HIGGS BUNDLES — EXISTENCE OF SOLUTIONS

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Abstract. In this talk we introduce the stability condition for Higgs bundles and prove the Hitchin–Kobayashi correspondence. The main result is [Hit87, Theorem 4.3]. Relevant literature is [Hit87, §3 and §4] and [Wen14, §2 and §3]. Maybe we will also use [Wen16] every now and then.

This talk is related to Tanuj's talk on *Stable vector bundles*, for which the main reference is [Kob87]. Therefore we will also use [Kob87] as a default reference for generalities on complex vector bundles.

Contents

Notation and conventions

We usually follow the notation of [Hit87]:

- M: compact Riemann surface of genus g.
- $O \to M$: trivial line bundle.
- $K \to M$: canonical line bundle.
- More generally, O_X and K_X denote the trivial and canonical line bundles over a complex manifold X.

Every now and then we will also use some other standard notation, for instance:

- For a vector bundle $E \to M$ we denote $\mu(E) := \deg E / \operatorname{rk} E$.
- Let X be a complex manifold and $E \to X$ a (holomorphic/algebraic) vector bundle. Then we denote by $\mathcal{E}$ its sheaf of sections. The assignement $E \mapsto \mathcal{E}$ defines an equivalence of categories between vector bundles on X and locally free

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sheaves of $\mathcal{O}_X$-modules of finite rank, which we will refer to simply as locally free sheaves. We can recover E from $\mathcal{E}$ either using cocycles [Voi02, Lemma 4.8] or by using the relative spectrum [Har77, Exercise II.5.18]. Following the second approach we would write

$$E = \operatorname{Spec}_X(S(\mathcal{E}^*)) =: \mathbb{V}(\mathcal{E}^*),$$

where $S(-)$ denotes the symmetric algebra.

- $\mathcal{O}$ and ω denote the trivial and canonical invertible sheaves on M. More generally, $\mathcal{O}_X$ and ω_X denote the trivial and canonical invertible sheaves on a complex manifold X,
- Let E be again a vector bundle on a complex manifold X. We will denote its projectivisation by $\mathbb{P}(E)$, which is obtained form E without its zero section by quotienting out the fibre-wise multiplication by scalars. Taking again the algebraic approach we can write

$$\mathbb{P}(E) = \operatorname{Proj}_X(S(\mathcal{E}^*)) =: \mathbb{P}(\mathcal{E}^*).$$

1. Stability

Definition 1.1 (Higgs bundle)**.** A *Higgs bundle* on M is a pair (E, Φ), where $E \to M$ is a rank 2 vector bundle and Φ is a global section of $\operatorname{End} E \otimes K$, called a *Higgs field* on E.

Remark 1.2. Using the canonical isomorphisms

$$H^0(M, \mathcal{E}nd(\mathcal{E}) \otimes \omega) \cong \operatorname{Hom}(\mathcal{O}, \mathcal{E}^* \otimes \mathcal{E} \otimes \omega) \cong \operatorname{Hom}(\mathcal{E}, \mathcal{E} \otimes \omega)$$

we may identify Φ with a morphism

$$\Phi \colon E \to E \otimes K.$$

Definition 1.3 (Stability)**.** A Higgs bundle (E, Φ) is said to be *stable* if for every Φ-invariant[1] line bundle $L \subseteq E$ we have $\mu(L) < \mu(E)$.

Remark 1.4. $(E, 0)$ is stable if and only if E is stable in the usual sense.

Exercise 1.5. There are no stable Higgs bundles on $\mathbb{P}^1$. [Hint: Grothedieck's theorem allows us to write Φ as a matrix. What can we say about each entry? Solution in [Hit87]]

Proposition 1.6. *Let* (E_1, Φ_1) *and* (E_2, Φ_2) *be stable pairs with* $\Lambda^2 E_1 \cong \Lambda^2 E_2$*. Let* $\Psi \colon E_1 \to E_2$ *be a non-zero morphism of constant rank such that* $(\Psi \otimes \mathrm{id}_K) \circ \Phi_1 = \Phi_2 \circ \Psi$*. Then* Ψ *is an isomorphism.*

[1] Meaning that $\Phi(L) \subseteq L \otimes K$.

Proof. We prove the result by contradiction. Suppose that Ψ is not an isomorphism. The rank $x \mapsto \dim_{\mathbb{C}} \Psi_x(E_{1,x})$ is upper semi-continuous [Ati89, Proposition 1.3.2], so the rank of Ψ cannot be generically zero. If the rank was generically 2, then Ψ would be generically an isomorphism. Taking determinants we obtain a morphism of line bundles

$$\Lambda^2 E_1 \xrightarrow{\Psi\wedge\Psi} \Lambda^2 E_2$$

which is generically an isomorphism. Since the two determinants are isomorphic by assumbtion, we have $\mathrm{Hom}(\Lambda^2 E_1, \Lambda^2 E_2) \cong \mathbb{C}$, hence the previous morphism must be given by a non-zero complex number. This shows that Ψ was in fact an isomorphism, because Ψ_x takes pairs of linearly independent vectors to pairs of linearly independent vectors for all $x \in M$. Therefore the rank is generically 1, only going down to 0 at special points.

Let $L_1 \subseteq E_1$ be the largest rank 1 subbundle of E_1 contained in the kernel of Ψ. Let $v_1 \in L_{1,x}$, and let z be a holomorphic coordinate around a general point $x \in M$. Then we can write $\Phi_1(v_1) = \phi_{1,x}(v_1) \otimes dz$ for some $\phi_{1,x} \in \mathrm{End}(E_{1,x})$. Then

$$0 = \Phi_{2,x}(\Psi_x(v_1)) = (\Psi \otimes \mathrm{id}_K)_x(\phi_{1,x}(v_1) \otimes dz) = \Psi_x(\phi_{1,x}(v_1)) \otimes dz,$$

so $\phi_{1,x}(v_1) \in \mathrm{Ker}(\Psi_x) = L_{1,x}$. Since it suffices to check $\Phi_1(L_1) \subseteq L_1 \otimes K$ generically, this shows that L_1 is Φ_1-invariant.

Let now $L_2 \subseteq E_2$ be the largest rank 1 subbundle of E_2 containing the image of Ψ. Let $v_2 = \Psi(v_1) \in L_{2,x}$ be a vector over a general point $x \in M$, which can thus be written as the image under Ψ of someone in E_1. Then

$$\Phi_{2,x}(\Psi_x(v_1)) = (\Psi \otimes \mathrm{id}_K)_x(\phi_{1,x}(v_1) \otimes dz) = \Psi_x(\phi_{1,x}(v_1)) \otimes dz,$$

so $\phi_{2,x}(v_2) \in \mathrm{Im}(\Psi_x) = L_{2,x}$. Since it suffices to check $\Phi_2(L_2) \subseteq L_2 \otimes K$ generically, this shows that L_2 is Φ_2-invariant.

Now we use that (E_i, Φ_i) are stable to deduce that

$$\deg(L_i) < \frac{d}{2}$$

for $i \in \{1, 2\}$, where $d := \deg(\Lambda^2 E_1) = \deg(\Lambda^2 E_2)$. By construction, Ψ induces a non-zero morphism of line bundles $E_1/L_1 \to L_2$, which corresponds to a non-zero global section of $(E_1/L_1)^* \otimes L_2$. Line bundles with negative degree do not have any non-zero global sections, so we must have $\deg(E_1/L_1) \leqslant \deg(L_2)$. Therefore

$$\frac{d}{2} < \deg(\Lambda^2 E_1) - \deg(L_1) = \deg(E_1/L_1) \leqslant \deg(L_2) < \frac{d}{2},$$

a contradiction. Hence Ψ must be an isomorphism. $\square$

Proposition 1.7. *Assume $g \geqslant 2$ and let $E \to M$ be a rank 2 vector bundle. Then there exists Higgs field Φ on E such that (E, Φ) is stable if and only if there exists a dense Zariski open subset $U \subseteq H^0(M, \mathcal{E}nd(\mathcal{E}) \otimes \omega)$ such that all $\Phi' \in U$ have the property that no line bundle $L \subseteq E$ is Φ'-invariant.*

Proof. We define the following sets of rank 2 vector bundles on M:

- $\mathbf{S} := \{E \in \mathbf{Vec}_2(M) \mid \exists \Phi \text{ such that } (E, \Phi) \text{ is stable}\}$.
- $\mathbf{A} := \{E \in \mathbf{Vec}_2(M) \mid \text{a general } \Phi \text{ has no invariant } L\}$.
- $\mathbf{B} := \{E \in \mathbf{Vec}_2(M) \mid \exists L \text{ invariant for all } \Phi\}$.

Our goal is to show that $\mathbf{S} = \mathbf{A}$. If Φ has no invariant L, then (E, Φ) is automatically stable. Hence $\mathbf{A} \subseteq \mathbf{S}$. The plan to show the other inclusion is to see that

$$\mathbf{Vec}_2(M) = \mathbf{A} \sqcup \mathbf{B}$$

and that $\mathbf{B} \subseteq \mathbf{Vec}_2(M) \setminus \mathbf{S}$.

Let us start by showing that $\mathbf{Vec}_2(M) = \mathbf{A} \sqcup \mathbf{B}$.

Let $p\colon \mathbb{P}(E) \to M$ be the projectivisation of our rank 2 vector bundle, which is a ruled surface in the sense of [Har77, §V.2]. Let $O(-1)$ denote the tautological line bundle on $\mathbb{P}(E)$, whose fibre over $[v] \in \mathbb{P}(E)$ is the line $\{\lambda v \mid \lambda \in \mathbb{C}\} \subseteq E_{p([v])}$. Let $O(l) := O(1)^{\otimes l}$ for all $l \in \mathbb{Z}$, and if $F \to \mathbb{P}(E)$ is another vector bundle, denote by $F(l)$ the tensor product $F \otimes O(l)$. We have then $p_*\mathcal{O}(l) = S^l(\mathcal{E}^*)$ for all $l \geqslant 0$ [Har77, Exercise III.8.4].

Let $x \in M$. Then every endomorphism $A \in \operatorname{End}(E_x)$ defines a quadratic map $E_x \to \Lambda^2 E_x$ sending v to $Av \wedge v$. Such a quadratic map can be naturally regarded as a degree 2 homogeneous polynomial on the coordinates of e with coefficients in $\Lambda^2 E_x$. Hence we have a morphism $\operatorname{End}(E) \to S^2 E^* \otimes \Lambda^2 E$, which vanishes precisely along the trivial line subbundle of $\operatorname{End}(E)$ consisting over each fibre of scalar multiples of the identity. Sending $A \mapsto A - \frac{\operatorname{tr}(A)}{2} \operatorname{id}_{E_x}$ on each fibre allows us to identify $\operatorname{End}_0(E)$ as the quotient of $\operatorname{End}(E)$ by this trivial line subbundle, so we obtain an injective morphism $\operatorname{End}_0(E) \to S^2E^* \otimes \Lambda^2 E$. Counting ranks we see that we have in fact an isomorphism $\operatorname{End}_0(E) \cong S^2 E^* \otimes \Lambda^2 E$, and therefore

$$\mathcal{E}nd_0(\mathcal{E}) \otimes \omega \cong p_*\mathcal{O}(2) \otimes \omega \otimes \Lambda^2 \mathcal{E}.$$

By the projection formula, $\mathcal{E}nd_0(\mathcal{E}) \otimes \omega \cong p_*(p^*(\omega \otimes \Lambda^2 \mathcal{E})(2))$. Therefore we have an isomorphism

$$\alpha\colon H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \omega) \cong H^0(\mathbb{P}(E), p^*(\omega \otimes \Lambda^2 \mathcal{E})(2)).$$

Let now Φ be a traceless Higgs field on E, and assume it is non-zero. By construction, a non-zero vector $v \in E$ is an eigenvector of the twisted endomorphism over the corresponding fibre if and only if the section $\alpha(\Phi)$ vanishes at the point $[v] \in \mathbb{P}(E)$, i.e. if and only if $[v]$ is in the divisor of zeros of the global section $\alpha(\Phi)$, which we denote $\operatorname{div}(\alpha(\Phi))$. Let $L \subseteq E$ be a Φ-invariant subbundle, which defines a section of $p\colon \mathbb{P}(E) \to M$ by functoriality of projectivisation on injective morphisms of vector bundles:

$$\begin{tikzcd} \mathbb{P}(L) \arrow[r, "\sigma"] \arrow[d, equal] & \mathbb{P}(E) \arrow[dl, "p"] \\ M & \end{tikzcd}$$

Being Φ-invariant means precisely that $\sigma(M) \subseteq \operatorname{div}(s(\Phi))$. But then any non-zero $v \in L$ is a non-zero eigenvector corresponding to some eigenvalue of the endomorphism over the corresponding fibre. Since Φ is traceless and non-zero, the other eigenvalue must be different, and there must be some non-zero eigenvector outside of L, call it $v' \in V$. Since v' is a non-zero eigenvector, $[v'] \in \operatorname{div}(\alpha(\Phi))$. And since $v' \notin L$, $[v'] \notin \sigma(M)$. Therefore $\sigma(M)$ is a proper irreducible component of the divisor $\operatorname{div}(\alpha(\Phi))$. So if $\operatorname{div}(\alpha(\Phi))$ is irreducible, then no line bundle $L \subseteq V$ is Φ-invariant and (E, Φ) is automatically stable.

Next we give a lower bound for the dimension of the linear system $|p^*(\omega \otimes \Lambda^2 \mathcal{E})(2)|$, which is one less than the dimension of the vector space $H^0(\mathbb{P}(E), p^*(\omega \otimes \Lambda^2\mathcal{E})(2))$. Using the previous isomorphism it suffices to gain control over the dimension of the global sections of $\mathcal{E}nd_0(\mathcal{E}) \otimes \omega$ on M, for which we can apply Hirzebruch–Riemann–Roch [Har77, Theorem A.4.1]. From [Har77, Example A.4.1.1] we get

$$\operatorname{td}(\omega^*) = 1 - \frac{c_1(\omega)}{2}.$$

Using the short exact sequence used earlier

$$0 \to \mathcal{O} \to \mathcal{E}nd(\mathcal{E}) \to \mathcal{E}nd_0(\mathcal{E}) \to 0$$

we see that $c_1(\mathcal{E}nd_0(\mathcal{E})) = c_1(\mathcal{E}nd\mathcal{E}) = 0$. Therefore

$$\operatorname{ch}(\mathcal{E}nd_0(\mathcal{E}) \otimes \omega) = 3 + c_1(\mathcal{E}nd_0(\mathcal{E})) + 3c_1(\omega) = 3 + 3c_1(\omega).$$

Multiplying the two expressions we obtain

$$\operatorname{ch}(\mathcal{E}nd_0(\mathcal{E}) \otimes \omega)\operatorname{td}(\omega^*) = 3 + \frac{3}{2}c_1(\omega),$$

whose codimension 1 part has degree $3g - 3 \geqslant 3$. So Hirzebruch–Riemann–Roch tells us that

$$h^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \omega) - h^1(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \omega) = 3g - 3 \geqslant 3,$$

which implies that $h^0(\mathbb{P}(E), p^*(\omega \otimes \Lambda^2\mathcal{E})(2)) = h^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \omega) \geqslant 3$.

Thus, our linear system $|p^*(\omega \otimes \Lambda^2\mathcal{E})(2)|$ has dimension 2. If it does not have any fixed component, then Bertini's theorem [Iit82, Theorem 7.19] and the discussion above imply that a general $\Phi \in H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \omega)$ leaves no line bundle $L \subseteq E$ invariant, i.e. $E \in \mathbf{A}$.

Let us see what happens if it does have some fixed divisor. By definition, a fixed divisor corresponds to a non-zero global section $s_0 \in H^0(\mathbb{P}(E), \mathcal{M}_1)$ for some invertible sheaf $\mathcal{M}_1$ such that there exists another invertible sheaf $\mathcal{M}_2$ with $\mathcal{M}_1 \otimes \mathcal{M}_2 \cong p^*(\omega \otimes \Lambda^2\mathcal{E})(2)$. Being a fixed divisor translates into saying that every global section $s \in H^0(\mathbb{P}(E), p^*(\omega \otimes \Lambda^2\mathcal{E})(2))$ can be written as ts_0 for some $t \in H^0(\mathbb{P}(E), \mathcal{M}_2)$. In particular, both $H^0(\mathbb{P}(E), \mathcal{M}_1)$ and $H^0(\mathbb{P}(E), \mathcal{M}_2)$ have to be non-zero. By [Har77, Exercise II.7.4] we can write $\mathcal{M}_i \cong p^*\mathcal{L}_i(l_i)$ with $l_1 + l_2 = 2$. In fact, we must have $0 \leqslant l_i \leqslant 2$, because using again the projection formula we have

$$H^0(\mathbb{P}(E), p^*\mathcal{L}_i(l_i)) \cong H^0(M, \mathcal{L}_i \otimes p_*\mathcal{O}(l_i))$$

and $p_*\mathcal{O}(l) = 0$ for all $l < 0$ [Har77, Exercise III.8.4]. So we only have the following three possibilities:

a) $l_1 = 0$;
b) $l_1 = 1$;
c) $l_1 = 2$.

Let us start with case a). Let $p^*s \in H^0(\mathbb{P}(E), p^*\mathcal{L}) \cong H^0(M, \mathcal{L})$ be a global section corresponding to the fixed component of our linear system. Dividing all global sections by s and by p^*s respectively we obtain the following commutative diagram:

$$\begin{tikzcd}
H^0(M, \mathrm{End}_0(E) \otimes K) \arrow[r, "\cong"] \arrow[d, "/s"', "\cong"] & H^0(\mathbb{P}(E), p^*(K \otimes \Lambda^2 E)(2)) \arrow[d, "/p^*s"', "\cong"] \\
H^0(M, \mathrm{End}_0(E) \otimes K \otimes L^*) \arrow[r, "\cong"] & H^0(\mathbb{P}(E), p^*(L^* \otimes K \otimes \Lambda^2 E)(2))
\end{tikzcd}$$

By definition, the new linear system does not have any fixed divisors and has the same dimension. Hence we can apply Bertini to conclude that a generic $\Phi' \in H^0(M, \mathrm{End}_0(E) \otimes K \otimes L^*)$ does not have invariant line bundles, which in this case are defined as line bundles $N \subseteq E$ such that $\Phi'(N) \subseteq N \otimes K \otimes L^*$. But a line bundle $N \subseteq E$ is Φ'-invariant if and only if it is $s\Phi'$-invariant, so we have $E \in \mathbf{A}$ in this case.

We move on to case b). Assume that the fixed divisor corresponds to a non-zero global section $s \in H^0(\mathbb{P}(E), p^*\mathcal{L}(1))$. This corresponds to a non-zero morphism $\mathcal{E} \to \mathcal{L}$, whose kernel $\mathcal{N} \subseteq \mathcal{E}$ must then be an invertible sheaf because it is torsion-free of rank 1 over the algebraic curve M. □

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