HIGGS BUNDLES — EXISTENCE OF SOLUTIONS

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Abstract. In this talk we introduce the stability condition for Higgs bundles and prove the Hitchin–Kobayashi correspondence. The main result is [Hit87, Theorem 4.3]. Relevant literature is [Hit87, §3 and §4] and [Wen14, §2 and §3]. Maybe we will also use [Wen16] every now and then.

This talk is related to Tanuj's talk on *Stable vector bundles*, for which the main reference is [Kob87]. Therefore we will also use [Kob87] as a main reference for generalities on complex vector bundles.

Contents

Notation and conventions

We will try to follow the notation and conventions of the main reference [Hit87] most of the time. M will denote a compact Riemann surface genus g. We denote by O the trivial line bundle on M and by K the canonical line bundle on M. All morphisms, vector bundles and sections are holomorphic unless otherwise specified.

The slope of a vector bundle V on M will be denoted by

$$\mu(V) := \frac{\deg V}{\operatorname{rk} V}.$$

The dimension of $H^i(M,V)$ will be denoted by $h^i(M,V)$ or simply by $h^i(V)$ if it is clear over which space we are taking the sections.

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When tensoring several bundles we will omit the symbols $\otimes$ unless confusion may result from other operators involved, for example

$$AB \otimes \operatorname{End}(C) \text{ or } \Lambda^2 A \otimes BC.$$

1. Stability

Definition 1.1 (Higgs bundles)**.** A *Higgs bundle* on M is a pair (V, Φ), where V is a rank 2 vector bundle on M and Φ is a global section of $\operatorname{End} V \otimes K$, called a *Higgs field* on V.

Remark 1.2. Using the canonical isomorphisms

$$H^0(M, \operatorname{End} V \otimes K) \cong \operatorname{Hom}(O, V^* \otimes V \otimes K) \cong \operatorname{Hom}(V, V \otimes K)$$

we will identify Φ with a morphism of vector bundles

$$\Phi \colon V \to V \otimes K.$$

Remark 1.3. We are mainly interested in the subbundle of traceless endomorphisms, denoted $\operatorname{End}_0 V$.

Example 1.4. Assume $g \geqslant 2$. Then $\deg K = 2g - 2 > 0$, so we can find a line bundle $K^{\frac{1}{2}}$ such that $K^{\frac{1}{2}} \otimes K^{\frac{1}{2}} \cong K$. Let $V = K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}$, where $K^{-\frac{1}{2}} = (K^{\frac{1}{2}})^{-1}$. We consider the Higgs field $\Phi_w \colon K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}} \to (K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}) \otimes K$ given by a matrix

$$\begin{pmatrix} 0 & w \\ 1 & 0 \end{pmatrix},$$

where $w \in \operatorname{Hom}(K^{-\frac{1}{2}}, K^{\frac{1}{2}} \otimes K) \cong H^0(M, K^2)$ can be regarded as a quadratic differential.

Definition 1.5 (Stability of Higgs bundles)**.** A Higgs bundle (V, Φ) on M is said to be *stable* if for every Φ-invariant[1] line bundle $L \subseteq V$ we have $\mu(L) < \mu(V)$.

Remark 1.6. $(V, 0)$ is stable if and only if V is stable in the usual sense.

Example 1.7 (Omit during the talk)**.** On $\mathbb{P}^1$, we can write every rank 2 vector bundle as $V \cong O(a) \oplus O(b)$ for some integers $a \geqslant b$, where $O(-1)$ is the tautological line bundle on $\mathbb{P}^1$ and $O(a) := O(-1)^{-a}$. Let $\Phi \in \operatorname{Hom}(V, V \otimes K)$ be given by the matrix

$$\begin{pmatrix} 0 & \theta_1 \\ \theta_2 & 0 \end{pmatrix}$$

with $\theta_1 \in \operatorname{Hom}(O(b), O(a) \otimes K)$ and $\theta_2 \in \operatorname{Hom}(O(a), O(b) \otimes K)$. Since $K \cong O(-2)$, we can also regard θ_1 as a global section of $O(a - b - 2)$

[1] Meaning that $\Phi(L) \subseteq L \otimes K$.

and θ_2 as a global section of $O(b-a-2)$. Since $a \geqslant b$, the line bundle $O(b-a-2)$ does not have any global sections. Hence $\theta_2 = 0$ and $O(a) \subseteq V$ is Φ-invariant. But

$$\deg O(a) = a \geqslant \frac{a+b}{2} = \frac{\deg V}{2},$$

so (V, Φ) cannot be a stable Higgs bundle.

Example 1.8. Assume $g \geqslant 2$ and consier $V = K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}$ again. Then Φ_0 is stable, because $K^{-\frac{1}{2}}$ is the only Φ_0-invariant line bundle and

$$\deg K^{-\frac{1}{2}} = 1 - g < 0 = \frac{\deg V}{2}.$$

Proposition 1.9. *Assume $g \geqslant 2$. A rank 2 vector bundle V occurs in a stable Higgs bundle (V, Φ) if and only if one of the following holds:*

i) V is stable;
ii) V is semi-stable and $g > 2$;
iii) V is semi-stgable, $g = 2$ and $V \cong U \otimes L$, where U is either a direct sum of line bundles or an extension of the form

$$0 \to O \to U \to O \to 0.$$

iv) V is not semi-stable and $h^0(L_V^{-2} \otimes K \otimes \det V) > 1$, where $L_V \subseteq V$ is the unique rank 1 subbundle with $\mu(L_V) \geqslant \mu(V)$;
v) V is a direct sum of line bundles of the form

$$V \cong L_V \oplus (L_V^{-1} \otimes \det V)$$

and $h^0(L_V^{-2} \otimes K \otimes \det V) = 1$, where $L_V \subseteq V$ is again the unique rank 1 subbundle with $\mu(L_V) \geqslant \mu(V)$.

Proof. Let $p\colon P(V) \to M$ be the projectivisation of our rank 2 vector bundle, which is the $\mathbb{P}^1$-bundle obtained by replacing each fibre V_x by its projectivisation $(V_x \setminus \{0\})/\mathbb{C}^\times$. Let $S \subseteq p^*V$ be the tautological line bundle on $P(V)$, whose fibre over a point $[v] \in p^{-1}(x)$ is given by the line $\{\lambda v \mid \lambda \in \mathbb{C}\} \subseteq V_x$. Let $H := S^*$ be its dual, which fits into a short exact sequence

$$0 \to Q^* \to p^*V^* \to H \to 0.$$

Let $U \subseteq M$ be an open subset trivialising V. Then the quotient map $p^*V^* \to H$ in the previous short exact sequence induces an isomorphism

$$H^0(p^{-1}(U), p^*V^*) \cong H^0(p^{-1}(U), H),$$

so the pushforward of the sheaf of sections of p^*V^* is isomorphic to the pushforward of the sheaf of sections of H. Since p has connected fibres we have $p_*\mathscr{O}_{P(V)} \cong \mathscr{O}_M$, so applying the projection formula [Har77, Exercise II.5.1.d] we deduce that the pushforward of the sheaf of sections of H is isomorphic to the sheaf of sections of V^*. Abusing slightly the notation we will express this as $V^* \cong p_* H$, and similarly we have $\operatorname{Sym}^2 V^* \cong p_* H^2$.

Let $x \in M$. Then every endomorphism $A \in \operatorname{End}(V_x)$ defines a quadratic map $V_x \to \Lambda^2 V_x$ sending v to $Av \wedge v$. Such a quadratic map can be naturally regarded as a degree 2 homogeneous polynomial on the coordinates coordinates of v with coefficients in $\Lambda^2 V_x$. Hence we have a vector bundle morphism $\operatorname{End}(V) \to S^2 V \otimes \Lambda^2 V$. Restricting to $\operatorname{End}_0(V)$ we get an injective morphism, because $A \in \operatorname{End}(V_x)$ is sent to $0 \in S^2 V_x \otimes \Lambda^2 V_x$ if and only if A is a multiple of the identity. Counting dimensions we see that we have in fact an isomorphism of vector bundles $\operatorname{End}_0(V) \cong S^2 V \otimes \Lambda^2 V$, and therefore

$$\operatorname{End}_0(V) \otimes K \cong p_* H^2 K \otimes \Lambda^2 V.$$

Using again that $p_* p^*(-) \cong (-)$ for vector bundles we have a $\mathbb{C}$-linear isomorphism

$$s \colon H^0(M, \operatorname{End}_0(V) \otimes K) \cong H^0(P(V), H^2 p^*(K \otimes \Lambda^2 V)).$$

$\square$

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