SHORT NOTES ON GENERALIZED KUMMER THEORY

1. Preliminaries

Goal: given a field K and a non-zero natural number n, characterize all Galois extensions of K whose Galois group is abelian with exponent $d \mid n$.

Language: by *abelian* extension we mean a Galois extension L/K with abelian Galois group; by *cyclic* extension we mean a Galois extension L/K with cyclic Galois group.

Reference: [Bos18, §4.10].

2. Setting

(1) Let K be a field and fix a separable closure K_s.
(2) Let $n \in \mathbb{N}$ be a non-zero natural number.
(3) Let $G := \mathrm{Gal}(K_s/K)$ be the absolute Galois group.
(4) Let A be an abelian group endowed with the discrete topology and a continuous action of G on A via group automorphisms, which we will denote by $\sigma \cdot a =: \sigma(a)$.
(5) For each intermediate field $K \subseteq L \subseteq K_s$ we denote
$$A_L := \{a \in A \mid \sigma(a) = a \text{ for all } \sigma \in \mathrm{Gal}(K_s/L)\}.$$
(6) Let $\wp\colon A \to A$ be a G-equivariant surjective homomorphism whose kernel, denoted μ_n, is a cyclic subgroup of order n of A_K.

Continuity of the action of G on A ensures that for all $a \in A$ we have
$$G(A/a) := \{\sigma \in G \mid \sigma(a) = a\} \overset{\circ}{\hookrightarrow} G.$$
Hence $G(A/a)$ is also closed in G and corresponds to an intermediate field $K \subseteq K_s^{G(A/a)} \subseteq K_s$ [Bos18, 4.2/3], let's denote it $K(a)$.

Lemma 1. *The intermediate field $K(a)$ is a finite extension of K.*

Proof. Let $\{L_i\}_{i\in I}$ be the direct system of all subfields of K_s which are finite field extensions of K. For each $i \in I$, let us denote by
$$f_i\colon G \to \mathrm{Gal}(L_i/K)$$
the restriction morphism. The topology in G is the coarsest one making all the f_i continuous. Since each $\mathrm{Gal}(L_i/K)$ is a finite group, endowed with the discrete topology, it follows that the topology on G should be the smallest

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topology in which all fibres of the morphisms f_i are open. But the fibres of all the f_i already form a basis for some topology on G, so the topology on G can be explicitly described in terms of this basis.

Since $G(A/a)$ is open and $\mathrm{id}_{K_s} \in G(A/a)$, there is some $i \in I$ such that

$$f_i^{-1}(f_i(\mathrm{id}_{K_s})) = \mathrm{Gal}(K_s/L_i) \subseteq G(A/a).$$

From Galois correspondence we deduce now that

$$K \subseteq K(a) \subseteq L_i,$$

hence $K(a)$ is also finite over K. □

More generally, given a subset $\Delta \subseteq A$ we may consider the subgroup

$$G(A/\Delta) := \{\sigma \in G \mid \sigma(a) = a \text{ for all } a \in \Delta\} = \cap_{a\in\Delta} G(A/a),$$

which is then a closed subgroup but not necessarily an open subgroup. In any case we obtain an intermediate field $K \subseteq K_s^{G(A/\Delta)} \subseteq K_s$, which we will denote by $K(\Delta)$.

If L/K is Galois, then the action of G on A restricts to an action of G on A_L. Indeed, let $\tau \in G$, $\sigma \in \mathrm{Gal}(K_s/L)$ and $a \in A_L$. Since $\mathrm{Gal}(K_s/L) \trianglelefteq G$, there is some $\sigma' \in \mathrm{Gal}(K_s/L)$ such that

$$\sigma\tau(a) = \tau\sigma'(a) = \tau(a),$$

hence $\tau(a) \in A_L$. And by definition $\mathrm{Gal}(K_s/L)$ acts trivially on A_L, so we get an induced action of $G/\mathrm{Gal}(K_s/L)$ on A_L. Using again that L/K is Galois, we may identify this quotient group with $\mathrm{Gal}(L/K)$, obtaining an action of $\mathrm{Gal}(L/K)$ on A_L. We can then talk about the cohomology group $H^1(\mathrm{Gal}(L/K), A_L)$. A function $f\colon \mathrm{Gal}(L/K) \to A_L$ is called a *crossed homomorphism* if for all $\sigma, \tau \in \mathrm{Gal}(L/K)$ we have

$$f(\sigma\tau) = f(\sigma) + \sigma(f(\tau)).$$

A function $f\colon \mathrm{Gal}(L/K) \to A_L$ is called a *principal crossed homomorphism* if there exists some $a \in A_L$ such that for all $\sigma \in \mathrm{Gal}(L/K)$ we have

$$f(\sigma) = \sigma(a) - a.$$

Principal crossed homomorphisms form a subgroup of the group of crossed homomorphisms, and the quotient group is then our first cohomology group $H^1(\mathrm{Gal}(L/K), A_L)$.

We are ready now to state the main assumption on which we will rely:

Axiom 2. *For every cyclic extension L/K whose degree divides n we have*

$$H^1(\mathrm{Gal}(L/K), A_L) = 0.$$

3. The pairing associated to a subgroup

Let $C \subseteq A_K$ be a subgroup and consider $\wp^{-1}(C) \subseteq A$. By G-equivariance of $\wp$ and our assumption that $C \subseteq A_K$, any $\sigma \in G$ restricts to a homomorphism $\sigma\colon \wp^{-1}(C) \to \wp^{-1}(C)$. If $\sigma(a) = 0$ for $a \in \wp^{-1}(C)$, then

$$\wp(\sigma(a)) = \sigma(\wp(a)) = \wp(a) = 0,$$

because $\wp(a) \in C \subseteq A_K$. Therefore $a \in \mu_n \subseteq A_K$, and this implies in turn that $\sigma(a) = a = 0$. So the restriction of σ is an injective homomorphism $\wp^{-1}(C) \to \wp^{-1}(C)$. For $a \in \wp^{-1}(C)$ we have

$$\sigma(a) - a \in \mu_n$$

again by G-equivariance of $\wp$ and our assumption that $C \subseteq A_K$. So if $\sigma(a) \in \wp^{-1}(C)$, then

$$\wp(\sigma(a)) = \wp(a) \in C$$

and $a \in \wp^{-1}(C)$ as well, showing that the restriction of σ is also surjective. Hence σ restricts to a bijection $\wp^{-1}(C) \to \wp^{-1}(C)$. We obtain in this manner a group homomorphism

$$G \to \mathrm{Aut}(\wp^{-1}(C)).$$

The kernel of this group homomorphism is $G(A/\wp^{-1}(C))$ by definition. It is therefore a normal subgroup of G, which means in turn that $K(\wp^{-1}(C))/K$ is a Galois extension with Galois group $G_C \cong G/G(A/\wp^{-1}(C))$.

We define now a pairing

$$\begin{aligned} G_C \times C &\longrightarrow \mu_n \\ (\sigma, c) &\longmapsto \sigma(a) - a, \text{ for } a \in \wp^{-1}(c). \end{aligned}$$

To check that it is well-defined, pick some other $a' \in \wp^{-1}(c)$. This elemnt will differ from the previous a by some $b \in \mu_n$, hence

$$\sigma(a') - a' = \sigma(a) + \sigma(b) - a - b = \sigma(a) - a.$$

All good then. Assume from now on that $\wp(A_K) \subseteq C$. We factor then the previous pairing into the pairing that we are interested in:

$$\begin{aligned} \langle\cdot,\cdot\rangle\colon G_C \times C/\wp(A_K) &\longrightarrow \mu_n \\ (\sigma, \bar{c}) &\longmapsto \sigma(a) - a, \text{ for } a \in \wp^{-1}(c). \end{aligned}$$

Lemma 3. *The pairing $\langle\cdot,\cdot\rangle$ is non-degenerate.*

Proof. We have to show that the induced morphisms

$$\varphi_1\colon G_C \to \mathrm{Hom}(C/\wp(A_K), \mu_n) \quad \text{and} \quad \varphi_2\colon C/\wp(A_K) \to \mathrm{Hom}(G_C, \mu_n)$$

are injective.

Suppose that $\sigma \in G_C$ is such that $\langle \sigma, \bar{c} \rangle = 0$ for all $\bar{c} \in C/\wp(A_K)$. In particular, if $\sigma' \in G$ is a preimage of σ, then $\sigma(a) = a$ for all $a \in \wp^{-1}(C)$. This means precisely that $\sigma' \in G(A/\wp^{-1}(C))$, hence $\sigma = 1_{G_C}$.

Suppose now that $c \in C$ is such that $\langle \sigma, \bar{c} \rangle = 0$ for all $\sigma \in G_C$. We want to show that $c \in \wp(A_K)$, so let $a \in \wp^{-1}(c)$. For all $\sigma' \in G$ we have $\sigma'(a) = a$, which means that $a \in A_K$ and therefore $\bar{c} = 0$.

$\square$

References

[Bos18] Siegfried Bosch. *Algebra—from the viewpoint of Galois theory*. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser/Springer, Cham, german edition, 2018.

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