \mathbb{N}} \left(F^p A / F^{p+1} A \right).$$

GOAL: compute cohomology with the help of a filtration



Filtration induced in cohomology

A filtration on the complx $F^\bullet C$ induces a filtration in cohomology

$$F^p H := \operatorname{im}(H(F^p C) \longrightarrow H(C)) \subseteq H(C) = H,$$

where $H(F^p C) \longrightarrow H(C)$ is induced by the inclusion map $F^p C \hookrightarrow C$.

Baby Example: 2 step filtration on a length 1 complex

$$
\begin{tikzcd}
C \arrow[d, phantom, "\shortparallel"] & & & & & & \\
F^0C \arrow[d, phantom, "\cup|"] & \cdots \arrow[r] & 0 \arrow[r] & C^0 \arrow[r, "d^0"] \arrow[d, phantom, "\cup|"] & C^1 \arrow[r] \arrow[d, phantom, "\cup|"] & 0 \arrow[r] & \cdots \\
F^1C \arrow[d, phantom, "\cup|"] & \cdots \arrow[r] & 0 \arrow[r] & F^1C^0 \arrow[r, "d^0|_{F^1C^0}"] \arrow[d, phantom, "\cup|"] & F^1C^1 \arrow[r] \arrow[d, phantom, "\cup|"] & 0 \arrow[r] & \cdots \\
F^2C \arrow[d, phantom, "\shortparallel"] & \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \\
0 & & & & & &
\end{tikzcd}
$$

Baby Example: filtration induced in cohomology

We first look at the map induced by $F^1C \hookrightarrow C$ in cohomology:

$$
\begin{array}{ccc}
H(F^1C) & = & \ker\left(d^0|_{F^1C}\right)[0] \oplus \left(\frac{F^1C^1}{\mathrm{im}\left(d^0|_{F^1C}\right)}\right)[-1] \\
\downarrow & & \downarrow \\
H(C) & = & \ker(d^0)[0] \oplus \left(\frac{C^1}{\mathrm{im}(d^0)}\right)[-1]
\end{array}
$$

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\Big\downarrow & & \Big\downarrow \\
H(C) & = & \ker(d^0)[0] \oplus \left(\dfrac{C^1}{\mathrm{im}(d^0)}\right)[-1]
\end{array}$$

The filtration on cohomology was given by its image, hence

$$F^1H = \ker\left(d^0|_{F^1C}\right)[0] \oplus \left(\frac{F^1C^1 + \mathrm{im}(d^0)}{\mathrm{im}(d^0)}\right)[-1]$$

Baby Example: graded cohomology pieces

Recall that $\mathrm{gr}^0 H := F^0 H / F^1 H = H / F^1 H$, hence (after $\cong$-theorem)

$$\mathrm{gr}^0 H = \left(\frac{\ker(d^0)}{\ker\left(d^0|_{F^1 C}\right)} \right) [0] \oplus \left(\frac{C^1}{F^1 C^1 + \mathrm{im}(d^0)} \right) [-1]$$

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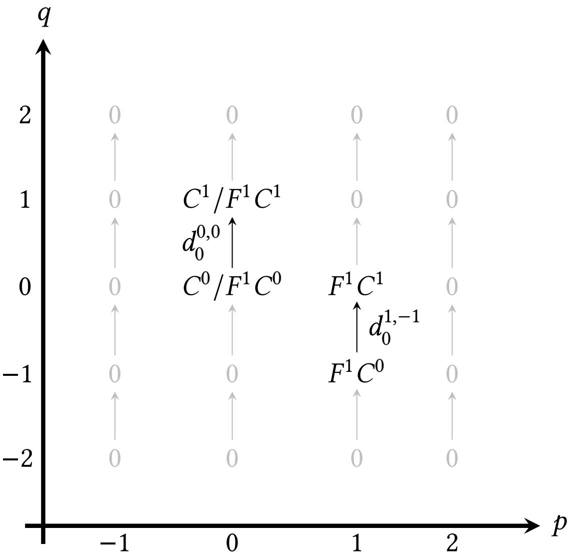
Similarly, $\mathrm{gr}^1 H = F^1 H / 0 = F^1 H$, thus

$$\mathrm{gr}^1 H = \ker\left(d^0|_{F^1 C}\right) [0] \oplus \left(\frac{F^1 C^1 + \mathrm{im}(d^0)}{\mathrm{im}(d^0)} \right) [-1]$$

Baby Example: define $E_0^{p,q} := (\mathrm{gr}^p\, C)^{p+q}$ and visualize in $\mathbb{Z}^2$



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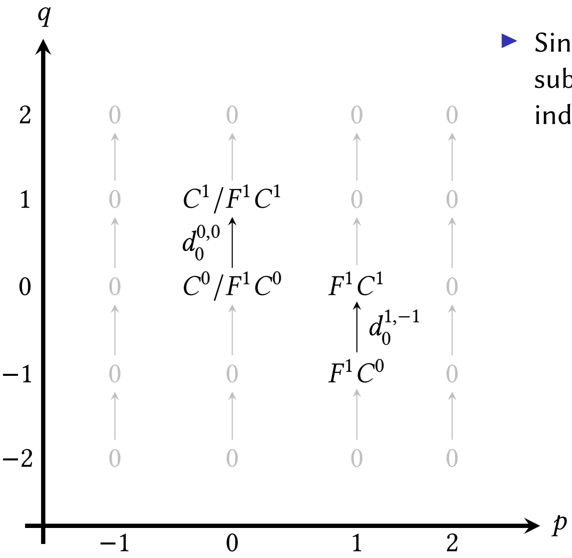

- Since $F^1C \subseteq C$ is a subcomplex, $d^0 \colon C^0 \to C^1$ induces the differential $d_0^{0,0}$.

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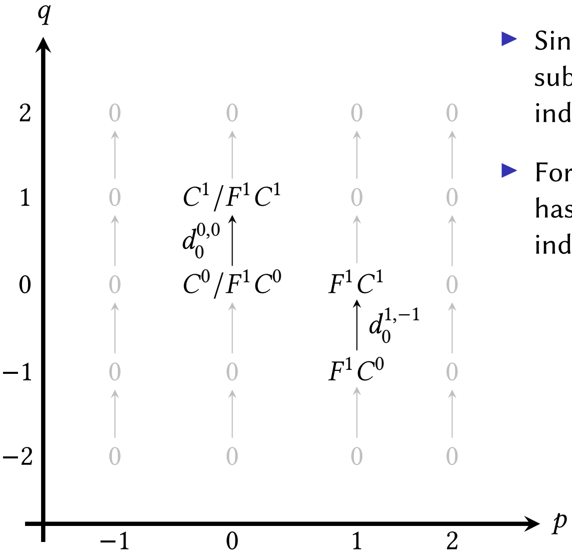

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- For the same reason, $d^0|_{F^1C}$ has image in F^1C^1, so it induces the differential $d_0^{1,-1}$.

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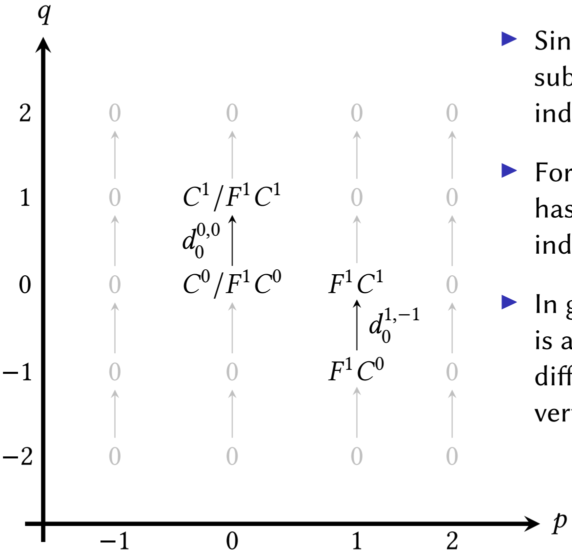

- Since $F^1C \subseteq C$ is a subcomplex, $d^0 : C^0 \to C^1$ induces the differential $d_0^{0,0}$.
- For the same reason, $d^0|_{F^1C}$ has image in F^1C^1, so it induces the differential $d_0^{1,-1}$.
- In general, since $F^{p+1}C \subseteq F^pC$ is a subcomplex, the original differentials induce the vertical differentials.

Baby Example: compute cohomology of the columns $\mathrm{gr}^p\, C$

From the $p = 0$ column in the "E_0-page" we compute

$$H(\mathrm{gr}^0\, C) = \left(\frac{\ker(d^0) + (d^0)^{-1}(F^1C^1)}{F^1C^0} \right)[0] \oplus \left(\frac{C^1}{F^1C^1 + \mathrm{im}(d^0)} \right)[-1]$$

Baby Example: compute cohomology of the columns $\mathrm{gr}^p\, C$

From the $p = 0$ column in the "E_0-page" we compute

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And from the $p = 1$ column in the "E_0-page" we compute

$$H(\mathrm{gr}^1\, C) = \ker\left(d^0|_{F^1C}\right)[0] \oplus \left(\frac{F^1C^1}{\mathrm{im}(d^0|_{F^1C})}\right)[-1]$$

Baby Example: 1$^{\text{st}}$ approximation to the $\mathrm{gr}^0 H$ part

We compare $\mathrm{gr}^0 H$ (left column) to $H(\mathrm{gr}^0 C)$ (right column):

$$\left(\frac{\ker(d^0)}{\ker(d^0|_{F^1 C})}\right)[0] \qquad \left(\frac{\ker(d^0)+(d^0)^{-1}(F^1C^1)}{F^1C^0}\right)[0]$$

$$\oplus \qquad \oplus$$

$$\left(\frac{C^1}{F^1C^1+\mathrm{im}(d^0)}\right)[-1] \qquad \left(\frac{C^1}{F^1C^1+\mathrm{im}(d^0)}\right)[-1]$$

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- $H^1(\operatorname{gr}^0 C)$ does compute $\operatorname{gr}^0 H^1$ (bottom row),

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- $H^1(\operatorname{gr}^0 C)$ does compute $\operatorname{gr}^0 H^1$ (bottom row),
- but $H^0(\operatorname{gr}^0 C)$ is not isomorphic to $\operatorname{gr}^0 H^0$ (top row).

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We compare now $\operatorname{gr}^1 H$ (left column) to $H(\operatorname{gr}^1 C)$ (right column):

$$\ker\left(d^0|_{F^1C}\right)[0] \qquad\qquad \ker\left(d^0|_{F^1C}\right)[0]$$

$$\oplus \qquad\qquad\qquad\qquad \oplus$$

$$\left(\frac{F^1C^1+\operatorname{im}(d^0)}{\operatorname{im}(d^0)}\right)[-1] \qquad\qquad \left(\frac{F^1C^1}{\operatorname{im}\left(d^0|_{F^1C}\right)}\right)[-1]$$

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- In this case $H^0(\operatorname{gr}^1 C)$ does compute $\operatorname{gr}^1 H^0$ (top row),
- but $H^1(\operatorname{gr}^1 C)$ is not isomorphic to $\operatorname{gr}^1 H^1$ (bottom row).

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$$
\begin{array}{c|ccc}
q & & & \\
1 & \dfrac{C^1}{F^1C^1+\mathrm{im}(d^0)} & \longrightarrow & 0 \\[2ex]
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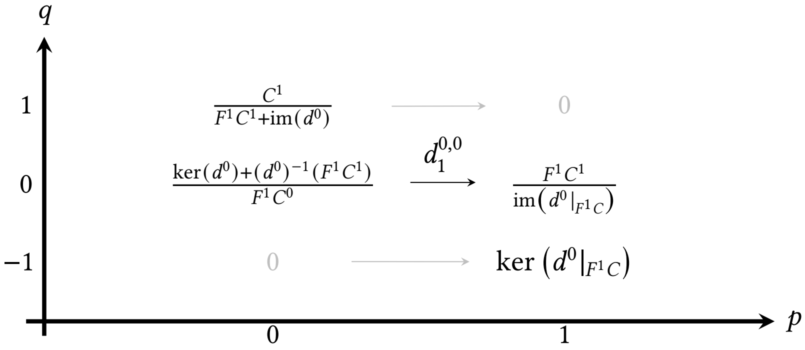

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\hline
 & 0 & & 1 & p
\end{array}
$$

- The original differential d^0 induces the differential $d_1^{0,0}$.
- More generally, if we had started from a longer complex, the original differentials would induce the horizontal differentials.

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- We obtained the 1^{st} approximation by taking cohomologies on the "E_0-page".

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- We obtained the 1^{st} approximation by taking cohomologies on the "E_0-page".
- These 1^{st} approximations were then organized into the "E_1-page" in such a way that $\operatorname{gr}^p H^n$ would correspond to $E_1^{p,n-p}$.

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  - $\operatorname{gr}^1 H^0 = E_1^{1,-1}$, although $\operatorname{gr}^1 H^1 \neq E_1^{1,0}$.
- Continuing with this way to arrange things, we define the "E_2-page" by taking cohomologies at each point of the "E_1-page", that is
$$E_2^{p,q} := H^p(E_1^{\bullet,q}),$$
so that $\operatorname{gr}^p H^n$ would again correspond to $E_2^{p,n-p}$.

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We compare then $\mathrm{gr}^0 H$ (left column) to $E_2^{0,\bullet}$ (right column):

$$\left(\frac{\ker(d^0)}{\ker(d^0|_{F^1C})}\right)[0] \qquad\qquad \left(\frac{\ker(d^0)+F^1C^0}{F^1C^0}\right)[0]$$

$$\oplus \qquad\qquad\qquad\qquad \oplus$$

$$\left(\frac{C^1}{F^1C^1+\mathrm{im}(d^0)}\right)[-1] \qquad\qquad \left(\frac{C^1}{F^1C^1+\mathrm{im}(d^0)}\right)[-1]$$

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After applying an $\cong$-theorem we see that the two agree!

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We compare now $\operatorname{gr}^1 H$ (left column) to $E_2^{1,\bullet-1}$ (right column):

$$\ker\left(d^0|_{F^1C}\right)[0] \qquad\qquad \ker\left(d^0|_{F^1C}\right)[0]$$

$$\oplus \qquad\qquad \oplus$$

$$\left(\frac{F^1C^1+\operatorname{im}(d^0)}{\operatorname{im}(d^0)}\right)[-1] \qquad\qquad \left(\frac{F^1C^1}{\operatorname{im}(d^0)\cap F^1C^1}\right)[-1]$$

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We compare now $\mathrm{gr}^1 H$ (left column) to $E_2^{1,\bullet-1}$ (right column):

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After applying an $\cong$-theorem we see that the two sides agree again!

Baby Example: reading off the result from the "E_2-page"

$$
\begin{array}{c|cc}
q & & \\
1 & \dfrac{C^1}{F^1C^1+\mathrm{im}(d^0)} & 0 \\[2ex]
0 & \dfrac{\ker(d^0)}{\ker\left(d^0|_{F^1C}\right)} & \dfrac{F^1C^1+\mathrm{im}(d^0)}{\mathrm{im}(d^0)} \\[2ex]
-1 & 0 & \ker\left(d^0|_{F^1C}\right) \\
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- $\mathrm{gr}^0 H \cong E_2^{0,0}[0] \oplus E_2^{0,1}[-1]$;

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$$

- $\mathrm{gr}^0 H \cong E_2^{0,0}[0] \oplus E_2^{0,1}[-1]$;
- $\mathrm{gr}^1 H \cong E_2^{1,-1}[0] \oplus E_2^{1,0}[-1]$.

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$$

- $\mathrm{gr}^0 H \cong E_2^{0,0}[0] \oplus E_2^{0,1}[-1]$;
- $\mathrm{gr}^1 H \cong E_2^{1,-1}[0] \oplus E_2^{1,0}[-1]$.

$$
\begin{cases}
H^0 \cong \mathrm{gr}^0 H^0 \oplus \mathrm{gr}^1 H^0 \cong E_2^{0,0} \oplus E_2^{1,-1} \\
H^1 \cong \mathrm{gr}^1 H^0 \oplus \mathrm{gr}^1 H^1 \cong E_2^{0,1} \oplus E_2^{1,0}
\end{cases}
$$

Definition [Sta20, Tag 011N]

A *spectral sequence* in an abelian category $\mathcal{A}$ is given by a system $(E_r, d_r)_{r \in \mathbb{N}}$ such that for all $r \in \mathbb{N}$ we have:

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- $E_0^{p,q} = \operatorname{gr}^p C^{p+q}$,
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If the filtration induced on each C^n was finite, then letting $r \longrightarrow \infty$ we would obtain the *limit* E_∞ of the spectral sequence, given by

$$E_\infty^{p,q} = (F^pC^{p+q} \cap \ker(d))/(F^pC^{p+q} \cap \mathrm{im}(d)).$$

Convergence: $E_r^{p,q} \Rightarrow H^{p+q}(C)$ [McC01, Theorem 2.6]

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Let $F^\bullet C$ be a filtered complex in an abelian category $\mathcal{A}$ and assume that the filtration induced on each C^n is finite. Then:

- The induced filtration $F^\bullet H$ is finite.
- For each $p, q \in \mathbb{Z}$ there exists $r_0 \in \mathbb{N}$ such that

$$E_\infty^{p,q} = E_r^{p,q} \text{ for all } r \geqslant r_0.$$

- The spectral sequence *converges* to $F^\bullet H$, i.e.

$$E_\infty^{p,q} \cong \operatorname{gr}^p H^{p+q} \text{ for all } p, q \in \mathbb{Z}.$$

In particular, modulo extension problems (e.g. if we are working with vector spaces), we can recover H as

$$H^n(C) \cong \bigoplus_{p \in \mathbb{Z}} E_\infty^{p,n-p} \text{ for all } n \in \mathbb{Z}.$$

Applications of spectral sequences from filtrations

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- Two different ways to compute the cohomology of the total complex of a double complex.

To avoid getting too much into specific topics, we will focus on the third application. We will do this again with an example.

Snake Lemma via spectral sequences

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