

HIGGS BUNDLES — EXISTENCE OF SOLUTIONS

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ABSTRACT. In this talk we introduce the stability condition for Higgs bundles and prove the Hitchin–Kobayashi correspondence. The main result is [Hit87, Theorem 4.3]. Relevant literature is [Hit87, §3 and §4] and [Wen14, §2 and §3]. Maybe we will also use [Wen16] every now and then.

This talk is related to Tanuj’s talk on *Stable vector bundles*, for which the main reference is [Kob87]. Therefore we will also use [Kob87] as a default reference for generalities on complex vector bundles.

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NOTATION AND CONVENTIONS

We usually follow the notation of [Hit87]:

- M : compact Riemann surface of genus g .
- $O \rightarrow M$: trivial line bundle.
- $K \rightarrow M$: canonical line bundle.
- More generally, O_X and K_X denote the trivial and canonical line bundles over a complex manifold X .

Every now and then we will also use some other standard notation, for instance:

- For a vector bundle $E \rightarrow M$ we denote $\mu(E) := \deg E / \operatorname{rk} E$.
- Let X be a complex manifold and $E \rightarrow X$ a (holomorphic/algebraic) vector bundle. Then we denote by \mathcal{E} its sheaf of sections. The assignment $E \mapsto \mathcal{E}$ defines an equivalence of categories between vector bundles on X and locally free

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sheaves of \mathcal{O}_X -modules of finite rank, which we will refer to simply as locally free sheaves. We can recover E from \mathcal{E} either using cocycles [Voi02, Lemma 4.8] or by using the relative spectrum [Har77, Exercise II.5.18]. Following the second approach we would write

$$E = \operatorname{Spec}_X(S(\mathcal{E}^*)) =: \mathbb{V}(\mathcal{E}^*),$$

where $S(-)$ denotes the symmetric algebra.

- \mathcal{O} and ω denote the trivial and canonical invertible sheaves on M . More generally, \mathcal{O}_X and ω_X denote the trivial and canonical invertible sheaves on a complex manifold X ,
- Let E be again a vector bundle on a complex manifold X . We will denote its projectivisation by $\mathbb{P}(E)$, which is obtained from E without its zero section by quotienting out the fibre-wise multiplication by scalars. Taking again the algebraic approach we can write

$$\mathbb{P}(E) = \operatorname{Proj}_X(S(\mathcal{E}^*)) =: \mathbb{P}(\mathcal{E}^*).$$

1. STABILITY

Definition 1.1 (Higgs bundle). A *Higgs bundle* on M is a pair (E, Φ) , where $E \rightarrow M$ is a rank 2 vector bundle and Φ is a global section of $\operatorname{End} E \otimes K$, called a *Higgs field* on E .

Remark 1.2. Using the canonical isomorphisms

$$H^0(M, \operatorname{End}(\mathcal{E}) \otimes \omega) \cong \operatorname{Hom}(\mathcal{O}, \mathcal{E}^* \otimes \mathcal{E} \otimes \omega) \cong \operatorname{Hom}(\mathcal{E}, \mathcal{E} \otimes \omega)$$

we may identify Φ with a morphism

$$\Phi: E \rightarrow E \otimes K.$$

Example 1.3. Assume $g \geq 2$. Then $\deg K = 2g - 2 > 0$, so we can find a line bundle $K^{\frac{1}{2}}$ such that $K^{\frac{1}{2}} \otimes K^{\frac{1}{2}} \cong K$. Let $E = K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}$, where $K^{-\frac{1}{2}} = (K^{\frac{1}{2}})^{-1}$. We consider the Higgs field $\Phi_w: K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}} \rightarrow (K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}) \otimes K$ given by a matrix

$$\begin{pmatrix} 0 & w \\ 1 & 0 \end{pmatrix},$$

where $w \in \operatorname{Hom}(K^{-\frac{1}{2}}, K^{\frac{1}{2}} \otimes K) \cong H^0(M, K^2)$ can be regarded as a quadratic differential.

Definition 1.4 (Stability). A Higgs bundle (E, Φ) is said to be *stable* if for every Φ -invariant¹ line bundle $L \subseteq E$ we have $\mu(L) < \mu(E)$.

¹Meaning that $\Phi(L) \subseteq L \otimes K$.

Remark 1.5. $(E, 0)$ is stable if and only if E is stable in the usual sense.

Exercise 1.6. There are no stable Higgs bundles on \mathbb{P}^1 . [Hint: Grothendieck's theorem allows us to write Φ as a matrix. What can we say about each entry?] [Solution in [Hit87]]

Example 1.7. Assume $g \geq 2$ and consider $E = K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}$ again. Then Φ_0 is stable, because $K^{-\frac{1}{2}}$ is the only Φ_0 -invariant line bundle and

$$\deg K^{-\frac{1}{2}} = 1 - g < 0 = \frac{\deg E}{2}.$$

Proposition 1.8. Assume $g \geq 2$ and let $E \rightarrow M$ be a rank 2 vector bundle. Then there exists Higgs field Φ on E such that (E, Φ) is stable if and only if there exists a dense Zariski open subset $U \subseteq H^0(M, \text{End}(\mathcal{E}) \otimes \omega)$ such that all $\Phi' \in U$ have the property that no line bundle $L \subseteq E$ is Φ' -invariant.

Proof. We define the following sets of rank 2 vector bundles on M :

- $\mathbf{S} := \{E \in \mathbf{Vec}_2(M) \mid \exists \Phi \text{ such that } (E, \Phi) \text{ is stable}\}.$
- $\mathbf{A} := \{E \in \mathbf{Vec}_2(M) \mid \text{a general } \Phi \text{ has no invariant } L\}.$
- $\mathbf{B} := \{E \in \mathbf{Vec}_2(M) \mid \exists L \text{ invariant for all } \Phi\}.$

Our goal is to show that $\mathbf{S} = \mathbf{A}$. If Φ has no invariant L , then (E, Φ) is automatically stable. Hence $\mathbf{A} \subseteq \mathbf{S}$. The plan to show the other inclusion is to see that

$$\mathbf{Vec}_2(M) = \mathbf{A} \sqcup \mathbf{B}$$

and that $\mathbf{B} \subseteq \mathbf{Vec}_2(M) \setminus \mathbf{S}$.

Let us start by showing that $\mathbf{Vec}_2(M) = \mathbf{A} \sqcup \mathbf{B}$.

Let $p: \mathbb{P}(E) \rightarrow M$ be the projectivisation of our rank 2 vector bundle, which is a ruled surface in the sense of [Har77, §V.2]. Let $\mathcal{O}(-1)$ denote the tautological line bundle on $\mathbb{P}(E)$, whose fibre over $[v] \in \mathbb{P}(E)$ is the line $\{\lambda v \mid \lambda \in \mathbb{C}\} \subseteq E_{p([v])}$. Let $\mathcal{O}(l) := \mathcal{O}(1)^{\otimes l}$ for all $l \in \mathbb{Z}$, and if $F \rightarrow \mathbb{P}(E)$ is another vector bundle, denote by $F(l)$ the tensor product $F \otimes \mathcal{O}(l)$. We have then $p_* \mathcal{O}(l) = S^l(\mathcal{E}^*)$ for all $l \geq 0$ [Har77, Exercise III.8.4].

Let $x \in M$. Then every endomorphism $A \in \text{End}(E_x)$ defines a quadratic map $E_x \rightarrow \Lambda^2 E_x$ sending v to $Av \wedge v$. Such a quadratic map can be naturally regarded as a degree 2 homogeneous polynomial on the coordinates of e with coefficients in $\Lambda^2 E_x$. Hence we have a morphism $\text{End}(E) \rightarrow S^2 E^* \otimes \Lambda^2 E$, which vanishes precisely along the trivial line subbundle of $\text{End}(E)$ consisting over each fibre of scalar multiples of the identity. Sending $A \mapsto A - \frac{\text{tr}(A)}{2} \text{id}_{E_x}$ on each fibre allows us to identify $\text{End}_0(E)$ as the quotient of $\text{End}(E)$ by this trivial line subbundle, so we

obtain an injective morphism $\text{End}_0(E) \rightarrow S^2 E^* \otimes \Lambda^2 E$. Counting ranks we see that we have in fact an isomorphism $\text{End}_0(E) \cong S^2 E^* \otimes \Lambda^2 E$, and therefore

$$\text{End}_0(\mathcal{E}) \otimes \omega \cong p_* \mathcal{O}(2) \otimes \omega \otimes \Lambda^2 \mathcal{E}.$$

By the projection formula, $\text{End}_0(\mathcal{E}) \otimes \omega \cong p_*(p^*(\omega \otimes \Lambda^2 \mathcal{E})(2))$. Therefore we have an isomorphism

$$\alpha: H^0(M, \text{End}_0(\mathcal{E}) \otimes \omega) \cong H^0(\mathbb{P}(E), p^*(\omega \otimes \Lambda^2 \mathcal{E})(2)).$$

Let now Φ be a traceless Higgs field on E , and assume it is non-zero. By construction, a non-zero vector $v \in E$ is an eigenvector of the twisted endomorphism over the corresponding fibre if and only if the section $\alpha(\Phi)$ vanishes at the point $[v] \in \mathbb{P}(E)$, i.e. if and only if $[v]$ is in the divisor of zeros of the global section $\alpha(\Phi)$, which we denote $\text{div}(\alpha(\Phi))$. Let $L \subseteq E$ be a Φ -invariant subbundle, which defines a section of $p: \mathbb{P}(E) \rightarrow M$ by functoriality of projectivisation on injective morphisms of vector bundles:

$$\begin{array}{ccc} \mathbb{P}(L) & \xrightarrow{\sigma} & \mathbb{P}(E) \\ \parallel & \swarrow p & \\ M & & \end{array}$$

Being Φ -invariant means precisely that $\sigma(M) \subseteq \text{div}(\alpha(\Phi))$. But then any non-zero $v \in L$ is a non-zero eigenvector corresponding to some eigenvalue of the endomorphism over the corresponding fibre. Since Φ is traceless and non-zero, the other eigenvalue must be different, and there must be some non-zero eigenvector outside of L , call it $v' \in V$. Since v' is a non-zero eigenvector, $[v'] \in \text{div}(\alpha(\Phi))$. And since $v' \notin L$, $[v'] \notin \sigma(M)$. Therefore $\sigma(M)$ is a proper irreducible component of the divisor $\text{div}(\alpha(\Phi))$. So if $\text{div}(\alpha(\Phi))$ is irreducible, then no line bundle $L \subseteq V$ is Φ -invariant and (E, Φ) is automatically stable.

Next we give a lower bound for the dimension of the linear system $|p^*(\omega \otimes \Lambda^2 \mathcal{E})(2)|$, which is one less than the dimension of the vector space $H^0(\mathbb{P}(E), p^*(\omega \otimes \Lambda^2 \mathcal{E})(2))$. Using the previous isomorphism it suffices to gain control over the dimension of the global sections of $\text{End}_0(\mathcal{E}) \otimes \omega$ on M , for which we can apply Hirzebruch–Riemann–Roch [Har77, Theorem A.4.1]. From [Har77, Example A.4.1.1] we get

$$\text{td}(\omega^*) = 1 - \frac{c_1(\omega)}{2}.$$

Using the short exact sequence used earlier

$$0 \rightarrow \mathcal{O} \rightarrow \text{End}(\mathcal{E}) \rightarrow \text{End}_0(\mathcal{E}) \rightarrow 0$$

we see that $c_1(\text{End}_0(\mathcal{E})) = c_1(\text{End}\mathcal{E}) = 0$. Therefore

$$\text{ch}(\text{End}_0(\mathcal{E}) \otimes \omega) = 3 + c_1(\text{End}_0(\mathcal{E})) + 3c_1(\omega) = 3 + 3c_1(\omega).$$

Multiplying the two expressions we obtain

$$\text{ch}(\text{End}_0(\mathcal{E}) \otimes \omega) \text{td}(\omega^*) = 3 + \frac{3}{2}c_1(\omega),$$

whose codimension 1 part has degree $3g - 3 \geq 3$. So Hirzebruch–Riemann–Roch tells us that

$$h^0(M, \text{End}_0(\mathcal{E}) \otimes \omega) - h^1(M, \text{End}_0(\mathcal{E}) \otimes \omega) = 3g - 3 \geq 3,$$

which implies that $h^0(\mathbb{P}(E), p^*(\omega \otimes \Lambda^2 \mathcal{E})(2)) = h^0(M, \text{End}_0(\mathcal{E}) \otimes \omega) \geq 3$.

Thus, our linear system $|p^*(\omega \otimes \Lambda^2 \mathcal{E})(2)|$ has dimension 2. If it does not have any fixed component, then Bertini's theorem [Lit82, Theorem 7.19] and the discussion above imply that a general $\Phi \in H^0(M, \text{End}_0(\mathcal{E}) \otimes \omega)$ leaves no line bundle $L \subseteq E$ invariant, i.e. $E \in \mathbf{A}$.

Let us see what happens if it does have some fixed divisor. By definition, a fixed divisor corresponds to a non-zero global section $s_0 \in H^0(\mathbb{P}(E), \mathcal{M}_1)$ for some invertible sheaf \mathcal{M}_1 such that there exists another invertible sheaf \mathcal{M}_2 with $\mathcal{M}_1 \otimes \mathcal{M}_2 \cong p^*(\omega \otimes \Lambda^2 \mathcal{E})(2)$. Being a fixed divisor translates into saying that every global section $s \in H^0(\mathbb{P}(E), p^*(\omega \otimes \Lambda^2 \mathcal{E})(2))$ can be written as ts_0 for some $t \in H^0(\mathbb{P}(E), \mathcal{M}_2)$. In particular, both $H^0(\mathbb{P}(E), \mathcal{M}_1)$ and $H^0(\mathbb{P}(E), \mathcal{M}_2)$ have to be non-zero. By [Har77, Exercise II.7.4] we can write $\mathcal{M}_i \cong p^*\mathcal{L}_i(l_i)$ with $l_1 + l_2 = 2$. In fact, we must have $0 \leq l_i \leq 2$, because using again the projection formula we have

$$H^0(\mathbb{P}(E), p^*\mathcal{L}_i(l_i)) \cong H^0(M, \mathcal{L}_i \otimes p_*\mathcal{O}(l_i))$$

and $p_*\mathcal{O}(l) = 0$ for all $l < 0$ [Har77, Exercise III.8.4]. So we only have the following three possibilities:

- a) $l_1 = 0$;
- b) $l_1 = 1$;
- c) $l_1 = 2$.

Let us start with case a). Let $p^*s \in H^0(\mathbb{P}(E), p^*\mathcal{L}) \cong H^0(M, \mathcal{L})$ be a global section corresponding to the fixed component of our linear system. Dividing all global sections by s and by p^*s respectively we obtain the following commutative diagram:

$$\begin{array}{ccc} H^0(M, \text{End}_0(E) \otimes K) & \xrightarrow{\cong} & H^0(\mathbb{P}(E), p^*(K \otimes \Lambda^2 E)(2)) \\ \downarrow /s \cong & & \downarrow /p^*s \cong \\ H^0(M, \text{End}_0(E) \otimes K \otimes L^*) & \xrightarrow{\cong} & H^0(\mathbb{P}(E), p^*(L^* \otimes K \otimes \Lambda^2 E)(2)) \end{array}$$

By definition, the new linear system does not have any fixed divisors and has the same dimension. Hence we can apply Bertini to conclude that a generic $\Phi' \in H^0(M, \text{End}_0(E) \otimes K \otimes L^*)$ does not have invariant line bundles, which in this case are defined as line bundles $N \subseteq E$ such that $\Phi'(N) \subseteq N \otimes K \otimes L^*$. But a line bundle $N \subseteq E$ is Φ' -invariant if and only if it is $s\Phi'$ -invariant, so we have $E \in \mathbf{A}$ in this case. \square

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