HIGGS BUNDLES — EXISTENCE OF SOLUTIONS

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Abstract. We introduce the stability condition for Higgs bundles and discuss the Hitchin–Kobayashi correspondence [Hit87, §3 and §4].

Contents

—parts in gray will be omitted during the talk—

1. Stability of Higgs bundles

Notation. In this first section, all vector bundles, morphisms and sections are assumed to be holomorphic. We will often go back and forth between vector bundles E and their sheaves of sections $\mathcal{E}$.

Let M be a compact Riemann surface and K its canonical line bundle. We denote the set of rank 2 vector bundles on M by $\mathbf{Vec}_2(M)$.

Definition (Higgs bundle)**.** A *Higgs bundle* on M is a pair (E, Φ), where $E \in \mathbf{Vec}_2(M)$ and $\Phi \in H^0(M, \mathcal{E}nd(\mathcal{E}) \otimes \mathbf{K})$. We call Φ a *Higgs field* on E.

Remark. Using the canonical isomorphisms

$$H^0(M, \mathcal{E}nd(\mathcal{E}) \otimes \mathcal{K}) \cong \mathrm{Hom}(\mathcal{O}, \mathcal{E}^\vee \otimes \mathcal{E} \otimes \mathcal{K}) \cong \mathrm{Hom}(\mathcal{E}, \mathcal{E} \otimes \mathcal{K})$$

we may identify Φ with a morphism $\Phi\colon\ E \longrightarrow E \otimes K$.

Definition (Stability)**.** A Higgs bundle (E, Φ) is said to be *stable* if for every Φ-invariant line bundle $L \subseteq E$ we have $\mu(L) < \mu(E)$, where Φ-invariance means that $\Phi(L) \subseteq L \otimes K$.

Remark. $(E, 0)$ is stable if and only if E is stable in the usual sense.

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Exercise A. There are no stable Higgs bundles on $\mathbb{P}^1$. [*Hints below*[1]]

Notation. For $E \in \mathbf{Vec}_2(M)$, denote by $\mathrm{End}_0(E)$ the vector bundle of traceless endomorphisms.

Lemma 1. *Let $E \in \mathbf{Vec}_2(E)$. Then there is a natural projection* $\mathrm{pr}_0 : \mathrm{End}(E) \longrightarrow \mathrm{End}_0(E)$ *whose kernel is the trivial line bundle of multiples of the identity, yielding a short exact sequence*

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{E}nd(\mathcal{E}) \longrightarrow \mathcal{E}nd_0(\mathcal{E}) \longrightarrow 0.$$

In particular, $\deg(\mathrm{ch}(\mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K})\,\mathrm{td}(\mathcal{K}^\vee))_1 = 3g - 3$.

Proof. Over $x \in M$, the map $\mathrm{pr}_{0,x} : \mathrm{End}(E_x) \longrightarrow \mathrm{End}_0(E_x)$ is given by

$$\phi \mapsto \phi - \frac{\mathrm{tr}(\phi)}{2}\,\mathrm{id}_{E_x}.$$

The endomorphisms in the kernel are precisely the multiples of the identity, and this fibre-wise description globalizes to the desired short exact sequence. For the Chern class computation we use the axioms in [Har77, Appendix A]. We have

$$c_1(\mathcal{E}nd_0(\mathcal{E})) = c_1(\mathcal{E}nd(\mathcal{E})) = c_1(\mathcal{E}^\vee \otimes \mathcal{E}) = 0,$$

therefore

$$\mathrm{ch}(\mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K}) = 3 + c_1(\mathcal{E}nd_0(\mathcal{E})) + 3c_1(\mathcal{K}) = 3 + 3c_1(\mathcal{K}).$$

We also have

$$\mathrm{td}(\mathcal{K}^\vee) = 1 - \frac{c_1(\mathcal{K})}{2},$$

so multiplying the two expressions we obtain

$$\mathrm{ch}(\mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K})\,\mathrm{td}(\mathcal{K}^\vee) = 3 + \frac{3}{2}c_1(\mathcal{K}).$$

Since $\deg(c_1(\mathcal{K})) = 2g - 2$, the result follows. $\square$

Notation. We define the following subsets of $\mathbf{Vec}_2(M)$:

- $\mathbf{S} := \{E \in \mathbf{Vec}_2(M) \mid \exists \Phi \text{ such that } (E, \Phi) \text{ is stable}\}$.
- $\mathbf{A} := \{E \in \mathbf{Vec}_2(M) \mid \text{a general } \Phi \text{ has no invariant } L\}$.
- $\mathbf{B} := \{E \in \mathbf{Vec}_2(M) \mid \exists L \text{ invariant for all } \Phi\}$.

Lemma 2. *If $g \geqslant 2$, then*

$$\mathbf{Vec}_2(M) = \mathbf{A} \sqcup \mathbf{B}.$$

[1] Grothedieck's theorem allows us to write Φ as a matrix. What can we say about each entry? The solution can be found in [Hit87, Remark (3.2) (iii)]

Proof. Let $\pi : \mathbb{P}(E) \longrightarrow M$ be the projectivization of our rank 2 vector bundle and let $O(-1) \longrightarrow \mathbb{P}(E)$ denote the tautological line bundle, whose fiber over $[v] \in \mathbb{P}(E)$ is the line $\{\lambda v \mid \lambda \in \mathbb{C}\} \subseteq E_{\pi([v])}$ spanned by v. Denote also $O(l) := O(-1)^{\otimes(-l)}$. If $\mathcal{F}$ is a sheaf on $\mathbb{P}(E)$, we denote $\mathcal{F}(l) := \mathcal{F} \otimes \mathcal{O}(l)$, where $\mathcal{O}(l)$ denotes the sheaf of sections of $O(l)$. Using the relative Proj construction we can write $\mathbb{P}(E) = \mathbb{P}(\mathcal{E}^\vee) := \operatorname{Proj}_M(S(\mathcal{E}^\vee))$, so [Har77, Exercise III.8.4] implies that

$$\pi_*\mathcal{O}(l) = \begin{cases} S^l(\mathcal{E}^\vee) & \text{if } l \geqslant 0, \\ 0 & \text{if } l < 0. \end{cases}$$

Let $x \in M$. Given $A \in \operatorname{End}(E_x)$, we define the quadratic form $v \mapsto Av \wedge v$ with values in $\Lambda^2 E_x$, which can then be naturally regarded as an element in $S^2(E_x^\vee) \otimes \Lambda^2 E_x$. The resulting quadratic form is trivial precisely when $A = \lambda \operatorname{id}_{E_x}$ for some $\lambda \in \mathbb{C}$, so by Lemma 1 we obtain an injective homomorphism $\operatorname{End}_0(E_x) \longrightarrow S^2(E_x^\vee) \otimes \Lambda^2 E_x$. Both vector spaces have the same dimension, so this must be an isomorphism. These isomorphisms globalize to an isomorphism $\operatorname{End}_0(E) \cong S^2(E^\vee) \otimes \Lambda^2 E$, hence we obtain an isomorphism

$$\mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K} \cong \pi_*\mathcal{O}(2) \otimes \mathcal{K} \otimes \Lambda^2\mathcal{E}.$$

The projection formula yields now an isomorphism $\mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K} \cong \pi_*(\pi^*(\mathcal{K} \otimes \Lambda^2\mathcal{E})(2))$, hence an isomorphism

$$\psi : H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K}) \cong H^0(\mathbb{P}(E), \pi^*(\mathcal{K} \otimes \Lambda^2\mathcal{E})(2)).$$

Let now $\Phi \in H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K})$. A line bundle $L \subseteq E$ is then Φ-invariant precisely when $\psi(\Phi)$ vanishes at all $[v] \in \mathbb{P}(L) \subseteq \mathbb{P}(E)$. In other words, L is Φ-invariant if and only if $\sigma(M) \subseteq \operatorname{div}(\psi(\Phi))$, where $\operatorname{div}(-)$ denotes the divisor of zeros of a section and $\sigma : M = \mathbb{P}(L) \longrightarrow \mathbb{P}(E)$ is the section induced by $L \subseteq E$.

Suppose now that $\Phi \in H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K})$ is not nilpotent and let $L \subseteq E$ be a Φ-invariant line bundle. Then over a general point $x \in M$, the corresponding traceless endomorphism $\phi_x \in \operatorname{End}_0(E_x)$ is diagonalizable, so we can find some eigenvector $v \in E_x \setminus L_x$ in an eigenspace other than L_x. This gives us a point $[v] \in \mathbb{P}(E) \setminus \sigma(M)$ on which $\psi(\Phi)$ vanishes. Hence $\sigma(M)$ is a proper irreducible component of the divisor $\operatorname{div}(\psi(\Phi))$.

The previous discussion shows that if Φ is not nilpotent and $\operatorname{div}(\psi(\Phi))$ is irreducible, then there are no invariant line bundles $L \subseteq E$. By Hirzebruch–Riemann–Roch and Lemma 1 we have

$$h^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K}) \geqslant 3g - 3 \geqslant 3,$$

so the complete linear system defined by the invertible sheaf $\pi^*(\mathcal{K} \otimes \Lambda^2\mathcal{E})(2)$ has dimension at least 2. If this linear system does not have a fixed divisor, then Bertini's theorem [Iit82, Theorem 7.19] tells us that $\operatorname{div}(\psi(\Phi))$

is irreducible for a general $\Phi \in H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K})$. Since in our case $\Phi \in H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K})$ is nilpotent if and only if $\Phi^2 = 0$, a general Φ is not nilpotent. Therefore $E \in \mathbf{A}$ in this case.

Let us see what happens if the linear system has a fixed divisor. Such a fixed divisor must be the zero locus of a non-zero global section s of an invertible sheaf on $\mathbb{P}(E)$, which is up to isomorphism of the form $\pi^*\mathcal{L}(l)$ with $\mathcal{L}$ an invertible sheaf on M and $l \in \mathbb{Z}$ [Har77, Exercise II.7.9]. Being a fixed divisor means then that every other global section of our line bundle can be written as a product st, where $t \in H^0(\mathbb{P}(E), \pi^*\mathcal{N}(2-l))$. Since our line bundle had non-zero global sections, both $\pi^*\mathcal{L}(l)$ and $\pi^*\mathcal{N}(2-l)$ must have non-zero global sections. By the projection formula, this leaves us with only three possibilities:

a) $l = 0$;
b) $l = 1$;
c) $l = 2$.

We analyze each case separately to conclude that

$$E \in \begin{cases} \mathbf{A} & \text{if } l = 0, \\ \mathbf{B} & \text{if } l \in \{1, 2\}. \end{cases}$$

Let us start with case *a*). Let $\pi^*s \in H^0(\mathbb{P}(E), \pi^*\mathcal{L}) \cong H^0(M, \mathcal{L})$ be a global section corresponding to the fixed component of our linear system. Dividing all global sections by s and by π^*s respectively we obtain the following commutative diagram:

$$\begin{tikzcd} H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K}) \arrow[r, "\cong"] \arrow[d, "/s"', "\cong"] & H^0(\mathbb{P}(E), \pi^*(\mathcal{K} \otimes \Lambda^2\mathcal{E})(2)) \arrow[d, "/\pi^*s"', "\cong"] \\ H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K} \otimes \mathcal{L}^\vee) \arrow[r, "\cong"] & H^0(\mathbb{P}(E), \pi^*(\mathcal{L}^\vee \otimes \mathcal{K} \otimes \Lambda^2\mathcal{E})(2)) \end{tikzcd}$$

By definition, the new linear system does not have any fixed divisors and has the same dimension. Hence we can apply Bertini to conclude that a generic $\Phi' \in H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K} \otimes \mathcal{L}^\vee)$ does not have invariant line bundles, which in this case are defined as line bundles $N \subseteq E$ such that $\Phi'(N) \subseteq N \otimes K \otimes L^\vee$. But a line bundle $N \subseteq E$ is Φ'-invariant if and only if it is $s\Phi'$-invariant, so we have $E \in \mathbf{A}$ in this case.

We move on to case *b*). Assume that the fixed divisor corresponds to a non-zero global section $s \in H^0(\mathbb{P}(E), \pi^*\mathcal{L}(1))$. This corresponds to a non-zero morphism $E \longrightarrow L$. The fibre-wise kernel has then dimension 1 generically and 2 at special points by upper semi-continuity [Ati89, Proposition 1.3.2]. Hence we can find a line bundle $N \subseteq \mathrm{Ker}(s) \subseteq E$, which can be described as the largest line subbundle of E contained in the kernel of s. If $v \in N$ is a non-zero vector, then $s(v) = 0$ and so $[v] \in \mathrm{div}(s) \subseteq \mathrm{div}(\psi(\Phi))$.

Thus the corresponding section $\sigma(M) \subseteq \mathbb{P}(E)$ is contained in $\mathrm{div}(\psi(\Phi))$ for all Φ and N is Φ-invariant for all Φ. Hence $E \in \mathbf{B}$ in this case.

In case *c*), the fixed divisor corresponds to a non-zero global section of $\pi^*\mathcal{L}(2)$. We have

$$H^0(\mathbb{P}(E), \pi^*\mathcal{L}(2)) \cong H^0(M, \mathcal{L} \otimes S^2\mathcal{E}^\vee) \cong H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{L} \otimes \Lambda^2\mathcal{E}^\vee).$$

So we can think of the fixed global section s as a traceless endomorphism of E with coefficients in $L \otimes \Lambda^2 E^\vee$. With this point of view, s-invariance of a line bundle $N \subseteq E$ translates into $s\Phi'$-invariance of $N \subseteq E$ as before, where $s\Phi'$ is a Higgs field. Let us see that the fixed section s has some non-trivial kernel, hence defining a line bundle invariant under all Higgs fields as in the previous case. To show that there is some non-trivial kernel, it suffices to check that $\det(s) = 0$. Since s is traceless, it suffices in turn to check that $\mathrm{tr}(s^2) = 0$. Suppose on the contrary that $\mathrm{tr}(s^2) \neq 0$. Fix some non-zero $s_1 \in H^0(M, \mathcal{L}^\vee \otimes \mathcal{K} \otimes \Lambda^2\mathcal{E})$ and consider the linear map

$$\begin{aligned} \theta : \ H^0(M, \mathcal{L}^\vee \otimes \mathcal{K} \otimes \Lambda^2\mathcal{E}) &\longrightarrow H^0(M, \mathcal{K}^2) \\ \Phi' &\longmapsto \mathrm{tr}(s^2)s_1\Phi' \end{aligned}$$

Since $\mathrm{tr}(s^2)s_1$ can only vanish at finitely many points, the image of a non-zero Φ' can only vanish at finitely many points, hence θ is injective. From Hirzebruch–Riemann–Roch and Lemma 1 we know that

$$h^0(M, \mathcal{L}^\vee \otimes \mathcal{K} \otimes \Lambda^2\mathcal{E}) \geqslant 3g - 3 = h^0(M, \mathcal{K}^2),$$

so θ is an isomorphism. Since $\mathcal{L}^\vee \otimes \mathcal{K} \otimes \Lambda^2\mathcal{E}$ has global sections, its degree is non-negative. If it was zero, then this would be the trivial line bundle and we would have $h^0(M, \mathcal{L}^\vee \otimes \mathcal{K} \otimes \Lambda^2\mathcal{E}) = 1 < 3g - 3$, a contradiction. Hence $\deg(\mathcal{L}^\vee \otimes \mathcal{K} \otimes \Lambda^2\mathcal{E}) > 0$ and the non-zero global section s_1 has at least one zero. If θ was indeed an isomorphism, then each zero of s_1 would give a base point of the complete linear system corresponding to $\mathcal{K}^2$. But $\deg(\mathcal{K}^2) = 4g - 4 \geqslant 2g$, so this linear system has no base points [Har77, Corollary IV.3.2]. This contradiction shows that s has non-trivial kernel, which contains a line bundle $N \subseteq E$ invariant by all $\Phi \in H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K})$. Hence $E \in \mathbf{B}$ as well in this case. □

Exercise B. Assume $g \geqslant 2$. Let $K^{\frac{1}{2}}$ be a line bundle whose square is K and let $K^{-\frac{1}{2}}$ be its inverse. Does $K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}$ belong to $\mathbf{A}$ or to $\mathbf{B}$? [*Hints below*[2]]

Lemma 3. *Let $E \in \mathbf{Vec}_2(M)$ and $L \subseteq E$ a line bundle. We have the following short exact sequences:*

[2]Consider the family of traceless endomorphisms given by

$$\Phi_\alpha := \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix},$$

a) $0 \longrightarrow \mathcal{L}^2 \otimes (\Lambda^2\mathcal{E})^\vee \otimes \mathcal{K} \longrightarrow \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K} \longrightarrow \mathcal{E} \otimes \mathcal{L}^{-1} \otimes \mathcal{K} \longrightarrow 0.$
b) $0 \longrightarrow \mathcal{E}^\vee \otimes \mathcal{L} \otimes \mathcal{K} \longrightarrow \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K} \longrightarrow \mathcal{L}^{-2} \otimes (\Lambda^2\mathcal{E}) \otimes \mathcal{K} \longrightarrow 0.$

Moreover, the sections of the image of $\mathcal{K}\otimes\mathcal{L}\otimes\Lambda^2\mathcal{E}^\vee$ in a) *leave only L invariant; and the sections of the image of $\mathcal{E}^\vee \otimes \mathcal{K} \otimes \mathcal{L}$ in* b) *are those which leave L invariant.*

Proof. All the short exact sequences are the result of tensoring another short exact sequence with $\mathcal{K}$, so let us find the necessary short exact sequences without $\mathcal{K}$. Under the isomorphism $E \cong E^\vee \otimes \Lambda^2 E$ [Har77, Exercise II.5.16], the line bundle L is sent to linear forms with coefficients in $\Lambda^2 E$ vanishing along L, hence we have a short exact sequence

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{E} \longrightarrow \mathcal{L}^\vee \otimes \Lambda^2\mathcal{E} \longrightarrow 0.$$

Dualizing this short exact sequence we obtain an inclusion $L \otimes \Lambda^2 E^\vee \subseteq E^\vee$. Tensoring with L and composing with the inclusion $E^\vee \otimes L \subseteq E^\vee \otimes E$ we obtain an inclusion $L^2 \otimes \Lambda^2 E^\vee \subseteq \mathrm{End}(E)$. Choosing a basis on each fibre and chasing all the identifications we have made so far, we see that the image of $L^2 \otimes \Lambda^2 E^\vee$ lies actually in $\mathrm{End}_0(E)$. Indeed, let V be a two dimensional $\mathbb{C}$-vector space and let e_1 and e_2 be a basis. Let L be the line spanned by a non-zero vector l, which we may assume to be e_1. The first identification we have is $V \cong \mathrm{Hom}(V, \Lambda^2 V)$, sending v to the homomorphism $v' \mapsto v' \wedge v$. This corresponds to $\alpha_v \otimes (e_1 \wedge e_2) \in V^\vee \otimes \Lambda^2 V$, where $\alpha_v \in V^\vee$ is the linear form sending $e_1 \mapsto v_2$ and $e_2 \mapsto -v_1$. Denoting by $\overline{\alpha_v}$ its image in $L^\vee$, we can describe the morphism corresponding to the right hand side of the previous short exact sequence as

$$\begin{aligned} V &\longrightarrow L^\vee \otimes \Lambda^2 V \\ v &\longmapsto \overline{\alpha_v} \otimes (e_1 \wedge e_2) \end{aligned}$$

Let now $\beta \in (L^\vee \otimes \Lambda^2 V)^\vee$ and denote by λ_v^β the complex number such that

$$\overline{\alpha_v} \otimes (e_1 \wedge e_2) \overset{\beta}{\longmapsto} \lambda_v^\beta.$$

A point $\mu l \otimes \beta \in L \otimes (L^\vee \otimes \Lambda^2 V)^\vee$ corresponds then to the endomorphism

$$\begin{aligned} V &\longrightarrow V \\ v &\longmapsto \mu\lambda_v^\beta l \end{aligned}$$

parametrized by quadratic differentials $\alpha \in H^0(M, K^2)$. Use without proof the fact that an invariant line bundle exists if and only if the characteristic polynomial

$$\lambda^2 - \alpha$$

has a root in $H^0(M, K)$, i.e. if and only if the quadratic differential α can be written as a square $\alpha = \beta^2$ for some differential form $\beta \in H^0(M, K)$. If α was a square, its zeros would all have multiplicity at least two. Conclude that $K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}} \in \mathbf{A}$ using Bertini's theorem.

A basis for L is e_1, a basis for $L^\vee \otimes \Lambda^2 V$ is $\overline{\alpha_{e_2}} \otimes (e_1 \wedge e_2)$ and a basis for $L \otimes (L^\vee \otimes \Lambda^2)^\vee$ is $e_1 \otimes \beta_0$, where $\beta_0 \in (L^\vee \otimes \Lambda^2 V)^\vee$ is such that $\lambda_{e_2}^{\beta_0} = 1$. Writing the image of the basis $e_1 \otimes \beta_0$ under the map $L \otimes (L^\vee \otimes \Lambda^2 V)^\vee \longrightarrow \mathrm{End}(V)$ as a matrix with respect to our bases we obtain

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

because $\overline{\alpha_{e_1}} = 0$ and therefore $\lambda_{e_1}^{\beta} = 0$ for any β. We have thus the desired injective homomorphism

$$L^2 \otimes \Lambda^2 V^\vee \hookrightarrow \mathrm{End}_0(V)$$

whose image are the traceless endomorphisms which preserve only L. This is the morphism from which we obtain the short exact sequence in *a*).

We regard this as a homomorphism into $\mathrm{End}(V)$ for a moment and use the basis $e_{11}, e_{12}, e_{21}, e_{22}$ of $\mathrm{End}(V)$, where e_{ij} denotes the endomorphism which, represented as a matrix in terms of our basis, has zeros everywhere except for a 1 in the ij-th position. Then our homomorphism is given by

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Dualizing it we obtain a surjection

$$\mathrm{End}_0(V^\vee) \twoheadrightarrow L^{-2} \otimes \Lambda^2 V$$

given with respect to the dual bases by

$$\begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}.$$

Its kernel are the endomorphisms of $V^\vee$ represented with respect to the dual basis by a matrix of the form

$$\begin{pmatrix} a & 0 \\ b & -a \end{pmatrix}.$$

Hence, under the isomorphism $\mathrm{End}(V) \cong \mathrm{End}(V^\vee)$ given in coordinates by sending a matrix to its transpose, we obtain a surjection

$$\mathrm{End}_0(V) \twoheadrightarrow L^{-2} \otimes \Lambda^2 V$$

whose kernel are endomorphisms represented with respect to our basis by matrices of the form

$$\begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}.$$

Therefore the kernel of this surjection consists precisely of the traceless endomorphisms of V that leave at least L invariant. The inclusion of this kernel can be naturally regarded as the composition of the inclusion $\mathrm{Hom}(V, L) \subseteq \mathrm{Hom}(V, V)$ and the projection $\mathrm{pr}_0 : \ \mathrm{Hom}(V, V) \longrightarrow \mathrm{End}_0(V)$, which writing every homomorphism as a matrix with respect to the bases above has the form

$$\begin{pmatrix} a & b \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} \frac{a}{2} & b \\ 0 & -\frac{a}{2} \end{pmatrix}.$$

This gives us the short exact sequence in $b)$ and the right hand side of the short exact sequence in $a)$. □

Lemma 4. *If $g \geqslant 2$ and E is a stable rank 2 vector bundle, then $E \in \mathbf{A}$.*

Proof. By Lemma 2 it suffices to show that it is not in $\mathbf{B}$. So let E be a stable rank 2 vector bundle on M and assume $L \subseteq E$ is a line bundle which is Φ-invariant for all $\Phi \in H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K})$. Consider the short exact sequence $b)$ from Lemma 3

$$0 \longrightarrow \mathcal{E}^\vee \otimes \mathcal{L} \otimes \mathcal{K} \longrightarrow \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K} \longrightarrow \mathcal{L}^{-2} \otimes (\Lambda^2\mathcal{E}) \otimes \mathcal{K} \longrightarrow 0.$$

Since all $\Phi \in H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K})$ leave L invariant, we get an induced isomorphism on global sections $H^0(M, \mathcal{E}^\vee \otimes \mathcal{L} \otimes \mathcal{K}) \cong H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K})$. The left hand side of the short exact sequence $a)$ in Lemma 3 factors by construction into a short exact sequence of the form

$$0 \longrightarrow \mathcal{L}^2 \otimes (\Lambda^2\mathcal{E})^\vee \otimes \mathcal{K} \longrightarrow \mathcal{E}^\vee \otimes \mathcal{L} \otimes \mathcal{K} \longrightarrow \mathcal{Q} \otimes \mathcal{K} \longrightarrow 0,$$

where $\deg(\mathcal{Q})$ can be seen to be 0 by computing first Chern classes. Riemann–Roch says then that $h^0(\mathcal{Q} \otimes \mathcal{K}) \in \{g - 1, g\}$. Since E is stable, we have $\deg(\mathcal{L}^2 \otimes \Lambda^2\mathcal{E}^\vee) < 0$, and since the complete linear system corresponding to $\mathcal{K}$ is base-point free [Har77, Lemma IV.5.1] we have $h^0(M, \mathcal{L}^2 \otimes (\Lambda^2\mathcal{E})^\vee \otimes \mathcal{K}) \leqslant g - 1$ by [Har77, Proposition IV.3.1]. The long exact sequence of the previous short exact sequence gives then

$$h^0(M, \mathcal{E}^\vee \otimes \mathcal{L} \otimes \mathcal{K}) \leqslant 2g - 1.$$

The earlier Hirzebruch–Riemann–Roch computation showed that

$$h^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K}) \geqslant 3g - 3.$$

If we want the two dimensions to be equal we must have $g = 2$ and $h^0(M, \mathcal{K} \otimes \mathcal{L} \otimes \mathcal{E}^\vee) = 3$. From the same long exact sequence as before we deduce, using that $h^0(M, \mathcal{L}^2 \otimes (\Lambda^2\mathcal{E})^\vee \otimes \mathcal{K}) < g = 2$, that $h^0(M, \mathcal{L}^2 \otimes (\Lambda^2\mathcal{E})^\vee \otimes \mathcal{K}) = 1$. In particular, $\deg(\mathcal{L}^2 \otimes (\Lambda^2\mathcal{E})^\vee \otimes \mathcal{K}) \geqslant 0$. We have $\deg(\mathcal{K}) = 2$ and by stability we had $\deg(\mathcal{L}^2 \otimes \Lambda^2\mathcal{E}^\vee) < 0$, so we must have

$$\deg(\mathcal{L}^2 \otimes (\Lambda^2\mathcal{E})^\vee \otimes \mathcal{K}) \in \{0, 1\}.$$

If it is 0, then the existence of global sections implieis that it is the trivial line bundle, hence the previous short exact sequence becomes

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{E}^\vee \otimes \mathcal{L} \otimes \mathcal{K} \longrightarrow \mathcal{K} \longrightarrow 0.$$

Split short exact sequences are preserved by dualizing and tensoring with line bundles, so if this sequence was split then $\mathcal{E}$ would be decomposable as a direct sum of line bundles. But this would give more endomorphisms of $\mathcal{E}$ than there should be, since stable vector bundles are simple. So the previous short exact sequence is a non-trivial extension and the coboundary map $H^0(M, \mathcal{K}) \longrightarrow H^1(M, \mathcal{O})$ is non-zero. The long exact sequence implies then that $h^0(M, \mathcal{E}^\vee \otimes \mathcal{K} \otimes \mathcal{L}) \leqslant 2$, contradicting our previous conclusion that this dimension was 2.

If it is 1, again by the existence of a non-zero global secction we deduce that it is the line bundle corresponding to some point $x \in M$. The short exact sequence becomes

$$0 \longrightarrow \mathcal{O}(x) \longrightarrow \mathcal{E}^\vee \otimes \mathcal{L} \otimes \mathcal{K} \longrightarrow \mathcal{K} \longrightarrow 0,$$

and the coboundary map is again non-zero, yielding the same contradiction as before. □

Lemma 5. *If $g \geqslant 2$, then*

$$\mathbf{S} = \mathbf{A}.$$

Proof. By definition $\mathbf{A} \subseteq \mathbf{S}$, so let us see the other inclusion. Let $E \in \mathbf{S}$. If E is stable, then $E \in \mathbf{A}$ by Lemma 4. So assume there exists $L \subseteq E$ such that $\mu(L) \geqslant \mu(E)$. Then $\deg(\mathcal{L}^2 \otimes (\Lambda^2 \mathcal{E})^\vee \otimes \mathcal{K}) \geqslant 2g-2$, so Riemann–Roch implies that $h^0(\mathcal{L}^2 \otimes (\Lambda^2 \mathcal{E})^\vee \otimes \mathcal{K}) \in \{g-1, g\}$. In particular, from the short exact sequence *a*) in Lemma 3 we deduce that there exists a non-zero Higgs field leaving only L invariant. So if $E \in \mathbf{B}$, then this L is a line bundle invariant by all $\Phi \in H^0(M, \mathcal{E}nd_0(\mathcal{E}) \otimes \mathcal{K})$. Since $\mu(L) \geqslant \mu(E)$, this contradicts the assumption that $E \in \mathbf{S}$. □

2. Idea of the existence theorem

We take a step back from the algebraic/holomorphic realm into the smooth setting, following the discussion in [Wen14, §2.1.1]. Let $E \longrightarrow M$ be a smooth complex rank 2 vector bundle over a compact Riemann surface M of genus $g \geqslant 2$. A holomorphic structure structure on E is equivalent to the choice of a $\bar\partial$-operator [Huy05, Theorem 2.6.26] satisfying the Leibniz rule and an integrability condition, which in our case is trivially verified because $d\bar z \wedge d\bar z = 0$. Let h be a hermitian structure on E, which can be glued from standard ones using a partition of unity [Huy05, Proposition 4.1.4]. Then there is a unique hermitian connection compatible with the holomorphic structure [Huy05, Proposition 4.2.14], which we called the Chern connection.

Let now $\mathcal{A}_E$ be the space of hermitian connections on the hermitian bundle (E, h), which is an affine space over the infinite-dimensional real vector subspace $A^1(\mathfrak{u}_h(E)) \subseteq A^1(\mathrm{End}(E))$ consisting of 1-forms with values in skew-hermitian endomorphisms [Huy05, Corollary 4.2.11]. Chern connections allow us to identify $\mathcal{A}_E$ with the space of holomorphic structures on E.

Definition. We consider the *gauge group*

$$\mathcal{G}_E = A^0(\mathrm{U}_h(E)) = \{g \in A^0(\mathrm{End}(E)) \mid gg^* = \mathrm{id}_E\},$$

where g^* is the h-adjoint endomorphism of g.

We will be mainly interested in the special unitary subgroup of gauge transformations, whose complexification is the group $A^0(\mathrm{Aut}_0(E))$ of automorphisms of E with constant determinant 1, because $\mathrm{SU}(2)^{\mathbb{C}} = \mathrm{SL}(2, \mathbb{C})$ [Kna02, p. 376].

Definition. We consider now the Banach manifold

$$\mathcal{B}_E = \mathcal{A}_E \times A^{1,0}(\mathrm{End}(E))$$

with the symplectic structure given by

$$\mathcal{K}((A_1, \Phi_1), (A_2, \Phi_2)) = -\int_M \mathrm{tr}(A_1 \wedge A_2) + 2i\,\mathrm{Im}(\mathrm{tr}(\Phi_1 \Phi_2^*)),$$

for $A_1, A_2 \in A^1(\mathfrak{u}_h(E))$, $\Phi_1, \Phi_2 \in A^{1,0}(\mathrm{End}(E))$ and Φ^* denoting the h-adjoint of Φ.

Fact 6. *The gauge group $\mathcal{G}_E$ acts nicely on both factors of $\mathcal{B}_E$ by conjugation, and this action admits a momentum map*

$$\mu(\nabla, \Phi) = -F - [\Phi, \Phi^*] - 2\pi i \mu(E)\, \mathrm{id}_E\, \mathcal{K}_M,$$

where F denotes the curvature of ∇.

References for this fact. This is briefly discussed in [Hit87, §4]. A more detailed account can be found in [**?**, Proposition III.3.2]. □

We also restrict our attention to the subspace of $\mathcal{B}'_E \subseteq \mathcal{B}_E$ consisting of pairs (∇, Φ) such that $\nabla^{0,1}\Phi = 0$, i.e. $\Phi \in H^0(M, \mathrm{End}(E) \otimes K)$ is holomorphic with respect to the holomorphic structure defined by ∇. We want to solve the self-duality equation

$$\mu(\nabla, \Phi) = 0$$

for $(\nabla, \Phi) \in \mathcal{B}'_E$. If we fix a holomorphic structure on E, we can think of the gauge group acting on the hermitian metric

$$(g \cdot h)(s_1, s_2) = h(g \cdot s_1, g \cdot s_2).$$

From this perspective, we need to find a Higgs field $\Phi \in H^0(M, \mathrm{End}(E) \otimes K)$ and a hermitian metric h on E such that the the Chern connection ∇ corresponding to the hermitian metric and the fixed holomorphic structure satisfies $\mu(\nabla, \Phi) = 0$.

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