HIGGS BUNDLES — EXISTENCE OF SOLUTIONS

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Abstract. In this talk we introduce the stability condition for Higgs bundles and prove the Hitchin–Kobayashi correspondence. The main result is [Hit87, Theorem 4.3]. Relevant literature is [Hit87, §3 and §4].

Contents

1. Stability

Let M be a compact Riemann surface.

Definition (Higgs bundle)**.** A *Higgs bundle* on M is a pair (E, Φ), where $E \longrightarrow M$ is a rank 2 vector bundle and Φ is a global section of $\operatorname{End} E \otimes K$, called a *Higgs field* on E.

Remark. Using the canonical isomorphisms

$$H^0(M, \mathcal{E}nd(\mathcal{E}) \otimes \omega) \cong \operatorname{Hom}(\mathcal{O}, \mathcal{E}^\vee \otimes \mathcal{E} \otimes \omega) \cong \operatorname{Hom}(\mathcal{E}, \mathcal{E} \otimes \omega)$$

we may identify Φ with a morphism

$$\Phi : E \longrightarrow E \otimes K.$$

Definition (Stability)**.** A Higgs bundle (E, Φ) is said to be *stable* if for every Φ-invariant line bundle $L \subseteq E$ we have $\mu(L) < \mu(E)$, where Φ-invariance means that $\Phi(L) \subseteq L \otimes K$.

Remark. $(E, 0)$ is stable if and only if E is stable in the usual sense.

Date: 15 July 2020.

Supported by the DFG-Graduiertenkolleg GK1821 "Cohomological Methods in Geometry" at the University of Freiburg.

Exercise A. There are no stable Higgs bundles on $\mathbb{P}^1$. [*Hints below*[1]]

Lemma 1. *Let (E_1, Φ_1) and (E_2, Φ_2) be stable pairs with $\Lambda^2 E_1 \cong \Lambda^2 E_2$. Let $\Psi : E_1 \longrightarrow E_2$ be a non-zero morphism such that $(\Psi \otimes \mathrm{id}_K) \circ \Phi_1 = \Phi_2 \circ \Psi$. Then Ψ is an isomorphism.*

Proof. We prove the result by contradiction. Suppose that Ψ is not an isomorphism. The rank $x \mapsto \dim_{\mathbb{C}} \Psi_x(E_{1,x})$ is upper semi-continuous [Ati89, Proposition 1.3.2], so the rank of Ψ cannot be generically zero. If the rank was generically 2, then $\det(\Psi) \in H^0(M, \Lambda^2 \mathcal{E}_1^\vee \otimes \Lambda^2 \mathcal{E}_2)$ would be generically non-zero. But $\Lambda^2 E_1 \cong \Lambda^2 E_2$, so $\det(\Psi) \in H^0(M, \mathcal{O}) = \mathbb{C}$ must be a constant and Ψ would be an isomorphism. Therefore the rank is generically 1, only going down to 0 at special points.

Let $L_1 \subseteq E_1$ be the largest rank 1 subbundle of E_1 contained in the kernel of Ψ. Let $v_1 \in L_{1,x}$, and let z be a holomorphic coordinate around a general point $x \in M$. Then we can write $\Phi_1(v_1) = \phi_{1,x}(v_1) \otimes dz$ for some $\phi_{1,x} \in \mathrm{End}(E_{1,x})$. Then

$$0 = \Phi_{2,x}(\Psi_x(v_1)) = (\Psi \otimes \mathrm{id}_K)_x(\phi_{1,x}(v_1) \otimes dz) = \Psi_x(\phi_{1,x}(v_1)) \otimes dz,$$

so $\phi_{1,x}(v_1) \in \mathrm{Ker}(\Psi_x) = L_{1,x}$. Since it suffices to check $\Phi_1(L_1) \subseteq L_1 \otimes K$ generically, this shows that L_1 is Φ_1-invariant.

Let now $L_2 \subseteq E_2$ be the largest rank 1 subbundle of E_2 containing the image of Ψ. Let $v_2 = \Psi(v_1) \in L_{2,x}$ be a vector over a general point $x \in M$, which can thus be written as the image under Ψ of someone in E_1. Then

$$\Phi_{2,x}(\Psi_x(v_1)) = (\Psi \otimes \mathrm{id}_K)_x(\phi_{1,x}(v_1) \otimes dz) = \Psi_x(\phi_{1,x}(v_1)) \otimes dz,$$

so $\phi_{2,x}(v_2) \in \mathrm{Im}(\Psi_x) = L_{2,x}$. Since it suffices to check $\Phi_2(L_2) \subseteq L_2 \otimes K$ generically, this shows that L_2 is Φ_2-invariant.

Now we use that (E_i, Φ_i) are stable to deduce that

$$\deg(L_i) < \frac{d}{2}$$

for $i \in \{1, 2\}$, where $d := \deg(\Lambda^2 E_1) = \deg(\Lambda^2 E_2)$. Since L_1 is contained in the kernel of Ψ, Ψ induces a non-zero morphism of line bundles $E_1/L_1 \longrightarrow L_2$, which corresponds to a non-zero global section of $(E_1/L_1)^\vee \otimes L_2$. Line bundles with negative degree do not have any non-zero global sections, so we must have $\deg(E_1/L_1) \leqslant \deg(L_2)$. Therefore

$$\frac{d}{2} < \deg(\Lambda^2 E_1) - \deg(L_1) = \deg(E_1/L_1) \leqslant \deg(L_2) < \frac{d}{2},$$

a contradiction. Hence Ψ must be an isomorphism. □

[1] Grothedieck's theorem allows us to write Φ as a matrix. What can we say about each entry? The solution can be found in [Hit87, Remark (3.2) (iii)]

Lemma 2. *Let $E \longrightarrow M$ be a rank 2 vector bundle and denote by* $\mathrm{End}_0(E)$ *the vector bundle of traceless endomorphisms. Then there is a natural projection* $\mathrm{pr}_0 : \mathrm{End}(E) \longrightarrow \mathrm{End}_0(E)$ *whose kernel is the trivial line bundle of multiples of the identity, yielding a short exact sequence*

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{E}nd(\mathcal{E}) \longrightarrow \mathcal{E}nd_0(\mathcal{E}) \longrightarrow 0.$$

In particular, $\deg(\mathrm{ch}(\mathcal{E}nd_0(\mathcal{E}) \otimes \omega)\,\mathrm{td}(\omega^\vee))_1 = 3g - 3$.

Proof. Over $x \in M$, the map $\mathrm{pr}_{0,x} : \mathrm{End}(E_x) \longrightarrow \mathrm{End}_0(E_x)$ is given by

$$A \longmapsto A - \frac{\mathrm{tr}(A)}{2}\,\mathrm{id}_{E_x}.$$

The endomorphisms in the kernel are precisely the multiples of the identity. This fibre-wise description globalizes to the desired short exact sequence.

For the Chern class computation we use the axioms in [Har77, Appendix A]. We have

$$c_1(\mathcal{E}nd_0(\mathcal{E})) = c_1(\mathcal{E}nd(\mathcal{E})) = c_1(\mathcal{E}^\vee \otimes \mathcal{E}) = 0,$$

therefore

$$\mathrm{ch}(\mathcal{E}nd_0(\mathcal{E}) \otimes \omega) = 3 + c_1(\mathcal{E}nd_0(\mathcal{E})) + 3c_1(\omega) = 3 + 3c_1(\omega).$$

We also have

$$\mathrm{td}(\omega^\vee) = 1 - \frac{c_1(\omega)}{2},$$

so multiplying the two expressions we obtain

$$\mathrm{ch}(\mathcal{E}nd_0(\mathcal{E}) \otimes \omega)\,\mathrm{td}(\omega^\vee) = 3 + \frac{3}{2}c_1(\omega).$$

Since $\deg(c_1(\omega)) = 2g - 2$, the result follows. □

Notation. Let us denote by $\mathbf{Vec}_2(M)$ the set of rank 2 vector bundles on M. We define the following subsets:

- $\mathbf{S} := \{E \in \mathbf{Vec}_2(M) \mid \exists \Phi \text{ such that } (E, \Phi) \text{ is stable}\}$.
- $\mathbf{A} := \{E \in \mathbf{Vec}_2(M) \mid \text{a general } \Phi \text{ has no invariant } L\}$.
- $\mathbf{B} := \{E \in \mathbf{Vec}_2(M) \mid \exists L \text{ invariant for all } \Phi\}$.

Lemma 3. *If $g \geqslant 2$, then*

$$\mathbf{Vec}_2(M) = \mathbf{A} \sqcup \mathbf{B}.$$

Proof. Let $\pi : \mathbb{P}(E) \longrightarrow M$ be the projectivization of our rank 2 vector bundle and let $O(-1) \longrightarrow \mathbb{P}(E)$ denote the tautological line bundle, whose fiber over $[v] \in \mathbb{P}(E)$ is the line $\{\lambda v \mid \lambda \in \mathbb{C}\} \subseteq E_{\pi([v])}$ spanned by v. Denote also $O(l) := O(-1)^{\otimes(-l)}$. If $\mathcal{F}$ is a sheaf on $\mathbb{P}(E)$, we denote $\mathcal{F}(l) := \mathcal{F} \otimes \mathcal{O}(l)$, where $\mathcal{O}(l)$ denotes the sheaf of sections of $O(l)$. We have $\pi_*\mathcal{O}(l) = S^l(\mathcal{E}^\vee)$ for all $l \geqslant 0$ and $\pi_*\mathcal{O}(l) = 0$ for all $l < 0$ [Har77, Exercise III.8.4].

Let $x \in M$. Given $A \in \mathrm{End}(E_x)$, we define the quadratic form $v \mapsto Av \wedge v$ with values in $\Lambda^2 E_x$, which can then be naturally regarded as an element in $S^2(E_x^\vee) \otimes \Lambda^2 E_x$. The resulting quadratic form is trivial precisely when $A = \lambda \,\mathrm{id}_{E_x}$ for some $\lambda \in \mathbb{C}$, so by Lemma 2 we obtain an injective homomorphism $\mathrm{End}_0(E_x) \longrightarrow S^2(E_x^\vee) \otimes \Lambda^2 E_x$. Both vector spaces have the same dimension, so this must be an isomorphism. These isomorphisms globalize to an isomorphism $\mathrm{End}_0(E) \cong S^2(E^\vee) \otimes \Lambda^2 E$, hence we obtain an isomorphism

$$\mathcal{E}nd_0(\mathcal{E}) \otimes \omega \cong \pi_* \mathcal{O}(2) \otimes \omega \otimes \Lambda^2 \mathcal{E}.$$

The projection formula yields now an isomorphism $\mathcal{E}nd_0(\mathcal{E}) \otimes \omega \cong \pi_*(\pi^*(\omega \otimes \Lambda^2 \mathcal{E})(2))$, hence an isomorphism

$$\psi : H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \omega) \cong H^0(\mathbb{P}(E), \pi^*(\omega \otimes \Lambda^2 \mathcal{E})(2)).$$

Let now $\Phi \in H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \omega)$. A line bundle $L \subseteq E$ is then Φ-invariant precisely when $\psi(\Phi)$ vanishes at all $[v] \in \mathbb{P}(L) \subseteq \mathbb{P}(E)$. In other words, L is Φ-invariant if and only if $\sigma(M) \subseteq \mathrm{div}(\psi(\Phi))$, where $\mathrm{div}(-)$ denotes the divisor of zeros of a section and $\sigma : M = \mathbb{P}(L) \longrightarrow \mathbb{P}(E)$ is the section induced by $L \subseteq E$.

Suppose now that $\Phi \in H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \omega)$ is not nilpotent and let $L \subseteq E$ be a Φ-invariant line bundle. Then over a general point $x \in M$, the corresponding traceless endomorphism $\phi_x \in \mathrm{End}_0(E_x)$ is diagonalizable, so we can find some eigenvector $v \in E_x \setminus L_x$ in an eigenspace other than L_x. This gives us a point $[v] \in \mathbb{P}(E) \setminus \sigma(M)$ on which $\psi(\Phi)$ vanishes. Hence $\sigma(M)$ is a proper irreducible component of the divisor $\mathrm{div}(\psi(\Phi))$.

The previous discussion shows that if Φ is not nilpotent and $\mathrm{div}(\psi(\Phi))$ is irreducible, then there are no invariant line bundles $L \subseteq E$. By Hirzebruch–Riemann–Roch and Lemma 2 we have

$$h^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \omega) \geqslant 3g - 3 \geqslant 3,$$

so the complete linear system defined by the invertible sheaf $\pi^*(\omega \otimes \Lambda^2 \mathcal{E})(2)$ has dimension at least 2. If this linear system does not have a fixed divisor, then Bertini's theorem [Iit82, Theorem 7.19] tells us that $\mathrm{div}(\psi(\Phi))$ is irreducible for a general $\Phi \in H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \omega)$. Since in our case $\Phi \in H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \omega)$ is nilpotent if and only if $\Phi^2 = 0$, a general Φ is not nilpotent. Therefore $E \in \mathbf{A}$ in this case.

Let us see what happens if the linear system has a fixed divisor. Such a fixed divisor must be the zero locus of a non-zero global section s of an invertible sheaf on $\mathbb{P}(E)$, which is up to isomorphism of the form $\pi^*\mathcal{L}(l)$ with $\mathcal{L}$ an invertible sheaf on M and $l \in \mathbb{Z}$ [Har77, Exercise II.7.9]. Being a fixed divisor means then that every other global section of our line bundle can be written as a product st, where $t \in H^0(\mathbb{P}(E), \pi^*\mathcal{N}(2 - l))$. Since our line bundle had non-zero global sections, both $\pi^*\mathcal{L}(l)$ and $\pi^*\mathcal{N}(2 - l)$ must have non-zero global sections. By the projection formula, this leaves us with only three possibilities:

a) $l = 0$;
b) $l = 1$;
c) $l = 2$.

Let us start with case *a*). Let $\pi^* s \in H^0(\mathbb{P}(E), \pi^*\mathcal{L}) \cong H^0(M, \mathcal{L})$ be a global section corresponding to the fixed component of our linear system. Dividing all global sections by s and by $\pi^* s$ respectively we obtain the following commutative diagram:

$$\begin{tikzcd}
H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \omega) \arrow[r, "\cong"] \arrow[d, "/s"', "\cong"] & H^0(\mathbb{P}(E), \pi^*(\omega \otimes \Lambda^2\mathcal{E})(2)) \arrow[d, "/\pi^* s"', "\cong"] \\
H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \omega \otimes \mathcal{L}^\vee) \arrow[r, "\cong"] & H^0(\mathbb{P}(E), \pi^*(\mathcal{L}^\vee \otimes \omega \otimes \Lambda^2\mathcal{E})(2))
\end{tikzcd}$$

By definition, the new linear system does not have any fixed divisors and has the same dimension. Hence we can apply Bertini to conclude that a generic $\Phi' \in H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \omega \otimes \mathcal{L}^\vee)$ does not have invariant line bundles, which in this case are defined as line bundles $N \subseteq E$ such that $\Phi'(N) \subseteq N \otimes K \otimes L^\vee$. But a line bundle $N \subseteq E$ is Φ'-invariant if and only if it is $s\Phi'$-invariant, so we have $E \in \mathbf{A}$ in this case.

We move on to case *b*). Assume that the fixed divisor corresponds to a non-zero global section $s \in H^0(\mathbb{P}(E), \pi^*\mathcal{L}(1))$. This corresponds to a non-zero morphism $E \longrightarrow L$. The fibre-wise kernel has then dimension 1 generically and 2 at special points by upper semi-continuity [Ati89, Proposition 1.3.2]. Hence we can find a line bundle $N \subseteq \mathrm{Ker}(s) \subseteq E$, which can be described as the largest line subbundle of E contained in the kernel of s. If $v \in N$ is a non-zero vector, then $s(v) = 0$ and so $[v] \in \mathrm{div}(s) \subseteq \mathrm{div}(\psi(\Phi))$. Thus the corresponding section $\sigma(M) \subseteq \mathbb{P}(E)$ is contained in $\mathrm{div}(\psi(\Phi))$ for all Φ and N is Φ-invariant for all Φ. Hence $E \in \mathbf{B}$ in this case.

In case *c*), the fixed divisor corresponds to a non-zero global section of $\pi^*\mathcal{L}(2)$. We have

$$H^0(\mathbb{P}(E), \pi^*\mathcal{L}(2)) \cong H^0(M, \mathcal{L} \otimes S^2\mathcal{E}^\vee) \cong H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{L} \otimes \Lambda^2\mathcal{E}^\vee).$$

So we can think of the fixed global section s as a traceless endomorphism of E with coefficients in $L \otimes \Lambda^2 E^\vee$. With this point of view, s-invariance of a line bundle $N \subseteq E$ translates into $s\Phi'$-invariance of $N \subseteq E$ as before, where $s\Phi'$ is a Higgs field. Let us see that the fixed section s has some non-trivial kernel, hence defining a line bundle invariant under all Higgs fields as in the previous case. To show that there is some non-trivial kernel, it suffices to check that $\det(s) = 0$. Since s is traceless, it suffices in turn to check that $\mathrm{tr}(s^2) = 0$. Suppose on the contrary that $\mathrm{tr}(s^2) \neq 0$. Fix some non-zero

$s_1 \in H^0(M, \mathcal{L}^\vee \otimes \omega \otimes \Lambda^2\mathcal{E})$ and consider the linear map

$$\begin{aligned} \theta:\ H^0(M, \mathcal{L}^\vee \otimes \omega \otimes \Lambda^2\mathcal{E}) &\longrightarrow H^0(M, \omega^2) \\ \Phi' &\longmapsto \operatorname{tr}(s^2)s_1\Phi' \end{aligned}$$

Since $\operatorname{tr}(s^2)s_1$ can only vanish at finitely many points, the image of a non-zero Φ' can only vanish at finitely many points, hence θ is injective. From Hirzebruch–Riemann–Roch and Lemma 2 we know that

$$h^0(M, \mathcal{L}^\vee \otimes \omega \otimes \Lambda^2\mathcal{E}) \geqslant 3g - 3 = h^0(M, \omega^2),$$

so θ is an isomorphism. Since $\mathcal{L}^\vee \otimes \omega \otimes \Lambda^2\mathcal{E}$ has global sections, its degree is non-negative. If it was zero, then this would be the trivial line bundle and we would have $h^0(M, \mathcal{L}^\vee \otimes \omega \otimes \Lambda^2\mathcal{E}) = 1 < 3g - 3$, a contradiction. Hence $\deg(\mathcal{L}^\vee \otimes \omega \otimes \Lambda^2\mathcal{E}) > 0$ and the non-zero global section s_1 has at least one zero. If θ was indeed an isomorphism, then each zero of s_1 would give a base point of the complete linear system corresponding to ω^2. But $\deg(\omega^2) = 4g - 4 \geqslant 2g$, so this linear system has no base points [Har77, Corollary IV.3.2]. This contradiction shows that s has non-trivial kernel, which contains a line bundle $N \subseteq E$ invariant by all $\Phi \in H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \omega)$. Hence $E \in \mathbf{B}$ as well in this case. □

Exercise B. Assume $g \geqslant 2$. Let $K^{\frac{1}{2}}$ be a line bundle whose square is K and let $K^{-\frac{1}{2}}$ be its inverse. Does $K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}$ belong to $\mathbf{A}$ or to $\mathbf{B}$? [*Hints below*[2]]

Lemma 4. *Let $E \longrightarrow M$ be a rank 2 vector bundle and denote by $\operatorname{End}_0(E)$ the vector bundle of traceless endomorphisms. Let $L \subseteq E$ be a line bundle. Then there is an injective morphism*

$$L^2 \otimes \Lambda^2 E \hookrightarrow \operatorname{End}_0(E)$$

whose image are the traceless endomorphisms which preserve only the line bundle L. Dualizing this we obtain a surjection

$$\operatorname{End}_0(E) \cong \operatorname{End}_0(E^\vee) \twoheadrightarrow L^{-2} \otimes \Lambda^2 E$$

whose kernel are the traceless endomorphisms which preserve at least the line bundle L. We can realize this kernel as the inclusion $\operatorname{Hom}(E, L) \subseteq \operatorname{Hom}(E, E)$

[2]Consider the family of traceless endomorphisms given by

$$\Phi_\alpha := \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix},$$

parametrized by quadratic differentials $\alpha \in H^0(M, K^2)$. Use without proof the fact that an invariant line bundle exists if and only if the characteristic polynomial

$$\lambda^2 - \alpha$$

has a root in $H^0(M, K)$, i.e. if and only if the quadratic differential α can be written as a square $\alpha = \beta^2$ for some differential form $\beta \in H^0(M, K)$. If α was a square, its zeros would all have multiplicity at least two. Conclude that $K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}} \in \mathbf{A}$ using Bertini's theorem.

followed by the projection $\mathrm{pr}_0 : \ \mathrm{Hom}(E,E) \longrightarrow \mathrm{End}_0(E)$*, yielding a short exact sequence*

$$0 \longrightarrow \mathcal{E}^\vee \otimes \mathcal{L} \longrightarrow \mathcal{E}nd_0(\mathcal{E}) \longrightarrow \mathcal{L}^{-2} \otimes \Lambda^2 \mathcal{E} \longrightarrow 0.$$

Proof. Under the isomorphism $E \cong E^\vee \otimes \Lambda^2 E$ [Har77, Exercise II.5.16], the line bundle L is sent to linear forms with coefficients in $\Lambda^2 E$ vanishing along L, hence we have a short exact sequence

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{E} \longrightarrow \mathcal{L}^\vee \otimes \Lambda^2 \mathcal{E} \longrightarrow 0.$$

Dualizing this short exact sequence we obtain an inclusion $L \otimes \Lambda^2 E^\vee \subseteq E^\vee$. Tensoring with L and composing with the inclusion $E^\vee \otimes L \subseteq E^\vee \otimes E$ we obtain an inclusion $L^2 \otimes \Lambda^2 E^\vee \subseteq \mathrm{End}(E)$. Choosing a basis on each fibre and chasing all the identifications we have made so far, we see that the image of $L^2 \otimes \Lambda^2 E^\vee$ lies actually in $\mathrm{End}_0(E)$. Indeed, let V be a two dimensional $\mathbb{C}$-vector space and let e_1 and e_2 be a basis. Let L be the line spanned by a non-zero vector l, which we may assume to be e_1. The first identification we have is $V \cong \mathrm{Hom}(V, \Lambda^2 V)$, sending v to the homomorphism $v' \mapsto v' \wedge v$. This corresponds to $\alpha_v \otimes (e_1 \wedge e_2) \in V^\vee \otimes \Lambda^2 V$, where $\alpha_v \in V^\vee$ is the linear form sending $e_1 \mapsto v_2$ and $e_2 \mapsto -v_1$. Denoting by $\overline{\alpha_v}$ its image in $L^\vee$, we can describe the morphism corresponding to the right hand side of the previous short exact sequence as

$$\begin{aligned} V &\longrightarrow L^\vee \otimes \Lambda^2 V \\ v &\longmapsto \overline{\alpha_v} \otimes (e_1 \wedge e_2) \end{aligned}$$

Let now $\beta \in (L^\vee \otimes \Lambda^2 V)^\vee$ and denote by λ_v^β the complex number such that

$$\overline{\alpha_v} \otimes (e_1 \wedge e_2) \stackrel{\beta}{\longmapsto} \lambda_v^\beta.$$

A point $\mu l \otimes \beta \in L \otimes (L^\vee \otimes \Lambda^2 V)^\vee$ corresponds then to the endomorphism

$$\begin{aligned} V &\longrightarrow V \\ v &\longmapsto \mu \lambda_v^\beta l \end{aligned}$$

A basis for L is e_1, a basis for $L^\vee \otimes \Lambda^2 V$ is $\overline{\alpha_{e_2}} \otimes (e_1 \wedge e_2)$ and a basis for $L \otimes (L^\vee \otimes \Lambda^2)^\vee$ is $e_1 \otimes \beta_0$, where $\beta_0 \in (L^\vee \otimes \Lambda^2 V)^\vee$ is such that $\lambda_{e_2}^{\beta_0} = 1$. Writing the image of the basis $e_1 \otimes \beta_0$ under the map $L \otimes (L^\vee \otimes \Lambda^2 V)^\vee \longrightarrow \mathrm{End}(V)$ as a matrix with respect to our bases we obtain

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

because $\overline{\alpha_{e_1}} = 0$ and therefore $\lambda_{e_1}^\beta = 0$ for any β. We have thus the desired injective homomorphism

$$L^2 \otimes \Lambda^2 V^\vee \hookrightarrow \mathrm{End}_0(V)$$

whose image are the traceless endomorphisms which preserve only L.

We regard this as a homomorphism into $\mathrm{End}(V)$ for a moment and use the basis $e_{11}, e_{12}, e_{21}, e_{22}$ of $\mathrm{End}(V)$, where e_{ij} denotes the endomorphism which, represented as a matrix in terms of our basis, has zeros everywhere except for a 1 in the ij-th position. Then our homomorphism is given by

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Dualizing it we obtain a surjection

$$\mathrm{End}_0(V^\vee) \twoheadrightarrow L^{-2} \otimes \Lambda^2 V$$

given with respect to the dual bases by

$$\begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}.$$

Its kernel are the endomorphisms of $V^\vee$ represented with respect to the dual basis by a matrix of the form

$$\begin{pmatrix} a & 0 \\ b & -a \end{pmatrix}.$$

Hence, under the isomorphism $\mathrm{End}(V) \cong \mathrm{End}(V^\vee)$ given in coordinates by sending a matrix to its transpose, we obtain a surjection

$$\mathrm{End}_0(V) \twoheadrightarrow L^{-2} \otimes \Lambda^2 V$$

whose kernel are endomorphisms represented with respect to our basis by matrices of the form

$$\begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}.$$

Therefore the kernel of this surjection consists precisely of the traceless endomorphisms of V that leave at least L invariant. The inclusion of this kernel can be naturally regarded as the composition of the inclusion $\mathrm{Hom}(V, L) \subseteq \mathrm{Hom}(V, V)$ and the projection $\mathrm{pr}_0 : \mathrm{Hom}(V, V) \longrightarrow \mathrm{End}_0(V)$, which writing every homomorphism as a matrix with respect to the bases above has the form

$$\begin{pmatrix} a & b \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} \frac{a}{2} & b \\ 0 & -\frac{a}{2} \end{pmatrix}.$$

This gives us the desired short exact sequence

$$0 \longrightarrow \mathcal{E}^\vee \otimes \mathcal{L} \otimes \omega \longrightarrow \mathcal{E}nd_0(\mathcal{E}) \otimes \omega \longrightarrow \mathcal{L}^{-2} \otimes \omega \otimes \Lambda^2 \mathcal{E} \longrightarrow 0,$$

in which the global sections of $\mathcal{E}^\vee \otimes \mathcal{L} \otimes \omega$ correspond to Higgs fields which leave at least L invariant. $\square$

Lemma 5. *If $g \geqslant 2$, then*

$$\mathbf{S} \subseteq \mathbf{A}.$$

Proof. We want to show that if $E \in \mathbf{Vec}_2(M)$ is stable, then it is in $\mathbf{A}$. By Lemma 3 it suffices to show that it is not in $\mathbf{B}$. So let E be a stable rank 2 vector bundle on M and assume $L \subseteq E$ is a line bundle which is Φ-invariant for all $\Phi \in H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \omega)$. Consider the short exact sequence from Lemma 4

$$0 \longrightarrow \mathcal{E}^\vee \otimes \mathcal{L} \otimes \omega \longrightarrow \mathcal{E}nd_0(\mathcal{E}) \otimes \omega \longrightarrow \mathcal{L}^{-2} \otimes \omega \otimes \Lambda^2\mathcal{E} \longrightarrow 0.$$

Since all $\Phi \in H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \omega)$ leave L invariant, we get an induced isomorphism on global sections $H^0(M, \mathcal{E}^\vee \otimes \mathcal{L} \otimes \omega) \cong H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \omega)$. We consider now the short exact sequence

$$0 \longrightarrow \mathcal{L}^2 \otimes \omega \otimes \Lambda^2\mathcal{E}^\vee \longrightarrow \mathcal{E}^\vee \otimes \mathcal{L} \otimes \omega \longrightarrow \omega \longrightarrow 0,$$

more or less implicit in the proof of Lemma 4. Since E is stable, we have $\deg(\mathcal{L}^2 \otimes \Lambda^2\mathcal{E}^\vee) < 0$, and since the complete linear system corresponding to ω is base-point free[3] we have $h^0(M, \omega \otimes \mathcal{L}^2 \otimes \Lambda^2\mathcal{E}^\vee) < g$ by [Har77, Proposition IV.3.1]. The long exact sequence of the previous short exact sequence gives then

$$h^0(M, \omega \otimes \mathcal{L} \otimes \mathcal{E}^\vee) \leqslant 2g - 1.$$

The earlier Hirzebruch–Riemann–Roch computation showed that

$$h^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \omega) \geqslant 3g - 3.$$

If we want the two dimensions to be equal we must have $g = 2$ and $h^0(M, \omega \otimes \mathcal{L} \otimes \mathcal{E}^\vee) = 3$. From the same long exact sequence as before we deduce, using that $h^0(M, \omega \otimes \mathcal{L}^2 \otimes \Lambda^2\mathcal{E}^\vee) < g = 2$, that $h^0(M, \omega \otimes \mathcal{L}^2 \otimes \Lambda^2\mathcal{E}^\vee) = 1$. In particular, $\deg(\omega \otimes \mathcal{L}^2 \otimes \Lambda^2\mathcal{E}^\vee) \geqslant 0$. We have $\deg(\omega) = 2$ and by stability we had $\deg(\mathcal{L}^2 \otimes \Lambda^2\mathcal{E}^\vee) < 0$, so we must have

$$\deg(\omega \otimes \mathcal{L}^2 \otimes \Lambda^2\mathcal{E}^\vee) \in \{0, 1\}.$$

If it is 0, then the existence of global sections implieis that it is the trivial line bundle, hence the previous short exact sequence becomes

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{E}^\vee \otimes \mathcal{L} \otimes \omega \longrightarrow \omega \longrightarrow 0.$$

Split short exact sequences are preserved by dualizing and tensoring with line bundles, so if this sequence was split then $\mathcal{E}$ would be decomposable as a direct sum of line bundles. But this would give more endomorphisms of

[3] Suppose that $P \in M$ was a base-point of $|\omega|$. This would mean that $h^0(\omega) = h^0(\omega \otimes \mathcal{O}(-P)) = g$, so by Riemann–Roch we would have $h^0(\mathcal{O}(P)) = 2$. For any other $Q \in M$ we would have that $\mathcal{O}(P-Q)$ is a degree 0 line bundle with $h^0(\mathcal{O}(P-Q)) \in \{1, 2\}$ [Har77, Proof of Proposition IV.3.1], hence the trivial line bundle. This would imply that any two points in M are linearly equivalent, hence $M = \mathbb{P}^1$ [Har77, Example II.6.10.1], a contradiction.

$\mathcal{E}$ than there should be, since stable vector bundles are simple. So the previous short exact sequence is a non-trivial extension and the coboundary map $H^0(M,\omega) \longrightarrow H^1(M,\mathcal{O})$ is non-zero. The long exact sequence implies then that $h^0(M,\mathcal{E}^\vee\otimes\omega\otimes\mathcal{L}) \leqslant 2$, contradicting our previous conclusion that this dimension was 2. □

2. An existence theorem

If (A,Φ) is an irreducible solution of the SO(3) self-duality equations on M, then the associated pair (V,Φ) is stable by [Hit87, Theorem 2.1]. In this section we prove the converse:

Theorem 6. *Let A be a connection on a principal* SO(3)*-bundle P on the compact Riemann surface M of genus $g \geqslant 2$. Let $\Phi \in \Omega^{1,0}(M,\mathrm{ad}(P)\otimes\mathbb{C})$ be such that $d''_A\Phi = 0$ and V an associated rank 2 complex vector bundle with the holomorphic structure determined by A. If (V,Φ) is a stable Higgs bundle, then there exists an automorphism of V with determinant 1 which takes (A,Φ) to a solution of the equation $F(A)+[\Phi,\Phi^*]=0$. Moreover, this automorphism is unique up to gauge transformation.*

Appendix A. The self-duality equations

From now on, every space, morphism and action will be assumed to be smooth without explicitly saying so.

Let $G=\mathrm{SO}(3)$ be the Lie group of rotations in $\mathbb{R}^3$. Endow our Riemannian surface M with the trivial right action by G, making it a right G-manifold. Let $\pi : P \longrightarrow M$ be a principal G-bundle over M, which is a right G-manifold over M which, locally on M, looks lilke the projection from a product $M\times G \longrightarrow M$.

[picture]

Let $g\in G$ and denote by $R_g : P \longrightarrow P$ the right multiplication by g. Since $\pi\circ R_g = \pi$, we have a short exact sequence

$$0 \longrightarrow T^{\mathrm{vert}}P \longrightarrow TP \xrightarrow{d\pi} \pi^*TM \longrightarrow 0$$

of G-equivariant maps, where $T_p^{\mathrm{vert}}P = \mathrm{Ker}(d\pi_p)$ is the *vertical tangent space* of P at the point p, consisting of all tangent vectors which lie in the tangent space of the fibre $\pi^{-1}(\pi(p))$. A *connection* A on P is then a G-equivariant splitting $\sigma_A : \pi^*TM \longrightarrow TP$. Therefore we may think of a connection A on P as a choice of *horizontal tangent spaces* $T_p^{\mathrm{hor}}P \subseteq T_pP$ having for each $p\in P$ the following properties:

- $T_pP = T_p^{\mathrm{vert}}P \oplus T_p^{\mathrm{hor}}P$.
- $d\pi_p|_{T_p^{\mathrm{hor}}P} : T_p^{\mathrm{hor}}P \longrightarrow T_{\pi(p)}M$ is an isomorphism.
- $(dR_g)_p(T_p^{\mathrm{hor}}P) = T_{p\cdot g}^{\mathrm{hor}}P$ for all $g\in G$.

[picture]

Remark 7. Once we fix a connection A on P, we can lift any vector field $X : M \longrightarrow TM$ on M to a horizontal vector field $\sigma_A \circ \pi^* X : P \longrightarrow T^{\mathrm{hor}}P \subseteq TP$, yielding a map $h : \mathfrak{X}(M) \longrightarrow \mathfrak{X}(P)$.

Let $\mathfrak{g} = \mathfrak{so}(3)$ be the Lie algebra of our Lie group, which in our case consists of skew-symmetric 3×3-matrices with real coefficients. The action of G on P induces a Lie algebra homomorphism $a : \mathfrak{g} \longrightarrow \mathfrak{X}(P)$, called the *infinitesimal action* of $\mathfrak{g}$ on P and given by

$$a(B)_p = \frac{d}{dt}\Big|_{t=0} p \cdot e^{tB} \in T_pP,$$

where e^{tB} is in our case the usual exponential of the matrix $tB \in \mathfrak{g}$. This infinitesimal action induces in turn for each $p \in P$ an injective $\mathbb{R}$-linear homomorphism

$$a_p : \mathfrak{g} \longrightarrow T_pP.$$

Indeed, if $p \cdot e^{tB} = p$ for all $t \in (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$, then e^{tB} is the identity matrix in G for all $t \in (-\varepsilon, \varepsilon)$ because the action of G on P is free. But the exponential is a local diffeomorphism around $0 \in \mathfrak{g}$, so we must have that $B = 0$ is the zero matrix.

We consider also the *adjoint representation* $\rho : G \longrightarrow \mathrm{GL}(\mathfrak{g})$ given by

$$\rho(g) = d(g \cdot (-) \cdot g^{-1})_e,$$

where e is the identity matrix in G. We have now a right G-action on P and a linear left G-action on $\mathfrak{g}$, so the vector bundle $\mathfrak{g} \times P \longrightarrow P$ carries the natural right G-action $(B, p) \cdot g := (g^{-1} \cdot B, p \cdot g)$. The infinitesimal action induces then an injective G-equivariant map $a : \mathfrak{g} \times P \longrightarrow TP$, because the exponential map is compatible with differentials of Lie group homomorphisms. Moreover, its image is by definition contained in the vertical tangent space, so that $a : \mathfrak{g} \times P \longrightarrow T^{\mathrm{vert}}P$ is an isomorphism by dimensional reasons. We may therefore replace our previous short exact sequence by

$$0 \longrightarrow \mathfrak{g} \times P \xrightarrow{a} TP \xrightarrow{d\pi} \pi^* TM \longrightarrow 0.$$

A connection was defined to be a G-equivariant section of $d\pi$. But this is equivalent by the equivariant version of the splitting lemma to a G-equivariant retraction of a. Such a retraction is in particular a $\mathfrak{g}$-valued 1-form $\omega \in \Omega^1(P, \mathfrak{g})$, called the *connection* 1*-form* on P. Conversely, every G-invariant $\mathfrak{g}$-valued 1-form $\omega \in \Omega^1(P, \mathfrak{g})$ with $\omega(a(B)) = B$ for all $B \in \mathfrak{g}$ gives rise to a connection on P, where G-invariance of ω means that $\rho(g)(R_g^*(\omega)(-)) = \omega(-)$.

Let A be a connection on our principal G-bundle P and let $\omega \in \Omega^1(P, \mathfrak{g})$ be the corresponding 1-form. We define the *curvature* $F(A)$ as the $\mathfrak{g}$-valued

2-form given by

$$F(A)(X, Y) = d\omega(X, Y) + [\omega(X), \omega(Y)] \text{ for all } X, Y \in \mathfrak{X}(P).$$

The curvature is in fact a G-invariant horizontal 2-form, so it corresponds to a uniquely determined 2-form with values in the vector bundle $\mathrm{ad}(P) \longrightarrow M$ associated to the principal G-bundle P and the adjoint representation $\rho : G \longrightarrow \mathrm{GL}(\mathfrak{g})$, whose transition functions are given by the composition of the transition functions of $P \longrightarrow M$ with the group homomorphism ρ. We may therefore regard the curvature $F(A)$ as an $\mathrm{ad}(P)$-valued 2-form $F(A) \in \Omega^2(M, \mathrm{ad}(P))$.

Let us denote now by $G_{\mathbb{C}}$ the complex Lie group of special orthogonal complex-valued 3×3-matrices and by $\mathfrak{g}_{\mathbb{C}}$ its Lie algebra, which consists of complex valued skew-symmetric 3×3-matrices [?, Proposition 3.38]. We consider $\Phi \in \Omega^{1,0}(M, \mathrm{ad}(P) \otimes \mathbb{C})$, which can be regarded as a $\mathfrak{g}_{\mathbb{C}}$-valued $(1, 0)$-form. Let us denote by Φ^* the composition of Φ with the anti-involution of the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}}$. Similarly as before, we use the Lie bracket on $\mathfrak{g}_{\mathbb{C}}$ to define a bracket on $\mathfrak{g}_{\mathbb{C}}$-valued forms:

$$[\Phi, \Phi^*](X, Y) = [\Phi(X), \Phi^*(Y)] + [\Phi^*(X), \Phi(Y)] \text{ for all } X, Y \in \mathfrak{X}(M).$$

We can now state the first of the two equations for which we want to produce solutions:

$$F(A) = -[\Phi, \Phi^*].$$

That is, for all $X, Y \in \mathfrak{X}(M)$ we want

$$d\omega(X, Y) + [\omega(X), \omega(Y)] + [\Phi(X), \Phi^*(Y)] + [\Phi^*(X), \Phi(Y)] = 0.$$

For the second equation we need to recall the curvature induced by A in the associated vector bundle $\mathrm{ad}(P) \otimes \mathbb{C}$.

Notation and conventions

We usually follow the notation of [Hit87]:

- M: compact Riemann surface of genus g.
- $O \longrightarrow M$: trivial line bundle.
- $K \longrightarrow M$: canonical line bundle.
- More generally, O_X and K_X denote the trivial and canonical line bundles over a complex manifold X.

Every now and then we will also use some other standard notation, for instance:

- For a vector bundle $E \longrightarrow M$ we denote $\mu(E) := \deg E / \operatorname{rk} E$.
- Let X be a complex manifold and $E \longrightarrow X$ a (holomorphic/algebraic) vector bundle. Then we denote by $\mathcal{E}$ its sheaf of sections. The assignement $E \mapsto \mathcal{E}$ defines an equivalence of categories between

vector bundles on X and locally free sheaves of $\mathcal{O}_X$-modules of finite rank, which we will refer to simply as locally free sheaves. We can recover E from $\mathcal{E}$ either using cocycles [Voi02, Lemma 4.8] or by using the relative spectrum [Har77, Exercise II.5.18]. Following the second approach we would write

$$E = \mathrm{Spec}_X(S(\mathcal{E}^\vee)) =: \mathbb{V}(\mathcal{E}^\vee),$$

where $S(-)$ denotes the symmetric algebra.

- $\mathcal{O}$ and ω denote the trivial and canonical invertible sheaves on M. More generally, $\mathcal{O}_X$ and ω_X denote the trivial and canonical invertible sheaves on a complex manifold X,
- Let E be again a vector bundle on a complex manifold X. We will denote its projectivisation by $\mathbb{P}(E)$, which is obtained form E without its zero section by quotienting out the fibre-wise multiplication by scalars. Taking again the algebraic approach we can write

$$\mathbb{P}(E) = \mathrm{Proj}_X(S(\mathcal{E}^\vee)) =: \mathbb{P}(\mathcal{E}^\vee).$$

- Let N be a smooth manifold and $E \longrightarrow N$ a smooth vector bundle. Then we denote by $\mathfrak{X}(N)$ the Lie algebra of smooth vector fields on N and by $\Omega^k(N, E)$ the vector space of smooth k-differential forms with values in E, which can be thought of as smooth global sections of the vector bundle $\mathrm{Hom}(TN, E)$.
- Let N be a smooth manifold equipped with an almost complex structure $I : TN \longrightarrow TN$. Then we denote by $\Omega^{i,j}(N, E)$...

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