HIGGS BUNDLES — EXISTENCE OF SOLUTIONS

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Abstract. In this talk we introduce the stability condition for Higgs bundles and prove the Hitchin–Kobayashi correspondence. The main result is [Hit87, Theorem 4.3]. Relevant literature is [Hit87, §3 and §4] and [Wen14, §2 and §3]. Maybe we will also use [Wen16] every now and then.

This talk is related to Tanuj's talk on *Stable vector bundles*, for which the main reference is [Kob87]. Therefore we will also use [Kob87] as a default reference for generalities on complex vector bundles.

Contents

Notation and conventions

We usually follow the notation of [Hit87]:

- M: compact Riemann surface of genus g.
- $O \to M$: trivial line bundle.
- $K \to M$: canonical line bundle.
- More generally, O_X resp. K_X denote the trivial resp. canonical line bundle on a complex manifold X.
- For a vector bundle $V \to M$ we denote $\mu(V) := \deg V / \operatorname{rk} V$.

1. Stability

Definition 1 (Higgs bundle)**.** A *Higgs bundle* on M is a pair (V, Φ), where $V \to M$ is a rank 2 vector bundle and Φ is a global section of $\operatorname{End} V \otimes K$, called a *Higgs field* on V.

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Remark 2. Using the canonical isomorphisms

$$H^0(M, \operatorname{End} V \otimes K) \cong \operatorname{Hom}(O, V^* \otimes V \otimes K) \cong \operatorname{Hom}(V, V \otimes K)$$

we may identify Φ with a morphism

$$\Phi\colon V \to V \otimes K.$$

Example 3. Assume $g \geqslant 2$. Then $\deg K = 2g - 2 > 0$, so we can find a line bundle $K^{\frac{1}{2}}$ such that $K^{\frac{1}{2}} \otimes K^{\frac{1}{2}} \cong K$. Let $V = K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}$, where $K^{-\frac{1}{2}} = (K^{\frac{1}{2}})^{-1}$. We consider the Higgs field $\Phi_w\colon K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}} \to (K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}) \otimes K$ given by a matrix

$$\begin{pmatrix} 0 & w \\ 1 & 0 \end{pmatrix},$$

where $w \in \operatorname{Hom}(K^{-\frac{1}{2}}, K^{\frac{1}{2}} \otimes K) \cong H^0(M, K^2)$ can be regarded as a quadratic differential.

Definition 4 (Stability)**.** A Higgs bundle (V, Φ) is said to be *stable* if for every Φ-invariant[1] line bundle $L \subseteq V$ we have $\mu(L) < \mu(V)$.

Remark 5. $(V, 0)$ is stable if and only if V is stable in the usual sense.

Exercise 1. There are no stable Higgs bundles on $\mathbb{P}^1$. [Hint: Grothedieck's theorem allows us to write Φ as a matrix. What can we say about each entry?] [Solution in [Hit87]]

Example 6. Assume $g \geqslant 2$ and consier $V = K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}$ again. Then Φ_0 is stable, because $K^{-\frac{1}{2}}$ is the only Φ_0-invariant line bundle and

$$\deg K^{-\frac{1}{2}} = 1 - g < 0 = \frac{\deg V}{2}.$$

Proposition 7. *Assume $g \geqslant 2$ and let $V \to M$ be a rank 2 vector bundle. Then there exists Higgs field Φ on V such that (V, Φ) is stable if and only if there exists a dense Zariski open subset $U \subseteq H^0(M, \operatorname{End} V \otimes K)$ such that all $\Phi' \in U$ have the property that no line bundle $L \subseteq V$ is Φ'-invariant.*

Proof. Let $p\colon P(V) \to M$ be the projectivisation of our rank 2 vector bundle, which is the $\mathbb{P}^1$-bundle obtained by replacing each fibre V_x by its projectivisation $(V_x \setminus \{0\})/\mathbb{C}^\times$. Let $S \subseteq p^*V$ be the tautological line bundle on $P(V)$, whose fibre over a point $[v] \in p^{-1}(x)$ is given by the line $\{\lambda v \mid \lambda \in \mathbb{C}\} \subseteq V_x$. Let $H := S^*$ be its dual, which fits into a short exact sequence

$$0 \to Q^* \to p^*V^* \to H \to 0.$$

[1]Meaning that $\Phi(L) \subseteq L \otimes K$.

Let $U \subseteq M$ be an open subset trivialising V. Then the quotient map $p^*V^* \to H$ in the previous short exact sequence induces an isomorphism

$$H^0(p^{-1}(U), p^*V^*) \cong H^0(p^{-1}(U), H),$$

so the pushforward of the sheaf of sections of p^*V^* is isomorphic to the pushforward of the sheaf of sections of H. Since p has connected fibres we have $p_*\mathscr{O}_{P(V)} \cong \mathscr{O}_M$, so applying the projection formula [Har77, Exercise II.5.1.d] we deduce that the pushforward of the sheaf of sections of H is isomorphic to the sheaf of sections of V^*. Abusing slightly the notation we will express this as $V^* \cong p_*H$, and similarly we have $\operatorname{Sym}^2 V^* \cong p_*H^2$.

Let $x \in M$. Then every endomorphism $A \in \operatorname{End}(V_x)$ defines a quadratic map $V_x \to \Lambda^2 V_x$ sending v to $Av \wedge v$. Such a quadratic map can be naturally regarded as a degree 2 homogeneous polynomial on the coordinates coordinates of v with coefficients in $\Lambda^2 V_x$. Hence we have a vector bundle morphism $\operatorname{End}(V) \to S^2V \otimes \Lambda^2 V$. Restricting to $\operatorname{End}_0(V)$ we get an injective morphism, because $A \in \operatorname{End}(V_x)$ is sent to $0 \in S^2V_x \otimes \Lambda^2 V_x$ if and only if A is a multiple of the identity. Counting dimensions we see that we have in fact an isomorphism of vector bundles $\operatorname{End}_0(V) \cong S^2V \otimes \Lambda^2 V$, and therefore

$$\operatorname{End}_0(V) \otimes K \cong p_*H^2K \otimes \Lambda^2 V.$$

Using again that $p_*p^*(-) \cong (-)$ for vector bundles we have a $\mathbb{C}$-linear isomorphism

$$s\colon H^0(M, \operatorname{End}_0(V) \otimes K) \cong H^0(P(V), H^2 p^*(K \otimes \Lambda^2 V)).$$

Let now Φ be a traceless Higgs field on V, and assume it is non-zero. By construction, a non-zero vector $v \in V$ is an eigenvector of the twisted endomorphism over the corresponding fibre if and only if the section $s(\Phi)$ vanishes at the point $[v] \in P(V)$, i.e. if and only if $[v]$ is in the divisor of zeros of the section $s(\Phi)$, which we denote $\operatorname{div}(s(\Phi))$. Let $L \subseteq V$ be a Φ-invariant subbundle, which defines a section of $p\colon P(V) \to M$ by functoriality of projectivisation on injective morphisms of vector bundles:

$$\begin{tikzcd} P(L) \arrow[r, "\sigma"] \arrow[d, equal] & P(L) \arrow[dl, "p"] \\ M & \end{tikzcd}$$

Being Φ-invariant means precisely that $\sigma(M) \subseteq \operatorname{div}(s(\Phi))$. But then any non-zero $v \in L$ is a non-zero eigenvector of the endomorphism over the corresponding fibre. Since Φ was non-zero, we can assume that the corresponding eigenvalue is non-zero as well. Since Φ is traceless, the other eigenvalue must be different, and there must be some non-zero eigenvector outside of L, call it $u \in V$. Since u is a non-zero eigenvector, $[u] \in \operatorname{div}(s(\Phi))$. And since $u \notin L$, $[u] \notin \sigma(M)$. Therefore $\sigma(M)$ is a proper irreducible component of the divisor $\operatorname{div}(s(\Phi))$. In conclusion: if $\operatorname{div}(s(\Phi))$ is irreducible, then no line bundle $L \subseteq V$ is Φ-invariant. $\square$

References

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