Endomorphisms of abelian varieties

Remarks on Section I.10 of Milne's Abelian Varieties

University of Freiburg

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Recall isogenies

Definition

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\alpha \in \mathsf{Hom}(A,B) isogeny \Leftrightarrow surjective with finite kernel;
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- \Leftrightarrow surjective and dim $A = \dim B$;
- \Leftrightarrow finite kernel and dim $A = \dim B$;
- ⇔ finite and surjective (and flat).

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Examples

- ▶ \mathcal{L} ample $\Rightarrow \lambda_{\mathcal{L}} \colon A \to A^{\vee}$ isogeny [Mil08, Prop. 8.1].
- ▶ $n > 0 \Rightarrow n_A$: $a \mapsto na$ isogeny [Mil08, Thm. 7.2].

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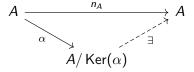
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Last example \Rightarrow *n*-torsion subgroup $A_n := \ker(a \mapsto na)$ is **finite**.

Abelian varieties up to isogeny

- **AV**(k): additive cat. of abelian varieties and regular homomorphisms over a field k.
- ▶ $AV^0(k)$: \mathbb{Q} -linear cat. with $Hom^0(A, B) := Hom(A, B) \otimes_{\mathbb{Z}} \mathbb{Q}$.
- ▶ $\alpha \in \text{Hom}(A, B)$ isogeny $\Rightarrow B \cong A / \text{Ker}(\alpha)$ and $\text{Ker}(\alpha) \subseteq A_n$ for some $n \in \mathbb{N}$. Hence α becomes an isomorphism in $AV^0(k)$:



- ▶ $\alpha \in \text{Hom}(A, B)$ isomorphism in $\text{Hom}^0(A, B) \Rightarrow \text{we can find}$ such a factorization.
- ▶ In fact $AV(k) \rightarrow AV^0(k)$ localizes AV(k) at all isogenies.

The category $AV^0(k)$

- It is abelian [encyclopediaofmath.org/wiki/Isogeny].
- ► Every object is **semisimple**: every *A* is isogenous to a finite direct sum of indecomposable *A_i*'s [Mil08, Prop. I.10.1].
 - Thus every short exact sequence in AV⁰(k) splits [Jeremy Rickard's comment on mathoverflow.net/a/327944/99436].
 - Thus all additive functors are already exact and its derived category, which is equivalent to the category of cochain complexes with trivial differentials, is abelian [GM03, III.2.4].
- ▶ All homs are finite dimensional over ℚ [Mil08, Thm. I.10.15].

Semisimple objects \Rightarrow to understand $AV^0(k)$, it suffices to study endomorphism algebras of simple objects [Mil08, p. 43].

References



Sergei I. Gelfand and Yuri I. Manin. Methods of homological algebra. Springer Monographs in Mathematics. Springer-Verlag, Berlin, second edition, 2003.



James S. Milne.

Abelian Varieties (v2.00), 2008.

Available at jmilne.org/math.