HIGGS BUNDLES — EXISTENCE OF SOLUTIONS

PEDRO NÚÑEZ

ABSTRACT. In this talk we introduce the stability condition for Higgs bundles and prove the Hitchin–Kobayashi correspondence. The main result is [Hit87, Theorem 4.3]. Relevant literature is [Hit87, §3 and §4].

CONTENTS

NOTATION AND CONVENTIONS

We usually follow the notation of [Hit87]:

- M: compact Riemann surface of genus g.
- $O \longrightarrow M$: trivial line bundle.
- $K \longrightarrow M$: canonical line bundle.
- More generally, O_X and K_X denote the trivial and canonical line bundles over a complex manifold X.

Every now and then we will also use some other standard notation, for instance:

- For a vector bundle $E \longrightarrow M$ we denote $\mu(E) := \deg E / \operatorname{rk} E$.
- Let X be a complex manifold and $E \longrightarrow X$ a (holomorphic/algebraic) vector bundle. Then we denote by $\mathcal{E}$ its sheaf of sections. The assignement $E \mapsto \mathcal{E}$ defines an equivalence of categories between vector bundles on X and locally free sheaves of $\mathcal{O}_X$-modules of finite rank, which we will refer to simply as locally free sheaves. We can recover E from $\mathcal{E}$ either using cocycles [Voi02, Lemma 4.8] or

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by using the relative spectrum [Har77, Exercise II.5.18]. Following the second approach we would write

$$E = \mathrm{Spec}_X(S(\mathcal{E}^\vee)) =: \mathbb{V}(\mathcal{E}^\vee),$$

where $S(-)$ denotes the symmetric algebra.

- $\mathcal{O}$ and ω denote the trivial and canonical invertible sheaves on M. More generally, $\mathcal{O}_X$ and ω_X denote the trivial and canonical invertible sheaves on a complex manifold X,
- Let E be again a vector bundle on a complex manifold X. We will denote its projectivisation by $\mathbb{P}(E)$, which is obtained form E without its zero section by quotienting out the fibre-wise multiplication by scalars. Taking again the algebraic approach we can write

$$\mathbb{P}(E) = \mathrm{Proj}_X(S(\mathcal{E}^\vee)) =: \mathbb{P}(\mathcal{E}^\vee).$$

1. Stability

Definition (Higgs bundle)**.** A *Higgs bundle* on M is a pair (E, Φ), where $E \longrightarrow M$ is a rank 2 vector bundle and Φ is a global section of $\mathrm{End}\, E \otimes K$, called a *Higgs field* on E.

Remark. Using the canonical isomorphisms

$$H^0(M, \mathcal{E}nd(\mathcal{E}) \otimes \omega) \cong \mathrm{Hom}(\mathcal{O}, \mathcal{E}^\vee \otimes \mathcal{E} \otimes \omega) \cong \mathrm{Hom}(\mathcal{E}, \mathcal{E} \otimes \omega)$$

we may identify Φ with a morphism

$$\Phi : \ E \longrightarrow E \otimes K.$$

Definition (Stability)**.** A Higgs bundle (E, Φ) is said to be *stable* if for every Φ-invariant line bundle $L \subseteq E$ we have $\mu(L) < \mu(E)$, where Φ-invariance means that $\Phi(L) \subseteq L \otimes K$.

Remark. $(E, 0)$ is stable if and only if E is stable in the usual sense.

Exercise A. There are no stable Higgs bundles on $\mathbb{P}^1$. [*Hints below*[1]]

Lemma 1. *Let* (E_1, Φ_1) *and* (E_2, Φ_2) *be stable pairs with* $\Lambda^2 E_1 \cong \Lambda^2 E_2$. *Let* $\Psi : \ E_1 \longrightarrow E_2$ *be a non-zero morphism such that* $(\Psi \otimes \mathrm{id}_K) \circ \Phi_1 = \Phi_2 \circ \Psi$. *Then* Ψ *is an isomorphism.*

Proof. We prove the result by contradiction. Suppose that Ψ is not an isomorphism. The rank $x \mapsto \dim_{\mathbb{C}} \Psi_x(E_{1,x})$ is upper semi-continuous [Ati89, Proposition 1.3.2], so the rank of Ψ cannot be generically zero. If the rank was generically 2, then $\det(\Psi) \in H^0(M, \Lambda^2 \mathcal{E}_1^\vee \otimes \Lambda^2 \mathcal{E}_2)$ would be generically non-zero. But $\Lambda^2 E_1 \cong \Lambda^2 E_2$, so $\det(\Psi) \in H^0(M, \mathcal{O}) = \mathbb{C}$ must be a constant

[1] Grothedieck's theorem allows us to write Φ as a matrix. What can we say about each entry? The solution can be found in [Hit87, Remark (3.2) (iii)]

and Ψ would be an isomorphism. Therefore the rank is generically 1, only going down to 0 at special points.

Let $L_1 \subseteq E_1$ be the largest rank 1 subbundle of E_1 contained in the kernel of Ψ. Let $v_1 \in L_{1,x}$, and let z be a holomorphic coordinate around a general point $x \in M$. Then we can write $\Phi_1(v_1) = \phi_{1,x}(v_1) \otimes dz$ for some $\phi_{1,x} \in \mathrm{End}(E_{1,x})$. Then

$$0 = \Phi_{2,x}(\Psi_x(v_1)) = (\Psi \otimes \mathrm{id}_K)_x(\phi_{1,x}(v_1) \otimes dz) = \Psi_x(\phi_{1,x}(v_1)) \otimes dz,$$

so $\phi_{1,x}(v_1) \in \mathrm{Ker}(\Psi_x) = L_{1,x}$. Since it suffices to check $\Phi_1(L_1) \subseteq L_1 \otimes K$ generically, this shows that L_1 is Φ_1-invariant.

Let now $L_2 \subseteq E_2$ be the largest rank 1 subbundle of E_2 containing the image of Ψ. Let $v_2 = \Psi(v_1) \in L_{2,x}$ be a vector over a general point $x \in M$, which can thus be written as the image under Ψ of someone in E_1. Then

$$\Phi_{2,x}(\Psi_x(v_1)) = (\Psi \otimes \mathrm{id}_K)_x(\phi_{1,x}(v_1) \otimes dz) = \Psi_x(\phi_{1,x}(v_1)) \otimes dz,$$

so $\phi_{2,x}(v_2) \in \mathrm{Im}(\Psi_x) = L_{2,x}$. Since it suffices to check $\Phi_2(L_2) \subseteq L_2 \otimes K$ generically, this shows that L_2 is Φ_2-invariant.

Now we use that (E_i, Φ_i) are stable to deduce that

$$\deg(L_i) < \frac{d}{2}$$

for $i \in \{1, 2\}$, where $d := \deg(\Lambda^2 E_1) = \deg(\Lambda^2 E_2)$. Since L_1 is contained in the kernel of Ψ, Ψ induces a non-zero morphism of line bundles $E_1/L_1 \longrightarrow L_2$, which corresponds to a non-zero global section of $(E_1/L_1)^\vee \otimes L_2$. Line bundles with negative degree do not have any non-zero global sections, so we must have $\deg(E_1/L_1) \leqslant \deg(L_2)$. Therefore

$$\frac{d}{2} < \deg(\Lambda^2 E_1) - \deg(L_1) = \deg(E_1/L_1) \leqslant \deg(L_2) < \frac{d}{2},$$

a contradiction. Hence Ψ must be an isomorphism. □

Lemma 2. *Let $E \longrightarrow M$ be a rank 2 vector bundle and denote by $\mathrm{End}_0(E)$ the vector bundle of traceless endomorphisms. Then there is a natural projection $\mathrm{pr}_0 : \mathrm{End}(E) \longrightarrow \mathrm{End}_0(E)$ whose kernel is the trivial line bundle of multiples of the identity, yielding a short exact sequence*

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{E}nd(\mathcal{E}) \longrightarrow \mathcal{E}nd_0(\mathcal{E}) \longrightarrow 0.$$

In particular, $\deg(\mathrm{ch}(\mathcal{E}nd_0(\mathcal{E}) \otimes \omega)\,\mathrm{td}(\omega^\vee))_1 = 3g - 3$.

Proof. Over $x \in M$, the map $\mathrm{pr}_{0,x} : \mathrm{End}(E_x) \longrightarrow \mathrm{End}_0(E_x)$ is given by

$$A \longmapsto A - \frac{\mathrm{tr}(A)}{2}\,\mathrm{id}_{E_x}.$$

The endomorphisms in the kernel are precisely the multiples of the identity. This fibre-wise description globalizes to the desired short exact sequence.

For the Chern class computation we use the axioms in [Har77, Appendix A]. We have

$$c_1(\mathcal{E}nd_0(\mathcal{E})) = c_1(\mathcal{E}nd(\mathcal{E})) = c_1(\mathcal{E}^\vee \otimes \mathcal{E}) = 0,$$

therefore

$$\mathrm{ch}(\mathcal{E}nd_0(\mathcal{E}) \otimes \omega) = 3 + c_1(\mathcal{E}nd_0(\mathcal{E})) + 3c_1(\omega) = 3 + 3c_1(\omega).$$

We also have

$$\mathrm{td}(\omega^\vee) = 1 - \frac{c_1(\omega)}{2},$$

so multiplying the two expressions we obtain

$$\mathrm{ch}(\mathcal{E}nd_0(\mathcal{E}) \otimes \omega)\,\mathrm{td}(\omega^\vee) = 3 + \frac{3}{2}c_1(\omega).$$

Since $\deg(c_1(\omega)) = 2g - 2$, the result follows. □

Notation. Let us denote by $\mathbf{Vec}_2(M)$ the set of rank 2 vector bundles on M. We define the following subsets:

- $\mathbf{S} := \{E \in \mathbf{Vec}_2(M) \mid \exists\Phi \text{ such that } (E, \Phi) \text{ is stable}\}$.
- $\mathbf{A} := \{E \in \mathbf{Vec}_2(M) \mid \text{a general } \Phi \text{ has no invariant } L\}$.
- $\mathbf{B} := \{E \in \mathbf{Vec}_2(M) \mid \exists L \text{ invariant for all } \Phi\}$.

Lemma 3. *If $g \geqslant 2$, then*

$$\mathbf{Vec}_2(M) = \mathbf{A} \sqcup \mathbf{B}.$$

Proof. Let $\pi : \mathbb{P}(E) \longrightarrow M$ be the projectivization of our rank 2 vector bundle and let $O(-1) \longrightarrow \mathbb{P}(E)$ denote the tautological line bundle, whose fiber over $[v] \in \mathbb{P}(E)$ is the line $\{\lambda v \mid \lambda \in \mathbb{C}\} \subseteq E_{\pi([v])}$ spanned by v. Denote also $O(l) := O(-1)^{\otimes(-l)}$. If $\mathcal{F}$ is a sheaf on $\mathbb{P}(E)$, we denote $\mathcal{F}(l) := \mathcal{F} \otimes \mathcal{O}(l)$, where $\mathcal{O}(l)$ denotes the sheaf of sections of $O(l)$. We have $\pi_*\mathcal{O}(l) = S^l(\mathcal{E}^\vee)$ for all $l \geqslant 0$ and $\pi_*\mathcal{O}(l) = 0$ for all $l < 0$ [Har77, Exercise III.8.4].

Let $x \in M$. Given $A \in \mathrm{End}(E_x)$, we define the quadratic form $v \mapsto Av \wedge v$ with values in $\Lambda^2 E_x$, which can then be naturally regarded as an element in $S^2(E_x^\vee) \otimes \Lambda^2 E_x$. The resulting quadratic form is trivial precisely when $A = \lambda\,\mathrm{id}_{E_x}$ for some $\lambda \in \mathbb{C}$, so by Lemma 2 we obtain an injective homomorphism $\mathrm{End}_0(E_x) \longrightarrow S^2(E_x^\vee) \otimes \Lambda^2 E_x$. Both vector spaces have the same dimension, so this must be an isomorphism. These isomorphisms globalize to an isomorphism $\mathrm{End}_0(E) \cong S^2(E^\vee) \otimes \Lambda^2 E$, hence we obtain an isomorphism

$$\mathcal{E}nd_0(\mathcal{E}) \otimes \omega \cong \pi_*\mathcal{O}(2) \otimes \omega \otimes \Lambda^2\mathcal{E}.$$

The projection formula yields now an isomorphism $\mathcal{E}nd_0(\mathcal{E})\otimes\omega \cong \pi_*(\pi^*(\omega\otimes\Lambda^2\mathcal{E})(2))$, hence an isomorphism

$$\psi:\ H^0(M,\mathcal{E}nd_0(\mathcal{E})\otimes\omega)\cong H^0(\mathbb{P}(E),\pi^*(\omega\otimes\Lambda^2\mathcal{E})(2)).$$

Let now $\Phi\in H^0(M,\mathcal{E}nd_0(\mathcal{E})\otimes\omega)$. A line bundle $L\subseteq E$ is then Φ-invariant precisely when $\psi(\Phi)$ vanishes at all $[v]\in\mathbb{P}(L)\subseteq\mathbb{P}(E)$. In other words, L is Φ-invariant if and only if $\sigma(M)\subseteq\operatorname{div}(\psi(\Phi))$, where $\operatorname{div}(-)$ denotes the divisor of zeros of a section and $\sigma: M=\mathbb{P}(L)\longrightarrow\mathbb{P}(E)$ is the section induced by $L\subseteq E$.

Suppose now that $\Phi\in H^0(M,\mathcal{E}nd_0(\mathcal{E})\otimes\omega)$ is not nilpotent and let $L\subseteq E$ be a Φ-invariant line bundle. Then over a general point $x\in M$, the corresponding traceless endomorphism $\phi_x\in\operatorname{End}_0(E_x)$ is diagonalizable, so we can find some eigenvector $v\in E_x\setminus L_x$ in an eigenspace other than L_x. This gives us a point $[v]\in\mathbb{P}(E)\setminus\sigma(M)$ on which $\psi(\Phi)$ vanishes. Hence $\sigma(M)$ is a proper irreducible component of the divisor $\operatorname{div}(\psi(\Phi))$.

The previous discussion shows that if Φ is not nilpotent and $\operatorname{div}(\psi(\Phi))$ is irreducible, then there are no invariant line bundles $L\subseteq E$. By Hirzebruch–Riemann–Roch and Lemma 2 we have

$$h^0(M,\mathcal{E}nd_0(\mathcal{E})\otimes\omega)\geqslant 3g-3\geqslant 3,$$

so the complete linear system defined by the invertible sheaf $\pi^*(\omega\otimes\Lambda^2\mathcal{E})(2)$ has dimension at least 2. If this linear system does not have a fixed divisor, then Bertini's theorem [Iit82, Theorem 7.19] tells us that $\operatorname{div}(\psi(\Phi))$ is irreducible for a general $\Phi\in H^0(M,\mathcal{E}nd_0(\mathcal{E})\otimes\omega)$. Since in our case $\Phi\in H^0(M,\mathcal{E}nd_0(\mathcal{E})\otimes\omega)$ is nilpotent if and only if $\Phi^2=0$, a general Φ is not nilpotent. Therefore $E\in\mathbf{A}$ in this case.

Let us see what happens if the linear system has a fixed divisor. Such a fixed divisor must be the zero locus of a non-zero global section s of an invertible sheaf on $\mathbb{P}(E)$, which is up to isomorphism of the form $\pi^*\mathcal{L}(l)$ with $\mathcal{L}$ an invertible sheaf on M and $l\in\mathbb{Z}$ [Har77, Exercise II.7.9]. Being a fixed divisor means then that every other global section of our line bundle can be written as a product st, where $t\in H^0(\mathbb{P}(E),\pi^*\mathcal{N}(2-l))$. Since our line bundle had non-zero global sections, both $\pi^*\mathcal{L}(l)$ and $\pi^*\mathcal{N}(2-l)$ must have non-zero global sections. By the projection formula, this leaves us with only three possibilities:

a) $l=0$;
b) $l=1$;
c) $l=2$.

Let us start with case *a*). Let $\pi^*s\in H^0(\mathbb{P}(E),\pi^*\mathcal{L})\cong H^0(M,\mathcal{L})$ be a global section corresponding to the fixed component of our linear system. Dividing all global sections by s and by π^*s respectively we obtain the following commutative diagram:

$$\begin{tikzcd}
H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \omega) \arrow[r, "\cong"] \arrow[d, "/s"', "\cong"] & H^0(\mathbb{P}(E), \pi^*(\omega \otimes \Lambda^2\mathcal{E})(2)) \arrow[d, "/\pi^* s"', "\cong"] \\
H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \omega \otimes \mathcal{L}^\vee) \arrow[r, "\cong"] & H^0(\mathbb{P}(E), \pi^*(\mathcal{L}^\vee \otimes \omega \otimes \Lambda^2\mathcal{E})(2))
\end{tikzcd}$$

By definition, the new linear system does not have any fixed divisors and has the same dimension. Hence we can apply Bertini to conclude that a generic $\Phi' \in H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \omega \otimes \mathcal{L}^\vee)$ does not have invariant line bundles, which in this case are defined as line bundles $N \subseteq E$ such that $\Phi'(N) \subseteq N \otimes K \otimes L^\vee$. But a line bundle $N \subseteq E$ is Φ'-invariant if and only if it is $s\Phi'$-invariant, so we have $E \in \mathbf{A}$ in this case.

We move on to case b). Assume that the fixed divisor corresponds to a non-zero global section $s \in H^0(\mathbb{P}(E), \pi^*\mathcal{L}(1))$. This corresponds to a non-zero morphism $E \longrightarrow L$. The fibre-wise kernel has then dimension 1 generically and 2 at special points by upper semi-continuity [Ati89, Proposition 1.3.2]. Hence we can find a line bundle $N \subseteq \mathrm{Ker}(s) \subseteq E$, which can be described as the largest line subbundle of E contained in the kernel of s. If $v \in N$ is a non-zero vector, then $s(v) = 0$ and so $[v] \in \mathrm{div}(s) \subseteq \mathrm{div}(\psi(\Phi))$. Thus the corresponding section $\sigma(M) \subseteq \mathbb{P}(E)$ is contained in $\mathrm{div}(\psi(\Phi))$ for all Φ and N is Φ-invariant for all Φ. Hence $E \in \mathbf{B}$ in this case.

In case c), the fixed divisor corresponds to a non-zero global section of $\pi^*\mathcal{L}(2)$. We have

$$H^0(\mathbb{P}(E), \pi^*\mathcal{L}(2)) \cong H^0(M, \mathcal{L} \otimes S^2\mathcal{E}^\vee) \cong H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{L} \otimes \Lambda^2\mathcal{E}^\vee).$$

So we can think of the fixed global section s as a traceless endomorphism of E with coefficients in $L \otimes \Lambda^2 E^\vee$. With this point of view, s-invariance of a line bundle $N \subseteq E$ translates into $s\Phi'$-invariance of $N \subseteq E$ as before, where $s\Phi'$ is a Higgs field. Let us see that the fixed section s has some non-trivial kernel, hence defining a line bundle invariant under all Higgs fields as in the previous case. To show that there is some non-trivial kernel, it suffices to check that $\det(s) = 0$. Since s is traceless, it suffices in turn to check that $\mathrm{tr}(s^2) = 0$. Suppose on the contrary that $\mathrm{tr}(s^2) \neq 0$. Fix some non-zero $s_1 \in H^0(M, \mathcal{L}^\vee \otimes \omega \otimes \Lambda^2\mathcal{E})$ and consider the linear map

$$\begin{aligned} \theta :\ H^0(M, \mathcal{L}^\vee \otimes \omega \otimes \Lambda^2\mathcal{E}) &\longrightarrow H^0(M, \omega^2) \\ \Phi' &\longmapsto \mathrm{tr}(s^2)s_1\Phi' \end{aligned}$$

Since $\mathrm{tr}(s^2)s_1$ can only vanish at finitely many points, the image of a non-zero Φ' can only vanish at finitely many points, hence θ is injective. From Hirzebruch–Riemann–Roch and Lemma 2 we know that

$$h^0(M, \mathcal{L}^\vee \otimes \omega \otimes \Lambda^2\mathcal{E}) \geqslant 3g - 3 = h^0(M, \omega^2),$$

so θ is an isomorphism. Since $\mathcal{L}^\vee \otimes \omega \otimes \Lambda^2\mathcal{E}$ has global sections, its degree is non-negative. If it was zero, then this would be the trivial line bundle and we would have $h^0(M, \mathcal{L}^\vee \otimes \omega \otimes \Lambda^2\mathcal{E}) = 1 < 3g - 3$, a contradiction. Hence $\deg(\mathcal{L}^\vee \otimes \omega \otimes \Lambda^2\mathcal{E}) > 0$ and the non-zero global section s_1 has at least one zero. If θ was indeed an isomorphism, then each zero of s_1 would give a base point of the complete linear system corresponding to ω^2. But $\deg(\omega^2) = 4g - 4 \geqslant 2g$, so this linear system has no base points [Har77, Corollary IV.3.2]. This contradiction shows that s has non-trivial kernel, which contains a line bundle $N \subseteq E$ invariant by all $\Phi \in H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \omega)$. Hence $E \in \mathbf{B}$ as well in this case. □

Exercise B. Assume $g \geqslant 2$. Let $K^{\frac{1}{2}}$ be a line bundle whose square is K and let $K^{-\frac{1}{2}}$ be its inverse. Does $K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}$ belong to $\mathbf{A}$ or to $\mathbf{B}$? [*Hints below* [2]]

Lemma 4. *If $g \geqslant 2$, then*

$$\mathbf{S} \subseteq \mathbf{A}.$$

Proof. We want to show that if $E \in \mathbf{Vec}_2(M)$ is stable, then it is in $\mathbf{A}$. By Lemma 3 it suffices to show that it is not in $\mathbf{B}$. So let E be a stable rank 2 vector bundle on M and assume $L \subseteq E$ is a line bundle which is Φ-invariant for all $\Phi \in H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \omega)$. Under the isomorphism $E \cong E^\vee \otimes \Lambda^2 E$ [Har77, Exercise II.5.16], the line bundle L is sent to linear forms with coefficients in $\Lambda^2 E$ vanishing along L, hence we have a short exact sequence

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{E} \longrightarrow \mathcal{L}^\vee \otimes \Lambda^2\mathcal{E} \longrightarrow 0.$$

Dualizing this short exact sequence we obtain an inclusion $L \otimes \Lambda^2 E^\vee \subseteq E^\vee$. Tensoring with L and composing with the inclusion $E^\vee \otimes L \subseteq E^\vee \otimes E$ we obtain an inclusion $L^2 \otimes \Lambda^2 E^\vee \subseteq \mathrm{End}(E)$. Choosing a basis on each fibre and chasing all the identifications we have made so far, we can see that the image of $L^2 \otimes \Lambda^2 E^\vee$ lies actually in $\mathrm{End}_0(E)$. Indeed, let V be a two dimensional $\mathbb{C}$-vector space and let e_1 and e_2 be a basis. Let L be the line spanned by a non-zero vector $l = l_1 e_1 + l_2 e_2$. The first identification we have is $V \cong \mathrm{Hom}(V, \Lambda^2 V)$, sending v to the homomorphism $v' \mapsto v' \wedge v$. This corresponds to $\alpha_v \otimes (e_1 \wedge e_2) \in V^\vee \otimes \Lambda^2 V$, where $\alpha_v \in V^\vee$ is the linear

[2] Consider the family of traceless endomorphisms given by

$$\Phi_\alpha := \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix},$$

parametrized by quadratic differentials $\alpha \in H^0(M, K^2)$. Use without proof the fact that an invariant line bundle exists if and only if the characteristic polynomial

$$\lambda^2 - \alpha$$

has a root in $H^0(M, K)$, i.e. if and only if the quadratic differential α can be written as a square $\alpha = \beta^2$ for some differential form $\beta \in H^0(M, K)$. If α was a square, its zeros would all have multiplicity at least two. Conclude that $K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}} \in \mathbf{A}$ using Bertini's theorem.

form sending $e_1 \mapsto v_2$ and $e_2 \mapsto -v_1$. Denoting by $\overline{\alpha_v}$ its image in $L^\vee$, we can describe the morphism corresponding to the right hand side of the previous short exact sequence as

$$\begin{aligned} V &\longrightarrow L^\vee \otimes \Lambda^2 V \\ v &\longmapsto \overline{\alpha_v} \otimes (e_1 \wedge e_2) \end{aligned}$$

Let now $\beta \in (L^\vee \otimes \Lambda^2 V)^\vee$ and denote by λ_v^β the complex number such that

$$\beta : \overline{\alpha_v} \otimes (e_1 \wedge e_2) \mapsto \lambda_v^\beta.$$

A point $\mu l \otimes \beta \in L \otimes (L^\vee \otimes \Lambda^2 V)^\vee$ corresponds then to the endomorphism

$$\begin{aligned} V &\longrightarrow V \\ v &\longmapsto \mu \lambda_v^\beta l \end{aligned}$$

Writing it as a matrix with respect to our basis we obtain

$$\begin{pmatrix} \mu\lambda_{e_1}^\beta l_1 & \mu\lambda_{e_2}^\beta l_1 \\ \mu\lambda_{e_1}^\beta l_2 & \mu\lambda_{e_2}^\beta l_2 \end{pmatrix}.$$

Therefore we have to show that $\lambda_{e_1}^\beta l_1 + \lambda_{e_2}^\beta l_2 = 0$. But we have $\lambda_{e_1}^\beta l_1 + \lambda_{e_2}^\beta l_2 = \lambda_{l_1 e_1 + l_2 e_2}^\beta = \lambda_l^\beta$, which must be zero for all β because $\alpha_l(l) = l_1 l_2 - l_2 l_1 = 0$ and therefore $\overline{\alpha_l} \otimes (e_1 \wedge e_2) = 0$.

So we have an inclusion

$$K \otimes L^2 \otimes \Lambda^2 E^\vee \subseteq \mathrm{End}_0(E) \otimes K.$$

A non-zero global section of this subbundle leaves only L invariant, because the inclusion $L^2 \otimes \Lambda^2 E^\vee \subseteq \mathrm{Hom}(E, E)$ factors through $\mathrm{Hom}(E, L) \subseteq \mathrm{Hom}(E, E)$, as we have seen a moment ago. □

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Pedro Núñez, Mathematisches Institut, Albert-Ludwigs-Universität Freiburg, Ernst-Zermelo-Strasse 1, 79104 Freiburg im Breisgau, Germany
Email address: pedro.nunez@math.uni-freiburg.de
URL: https://home.mathematik.uni-freiburg.de/nunez