ENUMERATIVE GEOMETRY WEDNESDAY SEMINAR

GK1821 "COHOMOLOGICAL METHODS IN GEOMETRY"

List of Talks (assuming 1h 30min / Talk)

Some suggestions that apply to all the talks:

- Working only over $\mathbb{C}$ sounds like a good idea, even if it is often unnecessary.
- The book *3264 and all that* written by Eisenbud and Harris [EH16] should provide a self-contained reference for the seminar. They often use schemes, but I think the book has a very geometric flavour and most arguments can be followed without knowing precisely what schemes are. There is also an introduction to schemes again with a very geometric flavour by the same authors [EH00].
- In fact I would try to avoid scheme-theoretic details altogether and instead try to draw many pictures and focus on examples with complex manifolds.
- There are connections and similarities with other areas, especially with singular homology. I think it would be nice to hear about them every now and then.

But these are only suggestions, feel free to do otherwise if you want/need at some point!

1. Talk 1 — The Chow ring. Affine spaces.

1.1. **Algebraic varieties.** These have already appeared often in past Wednesday seminars, so hopefully we can keep this to a very brief introduction or recollection. Possible things that may be useful to say:

- Huge open subsets (no strict Hausdorffness, no local $\mathbb{A}^n$-ness).
- The union of two intersecting lines is connected but not irreducible.
- Intersection of irreducible stuff is not necessarily irreducible.
- Dimension of varieties is defined by chains of irreducibles.

1.2. **Chow groups** [EH16, §1.2.1 and 1.2.2]**.** Cycles and rational equivalence. [EH16, Prop. 1.4] and [EH16, Prop. 1.10] are good to know. A nice picture to see what can happen is [EH16, Fig. 1.2].

1.3. **Ring structure** [EH16, Thm. 1.5]**.** Generic transversality and moving lemma [EH16, Thm. 1.6]. What goes wrong without the smoothness assumption? Example in [EH16, p. 20].

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1.4. **Chow groups of affine spaces.** For any variety X, the equivalence class $[X]$ is a free generator of $A^0(X)$. This can be argued using irreducibility and dimension. In the case of affine spaces, this free cyclic group is all there is [EH16, Prop. 1.13]. A nice picture to visualise the proof is [EH16, Fig. 1.7].

1.5. **Functoriality** [EH16, §1.3.6]**.** Proper pushforward and flat pullback without technical details. I wouldn't define properness and flatness too seriously, but it is good to know that inclusions of open subsets are flat morphisms and that any morphism between projective varieties is proper. Degree map [EH16, Prop. 1.21]. [EH16, Thm. 1.23] without details of the proof.

2. Talk 2 — Affine stratifications. Projective spaces.

2.1. **Brief recollection of previous talk.** Cycles, rational equivalence, functoriality.

2.2. **Mayer–Vietoris and excision** [EH16, §1.3.4]**.**

2.3. **Affine stratifications** [EH16, §1.3.5]**.** With examples of what is or isn't a stratification, examples of quasi-affine stratifications that are not affine, etc. Totaro's theorem [EH16, Thm. 1.18] is nice to know, although it won't be used later on.

2.4. **Chow ring of projective space.** [EH16, Thm. 2.1] and corollaries [EH16, Cor. 2.2 and Cor. 2.3]. Bézout's theorem as a consequence of [EH16, Thm. 2.1].

3. Talk 3 — Grassmannian of lines in space

3.1. **Kleiman's transversality** [EH16, Thm. 1.7]**.** Proof in the case of $\mathrm{GL}_n(\mathbb{C})$.

3.2. **Grassmannians.** Definition and projective spaces as a particular case. Plücker embedding and affine open cover already in the case of the Grassmannian of lines in $\mathbb{P}^3$. Schubert cycles and stratification of $\mathbb{G}(1,3)$.

3.3. **Computation of the Chow ring** [EH16, Thm. 3.10]**.** Mentioning explicitly how to use Kleiman's transversality and the method of undetermined coefficients during the proof.

As an immediate consequence of the previous point and transversality: how many lines in $\mathbb{P}^3$ intersect 4 general lines?

4. Talk 4 — Specialisation and Knutson–Tao puzzles

4.1. **Brief recollection of previous talk.** Using the method of undetermined coefficients to compute the square σ_1^2 of the Schubert class of lines intersecting a given line.

4.2. **(Static) specialisation** [EH16, §3.5.1]**.** This is another useful technique to compute products of Schubert classes. As an example we can use it to compute σ_1^2 in a different way. This will require describing the tangent space of the Schubert cycle of lines intersecting a given line L at a point different from L [EH16, Exe. 3.26].

4.3. **Knutson–Tao puzzles.** These give a nice visual tool to compute products of Schubert classes. Besides various articles by Knutson, Tao and others, see e.g. [KTW04], there is also a nice YouTube video discussing them https://youtu.be/U8sq3BplCfI. Again, one can use the computation of σ_1^2 as an example, as is done in the video already.

The speaker may want to discuss other combinatoric methods to compute products of Schubert classes as well/instead, e.g. Young diagrams [EH16, §4.5]. Whatever they prefer.

5. Talk 5 — Chern classes and lines on a cubic surface

5.1. **First Chern class of a line bundle** [EH16, §1.4]**.** Examples could include $c_1(K_{\mathbb{P}^n})$. Geometric motivation to generalise and define higher Chern classes [EH16, §5.2].

5.2. **Axiomatic definition of Chern classes.** Maybe mentioning some ideas in the existence proof but without getting into details [EH16, §5.3].

5.3. **The splitting principle** [EH16, §5.4]**.** And how to use it to derive various formulas [EH16, §5.5].

5.4. **Tautological bundles** [EH16, §5.6]**.** And computation of the Chern classes of the dual of the tautological subbundle of rank 2 on our Grassmannian $\mathbb{G}(1,3)$ as an example [EH16, §5.6.2].

5.5. **Counting lines on a cubic surface.** Following the argument in [EH16, Thm. 5.1] and using [EH16, §5.6.2] and [EH16, §6.2.1]. Emphasis on what the argument in [EH16, Thm. 5.1] does and does not show, i.e. there are some statements that remain to be shown in order to actually prove that there are exactly 27 lines. May be useful to compare this with the discussion in [EH16, §3.1]. If time permits and the speaker wants, maybe also briefly sketch the rest of the argument involving Fano varieties [EH16, §6.1–6.2].

References

[EH00] David Eisenbud and Joe Harris. *The geometry of schemes*, volume 197 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000.

[EH16] David Eisenbud and Joe Harris. *3264 and all that—a second course in algebraic geometry*. Cambridge University Press, Cambridge, 2016.

[KTW04] Allen Knutson, Terence Tao, and Christopher Woodward. The honeycomb model of $\mathrm{GL}_n(\mathbb{C})$ tensor products. II. Puzzles determine facets of the Littlewood-Richardson cone. *J. Amer. Math. Soc.*, 17(1):19–48, 2004.

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