Higgs bundles

Program of Wednesday's seminar

Summer Semester 2020

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GENERAL COMMENTS

(i) Each speaker should prepare a 1 hr. talk. Depending on the number of questions during the talk, the total duration may be somewhat longer than 1 hr., however the speaker should not assume that 30 more minutes are available for the main part of the talk.

(ii) Each speaker should prepare a set of exercises for the participants for their self-assessment.

(iii) The main objectives of the "summary and perspectives" talks are the following:

- Recapitulate the material of the corresponding block;
- Point out the most important results and their significance for the rest of the seminar (Andriy can help with this task);
- If time permits, the speaker may wish to provide indications where the material considered in the corresponding block can be used as well. In other words, it may be useful to put the corresponding block into a wider context.

PhD students should opt for one of the "summary and perspectives" talk if they feel to be especially qualified (for instance, if the topic of their PhD Thesis is closely related to the topic of the talk).

PART I: GEOMETRIC INVARIANT THEORY

(Program due to Lukas Braun)

Talk 1 (13.05). Quotients in algebraic geometry. Speaker: *Nicola Nesa.*

Motivation. Motivate by defining/recalling (topological) orbit spaces and categorical quotients. But orbit spaces and categorical quotients do not have to be compatible! Show that for example the orbit space of the diagonal action of $\mathbb{C}^*$ on $\mathbb{C}^n$ is *not* a categorical quotient in the category of algebraic varieties or complex analytic spaces - by noticing it is not even Hausdorff.

Invariant theory. Define algebraic groups. Give examples: finite groups, closed subgroups of $GL_n(\mathbb{C})$, unipotent groups,...

State [Bri10, Thm. 1.16] to show that in the case of a closed subgroup H of a linear algebraic group G the group theoretical quotient G/H is a nice quotient in the category of (smooth, quasiprojective) varieties (the third property [Bri10, Thm. 1.16] tells us it is a categorical quotient).

Take this as motivation to study the *ring of invariants*, first $\mathbb{C}[V]^G$ for a complex representation V of G, then $\mathbb{C}[X]^G$ for an affine variety X by [Bri10, Prop. 1.9]. State Hilbert's 14th Problem for invariant rings. Define reductive groups [Bri10, Def. 1.20] and state also some of the equivalent criteria of [Bri10, Thm. 1.23], at least semi-simplicity of G-modules. Then state the main theorem [Bri10, Thm. 1.24] ((i) for $X = \mathbb{C}^n$ is Hilbert's theorem). Give some examples of reductive groups: classical (complex) Lie groups, finite groups,...

Example: finite groups. Give an example where $\mathbb{C}[X]^G$ can be efficiently computed. E.g. the diagonal action of $G_1 := \mathbb{Z}/2$ on $\mathbb{C}^2$. The ring of invariants is generated by $g_i = x^2, y^2, xy$, and they satisfy $g_1 g_2 = g_3^2$ (this is the A_1-singularity!). Note that by a change of variables, this quotient is embedded in $\mathbb{C}^3$ as $X := V(x_1^2 + x_2^2 + x_3^2)$, which has an action by $G_2 := \mathbb{Z}/2 \times \mathbb{Z}/2$ induced by the action on $\mathbb{C}^3$ given by

$$(1,0) \mapsto \operatorname{diag}(-1,1,-1), \quad (0,1) \mapsto \operatorname{diag}(-1,-1,1).$$

Now note that the ring of invariants $\mathbb{C}[\mathbb{C}^3]^{G_2}$ is generated by $h_i = x_1^2, x_2^2, x_3^2, x_1x_2x_3$, and these satisfy $h_1h_2h_3 = h_4^3$. Pulling back the relation of X yields $h_1 + h_2 + h_3$, such that $\mathbb{C}[X]^{G_2} \cong \mathbb{C}[h_1, \dots, h_4]/\langle h_1h_2h_3 - h_4^3, h_1 + h_2 + h_3\rangle$ and eliminating h_1 yields $\mathbb{C}[h_2, h_3, h_4]/\langle h_2^2h_3 + h_3^2h_2 + h_4^3\rangle$, so that $X//G_2$ is the D_4-singularity. (In particular, the change of variables is responsible for D_4 being a solvable - not abelian - quotient of $\mathbb{C}^2$ and the computation shows how easy it is to compute solvable quotients this way)

You can of course adjust the details of the example according to your time limit!

Geometric quotients. Define geometric quotients [Bri10, Def. 1.18], compare to the group theoretical quotient G/H from above and note that it can be identified with the topological orbit space. Then note that for finite groups, the quotient $X//G$ from the main theorem above is geometric.

Show that this fails in general, in particular in the example of the diagonal action of $\mathbb{C}^*$ on $\mathbb{C}^n$ from the introduction/motivation. Here $X//G = \mathrm{pt}$ (since there are no nontrivial invariant polynomials).

In general, only closed orbits are parameterized by $X//G$, cf [Muk03, Cor. 5.4]. Define stable points [Bri10, Def. 1.25] and state [Bri10, Prop. 1.26], i.e. the restriction of $X \to X//G$ to the set of stable points yields a geometric quotient.

Finally show that in the example of the diagonal action of $\mathbb{C}^*$ on $\mathbb{C}^n$ from above, there are no stable points. If time permits, explain [Muk03, Ex. 5.1], where the set of stable points is nonempty.

Transition to the next talk. Note that we still have not obtained projective space from the example of the diagonal action of $\mathbb{C}^*$ on $\mathbb{C}^n$ and that we may have to identify "good" open sets allowing a categorical quotient that reasonably reflects the orbits of the group action.

Main results: [Bri10, Thm. 1.24].
Literature: [Bri10, Sec. 1.1,1.2], [Muk03, Ch. 5], [Soe20, Sec.'s 8.2, 8.3, 8.6] (German).

Talk 2 (20.05). Geometric invariant theory. Speaker: *Luca Terenzi.*

Motivation and connection to the first talk. Recall (or explain, if this has not been done in the first talk) [Muk03, Ex. 5.1] in contrast to the diagonal action of $\mathbb{C}^*$ on $\mathbb{C}^n$. State "The Italian Problem" and the notion of birational quotient [Muk03, p. 183]. Note that a birational quotient does not have to be unique, but different birational quotients are birational. State [Muk03, Prop. 6.2]. Which of the two examples is a birational quotient? We also want a birational quotient for the second one!

GIT-quotients. Define linearization of invertible line bundles and the sets of semi-stable and stable points with respect to such a line bundle as in [Bri10, Prop. 1.35] or [MFK94, Def. 1.7(b),(c)]. Note that the notion of G-invariant sections depends on the linearization of the line bundle.

Take as an example the diagonal action of $\mathbb{C}^*$ on $\mathbb{C}^n$ and the trivial bundle on $\mathbb{C}^n$. Firstly, take the trivial linearization of $\mathcal{O}_{\mathbb{C}^n}$, here the constant functions are invariant on the whole of $X = \mathbb{C}^n$, so $X^{ss}(L) = X$. For the linearization of $\mathcal{O}_{\mathbb{C}^n}$ given by the standard action of $\mathbb{C}^*$ on $\mathbb{C}$, the constant functions are not any more invariant, and thus $X^{ss}(L) = X \setminus \{0\}$.

State [MFK94, Thm. 1.10] (you can replace "scheme" and "pre-scheme" by "variety", compare [Hos, Thm. 5.31], but X does not have to be quasiprojective in general, while the GIT-quotient in the sense of Mumford is always quasiprojective!). The first point of [MFK94, Thm. 1.10] means that the GIT-quotient $X^{ss}(L)//G$ is a good quotient, i.e. a categorical quotient with an affine quotient map, cf. eg. [ADHL14, Ch. I, Def. 2.3.1].

Note that in our example from above with $X^{ss}(L) = \mathbb{C}^n \setminus \{0\}$, we finally get $\mathbb{P}^{n-1}(\mathbb{C})$ as a GIT-quotient $X^{ss}(L)//\mathbb{C}^*$ and even $X^s(L) = X^{ss}(L)$ in this case, so it is a geometric quotient.

Projective GIT-quotients. Discuss briefly [Bri10, Sec. 1.3], leading to [Bri10, Prop. 1.35], the projective version of [MFK94, Thm. 1.10]. Also [Hos, Sec. 5] is a good resource. Prove [Bri10, Prop. 1.35] if time permits. You should note (one sees this eg. through the proof), that the GIT-quotient can be explicitly computed by gluing the affine invariant theoretic quotients from the first talk.

If you do not have enough time, you may ignore [MFK94, Thm. 1.10] and just state and prove [Bri10, Prop. 1.35].

Variation of GIT. We note that there may be several GIT-quotients for one action of an algebraic group on a (projective) variety, corresponding to different linearized line bundles. Very briefly and informally state the (already informal) [Hu05, Thm. 3.1].

Optional: If you are familiar with toric varieties, you can explain how birational toric varieties arise as GIT-quotient of the same total coordinate space and how this reflects in the fan of these varieties (I think last summer semester there was a seminar on toric varieties, so there might be a chance that this is a good example).

Final remarks, generalizations. Optional, only if time permits: Remark that the GIT-quotients from above do provide all quasiprojective good quotients of open subsets *only if* X is smooth. For a singular counterexample, see [Hau04, Prop. 3.6]. Again, if you are familiar with toric varieties, you may be able to adapt and simplify the counterexample in the proof of [Hau04, Prop. 3.6]. Then mention [Hau04, Thm. 3.3], i.e. that in order to get all quasiprojective good quotients of a singular normal variety one has to take G-linearized Weil divisors instead of line bundles. Finally you can mention that by [Hau04, Thm. 3.5], taking G-linearizations of *groups of* Weil divisors instead of single Weil divisors, leads to all *divisorial quotient spaces*, which in particular includes *all smooth quotients*.

Main results: [MFK94, Thm. 1.10], [Bri10, Prop. 1.35].
Literature: [Bri10, Sec. 1.3], [Muk03, Ch. 5,6], [MFK94, Ch. 1], [Hos, Sec. 5] [Soe20, Sec. 8.7] (German); optionally bits of [Hu05] and [Hau04].

Talk 3 (27.05). GIT and symplectic quotients. Speaker: *Vincent Gajda.*

Motivation. State informally the Kempf-Ness theorem: given a smooth subvariety $X \subseteq \mathbb{P}^n$ of complex projective space, it has a symplectic structure coming from the restriction of the Fubini-Study metric. Now if the complex reductive group G acts on X via a representation in $GL_{n+1}(\mathbb{C})$, we have seen in the previous talks that there is a GIT-quotient $X^{ss}//G$. If a maximal compact subgroup K of G is connected and acts by symplectomorphisms on X, then there is a *symplectic quotient* $X//_sG$ (of an open subset of X) and the Kempf-Ness theorem says that both quotients agree, see [MFK94, Thm. 8.3]. We want to prove this theorem in this talk.

Actions of (real) Lie groups. Recall actions of (real) Lie groups K on (real) manifolds X, the Lie algebra $\mathfrak{k}$ associated to K, the exponential map, the Lie algebra homomorphism from $\mathfrak{k}$ to the vector fields on X, cf [Woo10, Sec. 2].

Symplectic manifolds and Hamiltonian group actions. Define symplectic manifolds, symplectic actions of Lie groups, Hamiltonian actions of Lie groups, the moment map [Woo10, Sec. 3]. Note that Kähler manifolds are symplectic.

Show by [MFK94, Ex. 8.1(ii)], that a smooth complex projective variety $X \subseteq \mathbb{P}^n(\mathbb{C})$ allows a moment map.

Symplectic quotients. Basically state [Woo10, Thm. 3.3.1] on the symplectic quotient.

The Kempf–Ness theorem. State the Kempf–Ness theorem, e.g. in the form of [MFK94, Thm. 8.3], with necessary preliminaries from the preceding paragraph in [MFK94, Ch. 8,§ 2]. Prove the theorem, e.g. take the proof of [MFK94, Thm. 8.3] (here you have to consider in addition the first part of [MFK94, Prop. 2.2], but this is not hard).

Main results: [MFK94, Thm. 8.3].
Literature: [MFK94, Ch 8.1-8.2], [Woo10, Sec. 2,3,5].
Note: This talk will be followed by the summary talk on GIT. The speaker should finish the talk by 15:30. We may wish to shorten the break as an exception to 20 Min.

Talk 4 (27.05 at 15:30). Summary and perspectives on GIT. Speaker: *Severin Barmeier.*

Note: This is a shorter 25 Min. talk.

Presumably the time suffices for the summary only. However, should there be some spare time, the speaker may wish to discuss for example the following:

- Representation variety vs. character variety.

Part II: Geometry of complex vector bundles

Talk 5 (03.06). Complex line bundles. Speaker: *Vera Gahlen.*

Introduce the notion of a classifying map for complex vector bundles. Argue that $\mathbb{C}P^\infty$ is a classifying space for line bundles. Explain the following fact: For Riemann surfaces any vector bundle is a pull-back of the tautological bundle on $\mathbb{C}P^1$.

Define the (topological) degree of a complex line bundle (throughout the talk one can assume that the base manifold is a Riemann surface). Recall the notion of the first Chern class of a complex line bundle and show that on a Riemann surface this essentially coincides with the degree.

Recall the notion of a holomorphic line bundle and its divisor as well as the relation between the degree of a divisor and the topological degree of the corresponding line bundle. Describe the divisor – line bundle correspondence.

Note: In this talk the speaker can assume that the audience is familiar with most of the notions to be introduced at least in some incarnations. The main purpose it to show how the same object appears in different contexts.

Main results: [Kob87, Thm. II.2.16, Prop. II.3.1], [Nei19, Prop. 5.3]
Literature: [Kob87, II.1–II.3], [Nei19, 2.6, 5.1], [Jos06, 5.4 (to the Riemann–Roch thm)].

Talk 6 (10.06). Connections on complex vector bundles. Speaker: *Giovanni Zaccanelli.*

Introduce the notion of a connection on a vector bundle and discuss the curvature of a connection. Mention that a connection is *never* unique (as long as the rank of the bundle is at least 1) and mention gauge-equivalence relation.

Prove that the degree of a complex line bundle can be expressed in terms of the curvature of a connection.

Mention how the curvature on a vector bundle E induces connections on related bundles such as $\det E = \Lambda^{\mathrm{top}} E$, $\mathrm{End}(E)$ etc.

Discuss flat connections and their relation to representations of the fundamental group. Describe the abelian case and the symplectic (Kähler) structure on the Jacobian.

Main results: [Kob87, Prop. I.2.5].
Literature: [Kob87, I.1, I.2], [Nei19, 5.4], [BGPGW07, 1.4, 2.1–2.5].

Talk 7 (17.06). Holomorphic structures on complex vector bundles. Speaker: *Yuhang Hou*

Discuss briefly holomorphic structures on complex bundles of rank ≥ 2 taking into account that holomorphic structures on line bundles should have been discussed in Talk 5.

State the existence and uniqueness of the Chern connection [Kob87, Prop. I.4.9] (the term *Hermitian connection* used by Kobayashi is misleading and should not be used in this context). It is not essential to prove this fact, just the statement should suffice.

Taking the Hodge theorem as granted, prove that any holomorphic line bundle admits a distinguished connection, which is given by [Nei19, Lemma 5.19]. Prove also [Nei19, Lemma 5.20], which will serve as a model for the proof of the Narasimhan–Seshadri theorem in the next talk.

Note: The second part of the talk is mainly a preparation for the next one. This should be made clear to the audience.

Main result: [Kob87, Prop. I.2.5].
Literature: [Kob87, I.1, I.2], [Nei19, 2.6, 5.4], [BGPGW07, 1.4, 2.1–2.5].

Talk 8 (24.06). Stable vector bundles. Speaker: *Tanuj Gomez.*

Explain that the Einstein condition of [Nei19, Lemma 5.19], which appeared in the previous talk, is in essence the requirement that the moment map for the gauge group action takes a prescribed value [Nei19, Sect. 5.2, from (5.7) to Ex. 5.8]. This can and should be done in the smooth category, i.e., without mentioning Sobolev completions.

State the theorem of Narasimhans-Seshadri and indicate how Donaldson's proof proceeds without going too much into analytic details, i.e., the strategy of the proof as indicated in [Don83, Sect. 1-2] and a selection of the rest of Donaldson's paper should suffice.

Main results: [Kob87, I.3.7, I.4.9, I.4.17, I.4.21].
Literature: [Kob87, I.3, I.4, II.1, II.2, II.3], [Nei19, 2.6].

Talk 9 (01.07). Summary and perspectives on the geometry of complex vector bundles. Speaker: *Andreas Demleitner.*

Possible extra topics to elaborate:

- Stable vector bundles on Kähler surfaces and anti-self-duality equations.
- Donaldson–Thomas invariants of Calabi–Yau three-folds.

PART III: HIGGS BUNDLES

Talk 10 (08.07). Hitchin's equations. Speaker: *Jin Li.*

Introduce Hitchin's equations (these are Eq. (1.3) in [Hit87]) and describe their origin. Show that Hitchin's moduli space has a natural smooth structure.

Note: The part on the smooth structure should be treated informally, i.e., we do not want to go carefully through the details of Sobolev spaces and elliptic operators, however it would be good just to state what is needed for the formal proof without details. The speaker should be familiar with basics of elliptic operators on manifolds (for example as used in Hodge theory).

Main Results: [Hit87, Thm. 2.7, Thm. 5.7].
Literature: [Hit87, Sect. 1, 2, 5].

Talk 11 (15.07). Existence of solutions. Speaker: *Pedro Núñez.*

Introduce the stability condition for Higgs bundles and prove the Hitchin–Kobayashi correspondence [Hit87, Thm. 4.3]. This is very much related to the talk on *Stable vector bundles*.

Main result: [Hit87, Thm. 4.3].
Literature: [Hit87, Sect. 3, 4], [Wen14, Sect. 2.3].

Talk 12 (22.07). Hitchin's equations and irreducible representations of the fundamental group. Speaker: *Fabian Kertels.*

Show that irreducible solutions of Hitchin's equations determines an irreducible representation of the fundamental group. Present the proof of Donaldson's theorem, which states the converse.

Main Result: [Don87, Thm, P.128],[Hit87, Prop. 9.18],
Literature: [Don87], [Hit87, Sect. 9, from below Remarks on P.109], [Nei19, 6.8–6.10].
Notes: The proof of Theorem of [Don87] should be presented in detail perhaps up to the proof of the existence of twisted harmonic maps; This can be done schematically just like in Donaldson's article.

Talk 13 (29.07). Summary and perspectives on Higgs bundles. Speaker: *N.N.*

Possible extra topics to elaborate:

- Topology of Hitchins moduli space.
- HyperKähler geometry of the Hitchin's moduli space.
- Integrable systems and Hitchin's moduli space.
- Higgs bundle and physics.

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