Endomorphisms of abelian varieties

Remarks on Section I.10 of Milne's *Abelian Varieties*

University of Freiburg

23th June 2020

Recall isogenies

Definition

$\alpha \in \mathrm{Hom}(A, B)$ **isogeny** $\Leftrightarrow$ surjective with finite kernel;

$\Leftrightarrow$ surjective and $\dim A = \dim B$;

$\Leftrightarrow$ finite kernel and $\dim A = \dim B$;

$\Leftrightarrow$ finite and surjective (and flat).

Recall isogenies

Definition

$\alpha \in \mathrm{Hom}(A, B)$ **isogeny** $\Leftrightarrow$ surjective with finite kernel;
$\Leftrightarrow$ surjective and $\dim A = \dim B$;
$\Leftrightarrow$ finite kernel and $\dim A = \dim B$;
$\Leftrightarrow$ finite and surjective (and flat).

Examples

- $\mathcal{L}$ ample $\Rightarrow \lambda_{\mathcal{L}} \colon A \to A^\vee$ isogeny [Mil08, Prop. 8.1].
- $n > 0 \Rightarrow n_A \colon a \mapsto na$ isogeny [Mil08, Thm. 7.2].

Recall isogenies

Definition

$$\begin{aligned}\alpha \in \mathrm{Hom}(A, B) \textbf{ isogeny} &\Leftrightarrow \text{surjective with finite kernel;}\\ &\Leftrightarrow \text{surjective and } \dim A = \dim B;\\ &\Leftrightarrow \text{finite kernel and } \dim A = \dim B;\\ &\Leftrightarrow \text{finite and surjective (and flat).}\end{aligned}$$

Examples

- $\mathcal{L}$ ample $\Rightarrow \lambda_{\mathcal{L}} \colon A \to A^\vee$ isogeny [Mil08, Prop. 8.1].
- $n > 0 \Rightarrow n_A \colon a \mapsto na$ isogeny [Mil08, Thm. 7.2].

Last example $\Rightarrow$ n-torsion subgroup $A_n := \ker(a \mapsto na)$ is **finite**.

Abelian varieties up to isogeny

- **AV**(k): additive cat. of abelian varieties and homomorphisms.

Abelian varieties up to isogeny

- **AV**(k): additive cat. of abelian varieties and homomorphisms.
- $\mathbf{AV}^0(k)$: $\mathbb{Q}$-linear cat. with $\mathrm{Hom}^0(A, B) := \mathrm{Hom}(A, B) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Abelian varieties up to isogeny

- $\mathbf{AV}(k)$: additive cat. of abelian varieties and homomorphisms.
- $\mathbf{AV}^0(k)$: $\mathbb{Q}$-linear cat. with $\mathrm{Hom}^0(A,B) := \mathrm{Hom}(A,B) \otimes_{\mathbb{Z}} \mathbb{Q}$.
- $\alpha \in \mathrm{Hom}(A,B)$ isogeny $\Rightarrow B \cong A/\mathrm{Ker}(\alpha)$ and $\mathrm{Ker}(\alpha) \subseteq A_n$ for some $n \in \mathbb{N}$. Hence α becomes an isomorphism in $\mathbf{AV}^0(k)$:

$$\begin{tikzcd} A \arrow[rr, "n_A"] \arrow[dr, "\alpha"'] & & A \\ & A/\mathrm{Ker}(\alpha) \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

Abelian varieties up to isogeny

- $\mathbf{AV}(k)$: additive cat. of abelian varieties and homomorphisms.
- $\mathbf{AV}^0(k)$: $\mathbb{Q}$-linear cat. with $\mathrm{Hom}^0(A, B) := \mathrm{Hom}(A, B) \otimes_{\mathbb{Z}} \mathbb{Q}$.
- $\alpha \in \mathrm{Hom}(A, B)$ isogeny $\Rightarrow B \cong A/\mathrm{Ker}(\alpha)$ and $\mathrm{Ker}(\alpha) \subseteq A_n$ for some $n \in \mathbb{N}$. Hence α becomes an isomorphism in $\mathbf{AV}^0(k)$:

$$\begin{tikzcd} A \arrow[rr, "n_A"] \arrow[dr, "\alpha"'] & & A \\ & A/\mathrm{Ker}(\alpha) \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

- $\alpha \in \mathrm{Hom}(A, B)$ isomorphism in $\mathrm{Hom}^0(A, B) \Rightarrow$ we can find such a factorization.

Abelian varieties up to isogeny

- $\mathbf{AV}(k)$: additive cat. of abelian varieties and homomorphisms.
- $\mathbf{AV}^0(k)$: $\mathbb{Q}$-linear cat. with $\mathrm{Hom}^0(A, B) := \mathrm{Hom}(A, B) \otimes_{\mathbb{Z}} \mathbb{Q}$.
- $\alpha \in \mathrm{Hom}(A, B)$ isogeny $\Rightarrow B \cong A/\mathrm{Ker}(\alpha)$ and $\mathrm{Ker}(\alpha) \subseteq A_n$ for some $n \in \mathbb{N}$. Hence α becomes an isomorphism in $\mathbf{AV}^0(k)$:

$$\begin{tikzcd} A \arrow[rr, "n_A"] \arrow[dr, "\alpha"'] & & A \\ & A/\mathrm{Ker}(\alpha) \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

- $\alpha \in \mathrm{Hom}(A, B)$ isomorphism in $\mathrm{Hom}^0(A, B) \Rightarrow$ we can find such a factorization.
- Hence α is an isogeny $\Leftrightarrow$ α is an isomorphism in $\mathbf{AV}^0(k)$.

Abelian varieties up to isogeny

- $\mathbf{AV}(k)$: additive cat. of abelian varieties and homomorphisms.
- $\mathbf{AV}^0(k)$: $\mathbb{Q}$-linear cat. with $\mathrm{Hom}^0(A,B) := \mathrm{Hom}(A,B) \otimes_{\mathbb{Z}} \mathbb{Q}$.
- $\alpha \in \mathrm{Hom}(A,B)$ isogeny $\Rightarrow B \cong A/\operatorname{Ker}(\alpha)$ and $\operatorname{Ker}(\alpha) \subseteq A_n$ for some $n \in \mathbb{N}$. Hence α becomes an isomorphism in $\mathbf{AV}^0(k)$:

$$\begin{tikzcd} A \arrow[rr, "n_A"] \arrow[dr, "\alpha"'] & & A \\ & A/\operatorname{Ker}(\alpha) \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

- $\alpha \in \mathrm{Hom}(A,B)$ isomorphism in $\mathrm{Hom}^0(A,B) \Rightarrow$ we can find such a factorization.
- Hence α is an isogeny $\Leftrightarrow \alpha$ is an isomorphism in $\mathbf{AV}^0(k)$.
- In fact $\mathbf{AV}(k) \to \mathbf{AV}^0(k)$ localizes $\mathbf{AV}(k)$ at all isogenies.

The category $\mathbf{AV}^0(k)$

- It is **abelian** [encyclopediaofmath.org/wiki/Isogeny].

The category $\mathbf{AV}^0(k)$

- It is **abelian** [encyclopediaofmath.org/wiki/Isogeny].
- Every object is **semisimple**: every A is isogenous to a finite direct sum of indecomposable A_i's [Mil08, Prop. I.10.1].

The category $\mathbf{AV}^0(k)$

- It is **abelian** [encyclopediaofmath.org/wiki/Isogeny].
- Every object is **semisimple**: every A is isogenous to a finite direct sum of indecomposable A_i's [Mil08, Prop. I.10.1].
  - Thus every short exact sequence in $\mathbf{AV}^0(k)$ splits [Jeremy Rickard's comment on mathoverflow.net/a/327944/99436].

The category $\mathbf{AV}^0(k)$

- It is **abelian** [encyclopediaofmath.org/wiki/Isogeny].
- Every object is **semisimple**: every A is isogenous to a finite direct sum of indecomposable A_i's [Mil08, Prop. I.10.1].
  - Thus every short exact sequence in $\mathbf{AV}^0(k)$ splits [Jeremy Rickard's comment on mathoverflow.net/a/327944/99436].
  - Thus all additive functors are already exact and its derived category, which is equivalent to the category of cochain complexes with trivial differentials, is abelian [GM03, III.2.4].

The category $\mathbf{AV}^0(k)$

- It is **abelian** [encyclopediaofmath.org/wiki/Isogeny].
- Every object is **semisimple**: every A is isogenous to a finite direct sum of indecomposable A_i's [Mil08, Prop. I.10.1].
  - Thus every short exact sequence in $\mathbf{AV}^0(k)$ splits [Jeremy Rickard's comment on mathoverflow.net/a/327944/99436].
  - Thus all additive functors are already exact and its derived category, which is equivalent to the category of cochain complexes with trivial differentials, is abelian [GM03, III.2.4].
- All homs are finite dimensional over $\mathbb{Q}$ [Mil08, Thm. I.10.15].

The category $\mathbf{AV}^0(k)$

- It is **abelian** [encyclopediaofmath.org/wiki/Isogeny].
- Every object is **semisimple**: every A is isogenous to a finite direct sum of indecomposable A_i's [Mil08, Prop. I.10.1].
  - Thus every short exact sequence in $\mathbf{AV}^0(k)$ splits [Jeremy Rickard's comment on mathoverflow.net/a/327944/99436].
  - Thus all additive functors are already exact and its derived category, which is equivalent to the category of cochain complexes with trivial differentials, is abelian [GM03, III.2.4].
- All homs are finite dimensional over $\mathbb{Q}$ [Mil08, Thm. I.10.15].

Semisimple objects $\Rightarrow$ to understand $\mathbf{AV}^0(k)$, it suffices to study endomorphism algebras of simple objects [Mil08, p. 43].

Idea in the complex case $[A_i = \mathbb{C}^{g_i}/\Lambda_i]$

- $\mathrm{Hom}(A_1, A_2) \hookrightarrow \mathrm{Hom}(\Lambda_1, \Lambda_2) = \mathrm{Hom}(H_1(A_1, \mathbb{Z}), H_1(A_2, \mathbb{Z}))$, as we saw in [Mil08, §I.2]. Fixing bases for homology we get:

$$\mathrm{Hom}(A_1, A_2) \hookrightarrow M_{2g_1 \times 2g_2}(\mathbb{Z}) \cong \mathbb{Z}^{\oplus 4g_1 g_2}.$$

Idea in the complex case $[A_i = \mathbb{C}^{g_i}/\Lambda_i]$

- $\mathrm{Hom}(A_1, A_2) \hookrightarrow \mathrm{Hom}(\Lambda_1, \Lambda_2) = \mathrm{Hom}(H_1(A_1, \mathbb{Z}), H_1(A_2, \mathbb{Z}))$, as we saw in [Mil08, §I.2]. Fixing bases for homology we get:

$$\mathrm{Hom}(A_1, A_2) \hookrightarrow M_{2g_1 \times 2g_2}(\mathbb{Z}) \cong \mathbb{Z}^{\oplus 4g_1 g_2}.$$

- Hence $\mathrm{Hom}(A_1, A_2) \cong \mathbb{Z}^{\oplus r}$ with $r \leqslant 4 \dim(A_1) \dim(A_2)$.

Idea in the complex case $[A_i = \mathbb{C}^{g_i}/\Lambda_i]$

- $\mathrm{Hom}(A_1, A_2) \hookrightarrow \mathrm{Hom}(\Lambda_1, \Lambda_2) = \mathrm{Hom}(H_1(A_1, \mathbb{Z}), H_1(A_2, \mathbb{Z}))$, as we saw in [Mil08, §I.2]. Fixing bases for homology we get:

$$\mathrm{Hom}(A_1, A_2) \hookrightarrow M_{2g_1 \times 2g_2}(\mathbb{Z}) \cong \mathbb{Z}^{\oplus 4g_1 g_2}.$$

- Hence $\mathrm{Hom}(A_1, A_2) \cong \mathbb{Z}^{\oplus r}$ with $r \leqslant 4 \dim(A_1) \dim(A_2)$.
- $\alpha \in \mathrm{End}(A_1)$ defines a linear $H_1(\alpha) \in \mathrm{End}(H_1(A_1, \mathbb{Q}))$ with characteristic polynomial

$$P_\alpha(t) = \det(H_1(\alpha) - t) = t^{2g} + a_1 t^{2g-1} + \ldots + a_{2g} \in \mathbb{Z}[t].$$

Idea in the complex case $[A_i = \mathbb{C}^{g_i}/\Lambda_i]$

- $\mathrm{Hom}(A_1, A_2) \hookrightarrow \mathrm{Hom}(\Lambda_1, \Lambda_2) = \mathrm{Hom}(H_1(A_1, \mathbb{Z}), H_1(A_2, \mathbb{Z}))$, as we saw in [Mil08, §I.2]. Fixing bases for homology we get:

$$\mathrm{Hom}(A_1, A_2) \hookrightarrow M_{2g_1 \times 2g_2}(\mathbb{Z}) \cong \mathbb{Z}^{\oplus 4g_1g_2}.$$

- Hence $\mathrm{Hom}(A_1, A_2) \cong \mathbb{Z}^{\oplus r}$ with $r \leqslant 4\dim(A_1)\dim(A_2)$.
- $\alpha \in \mathrm{End}(A_1)$ defines a linear $H_1(\alpha) \in \mathrm{End}(H_1(A_1, \mathbb{Q}))$ with characteristic polynomial

$$P_\alpha(t) = \det(H_1(\alpha) - t) = t^{2g} + a_1 t^{2g-1} + \ldots + a_{2g} \in \mathbb{Z}[t].$$

- We can then define $\mathrm{Tr}(\alpha) = -a_1$ and $\mathrm{Nm}(\alpha) = a_{2g} = \deg(\alpha)$ and use them to prove that, for A simple of dimension g:

$$\boxed{[Z : \mathbb{Q}]\sqrt{[\mathrm{End}^0(A) : Z]} \mid 2g, \text{ where } Z := Z(\mathrm{End}^0(A)).}$$

Extending previous ideas to arbitrary base fields [$\ell \neq p$]

- Replace singular homology $H_1(A, \mathbb{Q})$ by Tate module $T_\ell A$, which in this case is the étale homology $\varprojlim A_{\ell^n} \cong H^1(A, \mathbb{Z}_\ell)^\vee$.

Extending previous ideas to arbitrary base fields [$\ell \neq p$]

- Replace singular homology $H_1(A, \mathbb{Q})$ by Tate module $T_\ell A$, which in this case is the étale homology $\varprojlim A_{\ell^n} \cong H^1(A, \mathbb{Z}_\ell)^\vee$.
- [Mil08, Lemma I.10.6]: $\mathrm{Hom}(A_1, A_2)$ torsion-free, because

$$\mathrm{Hom}(A_1, A_2) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}_\ell}(T_\ell A_1, T_\ell A_2) \cong \mathbb{Z}_\ell^{\oplus 4 g_1 g_2}.$$

Extending previous ideas to arbitrary base fields [$\ell \neq p$]

- Replace singular homology $H_1(A, \mathbb{Q})$ by Tate module $T_\ell A$, which in this case is the étale homology $\varprojlim A_{\ell^n} \cong H^1(A, \mathbb{Z}_\ell)^\vee$.
- [Mil08, Lemma I.10.6]: $\mathrm{Hom}(A_1, A_2)$ torsion-free, because

$$\mathrm{Hom}(A_1, A_2) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}_\ell}(T_\ell A_1, T_\ell A_2) \cong \mathbb{Z}_\ell^{\oplus 4 g_1 g_2}.$$

- [Mil08, Thm. I.10.9 and Prop. I.10.20]: $\alpha \in \mathrm{End}(A)$ gives $T_\ell(\alpha) \in \mathrm{End}_{\mathbb{Q}_\ell}(T_\ell A \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell)$ with characteristic polynomial $P_\alpha(t) = \det(T_\ell(\alpha) - t) \in \mathbb{Z}[t]$ and independent of $\ell \neq p$.

Extending previous ideas to arbitrary base fields [$\ell \neq p$]

- Replace singular homology $H_1(A, \mathbb{Q})$ by Tate module $T_\ell A$, which in this case is the étale homology $\varprojlim A_{\ell^n} \cong H^1(A, \mathbb{Z}_\ell)^\vee$.
- [Mil08, Lemma I.10.6]: $\mathrm{Hom}(A_1, A_2)$ torsion-free, because

$$\mathrm{Hom}(A_1, A_2) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}_\ell}(T_\ell A_1, T_\ell A_2) \cong \mathbb{Z}_\ell^{\oplus 4g_1 g_2}.$$

- [Mil08, Thm. I.10.9 and Prop. I.10.20]: $\alpha \in \mathrm{End}(A)$ gives $T_\ell(\alpha) \in \mathrm{End}_{\mathbb{Q}_\ell}(T_\ell A \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell)$ with characteristic polynomial $P_\alpha(t) = \det(T_\ell(\alpha) - t) \in \mathbb{Z}[t]$ and independent of $\ell \neq p$.
- [Mil08, Thm. I.10.15]: $\mathrm{Hom}(A_1, A_2)$ f.g. over $\mathbb{Z}$ and

$$\mathrm{Hom}(A_1, A_2) \otimes \mathbb{Z}_\ell \hookrightarrow \mathrm{Hom}_{\mathbb{Z}_\ell}(T_\ell A_1, T_\ell A_2).$$

Rosati involution and Albert classification

Let $\mathcal{L}$ be an ample line bundle on A and $\lambda_{\mathcal{L}} \colon A \to A^{\vee}$ the corresponding isogeny. Define the *Rosati involution* on $\mathrm{End}^0(A)$ as

$$\alpha \mapsto \alpha' := \lambda_{\mathcal{L}}^{-1} \circ \alpha^{\vee} \circ \lambda_{\mathcal{L}}.$$

Rosati involution and Albert classification

Let $\mathcal{L}$ be an ample line bundle on A and $\lambda_{\mathcal{L}}\colon A \to A^\vee$ the corresponding isogeny. Define the *Rosati involution* on $\mathrm{End}^0(A)$ as

$$\alpha \mapsto \alpha' := \lambda_{\mathcal{L}}^{-1} \circ \alpha^\vee \circ \lambda_{\mathcal{L}}.$$

Then $\alpha \mapsto \mathrm{Tr}(\alpha'\alpha)$ is a positive definite quadratic form on $\mathrm{End}^0(A)$ [Mum70, Thm. 21.1].

Rosati involution and Albert classification

Let $\mathcal{L}$ be an ample line bundle on A and $\lambda_{\mathcal{L}} \colon A \to A^\vee$ the corresponding isogeny. Define the *Rosati involution* on $\mathrm{End}^0(A)$ as

$$\alpha \mapsto \alpha' := \lambda_{\mathcal{L}}^{-1} \circ \alpha^\vee \circ \lambda_{\mathcal{L}}.$$

Then $\alpha \mapsto \mathrm{Tr}(\alpha'\alpha)$ is a positive definite quadratic form on $\mathrm{End}^0(A)$ [Mum70, Thm. 21.1]. This has several applications [Mum70, §21]:

- Albert classification of $\mathrm{End}^0(A)$ for A simple [Thm. 21.2].
- (Weil's) Riemann hypothesis for abelian varieties [Thm. 21.4].
- Structure of $\mathrm{NS}^0(A)$ [Thm. 21.6].

Rosati involution and Albert classification

Let $\mathcal{L}$ be an ample line bundle on A and $\lambda_{\mathcal{L}} \colon A \to A^\vee$ the corresponding isogeny. Define the *Rosati involution* on $\mathrm{End}^0(A)$ as

$$\alpha \mapsto \alpha' := \lambda_{\mathcal{L}}^{-1} \circ \alpha^\vee \circ \lambda_{\mathcal{L}}.$$

Then $\alpha \mapsto \mathrm{Tr}(\alpha'\alpha)$ is a positive definite quadratic form on $\mathrm{End}^0(A)$ [Mum70, Thm. 21.1]. This has several applications [Mum70, §21]:

- Albert classification of $\mathrm{End}^0(A)$ for A simple [Thm. 21.2].
- (Weil's) Riemann hypothesis for abelian varieties [Thm. 21.4].
- Structure of $\mathrm{NS}^0(A)$ [Thm. 21.6].

Mumford studies two cases of the Albert classification in more detail [Mum70, §22], namely:

- CM-abelian varieties over $\mathbb{C}$.
- Elliptic curves in positive characteristic.

References

Sergei I. Gelfand and Yuri I. Manin.
Methods of homological algebra.
Springer Monographs in Mathematics. Springer-Verlag, Berlin, second edition, 2003.

James S. Milne.
Abelian Varieties (v2.00), 2008.
Available at jmilne.org/math.

David Mumford.
Abelian varieties.
Tata Institute of Fundamental Research Studies in Mathematics, No. 5. Published for the Tata Institute of Fundamental Research, Bombay; Oxford University Press, London, 1970.