Seminar on Condensed Mathematics
Talk 2: Condensed Abelian Groups

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1 The category of condensed abelian groups

The *pro-étale site* of a point, denoted $*_{pro\acute{e}t}$, consists of:

- The category[1] whose objects are profinite sets S (topological spaces homeomorphic to an inverse limit of finite discrete topological spaces, a.k.a. totally disconnected compact Hausdorff spaces) and whose morphisms are continuous maps.
- For a profinite set S, its covering sieves are the ones generated[2] by finite families $\{S_j \to S\}_{j \in J}$ of jointly surjective maps (see [Sta19, Tag 00Z1]).

Definition 1.1. Let $\mathcal{C}$ be any category. The category $\mathrm{Cond}(\mathcal{C})$ of *condensed objects* of $\mathcal{C}$ is the category of $\mathcal{C}$-valued sheaves on $*_{pro\acute{e}t}$.

As a warm up let us first characterize sheaves of abelian groups on $*_{pro\acute{e}t}$:

Lemma 1.2. *A contravariant functor F from the category of profinite sets to the category $\mathcal{A}b$ of abelian groups is a sheaf on $*_{pro\acute{e}t}$ if and only if the following conditions hold:*

[1] This is a large category, which translates into some set-theoretical issues later on. We will ignore these issues during the talk.

[2] Recall that a sieve on S is a subfunctor F of h_S, so for each profinite set S' we need to give a subset $F(S') \subseteq \mathrm{Hom}(S', S)$ with the only condition that $S_2 \to S_1 \to S$ is in $F(S_2)$ as soon as $S_1 \to S$ is in $F(S_1)$.

i) $F(\varnothing) = 0$,

ii) *The natural map* $F(S \sqcup S') \to F(S) \oplus F(S')$ *is an isomorphism, and*

iii) *for any surjection* $S' \to S$ *with projections* $p_1, p_2 \colon S' \times_S S' \to S'$ *from the fibre product, the map* $F(S) \to F(S')$ *is a monomorphism with image*

$$\{g \in F(S') \mid p_1^*(g) = p_2^*(g) \in F(S' \times_S S'\}.$$

Proof. Let $\{f_i \colon S_i \to S\}_{i \in I}$ be a finite fimily of jointly surjective morphisms. We identify each profinite set with its image under the Yoneda-embedding and denote $S_{i,j} = S_i \times_S S_j$, which on an object T is given by the set of pairs (g_i, g_j) (with $g_k \colon T \to S_k$ for $k \in \{i, j\}$) such that $f_i \circ g_i = f_2 \circ g_2$. Then the covering sieve generated by this family can be described as the coequalizer of the diagram

$$\bigsqcup_{i,j} S_{i,j} \rightrightarrows \bigsqcup_i S_i,$$

which on an object T is given by maps □

With this characterization we can already cook up some examples of condensed abelian groups:

Lemma 1.3. *Let* G *be an abelian topolgoical group. Then the functor* $\underline{G}$ *sending a condensed set* S *to the group of continuous maps* $S \to G$ *is a condensed abelian group.*

Proof. We need to show that it is a sheaf on $*_{proét}$. □

References

[Sta19] The Stacks project authors. The stacks project. `https://stacks.math.columbia.edu`, 2019.