Seminar on Condensed Mathematics
Talk 2: Condensed Abelian Groups

Pedro Núñez

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1 The category of condensed abelian groups

Recall from the previous talk:

Definition 1.1. The *pro-étale site* of a point, denoted $*_{prot}$, consists of:

- The category whose objects are profinite sets S (topological spaces homeomorphic to an inverse limit of finite discrete topological spaces, a.k.a. totally disconnected compact Hausdorff spaces) and whose morphisms are continuous maps.
- The coverings of a profinite set S are all finite families of jointly surjective maps, i.e. all families of morphisms $\{S_i \to S\}_{i \in I}$ indexed by finite sets J such that $\sqcup_{i \in I} S_i \to S$ is surjective.

It is easy to check that the axioms of a covering family are satisfied (see [Sta19, Tag 00VH]), so we have a well-defined site. Let us start by characterizing sheaves on this site:

Lemma 1.2. *A contravariant functor T from $*_{\text{proét}}$ to the category of abelian groups $\mathcal{A}b$ is a sheaf if and only if the following hold:*

a) $T(\varnothing) = 0$,

b) $T(S_1 \sqcup S_2) \cong T(S_1) \oplus T(S_2)$, *and* *c) For any surjection $S' \twoheadrightarrow S$, T induces a group isomorphism*

$$T(S) \to \{g \in T(S') \mid p_1^*(g) = p_2^*(g) \in T(S' \times_S S')\},$$

where $p_1, p_2 \colon S' \times_S S' \to S'$ denote the projections.

Proof. By definition, a functor T is a sheaf on $*_{pro\acute{e}t}$ if and only if for every finite family $\{S_i \to S\}_{i\in I}$ of jointly surjective morphisms the diagram

$$T(S) \to \prod_{i\in I} T(S_i) \overset{p_1^*}{\underset{p_2^*}{\rightrightarrows}} \prod_{(i,j)\in I^2} T(S_i \times_S S_j)$$

is exact, meaning that the left arrow is an equalizer of the two arrows on the right. Since we are in $\mathcal{A}b$ and I is a finite set, we can reformulate this as the sequence

$$0 \to T(S) \to \bigoplus_{i\in I} T(S_i) \xrightarrow{p_1^* - p_2^*} \bigoplus_{(i,j)\in I^2} T(S_i \times_S S_j) \tag{1}$$

being exact.

So assume first that this is the case. The empty set is covered by the empty family, so the $T(\varnothing)$ must be a subgroup of 0, hence 0 itself. This shows a). The natural inclusions $S_i \to S_1 \sqcup S_2 = S$ cover S for $i \in I = \{1,2\}$, so it suffices to show that $p_1^* = p_2^*$ to verify b). Since $i_1(S_1) \cap i_2(S_2) = \varnothing$, we have $S_1 \times_S S_2 = S_2 \times_S S_1 = \varnothing$. So if $(g,h) \in T(S_1) \oplus T(S_2)$, then

$$\begin{aligned} p_1^*(g,h) &= (T(p_1^{S_1\times_S S_1})(g), T(p_1^{S_1\times_S S_2})(g), T(p_1^{S_2\times_S S_1})(h), T(p_1^{S_2\times_S S_2})(h)) \\ &= (T(p_1^{S_1\times_S S_1})(g), 0, 0, T(p_1^{S_2\times_S S_2})(h)) \\ &= (T(p_2^{S_1\times_S S_1})(g), T(p_2^{S_1\times_S S_2})(h), T(p_2^{S_2\times_S S_1})(g), T(p_2^{S_2\times_S S_2})(h)) = p_2^*(g,h). \end{aligned}$$

To show c), note that $S' \twoheadrightarrow S$ is already a cover, so equation (1) immediately implies the result.

Let us now see the converse by induction on the cardinality of I. If $I = \{1\}$, then we have a surjection $S_1 \twoheadrightarrow S$ and we are done by c). If $I = \{1,\ldots,m\}$, then we have m morphisms $f_i \colon S_i \to S$ such that $\sqcup_{i=1}^m S_i \twoheadrightarrow S$ is surjective. Write $S' = f_1(S_1) \subseteq S$ and $S'' = S \setminus S'$. By b) we have $T(S) = T(S') \oplus T(S'')$. By c) we have an exact sequence

$$0 \to T(S') \to T(S_1) \to T(S_1 \times_{S'} S_1) = T(S_1 \times_S S_1),$$

and since S'' is covered by the remaining $m-1$ morphisms, by induction hypothesis we have an exact sequence

$$0 \to T(S'') \to \bigoplus_{i=2}^{m} T(S_i) \to \bigoplus_{(i,j)\in I^2\setminus\{(1,1)\}} T(S_i \times_S S_j).$$

Taking the direct sum of these two exact sequences finishes the proof. $\square$

References

[Sta19] The Stacks project authors. The stacks project. `https://stacks.math.columbia.edu`, 2019.