HIGGS BUNDLES — EXISTENCE OF SOLUTIONS

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Abstract. In this talk we introduce the stability condition for Higgs bundles and prove the Hitchin–Kobayashi correspondence. The main result is [Hit87, Theorem 4.3]. Relevant literature is [Hit87, §3 and §4] and [Wen14, §2 and §3]. Maybe we will also use [Wen16] every now and then.

This talk is related to Tanuj's talk on *Stable vector bundles*, for which the main reference is [Kob87]. Therefore we will also use [Kob87] as a main reference for generalities on complex vector bundles.

Contents

Notation and conventions

- **Top** is the category of topological spaces.
- **Diff** is the category of smooth manifolds.
- If $\mathbf{C}$ is a category and $X \in \mathbf{C}$ is an object, then $\mathbf{C}/X$ denotes the category of objects of $\mathbf{C}$ over X, i.e. the category whose objects are morphisms $f\colon Y \to X$ in $\mathbf{C}$ and whose morphisms are commutative triangles in $\mathbf{C}$ of the form

$$\begin{tikzcd} Y_1 \arrow[rr, "f"] \arrow[dr, "f_1"'] & & Y_2 \arrow[dl, "f_2"] \\ & X & \end{tikzcd}$$

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We will often talk about Y instead of $f\colon Y \to X$, leaving this structure morphism implicit.

- Let **C** be a category which has **Top** as an underlying category, e.g. **Diff**. Let $X \in \mathbf{C}$ and $f \in \mathbf{C}/X$ and let **P** be a property of morphisms in **C**. We will say that f has some property **P** *locally on* X if every point $x \in X$ has an open neighbourhood $x \in U \subseteq X$ in X such that the morphis $f|_{f^{-1}(U)}\colon f^{-1}(U) \to U$ has the property **P**.
- The category of group objects in a category **C** will be denoted by **CGrp**.
- Let **C** be a category and let $X \in \mathbf{C}$ and $G \in \mathbf{CGrp}$. A *left action* of G on X, denoted $G \circlearrowright X$, is a morphism
$$\rho\colon G \times X \to X$$
such that the following diagrams commute:
$$\begin{tikzcd} G \times G \times X \arrow[r, "m \times \mathrm{id}_X"] \arrow[d, "\mathrm{id}_G \times \rho"'] & G \times X \arrow[d, "\rho"] \\ G \times X \arrow[r, "\rho"] & X \end{tikzcd} \qquad \begin{tikzcd} * \times X \arrow[r, "e \times \mathrm{id}_X"] \arrow[dr, equal] & G \times X \arrow[d, "\rho"] \\ & X \end{tikzcd}$$
We can similarly define right actions.
- Let **C** be a category. Let $X \in \mathbf{C}$ and $G \in \mathbf{CGrp}$. Then we say that X is a G-object of **C**. A morphism of G-objects of **C** is a G-equivariant morphism $f\colon X_1 \to X_2$, meaning that the following diagram commutes:
$$\begin{tikzcd} G \times X_1 \arrow[r, "\mathrm{id}_G \times f"] \arrow[d, "\rho_1"'] & G \times X_2 \arrow[d, "\rho_2"] \\ X_1 \arrow[r, "f"] & X_2 \end{tikzcd}$$
The category of G-objects of **C** is denoted G-**C**.

1. Self-duality

We consider $\mathbb{R}^4$ with its standard smooth structure [Lee13, Example 1.22].

Appendix A. Complex vector bundles, connections and curvature

Definition A.1 (Complex vector bundle)**.** Let $M \in \mathbf{Diff}$. A *complex vector bundle* on M consists of a family $\{E_x\}_{x\in M}$ of complex vector spaces parametrized by M, together with a smooth manifold structure on $E := \sqcup_{x\in M} E_x$ such that

i) The projection map $\pi\colon E \to M$ taking E_x to x is smooth, and
ii) For every $x_0 \in M$, there exists an open set U in M containing x_0 and a diffeomorphism

$$\varphi_U\colon \pi^{-1}(U) \to U \times \mathbb{C}^k$$

taking the vector space E_x isomorphically onto $\{x\} \times \mathbb{C}^k$ for each $x \in U$; φ_U is called a *trivialization of E over U*.

Remark A.2. If M is a complex manifold, we can also talk about *holomorphic vector bundles*. These are complex vector bundles $\pi\colon E \to M$ together with a structure of complex manifold on E such that we can find around each point a biholomorphic local trivialization φ_U.

Definition A.3 (Complex differential forms)**.** Let $M \in \mathbf{Diff}$ and let T_M be its tangent bundle. Let $E \to M$ be a complex vector bundle on M. Then the bundle of *complex p-forms with values in E* is defined as

$$\Omega^p_{M,\mathbb{C}}(E) := \bigwedge^p \mathrm{Hom}_M(T_M, E).$$

A *complex p-form with values in E* is then a smooth global section of $\Omega^p_{M,\mathbb{C}}(E)$. The $\mathbb{C}$-vector space of complex p-forms with values in E will be denoted by $A^p(E)$.

Remark A.4. In the particular case in which $E = M \times \mathbb{C}$ is the trivial complex line bundle on M, we simply talk about the bundle of complex p-forms on M, denoted $\Omega^p_{M,\mathbb{C}}$. Similarly, a smooth global section of $\Omega^p_{M,\mathbb{C}}$ will be simply called a complex p-form on M, and the $\mathbb{C}$-vector space of complex p-forms on M will be denoted by A^p.

Definition A.5 (Connection)**.** Let $M \in \mathbf{Diff}$ and $E \to M$ a complex vector bundle. A *connection* D in E is a $\mathbb{C}$-linear homomorphism

$$D\colon A^0(E) \to A^1(E)$$

such that

$$D(f\sigma) = \sigma df + f \cdot D\sigma$$

for $f \in A^0 = C^\infty(M, \mathbb{C})$ and $\sigma \in A^0(E) = \Gamma(M, E)$.

Appendix B. Principal bundles on smooth manifolds

In this appendix we recall the basics of principal G-bundles on smooth manifolds, where G is a Lie group.

Definition B.1 (Lie group)**.** A *Lie group* is a group object in the category **Diff** of smooth manifolds.

Remark B.2. $G \in \mathbf{DiffGrp}$ if and only if its underlying set is equipped with a group structure such that the map $G \times G \to G$ given by $(g, h) \to gh^{-1}$ is smooth [Lee13, Proposition 7.1].

Recall that for $M \in \mathbf{Diff}$, the $\mathbb{R}$-vector space $\mathcal{X}(M)$ of smooth vector fields on M forms a Lie algebra under the Lie bracket [Lee13, Proposition 8.28].

Let $M \in \mathbf{Diff}$ and $G \in \mathbf{DiffGrp}$. Then the projection $\pi \colon M \times G \to M$ has some nice properties, namely:

- $G \circlearrowright M \times G$ smoothly and fibrewise via $(x, g) \cdot h \mapsto (x, gh)$.
- For all $x \in M$, $G \circlearrowright \pi^{-1}(x)$ induces $G \cong \{x\} \times G \cong \pi^{-1}(x)$.

The smooth manifold $M \times G$ over M equipped with this right fibrewise action is called the *trivial principal G-bundle* on M. We can encode all this structure by saying that

$$M \times G \in (G\text{-}\mathbf{Diff})/M,$$

where we consider M with the trivial G-action.

Definition B.3 (Principal bundle)**.** Let $M \in \mathbf{Diff}$ and $G \in \mathbf{DiffGrp}$. Consider $M \in G$-$\mathbf{Diff}$ with the trivial action. A *principal G-bundle* on M is an object $P \in (G\text{-}\mathbf{Diff})/M$ which is trivial locally on M.

Example B.4. Let $M \in \mathbf{Diff}$ and $G := \mathrm{GL}(n, \mathbb{R}) \in \mathbf{DiffGrp}$. Then the *frame bundle* of M, denoted $\mathrm{GL}(M)$, is the principal G-bundle whose fibra over $x \in M$ is the set of all frames for the tangent space T_xM.

Example B.5. Let $G \in \mathbf{Diff}$ and $H \subseteq G$ a closed subgroup. Then G is a principal H-bundle over the left coset space G/H.

Some nice properties in the topological category, which probably extend to the smooth category (check!):

Proposition B.6.
i) Any morphism of principal G-bundles is an isomorphism.
ii) A principal G-bundle is trivial if and only if it admits a section, where trivial means isomorphic to a trivial principal G-bundle.

References

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