Seminar on Condensed Mathematics
Talk 2: Condensed Abelian Groups

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1 The category of condensed abelian groups

Recall from the previous talk:

Definition 1.1. The *pro-étale site* of a point, denoted $*_{prot}$, consists of:

- The category whose objects are profinite sets S (topological spaces homeomorphic to an inverse limit of finite discrete topological spaces, a.k.a. totally disconnected compact Hausdorff spaces) and whose morphisms are continuous maps.
- The coverings of a profinite set S are all finite families of jointly surjective maps, i.e. all families of morphisms $\{S_i \to S\}_{i \in I}$ indexed by finite sets J such that $\sqcup_{i \in I} S_i \to S$ is surjective.

It is easy to check that the axioms of a covering family are satisfied (see [Sta19, Tag 00VH]), so we have a well-defined site. Let us start by characterizing sheaves on this site:

Lemma 1.2. *A contravariant functor T from $*_{pro\acute{e}t}$ to the category of abelian groups $\mathcal{A}b$ is a sheaf if and only if the following three conditions hold:*

a) $T(\varnothing) = 0$.

b) For all profinite sets S_1 and S_2, the inclusions $i_1 \colon S_1 \to S_1 \sqcup S_2$ and $i_2 \colon S_2 \to S_1 \sqcup S_2$ induce a group isomorphism

$$T(S_1 \sqcup S_2) \xrightarrow{T(i_1) \times T(i_2)} T(S_1) \oplus T(S_2).$$

c) Every surjection $f\colon S' \twoheadrightarrow S$ induces a group isomorphism

$$T(S) \xrightarrow{T(f)} \{g \in T(S') \mid T(p_1)(g) = T(p_2)(g) \in T(S' \times_S S')\},$$

where $p_1, p_2\colon S' \times_S S' \to S'$ denote the projections.

Proof. By definition, a functor T is a sheaf on $*_{pro\acute{e}t}$ if and only if for every finite family $\{S_i \to S\}_{i\in I}$ of jointly surjective morphisms the diagram

$$T(S) \to \prod_{i\in I} T(S_i) \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} \prod_{(i,j)\in I^2} T(S_i \times_S S_j)$$

is exact, meaning that the left arrow is an equalizer of the two arrows on the right, which are explicitly described below. Since we are in $\mathcal{A}b$ and I is a finite set, we can reformulate this as the sequence

$$0 \to T(S) \to \bigoplus_{i\in I} T(S_i) \xrightarrow{p_1^* - p_2^*} \bigoplus_{(i,j)\in I^2} T(S_i \times_S S_j) \tag{1}$$

being exact.

So assume first that 1 is exact. The empty set is covered by the empty family, so the $T(\varnothing)$ must be a subgroup of 0, hence 0 itself. The natural inclusions $S_i \to S_1 \sqcup S_2 = S$ cover S for $i \in I = \{1, 2\}$, so it suffices to show that $T(p_1) = T(p_2)$ to verify b). Since $i_1(S_1) \cap i_2(S_2) = \varnothing$, we have $S_1 \times_S S_2 = S_2 \times_S S_1 = \varnothing$. So if $(g, h) \in T(S_1) \oplus T(S_2)$, then

$$\begin{aligned} p_1^*(g,h) &= (T(p_1^{S_1\times_S S_1})(g), T(p_1^{S_1\times_S S_2})(g), T(p_1^{S_2\times_S S_1})(h), T(p_1^{S_2\times_S S_2})(h)) \\ &= (T(p_1^{S_1\times_S S_1})(g), 0, 0, T(p_1^{S_2\times_S S_2})(h)) \\ &= (T(p_2^{S_1\times_S S_1})(g), T(p_2^{S_1\times_S S_2})(h), T(p_2^{S_2\times_S S_1})(g), T(p_2^{S_2\times_S S_2})(h)) = p_2^*(g,h). \end{aligned}$$

To show c), note that $S' \twoheadrightarrow S$ is already a cover, so the exactness of 1 immediately implies the result.

For the converse, suppose that $\{f_j\colon S_j \to S\}_{j\in J}$ is a collection of jointly surjective morphisms indexed by $J = \{1, \dots, m\}$. We have then a surjection $f\colon S \twoheadrightarrow S$, where $S' := \sqcup_{j=1}^m S_j$ and $f := \sqcup_{j=1}^m f_j$. By c) we can write $T(S) = \{g \in T(S') \mid T(p_1^{S'\times_S S'})(g) = T(p_2^{S'\times_S S'})(g)\}$. But $S' \times_S S' = \sqcup_{(a,b)\in J^2} S_a \times_S S_b$ and $p_\varepsilon^{S'\times_S S'} = \sqcup_{(a,b)\in J^2} i_a \circ p_\varepsilon^{S_a\times_S S_b}$ for $\varepsilon \in \{1, 2\}$ and for $i_a\colon S_a \to S'$ the inclusions, so condition b) allows us to rewrite this as

$$T(S) = \bigcap_{(a,b)\in J^2} \{(g_1, \dots, g_m) \in T(S_1) \oplus \cdots \oplus T(S_m) \mid P_{a,b}(g_a, g_b)\}$$

with $P_{a,b}(g_a, g_b) :\equiv T(p_1^{S_a \times_S S_b})(g_a) = T(p_2^{S_a \times_S S_b})(g_b)$, which is what we wanted. $\square$

With this characterization we can already produce some examples:

Example 1.3. Let G be a topological abelian group. Then the functor $\underline{G} = \mathrm{Hom}_{\mathcal{T}op}(-, G)\colon *_{pro\acute{e}t} \to \mathcal{A}b$ is a condensed abelian group. Conditions a) and b) in lemma 1.2 are clear. For condition c), let $f\colon S' \twoheadrightarrow S$ be a surjection. We want to show that $f^*\colon \underline{G}(S) \to \underline{G}(S')$ induces an isomorphism between the set of continuous maps $g\colon S \to G$ and the set of continous maps $h\colon S' \to G$ such that $h \circ p_1 = h \circ p_2$. The image of f^* is indeed contained in the set of all such maps, because $f \circ p_1 = f \circ p_2$. Moreover, since f is an epimorphism, f^* is injective. So it only remains to show surjectivity. Let $h\colon S' \to G$ be a map such that $h \circ p_1 = h \circ p_2$. For each $s \in S$, let $g(s) := h(s')$ for some $s' \in f^{-1}(\{s\})$. This is well-defined in $\mathcal{S}et$, because if $f(s') = f(s'') = s$, then $(s', s'') \in S' \times_S S'$, and thus we can write

$$g(s) = h(s') = h \circ p_1(s', s'') = h \circ p_2(s', s'') = h(s'').$$

But in fact it is a morphism in $\mathcal{T}op$. Indeed, S carries the quotient topology induced by f, so $f^{-1}(g^{-1}(U)) = h^{-1}(U)$ being open in S' for all U open in G implies that $g^{-1}(U)$ is open in S for all U open in G.

References

[Sta19] The Stacks project authors. The stacks project. `https://stacks.math.columbia.edu`, 2019.