HIGGS BUNDLES — EXISTENCE OF SOLUTIONS

PEDRO NÚÑEZ

Abstract. In this talk we introduce the stability condition for Higgs bundles and prove the Hitchin–Kobayashi correspondence. The main result is [Hit87, Theorem 4.3]. Relevant literature is [Hit87, §3 and §4] and [Wen14, §2 and §3]. Maybe we will also use [Wen16] every now and then.

This talk is related to Tanuj's talk on *Stable vector bundles*, for which the main reference is [Kob87]. Therefore we will also use [Kob87] as a main reference for generalities on complex vector bundles.

Contents

Notation and conventions

All morphisms, vector bundles and sections are holomorphic unless otherwise specified.

1. Stability

[explain that the self-duality equations were originally talking about some principal G-bundle on space time $\mathbb{R}^4$, but after a series of simplifications [Hit87, §1] we ended up talking about rank 2 complex vector bundles on a compact Riemann surface]

[picture of moduli space of holomorphic rank 2 vector bundles]

[Upshot: a rank 2 holomorphic vector bundle doesn't quite determine a solution to the self-duality equation, but almost! Vector bundle + tangent vector in its moduli space does.]

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[Use fact from moduli space + Serre duality to translate this tangent vector into a Higgs field]

From now on let M be a compcat Riemann surface (of genus $g \geqslant 2$) and $K \to M$ its canonical line bundle.

Definition 1.1 (Higgs bundles)**.** A *Higgs bundle* on M is a pair (V, Φ), where $V \to M$ is a vector bundle on M and $\Phi \colon M \to K \otimes \operatorname{End}(V)$ is a global section of $K \otimes \operatorname{End} V$. We call Φ a *Higgs field* on V.

Remark 1.2. Φ can be regarded as morphism

$$\Phi \colon V \to V \otimes K$$

because $K \otimes \operatorname{End} V \cong K \otimes V \otimes V^\vee \cong \operatorname{Hom}(V, V \otimes K)$.

Remark 1.3. We are mainly interested in rank 2 vector bundles and traceless endomorphisms.

Example 1.4. Let M have genus $g > 1$. Then $\deg K = 2g - 2 > 0$, so we can find a line bundle $K^{\frac{1}{2}}$ such that

$$K^{\frac{1}{2}} \otimes K^{\frac{1}{2}} \cong K.$$

Let $V = K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}$, where $K^{-\frac{1}{2}} = (K^{\frac{1}{2}})^{-1}$. We consider the Higgs field $\Phi_w \colon K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}} \to (K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}) \otimes K$ given by a matrix

$$\begin{pmatrix} 0 & w \\ 1 & 0 \end{pmatrix},$$

where $w \in \operatorname{Hom}(K^{-\frac{1}{2}}, K^{\frac{1}{2}} \otimes K) \cong H^0(M, K^2)$ can be regarded as a quadratic differential.

Therefore we have a family $\{(V, \Phi_w)\}_w$ of Higgs bundles on M parametrised by quadratic differential forms w on M.

Let us denote the *slope* of a vector bundle $V \to M$ as

$$\mu(V) := \frac{\deg V}{\operatorname{rk} V}.$$

Definition 1.5 (Stability of Higgs bundles)**.** Let (V, Φ) be a Higgs bundle on M. A vector subbundle $W \subseteq V$ is said to be Φ-*invariant* if $\Phi(W) \subseteq W \otimes K$. We say that (V, Φ) is...

i) *stable* if for every Φ-invariant subbundle $0 \subsetneq W \subsetneq V$ we have $\mu(W) < \mu(V)$.
ii) *semi-stable* if for every Φ-invariant subbundle $0 \subsetneq W \subsetneq V$ we have $\mu(W) \leqslant \mu(V)$.

Remark 1.6. The Higgs bundle $(V, 0)$ is (semi-)stable if and only if V is (semi-)stable in the usual sense, because every vector subbundle is 0-invariant.

Example 1.7 (Omit during the talk)**.** On $\mathbb{P}^1$, we can write every rank 2 vector bundle as

$$V \cong O(a) \oplus O(b)$$

for some integers $a \geqslant b$, where $O(-1)$ is the tautological line bundle on $\mathbb{P}^1$ and $O(a) := O(-1)^{-a}$. Let $\Phi \in \operatorname{Hom}(V, V \otimes K)$ be given by the matrix

$$\begin{pmatrix} 0 & \theta_1 \\ \theta_2 & 0 \end{pmatrix}$$

with $\theta_1 \in \operatorname{Hom}(O(b), O(a) \otimes K)$ and $\theta_2 \in \operatorname{Hom}(O(a), O(b) \otimes K)$. Since $K \cong O(-2)$, we can also regard

$$\theta_1 \in H^0(\mathbb{P}^1, O(a-b-2)) \text{ and } \theta_2 \in H^0(\mathbb{P}^1, O(b-a-2)).$$

Since $a \geqslant b$, the line bundle $O(b-a-2)$ does not have any global sections. Hence $\theta_2 = 0$ and $O(a) \subseteq V$ is Φ-invariant. But

$$\deg O(a) = a \geqslant \frac{a+b}{2} = \frac{\deg V}{2},$$

so (V, Φ) cannot be a stable Higgs bundle.

Example 1.8. Let M have genus $g \geqslant 2$ and consier $V = K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}$ again. Then Φ_0 is stable, because $K^{-\frac{1}{2}}$ is the only Φ_0-invariant line bundle and

$$\deg K^{-\frac{1}{2}} = 1 - g < 0 = \frac{\deg V}{2}.$$

Proposition 1.9. *Let M have genus $g \geqslant 2$. a rank 2 vector bundle V occurs in a stable Higgs bundle (V, Φ) if and only if one of the following holds:*

i) V is stable;
ii) V is semi-stable and $g > 2$;
iii) V is semi-stgable, $g = 2$ and $V \cong U \otimes L$, where U is either a direct sum of line bundles or an extension of the form

$$0 \to O \to U \to O \to 0,$$

where $O = M \times \mathbb{C} \to M$ is the trivial line bundle.
iv) V is not semi-stable and $h^0(M, L_V^{-2} \otimes K \otimes \det V) > 1$, where $L_V \subseteq V$ is the unique rank 1 subbundle with $\mu(L_V) \geqslant \mu(V)$;
v) V is a direct sum of line bundles of the form

$$V \cong L_V \oplus (L_V^{-1} \otimes \det V)$$

and $h^0(M, L_V^{-2} \otimes K \otimes \det V) = 1$, where $L_V \subseteq V$ is again the unique rank 1 subbundle with $\mu(L_V) \geqslant \mu(V)$.

Proof. Let $p\colon \mathbb{P}(V) \to M$ be the projectivisation of our rank 2 vector bundle, which is the $\mathbb{P}^1$-bundle obtained by replacing each fibre V_x of $V \to M$ by its projectivisation $\mathbb{P}(V_x) := (V_x \setminus \{0\})/\mathbb{C}^\times$. Let $S \subseteq p^*V$ be the tautological line bundle on $\mathbb{P}(V)$, whose fibre over a point $([v], x) \in p^{-1}(x)$ is given by the line

$$\{(\lambda v, x) \in V_x \times \{x\} \mid \lambda \in \mathbb{C}\}.$$

Let $H := S^\vee$ be its dual, which fits into a short exact sequence

$$0 \to Q^\vee \to p^*V^\vee \to H \to 0.$$

Let $\varnothing \neq U \subseteq M$ be an open subset trivialising V. Then the quotient map $p^*V^\vee \to H$ induces an isomorphism

$$H^0(p^{-1}(U), p^*V^\vee) \cong H^0(p^{-1}(U), H),$$

so the pushforward of the sheaf of sections of $p^*V^\vee$ is isomorphic to the pushforward of the sheaf of sections of H. Since p has connected fibres we have $p_*\mathscr{O}_{\mathbb{P}(V)} \cong \mathscr{O}_M$, so applying the projection formula [Har77, Exercise II.5.1.d] we deduce that the pushforward of the sheaf of sections of H is isomorphic to the sheaf of sections of $V^\vee$. Abusing slightly the notation, from now on we will write these kind of statements as

$$V^\vee \cong p_*H.$$

With the same argument we can show that $\operatorname{Sym}^2 V^\vee \cong p_*H^2$. □

References

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Pedro Núñez, Mathematisches Institut, Albert-Ludwigs-Universität Freiburg, Ernst-Zermelo-Strasse 1, 79104 Freiburg im Breisgau, Germany

Email address: pedro.nunez@math.uni-freiburg.de

URL: https://home.mathematik.uni-freiburg.de/nunez