ENUMERATIVE GEOMETRY WEDNESDAY SEMINAR

GK1821 "COHOMOLOGICAL METHODS IN GEOMETRY"

List of Talks (assuming 1h 30min / Talk)

Some suggestions that apply to all the talks:

- Working only over $\mathbb{C}$ sounds like a good idea, even if it is often unnecessary.
- The book *3264 and all that* written by Eisenbud and Harris [EH16] should provide a self-contained reference for the seminar. They often use schemes, but I think the book has a very geometric flavour and most arguments can be followed without knowing precisely what schemes are. There is also an introduction to schemes again with a very geometric flavour by the same authors [EH00].
- In fact I would try to avoid scheme-theoretic details altogether and instead try to draw many pictures and focus on examples with complex manifolds.
- There are connections and similarities with other areas, especially with singular homology in the first talks and with differential geometry in the last talks. I think it would be nice to hear about them every now and then.
- Every talk —except the first one— is meant to start with a brief recollection of previous talks. The idea is to refresh some important results and definitions and then apply them in some specific situation.

But these are only suggestions, feel free to do otherwise if you want/need at some point!

1. Talk 1 — Introduction and the Chow ring

1.1. **Introduction and motivation.** What are enumerative problems and how can we go about solving them? See [EH16, §3.1.1]. What are —roughly speaking— intersection theory and the Chow ring, and what is their role in the process of solving an enumerative problem? See [EH16, §1.1] and [EH16, §3.1.1]. The problem that we will take as an example to introduce most of the concepts during the first four talks is the following

Question 1.1. How many lines in $\mathbb{P}^3$ intersect four given lines in general position?

Date: Winter Semester 2020/2021.

Caveat 1.2. One should not jump to precise conclusions immediately after some numerical computation, see [EH16, §3.1.2]. The meaning of this caveat will become clearer with an example that we will discuss during the last two talks, namely

Question 1.3. How many lines does a smooth cubic surface $S \subseteq \mathbb{P}^3$ contain?

Using Chern classes we will see that the relevant class in the relevant Chow ring has degree 27, but this is not enough to conclude that there are precisely 27 lines.

1.2. **Algebraic varieties.** These have already appeared often in past Wednesday seminars, so hopefully we can keep this to a very brief introduction or recollection. Something that we will use in later talks is the degree of a projective variety. I think having the intuitive definition with hyperplane cuts in mind is enough, but perhaps mentioning at least that there is a definition using the Hilbert polynomial is a good idea. Other things that may be useful to mention:

- Huge open subsets (no strict Hausdorffness, no strict local $\mathbb{A}^n$-ness).
- The union of two intersecting lines is connected but not irreducible.
- Intersection of irreducible stuff is not necessarily irreducible.
- Dimension of varieties is defined by chains of irreducibles.

1.3. **Chow groups** [EH16, §1.2.1 and 1.2.2]**.** Cycles and rational equivalence. [EH16, Prop. 1.4] and [EH16, Prop. 1.10] are good to know. A nice picture to see what can happen is [EH16, Fig. 1.2].

1.4. **Ring structure** [EH16, Thm. 1.5]**.** Generic transversality and moving lemma [EH16, Thm. 1.6]. What goes wrong without the smoothness assumption? Example in [EH16, p. 20].

2. Talk 2 — Basic computational tools

2.1. **Brief recollection of previous talk.** Cycles, rational equivalence and intersection product.

2.2. **Chow groups of affine spaces using what we saw in the previous talk.** For any variety X, the equivalence class $[X]$ is a free generator of $A^0(X)$. This can be argued using irreducibility and dimension. In the case of affine spaces, this free cyclic group is all there is [EH16, Prop. 1.13]. A nice picture to visualise the proof is [EH16, Fig. 1.7].

2.3. **Functoriality** [EH16, §1.3.6]**.** Proper pushforward and flat pullback without technical details. I wouldn't define properness and flatness too seriously, but it is good to know that inclusions of open subsets are flat morphisms and that any morphism between projective varieties is proper. Degree map [EH16, Prop. 1.21]. [EH16, Thm. 1.23] without details of the proof.

2.4. **Mayer–Vietoris and excision** [EH16, §1.3.4]**.**

2.5. **Affine stratifications** [EH16, §1.3.5]**.** With examples of what is or isn't a stratification, examples of quasi-affine stratifications that are not affine, etc. Totaro's theorem [EH16, Thm. 1.18] is nice to know, although it won't be used later on.

3. Talk 3 — Grassmannians and Schubert cycles

3.1. **Brief recollection of the previous talk.** Proper pushforward, degree map and stratifications.

3.2. **Chow ring of projective space using what we saw in previous talks.** [EH16, Thm. 2.1] and corollaries [EH16, Cor. 2.2 and Cor. 2.3]. Bézout's theorem as a consequence of [EH16, Thm. 2.1].

3.3. **Grassmannians.** Definition and projective spaces as a particular case. Plücker embedding [EH16, §3.2.1] and affine open cover [EH16, §3.2.2]. I would suggest to focus just on the Grassmannian of lines in $\mathbb{P}^3$ to discuss these notions, since it is the one we will be using later on and I think it would make the notation more explicit and clear.

3.4. **Schubert cycles.** Again, I would stick to the case of $\mathbb{G}(1,3)$, discussed in detail in [EH16, §3.3.1]. In this case we can also draw nice pictures [EH16, Fig. 3.3]. State Kleiman's transversality [EH16, Thm. 1.7] and explain what it means for Schubert cycles with respect to different and generically positioned flags [EH16, pp. 105–106].

4. Talk 4 — How many lines intersect 4 random lines in space?

4.1. **Brief recollection of previous talks.** Stratifications, $\mathbb{G}(1,3)$ and its Schubert cycles.

4.2. **Computation of the Chow ring using what we saw in previous talks.** This computation is carried out in [EH16, Thm. 3.10]. I would stress how to use Kleiman's transversality [EH16, Thm. 1.7] and the method of undetermined coefficients during the proof.

After this computation we can already answer Question 1.1. But this also seems like a good time to recall Caveat 1.2. Why is the precise answer to the question justified in this case?

4.3. **(Static) specialisation** [EH16, §3.5.1]**.** This is another useful technique to compute products of Schubert classes. As an example we can use it to compute σ_1^2 in a different way. This will require describing the tangent space of the Schubert cycle of lines intersecting a given line L at a point different from L [EH16, Exe. 3.26].

4.4. **Geometry behind the answer to Question 1.1.** Paraphrasing İzzet Coşkun: What shape do you produce when you want to cook spaguetti and you put them in a pot? What does the answer to this question have to do with Question 1.1? See [EH16, §3.4.1].

5. Talk 5 — Knutson–Tao puzzles and Chern classes

The two topics in the title of this talk are not directly related to each other and they are only in the same talk because I couldn't figure out a better way to organise the contents. So this talk will probably have two indepenedent parts:

(1) Recollections from previous talk and some more Schubert calculus.
(2) Introduction to Chern classes.

5.1. **Brief recollection of the previous talk.** Using the methods of undetermined coefficients and static specialisation to compute σ_1^2.

5.2. **Knutson–Tao puzzles.** These give a nice visual tool to compute products of Schubert classes. Besides various articles by Knutson, Tao and others, see e.g. [KTW04], there is also a nice YouTube video discussing them https://youtu.be/U8sq3BplCfI. Again, one can use the computation of σ_1^2 as an example, as is done in the video already.

The speaker may want to discuss other combinatoric methods to compute products of Schubert classes as well/instead, e.g. Young diagrams [EH16, §4.5]. Whatever they prefer.

5.3. **First Chern class of a line bundle** [EH16, §1.4]**.** A nice explicit example of computation could be $c_1(K_{\mathbb{P}^n})$. Geometric motivation to generalise and define higher Chern classes [EH16, §5.2].

5.4. **Axiomatic definition of Chern classes.** Maybe mentioning some ideas in the existence proof but without getting into details [EH16, §5.3]. It would also be nice to mention here that one can use Chern–Weil theory for a differential-geometric approach to the existence of Chern classes.

5.5. **The splitting principle** [EH16, §5.4]**.**

6. Talk 6 — How many lines does a smooth cubic surface contain?

6.1. **Brief recollection from the previous talk.** Axiomatic definition of Chern classes and splitting principle.

6.2. **Derivation of some formulas using what we saw in the previous talk** [EH16, §5.5]**.**

6.3. **Tautological bundles** [EH16, §5.6]**.** Describing them just for $\mathbb{G}(1,3)$ and projective spaces should be enough. The example relevant for Question 1.3 is the dual of the tautological subbundle of rank 2 on our Grassmannian $\mathbb{G}(1,3)$. What are its Chern classes? See [EH16, §5.6.2]. Another nice example could be to compute explicitly —with coordinates— that the tautological line bundle on $\mathbb{P}^n$ is $\mathscr{O}(-1)$.

6.4. **Counting lines on a cubic surface.** Following the argument in [EH16, Thm. 5.1] and using [EH16, §5.6.2] and [EH16, §6.2.1]. Recall Caveat 1.2 and the discussion in [EH16, §3.1.2]. What does the argument in [EH16, Thm. 5.1] show and what remains to be shown in order to give a precise answer to Question 1.3? If time permits and the speaker wants, it might be nice to also briefly sketch the rest of the argument involving the Fano variety of lines [EH16, §6.1–6.2].

References

[EH00] David Eisenbud and Joe Harris. *The geometry of schemes*, volume 197 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000.

[EH16] David Eisenbud and Joe Harris. *3264 and all that—a second course in algebraic geometry*. Cambridge University Press, Cambridge, 2016.

[KTW04] Allen Knutson, Terence Tao, and Christopher Woodward. The honeycomb model of $\mathrm{GL}_n(\mathbb{C})$ tensor products. II. Puzzles determine facets of the Littlewood-Richardson cone. *J. Amer. Math. Soc.*, 17(1):19–48, 2004.

Pedro Núñez
Albert-Ludwigs-Universität Freiburg, Mathematisches Institut
Ernst-Zermelo-Strasse 1, 79104 Freiburg im Breisgau (Germany)
Email address: pedro.nunez@math.uni-freiburg.de