Endomorphisms of abelian varieties

Remarks on Section I.10 of Milne's *Abelian Varieties*

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Recall isogenies

Definition

$$
\begin{aligned}
\alpha \in \mathrm{Hom}(A, B) \textbf{ isogeny} &\Leftrightarrow \text{surjective with finite kernel;} \\
&\Leftrightarrow \text{surjective and } \dim A = \dim B; \\
&\Leftrightarrow \text{finite kernel and } \dim A = \dim B; \\
&\Leftrightarrow \text{finite and surjective (and flat).}
\end{aligned}
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Examples

- $\mathcal{L}$ ample $\Rightarrow \lambda_{\mathcal{L}} \colon A \to A^\vee$ isogeny [Mil08, Prop. 8.1].
- $n > 0 \Rightarrow n_A \colon a \mapsto na$ isogeny [Mil08, Thm. 7.2].

Recall isogenies

Definition

$\alpha \in \mathrm{Hom}(A, B)$ **isogeny** $\Leftrightarrow$ surjective with finite kernel;
$\Leftrightarrow$ surjective and $\dim A = \dim B$;
$\Leftrightarrow$ finite kernel and $\dim A = \dim B$;
$\Leftrightarrow$ finite and surjective (and flat).

Examples

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Last example $\Rightarrow$ n-torsion subgroup $A_n := \ker(a \mapsto na)$ is **finite**.

Abelian varieties up to isogeny

- $\mathbf{AV}(k)$: additive cat. of abelian varieties and regular homomorphisms over a field k.
- $\mathbf{AV}^0(k)$: $\mathbb{Q}$-linear cat. with $\mathrm{Hom}^0(A, B) := \mathrm{Hom}(A, B) \otimes_{\mathbb{Z}} \mathbb{Q}$.
- $\alpha \in \mathrm{Hom}(A, B)$ isogeny $\Rightarrow B \cong A/\mathrm{Ker}(\alpha)$ and $\mathrm{Ker}(\alpha) \subseteq A_n$ for some $n \in \mathbb{N}$. Hence α becomes an isomorphism in $\mathbf{AV}^0(k)$:

$$\begin{tikzcd} A \arrow[rr, "n_A"] \arrow[dr, "\alpha"'] & & A \\ & A/\mathrm{Ker}(\alpha) \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

- $\alpha \in \mathrm{Hom}(A, B)$ isomorphism in $\mathrm{Hom}^0(A, B) \Rightarrow$ we can find such a factorization.
- In fact $\mathbf{AV}(k) \to \mathbf{AV}^0(k)$ localizes $\mathbf{AV}(k)$ at all isogenies.

The category $\mathbf{AV}^0(k)$

- It is **abelian** [encyclopediaofmath.org/wiki/Isogeny].
- Every object is **semisimple**: every A is isogenous to a finite direct sum of indecomposable A_i's [Mil08, Prop. I.10.1].
  - Thus every short exact sequence in $\mathbf{AV}^0(k)$ splits [Jeremy Rickard's comment on mathoverflow.net/a/327944/99436].
  - Thus all additive functors are already exact and its derived category, which is equivalent to the category of cochain complexes with trivial differentials, is abelian [GM03, III.2.4].
- All homs are finite dimensional over $\mathbb{Q}$ [Mil08, Thm. I.10.15].

Semisimple objects $\Rightarrow$ to understand $\mathbf{AV}^0(k)$, it suffices to study endomorphism algebras of simple objects [Mil08, p. 43].

References

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