Seesaw Principle and Theorem of the Cube

Remarks on Section I.5 of Milne's *Abelian Varieties*

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Seesaw Principle — Idea

Theorem (Weil)

A limit of trivial line bundles on a complete variety is again trivial.

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$\downarrow$ making this idea more precise

Theorem (Statement for varieties over $k = \bar{k}$)

V complete, T any ("parameter space"), $\mathcal{L} \in \mathrm{Pic}(V \times T)$. Then

$$Z := \{t \in T \mid \mathcal{L}_t \text{ is trivial}\} \lhook\joinrel\not\longrightarrow T$$

is closed and $\mathcal{L}|_{V \times Z}$ is the pullback of a line bundle on Z.

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This is the conclusion of the subsection on the seesaw principle in [Mil08], which can be read independently. This statement will be used to **reduce** the Theorem of the Cube to an easier case.

Seesaw Principle — Generalisation

Theorem ([Bha17, Theorem 6.3] or [Fra18, Theorem 5])

Let $f : X \to S$ be a proper flat morphism of (locally) noetherian schemes with geometrically integral fibres and let $\mathcal{L} \in X$. Then

$$Z := \{ s \in S \mid \mathcal{L}_s \text{ is trivial} \} \lhook\joinrel\not\longrightarrow S$$

and $\mathcal{L}|_{f^{-1}(Z_{\mathrm{red}})} = f^*\mathcal{M}_0$ for some $\mathcal{M}_0 \in \mathrm{Pic}(Z_{\mathrm{red}})$. Moreover, there exists a (unique) closed subscheme structure on Z such that $\mathcal{L}|_{f^{-1}(Z)} = f^*\mathcal{M}$ for some $\mathcal{M} \in \mathrm{Pic}(Z)$ and Z is universal amongst all (locally) noetherian S-schemes with this property.

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Brian Conrad proves this using existence and separatedness of $\mathrm{Pic}_{X/S}$ [Con15, Theorem 3.1.1]: $\mathscr{O}_X$ gives us $[\mathscr{O}_X]\colon S \to \mathrm{Pic}_{X/S}$, whose image we identify with S, and then $Z = [\mathcal{L}](S) \cap [\mathscr{O}_X](S)$. Conversely, assuming $\mathrm{Pic}_{X/S}$ exists, the seesaw theorem implies that it is separated [Fra18, Remark 2.4.2] (automatic if $S = k$).

Seesaw Principle — Particular case in a picture

$$
\begin{tikzcd}
X & & Y \arrow[dd, dashed, bend left] \\
& X \times Y \arrow[ul] \arrow[ur, dashed] \arrow[dl, dashed] \arrow[dr] & \\
X \arrow[uu, dashed, bend left] & & Y
\end{tikzcd}
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We have some $\mathcal{L} \in \mathrm{Pic}(X \times Y)$. We can think of $\mathcal{L}$ as a weight distribution on the two extremes X and Y of this seesaw: the more points $x \in X$ such that $\mathcal{L}|_{\{x\}\times Y}$ is trivial, the heavier $\mathcal{L}$ sits on X.

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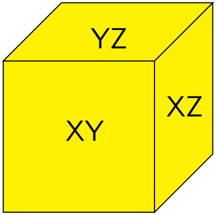
Theorem of the Cube — Statement

Theorem (Statement for varieties over $k = \bar{k}$)

X, Y complete, Z any and $\mathcal{L} \in \mathrm{Pic}(X \times Y \times Z)$ such that

$$\mathcal{L}|_{X \times Y \times \{z_0\}}, \mathcal{L}|_{X \times \{y_0\} \times Z} \text{ and } \mathcal{L}|_{\{x_0\} \times Y \times Z}$$

are trivial. Then $\mathcal{L}$ is trivial.



If the restriction of $\mathcal{L} \in \mathrm{Pic}(\mathrm{Cube})$ to each face XY, YZ and XZ is trivial, then $\mathcal{L}$ is trivial.

Theorem of the Cube — Generalisation

As with the Seesaw principle, it is possible to generalise the Theorem of the Cube to the relative setting, see for example:

- [Bha17, Corollary 6.8] for proper flat schemes with geometrically integral fibres over a noetherian base.
- [Fra18, Theorem 9] for an abelian scheme over a locally noetherian base.

Theorem of the Cube — An aplication

One application deserving its own name is [Mil08, Theorem I.5.5]:

Theorem (Theorem of the Square for varieties over $k = \bar{k}$)

Let A be an abelian variety and $\mathcal{L} \in \mathrm{Pic}(A)$. For all $a \in A$ denote $t_a \colon A \to A$ the translation by a. Then for all $a, b \in A$ we have

$$\mathcal{L} \otimes t_{a+b}^*\mathcal{L} \cong t_a^*\mathcal{L} \otimes t_b^*\mathcal{L}.$$

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Let A be an abelian variety and $\mathcal{L} \in \operatorname{Pic}(A)$. For all $a \in A$ denote $t_a \colon A \to A$ the translation by a. Then for all $a, b \in A$ we have

$$\mathcal{L} \otimes t_{a+b}^* \mathcal{L} \cong t_a^* \mathcal{L} \otimes t_b^* \mathcal{L}.$$

In the relative setting, LHS and RHS will differ by the pullback of a line bundle on the base scheme, so if $\operatorname{Pic}(S) = 0$ we still have such an isomorphism but it is not canonical. On the other hand, the isomorphisms in [Mil08, Corollaries I.5.2–I.5.4] are canonical even in the relative setting, cf. [Fra18, Corollaries to Theorem 9].

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