Seminar on Condensed Mathematics
Talk 2: Condensed Abelian Groups

Pedro Núñez

14th October 2019

Contents

1 The category of condensed abelian groups

Recall from the previous talk:

Definition 1.1. The *pro-étale site* of a point, denoted $*_{prot}$, consists of:

- The category whose objects are profinite sets S (topological spaces homeomorphic to an inverse limit of finite discrete topological spaces, a.k.a. totally disconnected compact Hausdorff spaces) and whose morphisms are continuous maps.
- The coverings of a profinite set S are all finite families of jointly surjective maps, i.e. all families of morphisms $\{S_i \to S\}_{i \in I}$ indexed by finite sets J such that $\sqcup_{i \in I} S_i \to S$ is surjective.

It is easy to check that the axioms of a covering family are satisfied (see [**?**, Tag 00VH]), so we have a well-defined site. Let us start by characterizing sheaves on this site:

Lemma 1.2. *A contravariant functor T from proét to the category of abelian groups $\mathcal{A}b$ is a sheaf if and only if the following hold:*

i) $T(\varnothing) = 0$,

ii) $T(S_1 \sqcup S_2) \cong T(S_1) \oplus T(S_2)$, *and* *iii) For any surjection $S' \twoheadrightarrow S$, T induces a group isomorphism*

$$T(S) \to \{g \in T(S') \mid p_1^*(g) = p_2^*(g) \in T(S' \times_S S')\},$$

where $p_1, p_2 \colon S' \times_S S' \to S'$ denote the projections.

Proof. By definition, a functor T is a sheaf on *proét* if and only if for every finite family $\{S_i \to S\}_{i \in I}$ of jointly surjective morphisms the diagram

$$T(S) \to \prod_{i \in I} T(S_i) \overset{p_1^*}{\underset{p_2^*}{\rightrightarrows}} \prod_{i,j \in I} T(S_i \times_S S_j)$$

is exact, meaning that the left arrow is an equalizer of the two arrows on the right. □

References