HIGGS BUNDLES — EXISTENCE OF SOLUTIONS

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Abstract. In this talk we introduce the stability condition for Higgs bundles and prove the Hitchin–Kobayashi correspondence. The main result is [Hit87, Theorem 4.3]. Relevant literature is [Hit87, §3 and §4] and [Wen14, §2 and §3]. Maybe we will also use [Wen16] every now and then.

This talk is related to Tanuj's talk on *Stable vector bundles*, for which the main reference is [Kob87]. Therefore we will also use [Kob87] as a default reference for generalities on complex vector bundles.

Contents

Notation and conventions

We usually follow the notation of [Hit87]:

- M: compact Riemann surface of genus g.
- $O \to M$: trivial line bundle.
- $K \to M$: canonical line bundle.
- More generally, O_X resp. K_X denote the trivial resp. canonical line bundle on a complex manifold X.
- For a vector bundle $E \to M$ we denote $\mu(E) := \deg E / \operatorname{rk} E$.

Every now and then we will also use some other standard notation, for instance:

- Let X be a complex manifold and $E \to X$ a (holomorphic/algebraic) vector bundle. Then we denote by $\mathscr{E}$ its sheaf of sections. The assignement $E \mapsto \mathscr{E}$ defines an equivalence of categories between vector bundles on X and locally free

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sheaves of $\mathscr{O}_X$-modules of finite rank. We can recover E from $\mathscr{E}$ either using cocycles [Voi02, Lemma 4.8] or by using the relative spectrum [Har77, Exercise II.5.18]. Following the second approach we would write

$$E = \operatorname{Spec}_X(S(\mathscr{E}^*)) =: \mathbb{V}(\mathscr{E}^*),$$

where $S(-)$ denotes the symmetric algebra.

- Let E be again a vector bundle on a complex manifold X. We will denote its projectivisation by $\mathbb{P}(E)$, which is obtained form E without its zero section by quotienting out the fibre-wise multiplication by scalars. Taking again the algebraic approach we can write

$$\mathbb{P}(E) = \operatorname{Proj}_X(S(\mathscr{E}^*)) =: \mathbb{P}(\mathscr{E}^*).$$

1. Stability

Definition 1 (Higgs bundle)**.** A *Higgs bundle* on M is a pair (E, Φ), where $E \to M$ is a rank 2 vector bundle and Φ is a global section of $\operatorname{End} E \otimes K$, called a *Higgs field* on E.

Remark 2. Using the canonical isomorphisms

$$H^0(M, \operatorname{End}(E) \otimes K) \cong \operatorname{Hom}(O, E^* \otimes E \otimes K) \cong \operatorname{Hom}(E, E \otimes K)$$

we may identify Φ with a morphism

$$\Phi\colon E \to E \otimes K.$$

Example 3. Assume $g \geqslant 2$. Then $\deg K = 2g - 2 > 0$, so we can find a line bundle $K^{\frac{1}{2}}$ such that $K^{\frac{1}{2}} \otimes K^{\frac{1}{2}} \cong K$. Let $E = K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}$, where $K^{-\frac{1}{2}} = (K^{\frac{1}{2}})^{-1}$. We consider the Higgs field $\Phi_w\colon K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}} \to (K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}) \otimes K$ given by a matrix

$$\begin{pmatrix} 0 & w \\ 1 & 0 \end{pmatrix},$$

where $w \in \operatorname{Hom}(K^{-\frac{1}{2}}, K^{\frac{1}{2}} \otimes K) \cong H^0(M, K^2)$ can be regarded as a quadratic differential.

Definition 4 (Stability)**.** A Higgs bundle (E, Φ) is said to be *stable* if for every Φ-invariant[1] line bundle $L \subseteq E$ we have $\mu(L) < \mu(E)$.

Remark 5. $(E, 0)$ is stable if and only if E is stable in the usual sense.

Exercise 1. There are no stable Higgs bundles on $\mathbb{P}^1$. [Hint: Grothedieck's theorem allows us to write Φ as a matrix. What can we say about each entry?] [Solution in [Hit87]]

[1] Meaning that $\Phi(L) \subseteq L \otimes K$.

Example 6. Assume $g \geqslant 2$ and consier $E = K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}$ again. Then Φ_0 is stable, because $K^{-\frac{1}{2}}$ is the only Φ_0-invariant line bundle and

$$\deg K^{-\frac{1}{2}} = 1 - g < 0 = \frac{\deg E}{2}.$$

Proposition 7. *Assume $g \geqslant 2$ and let $E \to M$ be a rank 2 vector bundle. Then there exists Higgs field Φ on E such that (E, Φ) is stable if and only if there exists a dense Zariski open subset $U \subseteq H^0(M, \mathrm{End}(E) \otimes K)$ such that all $\Phi' \in U$ have the property that no line bundle $L \subseteq E$ is Φ'-invariant.*

Proof. Let $p\colon \mathbb{P}(E) \to M$ be the projectivisation of our rank 2 vector bundle, which is a ruled surface in the sense of [Har77, §V.2]. Let $O(-1)$ denote the tautological line bundle on $\mathbb{P}(E)$, whose fibre over $[v] \in \mathbb{P}(E)$ is the line $\{\lambda v \mid \lambda \in \mathbb{C}\} \subseteq E_{p([v])}$. Let $O(l) := O(1)^{\otimes l}$ for all $l \in \mathbb{Z}$, and if $F \to \mathbb{P}(E)$ is another vector bundle, denote by $F(l)$ the tensor product $F \otimes O(l)$. We have then $p_*\mathscr{O}(l) = S^l(\mathscr{E}^*)$ for all $l \geqslant 0$ [Har77, Exercise III.8.4]. Abusing slightly the notation we will write $p_*O(l) = S^l E^*$ for all $l \geqslant 0$.

Let $x \in M$. Then every endomorphism $A \in \mathrm{End}(E_x)$ defines a quadratic map $E_x \to \Lambda^2 E_x$ sending v to $Av \wedge v$. Such a quadratic map can be naturally regarded as a degree 2 homogeneous polynomial on the coordinates of e with coefficients in $\Lambda^2 E_x$. Hence we have a vector bundle morphism $\mathrm{End}(E) \to S^2E^* \otimes \Lambda^2 E$, which vanishes precisely along the trivial line subbundle of $\mathrm{End}(E)$ consisting over each fibre of scalar multiples of the identity. Sending $A \mapsto A - \frac{\mathrm{tr}(A)}{2}\,\mathrm{id}_{E_x}$ on each fibre allows us to identify $\mathrm{End}_0(E)$ as the quotient of $\mathrm{End}(E)$ by this trivial line subbundle, so we obtain an injective morphism $\mathrm{End}_0(E) \to S^2E^* \otimes \Lambda^2 E$. Counting ranks we see that we have in fact an isomorphism of vector bundles $\mathrm{End}_0(E) \cong S^2E^* \otimes \Lambda^2 E$, and therefore

$$\mathrm{End}_0(E) \otimes K \cong p_*O(2) \otimes K \otimes \Lambda^2 E.$$

Since $\mathscr{O}_M \cong p_*\mathscr{O}_{\mathbb{P}(E)}$, the projection formula implies that $p_*p^*(-) \cong (-)$ for vector bundles on M. The adjunction $p^* \dashv p_*$ induces then a $\mathbb{C}$-linear isomorphism

$$\alpha\colon H^0(M, \mathrm{End}_0(E) \otimes K) \cong H^0(\mathbb{P}(E), p^*(K \otimes \Lambda^2 E)(2)).$$

Let now Φ be a traceless Higgs field on E, and assume it is non-zero. By construction, a non-zero vector $e \in E$ is an eigenvector of the twisted endomorphism over the corresponding fibre if and only if the section $\alpha(\Phi)$ vanishes at the point $[e] \in \mathbb{P}(E)$, i.e. if and only if $[e]$ is in the divisor of zeros of the section $\alpha(\Phi)$, which we denote $\mathrm{div}(\alpha(\Phi))$. Let $L \subseteq E$ be a Φ-invariant subbundle, which defines a section of $p\colon \mathbb{P}(E) \to M$ by functoriality of projectivisation on injective morphisms of vector bundles:

$$\begin{tikzcd} \mathbb{P}(L) \arrow[r, "\sigma"] \arrow[d, equal] & \mathbb{P}(E) \arrow[dl, "p"] \\ M & \end{tikzcd}$$

Being Φ-invariant means precisely that $\sigma(M) \subseteq \operatorname{div}(s(\Phi))$. But then any non-zero $v \in L$ is a non-zero eigenvector corresponding to some eigenvalue of the endomorphism over the corresponding fibre. Since Φ is traceless and non-zero, the other eigenvalue must be different, and there must be some non-zero eigenvector outside of L, call it $v' \in V$. Since v' is a non-zero eigenvector, $[v'] \in \operatorname{div}(\alpha(\Phi))$. And since $v' \notin L$, $[v'] \notin \sigma(M)$. Therefore $\sigma(M)$ is a proper irreducible component of the divisor $\operatorname{div}(\alpha(\Phi))$. So if $\operatorname{div}(\alpha(\Phi))$ is irreducible, then no line bundle $L \subseteq V$ is Φ-invariant.

Our next goal is to show that $\operatorname{div}(\alpha(\Phi))$ is irreducible for generic Φ. We will show that the linear system $|p^*(K \otimes \Lambda^2 E)(2)|$ has dimension 2 and has no fixed component, and then irreducibility follows from Bertini's theorem [Iit82, Theorem 7.19].

The dimension of the linear system is one less than the dimension of the vector space of global secitons. Using the previous isomorphism it suffices to gain control over the dimension of the global sections of $\operatorname{End}_0(E) \otimes K$ on M, for which we can apply Hirzebruch–Riemann–Roch [Har77, Theorem A.4.1]. From [Har77, Example A.4.1.1] we get

$$\operatorname{td}(K^*) = 1 - \frac{c_1(K)}{2}.$$

Using the short exact sequence used earlier

$$0 \to O \to \operatorname{End}(E) \to \operatorname{End}_0(E) \to 0$$

we see that $c_1(\operatorname{End}_0(E)) = c_1(\operatorname{End} E) = 0$. Therefore

$$\operatorname{ch}(\operatorname{End}_0 E \otimes K) = 3 + c_1(\operatorname{End}_0 E) + 3c_1(K) = 3 + 3c_1(K).$$

Multiplying the two expressions we obtain

$$\operatorname{ch}(\operatorname{End}_0 E \otimes K) \operatorname{td}(K^*) = 3 + \frac{3}{2} c_1(K),$$

whose codimension 1 part has degree $3g - 3 \geqslant 3$. So Hirzebruch–Riemann–Roch tells us that

$$h^0(M, \operatorname{End}_0(E) \otimes K) - h^1(M, \operatorname{End}_0(E) \otimes K) = 3g - 3 \geqslant 3,$$

which implies that $h^0(\mathbb{P}(E), p^*(K \otimes \Lambda^2 E)(2)) = h^0(M, \operatorname{End}_0(E) \otimes K) \geqslant 3$, as we wanted to show.

So if $|p^*(K \otimes \Lambda^2 E)(2)|$ has no fixed component, then Bertini's theorem and the discussion above imply that a generic $\Phi \in H^0(M, \mathrm{End}_0(E) \otimes K)$ leaves no line bundle invariant, making (E, Φ) automatically stable. Let us see what happens if it does have some fixed divisor.

Write $H \subseteq \mathbb{P}(E)$ for an effective Cartier divisor corresponding to the line bundle $O(1)$ and $D_0 \subseteq M$ for an effective Cartier divisor corresponding to the line bundle $K \otimes \Lambda^2 E$. Since $\mathrm{Pic}(\mathbb{P}(E)) \cong \{\mathscr{O}(l) \otimes p^*\mathscr{L} \mid l \in \mathbb{Z}, \mathscr{L} \in \mathrm{Pic}(M)\}$ [Har77, Exercise II.7.9] and pullback of line bundles commutes with scheme-theoretic preimage of Cartier divisors, we can write the linear system $|p^*(K \otimes \Lambda^2 E)(2)|$ as

$$\{D \geqslant 0 \mid D \sim 2H + p^{-1}D_0\}.$$

Let $F \geqslant 0$ be a fixed divisor of this linear system, so that in particulat we have

$$F \leqslant 2H + p^{-1}D_0.$$

Suppose first that $F \leqslant p^{-1}D_0$. Then we can write F as $\mathrm{div}(p^*s)$ with $s \in H^0(M, L)$ for some line bundle $L \to M$, so that divividing by s on the left and by p^*s on the right gives isomorphisms

$$\begin{tikzcd}
H^0(M, \mathrm{End}_0(E) \otimes K) \arrow[r, "\cong"] \arrow[d, "/s"', "\cong"] & H^0(\mathbb{P}(E), p^*(K \otimes \Lambda^2 E)(2)) \arrow[d, "/p^*s"', "\cong"] \\
H^0(M, \mathrm{End}_0(E) \otimes K \otimes L^*) \arrow[r, "\cong"] & H^0(\mathbb{P}(E), p^*(L^* \otimes K \otimes \Lambda^2 E)(2))
\end{tikzcd}$$

By definition, the new linear system does not have any fixed divisors, hence we can apply Bertini to conclude that a generic $\Phi' \in H^0(M, \mathrm{End}_0(E) \otimes K \otimes L^*)$ does not have invariant line bundles. The difference between a global section of $\mathrm{End}_0(E) \otimes K$ and a global section of $\mathrm{End}_0(E) \otimes K \otimes L^*$ is that our endomorphism has different coefficients, it is twisted by a different line bundle. But our notion of invariance is the same in both cases, so $L \subseteq E$ is $\Phi' \in H^0(M, \mathrm{End}_0(E) \otimes K \otimes L^*)$ invariant if and only if it is $s\Phi' \in H^0(M, \mathrm{End}_0(E) \otimes K)$ invariant. So if $F \leqslant p^{-1}D_0$, then it is still true that a generic $\Phi \in H^0(M, \mathrm{End}_0(E) \otimes K)$ does not have invariant line bundles, again making (E, Φ) automatically stable in those cases. $\square$

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