ENUMERATIVE GEOMETRY

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Contents

Some suggestions that apply to all the talks:

- Follow *3264 and all that*, by Eisenbud and Harris [EH16].
- Work only over $\mathbb{C}$, even if it is not necessary at many points.
- Try to avoid scheme-theoretic technical details.
- Try to draw many pictures and focus on examples.

But feel free to do otherwise if you want/need at some point!

1. Talk 1 — The Chow ring. Affine spaces.

1.1. **Algebraic varieties.** These have appeared a number of times in past iterations of the Wednesday seminar already, so hopefully we can keep this as a one line introduction/recollection. Roughly speaking, they are spaces glued from zero loci of polynomials which globally satisfy a certain Hausdorffness property. They are called *projective* if they can be embedded as a closed subset in some projective space, and *irreducible* if as a topological space they cannot be expressed as a union of two proper closed subsets. Every variety has a unique decomposition into irreducible components, so the word variety is reserved for irreducible varieties in [EH16].

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1.2. The Chow group.

Definition 1.1 ([EH16, §1.2.1])**.** Let X be a variety. The *group of cycles* on X, denoted $Z(X)$, is the free abelian group generated by the set of subvarieties of X.

Definition 1.2 ([EH16, §1.2.2])**.** Denote by $\mathrm{Rat}(X) \subseteq Z(X)$ the subgroup generated by differences of the form

$$\Phi \cap (\{t_0\} \times X) - \Phi \cap (\{t_1\} \times X),$$

where $t_0, t_1 \in \mathbb{P}^1$ are two points and $\Phi \subseteq \mathbb{P}^1 \times X$ is a subvariety not contained in any fibre $\{t\} \times X$.

We say that two cycles $\xi_1, \xi_2 \in Z(X)$ are *rationally equivalent* if their difference is in $\mathrm{Rat}(X)$. The quotient group $Z(X)/\mathrm{Rat}(X)$ is called the *Chow group* of X, denoted by $A(X)$.

Remark 1.3.

(1) It follows from Krull's principal ideal theorem that $A(X)$ is still graded by dimension [EH16, Proposition 1.4], hence also by codimension. We denote by $A^c(X)$ the codimension c part.

(2) One can equivalently describe $\mathrm{Rat}(X)$ as the subgroup generated by all divisors of rational functions on all subvarieties of X [EH16, Proposition 1.10]. In particular, linear and rational equivalence agree for divisors and

$$A^1(X) = \mathrm{Cl}(X).$$

1.3. Ring structure on the Chow group.

Remark 1.4 ([EH16, §1.2.1]). To any closed subscheme —which we can think of as a bunch of subvarieties each with some multiplicity— we can associate an obvious cycle, and the other way around. We can then use fibre products to define the scheme-theoretic intersection of two such closed subschemes, making sense of the intersection of two cycles.

Definition 1.5 ([EH16, §1.2.3])**.** Let X be a variety and let $A, B \subseteq X$ be subvarieties. We say that A and B intersect *transversely* at a point $p \in X$ if A, B and X are all smooth at p and

$$T_pA + T_pB = T_pX.$$

We say that A and B intersect *generically transversely* if they meet transversely at a general point of each irreducible component of the intersection $A \cap B$. Note that the intersection of two subvarieties need not be again irreducible, e.g. a circle and a line in the plane intersecting in two points.

We can naturally extend this definition to cycles.

Lemma 1.6 ([EH16, Theorem 1.6]). *Let X be a smooth quasi-projective variety.*

a) Every pair of equivalence classes $\alpha, \beta \in A(X)$ admits a pair of generically transverse representing cycles $A, B \in Z(X)$.
b) The class $[A \cap B]$ is then independent of the choice of such representing cycles A and B.

Remark 1.7 ([EH16, p. 20]). The smoothness assumption is necessary.

Theorem 1.8 ([EH16, Theorem–Definition 1.5]). *Let X be a smooth quasi-projective variety. Then there is a unique product structure on $A(X)$ such that*

$$[A][B] = [A \cap B]$$

for every pair of generically transverse subvarieties $A, B \subseteq X$. Moreover, this product turns $A(X)$ into a commutative ring graded by codimension, called the Chow ring of X.

1.4. **Fundamental classes and Chow groups of affine spaces.** The equivalence class $[X]$ is called the *fundamental class* of X. We have $A^0(X) \cong \mathbb{Z} \cdot [X]$ [EH16, Prop. 1.8], and in particular $A(X) \neq 0$. With emphasis on how to use irreducibility and dimension to show that $[X]$ is a free generator of $A^0(X)$. For affine space $\mathbb{A}^n$ this is all there is: $A(\mathbb{A}^n) \cong \mathbb{Z} \cdot [\mathbb{A}^n]$ [EH16, Prop. 1.13].

1.5. **Functoriality:** proper pushforward and flat pullback without getting into details [EH16, §1.3.6]. Degree map [EH16, Prop. 1.21]. [EH16, Thm. 1.23] without getting into the proof.

1.6. **Relation to other famous invariants (only if time permits).** Relation to K-theory, relation to singular cohomology (Hodge conjecture), motives... The Wikipedia page on Chow groups discusses all these relations in a summarised way.

2. Talk 2 — Basic computational tools. Projective spaces.

2.1. **Brief recollection of previous talk:** cycles and rational equivalence, functoriality.

2.2. **Mayer–Vietoris and excision.** in some detail [EH16, §1.3.4].

2.3. **Affine stratifications.** in some detail [EH16, §1.3.5].

2.4. **Chow ring of projective space.** [EH16, Thm. 2.1] and cool corollaries [EH16, Cor. 2.2 and Cor. 2.3]. State also Bézout's theorem, which is an immediate consequence of [EH16, Thm. 2.1].

3. Talk 3 — Grassmannian of lines in space

3.1. **Kleiman's transversality:** [EH16, Thm. 1.7] and its proof in the case of $\mathrm{GL}_n(\mathbb{C})$.

3.2. **Definition of Grassmannians.** Projective spaces as a particular case.

3.3. **Plücker embedding and affine cover:** already in the case of the Grassmannian of lines in $\mathbb{P}^3$.

3.4. **Schubert cycles:** complete flags in $\mathbb{P}^3$ and stratification of $\mathbb{G}(1, 3)$.

3.5. **Computation of the Chow ring:** [EH16, Thm. 3.10]. With emphasis on how to use Kleiman's transversality during the proof.

As an immediate consequence of the previous point and transversality: how many lines in $\mathbb{P}^3$ intersect 4 general lines?

4. Talk 4 — Enumerative formulas

4.1. **What are they and where is the catch?** [EH16, §3.1.1 and 3.1.2].

References

[EH16] David Eisenbud and Joe Harris. *3264 and all that—a second course in algebraic geometry*. Cambridge University Press, Cambridge, 2016. ↑ (document), 1.1, 1.1, 1.2, 1, 2, 1.4, 1.5, 1.6, 1.7, 1.8, 1.4, 1.5, 2.2, 2.3, 2.4, 3.1, 3.5, 4.1

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