HIGGS BUNDLES — EXISTENCE OF SOLUTIONS

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Abstract. In this talk we introduce the stability condition for Higgs bundles and prove the Hitchin–Kobayashi correspondence. The main result is [Hit87, Theorem 4.3]. Relevant literature is [Hit87, §3 and §4] and [Wen14, §2 and §3]. Maybe we will also use [Wen16] every now and then.

This talk is related to Tanuj's talk on *Stable vector bundles*, for which the main reference is [Kob87]. Therefore we will also use [Kob87] as a main reference for generalities on complex vector bundles.

Contents

Notation and conventions

- **Top** is the category of topological spaces.
- **Diff** is the category of smooth manifolds.
- If $\mathbf{C}$ is a category and $X \in \mathbf{C}$ is an object, then $\mathbf{C}/X$ denotes the category of objects of $\mathbf{C}$ over X, i.e. the category whose objects are morphisms $f\colon Y \to X$ in $\mathbf{C}$ and whose morphisms are commutative triangles in $\mathbf{C}$ of the form

$$\begin{tikzcd} Y_1 \arrow[rr, "f"] \arrow[dr, "f_1"'] & & Y_2 \arrow[dl, "f_2"] \\ & X & \end{tikzcd}$$

  We will often talk about Y instead of $f\colon Y \to X$, leaving this structure morphism implicit.

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- Let **C** be a category which has **Top** as an underlying category, e.g. **Diff**. Let $X \in \mathbf{C}$ and $f \in \mathbf{C}/X$ and let **P** be a property of morphisms in **C**. We will say that f has some property **P** *locally on* X if every point $x \in X$ has an open neighbourhood $x \in U \subseteq X$ in X such that the morphis $f|_{f^{-1}(U)} \colon f^{-1}(U) \to U$ has the property **P**.
- The category of group objects in a category **C** will be denoted by **CGrp**.
- Let **C** be a category and let $X \in \mathbf{C}$ and $G \in \mathbf{CGrp}$. A *left action* of G on X, denoted $G \circlearrowright X$, is a morphism
$$\rho \colon G \times X \to X$$
such that the following diagrams commute:
$$\begin{tikzcd} G \times G \times X \arrow[r, "m\times \mathrm{id}_X"] \arrow[d, "\mathrm{id}_G \times \rho"'] & G \times X \arrow[d, "\rho"] \\ G \times X \arrow[r, "\rho"] & X \end{tikzcd} \qquad \begin{tikzcd} * \times X \arrow[r, "e\times \mathrm{id}_X"] \arrow[dr, equal] & G \times X \arrow[d, "\rho"] \\ & X \end{tikzcd}$$
We can similarly define right actions.
- Let **C** be a category. Let $X \in \mathbf{C}$ and $G \in \mathbf{CGrp}$. Then we say that X is a G-object of **C**. A morphism of G-objects of **C** is a G-equivariant morphism $f \colon X_1 \to X_2$, meaning that the following diagram commutes:
$$\begin{tikzcd} G \times X_1 \arrow[r, "\mathrm{id}_G \times f"] \arrow[d, "\rho_1"'] & G \times X_2 \arrow[d, "\rho_2"] \\ X_1 \arrow[r, "f"] & X_2 \end{tikzcd}$$
The category of G-objects of **C** is denoted G-**C**.

1. Self-duality

We consider $\mathbb{R}^4$ with its standard smooth structure [Lee13, Example 1.22].

Appendix A. Principal bundles on smooth manifolds

In this appendix we recall the basics of principal G-bundles on smooth manifolds, where G is a Lie group.

Definition A.1 (Lie group)**.** A *Lie group* is a group object in the category **Diff** of smooth manifolds.

Remark A.2. $G \in$ **DiffGrp** if and only if its underlying set is equipped with a group structure such that the map $G \times G \to G$ given by $(g, h) \to gh^{-1}$ is smooth [Lee13, Proposition 7.1].

Let $M \in \mathbf{Diff}$ and $G \in \mathbf{Diff\,Grp}$. Then the projection $\pi\colon M \times G \to M$ has some nice properties, namely:

- $G \circlearrowright M \times G$ smoothly and fibrewise via $(x, g) \cdot h \mapsto (x, gh)$.
- For all $x \in M$, $G \circlearrowright \pi^{-1}(x)$ induces $G \cong \{x\} \times G \cong \pi^{-1}(x)$.

The smooth manifold $M \times G$ over M equipped with this right fibrewise action is called the *trivial principal G-bundle* on M. We can encode all this structure by saying that

$$M \times G \in (G\text{-}\mathbf{Diff})/M,$$

where we consider M with the trivial G-action.

Definition A.3 (Principal bundle)**.** Let $M \in \mathbf{Diff}$ and $G \in \mathbf{Diff\,Grp}$. Consider $M \in G$-$\mathbf{Diff}$ with the trivial action. A *principal G-bundle* on M is an object $P \in (G\text{-}\mathbf{Diff})/M$ which is trivial locally on M.

Example A.4. Let $M \in \mathbf{Diff}$ and $G := \mathrm{GL}(n, \mathbb{R}) \in \mathbf{Diff\,Grp}$. Then the *frame bundle* of M, denoted $\mathrm{GL}(M)$, is the principal G-bundle whose fibra over $x \in M$ is the set of all frames for the tangent space T_xM.

Example A.5. Let $G \in \mathbf{Diff}$ and $H \subseteq G$ a closed subgroup. Then G is a principal H-bundle over the left coset space G/H.

Some nice properties in the topological category, which probably extend to the smooth category (check!):

Proposition A.6. *i) Any morphism of principal G-bundles is an isomorphism.*
ii) A principal G-bundle is trivial if and only if it admits a section, where trivial means isomorphic to a trivial principal G-bundle.

References

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