

# ENUMERATIVE GEOMETRY

PEDRO NÚÑEZ

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Some suggestions that apply to all the talks:

- Follow *3264 and all that*, by Eisenbud and Harris [EH16].
- Work only over  $\mathbb{C}$ , even if it is not necessary at many points.
- Try to avoid scheme-theoretic technical details.
- Try to draw many pictures and focus on examples.

But feel free to do otherwise if you want/need at some point!

## 1. TALK 1 – THE CHOW RING

**1.1. Algebraic varieties.** These have appeared a number of times in past iterations of the Wednesday seminar already, so hopefully we can keep this as a one line introduction/recollection. Roughly speaking, they are spaces glued from zero loci of polynomials which globally satisfy a certain Hausdorffness property. They are called *projective* if they can be embedded as a closed subset in some projective space, and *irreducible* if as a topological space they cannot be expressed as a union of two proper closed subsets. Every variety has a unique decomposition into irreducible components, so the word variety is reserved for irreducible varieties in [EH16].

### 1.2. The Chow group.

**Definition 1.1** ([EH16, §1.2.1]). Let  $X$  be a variety. The *group of cycles* on  $X$ , denoted  $Z(X)$ , is the free abelian group generated by the set of subvarieties of  $X$ .

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**Definition 1.2** ([EH16, §1.2.2]). Denote by  $\text{Rat}(X) \subseteq Z(X)$  the subgroup generated by differences of the form

$$\Phi \cap (\{t_0\} \times X) - \Phi \cap (\{t_1\} \times X),$$

where  $t_0, t_1 \in \mathbb{P}^1$  are two points and  $\Phi \subseteq \mathbb{P}^1 \times X$  is a subvariety not contained in any fibre  $\{t\} \times X$ .

We say that two cycles  $\xi_1, \xi_2 \in Z(X)$  are *rationally equivalent* if their difference is in  $\text{Rat}(X)$ . The quotient group  $Z(X)/\text{Rat}(X)$  is called the *Chow group* of  $X$ , denoted by  $A(X)$ .

*Remark 1.3.*

- (1) It follows from Krull's principal ideal theorem that  $A(X)$  is still graded by dimension [EH16, Proposition 1.4], hence also by codimension. We denote by  $A^c(X)$  the codimension  $c$  part.
- (2) One can equivalently describe  $\text{Rat}(X)$  as the subgroup generated by all divisors of rational functions on all subvarieties of  $X$  [EH16, Proposition 1.10]. In particular, linear and rational equivalence agree for divisors and

$$A^1(X) = \text{Cl}(X).$$

### 1.3. Ring structure on the Chow group.

*Remark 1.4* ([EH16, §1.2.1]). To any closed subscheme —which we can think of as a bunch of subvarieties each with some multiplicity— we can associate an obvious cycle, and the other way around. We can then use fibre products to define the scheme-theoretic intersection of two such closed subschemes, making sense of the intersection of two cycles.

**Definition 1.5** ([EH16, §1.2.3]). Let  $X$  be a variety and let  $A, B \subseteq X$  be subvarieties. We say that  $A$  and  $B$  intersect *transversely* at a point  $p \in X$  if  $A, B$  and  $X$  are all smooth at  $p$  and

$$T_p A + T_p B = T_p X.$$

We say that  $A$  and  $B$  intersect *generically transversely* if they meet transversely at a general point of each irreducible component of the intersection  $A \cap B$ . Note that the intersection of two subvarieties need not be again irreducible, e.g. a circle and a line in the plane intersecting in two points.

We can naturally extend this definition to cycles.

**Lemma 1.6** ([EH16, Theorem 1.6]). *Let  $X$  be a smooth quasi-projective variety.*

- a) *Every pair of equivalence classes  $\alpha, \beta \in A(X)$  admits a pair of generically transverse representing cycles  $A, B \in Z(X)$ .*
- b) *The class  $[A \cap B]$  is then independent of the choice of such representing cycles  $A$  and  $B$ .*

*Remark 1.7* ([EH16, p. 20]). The smoothness assumption is necessary.



**Theorem 1.8** ([EH16, Theorem–Definition 1.5]). *Let  $X$  be a smooth quasi-projective variety. Then there is a unique product structure on  $A(X)$  such that*

$$[A][B] = [A \cap B]$$

*for every pair of generically transverse subvarieties  $A, B \subseteq X$ . Moreover, this product turns  $A(X)$  into a commutative ring graded by codimension, called the Chow ring of  $X$ .*

**1.4. Fundamental classes and Chow groups of affine spaces.** The equivalence class  $[X]$  is called the *fundamental class* of  $X$ . We have  $A^0(X) \cong \mathbb{Z} \cdot [X]$  [EH16, Prop. 1.8], and in particular  $A(X) \neq 0$ . For affine space  $\mathbb{A}^n$  this is all there is:  $A(\mathbb{A}^n) \cong \mathbb{Z} \cdot [\mathbb{A}^n]$  [EH16, Prop. 1.13].

**1.5. Functoriality:** proper pushforward and flat pullback without getting into details [EH16, §1.3.6]. Degree map [EH16, Prop. 1.21]. [EH16, Thm. 1.23] without getting into the proof.

**1.6. Relation to other famous invariants (only if time permits).** Relation to  $K$ -theory, relation to singular cohomology (Hodge conjecture), motives... The Wikipedia page on Chow groups discusses all these relations in a summarised way.

## 2. TALK 2 — BASIC TOOLS TO COMPUTE CHOW RINGS

**2.1. Brief recollection of previous talk:** cycles and rational equivalence, functoriality.

**2.2. Mayer–Vietoris and excision.** in some detail [EH16, §1.3.4].

**2.3. Affine stratifications.** in some detail [EH16, §1.3.5].

**2.4. Chow ring of projective space.** [EH16, Thm. 2.1] and cool corollaries [EH16, Cor. 2.2 and Cor. 2.3]. State also Bézout’s theorem, which is an immediate consequence of [EH16, Thm. 2.1].

## REFERENCES

- [EH16] David Eisenbud and Joe Harris. *3264 and all that—a second course in algebraic geometry*. Cambridge University Press, Cambridge, 2016. ↑ (document), 1.1, 1.1, 1.2, 1, 2, 1.4, 1.5, 1.6, 1.7, 1.8, 1.4, 1.5, 2.2, 2.3, 2.4

PEDRO NÚÑEZ

ALBERT-LUDWIGS-UNIVERSITÄT FREIBURG, MATHEMATISCHES INSTITUT  
ERNST-ZERMELO-STRASSE 1, 79104 FREIBURG IM BREISGAU (GERMANY)

Email address: [pedro.nunez@math.uni-freiburg.de](mailto:pedro.nunez@math.uni-freiburg.de)

Homepage: <https://home.mathematik.uni-freiburg.de/nunez>