

HIGGS BUNDLES — EXISTENCE OF SOLUTIONS

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ABSTRACT. In this talk we introduce the stability condition for Higgs bundles and prove the Hitchin–Kobayashi correspondence. The main result is [Hit87, Theorem 4.3]. Relevant literature is [Hit87, §3 and §4].

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1. STABILITY

Let M be a compact Riemann surface.

Definition (Higgs bundle). A *Higgs bundle* on M is a pair (E, Φ) , where $E \rightarrow M$ is a rank 2 vector bundle and Φ is a global section of $\text{End } E \otimes K$, called a *Higgs field* on E .

Remark. Using the canonical isomorphisms

$$H^0(M, \text{End}(\mathcal{E}) \otimes \omega) \cong \text{Hom}(\mathcal{O}, \mathcal{E}^\vee \otimes \mathcal{E} \otimes \omega) \cong \text{Hom}(\mathcal{E}, \mathcal{E} \otimes \omega)$$

we may identify Φ with a morphism

$$\Phi : E \rightarrow E \otimes K.$$

Definition (Stability). A Higgs bundle (E, Φ) is said to be *stable* if for every Φ -invariant line bundle $L \subseteq E$ we have $\mu(L) < \mu(E)$, where Φ -invariance means that $\Phi(L) \subseteq L \otimes K$.

Remark. $(E, 0)$ is stable if and only if E is stable in the usual sense.

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Exercise A. There are no stable Higgs bundles on \mathbb{P}^1 . [*Hints below*¹]

Lemma 1. Let (E_1, Φ_1) and (E_2, Φ_2) be stable pairs with $\Lambda^2 E_1 \cong \Lambda^2 E_2$. Let $\Psi : E_1 \rightarrow E_2$ be a non-zero morphism such that $(\Psi \otimes \text{id}_K) \circ \Phi_1 = \Phi_2 \circ \Psi$. Then Ψ is an isomorphism.

Proof. We prove the result by contradiction. Suppose that Ψ is not an isomorphism. The rank $x \mapsto \dim_{\mathbb{C}} \Psi_x(E_{1,x})$ is upper semi-continuous [Ati89, Proposition 1.3.2], so the rank of Ψ cannot be generically zero. If the rank was generically 2, then $\det(\Psi) \in H^0(M, \Lambda^2 \mathcal{E}_1^\vee \otimes \Lambda^2 \mathcal{E}_2)$ would be generically non-zero. But $\Lambda^2 E_1 \cong \Lambda^2 E_2$, so $\det(\Psi) \in H^0(M, \mathcal{O}) = \mathbb{C}$ must be a constant and Ψ would be an isomorphism. Therefore the rank is generically 1, only going down to 0 at special points.

Let $L_1 \subseteq E_1$ be the largest rank 1 subbundle of E_1 contained in the kernel of Ψ . Let $v_1 \in L_{1,x}$, and let z be a holomorphic coordinate around a general point $x \in M$. Then we can write $\Phi_1(v_1) = \phi_{1,x}(v_1) \otimes dz$ for some $\phi_{1,x} \in \text{End}(E_{1,x})$. Then

$$0 = \Phi_{2,x}(\Psi_x(v_1)) = (\Psi \otimes \text{id}_K)_x(\phi_{1,x}(v_1) \otimes dz) = \Psi_x(\phi_{1,x}(v_1)) \otimes dz,$$

so $\phi_{1,x}(v_1) \in \text{Ker}(\Psi_x) = L_{1,x}$. Since it suffices to check $\Phi_1(L_1) \subseteq L_1 \otimes K$ generically, this shows that L_1 is Φ_1 -invariant.

Let now $L_2 \subseteq E_2$ be the largest rank 1 subbundle of E_2 containing the image of Ψ . Let $v_2 = \Psi(v_1) \in L_{2,x}$ be a vector over a general point $x \in M$, which can thus be written as the image under Ψ of someone in E_1 . Then

$$\Phi_{2,x}(\Psi_x(v_1)) = (\Psi \otimes \text{id}_K)_x(\phi_{1,x}(v_1) \otimes dz) = \Psi_x(\phi_{1,x}(v_1)) \otimes dz,$$

so $\phi_{2,x}(v_2) \in \text{Im}(\Psi_x) = L_{2,x}$. Since it suffices to check $\Phi_2(L_2) \subseteq L_2 \otimes K$ generically, this shows that L_2 is Φ_2 -invariant.

Now we use that (E_i, Φ_i) are stable to deduce that

$$\deg(L_i) < \frac{d}{2}$$

for $i \in \{1, 2\}$, where $d := \deg(\Lambda^2 E_1) = \deg(\Lambda^2 E_2)$. Since L_1 is contained in the kernel of Ψ , Ψ induces a non-zero morphism of line bundles $E_1/L_1 \rightarrow L_2$, which corresponds to a non-zero global section of $(E_1/L_1)^\vee \otimes L_2$. Line bundles with negative degree do not have any non-zero global sections, so we must have $\deg(E_1/L_1) \leq \deg(L_2)$. Therefore

$$\frac{d}{2} < \deg(\Lambda^2 E_1) - \deg(L_1) = \deg(E_1/L_1) \leq \deg(L_2) < \frac{d}{2},$$

a contradiction. Hence Ψ must be an isomorphism. \square

¹Grothendieck's theorem allows us to write Φ as a matrix. What can we say about each entry? The solution can be found in [Hit87, Remark (3.2) (iii)]

Lemma 2. *Let $E \rightarrow M$ be a rank 2 vector bundle and denote by $\text{End}_0(E)$ the vector bundle of traceless endomorphisms. Then there is a natural projection $\text{pr}_0 : \text{End}(E) \rightarrow \text{End}_0(E)$ whose kernel is the trivial line bundle of multiples of the identity, yielding a short exact sequence*

$$0 \rightarrow \mathcal{O} \rightarrow \mathcal{E}nd(\mathcal{E}) \rightarrow \mathcal{E}nd_0(\mathcal{E}) \rightarrow 0.$$

In particular, $\deg(\text{ch}(\mathcal{E}nd_0(\mathcal{E}) \otimes \omega) \text{td}(\omega^\vee))_1 = 3g - 3$.

Proof. Over $x \in M$, the map $\text{pr}_{0,x} : \text{End}(E_x) \rightarrow \text{End}_0(E_x)$ is given by

$$A \mapsto A - \frac{\text{tr}(A)}{2} \text{id}_{E_x}.$$

The endomorphisms in the kernel are precisely the multiples of the identity. This fibre-wise description globalizes to the desired short exact sequence.

For the Chern class computation we use the axioms in [Har77, Appendix A]. We have

$$c_1(\mathcal{E}nd_0(\mathcal{E})) = c_1(\mathcal{E}nd(\mathcal{E})) = c_1(\mathcal{E}^\vee \otimes \mathcal{E}) = 0,$$

therefore

$$\text{ch}(\mathcal{E}nd_0(\mathcal{E}) \otimes \omega) = 3 + c_1(\mathcal{E}nd_0(\mathcal{E})) + 3c_1(\omega) = 3 + 3c_1(\omega).$$

We also have

$$\text{td}(\omega^\vee) = 1 - \frac{c_1(\omega)}{2},$$

so multiplying the two expressions we obtain

$$\text{ch}(\mathcal{E}nd_0(\mathcal{E}) \otimes \omega) \text{td}(\omega^\vee) = 3 + \frac{3}{2}c_1(\omega).$$

Since $\deg(c_1(\omega)) = 2g - 2$, the result follows. \square

Notation. Let us denote by $\mathbf{Vec}_2(M)$ the set of rank 2 vector bundles on M . We define the following subsets:

- $\mathbf{S} := \{E \in \mathbf{Vec}_2(M) \mid \exists \Phi \text{ such that } (E, \Phi) \text{ is stable}\}.$
- $\mathbf{A} := \{E \in \mathbf{Vec}_2(M) \mid \text{a general } \Phi \text{ has no invariant } L\}.$
- $\mathbf{B} := \{E \in \mathbf{Vec}_2(M) \mid \exists L \text{ invariant for all } \Phi\}.$

Lemma 3. *If $g \geq 2$, then*

$$\mathbf{Vec}_2(M) = \mathbf{A} \sqcup \mathbf{B}.$$

Proof. Let $\pi : \mathbb{P}(E) \rightarrow M$ be the projectivization of our rank 2 vector bundle and let $\mathcal{O}(-1) \rightarrow \mathbb{P}(E)$ denote the tautological line bundle, whose fiber over $[v] \in \mathbb{P}(E)$ is the line $\{\lambda v \mid \lambda \in \mathbb{C}\} \subseteq E_{\pi([v])}$ spanned by v . Denote also $\mathcal{O}(l) := \mathcal{O}(-1)^{\otimes(-l)}$. If \mathcal{F} is a sheaf on $\mathbb{P}(E)$, we denote $\mathcal{F}(l) := \mathcal{F} \otimes \mathcal{O}(l)$, where $\mathcal{O}(l)$ denotes the sheaf of sections of $\mathcal{O}(l)$. We have $\pi_* \mathcal{O}(l) = S^l(\mathcal{E}^\vee)$ for all $l \geq 0$ and $\pi_* \mathcal{O}(l) = 0$ for all $l < 0$ [Har77, Exercise III.8.4].

Let $x \in M$. Given $A \in \text{End}(E_x)$, we define the quadratic form $v \mapsto Av \wedge v$ with values in $\Lambda^2 E_x$, which can then be naturally regarded as an element in $S^2(E_x^\vee) \otimes \Lambda^2 E_x$. The resulting quadratic form is trivial precisely when $A = \lambda \text{id}_{E_x}$ for some $\lambda \in \mathbb{C}$, so by Lemma 2 we obtain an injective homomorphism $\text{End}_0(E_x) \rightarrow S^2(E_x^\vee) \otimes \Lambda^2 E_x$. Both vector spaces have the same dimension, so this must be an isomorphism. These isomorphisms globalize to an isomorphism $\text{End}_0(E) \cong S^2(E^\vee) \otimes \Lambda^2 E$, hence we obtain an isomorphism

$$\mathcal{E}nd_0(\mathcal{E}) \otimes \omega \cong \pi_* \mathcal{O}(2) \otimes \omega \otimes \Lambda^2 \mathcal{E}.$$

The projection formula yields now an isomorphism $\mathcal{E}nd_0(\mathcal{E}) \otimes \omega \cong \pi_*(\pi^*(\omega \otimes \Lambda^2 \mathcal{E})(2))$, hence an isomorphism

$$\psi : H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \omega) \cong H^0(\mathbb{P}(E), \pi^*(\omega \otimes \Lambda^2 \mathcal{E})(2)).$$

Let now $\Phi \in H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \omega)$. A line bundle $L \subseteq E$ is then Φ -invariant precisely when $\psi(\Phi)$ vanishes at all $[v] \in \mathbb{P}(L) \subseteq \mathbb{P}(E)$. In other words, L is Φ -invariant if and only if $\sigma(M) \subseteq \text{div}(\psi(\Phi))$, where $\text{div}(-)$ denotes the divisor of zeros of a section and $\sigma : M = \mathbb{P}(L) \rightarrow \mathbb{P}(E)$ is the section induced by $L \subseteq E$.

Suppose now that $\Phi \in H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \omega)$ is not nilpotent and let $L \subseteq E$ be a Φ -invariant line bundle. Then over a general point $x \in M$, the corresponding traceless endomorphism $\phi_x \in \text{End}_0(E_x)$ is diagonalizable, so we can find some eigenvector $v \in E_x \setminus L_x$ in an eigenspace other than L_x . This gives us a point $[v] \in \mathbb{P}(E) \setminus \sigma(M)$ on which $\psi(\Phi)$ vanishes. Hence $\sigma(M)$ is a proper irreducible component of the divisor $\text{div}(\psi(\Phi))$.

The previous discussion shows that if Φ is not nilpotent and $\text{div}(\psi(\Phi))$ is irreducible, then there are no invariant line bundles $L \subseteq E$. By Hirzebruch–Riemann–Roch and Lemma 2 we have

$$h^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \omega) \geq 3g - 3 \geq 3,$$

so the complete linear system defined by the invertible sheaf $\pi^*(\omega \otimes \Lambda^2 \mathcal{E})(2)$ has dimension at least 2. If this linear system does not have a fixed divisor, then Bertini's theorem [Lit82, Theorem 7.19] tells us that $\text{div}(\psi(\Phi))$ is irreducible for a general $\Phi \in H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \omega)$. Since in our case $\Phi \in H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \omega)$ is nilpotent if and only if $\Phi^2 = 0$, a general Φ is not nilpotent. Therefore $E \in \mathbf{A}$ in this case.

Let us see what happens if the linear system has a fixed divisor. Such a fixed divisor must be the zero locus of a non-zero global section s of an invertible sheaf on $\mathbb{P}(E)$, which is up to isomorphism of the form $\pi^* \mathcal{L}(l)$ with \mathcal{L} an invertible sheaf on M and $l \in \mathbb{Z}$ [Har77, Exercise II.7.9]. Being a fixed divisor means then that every other global section of our line bundle can be written as a product st , where $t \in H^0(\mathbb{P}(E), \pi^* \mathcal{N}(2 - l))$. Since our line bundle had non-zero global sections, both $\pi^* \mathcal{L}(l)$ and $\pi^* \mathcal{N}(2 - l)$ must

have non-zero global sections. By the projection formula, this leaves us with only three possibilities:

- a) $l = 0$;
- b) $l = 1$;
- c) $l = 2$.

Let us start with case *a*). Let $\pi^*s \in H^0(\mathbb{P}(E), \pi^*\mathcal{L}) \cong H^0(M, \mathcal{L})$ be a global section corresponding to the fixed component of our linear system. Dividing all global sections by s and by π^*s respectively we obtain the following commutative diagram:

$$\begin{array}{ccc} H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \omega) & \xrightarrow{\cong} & H^0(\mathbb{P}(E), \pi^*(\omega \otimes \Lambda^2 \mathcal{E})(2)) \\ \downarrow \cong /s & & \downarrow \cong / \pi^*s \\ H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \omega \otimes \mathcal{L}^\vee) & \xrightarrow{\cong} & H^0(\mathbb{P}(E), \pi^*(\mathcal{L}^\vee \otimes \omega \otimes \Lambda^2 \mathcal{E})(2)) \end{array}$$

By definition, the new linear system does not have any fixed divisors and has the same dimension. Hence we can apply Bertini to conclude that a generic $\Phi' \in H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \omega \otimes \mathcal{L}^\vee)$ does not have invariant line bundles, which in this case are defined as line bundles $N \subseteq E$ such that $\Phi'(N) \subseteq N \otimes K \otimes L^\vee$. But a line bundle $N \subseteq E$ is Φ' -invariant if and only if it is $s\Phi'$ -invariant, so we have $E \in \mathbf{A}$ in this case.

We move on to case *b*). Assume that the fixed divisor corresponds to a non-zero global section $s \in H^0(\mathbb{P}(E), \pi^*\mathcal{L}(1))$. This corresponds to a non-zero morphism $E \rightarrow L$. The fibre-wise kernel has then dimension 1 generically and 2 at special points by upper semi-continuity [Ati89, Proposition 1.3.2]. Hence we can find a line bundle $N \subseteq \text{Ker}(s) \subseteq E$, which can be described as the largest line subbundle of E contained in the kernel of s . If $v \in N$ is a non-zero vector, then $s(v) = 0$ and so $[v] \in \text{div}(s) \subseteq \text{div}(\psi(\Phi))$. Thus the corresponding section $\sigma(M) \subseteq \mathbb{P}(E)$ is contained in $\text{div}(\psi(\Phi))$ for all Φ and N is Φ -invariant for all Φ . Hence $E \in \mathbf{B}$ in this case.

In case *c*), the fixed divisor corresponds to a non-zero global section of $\pi^*\mathcal{L}(2)$. We have

$$H^0(\mathbb{P}(E), \pi^*\mathcal{L}(2)) \cong H^0(M, \mathcal{L} \otimes S^2 \mathcal{E}^\vee) \cong H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{L} \otimes \Lambda^2 \mathcal{E}^\vee).$$

So we can think of the fixed global section s as a traceless endomorphism of E with coefficients in $L \otimes \Lambda^2 E^\vee$. With this point of view, s -invariance of a line bundle $N \subseteq E$ translates into $s\Phi'$ -invariance of $N \subseteq E$ as before, where $s\Phi'$ is a Higgs field. Let us see that the fixed section s has some non-trivial kernel, hence defining a line bundle invariant under all Higgs fields as in the previous case. To show that there is some non-trivial kernel, it suffices to check that $\det(s) = 0$. Since s is traceless, it suffices in turn to check that $\text{tr}(s^2) = 0$. Suppose on the contrary that $\text{tr}(s^2) \neq 0$. Fix some non-zero

$s_1 \in H^0(M, \mathcal{L}^\vee \otimes \omega \otimes \Lambda^2 \mathcal{E})$ and consider the linear map

$$\begin{aligned} \theta : H^0(M, \mathcal{L}^\vee \otimes \omega \otimes \Lambda^2 \mathcal{E}) &\longrightarrow H^0(M, \omega^2) \\ \Phi' &\longmapsto \text{tr}(s^2)s_1\Phi' \end{aligned}$$

Since $\text{tr}(s^2)s_1$ can only vanish at finitely many points, the image of a non-zero Φ' can only vanish at finitely many points, hence θ is injective. From Hirzebruch–Riemann–Roch and Lemma 2 we know that

$$h^0(M, \mathcal{L}^\vee \otimes \omega \otimes \Lambda^2 \mathcal{E}) \geq 3g - 3 = h^0(M, \omega^2),$$

so θ is an isomorphism. Since $\mathcal{L}^\vee \otimes \omega \otimes \Lambda^2 \mathcal{E}$ has global sections, its degree is non-negative. If it was zero, then this would be the trivial line bundle and we would have $h^0(M, \mathcal{L}^\vee \otimes \omega \otimes \Lambda^2 \mathcal{E}) = 1 < 3g - 3$, a contradiction. Hence $\deg(\mathcal{L}^\vee \otimes \omega \otimes \Lambda^2 \mathcal{E}) > 0$ and the non-zero global section s_1 has at least one zero. If θ was indeed an isomorphism, then each zero of s_1 would give a base point of the complete linear system corresponding to ω^2 . But $\deg(\omega^2) = 4g - 4 \geq 2g$, so this linear system has no base points [Har77, Corollary IV.3.2]. This contradiction shows that s has non-trivial kernel, which contains a line bundle $N \subseteq E$ invariant by all $\Phi \in H^0(M, \text{End}_0(\mathcal{E}) \otimes \omega)$. Hence $E \in \mathbf{B}$ as well in this case. \square

Exercise B. Assume $g \geq 2$. Let $K^{\frac{1}{2}}$ be a line bundle whose square is K and let $K^{-\frac{1}{2}}$ be its inverse. Does $K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}$ belong to \mathbf{A} or to \mathbf{B} ? [Hints below²]

Lemma 4. Let $E \rightarrow M$ be a rank 2 vector bundle and denote by $\text{End}_0(E)$ the vector bundle of traceless endomorphisms. Let $L \subseteq E$ be a line bundle. Then there is an injective morphism

$$L^2 \otimes \Lambda^2 E \hookrightarrow \text{End}_0(E)$$

whose image are the traceless endomorphisms which preserve only the line bundle L . Dualizing this we obtain a surjection

$$\text{End}_0(E) \cong \text{End}_0(E^\vee) \twoheadrightarrow L^{-2} \otimes \Lambda^2 E$$

whose kernel are the traceless endomorphisms which preserve at least the line bundle L . We can realize this kernel as the inclusion $\text{Hom}(E, L) \subseteq \text{Hom}(E, E)$

²Consider the family of traceless endomorphisms given by

$$\Phi_\alpha := \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix},$$

parametrized by quadratic differentials $\alpha \in H^0(M, K^2)$. Use without proof the fact that an invariant line bundle exists if and only if the characteristic polynomial

$$\lambda^2 - \alpha$$

has a root in $H^0(M, K)$, i.e. if and only if the quadratic differential α can be written as a square $\alpha = \beta^2$ for some differential form $\beta \in H^0(M, K)$. If α was a square, its zeros would all have multiplicity at least two. Conclude that $K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}} \in \mathbf{A}$ using Bertini's theorem.

followed by the projection $\text{pr}_0 : \text{Hom}(E, E) \rightarrow \text{End}_0(E)$, yielding a short exact sequence

$$0 \rightarrow \mathcal{E}^\vee \otimes \mathcal{L} \rightarrow \mathcal{E}nd_0(\mathcal{E}) \rightarrow \mathcal{L}^{-2} \otimes \Lambda^2 \mathcal{E} \rightarrow 0.$$

Proof. Under the isomorphism $E \cong E^\vee \otimes \Lambda^2 E$ [Har77, Exercise II.5.16], the line bundle L is sent to linear forms with coefficients in $\Lambda^2 E$ vanishing along L , hence we have a short exact sequence

$$0 \rightarrow \mathcal{L} \rightarrow \mathcal{E} \rightarrow \mathcal{L}^\vee \otimes \Lambda^2 \mathcal{E} \rightarrow 0.$$

Dualizing this short exact sequence we obtain an inclusion $L \otimes \Lambda^2 E^\vee \subseteq E^\vee$. Tensoring with L and composing with the inclusion $E^\vee \otimes L \subseteq E^\vee \otimes E$ we obtain an inclusion $L^2 \otimes \Lambda^2 E^\vee \subseteq \text{End}(E)$. Choosing a basis on each fibre and chasing all the identifications we have made so far, we see that the image of $L^2 \otimes \Lambda^2 E^\vee$ lies actually in $\text{End}_0(E)$. Indeed, let V be a two dimensional \mathbb{C} -vector space and let e_1 and e_2 be a basis. Let L be the line spanned by a non-zero vector l , which we may assume to be e_1 . The first identification we have is $V \cong \text{Hom}(V, \Lambda^2 V)$, sending v to the homomorphism $v' \mapsto v' \wedge v$. This corresponds to $\alpha_v \otimes (e_1 \wedge e_2) \in V^\vee \otimes \Lambda^2 V$, where $\alpha_v \in V^\vee$ is the linear form sending $e_1 \mapsto v_2$ and $e_2 \mapsto -v_1$. Denoting by $\overline{\alpha_v}$ its image in L^\vee , we can describe the morphism corresponding to the right hand side of the previous short exact sequence as

$$\begin{aligned} V &\longrightarrow L^\vee \otimes \Lambda^2 V \\ v &\longmapsto \overline{\alpha_v} \otimes (e_1 \wedge e_2) \end{aligned}$$

Let now $\beta \in (L^\vee \otimes \Lambda^2 V)^\vee$ and denote by λ_v^β the complex number such that

$$\overline{\alpha_v} \otimes (e_1 \wedge e_2) \xrightarrow{\beta} \lambda_v^\beta.$$

A point $\mu l \otimes \beta \in L \otimes (L^\vee \otimes \Lambda^2 V)^\vee$ corresponds then to the endomorphism

$$\begin{aligned} V &\longrightarrow V \\ v &\longmapsto \mu \lambda_v^\beta l \end{aligned}$$

A basis for L is e_1 , a basis for $L^\vee \otimes \Lambda^2 V$ is $\overline{\alpha_{e_2}} \otimes (e_1 \wedge e_2)$ and a basis for $L \otimes (L^\vee \otimes \Lambda^2 V)^\vee$ is $e_1 \otimes \beta_0$, where $\beta_0 \in (L^\vee \otimes \Lambda^2 V)^\vee$ is such that $\lambda_{e_2}^{\beta_0} = 1$. Writing the image of the basis $e_1 \otimes \beta_0$ under the map $L \otimes (L^\vee \otimes \Lambda^2 V)^\vee \rightarrow \text{End}(V)$ as a matrix with respect to our bases we obtain

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

because $\overline{\alpha_{e_1}} = 0$ and therefore $\lambda_{e_1}^\beta = 0$ for any β . We have thus the desired injective homomorphism

$$L^2 \otimes \Lambda^2 V^\vee \hookrightarrow \text{End}_0(V)$$

whose image are the traceless endomorphisms which preserve only L .

We regard this as a homomorphism into $\text{End}(V)$ for a moment and use the basis $e_{11}, e_{12}, e_{21}, e_{22}$ of $\text{End}(V)$, where e_{ij} denotes the endomorphism which, represented as a matrix in terms of our basis, has zeros everywhere except for a 1 in the ij -th position. Then our homomorphism is given by

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Dualizing it we obtain a surjection

$$\text{End}_0(V^\vee) \twoheadrightarrow L^{-2} \otimes \Lambda^2 V$$

given with respect to the dual bases by

$$\begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}.$$

Its kernel are the endomorphisms of V^\vee represented with respect to the dual basis by a matrix of the form

$$\begin{pmatrix} a & 0 \\ b & -a \end{pmatrix}.$$

Hence, under the isomorphism $\text{End}(V) \cong \text{End}(V^\vee)$ given in coordinates by sending a matrix to its transpose, we obtain a surjection

$$\text{End}_0(V) \twoheadrightarrow L^{-2} \otimes \Lambda^2 V$$

whose kernel are endomorphisms represented with respect to our basis by matrices of the form

$$\begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}.$$

Therefore the kernel of this surjection consists precisely of the traceless endomorphisms of V that leave at least L invariant. The inclusion of this kernel can be naturally regarded as the composition of the inclusion $\text{Hom}(V, L) \subseteq \text{Hom}(V, V)$ and the projection $\text{pr}_0 : \text{Hom}(V, V) \rightarrow \text{End}_0(V)$, which writing every homomorphism as a matrix with respect to the bases above has the form

$$\begin{pmatrix} a & b \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} \frac{a}{2} & b \\ 0 & -\frac{a}{2} \end{pmatrix}.$$

This gives us the desired short exact sequence

$$0 \rightarrow \mathcal{E}^\vee \otimes \mathcal{L} \otimes \omega \rightarrow \mathcal{E}nd_0(\mathcal{E}) \otimes \omega \rightarrow \mathcal{L}^{-2} \otimes \omega \otimes \Lambda^2 \mathcal{E} \rightarrow 0,$$

in which the global sections of $\mathcal{E}^\vee \otimes \mathcal{L} \otimes \omega$ correspond to Higgs fields which leave at least L invariant. \square

Lemma 5. *If $g \geq 2$, then*

$$S \subseteq A.$$

Proof. We want to show that if $E \in \mathbf{Vec}_2(M)$ is stable, then it is in A . By Lemma 3 it suffices to show that it is not in B . So let E be a stable rank 2 vector bundle on M and assume $L \subseteq E$ is a line bundle which is Φ -invariant for all $\Phi \in H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \omega)$. Consider the short exact sequence from Lemma 4

$$0 \rightarrow \mathcal{E}^\vee \otimes \mathcal{L} \otimes \omega \rightarrow \mathcal{E}nd_0(\mathcal{E}) \otimes \omega \rightarrow \mathcal{L}^{-2} \otimes \omega \otimes \Lambda^2 \mathcal{E} \rightarrow 0.$$

Since all $\Phi \in H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \omega)$ leave L invariant, we get an induced isomorphism on global sections $H^0(M, \mathcal{E}^\vee \otimes \mathcal{L} \otimes \omega) \cong H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \omega)$. We consider now the short exact sequence

$$0 \rightarrow \mathcal{L}^2 \otimes \omega \otimes \Lambda^2 \mathcal{E}^\vee \rightarrow \mathcal{E}^\vee \otimes \mathcal{L} \otimes \omega \rightarrow \omega \rightarrow 0,$$

more or less implicit in the proof of Lemma 4. Since E is stable, we have $\deg(\mathcal{L}^2 \otimes \Lambda^2 \mathcal{E}^\vee) < 0$, and since the complete linear system corresponding to ω is base-point free³ we have $h^0(M, \omega \otimes \mathcal{L}^2 \otimes \Lambda^2 \mathcal{E}^\vee) < g$ by [Har77, Proposition IV.3.1]. The long exact sequence of the previous short exact sequence gives then

$$h^0(M, \omega \otimes \mathcal{L} \otimes \mathcal{E}^\vee) \leq 2g - 1.$$

The earlier Hirzebruch–Riemann–Roch computation showed that

$$h^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \omega) \geq 3g - 3.$$

If we want the two dimensions to be equal we must have $g = 2$ and $h^0(M, \omega \otimes \mathcal{L} \otimes \mathcal{E}^\vee) = 3$. From the same long exact sequence as before we deduce, using that $h^0(M, \omega \otimes \mathcal{L}^2 \otimes \Lambda^2 \mathcal{E}^\vee) < g = 2$, that $h^0(M, \omega \otimes \mathcal{L}^2 \otimes \Lambda^2 \mathcal{E}^\vee) = 1$. In particular, $\deg(\omega \otimes \mathcal{L}^2 \otimes \Lambda^2 \mathcal{E}^\vee) \geq 0$. We have $\deg(\omega) = 2$ and by stability we had $\deg(\mathcal{L}^2 \otimes \Lambda^2 \mathcal{E}^\vee) < 0$, so we must have

$$\deg(\omega \otimes \mathcal{L}^2 \otimes \Lambda^2 \mathcal{E}^\vee) \in \{0, 1\}.$$

If it is 0, then the existence of global sections implies that it is the trivial line bundle, hence the previous short exact sequence becomes

$$0 \rightarrow \mathcal{O} \rightarrow \mathcal{E}^\vee \otimes \mathcal{L} \otimes \omega \rightarrow \omega \rightarrow 0.$$

Split short exact sequences are preserved by dualizing and tensoring with line bundles, so if this sequence was split then \mathcal{E} would be decomposable as a direct sum of line bundles. But this would give more endomorphisms of

³Suppose that $P \in M$ was a base-point of $|\omega|$. This would mean that $h^0(\omega) = h^0(\omega \otimes \mathcal{O}(-P)) = g$, so by Riemann–Roch we would have $h^0(\mathcal{O}(P)) = 2$. For any other $Q \in M$ we would have that $\mathcal{O}(P - Q)$ is a degree 0 line bundle with $h^0(\mathcal{O}(P - Q)) \in \{1, 2\}$ [Har77, Proof of Proposition IV.3.1], hence the trivial line bundle. This would imply that any two points in M are linearly equivalent, hence $M = \mathbb{P}^1$ [Har77, Example II.6.10.1], a contradiction.

\mathcal{E} than there should be, since stable vector bundles are simple. So the previous short exact sequence is a non-trivial extension and the coboundary map $H^0(M, \omega) \rightarrow H^1(M, \mathcal{O})$ is non-zero. The long exact sequence implies then that $h^0(M, \mathcal{E}^\vee \otimes \omega \otimes \mathcal{L}) \leq 2$, contradicting our previous conclusion that this dimension was 2. \square

2. AN EXISTENCE THEOREM

From now on, every space, morphism and action will be assumed to be smooth without explicitly saying so.

2.1. Recall the self-duality equations. Let $G = \mathrm{SO}(3)$ be the Lie group of rotations in \mathbb{R}^3 . Endow our Riemannian surface M with the trivial right action by G , making it a right G -manifold. Let $\pi : P \rightarrow M$ be a principal G -bundle over M , which is a right G -manifold over M which, locally on M , looks like the projection from a product $M \times G \rightarrow M$.

[picture]

Let $g \in G$ and denote by $R_g : P \rightarrow P$ the right multiplication by g . Since $\pi \circ R_g = \pi$, we have a short exact sequence

$$0 \rightarrow T^{\mathrm{vert}}P \rightarrow TP \xrightarrow{d\pi} \pi^*TM \rightarrow 0$$

of G -equivariant maps, where $T_p^{\mathrm{vert}}P = \mathrm{Ker}(d\pi_p)$ is the *vertical tangent space* of P at the point p , consisting of all tangent vectors which lie in the tangent space of the fibre $\pi^{-1}(\pi(p))$. A *connection* A on P is then a G -equivariant splitting $\sigma_A : \pi^*TM \rightarrow TP$. Therefore we may think of a connection A on P as a choice of *horizontal tangent spaces* $T_p^{\mathrm{hor}}P \subseteq T_pP$ having for each $p \in P$ the following properties:

- $T_pP = T_p^{\mathrm{vert}}P \oplus T_p^{\mathrm{hor}}P$.
- $d\pi_p|_{T_p^{\mathrm{hor}}P} : T_p^{\mathrm{hor}}P \rightarrow T_{\pi(p)}M$ is an isomorphism.
- $(dR_g)_p(T_p^{\mathrm{hor}}P) = T_{p \cdot g}^{\mathrm{hor}}P$ for all $g \in G$.

[picture]

Remark 6. Once we fix a connection A on P , we can lift any vector field $X : M \rightarrow TM$ on M to a horizontal vector field $\sigma_A \circ \pi^*X : P \rightarrow T^{\mathrm{hor}}P \subseteq TP$, yielding a map $h : \mathfrak{X}(M) \rightarrow \mathfrak{X}(P)$.

Let $\mathfrak{g} = \mathfrak{so}(3)$ be the Lie algebra of our Lie group, which in our case consists of skew-symmetric 3×3 -matrices with real coefficients. The action of G on P induces a Lie algebra homomorphism $a : \mathfrak{g} \rightarrow \mathfrak{X}(P)$, called the *infinitesimal action* of \mathfrak{g} on P and given by

$$a(B)_p = \left. \frac{d}{dt} \right|_{t=0} p \cdot e^{tB} \in T_pP,$$

where e^{tB} is in our case the usual exponential of the matrix $tB \in \mathfrak{g}$. This infinitesimal action induces in turn for each $p \in P$ an injective \mathbb{R} -linear homomorphism

$$a_p : \mathfrak{g} \rightarrow T_p P.$$

Indeed, if $p \cdot e^{tB} = p$ for all $t \in (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$, then e^{tB} is the identity matrix in G for all $t \in (-\varepsilon, \varepsilon)$ because the action of G on P is free. But the exponential is a local diffeomorphism around $0 \in \mathfrak{g}$, so we must have that $B = 0$ is the zero matrix.

We consider also the *adjoint representation* $\rho : G \rightarrow \mathrm{GL}(\mathfrak{g})$ given by

$$\rho(g) = d(g \cdot (-) \cdot g^{-1})_e,$$

where e is the identity matrix in G . We have now a right G -action on P and a linear left G -action on \mathfrak{g} , so the vector bundle $\mathfrak{g} \times P \rightarrow P$ carries the natural right G -action $(B, p) \cdot g := (g^{-1} \cdot B, p \cdot g)$. The infinitesimal action induces then an injective G -equivariant map $a : \mathfrak{g} \times P \rightarrow TP$, because the exponential map is compatible with differentials of Lie group homomorphisms. Moreover, its image is by definition contained in the vertical tangent space, so that $a : \mathfrak{g} \times P \rightarrow T^{\mathrm{vert}} P$ is an isomorphism by dimensional reasons. We may therefore replace our previous short exact sequence by

$$0 \rightarrow \mathfrak{g} \times P \xrightarrow{a} TP \xrightarrow{d\pi} \pi^* TM \rightarrow 0.$$

A connection was defined to be a G -equivariant section of $d\pi$. But this is equivalent by the equivariant version of the splitting lemma to a G -equivariant retraction of a . Such a retraction is in particular a \mathfrak{g} -valued 1-form $\omega \in \Omega^1(P, \mathfrak{g})$, called the *connection 1-form* on P . Conversely, every G -invariant \mathfrak{g} -valued 1-form $\omega \in \Omega^1(P, \mathfrak{g})$ with $\omega(a(B)) = B$ for all $B \in \mathfrak{g}$ gives rise to a connection on P , where G -invariance of ω means that $\rho(g)(R_g^*(\omega)(-)) = \omega(-)$.

Let A be a connection on our principal G -bundle P and let $\omega \in \Omega^1(P, \mathfrak{g})$ be the corresponding 1-form. We define the *curvature* $F(A)$ as the \mathfrak{g} -valued 2-form given by

$$F(A)(X, Y) = d\omega(X, Y) + [\omega(X), \omega(Y)] \text{ for all } X, Y \in \mathfrak{X}(P).$$

The curvature is in fact a G -invariant horizontal 2-form, so it corresponds to a uniquely determined 2-form with values in the vector bundle $\mathrm{ad}(P) \rightarrow P$ associated to the principal G -bundle P and the adjoint representation $\rho : G \rightarrow \mathrm{GL}(\mathfrak{g})$, whose transition functions are given by the composition of the transition functions of $P \rightarrow M$ with the group homomorphism ρ . We may therefore regard the curvature $F(A)$ as an $\mathrm{ad}(P)$ -valued 2-form $F(A) \in \Omega^2(M, \mathrm{ad}(P))$.

Let us denote now by $G_{\mathbb{C}}$ the complex Lie group of special orthogonal complex-valued 3×3 -matrices and by $\mathfrak{g}_{\mathbb{C}}$ its Lie algebra, which consists of

complex valued skew-symmetric 3×3 -matrices [?, Proposition 3.38]. We consider $\Phi \in \Omega^{1,0}(M, \text{ad}(P) \otimes \mathbb{C})$, which can be regarded as a $\mathfrak{g}_{\mathbb{C}}$ -valued $(1, 0)$ -form. Let us denote by Φ^* the composition of Φ with the anti-involution of the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}}$. Similarly as before, we use the Lie bracket on $\mathfrak{g}_{\mathbb{C}}$ to define a bracket on $\mathfrak{g}_{\mathbb{C}}$ -valued forms:

$$[\Phi, \Phi^*](X, Y) = [\Phi(X), \Phi^*(Y)] + [\Phi^*(X), \Phi(Y)] \text{ for all } X, Y \in \mathfrak{X}(M).$$

NOTATION AND CONVENTIONS

We usually follow the notation of [Hit87]:

- M : compact Riemann surface of genus g .
- $O \rightarrow M$: trivial line bundle.
- $K \rightarrow M$: canonical line bundle.
- More generally, O_X and K_X denote the trivial and canonical line bundles over a complex manifold X .

Every now and then we will also use some other standard notation, for instance:

- For a vector bundle $E \rightarrow M$ we denote $\mu(E) := \deg E / \text{rk } E$.
- Let X be a complex manifold and $E \rightarrow X$ a (holomorphic/algebraic) vector bundle. Then we denote by \mathcal{E} its sheaf of sections. The assignment $E \mapsto \mathcal{E}$ defines an equivalence of categories between vector bundles on X and locally free sheaves of \mathcal{O}_X -modules of finite rank, which we will refer to simply as locally free sheaves. We can recover E from \mathcal{E} either using cocycles [Voi02, Lemma 4.8] or by using the relative spectrum [Har77, Exercise II.5.18]. Following the second approach we would write

$$E = \text{Spec}_X(S(\mathcal{E}^\vee)) =: \mathbb{V}(\mathcal{E}^\vee),$$

where $S(-)$ denotes the symmetric algebra.

- \mathcal{O} and ω denote the trivial and canonical invertible sheaves on M . More generally, \mathcal{O}_X and ω_X denote the trivial and canonical invertible sheaves on a complex manifold X ,
- Let E be again a vector bundle on a complex manifold X . We will denote its projectivisation by $\mathbb{P}(E)$, which is obtained from E without its zero section by quotienting out the fibre-wise multiplication by scalars. Taking again the algebraic approach we can write

$$\mathbb{P}(E) = \text{Proj}_X(S(\mathcal{E}^\vee)) =: \mathbb{P}(\mathcal{E}^\vee).$$

- Let N be a smooth manifold and $E \rightarrow N$ a smooth vector bundle. Then we denote by $\mathfrak{X}(N)$ the Lie algebra of smooth vector fields on N and by $\Omega^k(N, E)$ the vector space of smooth k -differential forms

with values in E , which can be thought of as smooth global sections of the vector bundle $\mathrm{Hom}(TN, E)$.

- Let N be a smooth manifold equipped with an almost complex structure $I : TN \rightarrow TN$. Then we denote by $\Omega^{i,j}(N, E)$...

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