Endomorphisms of abelian varieties

Remarks on Section I.10 of Milne's Abelian Varieties

University of Freiburg

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Recall isogenies

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\alpha \in \text{Hom}(A, B) isogeny \Leftrightarrow surjective with finite kernel;
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- \Leftrightarrow surjective and dim $A = \dim B$;
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Examples

- ▶ \mathcal{L} ample $\Rightarrow \lambda_{\mathcal{L}} \colon A \to A^{\vee}$ isogeny [Mil08, Prop. 8.1].
- ▶ $n > 0 \Rightarrow n_A$: $a \mapsto na$ isogeny [Mil08, Thm. 7.2].

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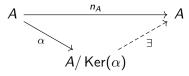
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Last example \Rightarrow *n*-torsion subgroup $A_n := \ker(a \mapsto na)$ is **finite**.

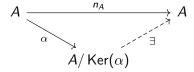
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- ▶ $\alpha \in \text{Hom}(A, B)$ isogeny $\Rightarrow B \cong A / \text{Ker}(\alpha)$ and $\text{Ker}(\alpha) \subseteq A_n$ for some $n \in \mathbb{N}$. Hence α becomes an isomorphism in $AV^0(k)$:

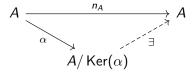


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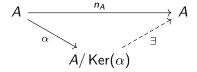
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- ▶ In fact $AV(k) \rightarrow AV^0(k)$ localizes AV(k) at all isogenies.

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Semisimple objects \Rightarrow to understand $AV^0(k)$, it suffices to study endomorphism algebras of simple objects [Mil08, p. 43].

▶ $\operatorname{\mathsf{Hom}}(A_1,A_2) \hookrightarrow \operatorname{\mathsf{Hom}}(\Lambda_1,\Lambda_2) = \operatorname{\mathsf{Hom}}(H_1(A_1,\mathbb{Z}),H_1(A_2,\mathbb{Z})),$ as we saw in [Mil08, §1.2]. Fixing bases for homology we get:

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- ▶ $\alpha \in \operatorname{End}(A_1)$ defines a linear $H_1(\alpha) \in \operatorname{End}(H_1(A_1, \mathbb{Q}))$ with characteristic polynomial

$$P_{\alpha}(t) = \det(H_1(\alpha) - t) = t^{2g} + a_1 t^{2g-1} + \ldots + a_{2g} \in \mathbb{Z}[t].$$



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▶ We can then define $Tr(\alpha) = -a_1$ and $Nm(\alpha) = a_{2g} = deg(\alpha)$ and use them to prove that, for A simple of dimension g:

$$[Z:\mathbb{Q}]\sqrt{[\operatorname{End}^0(A):Z]}\mid 2g$$
, where $Z:=Z(\operatorname{End}^0(A))$.

▶ Replace singular homology $H_1(A, \mathbb{Q})$ by Tate module $T_\ell A$, which in this case is the étale homology $\varprojlim A_{\ell^n} \cong H^1(A, \mathbb{Z}_\ell)^\vee$.

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- ▶ [Mil08, Lemma I.10.6]: $Hom(A_1, A_2)$ torsion-free, because

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- ▶ [Mil08, Thm. I.10.15]: Hom (A_1, A_2) f.g. over \mathbb{Z} and

$$\mathsf{Hom}(A_1,A_2)\otimes \mathbb{Z}_\ell \hookrightarrow \mathsf{Hom}_{\mathbb{Z}_\ell}(\mathcal{T}_\ell A_1,\mathcal{T}_\ell A_2).$$



Let \mathcal{L} be an ample line bundle on A and $\lambda_{\mathcal{L}} \colon A \to A^{\vee}$ the corresponding isogeny. Define the *Rosati involution* on $\operatorname{End}^0(A)$ as

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- ▶ Albert classification of $End^0(A)$ for A simple [Thm. 21.2].
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- Structure of $NS^0(A)$ [Thm. 21.6].

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Mumford studies two cases of the Albert classification in more detail [Mum70, §22], namely:

- CM-abelian varieties over C.
- ▶ Elliptic curves in positive characteristic.

References



Sergei I. Gelfand and Yuri I. Manin.

Methods of homological algebra.

Springer Monographs in Mathematics. Springer-Verlag, Berlin, second edition, 2003.



James S. Milne.

Abelian Varieties (v2.00), 2008.

Available at imilne.org/math.



David Mumford.

Abelian varieties.

Tata Institute of Fundamental Research Studies in Mathematics, No. 5, Published for the Tata Institute of Fundamental Research, Bombay; Oxford University Press, London, 1970.