Seminar on Condensed Mathematics
Talk 2: Condensed Abelian Groups

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22nd October 2019

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1 The category of condensed abelian groups

Recall from the previous talk:

Definition 1.1. The *pro-étale site* of a point, denoted $*_{prot}$, consists of:

- The category whose objects are profinite sets S (topological spaces homeomorphic to an inverse limit of finite discrete topological spaces, a.k.a. totally disconnected compact Hausdorff spaces) and whose morphisms are continuous maps.
- The coverings of a profinite set S are all finite families of jointly surjective maps, i.e. all families of morphisms $\{S_i \to S\}_{i \in I}$ indexed by finite sets J such that $\sqcup_{i \in I} S_i \to S$ is surjective.

It is easy to check that the axioms of a covering family are satisfied (see [Sta19, Tag 00VH]), so we have a well-defined site. Let us start by characterizing sheaves on this site:

Lemma 1.2. *A contravariant functor T from $*_{proét}$ to the category of abelian groups $\mathcal{A}b$ is a sheaf if and only if the following three conditions hold:*

a) $T(\varnothing) = 0$.

b) For all profinite sets S_1 *and* S_2*, the inclusions* $i_1 \colon S_1 \to S_1 \sqcup S_2$ *and* $i_2 \colon S_2 \to S_1 \sqcup S_2$ *induce a group isomorphism*

$$T(S_1 \sqcup S_2) \xrightarrow{T(i_1)\times T(i_2)} T(S_1) \oplus T(S_2).$$

c) Every surjection $f \colon S' \twoheadrightarrow S$ *induces a group isomorphism*

$$T(S) \xrightarrow{T(f)} \{g \in T(S') \mid T(p_1)(g) = T(p_2)(g) \in T(S' \times_S S')\},$$

where $p_1, p_2 \colon S' \times_S S' \to S'$ *denote the projections.*

Proof. By definition, a functor T is a sheaf on $*_{pro\acute{e}t}$ if and only if for every finite family $\{S_i \to S\}_{i\in I}$ of jointly surjective morphisms the diagram

$$T(S) \to \prod_{i\in I} T(S_i) \overset{p_1^*}{\underset{p_2^*}{\rightrightarrows}} \prod_{(i,j)\in I^2} T(S_i \times_S S_j)$$

is exact, meaning that the left arrow is an equalizer of the two arrows on the right, which are explicitly described below. Since we are in $\mathcal{A}b$ and I is a finite set, we can reformulate this as the sequence

$$0 \to T(S) \to \bigoplus_{i\in I} T(S_i) \xrightarrow{p_1^* - p_2^*} \bigoplus_{(i,j)\in I^2} T(S_i \times_S S_j) \tag{1}$$

being exact.

So assume first that 1 is exact. The empty set is covered by the empty family, so the $T(\varnothing)$ must be a subgroup of 0, hence 0 itself. The natural inclusions $S_i \to S_1 \sqcup S_2 = S$ cover S for $i \in I = \{1, 2\}$, so it suffices to show that $T(p_1) = T(p_2)$ to verify *b*). Since $i_1(S_1) \cap i_2(S_2) = \varnothing$, we have $S_1 \times_S S_2 = S_2 \times_S S_1 = \varnothing$. So if $(g, h) \in T(S_1) \oplus T(S_2)$, then

$$\begin{aligned} p_1^*(g,h) &= (T(p_1^{S_1\times_S S_1})(g), T(p_1^{S_1\times_S S_2})(g), T(p_1^{S_2\times_S S_1})(h), T(p_1^{S_2\times_S S_2})(h)) \\ &= (T(p_1^{S_1\times_S S_1})(g), 0, 0, T(p_1^{S_2\times_S S_2})(h)) \\ &= (T(p_2^{S_1\times_S S_1})(g), T(p_2^{S_1\times_S S_2})(h), T(p_2^{S_2\times_S S_1})(g), T(p_2^{S_2\times_S S_2})(h)) = p_2^*(g,h). \end{aligned}$$

To show *c*), note that $S' \twoheadrightarrow S$ is already a cover, so the exactness of 1 immediately implies the result.

For the converse, suppose that $\{f_j \colon S_j \to S\}_{j\in J}$ is a collection of jointly surjective morphisms indexed by $J = \{1, \ldots, m\}$. We have then a surjection

$f\colon S \twoheadrightarrow S$, where $S' := \sqcup_{j=1}^m S_j$ and $f := \sqcup_{j=1}^m f_j$. By $c)$ we can write $T(S) = \{g \in T(S') \mid T(p_1^{S'\times_S S'})(g) = T(p_2^{S'\times_S S'})(g)\}$. But $S' \times_S S' = \sqcup_{(a,b)\in J^2} S_a \times_S S_b$ and $p_\varepsilon^{S'\times_S S'} = \sqcup_{(a_1,a_2)\in J^2} i_{a_\varepsilon} \circ p_\varepsilon^{S_{a_1}\times_S S_{a_2}}$ for $\varepsilon \in \{1,2\}$ and for $i_a\colon S_a \to S'$ the inclusions, so condition $b)$ allows us to rewrite this as

$$T(S) = \bigcap_{(a,b)\in J^2} \{(g_1,\ldots,g_m) \in T(S_1) \oplus \cdots \oplus T(S_m) \mid P_{a,b}(g_a,g_b)\}$$

with $P_{a,b}(g_a,g_b) :\equiv T(p_1^{S_a\times_S S_b})(g_a) = T(p_2^{S_a\times_S S_b})(g_b)$, which is what we wanted. □

Definition 1.3. A *condensed abelian group* is a sheaf of abelian groups on $*_{pro\acute{e}t}$. We denote the category of condensed abelian groups by $\mathrm{Cond}(\mathcal{A}b)$.

Remark 1.4. Barwick and Haine have developed independently a notion very similar to the condensed objects of Clausen and Scholze in their recent paper [BH19]. This is the notion of *Pyknotic*[1] object, which differs from the notion of condensed object on set theoretical matters. Since we are not discussing set theoretical issues, we will also not discuss the differences between these two notions here.

From lemma 1.2 we deduce:

Example 1.5. Let G be a topological abelian group. Then the functor $\underline{G} = \mathrm{Hom}_{\mathcal{T}op}(-,G)\colon *_{pro\acute{e}t} \to \mathcal{A}b$ is a condensed abelian group. Conditions $a)$ and $b)$ in lemma 1.2 are clear. For condition $c)$, let $f\colon S' \twoheadrightarrow S$ be a surjection. We want to show that $f^*\colon \underline{G}(S) \to \underline{G}(S')$ induces an isomorphism between the set of continuous maps $g\colon S \to G$ and the set of continous maps $h\colon S' \to G$ such that $h \circ p_1 = h \circ p_2$. The image of f^* is indeed contained in the set of all such maps, because $f \circ p_1 = f \circ p_2$. Moreover, since f is an epimorphism, f^* is injective. So it only remains to show surjectivity. Let $h\colon S' \to G$ be a map such that $h \circ p_1 = h \circ p_2$. For each $s \in S$, let $g(s) := h(s')$ for some $s' \in f^{-1}(\{s\})$. This is well-defined in $\mathcal{S}et$, because if $f(s') = f(s'') = s$, then $(s',s'') \in S' \times_S S'$, and thus we can write

$$g(s) = h(s') = h \circ p_1(s',s'') = h \circ p_2(s',s'') = h(s'').$$

But in fact it is a morphism in $\mathcal{T}op$. Indeed, S carries the quotient topology induced by f, so $f^{-1}(g^{-1}(U)) = h^{-1}(U)$ being open in S' for all U open in G implies that $g^{-1}(U)$ is open in S for all U open in G.

[1]This name comes from the greek word *pykno*, which means dense, compact or thick.

Sometimes it will be necessary to consider sheafification of presheaves:

Definition 1.6. Let $\mathcal{C}$ be a site and let $\mathrm{Sh}(\mathcal{C}) \hookrightarrow \mathrm{PSh}(\mathcal{C})$ be the inclusion of the category of sheaves to the category of presheaves on $\mathcal{C}$. A *sheafification* functor is a left adjoint $(-)^a\colon \mathrm{PSh}(\mathcal{C}) \to \mathrm{Sh}(\mathcal{C})$ to this inclusion.

We can explicitly describe the sheafification of a presheaf F as follows. For a covering family $\mathcal{U} = \{f_j \colon U_j \to U\}_{j\in J}$ with $J = \{1, \dots, m\}$ we define the *zero Čech cohomology* of F with respect to $\mathcal{U}$ as

$$\check{H}^0(\mathcal{U}, F) = \bigcap_{(a,b)\in J^2} \{(g_1, \dots, g_m) \in \bigoplus_{j=1}^{m} F(U_j) \mid P_{a,b}(g_a, g_b)\}$$

with the notation from lemma 1.2. If $\mathcal{U}' \to \mathcal{U}$ is a morphism of covering families of U, then we obtain a pullback morphism $\check{H}^0(\mathcal{U}, F) \to \check{H}^0(\mathcal{U}', F)$ between zero Čech cohomology groups which does not depend on the particular morphism of covering families but only on the covering families themselves. Define then a presheaf F^+ by setting

$$F^+(U) := \varinjlim_{\mathcal{U}} \check{H}^0(\mathcal{U}, F)$$

where the direct limit runs over all covering families of U with $\mathcal{U} \leqslant \mathcal{U}'$ if and only if we have a morphism of covering families $\mathcal{U}' \to \mathcal{U}$ (we can think of $\mathcal{U}'$ being finer $\mathcal{U}$). This construction takes presheaves into separated presheaves and separated presheaves into sheaves, so taking $F^a := (F^+)^+$ we obtain the desired sheafification. See [Sta19, Tag 03NQ] for the precise statements and proofs.

Remark 1.7. Since $(-)^a$ has a right adjoint, it preserves all colimits, in particular all finite colimits. And since finite limits commute with direct limits in $\mathcal{A}b$, it follows from the previous description that $(-)^a$ also preserves finite limits. Hence sheafification is exact.

1.1 Grothendieck category

A category $\mathcal{C}$ is called *abelian* if it satisfies the following properties:

i) There exists a zero object $0 \in \mathrm{Ob}(\mathcal{C})$.

ii) For all $A, B \in \mathrm{Ob}(\mathcal{C})$, their product $A \xleftarrow{p_A} A \times B \xrightarrow{p_B} B$ and their coproduct $A \xrightarrow{i_A} A \sqcup B \xleftarrow{i_B} B$ exist in $\mathcal{C}$ and the canonical morphism $A \sqcup B \xrightarrow{(\mathrm{id}_A \times 0) \sqcup (0 \times \mathrm{id}_B)} A \times B$ is an isomorphism[2] in $\mathcal{C}$.

[2]We identify $A \sqcup B$ with $A \times B$ via this isomorphism and call the result the *direct sum* of A and B, denoted $A \oplus B$.

iii) Every morphisms has a kernel and a cokernel in $\mathcal{C}$.

iv) Every monomorphism in $\mathcal{C}$ is a kernel and every epimorphism in $\mathcal{C}$ is a cokernel.

In particular, $\mathcal{C}$ is naturally an additive category and all finite limits and colimits exist in $\mathcal{C}$.

In addition to these properties, Grothendieck introduced a series of extra axioms in his Tôhoku paper [Gro57]. The ones that we will be considering are (AB3) [*all colimits exist*], (AB3*) [*all limits exist*], (AB4) [*all colimits exist and coproducts are exact*], (AB4*) [*all limits exist and products are exact*], (AB5) [*all colimits exist and filtered colimits are exact*] and (AB6) [*all colimits exist and for any family* $\{I_j\}_{j\in J}$ *of filtered categories indexed by a set* J *with functors* $F_j\colon I_j \to \mathcal{C}$ *the canonical morphism*

$$\varinjlim_{(i_j\in I_j)_j} \prod_{j\in J} F_j(i_j) \to \prod_{j\in J} \varinjlim_{i_j\in I_j} F_j(i_j)$$

is an isomorphism in $\mathcal{C}$].

Remark 1.8. To check (AB3) on an abelian category it suffices to show that arbitrary coproducts exist, since we can build any colimit from coproducts and coequalizers, and similarly for the dual statement (AB3*). Colimits preserve colimits because the colimit functor is left adjoint to the diagonal functor. In particular coproducts are always right exact, so to check (AB4) on an abelian category satisfying (AB3) it suffices to check that coproducts preserve monomorphisms, and similarly for the dual statement (AB4*).

The main goal of this talk is to show that the category $\mathrm{Cond}(\mathcal{A}b)$ is an abelian category which verifies all the previous axioms. Except for axioms (AB4*) and (AB6), this is a general fact which holds on any category of sheaves of abelian groups on a site, but we will give a proof specific to our situation. This will be much easier after we express $\mathrm{Cond}(\mathcal{A}b)$ as the category of sheaves of abelian groups on a simpler site.

Definition 1.9. A compact Hausdorff[3] topological space is called *extremally disconnected* if the closure of every open set is again open.

Equivalently, a compact Hausdorff topological space is extremally disconnected if the closures of every pair of disjoint open sets are also disjoint. Therefore extremally disconnected spaces are totally disconnected compact Hausdorff spaces, hence profinite spaces. The converse is not true:

[3]If we do not require Hausdorffness, an extremally disconnected space could be very connected, e.g. any irreducible topological space.

Example 1.10. In an extremally disconnected space every convergent sequence is eventually constant (see [Gle58, Theorem 1.3]), so the p-adic integers are profinite but not extremally disconnected, because $(p^n)_{n=1}^{\infty}$ converges to 0.

Extremally disconnected spaces are precisely the projective objects in the category $\mathcal{CH}aus$ of compact Hausdorff spaces (see [Gle58, Theorem 2.5]). Since pullbacks exist and preserve epimorphisms (i.e. surjective continuous functions[4]) in $\mathcal{CH}aus$, a compact Hausdorff space S is extremally disconnected if and only if any surjection $S' \twoheadrightarrow S$ from a compact Hausdorff space admits a section:

$$\begin{tikzcd} & S' \arrow[d, two heads] \\ S \arrow[ur, dashed] \arrow[r, equal] & S \end{tikzcd}$$

The inclusion functor $U\colon \mathcal{CH}aus \to \mathcal{T}op$ has a left adjoint $\beta\colon \mathcal{T}op \to \mathcal{CH}aus$ called the *Stone-Čech compactification.* If X is a discrete topological space, then βX is extremally disconnected, because if $S \twoheadrightarrow \beta X$ is a continuous surjection from a compact Hausdorff topological space we may first lift the canonical map $X \to \beta X$ to S and then extend it to a section from βX by its universal property. In particular, every compact Hausdorff space admits a surjection $\beta(X_{disc}) \twoheadrightarrow X$ from an extremally disconnected space. This has the following consequence:

Proposition 1.11. Cond$(\mathcal{A}b)$ *is equivalent to the category of sheaves of abelian groups on the site of extremally disconnected spaces via the restriction from profinite sets.*

Proof. Let $T \in \mathrm{Cond}(\mathcal{A}b)$ and let S be a profinite set. Let $f\colon \tilde{S} \twoheadrightarrow S$ and $g\colon \tilde{\tilde{S}} \twoheadrightarrow \tilde{S} \times_S \tilde{S}$ be continuous surjections from extremally disconnected spaces. By lemma 1.2 we have $T(S) = \ker(T(p_1) - T(p_2))$, where $p_1, p_2 \colon \tilde{S} \times_S \tilde{S}$ denote the projections from the fiber product. Since $T(\tilde{S} \times_S \tilde{S}) \xrightarrow{T(g)} T(\tilde{\tilde{S}})$ is injective, the kernel of $T(\tilde{S}) \xrightarrow{T(p_1)-T(p_2)} T(\tilde{S} \times_S \tilde{S})$ is the same as the kernel of the composition $T(\tilde{S}) \xrightarrow{T(p_1 \circ g)-T(p_2 \circ g)} T(\tilde{\tilde{S}})$. Therefore the value of T at the profinite set S is completely determined by the value of T at the extremally disconnected sets $\tilde{S}$ and $\tilde{\tilde{S}}$. □

[4]Urysohn's lemma implies that epimorphisms in the category of compact Hausdorff spaces are precisely surjective continuous functions, which is not true in the whole category of Hausdorff spaces (e.g. the inclusion of the complement of a point in the real line).

Corollary 1.12. $\mathrm{Cond}(\mathcal{A}b)$ *is equivalent to the category of contravariant functors T from the category of extremally disconnected sets to the category of abelian groups such that:*

a) $T(\varnothing) = 0$.

b) For all extremally disconnected sets S_1 and S_2, the inclusions $i_1 \colon S_1 \to S_1 \sqcup S_2$ and $i_2 \colon S_2 \to S_1 \sqcup S_2$ induce a group isomorphism

$$T(S_1 \sqcup S_2) \xrightarrow{T(i_1) \times T(i_2)} T(S_1) \oplus T(S_2).$$

Proof. We need to show that condition $c)$ in lemma 1.2 is automatically satisfied. So let $f \colon S' \twoheadrightarrow S$ be a continuous surjection of extremally disconnected spaces. Then we can find a section $\sigma \colon S \to S'$ with $f \circ \sigma = \mathrm{id}_S$. This implies that $T(\sigma) \circ T(f) = \mathrm{id}_{T(S)}$, so $T(f)$ is injective. The image of $T(f)$ is contained in $\{g \in T(\tilde{S}) \mid T(p_1)(g) = T(p_2)(g)\}$, because $f \circ p_1 = f \circ p_2$. And conversely if $g \in T(\tilde{S})$ is such that $T(p_1)(g) = T(p_2)(g)$, then $T((\sigma \circ f) \times_S \mathrm{id}_{\tilde{S}})(T(p_1)(g)) = T((\sigma \circ f) \times_S \mathrm{id}_{\tilde{S}})(T(p_2)(g))$, so $T(f)(T(\sigma)(g)) = g$ and g is in the image of $T(f)$. $\square$

Remark 1.13. The following formal consequence of $b)$ will be useful later. Let $\eta \colon T_1 \to T_2$ be a morphism in $\mathrm{Cond}(\mathcal{A}b)$ and let S_1 and S_2 be two extremally disconnected sets. Then under the isomorphisms of corollary 1.12 $b)$ the component of η at $S_1 \sqcup S_2$ is given by $\eta_{S_1} \oplus \eta_{S_2}$, i.e. by

$$T_1(S_1) \oplus T_1(S_1) \xrightarrow{\begin{pmatrix} \eta_{S_1} & 0 \\ 0 & \eta_{S_2} \end{pmatrix}} T_2(S_1) \oplus T_2(S_2).$$

Indeed, $\eta_{S_1} \oplus \eta_{S_2} = (\eta_{S_1} \circ p_{T_1(S_1)}) \times (\eta_{S_2} \circ p_{T_1(S_2)})$, so this follows from the commutativity of the following diagram:

$$
\begin{tikzcd}
T_1(S_1) \arrow[rr, "\eta_{S_1}"] & & T_2(S_1) \\
T_1(S_1 \sqcup S_2) \arrow[u, "T_1(i_1)"'] \arrow[d, "T_1(i_1)\times T_1(i_2)"'] \arrow[rr, "\eta_{S_1 \sqcup S_2}"] & & T_2(S_1 \sqcup S_2) \arrow[u, "T_2(i_1)"] \arrow[d, "T_2(i_1)\times T_2(i_2)"] \\
T_1(S_1) \oplus T_1(S_2) \arrow[uu, bend left, "p_{T_1(S_1)}"] \arrow[dd, bend right, "p_{T_1(S_2)}"'] \arrow[rr, dashed] & & T_2(S_1) \oplus T_2(S_2) \arrow[uu, bend right, "p_{T_2(S_1)}"'] \arrow[dd, bend left, "p_{T_2(S_2)}"] \\
T_1(S_1 \sqcup S_2) \arrow[u, "T_1(i_1)\times T_1(i_2)"] \arrow[d, "T_1(i_2)"'] \arrow[rr, "\eta_{S_1 \sqcup S_2}"] & & T_2(S_1 \sqcup S_2) \arrow[u, "T_2(i_1)\times T_2(i_2)"'] \arrow[d, "T_2(i_2)"] \\
T_1(S_2) \arrow[rr, "\eta_{S_2}"'] & & T_2(S_2)
\end{tikzcd}
$$

With this nice description we are ready to prove the main result of this talk:

Theorem 1.14. Cond($\mathcal{A}b$) *is an abelian category satisfying axioms (AB3), (AB3*), (AB4), (AB4*), (AB5) and (AB6). Moreover, it is generated*[5] *by compact*[6] *projective objects.*

Proof. In corollary 1.12 we have described Cond($\mathcal{A}b$) as the category of contravariant functors from extremally disconnected spaces to $\mathcal{A}b$ sending finite disjoint unions to finite products. The idea is to use this description to see that we can construct limits and colimits pointwise. Once we have shown that pointwise limits and colimits are sheaves, checking the necessary universal properties and isomorphisms is reduced to checking the corresponding statements in $\mathcal{A}b$ pointwise, so we will not mention these computations explicitly.

Let us do the case of limits, for example. Let $F\colon \mathcal{J} \to \mathrm{Cond}(\mathcal{A}b)$ be a diagram in Cond($\mathcal{A}b$), i.e. a functor from some indexing category $\mathcal{J}$. For an extremally disconnected set S, let $L(S) = \lim(\mathrm{ev}_S \circ F)$ in $\mathcal{A}b$ and for all $\alpha \in \mathrm{Ob}(\mathcal{J})$ denote by $\varepsilon_S^\alpha\colon L(S) \to F_\alpha(S)$ the corresponding canonical map. If $f\colon S \to S'$ is a continuous map of extremally disconnected sets, we have a cone to $\mathrm{ev}_S \circ F$ given by the compositions $L(S') \to F_\alpha(S') \xrightarrow{F_\alpha(f)} F_\alpha(S)$

[5] A category $\mathcal{C}$ is said to be *generated* by a set $\mathcal{S} \subseteq \mathrm{Ob}(\mathcal{C})$ if for all pairs of distinct parallel arrows $f \neq g\colon A \rightrightarrows B$ we can find some $h\colon S \to A$ with $S \in \mathcal{S}$ such that $fh \neq gh$. As Mac Lane points out, the word *separated* would have been a better choice (see [ML78, Section V.7]).

[6] An object M on a category $\mathcal{C}$ is called *compact* if $\mathrm{Hom}_{\mathcal{C}}(M, -)$ commutes with filtered colimits.

for all $\alpha \in \mathrm{Ob}(\mathcal{J})$, so we get a canonical homomorphism to the terminal cone $L(f)\colon L(S') \to L(S)$ making the resulting diagram commute. To show that L is a sheaf we use corollary 1.12. Condition $a)$ is verified because $\mathrm{ev}_\varnothing$ is the constantly zero functor. For condition $b)$ let $i_1\colon S_1 \to S_1 \sqcup S_2$ and $i_2\colon S_2 \to S_1 \sqcup S_2$ be the inclusion of two extremally disconnected sets in their disjoint union and consider the following diagram:

$$\begin{tikzcd}
 & L(S_1 \sqcup S_2) \arrow[rr, "L(i_1)\times L(i_2)"] \arrow[dl, "\varepsilon^\alpha_{S_1 \sqcup S_2}"'] \arrow[d, "\varepsilon^\beta_{S_1 \sqcup S_2}"] & & L(S_1) \oplus L(S_2) \arrow[dl, "\varepsilon^\alpha_{S_1} \oplus \varepsilon^\alpha_{S_2}"] \arrow[d, "\varepsilon^\beta_{S_1} \oplus \varepsilon^\beta_{S_2}"] \\
T_\alpha(S_1 \sqcup S_2) \arrow[rr, "T_\alpha(i_1)\times T_\alpha(i_2)"'] \arrow[dr, "\eta_{S_1 \sqcup S_2}"'] & & T_\alpha(S_1) \oplus T_\alpha(S_2) \arrow[dr, "\eta_{S_1} \oplus \eta_{S_2}"] & \\
 & T_\beta(S_1 \sqcup S_2) \arrow[rr, "T_\beta(i_1)\times T_\beta(i_2)"'] & & T_\beta(S_1) \oplus T_\beta(S_2)
\end{tikzcd}$$

Since limits commute with finite direct sums, the right triangle is the cone corresponding to the limit of the direct sum of diagrams. The left triangle is by definition the cone corresponding to the limit of the diagram at its base. The bottom square commutes by remark 1.13, so the universal property of the limit $L(S_1) \oplus L(S_2)$ induces a unique group homomorphism $L(S_1 \sqcup S_2) \to L(S_1) \oplus L(S_2)$ making everything commute. Since all arrows going right at the bottom are isomorphisms, this unique morphism is a group isomorphism, so it suffices to show that $L(i_1) \times L(i_2)$ makes the whole diagram commute. Let us see for example that the β square commutes. By the universal property of the product, it suffices to show commutativity after composing with each projection. Composing with the first projection $p_{T_\beta(S_1)}$ we further reduce our problem to the commutativity of the following square:

$$\begin{tikzcd}
L(S_1 \sqcup S_2) \arrow[r, "L(i_1)"] \arrow[d, "\varepsilon^\beta_{S_1 \sqcup S_2}"'] & L(S_1) \arrow[d, "\varepsilon^\beta_{S_1}"] \\
T_\beta(S_1 \sqcup S_2) \arrow[r, "T_\beta(i_1)"] & T_\beta(S_1)
\end{tikzcd}$$

But this square commutes by construction of $L(i_1)$, so we are done proving that L is a sheaf.

Let us see now that $\mathrm{Cond}(\mathcal{A}b)$ is generated by compact projective objects. The forgetful functor $U\colon \mathrm{Cond}(\mathcal{A}b) \to \mathrm{Cond}(\mathcal{S}et)$ preserves all limits. This follows from the pointwise construction of limits in both cases and for the corresponding statement for the forgetful functor $\mathcal{A}b \to \mathcal{S}et$. Both $\mathrm{Cond}(\mathcal{A}b)$ and $\mathrm{Cond}(\mathcal{S}et)$ satisfy the necessary conditions for the adjoint functor theorem, so we have a left adjoint functor $\mathbb{Z}[-]\colon \mathrm{Cond}(\mathcal{S}et) \to \mathrm{Cond}(\mathcal{A}b)$. This functor attaches to a condensed set T the sheafification of the presheaf which sends an extremally disconnected set S to the free abelian group generated by $T(S)$, hence the notation. For an extremally disconnected set S consider the condensed set $\underline{S} = \mathrm{Hom}_{\mathcal{T}op}(S, -)$. A morphism between two sheaves in $\mathrm{Cond}(\mathcal{S}et)$ is just a natural transformation between them, so by the Yoneda lemma we have a natural bijection $\mathrm{Hom}_{\mathrm{Cond}(\mathcal{A}b)}(\underline{S}, U(M)) \cong M(S)$ for all condensed abelian groups M. Combining this with the previous adjunction we obtain natural bijections $\mathrm{Hom}_{\mathrm{Cond}(\mathcal{A}b)}(\mathbb{Z}[\underline{S}], M) \cong M(S)$ for all condensed abelian groups M. Since limits and colimits are constructed pointwise in $\mathrm{Cond}(\mathcal{A}b)$, the evaluation functor ev_S commutes with all limits and colimits, which implies by the previous natural bijection that $\mathbb{Z}[\underline{S}]$ is both compact and projective. Let us prove now that these objects generate $\mathrm{Cond}(\mathcal{A}b)$, for which it suffices to find for all condensed abelian group M a surjection $\bigoplus_{\alpha\in\Lambda} \mathbb{Z}[\underline{S_\alpha}] \twoheadrightarrow M$ for some collection of extremally disconnected sets $\{S_\alpha\}_{\alpha\in\Lambda}$.

So let $M \in \mathrm{Ob}(\mathrm{Cond}(\mathcal{A}b))$. By Zorn's lemma, there is a maximal subobject M' of M such that M' admits a surjection $f\colon \bigoplus_{\alpha\in\Lambda} \mathbb{Z}[\underline{S_\alpha}] \twoheadrightarrow M'$ for some collection of extremally disconnected sets $\{S_\alpha\}_{\alpha\in\Lambda}$. Suppose $M' \neq M$, i.e. suppose $M/M' \neq 0$. Then we can find some extremally disconnected set S such that $(M/M')(S) \neq 0$. By Yoneda this means that we can find at least one non-zero morphism $\bar{g}\colon \mathbb{Z}[\underline{S}] \to M/M'$, which by projectivity of $\mathbb{Z}[\underline{S}]$ can be lifted to a morphism $g\colon \mathbb{Z}[\underline{S}] \to M$ such that $\mathrm{im}(g) \not\subseteq M'$. Let then M'' be the smallest subobject of M containing M' and $\mathrm{im}(g)$, so that $M' \subsetneq M'' \subsetneq M$. Then we can find a surjection $(\bigoplus_{\alpha\in\Lambda} \mathbb{Z}[\underline{S_\alpha}]) \oplus \mathbb{Z}[\underline{S}] \overset{f\sqcup g}{\twoheadrightarrow} M''$ contradicting maximality of M'. This implies that $M' = M$ and finishes the proof. $\square$

Categories nice enough to have some of the previous properties deserve a name of their own. We say that $\mathcal{C}$ is a *Grothendieck category* if it is an (AB5) abelian category with a generator. By definition $G \in \mathrm{Ob}(\mathcal{C})$ is a generator precisely when the functor $\mathrm{Hom}_{\mathcal{C}}(G, -)\colon \mathcal{C} \to \mathcal{S}et$ is faithful. Note that if $\{G_i\}_{i\in I}$ is a set of generators and arbitrary coproducts exist in $\mathcal{C}$, then $G = \bigoplus_{i\in I} G_i$ is a generator.

Corollary 1.15. $\mathrm{Cond}(\mathcal{A}b)$ *is a Grothendieck category with generator* $G = \bigoplus_S \mathbb{Z}[\underline{S}]$*, where* S *ranges over all extremally disconnected sets.*

1.2 Closed symmetric monoidal

Roughly speaking, a (*symmetric*) *monoidal structure* on a category $\mathcal{C}$ consists of a functor $(-) \otimes (-)\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ which turns $\mathcal{C}$ into a (commutative up to natural isomorphism) monoid with associativity and unit up to natural isomorphism (see [nLa19] for the precise definition). When $\mathcal{C}$ is considered endowed with this extra structure we call it a (*symmetric*) *monoidal category.*

Example 1.16. $\mathcal{A}b$ is a symmetric monoidal category with respect to the usual tensor product and with unit object $\mathbb{Z}$.

Let us see now that $\mathrm{Cond}(\mathcal{A}b)$ is also a symmetric monoidal category:

Proposition 1.17. *The functor* $\otimes\colon \mathrm{Cond}(\mathcal{A}b) \times \mathrm{Cond}(\mathcal{A}b) \to \mathrm{Cond}(\mathcal{A}b)$ *with* $A \otimes B := (S \mapsto A(S) \otimes B(S))^a$ *makes* $\mathrm{Cond}(\mathcal{A}b)$ *a symmetric monoidal category with unit given by the constant sheaf* $\mathbb{Z} := (S \mapsto \mathbb{Z})^a$.

Proof. Note that the pointwise tensor product induces a symmetric monoidal structure on the category $\mathrm{PSh}(\mathcal{A}b)$ of presheaves of abelian groups in which associator, unitors and braiding are given pointwise by the correso-pnding associator and unitors from $\mathcal{A}b$. We can write the tensor product on $\mathrm{Cond}(\mathcal{A}b)$ as the composition $(-)^a \circ \otimes_{\mathrm{PSh}(\mathcal{A}b)} \circ i$ where i denotes the inclusion $\mathrm{Cond}(\mathcal{A}b) \times \mathrm{Cond}(\mathcal{A}b) \to \mathrm{PSh}(\mathcal{A}b) \times \mathrm{PSh}(\mathcal{A}b)$, which shows that $\otimes$ is indeed a functor. Associator, unitors and braiding are obtained by applying $(-)^a$ componentwise. Since functors preserve commutative diagrams, all coherence diagrams are still commutative. □

The category $\mathrm{Cond}(\mathcal{S}et)$ is also symmetric monoidal with respect to cartesian product. Relating these two symmetric monoidal categories we have the following:

Proposition 1.18. *The functor* $\mathbb{Z}[-]\colon \mathrm{Cond}(\mathcal{S}et) \to \mathrm{Cond}(\mathcal{A}b)$ *is symmetric monoidal, i.e. it sends cartesian products to tensor products.*

Proof. Recall from the proof of theorem 1.14 that $\mathbb{Z}[-]$ sends a condensed set T to the sheafification of the presheaf $S \mapsto \mathbb{Z}[T(S)]$. Hence we only have to show that $\mathbb{Z}[(T_1 \times T_2)(S)] \cong \mathbb{Z}[T_1(S)] \otimes \mathbb{Z}[T_2(S)]$. But the cartesian product is formed pointwise, so the result follows from the functor $\mathbb{Z}[-]\colon (\mathcal{S}et, \times) \to (\mathcal{A}b, \otimes)$ being monoidal. The word symmetric can be added since the isomorphisms $\mathbb{Z}[A \times B] \cong \mathbb{Z}[A] \otimes \mathbb{Z}[B]$ are compatible with the braiding natural transformations $A \times B \cong B \times A$ and $\mathbb{Z}[A] \otimes \mathbb{Z}[B] \cong \mathbb{Z}[B] \otimes \mathbb{Z}[A]$. □

It is also worth pointing out that the objects $\mathbb{Z}[T]$ are flat for all condensed sets T. Indeed, since sheafification is exact it suffices to check that tensoring pointwise with the presheaf $S \mapsto \mathbb{Z}[T(S)]$ is exact, which follows from all $\mathbb{Z}[T(S)]$ being free abelian groups.

One last natural step is to check that our symmetric monoidal category $\mathrm{Cond}(\mathcal{A}b)$ is closed:

Proposition 1.19. *For all $M \in \mathrm{Ob}(\mathrm{Cond}(\mathcal{A}b))$ the functor $(-) \otimes M$ has a right adjoint $[M, -]\colon \mathrm{Cond}(\mathcal{A}b) \to \mathrm{Cond}(\mathcal{A}b)$, called the internal hom in $\mathrm{Cond}(\mathcal{A}b)$.*

Proof. Pointwise tensor product preserves pointwise colimits, because the tensor product of abelian groups has a right adjoint. Sheafification also has a right adjoint, hence $(-) \otimes M$ preserves colimits. This implies the existence of a right adjoint functor $[M, -]$ by the adjoint functor theorem. $\square$

To get an explicit description of the internal-hom we can use the Yoneda isomorphisms in the proof of theorem 1.14 again. More precisely, let M and N be condensed abelian groups and let S be an extremally disconnected set. Then we have a natural isomorphism $[M, N](S) \cong \mathrm{Hom}(\mathbb{Z}[\underline{S}], [M, N])$ by Yoneda and the free-forgetful adjunction, so by the tensor-hom adjunction we get a natural isomorphism

$$[M, N](S) \cong \mathrm{Hom}(\mathbb{Z}[\underline{S}] \otimes M, N).$$

1.3 Derived category

Grothendieck categories are a particularly nice setting for homological algebra, e.g. because they have enough injectives. But in our case it gets even better, because $\mathrm{Cond}(\mathcal{A}b)$ also has enough projectives and is generated by its compact projective objects. So let us say a few words about the derived category of condensed abelian grouops.

Let $\mathcal{D}^\star = \mathcal{D}^\star(\mathrm{Cond}(\mathcal{A}b))$ be the derived categroy of $\mathrm{Cond}(\mathcal{A}b)$, where $\star \in \{\ \ , \mathrm{b}, +, -\}$ stands for unbounded, bounded above and below, bounded below and bounded above complexes. This can be constructed as usual, passing first to the homotopy category and then inverting the remaining quasi-isomorphisms with roofs (see e.g. [GM03]). We will see these things in detail in Tanuj's seminar this semester, and it does not seem necessary to get into these details now, so we will postpone them for future talks. The upshot is that the objects in $\mathcal{D}$ are cochain complexes of condesned abelian groups and we identify two complexes whenever they are quasi-isomorphic, i.e. whenever there is a morphism from one to the other inducing isomorphisms in cohomology.

Since $\mathrm{Cond}(\mathcal{A}b)$ has enough injectives and enough projectives, we can identify

$$\mathcal{D}^- = \mathcal{K}^-(\mathcal{P}) \quad \text{and} \quad \mathcal{D}^+ = \mathcal{K}^+(\mathcal{I}),$$

where $\mathcal{P}$ and $\mathcal{I}$ are the full subcategories of projective and injective objects respectively. This implies that we can construct left and right derived functors as usual on $\mathcal{D}^-$ and on $\mathcal{D}^+$ respectively, namely, replace a complex by a resolution and then apply the functor degree-wise. But in fact, thanks to Spaltenstein's resolutions of unbounded complexes, we can usually define left and right derived functors on the whole unbounded derived category:

Lemma 1.20 ([Sta19, Tag 0794])**.** *Let $\mathcal{C}$ be an abelian category in which colimits (resp. limits) indexed over $\mathbb{N}$ exist and are exact, and let $F\colon \mathrm{Cond}(\mathcal{A}b) \to \mathcal{C}$ be a right (resp. left) exact functor that commutes with colimits (resp. limits) indexed over $\mathbb{N}$. Then we can define LF (resp. RF) in the whole derived category $\mathcal{D}$.*

In the previous section we have seen that the tensor-hom adjunction $(-) \otimes M \dashv [M, -]$, which turns $\mathrm{Cond}(\mathcal{A}b)$ into a closed symmetric monoidal category. From this adjunction we deduce that $(-) \otimes M$ commutes with all colimits and that $[M, -]$ commutes with all limits. Hence we can form the derived functors $(-) \otimes^L M \colon \mathcal{D} \to \mathcal{D}$ and $R[M, -]\colon \mathcal{D} \to \mathcal{D}$. These still form an adjoint pair $(-) \otimes^L M \dashv R[M, -]$ making $\mathcal{D}$ a closed symmetric monoidal category as well (see [Sta19, Tag 09T5]).

The category $\mathcal{D}$ is not abelian anymore[7], but it does carry a natural triangulated structure. A *triangulated structure* on an additive category consists of an additive automorphism Σ, called the *suspension* functor, and a collection of *distinguished triangles*, which are diagrams of the form

$$A \to B \to C \to \Sigma(A)$$

satisfying some axioms (TR1) to (TR4) (see e.g. [GM03]). To describe this triangulated structure we need the following:

Definition 1.21. Let $A^\bullet$ be a cochain complex. Define its *shift* as the complex $A[p]^\bullet$ with $A[p]^i = A^{i+p}$ and with differential $d_{A[p]} = (-1)^p d_A$.

Define the *mapping cone* of a morphism of cochain complexes $f\colon A^\bullet \to B^\bullet$ as the complex $C(f)^\bullet$ with $C(f)^i = A^{i+1} \oplus B^i$ and with differential $d_{C(f)} = \left(\begin{smallmatrix} d_{A[1]} & 0 \\ f & d_B \end{smallmatrix}\right)$.

We take the shift [1] as our suspension functor and we take distinguished triangles to be all triangles isomorphic to one of the form

$$A^\bullet \xrightarrow{f} B^\bullet \to C(f)^\bullet \to A[1]^\bullet.$$

[7] $\mathcal{D}(\mathcal{A})$ is abelian if and only if $\mathcal{A}$ is semisimple, meaning that every short exact sequence splits.

Because of the cone short exact sequence (see e.g. [Wei94]) every distinguished triangle comes from a short exact sequence of complexes. And conversely, every short exact sequence of complexes gives rise to a distinguished triangle in $\mathcal{D}$ (but not on the homotopy category).

Remark 1.22. Triangulated categories are not a very good notion, and this will be our main reason to introduce ∞-categories later on. Philosophically they are not good because they are an extra structure on the category, unlike being an abelian category which is a property that any given category may or may not have. But in practice they are also very inconvenient because of their lack of stability with respect to many usual categorical constructions, which in turn usually goes back to the non-functoriality of cones. For instance, categories of functors on a triangulated category are usually not triangulated. This implies that $\mathcal{D}$ is probably not equivalent to $\mathrm{Cond}(\mathcal{D}(\mathcal{A}b))$, because one of them is triangulated and the other one probably is not[8].

Let us close this section with a few words on compact generation. This is particularly relevant in the context of triangulated categories. As Thomason said to Neeman (see [Nee01]), "compact objects are as necessary to this theory as air to breathe".

Let $P \in \mathrm{Cond}(\mathcal{A}b)$ be a compact projective object. We regard $P \in \mathcal{D}$ by seeing it as a complex concentrated on degree 0. Then P is a compact[9] object in the derived category. Since being compact is shift invariant and $\mathrm{Cond}(\mathcal{A}b)$ is compactly generated, so is $\mathcal{D}$. Hence our triangulated category $\mathcal{D}$ is compactly generated. This has the following advantage:

Theorem 1.23 (Brown Representability)**.** *Let $\mathcal{T}$ be a compactly generated triangulated category and let $F\colon \mathcal{T} \to \mathcal{S}$ be a triangulated functor which preserves coproducts. Then F has a right adjoint.*

Note that if we already know F to be triangulated, then this theorem is much stronger than the usual adjoint functor theorems, because we only need to check that the functor preserves coproducts.

[8]Since triangulated categories are an extra structure, this should not strike as a valid reason for them not to be equivalent at first glance. But admitting a triangulated structure has very strong consequences on the category, for instance, all monomorphisms and all epimorphisms split on triangulated categories, and with such properties we can discard equivalences between them.

[9]In the context of additive categories $\mathcal{C}$, an object C is called compact if $\mathrm{Hom}_{\mathcal{C}}(C,-)\colon \mathcal{C} \to \mathcal{A}b$ preserves coproducts, i.e. for every set $\{A_i\}_{i\in I} \subseteq \mathrm{Ob}(\mathcal{C})$ such that $\oplus_i A_i$ exists in $\mathcal{C}$, the canonical map $\oplus_i \mathrm{Hom}_{\mathcal{C}}(C, A_i) \to \mathrm{Hom}_{\mathcal{C}}(C, \oplus_i A_i)$ is a group isomorphism.

2 ∞-categories

Let X be a topological space. We can try to consider this as a category in which the objects are the points of X and the (1-)morphisms are the paths in X, which can be seen as homotopies between maps $\{*\} \to X$. We call a homotopy between paths a 2-morphism, a homotopy between homotopies of paths a 3-morphism, and so on. We obtain an object called the *fundamental* ∞-*grupoid* of X, denoted $\pi_{\leqslant\infty}(X)$. Even if we just consider its objects and its 1-morphisms, $\pi_{\leqslant\infty}(X)$ is not a category, because composition of paths is not uniquely defined, and even if we agree on a formula, it would not be associative. But it is uniquely defined and associative up to a 2-morphism, and similarly composition of n-morphisms is only uniquely defined and associative up to an $n+1$-morphism. Note also that n-morphisms are invertible up to an $n+1$-morphism for all $n \in \mathbb{N}_{>0}$.

Question 2.1. How similar or different is $\pi_{\leqslant\infty}(X)$ from a usual category?

If $\mathcal{C}$ is any category, then we can think of it as having no n-morphisms for any $n \in \mathbb{N}_{>1}$. Then uniqueness of composition and associativity up to higher morphisms is still true in this case, so we can say that this is something that $\mathcal{C}$ and $\pi_{\leqslant\infty}(X)$ have in common. On the other hand, the 1-morphisms in $\mathcal{C}$ may not be invertible, so this is a feature that $\pi_{\leqslant\infty}(X)$ has but $\mathcal{C}$ does not have.

We want to find a more general notion of which both $\mathcal{C}$ and $\pi_{\leqslant\infty}(X)$ are particular cases. This will be the notion of an ∞-category: a collection of objects and for each $n \in \mathbb{N}_{>0}$ a collection of n-morphisms such that composition of n-morphisms is uniquely defined and associative only up to $n+1$-morphism and such that all n-morphisms are invertible up to $n+1$-morphism for all $n \in \mathbb{N}_{>1}$.

Besides of being a natural generalization, ∞-categories also solve some issues that arise in classical category theory. The derived category of an abelian category carries a natural triangulated structure, but triangulated categories are not well-behaved with respect to many usual constructions in category theory due to the lack of functoriality of cones. In particular, despite $\mathcal{D}(\mathcal{A}b)$ carrying a natural triangulated structure, one should not expect $\mathrm{Cond}(\mathcal{D}(\mathcal{A}b))$ to be triangulated. Therefore we should neither expect to have an equivalence $\mathcal{D}(\mathrm{Cond}(\mathcal{A}b)) \cong \mathrm{Cond}(\mathcal{D}(\mathcal{A}b))$, because even if a triangulation is by definition an extra structure on our category, admitting such an extra structure has very strong implications on the properties of the underlying category, e.g. all monomorphisms and all epimorphisms have to split.

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