Seminar on Condensed Mathematics

Talk 2: Condensed Abelian Groups

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1 The category of condensed abelian groups

Recall from the previous talk:

Definition 1.1. The *pro-étale site* of a point, denoted $*_{prot}$, consists of:

- The category whose objects are profinite sets S (topological spaces homeomorphic to an inverse limit of finite discrete topological spaces, a.k.a. totally disconnected compact Hausdorff spaces) and whose morphisms are continuous maps.
- The coverings of a profinite set S are all finite families of jointly surjective maps, i.e. all families of morphisms $\{S_i \to S\}_{i\in I}$ indexed by finite sets J such that $\sqcup_{i\in I} S_i \to S$ is surjective.

It is easy to check that the axioms of a covering family are satisfied (see [Sta19, Tag 00VH]), so we have a well-defined site. Let us start by characterizing sheaves on this site:

Lemma 1.2. *A contravariant functor T from $*_{pro\acute{e}t}$ to the category of abelian groups $\mathcal{A}b$ is a sheaf if and only if the following hold:*

a) $T(\varnothing) = 0$*,*

b) $T(S_1 \sqcup S_2) \cong T(S_1) \oplus T(S_2)$*, and* *c) For any surjection $S' \twoheadrightarrow S$, T induces a group isomorphism*

$$T(S) \to \{g \in T(S') \mid p_1^*(g) = p_2^*(g) \in T(S' \times_S S')\},$$

where $p_1, p_2 \colon S' \times_S S' \to S'$ denote the projections.

Proof. By definition, a functor T is a sheaf on $*_{pro\acute{e}t}$ if and only if for every finite family $\{S_i \to S\}_{i \in I}$ of jointly surjective morphisms the diagram

$$T(S) \to \prod_{i \in I} T(S_i) \overset{p_1^*}{\underset{p_2^*}{\rightrightarrows}} \prod_{(i,j) \in I^2} T(S_i \times_S S_j)$$

is exact, meaning that the left arrow is an equalizer of the two arrows on the right. Since we are in $\mathcal{A}b$ and I is a finite set, we can reformulate this as the sequence

$$0 \to T(S) \to \bigoplus_{i \in I} T(S_i) \xrightarrow{p_1^* - p_2^*} \bigoplus_{(i,j) \in I^2} T(S_i \times_S S_j) \tag{1}$$

being exact.

So assume first that this is the case. The empty set is covered by the empty family, so the $T(\varnothing)$ must be a subgroup of 0, hence 0 itself. This shows $a)$. The natural inclusions $S_i \to S_1 \sqcup S_2 = S$ cover S for $i \in I = \{1, 2\}$, so it suffices to show that $p_1^* = p_2^*$ to verify $b)$. Since $i_1(S_1) \cap i_2(S_2) = \varnothing$, we have $S_1 \times_S S_2 = S_2 \times_S S_1 = \varnothing$. So if $(g, h) \in T(S_1) \oplus T(S_2)$, then

$$\begin{aligned} p_1^*(g,h) &= (T(p_1^{S_1 \times_S S_1})(g), T(p_1^{S_1 \times_S S_2})(g), T(p_1^{S_2 \times_S S_1})(h), T(p_1^{S_2 \times_S S_2})(h)) \\ &= (T(p_1^{S_1 \times_S S_1})(g), 0, 0, T(p_1^{S_2 \times_S S_2})(h)) \\ &= (T(p_2^{S_1 \times_S S_1})(g), T(p_2^{S_1 \times_S S_2})(h), T(p_2^{S_2 \times_S S_1})(g), T(p_2^{S_2 \times_S S_2})(h)) = p_2^*(g,h). \end{aligned}$$

To show $c)$, note that $S' \twoheadrightarrow S$ is already a cover, so equation (1) immediately implies the result.

Let us now see the converse by induction on the cardinality of I. If $I = \{1\}$, then we have a surjection $S_1 \twoheadrightarrow S$ and we are done by $c)$. If $I = \{1, 2\}$, then we consider $S' = S_1 \sqcup S_2 \twoheadrightarrow S$. By the previous case and $b)$ we have

$$T(S) = \{(g_1, g_2) \in T(S_1) \oplus T(S_2) \mid T(p_1^{S' \times_S S'})(g_1, g_2) = T(p_2^{S' \times_S S'})(g_1, g_2)\}.$$

Since $S' \times_S S' = \sqcup_{(i,j) \in I^2} S_i \times_S S_j$, the previous condition is equivalent to the conditions

$$P_{i,j}(g_i, g_j) :\equiv T(p_1^{S_i \times_S S_j})(g_i) = T(p_2^{S_i \times_S S_j})(g_j)$$

for $(i,j) \in I^2$, which is what we wanted. If $I = \{1,\dots,m\}$ with $m \geqslant 3$, then we have m morphisms $f_i\colon S_i \to S$ such that $\sqcup_{i=1}^m S_i \twoheadrightarrow S$ is surjective. Write $S' = f_1(S_1) \subseteq S$ and $S'' = \cup_{i=2}^m f_i(S_i) \subseteq S$. These are compact subsets of a profinite set, hence profinite sets themselves. By c) and using that $S_i \times_{S'} S_j = S_i \times_S S_j$ we have an exact sequence

$$0 \to T(S') \to T(S_1) \to T(S_1 \times_S S_1),$$

and since S'' is covered by the remaining $m-1$ morphisms, by induction hypothesis and the same identification as before we have an exact sequence

$$0 \to T(S'') \to \bigoplus_{i=2}^m T(S_i) \to \bigoplus_{(i,j)\in J^2} T(S_i \times_S S_j),$$

where $J = \{2,\dots,m\}$. This means that $T(S') = \{g_1 \in T(S_1) \mid P_{1,1}(g_1,g_1)\}$ and that $T(S'') = \bigcap_{(i,j)\in J^2}\{(g_2,\dots,g_m) \in \oplus_{i=2}^m T(S_i) \mid P_{i,j}(g_i,g_j)\}$. We need to show $T(S) = \bigcap_{(i,j)\in I^2}\{(g_1,\dots,g_m) \in \oplus_{i=1}^m T(S_i) \mid P_{i,j}(g_i,g_j)\}$. For this we use the cover given by the two inclusions $i'\colon S' \to S$ and $i''\colon S'' \to S$, which by induction hypothesis yields

$$T(S) = \{(g_1; g_2,\dots,g_m) \in T(S') \oplus T(S'') \mid \\ T(p_1^{(S'\cap S'')^2})(g_1) = T(p_2^{(S'\cap S'')^2})(g_2,\dots,g_m)\}.$$

Besides of the conditions $P_{i,j}$ already present in $T(S')$ and in $T(S'')$ for $(i,j) \in I^2\backslash(\{1\}\times J \cup J \times \{1\})$, the previous equality shows that the conditions $P_{i,j}$ for $(i,j) \in \{1\}\times J \cup J \times\{1\}$ are also satisfied. $\square$

With this characterization we can already produce some examples:

Example 1.3. Let G be a topological abelian group. Then the functor $\underline{G} = \mathrm{Hom}_{\mathcal{T}op}(-,G)\colon *_{pro\acute{e}t} \to \mathcal{A}b$ is a condensed abelian group. Conditions a) and b) in lemma 1.2 are clear. For condition c), let $f\colon S' \twoheadrightarrow S$ be a surjection. We want to show that $f^*\colon \underline{G}(S) \to \underline{G}(S')$ induces an isomorphism between the set of continuous maps $g\colon S \to G$ and the set of continous maps $h\colon S' \to G$ such that $h \circ p_1 = h \circ p_2$. The image of f^* is indeed contained in the set of all such maps, because $f \circ p_1 = f \circ p_2$. Moreover, since f is an epimorphism, f^* is injective. So it only remains to show surjectivity. Let $h\colon S' \to G$ be a map such that $h \circ p_1 = h \circ p_2$. For each $s \in S$, let $g(s) := h(s')$ for some $s' \in f^{-1}(\{s\})$. This is well-defined in $\mathcal{S}et$, because if $f(s') = f(s'') = s$, then $(s',s'') \in S' \times_S S'$, and thus we can write

$$g(s) = h(s') = h \circ p_1(s',s'') = h \circ p_2(s',s'') = h(s'').$$

But in fact it is a morphism in $\mathcal{T}op$. Indeed, S carries the quotient topology induced by f, so $f^{-1}(g^{-1}(U)) = h^{-1}(U)$ being open in S' for all U open in G implies that $g^{-1}(U)$ is open in S for all U open in G.

References

[Sta19] The Stacks project authors. The stacks project. `https://stacks.math.columbia.edu`, 2019.