Endomorphisms of abelian varieties

Remarks on Section I.10 of Milne's *Abelian Varieties*

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Recall isogenies

Definition

$\alpha \in \mathrm{Hom}(A, B)$ **isogeny** $\Leftrightarrow$ surjective with finite kernel;

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$\Leftrightarrow$ finite and surjective (and flat).

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Examples

- $\mathcal{L}$ ample $\Rightarrow \lambda_{\mathcal{L}} \colon A \to A^{\vee}$ isogeny [Mil08, Prop. 8.1].
- $a \mapsto na$ isogeny, étale in char. zero [Mil08, Thm. 7.2].

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Last example $\Rightarrow A_n := \ker(a \mapsto na)$ is finite, and being isogenous is an equivalence relation [Mil08, Rem. 8.6].

References

James S. Milne.
Abelian Varieties (v2.00), 2008.
Available at jmilne.org/math.