

# HIGGS BUNDLES — EXISTENCE OF SOLUTIONS

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ABSTRACT. In this talk we introduce the stability condition for Higgs bundles and prove the Hitchin–Kobayashi correspondence. The main result is [Hit87, Theorem 4.3]. Relevant literature is [Hit87, §3 and §4] and [Wen14, §2 and §3]. Maybe we will also use [Wen16] every now and then.

This talk is related to Tanuj’s talk on *Stable vector bundles*, for which the main reference is [Kob87]. Therefore we will also use [Kob87] as a default reference for generalities on complex vector bundles.

## CONTENTS

Notation and conventions	1
1. Stability	2
References	7

## NOTATION AND CONVENTIONS

We usually follow the notation of [Hit87]:

- $M$ : compact Riemann surface of genus  $g$ .
- $O \rightarrow M$ : trivial line bundle.
- $K \rightarrow M$ : canonical line bundle.
- More generally,  $O_X$  and  $K_X$  denote the trivial and canonical line bundles over a complex manifold  $X$ .

Every now and then we will also use some other standard notation, for instance:

- For a vector bundle  $E \rightarrow M$  we denote  $\mu(E) := \deg E / \operatorname{rk} E$ .
- Let  $X$  be a complex manifold and  $E \rightarrow X$  a (holomorphic/algebraic) vector bundle. Then we denote by  $\mathcal{E}$  its sheaf of sections. The assignment  $E \mapsto \mathcal{E}$  defines an equivalence of categories between vector bundles on  $X$  and locally free

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sheaves of  $\mathcal{O}_X$ -modules of finite rank, which we will refer to simply as locally free sheaves. We can recover  $E$  from  $\mathcal{E}$  either using cocycles [Voi02, Lemma 4.8] or by using the relative spectrum [Har77, Exercise II.5.18]. Following the second approach we would write

$$E = \operatorname{Spec}_X(S(\mathcal{E}^*)) =: \mathbb{V}(\mathcal{E}^*),$$

where  $S(-)$  denotes the symmetric algebra.

- $\mathcal{O}$  and  $\omega$  denote the trivial and canonical invertible sheaves on  $M$ . More generally,  $\mathcal{O}_X$  and  $\omega_X$  denote the trivial and canonical invertible sheaves on a complex manifold  $X$ ,
- Let  $E$  be again a vector bundle on a complex manifold  $X$ . We will denote its projectivisation by  $\mathbb{P}(E)$ , which is obtained from  $E$  without its zero section by quotienting out the fibre-wise multiplication by scalars. Taking again the algebraic approach we can write

$$\mathbb{P}(E) = \operatorname{Proj}_X(S(\mathcal{E}^*)) =: \mathbb{P}(\mathcal{E}^*).$$

## 1. STABILITY

**Definition 1.1** (Higgs bundle). A *Higgs bundle* on  $M$  is a pair  $(E, \Phi)$ , where  $E \rightarrow M$  is a rank 2 vector bundle and  $\Phi$  is a global section of  $\operatorname{End} E \otimes K$ , called a *Higgs field* on  $E$ .

*Remark 1.2.* Using the canonical isomorphisms

$$H^0(M, \operatorname{End}(\mathcal{E}) \otimes \omega) \cong \operatorname{Hom}(\mathcal{O}, \mathcal{E}^* \otimes \mathcal{E} \otimes \omega) \cong \operatorname{Hom}(\mathcal{E}, \mathcal{E} \otimes \omega)$$

we may identify  $\Phi$  with a morphism

$$\Phi: E \rightarrow E \otimes K.$$

**Definition 1.3** (Stability). A Higgs bundle  $(E, \Phi)$  is said to be *stable* if for every  $\Phi$ -invariant<sup>1</sup> line bundle  $L \subseteq E$  we have  $\mu(L) < \mu(E)$ .

*Remark 1.4.*  $(E, 0)$  is stable if and only if  $E$  is stable in the usual sense.

*Exercise 1.5.* There are no stable Higgs bundles on  $\mathbb{P}^1$ . [Hint: Grothendieck's theorem allows us to write  $\Phi$  as a matrix. What can we say about each entry? Solution in [Hit87]]

**Proposition 1.6.** Let  $(E_1, \Phi_1)$  and  $(E_2, \Phi_2)$  be stable pairs with  $\Lambda^2 E_1 \cong \Lambda^2 E_2$ . Let  $\Psi: E_1 \rightarrow E_2$  be a non-zero morphism of constant rank such that  $(\Psi \otimes \operatorname{id}_K) \circ \Phi_1 = \Phi_2 \circ \Psi$ . Then  $\Psi$  is an isomorphism.

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<sup>1</sup>Meaning that  $\Phi(L) \subseteq L \otimes K$ .

*Proof.* We prove the result by contradiction. Suppose that  $\Psi$  is not an isomorphism. The rank  $x \mapsto \dim_{\mathbb{C}} \Psi_x(E_{1,x})$  is upper semi-continuous [Ati89, Proposition 1.3.2], so the rank of  $\Psi$  cannot be generically zero. If the rank was generically 2, then  $\Psi$  would be generically an isomorphism. Taking determinants we obtain a morphism of line bundles

$$\Lambda^2 E_1 \xrightarrow{\Psi \wedge \Psi} \Lambda^2 E_2$$

which is generically an isomorphism. Since the two determinants are isomorphic by assumption, we have  $\text{Hom}(\Lambda^2 E_1, \Lambda^2 E_2) \cong \mathbb{C}$ , hence the previous morphism must be given by a non-zero complex number. This shows that  $\Psi$  was in fact an isomorphism, because  $\Psi_x$  takes pairs of linearly independent vectors to pairs of linearly independent vectors for all  $x \in M$ . Therefore the rank is generically 1, only going down to 0 at special points.

Let  $L_1 \subseteq E_1$  be the largest rank 1 subbundle of  $E_1$  contained in the kernel of  $\Psi$ . Let  $v_1 \in L_{1,x}$ , and let  $z$  be a holomorphic coordinate around a general point  $x \in M$ . Then we can write  $\Phi_1(v_1) = \phi_{1,x}(v_1) \otimes dz$  for some  $\phi_{1,x} \in \text{End}(E_{1,x})$ . Then

$$0 = \Phi_{2,x}(\Psi_x(v_1)) = (\Psi \otimes \text{id}_K)_x(\phi_{1,x}(v_1) \otimes dz) = \Psi_x(\phi_{1,x}(v_1)) \otimes dz,$$

so  $\phi_{1,x}(v_1) \in \text{Ker}(\Psi_x) = L_{1,x}$ . Since it suffices to check  $\Phi_1(L_1) \subseteq L_1 \otimes K$  generically, this shows that  $L_1$  is  $\Phi_1$ -invariant.

Let now  $L_2 \subseteq E_2$  be the largest rank 1 subbundle of  $E_2$  containing the image of  $\Psi$ . Let  $v_2 = \Psi(v_1) \in L_{2,x}$  be a vector over a general point  $x \in M$ , which can thus be written as the image under  $\Psi$  of someone in  $E_1$ . Then

$$\Phi_{2,x}(\Psi_x(v_1)) = (\Psi \otimes \text{id}_K)_x(\phi_{1,x}(v_1) \otimes dz) = \Psi_x(\phi_{1,x}(v_1)) \otimes dz,$$

so  $\phi_{2,x}(v_2) \in \text{Im}(\Psi_x) = L_{2,x}$ . Since it suffices to check  $\Phi_2(L_2) \subseteq L_2 \otimes K$  generically, this shows that  $L_2$  is  $\Phi_2$ -invariant.

Now we use that  $(E_i, \Phi_i)$  are stable to deduce that

$$\deg(L_i) < \frac{d}{2}$$

for  $i \in \{1, 2\}$ , where  $d := \deg(\Lambda^2 E_1) = \deg(\Lambda^2 E_2)$ . By construction,  $\Psi$  induces a non-zero morphism of line bundles  $E_1/L_1 \rightarrow L_2$ , which corresponds to a non-zero global section of  $(E_1/L_1)^* \otimes L_2$ . Line bundles with negative degree do not have any non-zero global sections, so we must have  $\deg(E_1/L_1) \leq \deg(L_2)$ . Therefore

$$\frac{d}{2} < \deg(\Lambda^2 E_1) - \deg(L_1) = \deg(E_1/L_1) \leq \deg(L_2) < \frac{d}{2},$$

a contradiction. Hence  $\Psi$  must be an isomorphism.  $\square$

**Proposition 1.7.** *Assume  $g \geq 2$  and let  $E \rightarrow M$  be a rank 2 vector bundle. Then there exists Higgs field  $\Phi$  on  $E$  such that  $(E, \Phi)$  is stable if and only if there exists a dense Zariski open subset  $U \subseteq H^0(M, \text{End}(\mathcal{E}) \otimes \omega)$  such that all  $\Phi' \in U$  have the property that no line bundle  $L \subseteq E$  is  $\Phi'$ -invariant.*

*Proof.* We define the following sets of rank 2 vector bundles on  $M$ :

- $\mathbf{S} := \{E \in \mathbf{Vec}_2(M) \mid \exists \Phi \text{ such that } (E, \Phi) \text{ is stable}\}.$
- $\mathbf{A} := \{E \in \mathbf{Vec}_2(M) \mid \text{a general } \Phi \text{ has no invariant } L\}.$
- $\mathbf{B} := \{E \in \mathbf{Vec}_2(M) \mid \exists L \text{ invariant for all } \Phi\}.$

Our goal is to show that  $\mathbf{S} = \mathbf{A}$ . If  $\Phi$  has no invariant  $L$ , then  $(E, \Phi)$  is automatically stable. Hence  $\mathbf{A} \subseteq \mathbf{S}$ . The plan to show the other inclusion is to see that

$$\mathbf{Vec}_2(M) = \mathbf{A} \sqcup \mathbf{B}$$

and that  $\mathbf{B} \subseteq \mathbf{Vec}_2(M) \setminus \mathbf{S}$ .

Let us start by showing that  $\mathbf{Vec}_2(M) = \mathbf{A} \sqcup \mathbf{B}$ .

Let  $p: \mathbb{P}(E) \rightarrow M$  be the projectivisation of our rank 2 vector bundle, which is a ruled surface in the sense of [Har77, §V.2]. Let  $\mathcal{O}(-1)$  denote the tautological line bundle on  $\mathbb{P}(E)$ , whose fibre over  $[v] \in \mathbb{P}(E)$  is the line  $\{\lambda v \mid \lambda \in \mathbb{C}\} \subseteq E_{p([v])}$ . Let  $\mathcal{O}(l) := \mathcal{O}(1)^{\otimes l}$  for all  $l \in \mathbb{Z}$ , and if  $F \rightarrow \mathbb{P}(E)$  is another vector bundle, denote by  $F(l)$  the tensor product  $F \otimes \mathcal{O}(l)$ . We have then  $p_*\mathcal{O}(l) = S^l(\mathcal{E}^*)$  for all  $l \geq 0$  [Har77, Exercise III.8.4].

Let  $x \in M$ . Then every endomorphism  $A \in \text{End}(E_x)$  defines a quadratic map  $E_x \rightarrow \Lambda^2 E_x$  sending  $v$  to  $Av \wedge v$ . Such a quadratic map can be naturally regarded as a degree 2 homogeneous polynomial on the coordinates of  $e$  with coefficients in  $\Lambda^2 E_x$ . Hence we have a morphism  $\text{End}(E) \rightarrow S^2 E^* \otimes \Lambda^2 E$ , which vanishes precisely along the trivial line subbundle of  $\text{End}(E)$  consisting over each fibre of scalar multiples of the identity. Sending  $A \mapsto A - \frac{\text{tr}(A)}{2} \text{id}_{E_x}$  on each fibre allows us to identify  $\text{End}_0(E)$  as the quotient of  $\text{End}(E)$  by this trivial line subbundle, so we obtain an injective morphism  $\text{End}_0(E) \rightarrow S^2 E^* \otimes \Lambda^2 E$ . Counting ranks we see that we have in fact an isomorphism  $\text{End}_0(E) \cong S^2 E^* \otimes \Lambda^2 E$ , and therefore

$$\text{End}_0(\mathcal{E}) \otimes \omega \cong p_*\mathcal{O}(2) \otimes \omega \otimes \Lambda^2 \mathcal{E}.$$

By the projection formula,  $\text{End}_0(\mathcal{E}) \otimes \omega \cong p_*(p^*(\omega \otimes \Lambda^2 \mathcal{E}))(2)$ . Therefore we have an isomorphism

$$\alpha: H^0(M, \text{End}_0(\mathcal{E}) \otimes \omega) \cong H^0(\mathbb{P}(E), p^*(\omega \otimes \Lambda^2 \mathcal{E}))(2).$$

Let now  $\Phi$  be a traceless Higgs field on  $E$ , and assume it is non-zero. By construction, a non-zero vector  $v \in E$  is an eigenvector of

the twisted endomorphism over the corresponding fibre if and only if the section  $\alpha(\Phi)$  vanishes at the point  $[v] \in \mathbb{P}(E)$ , i.e. if and only if  $[v]$  is in the divisor of zeros of the global section  $\alpha(\Phi)$ , which we denote  $\text{div}(\alpha(\Phi))$ . Let  $L \subseteq E$  be a  $\Phi$ -invariant subbundle, which defines a section of  $p: \mathbb{P}(E) \rightarrow M$  by functoriality of projectivisation on injective morphisms of vector bundles:

$$\begin{array}{ccc} \mathbb{P}(L) & \xrightarrow{\sigma} & \mathbb{P}(E) \\ \parallel & \swarrow p & \\ M & & \end{array}$$

Being  $\Phi$ -invariant means precisely that  $\sigma(M) \subseteq \text{div}(\alpha(\Phi))$ . But then any non-zero  $v \in L$  is a non-zero eigenvector corresponding to some eigenvalue of the endomorphism over the corresponding fibre. Since  $\Phi$  is traceless and non-zero, the other eigenvalue must be different, and there must be some non-zero eigenvector outside of  $L$ , call it  $v' \in V$ . Since  $v'$  is a non-zero eigenvector,  $[v'] \in \text{div}(\alpha(\Phi))$ . And since  $v' \notin L$ ,  $[v'] \notin \sigma(M)$ . Therefore  $\sigma(M)$  is a proper irreducible component of the divisor  $\text{div}(\alpha(\Phi))$ . So if  $\text{div}(\alpha(\Phi))$  is irreducible, then no line bundle  $L \subseteq V$  is  $\Phi$ -invariant and  $(E, \Phi)$  is automatically stable.

Next we give a lower bound for the dimension of the linear system  $|p^*(\omega \otimes \Lambda^2 \mathcal{E})(2)|$ , which is one less than the dimension of the vector space  $H^0(\mathbb{P}(E), p^*(\omega \otimes \Lambda^2 \mathcal{E})(2))$ . Using the previous isomorphism it suffices to gain control over the dimension of the global sections of  $\text{End}_0(\mathcal{E}) \otimes \omega$  on  $M$ , for which we can apply Hirzebruch–Riemann–Roch [Har77, Theorem A.4.1]. From [Har77, Example A.4.1.1] we get

$$\text{td}(\omega^*) = 1 - \frac{c_1(\omega)}{2}.$$

Using the short exact sequence used earlier

$$0 \rightarrow \mathcal{O} \rightarrow \text{End}(\mathcal{E}) \rightarrow \text{End}_0(\mathcal{E}) \rightarrow 0$$

we see that  $c_1(\text{End}_0(\mathcal{E})) = c_1(\text{End} \mathcal{E}) = 0$ . Therefore

$$\text{ch}(\text{End}_0(\mathcal{E}) \otimes \omega) = 3 + c_1(\text{End}_0(\mathcal{E})) + 3c_1(\omega) = 3 + 3c_1(\omega).$$

Multiplying the two expressions we obtain

$$\text{ch}(\text{End}_0(\mathcal{E}) \otimes \omega) \text{td}(\omega^*) = 3 + \frac{3}{2}c_1(\omega),$$

whose codimension 1 part has degree  $3g - 3 \geq 3$ . So Hirzebruch–Riemann–Roch tells us that

$$h^0(M, \text{End}_0(\mathcal{E}) \otimes \omega) - h^1(M, \text{End}_0(\mathcal{E}) \otimes \omega) = 3g - 3 \geq 3,$$

which implies that  $h^0(\mathbb{P}(E), p^*(\omega \otimes \Lambda^2 \mathcal{E})(2)) = h^0(M, \text{End}_0(\mathcal{E}) \otimes \omega) \geq 3$ .

Thus, our linear system  $|p^*(\omega \otimes \Lambda^2 \mathcal{E})(2)|$  has dimension 2. If it does not have any fixed component, then Bertini's theorem [Lit82, Theorem 7.19] and the discussion above imply that a general  $\Phi \in H^0(M, \text{End}_0(\mathcal{E}) \otimes \omega)$  leaves no line bundle  $L \subseteq E$  invariant, i.e.  $E \in \mathbf{A}$ .

Let us see what happens if it does have some fixed divisor. By definition, a fixed divisor corresponds to a non-zero global section  $s_0 \in H^0(\mathbb{P}(E), \mathcal{M}_1)$  for some invertible sheaf  $\mathcal{M}_1$  such that there exists another invertible sheaf  $\mathcal{M}_2$  with  $\mathcal{M}_1 \otimes \mathcal{M}_2 \cong p^*(\omega \otimes \Lambda^2 \mathcal{E})(2)$ . Being a fixed divisor translates into saying that every global section  $s \in H^0(\mathbb{P}(E), p^*(\omega \otimes \Lambda^2 \mathcal{E})(2))$  can be written as  $ts_0$  for some  $t \in H^0(\mathbb{P}(E), \mathcal{M}_2)$ . In particular, both  $H^0(\mathbb{P}(E), \mathcal{M}_1)$  and  $H^0(\mathbb{P}(E), \mathcal{M}_2)$  have to be non-zero. By [Har77, Exercise II.7.4] we can write  $\mathcal{M}_i \cong p^*\mathcal{L}_i(l_i)$  with  $l_1 + l_2 = 2$ . In fact, we must have  $0 \leq l_i \leq 2$ , because using again the projection formula we have

$$H^0(\mathbb{P}(E), p^*\mathcal{L}_i(l_i)) \cong H^0(M, \mathcal{L}_i \otimes p_*\mathcal{O}(l_i))$$

and  $p_*\mathcal{O}(l) = 0$  for all  $l < 0$  [Har77, Exercise III.8.4]. So we only have the following three possibilities:

- a)  $l_1 = 0$ ;
- b)  $l_1 = 1$ ;
- c)  $l_1 = 2$ .

Let us start with case a). Let  $p^*s \in H^0(\mathbb{P}(E), p^*\mathcal{L}) \cong H^0(M, \mathcal{L})$  be a global section corresponding to the fixed component of our linear system. Dividing all global sections by  $s$  and by  $p^*s$  respectively we obtain the following commutative diagram:

$$\begin{array}{ccc} H^0(M, \text{End}_0(\mathcal{E}) \otimes \omega) & \xrightarrow{\cong} & H^0(\mathbb{P}(E), p^*(\omega \otimes \Lambda^2 \mathcal{E})(2)) \\ \downarrow /s \cong & & \downarrow /p^*s \cong \\ H^0(M, \text{End}_0(\mathcal{E}) \otimes \omega \otimes \mathcal{L}^*) & \xrightarrow{\cong} & H^0(\mathbb{P}(E), p^*(\mathcal{L}^* \otimes \omega \otimes \Lambda^2 \mathcal{E})(2)) \end{array}$$

By definition, the new linear system does not have any fixed divisors and has the same dimension. Hence we can apply Bertini to conclude that a generic  $\Phi' \in H^0(M, \text{End}_0(\mathcal{E}) \otimes \omega \otimes \mathcal{L}^*)$  does not have invariant line bundles, which in this case are defined as line bundles  $N \subseteq E$  such that  $\Phi'(N) \subseteq N \otimes K \otimes L^*$ . But a line bundle  $N \subseteq E$  is  $\Phi'$ -invariant if and only if it is  $s\Phi'$ -invariant, so we have  $E \in \mathbf{A}$  in this case.

We move on to case b). Assume that the fixed divisor corresponds to a non-zero global section  $s \in H^0(\mathbb{P}(E), p^*\mathcal{L}(1))$ . This corresponds to a non-zero morphism  $E \rightarrow L$ . The fibre-wise kernel has then dimension 1 generically and 2 at special points by upper semi-continuity [Ati89, Proposition 1.3.2]. Hence we can find a line bundle  $N \subseteq \text{Ker}(s) \subseteq E$ ,

which can be described as the largest line subbundle of  $E$  contained in the kernel of  $s$ . If  $v \in N$  is a non-zero vector, then  $s(v) = 0$  and so  $[v] \in \operatorname{div}(s) \subseteq \operatorname{div}(\alpha(\Phi))$ . Thus the corresponding section  $\sigma(M) \subseteq \mathbb{P}(E)$  is contained in  $\operatorname{div}(\alpha(\Phi))$  for all  $\Phi$  and  $N$  is  $\Phi$ -invariant for all  $\Phi$ .  $\square$

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