

# HIGGS BUNDLES — EXISTENCE OF SOLUTIONS

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ABSTRACT. In this talk we introduce the stability condition for Higgs bundles and prove the Hitchin–Kobayashi correspondence. The main result is [Hit87, Theorem 4.3]. Relevant literature is [Hit87, §3 and §4].

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## 1. STABILITY

Let  $M$  be a compact Riemann surface.

**Definition** (Higgs bundle). A *Higgs bundle* on  $M$  is a pair  $(E, \Phi)$ , where  $E \rightarrow M$  is a rank 2 vector bundle and  $\Phi$  is a global section of  $\text{End } E \otimes K$ , called a *Higgs field* on  $E$ .

*Remark.* Using the canonical isomorphisms

$$H^0(M, \text{End}(\mathcal{E}) \otimes \omega) \cong \text{Hom}(\mathcal{O}, \mathcal{E}^\vee \otimes \mathcal{E} \otimes \omega) \cong \text{Hom}(\mathcal{E}, \mathcal{E} \otimes \omega)$$

we may identify  $\Phi$  with a morphism

$$\Phi : E \rightarrow E \otimes K.$$

**Definition** (Stability). A Higgs bundle  $(E, \Phi)$  is said to be *stable* if for every  $\Phi$ -invariant line bundle  $L \subseteq E$  we have  $\mu(L) < \mu(E)$ , where  $\Phi$ -invariance means that  $\Phi(L) \subseteq L \otimes K$ .

*Remark.*  $(E, 0)$  is stable if and only if  $E$  is stable in the usual sense.

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**Exercise A.** There are no stable Higgs bundles on  $\mathbb{P}^1$ . [*Hints below*<sup>1</sup>]

**Lemma 1.** Let  $(E_1, \Phi_1)$  and  $(E_2, \Phi_2)$  be stable pairs with  $\Lambda^2 E_1 \cong \Lambda^2 E_2$ . Let  $\Psi : E_1 \rightarrow E_2$  be a non-zero morphism such that  $(\Psi \otimes \text{id}_K) \circ \Phi_1 = \Phi_2 \circ \Psi$ . Then  $\Psi$  is an isomorphism.

*Proof.* We prove the result by contradiction. Suppose that  $\Psi$  is not an isomorphism. The rank  $x \mapsto \dim_{\mathbb{C}} \Psi_x(E_{1,x})$  is upper semi-continuous [Ati89, Proposition 1.3.2], so the rank of  $\Psi$  cannot be generically zero. If the rank was generically 2, then  $\det(\Psi) \in H^0(M, \Lambda^2 \mathcal{E}_1^\vee \otimes \Lambda^2 \mathcal{E}_2)$  would be generically non-zero. But  $\Lambda^2 E_1 \cong \Lambda^2 E_2$ , so  $\det(\Psi) \in H^0(M, \mathcal{O}) = \mathbb{C}$  must be a constant and  $\Psi$  would be an isomorphism. Therefore the rank is generically 1, only going down to 0 at special points.

Let  $L_1 \subseteq E_1$  be the largest rank 1 subbundle of  $E_1$  contained in the kernel of  $\Psi$ . Let  $v_1 \in L_{1,x}$ , and let  $z$  be a holomorphic coordinate around a general point  $x \in M$ . Then we can write  $\Phi_1(v_1) = \phi_{1,x}(v_1) \otimes dz$  for some  $\phi_{1,x} \in \text{End}(E_{1,x})$ . Then

$$0 = \Phi_{2,x}(\Psi_x(v_1)) = (\Psi \otimes \text{id}_K)_x(\phi_{1,x}(v_1) \otimes dz) = \Psi_x(\phi_{1,x}(v_1)) \otimes dz,$$

so  $\phi_{1,x}(v_1) \in \text{Ker}(\Psi_x) = L_{1,x}$ . Since it suffices to check  $\Phi_1(L_1) \subseteq L_1 \otimes K$  generically, this shows that  $L_1$  is  $\Phi_1$ -invariant.

Let now  $L_2 \subseteq E_2$  be the largest rank 1 subbundle of  $E_2$  containing the image of  $\Psi$ . Let  $v_2 = \Psi(v_1) \in L_{2,x}$  be a vector over a general point  $x \in M$ , which can thus be written as the image under  $\Psi$  of someone in  $E_1$ . Then

$$\Phi_{2,x}(\Psi_x(v_1)) = (\Psi \otimes \text{id}_K)_x(\phi_{1,x}(v_1) \otimes dz) = \Psi_x(\phi_{1,x}(v_1)) \otimes dz,$$

so  $\phi_{2,x}(v_2) \in \text{Im}(\Psi_x) = L_{2,x}$ . Since it suffices to check  $\Phi_2(L_2) \subseteq L_2 \otimes K$  generically, this shows that  $L_2$  is  $\Phi_2$ -invariant.

Now we use that  $(E_i, \Phi_i)$  are stable to deduce that

$$\deg(L_i) < \frac{d}{2}$$

for  $i \in \{1, 2\}$ , where  $d := \deg(\Lambda^2 E_1) = \deg(\Lambda^2 E_2)$ . Since  $L_1$  is contained in the kernel of  $\Psi$ ,  $\Psi$  induces a non-zero morphism of line bundles  $E_1/L_1 \rightarrow L_2$ , which corresponds to a non-zero global section of  $(E_1/L_1)^\vee \otimes L_2$ . Line bundles with negative degree do not have any non-zero global sections, so we must have  $\deg(E_1/L_1) \leq \deg(L_2)$ . Therefore

$$\frac{d}{2} < \deg(\Lambda^2 E_1) - \deg(L_1) = \deg(E_1/L_1) \leq \deg(L_2) < \frac{d}{2},$$

a contradiction. Hence  $\Psi$  must be an isomorphism.  $\square$

<sup>1</sup>Grothendieck's theorem allows us to write  $\Phi$  as a matrix. What can we say about each entry? The solution can be found in [Hit87, Remark (3.2) (iii)]

**Lemma 2.** *Let  $E \rightarrow M$  be a rank 2 vector bundle and denote by  $\text{End}_0(E)$  the vector bundle of traceless endomorphisms. Then there is a natural projection  $\text{pr}_0 : \text{End}(E) \rightarrow \text{End}_0(E)$  whose kernel is the trivial line bundle of multiples of the identity, yielding a short exact sequence*

$$0 \rightarrow \mathcal{O} \rightarrow \mathcal{E}nd(\mathcal{E}) \rightarrow \mathcal{E}nd_0(\mathcal{E}) \rightarrow 0.$$

*In particular,  $\deg(\text{ch}(\mathcal{E}nd_0(\mathcal{E}) \otimes \omega) \text{td}(\omega^\vee))_1 = 3g - 3$ .*

*Proof.* Over  $x \in M$ , the map  $\text{pr}_{0,x} : \text{End}(E_x) \rightarrow \text{End}_0(E_x)$  is given by

$$A \mapsto A - \frac{\text{tr}(A)}{2} \text{id}_{E_x}.$$

The endomorphisms in the kernel are precisely the multiples of the identity. This fibre-wise description globalizes to the desired short exact sequence.

For the Chern class computation we use the axioms in [Har77, Appendix A]. We have

$$c_1(\mathcal{E}nd_0(\mathcal{E})) = c_1(\mathcal{E}nd(\mathcal{E})) = c_1(\mathcal{E}^\vee \otimes \mathcal{E}) = 0,$$

therefore

$$\text{ch}(\mathcal{E}nd_0(\mathcal{E}) \otimes \omega) = 3 + c_1(\mathcal{E}nd_0(\mathcal{E})) + 3c_1(\omega) = 3 + 3c_1(\omega).$$

We also have

$$\text{td}(\omega^\vee) = 1 - \frac{c_1(\omega)}{2},$$

so multiplying the two expressions we obtain

$$\text{ch}(\mathcal{E}nd_0(\mathcal{E}) \otimes \omega) \text{td}(\omega^\vee) = 3 + \frac{3}{2}c_1(\omega).$$

Since  $\deg(c_1(\omega)) = 2g - 2$ , the result follows.  $\square$

*Notation.* Let us denote by  $\mathbf{Vec}_2(M)$  the set of rank 2 vector bundles on  $M$ . We define the following subsets:

- $\mathbf{S} := \{E \in \mathbf{Vec}_2(M) \mid \exists \Phi \text{ such that } (E, \Phi) \text{ is stable}\}.$
- $\mathbf{A} := \{E \in \mathbf{Vec}_2(M) \mid \text{a general } \Phi \text{ has no invariant } L\}.$
- $\mathbf{B} := \{E \in \mathbf{Vec}_2(M) \mid \exists L \text{ invariant for all } \Phi\}.$

**Lemma 3.** *If  $g \geq 2$ , then*

$$\mathbf{Vec}_2(M) = \mathbf{A} \sqcup \mathbf{B}.$$

*Proof.* Let  $\pi : \mathbb{P}(E) \rightarrow M$  be the projectivization of our rank 2 vector bundle and let  $\mathcal{O}(-1) \rightarrow \mathbb{P}(E)$  denote the tautological line bundle, whose fiber over  $[v] \in \mathbb{P}(E)$  is the line  $\{\lambda v \mid \lambda \in \mathbb{C}\} \subseteq E_{\pi([v])}$  spanned by  $v$ . Denote also  $\mathcal{O}(l) := \mathcal{O}(-1)^{\otimes(-l)}$ . If  $\mathcal{F}$  is a sheaf on  $\mathbb{P}(E)$ , we denote  $\mathcal{F}(l) := \mathcal{F} \otimes \mathcal{O}(l)$ , where  $\mathcal{O}(l)$  denotes the sheaf of sections of  $\mathcal{O}(l)$ . We have  $\pi_* \mathcal{O}(l) = S^l(\mathcal{E}^\vee)$  for all  $l \geq 0$  and  $\pi_* \mathcal{O}(l) = 0$  for all  $l < 0$  [Har77, Exercise III.8.4].

Let  $x \in M$ . Given  $A \in \text{End}(E_x)$ , we define the quadratic form  $v \mapsto Av \wedge v$  with values in  $\Lambda^2 E_x$ , which can then be naturally regarded as an element in  $S^2(E_x^\vee) \otimes \Lambda^2 E_x$ . The resulting quadratic form is trivial precisely when  $A = \lambda \text{id}_{E_x}$  for some  $\lambda \in \mathbb{C}$ , so by Lemma 2 we obtain an injective homomorphism  $\text{End}_0(E_x) \rightarrow S^2(E_x^\vee) \otimes \Lambda^2 E_x$ . Both vector spaces have the same dimension, so this must be an isomorphism. These isomorphisms globalize to an isomorphism  $\text{End}_0(E) \cong S^2(E^\vee) \otimes \Lambda^2 E$ , hence we obtain an isomorphism

$$\mathcal{E}nd_0(\mathcal{E}) \otimes \omega \cong \pi_* \mathcal{O}(2) \otimes \omega \otimes \Lambda^2 \mathcal{E}.$$

The projection formula yields now an isomorphism  $\mathcal{E}nd_0(\mathcal{E}) \otimes \omega \cong \pi_*(\pi^*(\omega \otimes \Lambda^2 \mathcal{E})(2))$ , hence an isomorphism

$$\psi : H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \omega) \cong H^0(\mathbb{P}(E), \pi^*(\omega \otimes \Lambda^2 \mathcal{E})(2)).$$

Let now  $\Phi \in H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \omega)$ . A line bundle  $L \subseteq E$  is then  $\Phi$ -invariant precisely when  $\psi(\Phi)$  vanishes at all  $[v] \in \mathbb{P}(L) \subseteq \mathbb{P}(E)$ . In other words,  $L$  is  $\Phi$ -invariant if and only if  $\sigma(M) \subseteq \text{div}(\psi(\Phi))$ , where  $\text{div}(-)$  denotes the divisor of zeros of a section and  $\sigma : M = \mathbb{P}(L) \rightarrow \mathbb{P}(E)$  is the section induced by  $L \subseteq E$ .

Suppose now that  $\Phi \in H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \omega)$  is not nilpotent and let  $L \subseteq E$  be a  $\Phi$ -invariant line bundle. Then over a general point  $x \in M$ , the corresponding traceless endomorphism  $\phi_x \in \text{End}_0(E_x)$  is diagonalizable, so we can find some eigenvector  $v \in E_x \setminus L_x$  in an eigenspace other than  $L_x$ . This gives us a point  $[v] \in \mathbb{P}(E) \setminus \sigma(M)$  on which  $\psi(\Phi)$  vanishes. Hence  $\sigma(M)$  is a proper irreducible component of the divisor  $\text{div}(\psi(\Phi))$ .

The previous discussion shows that if  $\Phi$  is not nilpotent and  $\text{div}(\psi(\Phi))$  is irreducible, then there are no invariant line bundles  $L \subseteq E$ . By Hirzebruch–Riemann–Roch and Lemma 2 we have

$$h^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \omega) \geq 3g - 3 \geq 3,$$

so the complete linear system defined by the invertible sheaf  $\pi^*(\omega \otimes \Lambda^2 \mathcal{E})(2)$  has dimension at least 2. If this linear system does not have a fixed divisor, then Bertini's theorem [Lit82, Theorem 7.19] tells us that  $\text{div}(\psi(\Phi))$  is irreducible for a general  $\Phi \in H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \omega)$ . Since in our case  $\Phi \in H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \omega)$  is nilpotent if and only if  $\Phi^2 = 0$ , a general  $\Phi$  is not nilpotent. Therefore  $E \in \mathbf{A}$  in this case.

Let us see what happens if the linear system has a fixed divisor. Such a fixed divisor must be the zero locus of a non-zero global section  $s$  of an invertible sheaf on  $\mathbb{P}(E)$ , which is up to isomorphism of the form  $\pi^* \mathcal{L}(l)$  with  $\mathcal{L}$  an invertible sheaf on  $M$  and  $l \in \mathbb{Z}$  [Har77, Exercise II.7.9]. Being a fixed divisor means then that every other global section of our line bundle can be written as a product  $st$ , where  $t \in H^0(\mathbb{P}(E), \pi^* \mathcal{N}(2 - l))$ . Since our line bundle had non-zero global sections, both  $\pi^* \mathcal{L}(l)$  and  $\pi^* \mathcal{N}(2 - l)$  must

have non-zero global sections. By the projection formula, this leaves us with only three possibilities:

- a)  $l = 0$ ;
- b)  $l = 1$ ;
- c)  $l = 2$ .

Let us start with case *a*). Let  $\pi^*s \in H^0(\mathbb{P}(E), \pi^*\mathcal{L}) \cong H^0(M, \mathcal{L})$  be a global section corresponding to the fixed component of our linear system. Dividing all global sections by  $s$  and by  $\pi^*s$  respectively we obtain the following commutative diagram:

$$\begin{array}{ccc} H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \omega) & \xrightarrow{\cong} & H^0(\mathbb{P}(E), \pi^*(\omega \otimes \Lambda^2 \mathcal{E})(2)) \\ \downarrow \cong /s & & \downarrow \cong / \pi^*s \\ H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \omega \otimes \mathcal{L}^\vee) & \xrightarrow{\cong} & H^0(\mathbb{P}(E), \pi^*(\mathcal{L}^\vee \otimes \omega \otimes \Lambda^2 \mathcal{E})(2)) \end{array}$$

By definition, the new linear system does not have any fixed divisors and has the same dimension. Hence we can apply Bertini to conclude that a generic  $\Phi' \in H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \omega \otimes \mathcal{L}^\vee)$  does not have invariant line bundles, which in this case are defined as line bundles  $N \subseteq E$  such that  $\Phi'(N) \subseteq N \otimes K \otimes L^\vee$ . But a line bundle  $N \subseteq E$  is  $\Phi'$ -invariant if and only if it is  $s\Phi'$ -invariant, so we have  $E \in \mathbf{A}$  in this case.

We move on to case *b*). Assume that the fixed divisor corresponds to a non-zero global section  $s \in H^0(\mathbb{P}(E), \pi^*\mathcal{L}(1))$ . This corresponds to a non-zero morphism  $E \rightarrow L$ . The fibre-wise kernel has then dimension 1 generically and 2 at special points by upper semi-continuity [Ati89, Proposition 1.3.2]. Hence we can find a line bundle  $N \subseteq \text{Ker}(s) \subseteq E$ , which can be described as the largest line subbundle of  $E$  contained in the kernel of  $s$ . If  $v \in N$  is a non-zero vector, then  $s(v) = 0$  and so  $[v] \in \text{div}(s) \subseteq \text{div}(\psi(\Phi))$ . Thus the corresponding section  $\sigma(M) \subseteq \mathbb{P}(E)$  is contained in  $\text{div}(\psi(\Phi))$  for all  $\Phi$  and  $N$  is  $\Phi$ -invariant for all  $\Phi$ . Hence  $E \in \mathbf{B}$  in this case.

In case *c*), the fixed divisor corresponds to a non-zero global section of  $\pi^*\mathcal{L}(2)$ . We have

$$H^0(\mathbb{P}(E), \pi^*\mathcal{L}(2)) \cong H^0(M, \mathcal{L} \otimes S^2 \mathcal{E}^\vee) \cong H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{L} \otimes \Lambda^2 \mathcal{E}^\vee).$$

So we can think of the fixed global section  $s$  as a traceless endomorphism of  $E$  with coefficients in  $L \otimes \Lambda^2 E^\vee$ . With this point of view,  $s$ -invariance of a line bundle  $N \subseteq E$  translates into  $s\Phi'$ -invariance of  $N \subseteq E$  as before, where  $s\Phi'$  is a Higgs field. Let us see that the fixed section  $s$  has some non-trivial kernel, hence defining a line bundle invariant under all Higgs fields as in the previous case. To show that there is some non-trivial kernel, it suffices to check that  $\det(s) = 0$ . Since  $s$  is traceless, it suffices in turn to check that  $\text{tr}(s^2) = 0$ . Suppose on the contrary that  $\text{tr}(s^2) \neq 0$ . Fix some non-zero

$s_1 \in H^0(M, \mathcal{L}^\vee \otimes \omega \otimes \Lambda^2 \mathcal{E})$  and consider the linear map

$$\begin{aligned} \theta : H^0(M, \mathcal{L}^\vee \otimes \omega \otimes \Lambda^2 \mathcal{E}) &\longrightarrow H^0(M, \omega^2) \\ \Phi' &\longmapsto \text{tr}(s^2)s_1\Phi' \end{aligned}$$

Since  $\text{tr}(s^2)s_1$  can only vanish at finitely many points, the image of a non-zero  $\Phi'$  can only vanish at finitely many points, hence  $\theta$  is injective. From Hirzebruch–Riemann–Roch and Lemma 2 we know that

$$h^0(M, \mathcal{L}^\vee \otimes \omega \otimes \Lambda^2 \mathcal{E}) \geq 3g - 3 = h^0(M, \omega^2),$$

so  $\theta$  is an isomorphism. Since  $\mathcal{L}^\vee \otimes \omega \otimes \Lambda^2 \mathcal{E}$  has global sections, its degree is non-negative. If it was zero, then this would be the trivial line bundle and we would have  $h^0(M, \mathcal{L}^\vee \otimes \omega \otimes \Lambda^2 \mathcal{E}) = 1 < 3g - 3$ , a contradiction. Hence  $\deg(\mathcal{L}^\vee \otimes \omega \otimes \Lambda^2 \mathcal{E}) > 0$  and the non-zero global section  $s_1$  has at least one zero. If  $\theta$  was indeed an isomorphism, then each zero of  $s_1$  would give a base point of the complete linear system corresponding to  $\omega^2$ . But  $\deg(\omega^2) = 4g - 4 \geq 2g$ , so this linear system has no base points [Har77, Corollary IV.3.2]. This contradiction shows that  $s$  has non-trivial kernel, which contains a line bundle  $N \subseteq E$  invariant by all  $\Phi \in H^0(M, \text{End}_0(\mathcal{E}) \otimes \omega)$ . Hence  $E \in \mathbf{B}$  as well in this case.  $\square$

**Exercise B.** Assume  $g \geq 2$ . Let  $K^{\frac{1}{2}}$  be a line bundle whose square is  $K$  and let  $K^{-\frac{1}{2}}$  be its inverse. Does  $K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}$  belong to  $\mathbf{A}$  or to  $\mathbf{B}$ ? [Hints below<sup>2</sup>]

**Lemma 4.** Let  $E \rightarrow M$  be a rank 2 vector bundle and denote by  $\text{End}_0(E)$  the vector bundle of traceless endomorphisms. Let  $L \subseteq E$  be a line bundle. Then there is an injective morphism

$$L^2 \otimes \Lambda^2 E \hookrightarrow \text{End}_0(E)$$

whose image are the traceless endomorphisms which preserve only the line bundle  $L$ . Dualizing this we obtain a surjection

$$\text{End}_0(E) \cong \text{End}_0(E^\vee) \twoheadrightarrow L^{-2} \otimes \Lambda^2 E$$

whose kernel are the traceless endomorphisms which preserve at least the line bundle  $L$ . We can realize this kernel as the inclusion  $\text{Hom}(E, L) \subseteq \text{Hom}(E, E)$

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<sup>2</sup>Consider the family of traceless endomorphisms given by

$$\Phi_\alpha := \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix},$$

parametrized by quadratic differentials  $\alpha \in H^0(M, K^2)$ . Use without proof the fact that an invariant line bundle exists if and only if the characteristic polynomial

$$\lambda^2 - \alpha$$

has a root in  $H^0(M, K)$ , i.e. if and only if the quadratic differential  $\alpha$  can be written as a square  $\alpha = \beta^2$  for some differential form  $\beta \in H^0(M, K)$ . If  $\alpha$  was a square, its zeros would all have multiplicity at least two. Conclude that  $K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}} \in \mathbf{A}$  using Bertini's theorem.

followed by the projection  $\text{pr}_0 : \text{Hom}(E, E) \rightarrow \text{End}_0(E)$ , yielding a short exact sequence

$$0 \rightarrow \mathcal{E}^\vee \otimes \mathcal{L} \rightarrow \mathcal{E}nd_0(\mathcal{E}) \rightarrow \mathcal{L}^{-2} \otimes \Lambda^2 \mathcal{E} \rightarrow 0.$$

*Proof.* Under the isomorphism  $E \cong E^\vee \otimes \Lambda^2 E$  [Har77, Exercise II.5.16], the line bundle  $L$  is sent to linear forms with coefficients in  $\Lambda^2 E$  vanishing along  $L$ , hence we have a short exact sequence

$$0 \rightarrow \mathcal{L} \rightarrow \mathcal{E} \rightarrow \mathcal{L}^\vee \otimes \Lambda^2 \mathcal{E} \rightarrow 0.$$

Dualizing this short exact sequence we obtain an inclusion  $L \otimes \Lambda^2 E^\vee \subseteq E^\vee$ . Tensoring with  $L$  and composing with the inclusion  $E^\vee \otimes L \subseteq E^\vee \otimes E$  we obtain an inclusion  $L^2 \otimes \Lambda^2 E^\vee \subseteq \text{End}(E)$ . Choosing a basis on each fibre and chasing all the identifications we have made so far, we see that the image of  $L^2 \otimes \Lambda^2 E^\vee$  lies actually in  $\text{End}_0(E)$ . Indeed, let  $V$  be a two dimensional  $\mathbb{C}$ -vector space and let  $e_1$  and  $e_2$  be a basis. Let  $L$  be the line spanned by a non-zero vector  $l$ , which we may assume to be  $e_1$ . The first identification we have is  $V \cong \text{Hom}(V, \Lambda^2 V)$ , sending  $v$  to the homomorphism  $v' \mapsto v' \wedge v$ . This corresponds to  $\alpha_v \otimes (e_1 \wedge e_2) \in V^\vee \otimes \Lambda^2 V$ , where  $\alpha_v \in V^\vee$  is the linear form sending  $e_1 \mapsto v_2$  and  $e_2 \mapsto -v_1$ . Denoting by  $\overline{\alpha_v}$  its image in  $L^\vee$ , we can describe the morphism corresponding to the right hand side of the previous short exact sequence as

$$\begin{aligned} V &\longrightarrow L^\vee \otimes \Lambda^2 V \\ v &\longmapsto \overline{\alpha_v} \otimes (e_1 \wedge e_2) \end{aligned}$$

Let now  $\beta \in (L^\vee \otimes \Lambda^2 V)^\vee$  and denote by  $\lambda_v^\beta$  the complex number such that

$$\overline{\alpha_v} \otimes (e_1 \wedge e_2) \xrightarrow{\beta} \lambda_v^\beta.$$

A point  $\mu l \otimes \beta \in L \otimes (L^\vee \otimes \Lambda^2 V)^\vee$  corresponds then to the endomorphism

$$\begin{aligned} V &\longrightarrow V \\ v &\longmapsto \mu \lambda_v^\beta l \end{aligned}$$

A basis for  $L$  is  $e_1$ , a basis for  $L^\vee \otimes \Lambda^2 V$  is  $\overline{\alpha_{e_2}} \otimes (e_1 \wedge e_2)$  and a basis for  $L \otimes (L^\vee \otimes \Lambda^2 V)^\vee$  is  $e_1 \otimes \beta_0$ , where  $\beta_0 \in (L^\vee \otimes \Lambda^2 V)^\vee$  is such that  $\lambda_{e_2}^{\beta_0} = 1$ . Writing the image of the basis  $e_1 \otimes \beta_0$  under the map  $L \otimes (L^\vee \otimes \Lambda^2 V)^\vee \rightarrow \text{End}(V)$  as a matrix with respect to our bases we obtain

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

because  $\overline{\alpha_{e_1}} = 0$  and therefore  $\lambda_{e_1}^\beta = 0$  for any  $\beta$ . We have thus the desired injective homomorphism

$$L^2 \otimes \Lambda^2 V^\vee \hookrightarrow \text{End}_0(V)$$

whose image are the traceless endomorphisms which preserve only  $L$ .

We regard this as a homomorphism into  $\text{End}(V)$  for a moment and use the basis  $e_{11}, e_{12}, e_{21}, e_{22}$  of  $\text{End}(V)$ , where  $e_{ij}$  denotes the endomorphism which, represented as a matrix in terms of our basis, has zeros everywhere except for a 1 in the  $ij$ -th position. Then our homomorphism is given by

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Dualizing it we obtain a surjection

$$\text{End}_0(V^\vee) \twoheadrightarrow L^{-2} \otimes \Lambda^2 V$$

given with respect to the dual bases by

$$\begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}.$$

Its kernel are the endomorphisms of  $V^\vee$  represented with respect to the dual basis by a matrix of the form

$$\begin{pmatrix} a & 0 \\ b & -a \end{pmatrix}.$$

Hence, under the isomorphism  $\text{End}(V) \cong \text{End}(V^\vee)$  given in coordinates by sending a matrix to its transpose, we obtain a surjection

$$\text{End}_0(V) \twoheadrightarrow L^{-2} \otimes \Lambda^2 V$$

whose kernel are endomorphisms represented with respect to our basis by matrices of the form

$$\begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}.$$

Therefore the kernel of this surjection consists precisely of the traceless endomorphisms of  $V$  that leave at least  $L$  invariant. The inclusion of this kernel can be naturally regarded as the composition of the inclusion  $\text{Hom}(V, L) \subseteq \text{Hom}(V, V)$  and the projection  $\text{pr}_0 : \text{Hom}(V, V) \rightarrow \text{End}_0(V)$ , which writing every homomorphism as a matrix with respect to the bases above has the form

$$\begin{pmatrix} a & b \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} \frac{a}{2} & b \\ 0 & -\frac{a}{2} \end{pmatrix}.$$

This gives us the desired short exact sequence

$$0 \rightarrow \mathcal{E}^\vee \otimes \mathcal{L} \otimes \omega \rightarrow \mathcal{E}nd_0(\mathcal{E}) \otimes \omega \rightarrow \mathcal{L}^{-2} \otimes \omega \otimes \Lambda^2 \mathcal{E} \rightarrow 0,$$

in which the global sections of  $\mathcal{E}^\vee \otimes \mathcal{L} \otimes \omega$  correspond to Higgs fields which leave at least  $L$  invariant.  $\square$



**Lemma 5.** *If  $g \geq 2$ , then*

$$\mathbf{S} \subseteq \mathbf{A}.$$

*Proof.* We want to show that if  $E \in \mathbf{Vec}_2(M)$  is stable, then it is in  $\mathbf{A}$ . By Lemma 3 it suffices to show that it is not in  $\mathbf{B}$ . So let  $E$  be a stable rank 2 vector bundle on  $M$  and assume  $L \subseteq E$  is a line bundle which is  $\Phi$ -invariant for all  $\Phi \in H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \omega)$ . Consider the short exact sequence from Lemma 4

$$0 \rightarrow \mathcal{E}^\vee \otimes \mathcal{L} \otimes \omega \rightarrow \mathcal{E}nd_0(\mathcal{E}) \otimes \omega \rightarrow \mathcal{L}^{-2} \otimes \omega \otimes \Lambda^2 \mathcal{E} \rightarrow 0.$$

Since all  $\Phi \in H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \omega)$  leave  $L$  invariant, we get an induced isomorphism on global sections  $H^0(M, \mathcal{E}^\vee \otimes \mathcal{L} \otimes \omega) \cong H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \omega)$ . We consider now the short exact sequence

$$0 \rightarrow \mathcal{L}^2 \otimes \omega \otimes \Lambda^2 \mathcal{E}^\vee \rightarrow \mathcal{E}^\vee \otimes \mathcal{L} \otimes \omega \rightarrow \omega \rightarrow 0,$$

more or less implicit in the proof of Lemma 4. Since  $E$  is stable, we have  $\deg(\mathcal{L}^2 \otimes \Lambda^2 \mathcal{E}^\vee) < 0$ , and since the complete linear system corresponding to  $\omega$  is base-point free<sup>3</sup> we have  $h^0(M, \omega \otimes \mathcal{L}^2 \otimes \Lambda^2 \mathcal{E}^\vee) < g$  by [Har77, Proposition IV.3.1]. The long exact sequence of the previous short exact sequence gives then

$$h^0(M, \omega \otimes \mathcal{L} \otimes \mathcal{E}^\vee) \leq 2g - 1.$$

The earlier Hirzebruch–Riemann–Roch computation showed that

$$h^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \omega) \geq 3g - 3.$$

If we want the two dimensions to be equal we must have  $g = 2$  and  $h^0(M, \omega \otimes \mathcal{L} \otimes \mathcal{E}^\vee) = 3$ . From the same long exact sequence as before we deduce, using that  $h^0(M, \omega \otimes \mathcal{L}^2 \otimes \Lambda^2 \mathcal{E}^\vee) < g = 2$ , that  $h^0(M, \omega \otimes \mathcal{L}^2 \otimes \Lambda^2 \mathcal{E}^\vee) = 1$ . In particular,  $\deg(\omega \otimes \mathcal{L}^2 \otimes \Lambda^2 \mathcal{E}^\vee) \geq 0$ . We have  $\deg(\omega) = 2$  and by stability we had  $\deg(\mathcal{L}^2 \otimes \Lambda^2 \mathcal{E}^\vee) < 0$ , so we must have

$$\deg(\omega \otimes \mathcal{L}^2 \otimes \Lambda^2 \mathcal{E}^\vee) \in \{0, 1\}.$$

If it is 0, then the existence of global sections implies that it is the trivial line bundle, hence the previous short exact sequence becomes

$$0 \rightarrow \mathcal{O} \rightarrow \mathcal{E}^\vee \otimes \mathcal{L} \otimes \omega \rightarrow \omega \rightarrow 0.$$

Split short exact sequences are preserved by dualizing and tensoring with line bundles, so if this sequence was split then  $\mathcal{E}$  would be decomposable as a direct sum of line bundles. But this would give more endomorphisms of

<sup>3</sup>Suppose that  $P \in M$  was a base-point of  $|\omega|$ . This would mean that  $h^0(\omega) = h^0(\omega \otimes \mathcal{O}(-P)) = g$ , so by Riemann–Roch we would have  $h^0(\mathcal{O}(P)) = 2$ . For any other  $Q \in M$  we would have that  $\mathcal{O}(P - Q)$  is a degree 0 line bundle with  $h^0(\mathcal{O}(P - Q)) \in \{1, 2\}$  [Har77, Proof of Proposition IV.3.1], hence the trivial line bundle. This would imply that any two points in  $M$  are linearly equivalent, hence  $M = \mathbb{P}^1$  [Har77, Example II.6.10.1], a contradiction.

$\mathcal{E}$  than there should be, since stable vector bundles are simple. So the previous short exact sequence is a non-trivial extension and the coboundary map  $H^0(M, \omega) \rightarrow H^1(M, \mathcal{O})$  is non-zero. The long exact sequence implies then that  $h^0(M, \mathcal{E}^\vee \otimes \omega \otimes \mathcal{L}) \leq 2$ , contradicting our previous conclusion that this dimension was 2.  $\square$

## 2. AN EXISTENCE THEOREM

Let  $E$  be a smooth complex vector bundle on a compact Riemann surface  $M$  of genus  $g \geq 2$ . Assume  $E$  has rank 2, so that it is determined up to smooth isomorphism by its degree  $d = \deg(\Lambda^2 E)$ . Let  $h$  be a Hermitian metric on  $E$ , that is, a smooth choice of Hermitian inner product  $\langle -, - \rangle$  on each fibre  $E_x$ . Let  $\mathcal{A}_h$  be the set of  $h$ -unitary connections on  $E$ , that is, the set of connections  $\nabla : \Gamma(E) \rightarrow \Omega^1(M, E)$  such that

$$d\langle s_1, s_2 \rangle = \langle \nabla(s_1), s_2 \rangle + \langle s_1, \nabla(s_2) \rangle.$$

Let  $F = \nabla \circ \nabla : \Gamma(E) \rightarrow \Omega^2(M, E)$  be the curvature of the connection, where we interpret the second  $\nabla$  as the extension of the induced  $\mathbb{C}$ -linear map  $\nabla : \Omega^1(E) \rightarrow \Omega^2(E)$  defined as

$$\nabla(\sigma \cdot \varphi) := \nabla(\sigma) \wedge \varphi + \sigma \cdot d\varphi.$$

We may regard  $F \in \Omega^2(M, \text{End}(E))$ , and possibly after normalizing to obtain an integer cohomology class we have then an equality

$$[\text{tr}(F)] = [F_{\Lambda^2 E}] = -2\pi i c_1(E) \in H^2(M, \mathbb{Z}),$$

where  $F_{\Lambda^2 E}$  is the curvature of the induced connection on  $\Lambda^2 E$ .

## NOTATION AND CONVENTIONS

We usually follow the notation of [Hit87]:

- $M$ : compact Riemann surface of genus  $g$ .
- $\mathcal{O} \rightarrow M$ : trivial line bundle.
- $K \rightarrow M$ : canonical line bundle.
- More generally,  $\mathcal{O}_X$  and  $K_X$  denote the trivial and canonical line bundles over a complex manifold  $X$ .

Every now and then we will also use some other standard notation, for instance:

- For a vector bundle  $E \rightarrow M$  we denote  $\mu(E) := \deg E / \text{rk } E$ .
- Let  $X$  be a complex manifold and  $E \rightarrow X$  a (holomorphic/algebraic) vector bundle. Then we denote by  $\mathcal{E}$  its sheaf of sections. The assignment  $E \mapsto \mathcal{E}$  defines an equivalence of categories between vector bundles on  $X$  and locally free sheaves of  $\mathcal{O}_X$ -modules of finite rank, which we will refer to simply as locally free sheaves. We

can recover  $E$  from  $\mathcal{E}$  either using cocycles [Voi02, Lemma 4.8] or by using the relative spectrum [Har77, Exercise II.5.18]. Following the second approach we would write

$$E = \operatorname{Spec}_X(S(\mathcal{E}^\vee)) =: \mathbb{V}(\mathcal{E}^\vee),$$

where  $S(-)$  denotes the symmetric algebra.

- $\mathcal{O}$  and  $\omega$  denote the trivial and canonical invertible sheaves on  $M$ . More generally,  $\mathcal{O}_X$  and  $\omega_X$  denote the trivial and canonical invertible sheaves on a complex manifold  $X$ ,
- Let  $E$  be again a vector bundle on a complex manifold  $X$ . We will denote its projectivisation by  $\mathbb{P}(E)$ , which is obtained from  $E$  without its zero section by quotienting out the fibre-wise multiplication by scalars. Taking again the algebraic approach we can write

$$\mathbb{P}(E) = \operatorname{Proj}_X(S(\mathcal{E}^\vee)) =: \mathbb{P}(\mathcal{E}^\vee).$$

- Let  $N$  be a smooth manifold and  $E \rightarrow N$  a smooth vector bundle. Then we denote by  $\mathfrak{X}(N)$  the Lie algebra of smooth vector fields on  $N$  and by  $\Omega^k(N, E)$  the vector space of smooth  $k$ -differential forms with values in  $E$ , which can be thought of as smooth global sections of the vector bundle  $\operatorname{Hom}(TN, E)$ .
- Let  $N$  be a smooth manifold equipped with an almost complex structure  $I : TN \rightarrow TN$ . Then we denote by  $\Omega^{i,j}(N, E) \dots$

## REFERENCES

- [Ati89] M. F. Atiyah. *K-theory*. Advanced Book Classics. Addison-Wesley Publishing Company, Advanced Book Program, Redwood City, CA, second edition, 1989. Notes by D. W. Anderson. [↑](#) [2](#), [5](#)
- [Har77] R. Hartshorne. *Algebraic geometry*. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52. [↑](#) [3](#), [4](#), [6](#), [7](#), [9](#), [11](#)
- [Hit87] N. J. Hitchin. The self-duality equations on a Riemann surface. *Proc. London Math. Soc.* (3), 55(1):59–126, 1987. [↑](#) [1](#), [2](#), [10](#)
- [Lit82] Shigeru Iitaka. *Algebraic geometry*, volume 76 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1982. An introduction to birational geometry of algebraic varieties, North-Holland Mathematical Library, 24. [↑](#) [4](#)
- [Voi02] Claire Voisin. *Hodge theory and complex algebraic geometry. I*, volume 76 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2002. Translated from the French original by Leila Schneps. [↑](#) [11](#)

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