HIGGS BUNDLES — EXISTENCE OF SOLUTIONS

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Abstract. In this talk we introduce the stability condition for Higgs bundles and discuss the Hitchin–Kobayashi correspondence. The main result is [Hit87, Theorem 4.3]. Relevant literature is [Hit87, §3 and §4] and [Wen14].

Contents

1. Stability of Higgs bundles

Let M be a compact Riemann surface. In this first section, all vector bundles, morphisms and sections are assumed to be holomorphic.

Definition (Higgs bundle)**.** A *Higgs bundle* on M is a pair (E, Φ), where $E \longrightarrow M$ is a rank 2 vector bundle and Φ is a global section of $\mathrm{End}(E) \otimes K$, called a *Higgs field* on E.

Remark. Using the canonical isomorphisms

$$H^0(M, \mathcal{E}nd(\mathcal{E}) \otimes \mathcal{K}) \cong \mathrm{Hom}(\mathcal{O}, \mathcal{E}^\vee \otimes \mathcal{E} \otimes \mathcal{K}) \cong \mathrm{Hom}(\mathcal{E}, \mathcal{E} \otimes \mathcal{K})$$

we may identify Φ with a morphism

$$\Phi : \ E \longrightarrow E \otimes K.$$

Definition (Stability)**.** A Higgs bundle (E, Φ) is said to be *stable* if for every Φ-invariant line bundle $L \subseteq E$ we have $\mu(L) < \mu(E)$, where Φ-invariance means that $\Phi(L) \subseteq L \otimes K$.

Remark. $(E, 0)$ is stable if and only if E is stable in the usual sense.

Date: 15 July 2020.

Supported by the DFG-Graduiertenkolleg GK1821 “Cohomological Methods in Geometry” at the University of Freiburg.

Exercise A. There are no stable Higgs bundles on $\mathbb{P}^1$. [*Hints below*[1]]

Lemma 1. *Let $E \longrightarrow M$ be a rank 2 vector bundle and denote by $\mathrm{End}_0(E)$ the vector bundle of traceless endomorphisms. Then there is a natural projection $\mathrm{pr}_0 : \mathrm{End}(E) \longrightarrow \mathrm{End}_0(E)$ whose kernel is the trivial line bundle of multiples of the identity, yielding a short exact sequence*

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{E}nd(\mathcal{E}) \longrightarrow \mathcal{E}nd_0(\mathcal{E}) \longrightarrow 0.$$

In particular, $\deg(\mathrm{ch}(\mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K})\,\mathrm{td}(\mathcal{K}^\vee))_1 = 3g - 3$.

Proof. Over $x \in M$, the map $\mathrm{pr}_{0,x} : \mathrm{End}(E_x) \longrightarrow \mathrm{End}_0(E_x)$ is given by

$$A \longmapsto A - \frac{\mathrm{tr}(A)}{2}\,\mathrm{id}_{E_x}.$$

The endomorphisms in the kernel are precisely the multiples of the identity. This fibre-wise description globalizes to the desired short exact sequence.

For the Chern class computation we use the axioms in [Har77, Appendix A]. We have

$$c_1(\mathcal{E}nd_0(\mathcal{E})) = c_1(\mathcal{E}nd(\mathcal{E})) = c_1(\mathcal{E}^\vee \otimes \mathcal{E}) = 0,$$

therefore

$$\mathrm{ch}(\mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K}) = 3 + c_1(\mathcal{E}nd_0(\mathcal{E})) + 3c_1(\mathcal{K}) = 3 + 3c_1(\mathcal{K}).$$

We also have

$$\mathrm{td}(\mathcal{K}^\vee) = 1 - \frac{c_1(\mathcal{K})}{2},$$

so multiplying the two expressions we obtain

$$\mathrm{ch}(\mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K})\,\mathrm{td}(\mathcal{K}^\vee) = 3 + \frac{3}{2}c_1(\mathcal{K}).$$

Since $\deg(c_1(\mathcal{K})) = 2g - 2$, the result follows. □

Notation. Let us denote by $\mathbf{Vec}_2(M)$ the set of rank 2 vector bundles on M. We define the following subsets:

- $\mathbf{S} := \{E \in \mathbf{Vec}_2(M) \mid \exists\Phi \text{ such that } (E, \Phi) \text{ is stable}\}$.
- $\mathbf{A} := \{E \in \mathbf{Vec}_2(M) \mid \text{a general } \Phi \text{ has no invariant } L\}$.
- $\mathbf{B} := \{E \in \mathbf{Vec}_2(M) \mid \exists L \text{ invariant for all } \Phi\}$.

Lemma 2. *If $g \geqslant 2$, then*

$$\mathbf{Vec}_2(M) = \mathbf{A} \sqcup \mathbf{B}.$$

[1] Grothedieck's theorem allows us to write Φ as a matrix. What can we say about each entry? The solution can be found in [Hit87, Remark (3.2) (iii)]

Proof. Let $\pi : \mathbb{P}(E) \longrightarrow M$ be the projectivization of our rank 2 vector bundle and let $O(-1) \longrightarrow \mathbb{P}(E)$ denote the tautological line bundle, whose fiber over $[v] \in \mathbb{P}(E)$ is the line $\{\lambda v \mid \lambda \in \mathbb{C}\} \subseteq E_{\pi([v])}$ spanned by v. Denote also $O(l) := O(-1)^{\otimes(-l)}$. If $\mathcal{F}$ is a sheaf on $\mathbb{P}(E)$, we denote $\mathcal{F}(l) := \mathcal{F} \otimes \mathcal{O}(l)$, where $\mathcal{O}(l)$ denotes the sheaf of sections of $O(l)$. We have $\pi_* \mathcal{O}(l) = S^l(\mathcal{E}^\vee)$ for all $l \geqslant 0$ and $\pi_* \mathcal{O}(l) = 0$ for all $l < 0$ [Har77, Exercise III.8.4].

Let $x \in M$. Given $A \in \mathrm{End}(E_x)$, we define the quadratic form $v \mapsto Av \wedge v$ with values in $\Lambda^2 E_x$, which can then be naturally regarded as an element in $S^2(E_x^\vee) \otimes \Lambda^2 E_x$. The resulting quadratic form is trivial precisely when $A = \lambda \, \mathrm{id}_{E_x}$ for some $\lambda \in \mathbb{C}$, so by Lemma 1 we obtain an injective homomorphism $\mathrm{End}_0(E_x) \longrightarrow S^2(E_x^\vee) \otimes \Lambda^2 E_x$. Both vector spaces have the same dimension, so this must be an isomorphism. These isomorphisms globalize to an isomorphism $\mathrm{End}_0(E) \cong S^2(E^\vee) \otimes \Lambda^2 E$, hence we obtain an isomorphism

$$\mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K} \cong \pi_* \mathcal{O}(2) \otimes \mathcal{K} \otimes \Lambda^2 \mathcal{E}.$$

The projection formula yields now an isomorphism $\mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K} \cong \pi_*(\pi^*(\mathcal{K} \otimes \Lambda^2 \mathcal{E})(2))$, hence an isomorphism

$$\psi : H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K}) \cong H^0(\mathbb{P}(E), \pi^*(\mathcal{K} \otimes \Lambda^2 \mathcal{E})(2)).$$

Let now $\Phi \in H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K})$. A line bundle $L \subseteq E$ is then Φ-invariant precisely when $\psi(\Phi)$ vanishes at all $[v] \in \mathbb{P}(L) \subseteq \mathbb{P}(E)$. In other words, L is Φ-invariant if and only if $\sigma(M) \subseteq \mathrm{div}(\psi(\Phi))$, where $\mathrm{div}(-)$ denotes the divisor of zeros of a section and $\sigma : M = \mathbb{P}(L) \longrightarrow \mathbb{P}(E)$ is the section induced by $L \subseteq E$.

Suppose now that $\Phi \in H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K})$ is not nilpotent and let $L \subseteq E$ be a Φ-invariant line bundle. Then over a general point $x \in M$, the corresponding traceless endomorphism $\phi_x \in \mathrm{End}_0(E_x)$ is diagonalizable, so we can find some eigenvector $v \in E_x \setminus L_x$ in an eigenspace other than L_x. This gives us a point $[v] \in \mathbb{P}(E) \setminus \sigma(M)$ on which $\psi(\Phi)$ vanishes. Hence $\sigma(M)$ is a proper irreducible component of the divisor $\mathrm{div}(\psi(\Phi))$.

The previous discussion shows that if Φ is not nilpotent and $\mathrm{div}(\psi(\Phi))$ is irreducible, then there are no invariant line bundles $L \subseteq E$. By Hirzebruch–Riemann–Roch and Lemma 1 we have

$$h^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K}) \geqslant 3g - 3 \geqslant 3,$$

so the complete linear system defined by the invertible sheaf $\pi^*(\mathcal{K} \otimes \Lambda^2 \mathcal{E})(2)$ has dimension at least 2. If this linear system does not have a fixed divisor, then Bertini's theorem [Iit82, Theorem 7.19] tells us that $\mathrm{div}(\psi(\Phi))$ is irreducible for a general $\Phi \in H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K})$. Since in our case $\Phi \in H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K})$ is nilpotent if and only if $\Phi^2 = 0$, a general Φ is not nilpotent. Therefore $E \in \mathbf{A}$ in this case.

Let us see what happens if the linear system has a fixed divisor. Such a fixed divisor must be the zero locus of a non-zero global section s of an invertible sheaf on $\mathbb{P}(E)$, which is up to isomorphism of the form $\pi^*\mathcal{L}(l)$ with $\mathcal{L}$ an invertible sheaf on M and $l \in \mathbb{Z}$ [Har77, Exercise II.7.9]. Being a fixed divisor means then that every other global section of our line bundle can be written as a product st, where $t \in H^0(\mathbb{P}(E), \pi^*\mathcal{N}(2-l))$. Since our line bundle had non-zero global sections, both $\pi^*\mathcal{L}(l)$ and $\pi^*\mathcal{N}(2-l)$ must have non-zero global sections. By the projection formula, this leaves us with only three possibilities:

a) $l = 0$;
b) $l = 1$;
c) $l = 2$.

Let us start with case *a*). Let $\pi^*s \in H^0(\mathbb{P}(E), \pi^*\mathcal{L}) \cong H^0(M, \mathcal{L})$ be a global section corresponding to the fixed component of our linear system. Dividing all global sections by s and by π^*s respectively we obtain the following commutative diagram:

$$\begin{tikzcd}
H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K}) \arrow[r, "\cong"] \arrow[d, "/s"', "\cong"] & H^0(\mathbb{P}(E), \pi^*(\mathcal{K} \otimes \Lambda^2\mathcal{E})(2)) \arrow[d, "/\pi^* s"', "\cong"] \\
H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K} \otimes \mathcal{L}^\vee) \arrow[r, "\cong"] & H^0(\mathbb{P}(E), \pi^*(\mathcal{L}^\vee \otimes \mathcal{K} \otimes \Lambda^2\mathcal{E})(2))
\end{tikzcd}$$

By definition, the new linear system does not have any fixed divisors and has the same dimension. Hence we can apply Bertini to conclude that a generic $\Phi' \in H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K} \otimes \mathcal{L}^\vee)$ does not have invariant line bundles, which in this case are defined as line bundles $N \subseteq E$ such that $\Phi'(N) \subseteq N \otimes K \otimes L^\vee$. But a line bundle $N \subseteq E$ is Φ'-invariant if and only if it is $s\Phi'$-invariant, so we have $E \in \mathbf{A}$ in this case.

We move on to case *b*). Assume that the fixed divisor corresponds to a non-zero global section $s \in H^0(\mathbb{P}(E), \pi^*\mathcal{L}(1))$. This corresponds to a non-zero morphism $E \longrightarrow L$. The fibre-wise kernel has then dimension 1 generically and 2 at special points by upper semi-continuity [Ati89, Proposition 1.3.2]. Hence we can find a line bundle $N \subseteq \mathrm{Ker}(s) \subseteq E$, which can be described as the largest line subbundle of E contained in the kernel of s. If $v \in N$ is a non-zero vector, then $s(v) = 0$ and so $[v] \in \mathrm{div}(s) \subseteq \mathrm{div}(\psi(\Phi))$. Thus the corresponding section $\sigma(M) \subseteq \mathbb{P}(E)$ is contained in $\mathrm{div}(\psi(\Phi))$ for all Φ and N is Φ-invariant for all Φ. Hence $E \in \mathbf{B}$ in this case.

In case *c*), the fixed divisor corresponds to a non-zero global section of $\pi^*\mathcal{L}(2)$. We have

$$H^0(\mathbb{P}(E), \pi^*\mathcal{L}(2)) \cong H^0(M, \mathcal{L} \otimes S^2\mathcal{E}^\vee) \cong H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{L} \otimes \Lambda^2\mathcal{E}^\vee).$$

So we can think of the fixed global section s as a traceless endomorphism of E with coefficients in $L \otimes \Lambda^2 E^\vee$. With this point of view, s-invariance of a line bundle $N \subseteq E$ translates into $s\Phi'$-invariance of $N \subseteq E$ as before, where $s\Phi'$ is a Higgs field. Let us see that the fixed section s has some non-trivial kernel, hence defining a line bundle invariant under all Higgs fields as in the previous case. To show that there is some non-trivial kernel, it suffices to check that $\det(s) = 0$. Since s is traceless, it suffices in turn to check that $\mathrm{tr}(s^2) = 0$. Suppose on the contrary that $\mathrm{tr}(s^2) \neq 0$. Fix some non-zero $s_1 \in H^0(M, \mathcal{L}^\vee \otimes \mathcal{K} \otimes \Lambda^2 \mathcal{E})$ and consider the linear map

$$\begin{aligned} \theta :\ H^0(M, \mathcal{L}^\vee \otimes \mathcal{K} \otimes \Lambda^2 \mathcal{E}) &\longrightarrow H^0(M, \mathcal{K}^2) \\ \Phi' &\longmapsto \mathrm{tr}(s^2) s_1 \Phi' \end{aligned}$$

Since $\mathrm{tr}(s^2)s_1$ can only vanish at finitely many points, the image of a non-zero Φ' can only vanish at finitely many points, hence θ is injective. From Hirzebruch–Riemann–Roch and Lemma 1 we know that

$$h^0(M, \mathcal{L}^\vee \otimes \mathcal{K} \otimes \Lambda^2 \mathcal{E}) \geqslant 3g - 3 = h^0(M, \mathcal{K}^2),$$

so θ is an isomorphism. Since $\mathcal{L}^\vee \otimes \mathcal{K} \otimes \Lambda^2 \mathcal{E}$ has global sections, its degree is non-negative. If it was zero, then this would be the trivial line bundle and we would have $h^0(M, \mathcal{L}^\vee \otimes \mathcal{K} \otimes \Lambda^2 \mathcal{E}) = 1 < 3g - 3$, a contradiction. Hence $\deg(\mathcal{L}^\vee \otimes \mathcal{K} \otimes \Lambda^2 \mathcal{E}) > 0$ and the non-zero global section s_1 has at least one zero. If θ was indeed an isomorphism, then each zero of s_1 would give a base point of the complete linear system corresponding to $\mathcal{K}^2$. But $\deg(\mathcal{K}^2) = 4g - 4 \geqslant 2g$, so this linear system has no base points [Har77, Corollary IV.3.2]. This contradiction shows that s has non-trivial kernel, which contains a line bundle $N \subseteq E$ invariant by all $\Phi \in H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K})$. Hence $E \in \mathbf{B}$ as well in this case. □

Exercise B. Assume $g \geqslant 2$. Let $K^{\frac{1}{2}}$ be a line bundle whose square is K and let $K^{-\frac{1}{2}}$ be its inverse. Does $K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}$ belong to $\mathbf{A}$ or to $\mathbf{B}$? [*Hints below*[2]]

[2] Consider the family of traceless endomorphisms given by

$$\Phi_\alpha := \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix},$$

parametrized by quadratic differentials $\alpha \in H^0(M, K^2)$. Use without proof the fact that an invariant line bundle exists if and only if the characteristic polynomial

$$\lambda^2 - \alpha$$

has a root in $H^0(M, K)$, i.e. if and only if the quadratic differential α can be written as a square $\alpha = \beta^2$ for some differential form $\beta \in H^0(M, K)$. If α was a square, its zeros would all have multiplicity at least two. Conclude that $K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}} \in \mathbf{A}$ using Bertini's theorem.

Lemma 3. *Let $E \longrightarrow M$ be a rank 2 vector bundle and denote by* $\mathrm{End}_0(E)$ *the vector bundle of traceless endomorphisms. Let $L \subseteq E$ be a line bundle. We have the following short exact sequences of vector bundles:*

a) $0 \longrightarrow \mathcal{L}^2 \otimes (\Lambda^2\mathcal{E})^\vee \otimes \mathcal{K} \longrightarrow \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K} \longrightarrow \mathcal{E} \otimes \mathcal{L}^{-1} \otimes \mathcal{K} \longrightarrow 0.$
b) $0 \longrightarrow \mathcal{E}^\vee \otimes \mathcal{L} \otimes \mathcal{K} \longrightarrow \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K} \longrightarrow \mathcal{L}^{-2} \otimes (\Lambda^2\mathcal{E}) \otimes \mathcal{K} \longrightarrow 0.$

Moreover, the sections of the image of $\mathcal{K}\otimes\mathcal{L}\otimes\Lambda^2\mathcal{E}^\vee$ in a) *leave only L invariant; and the sections of the image of $\mathcal{E}^\vee \otimes \mathcal{K} \otimes \mathcal{L}$ in* b) *are those which leave L invariant.*

Proof. All the short exact sequences are the result of tensoring another short exact sequence with $\mathcal{K}$, so let us find the necessary short exact sequences without $\mathcal{K}$. Under the isomorphism $E \cong E^\vee \otimes \Lambda^2 E$ [Har77, Exercise II.5.16], the line bundle L is sent to linear forms with coefficients in $\Lambda^2 E$ vanishing along L, hence we have a short exact sequence

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{E} \longrightarrow \mathcal{L}^\vee \otimes \Lambda^2\mathcal{E} \longrightarrow 0.$$

Dualizing this short exact sequence we obtain an inclusion $L \otimes \Lambda^2 E^\vee \subseteq E^\vee$. Tensoring with L and composing with the inclusion $E^\vee \otimes L \subseteq E^\vee \otimes E$ we obtain an inclusion $L^2 \otimes \Lambda^2 E^\vee \subseteq \mathrm{End}(E)$. Choosing a basis on each fibre and chasing all the identifications we have made so far, we see that the image of $L^2 \otimes \Lambda^2 E^\vee$ lies actually in $\mathrm{End}_0(E)$. Indeed, let V be a two dimensional $\mathbb{C}$-vector space and let e_1 and e_2 be a basis. Let L be the line spanned by a non-zero vector l, which we may assume to be e_1. The first identification we have is $V \cong \mathrm{Hom}(V, \Lambda^2 V)$, sending v to the homomorphism $v' \longmapsto v' \wedge v$. This corresponds to $\alpha_v \otimes (e_1 \wedge e_2) \in V^\vee \otimes \Lambda^2 V$, where $\alpha_v \in V^\vee$ is the linear form sending $e_1 \longmapsto v_2$ and $e_2 \longmapsto -v_1$. Denoting by $\overline{\alpha_v}$ its image in $L^\vee$, we can describe the morphism corresponding to the right hand side of the previous short exact sequence as

$$\begin{aligned} V &\longrightarrow L^\vee \otimes \Lambda^2 V \\ v &\longmapsto \overline{\alpha_v} \otimes (e_1 \wedge e_2) \end{aligned}$$

Let now $\beta \in (L^\vee \otimes \Lambda^2 V)^\vee$ and denote by λ_v^β the complex number such that

$$\overline{\alpha_v} \otimes (e_1 \wedge e_2) \overset{\beta}{\longmapsto} \lambda_v^\beta.$$

A point $\mu l \otimes \beta \in L \otimes (L^\vee \otimes \Lambda^2 V)^\vee$ corresponds then to the endomorphism

$$\begin{aligned} V &\longrightarrow V \\ v &\longmapsto \mu\lambda_v^\beta l \end{aligned}$$

A basis for L is e_1, a basis for $L^\vee \otimes \Lambda^2 V$ is $\overline{\alpha_{e_2}} \otimes (e_1 \wedge e_2)$ and a basis for $L \otimes (L^\vee \otimes \Lambda^2)^\vee$ is $e_1 \otimes \beta_0$, where $\beta_0 \in (L^\vee \otimes \Lambda^2 V)^\vee$ is such that $\lambda_{e_2}^{\beta_0} = 1$. Writing the image of the basis $e_1 \otimes \beta_0$ under the map $L \otimes (L^\vee \otimes \Lambda^2 V)^\vee \longrightarrow \mathrm{End}(V)$ as a matrix with respect to our bases we obtain

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

because $\overline{\alpha_{e_1}} = 0$ and therefore $\lambda_{e_1}^{\beta} = 0$ for any β. We have thus the desired injective homomorphism

$$L^2 \otimes \Lambda^2 V^\vee \hookrightarrow \mathrm{End}_0(V)$$

whose image are the traceless endomorphisms which preserve only L. This is the morphism from which we obtain the short exact sequence in a).

We regard this as a homomorphism into $\mathrm{End}(V)$ for a moment and use the basis $e_{11}, e_{12}, e_{21}, e_{22}$ of $\mathrm{End}(V)$, where e_{ij} denotes the endomorphism which, represented as a matrix in terms of our basis, has zeros everywhere except for a 1 in the ij-th position. Then our homomorphism is given by

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Dualizing it we obtain a surjection

$$\mathrm{End}_0(V^\vee) \twoheadrightarrow L^{-2} \otimes \Lambda^2 V$$

given with respect to the dual bases by

$$\begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}.$$

Its kernel are the endomorphisms of $V^\vee$ represented with respect to the dual basis by a matrix of the form

$$\begin{pmatrix} a & 0 \\ b & -a \end{pmatrix}.$$

Hence, under the isomorphism $\mathrm{End}(V) \cong \mathrm{End}(V^\vee)$ given in coordinates by sending a matrix to its transpose, we obtain a surjection

$$\mathrm{End}_0(V) \twoheadrightarrow L^{-2} \otimes \Lambda^2 V$$

whose kernel are endomorphisms represented with respect to our basis by matrices of the form

$$\begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}.$$

Therefore the kernel of this surjection consists precisely of the traceless endomorphisms of V that leave at least L invariant. The inclusion of this kernel can be naturally regarded as the composition of the inclusion $\mathrm{Hom}(V, L) \subseteq \mathrm{Hom}(V, V)$ and the projection $\mathrm{pr}_0 : \mathrm{Hom}(V, V) \longrightarrow \mathrm{End}_0(V)$, which writing every homomorphism as a matrix with respect to the bases above has the form

$$\begin{pmatrix} a & b \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} \frac{a}{2} & b \\ 0 & -\frac{a}{2} \end{pmatrix}.$$

This gives us the short exact sequence in b) and the right hand side of the short exact sequence in a). ☐

Lemma 4. *If $g \geqslant 2$ and E is a stable rank 2 vector bundle, then $E \in \mathbf{A}$.*

Proof. By Lemma 2 it suffices to show that it is not in $\mathbf{B}$. So let E be a stable rank 2 vector bundle on M and assume $L \subseteq E$ is a line bundle which is Φ-invariant for all $\Phi \in H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K})$. Consider the short exact sequence b) from Lemma 3

$$0 \longrightarrow \mathcal{E}^\vee \otimes \mathcal{L} \otimes \mathcal{K} \longrightarrow \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K} \longrightarrow \mathcal{L}^{-2} \otimes (\Lambda^2\mathcal{E}) \otimes \mathcal{K} \longrightarrow 0.$$

Since all $\Phi \in H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K})$ leave L invariant, we get an induced isomorphism on global sections $H^0(M, \mathcal{E}^\vee \otimes \mathcal{L} \otimes \mathcal{K}) \cong H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K})$. The left hand side of the short exact sequence a) in Lemma 3 factors by construction into a short exact sequence of the form

$$0 \longrightarrow \mathcal{L}^2 \otimes (\Lambda^2\mathcal{E})^\vee \otimes \mathcal{K} \longrightarrow \mathcal{E}^\vee \otimes \mathcal{L} \otimes \mathcal{K} \longrightarrow \mathcal{Q} \otimes \mathcal{K} \longrightarrow 0,$$

where $\deg(\mathcal{Q})$ can be seen to be 0 by computing first Chern classes. Riemann–Roch says then that $h^0(\mathcal{Q} \otimes \mathcal{K}) \in \{g-1, g\}$. Since E is stable, we have $\deg(\mathcal{L}^2 \otimes \Lambda^2\mathcal{E}^\vee) < 0$, and since the complete linear system corresponding to $\mathcal{K}$ is base-point free[3] we have $h^0(M, \mathcal{L}^2 \otimes (\Lambda^2\mathcal{E})^\vee \otimes \mathcal{K}) \leqslant g-1$ by [Har77, Proposition IV.3.1]. The long exact sequence of the previous short exact sequence gives then

$$h^0(M, \mathcal{E}^\vee \otimes \mathcal{L} \otimes \mathcal{K}) \leqslant 2g-1.$$

The earlier Hirzebruch–Riemann–Roch computation showed that

$$h^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K}) \geqslant 3g-3.$$

If we want the two dimensions to be equal we must have $g = 2$ and $h^0(M, \mathcal{K} \otimes \mathcal{L} \otimes \mathcal{E}^\vee) = 3$. From the same long exact sequence as before we deduce, using that $h^0(M, \mathcal{L}^2 \otimes (\Lambda^2\mathcal{E})^\vee \otimes \mathcal{K}) < g = 2$, that $h^0(M, \mathcal{L}^2 \otimes (\Lambda^2\mathcal{E})^\vee \otimes \mathcal{K}) = 1$. In particular, $\deg(\mathcal{L}^2 \otimes (\Lambda^2\mathcal{E})^\vee \otimes \mathcal{K}) \geqslant 0$. We have $\deg(\mathcal{K}) = 2$ and by stability we had $\deg(\mathcal{L}^2 \otimes \Lambda^2\mathcal{E}^\vee) < 0$, so we must have

$$\deg(\mathcal{L}^2 \otimes (\Lambda^2\mathcal{E})^\vee \otimes \mathcal{K}) \in \{0, 1\}.$$

[3] Suppose that $x \in M$ was a base-point of $|\mathcal{K}|$. This would mean that $h^0(\mathcal{K}) = h^0(\mathcal{K} \otimes \mathcal{O}(-x)) = g$, so by Riemann–Roch we would have $h^0(\mathcal{O}(x)) = 2$. For any other $x' \in M$ we would have that $\mathcal{O}(x-x')$ is a degree 0 line bundle with $h^0(\mathcal{O}(x-x')) \in \{1, 2\}$ [Har77, Proof of Proposition IV.3.1], hence the trivial line bundle. This would imply that any two points in M are linearly equivalent, hence $M = \mathbb{P}^1$ [Har77, Example II.6.10.1], a contradiction.

If it is 0, then the existence of global sections implieis that it is the trivial line bundle, hence the previous short exact sequence becomes

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{E}^\vee \otimes \mathcal{L} \otimes \mathcal{K} \longrightarrow \mathcal{K} \longrightarrow 0.$$

Split short exact sequences are preserved by dualizing and tensoring with line bundles, so if this sequence was split then $\mathcal{E}$ would be decomposable as a direct sum of line bundles. But this would give more endomorphisms of $\mathcal{E}$ than there should be, since stable vector bundles are simple. So the previous short exact sequence is a non-trivial extension and the coboundary map $H^0(M, \mathcal{K}) \longrightarrow H^1(M, \mathcal{O})$ is non-zero. The long exact sequence implies then that $h^0(M, \mathcal{E}^\vee \otimes \mathcal{K} \otimes \mathcal{L}) \leqslant 2$, contradicting our previous conclusion that this dimension was 2.

If it is 1, again by the existence of a non-zero global secction we deduce that it is the line bundle corresponding to some point $x \in M$. The short exact sequence becomes

$$0 \longrightarrow \mathcal{O}(x) \longrightarrow \mathcal{E}^\vee \otimes \mathcal{L} \otimes \mathcal{K} \longrightarrow \mathcal{K} \longrightarrow 0,$$

and the coboundary map is again non-zero, yielding the same contradiction as before. □

Lemma 5. *If $g \geqslant 2$, then*

$$\mathbf{S} = \mathbf{A}.$$

Proof. By definition $\mathbf{A} \subseteq \mathbf{S}$, so let us see the other inclusion. Let $E \in \mathbf{S}$. If E is stable, then $E \in \mathbf{A}$ by Lemma 4. So assume there exists $L \subseteq E$ such that $\mu(L) \geqslant \mu(E)$. Then $\deg(\mathcal{L}^2 \otimes (\Lambda^2\mathcal{E})^\vee \otimes \mathcal{K}) \geqslant 2g-2$, so Riemann–Roch implies that $h^0(\mathcal{L}^2 \otimes (\Lambda^2\mathcal{E})^\vee \otimes \mathcal{K}) \in \{g-1, g\}$. In particular, from the short exact sequence *a*) in Lemma 3 we deduce that there exists a non-zero Higgs field leaving only L invariant. So if $E \in \mathbf{B}$, then this L is a line bundle invariant by all $\Phi \in H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K})$. Since $\mu(L) \geqslant \mu(E)$, this contradicts the assumption that $E \in \mathbf{S}$. □

2. Idea of the existence theorem

We take a step back from the algebraic/holomorphic realm into the smooth setting, following the discussion in [Wen14, §2.1.1]. Let $E \longrightarrow M$ be a smooth complex rank 2 vector bundle over a compact Riemann surface M of genus $g \geqslant 2$. A holomorphic structure structure on E is equivalent to the choice of a $\bar\partial$-operator [Huy05, Theorem 2.6.26] satisfying the Leibniz rule and an integrability condition, which in our case is trivially verified because $d\bar z \wedge d\bar z = 0$. Let h be a hermitian structure on E, which can be glued from standard ones using a partition of unity [Huy05, Proposition 4.1.4]. Then there is a unique hermitian connection compatible with the holomorphic structure [Huy05, Proposition 4.2.14], which we called the Chern connection.

Let now $\mathcal{A}_E$ be the space of hermitian connections on the hermitian bundle (E, h), which is an affine space over the infinite-dimensional real vector subspace $A^1(\mathfrak{u}_h(E)) \subseteq A^1(\mathrm{End}(E))$ consisting of 1-forms with values in skew-hermitian endomorphisms [Huy05, Corollary 4.2.11]. Chern connections allow us to identify $\mathcal{A}_E$ with the space of holomorphic structures on E.

Definition. We consider the *gauge group*

$$\mathcal{G}_E = A^0(\mathrm{U}_h(E)) = \{g \in A^0(\mathrm{End}(E)) \mid gg^* = \mathrm{id}_E\},$$

where g^* is the h-adjoint endomorphism of g.

We will be mainly interested in the special unitary subgroup of gauge transformations, whose complexification is the group $A^0(\mathrm{Aut}_0(E))$ of automorphisms of E with constant determinant 1, because $\mathrm{SU}(2)^{\mathbb{C}} = \mathrm{SL}(2, \mathbb{C})$ [Kna02, p. 376].

Definition. We consider now the Banach manifold

$$\mathcal{B}_E = \mathcal{A}_E \times A^{1,0}(\mathrm{End}(E))$$

with the symplectic structure given by

$$\mathcal{K}((A_1, \Phi_1), (A_2, \Phi_2)) = -\int_M \mathrm{tr}(A_1 \wedge A_2) + 2i\,\mathrm{Im}(\mathrm{tr}(\Phi_1 \Phi_2^*)),$$

for $A_1, A_2 \in A^1(\mathfrak{u}_h(E))$, $\Phi_1, \Phi_2 \in A^{1,0}(\mathrm{End}(E))$ and Φ^* denoting the h-adjoint of Φ.

Fact 6. *The gauge group $\mathcal{G}_E$ acts nicely on both factors of $\mathcal{B}_E$ by conjugation, and this action admits a momentum map*

$$\mu(\nabla, \Phi) = -F - [\Phi, \Phi^*] - 2\pi i \mu(E)\,\mathrm{id}_E\,\mathcal{K}_M,$$

where F denotes the curvature of ∇.

References for this fact. This is briefly discussed in [Hit87, §4]. A more detailed account can be found in [**?**, Proposition III.3.2]. □

We also restrict our attention to the subspace of $\mathcal{B}'_E \subseteq \mathcal{B}_E$ consisting of pairs (∇, Φ) such that $\nabla^{0,1}\Phi = 0$, i.e. $\Phi \in H^0(M, \mathrm{End}(E) \otimes K)$ is holomorphic with respect to the holomorphic structure defined by ∇. We want to solve the self-duality equation

$$\mu(\nabla, \Phi) = 0$$

for $(\nabla, \Phi) \in \mathcal{B}'_E$. If we fix a holomorphic structure on E, we can think of the gauge group acting on the hermitian metric

$$(g \cdot h)(s_1, s_2) = h(g \cdot s_1, g \cdot s_2).$$

From this perspective, we need to find a Higgs field $\Phi \in H^0(M, \mathrm{End}(E) \otimes K)$ and a hermitian metric h on E such that the the Chern connection ∇ corresponding to the hermitian metric and the fixed holomorphic structure satisfies $\mu(\nabla, \Phi) = 0$.

Notation and conventions

We usually follow the notation of [Hit87]:

- M: compact Riemann surface of genus g.
- $O \longrightarrow M$: trivial line bundle.
- $K \longrightarrow M$: canonical line bundle.
- More generally, O_X and K_X denote the trivial and canonical line bundles over a complex manifold X.

Every now and then we will also use some other standard notation, for instance:

- For a vector bundle $E \longrightarrow M$ we denote $\mu(E) := \deg E / \operatorname{rk} E$.
- Let X be a complex manifold and $E \longrightarrow X$ a (holomorphic/algebraic) vector bundle. Then we denote by $\mathcal{E}$ its sheaf of sections. The assignement $E \longmapsto \mathcal{E}$ defines an equivalence of categories between vector bundles on X and locally free sheaves of $\mathcal{O}_X$-modules of finite rank, which we will refer to simply as locally free sheaves. We can recover E from $\mathcal{E}$ either using cocycles [Voi02, Lemma 4.8] or by using the relative spectrum [Har77, Exercise II.5.18]. Following the second approach we would write
$$E = \mathrm{Spec}_X(S(\mathcal{E}^\vee)) =: \mathbb{V}(\mathcal{E}^\vee),$$
where $S(-)$ denotes the symmetric algebra.
- $\mathcal{O}$ and ω denote the trivial and canonical invertible sheaves on M. More generally, $\mathcal{O}_X$ and ω_X denote the trivial and canonical invertible sheaves on a complex manifold X,
- Let E be again a vector bundle on a complex manifold X. We will denote its projectivisation by $\mathbb{P}(E)$, which is obtained form E without its zero section by quotienting out the fibre-wise multiplication by scalars. Taking again the algebraic approach we can write
$$\mathbb{P}(E) = \mathrm{Proj}_X(S(\mathcal{E}^\vee)) =: \mathbb{P}(\mathcal{E}^\vee).$$
- Let N be a smooth manifold and $E \longrightarrow N$ a smooth vector bundle. Then we denote by $\mathfrak{X}(N)$ the Lie algebra of smooth vector fields on N and by $\Omega^k(N, E)$ the vector space of smooth k-differential forms with values in E, which can be thought of as smooth global sections of the vector bundle $\mathrm{Hom}(TN, E)$.

- Let N be a smooth manifold equipped with an almost complex structure $I\colon\ TN \longrightarrow TN$. Then we denote by $\Omega^{i,j}(N, E)$...

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