# HIGGS BUNDLES — EXISTENCE OF SOLUTIONS

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ABSTRACT. In this talk we introduce the stability condition for Higgs bundles and prove the Hitchin–Kobayashi correspondence. The main result is [Hit87, Theorem 4.3]. Relevant literature is [Hit87, §3 and §4] and [Wen14, §2 and §3]. Maybe we will also use [Wen16] every now and then.

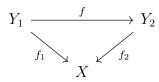
This talk is related to Tanuj's talk on *Stable vector bundles*, for which the main reference is [Kob87]. Therefore we will also use [Kob87] as a main reference for generalities on complex vector bundles.

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### NOTATION AND CONVENTIONS

- **Top** is the category of topological spaces.
- **Diff** is the category of smooth manifolds.
- If **C** is a category and  $X \in \mathbf{C}$  is an object, then  $\mathbf{C}/X$  denotes the category of objects of **C** over X, i.e. the category whose objects are morphisms  $f: Y \to X$  in **C** and whose morphisms are commutative triangles in **C** of the form



We will often talk about Y instead of  $f: Y \to X$ , leaving this structure morphism implicit.

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- Let C be a category which has **Top** as an underlying category, e.g. **Diff**. Let  $X \in C$  and  $f \in C/X$  and let P be a property of morphisms in C. We will say that f has some property P locally on X if every point  $x \in X$  has an open neighbourhood  $x \in U \subseteq X$  in X such that the morphis  $f|_{f^{-1}(U)}: f^{-1}(U) \to U$  has the property P.
- The category of group objects in a category **C** will be denoted by **CGrp**.
- Let C be a category and let  $X \in \mathbb{C}$  and  $G \in \mathbb{CGrp}$ . A left action of G on X, denoted  $G \odot X$ , is a morphism

$$\rho \colon G \times X \to X$$

such that the following diagrams commute:

We can similarly define right actions.

• Let **C** be a category. Let  $X \in \mathbf{C}$  and  $G \in \mathbf{CGrp}$ . Then we say that X is a G-object of **C**. A morphism of G-objects of **C** is a G-equivariant morphism  $f: X_1 \to X_2$ , meaning that the following diagram commutes:

$$G \times X_1 \xrightarrow{\operatorname{id}_G \times f} G \times X_2$$

$$\downarrow^{\rho_1} \qquad \qquad \downarrow^{\rho_2}$$

$$X_1 \xrightarrow{f} X_2$$

The category of G-objects of C is denoted G-C.

## 1. Self-duality

We consider  $\mathbb{R}^4$  with its standard smooth structure [Lee13, Example 1.22].

#### APPENDIX A. PRINCIPAL BUNDLES ON SMOOTH MANIFOLDS

In this appendix we recall the basics of principal G-bundles on smooth manifolds, where G is a Lie group.

**Definition A.1** (Lie group). A *Lie group* is a group object in the category **Diff** of smooth manifolds.

Remark A.2.  $G \in \mathbf{DiffGrp}$  if and only if its underlying set is equipped with a group structure such that the map  $G \times G \to G$  given by  $(g, h) \to gh^{-1}$  is smooth [Lee13, Proposition 7.1].

Let  $M \in \mathbf{Diff}$  and  $G \in \mathbf{DiffGrp}$ . Then the projection  $\pi \colon M \times G \to M$  has some nice properties, namely:

- $G \supseteq M \times G$  smoothly and fibrewise via  $(x, g) \cdot h \mapsto (x, gh)$ .
- For all  $x \in M$ ,  $G \supseteq \pi^{-1}(x)$  induces  $G \cong \{x\} \times G \cong \pi^{-1}(x)$ .

The smooth manifold  $M \times G$  over M equipped with this right fibrewise action is called the *trivial principal G-bundle* on M. We can encode all this structure by saying that

$$M \times G \in (G\text{-Diff})/M$$
,

where we consider M with the trivial G-action.

**Definition A.3** (Principal bundle). Let  $M \in \mathbf{Diff}$  and  $G \in \mathbf{DiffGrp}$ . Consider  $M \in G$ -**Diff** with the trivial action. A *principal G-bundle* on M is an object  $P \in (G$ -**Diff**)/M which is trivial locally on M.

**Example A.4.** Let  $M \in \mathbf{Diff}$  and  $G := \mathrm{GL}(n, \mathbb{R}) \in \mathbf{DiffGrp}$ . Then the *frame bundle* of M, denoted  $\mathrm{GL}(M)$ , is the principal G-bundle whose fibra over  $x \in M$  is the set of all frames for the tangent space  $T_xM$ .

**Example A.5.** Let  $G \in \mathbf{Diff}$  and  $H \subseteq G$  a closed subgroup. Then G is a principal H-bundle over the left coset space G/H.

Some nice properties in the topological category, which probably extend to the smooth category (check!):

- **Proposition A.6.** i) Any morphism of principal G-bundles is an isomorphism.
  - ii) A principal G-bundle is trivial if and only if it admits a section, where trivial means isomorphic to a trivial principal G-bundle.

### References

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