SHORT NOTES ON GENERALIZED KUMMER THEORY

1. **Reference:** [Bos18, §4.10].
2. **Data:**
   (1) Let K be a field and fix a separable closure K_s.
   (2) Let $G := \mathrm{Gal}(K_s/K)$ be the absolute Galois group.
   (3) Let A be an abelian group endowed with the discrete topology and a continuous action of G on A via group automorphisms, which we will denote by $\sigma \cdot a =: \sigma(a)$.
   (4) Continuity ensures that for all $a \in A$ we have

$$G(A/a) := \{\sigma \in G \mid \sigma(a) = a\} \hookrightarrow\!\!\!\!\circ\!\!\!\rightarrow G.$$

It follows from (4) that $G(A/a)$ is also closed in G, hence corresponds to an intermediate field $K \subseteq K_s^{G(A/a)} \subseteq K_s$ [Bos18, 4.2/3], let's denote it $K(a)$.

Lemma 1. *The intermediate field $K(a)$ is a finite extension of K.*

Proof. Let $\{L_i\}_{i \in I}$ be the direct system of all subfields of K_s which are finite field extensions of K. For each $i \in I$, let us denote by

$$f_i \colon G \to \mathrm{Gal}(L_i/K)$$

the restriction morphism. The topology in G is the coarsest one making all the f_i continuous. Since each $\mathrm{Gal}(L_i/K)$ is a finite group, endowed with the discrete topology, it follows that the topology on G should be the smallest topology in which all fibres of the morphisms f_i are open. But the fibres of all the f_i already form a basis for some topology on G, so the topology on G can be explicitly described in terms of this basis.

Since $G(A/a)$ is open and $\mathrm{id}_{K_s} \in G(A/a)$, there is some $i \in I$ such that

$$f_i^{-1}(f_i(\mathrm{id}_{K_s})) = \mathrm{Gal}(K_s/L_i) \subseteq G(A/a).$$

From Galois correspondence we deduce now that

$$K \subseteq K(a) \subseteq L_i,$$

hence $K(a)$ is also finite over K. □

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References

[Bos18] Siegfried Bosch. *Algebra—from the viewpoint of Galois theory*. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser/Springer, Cham, german edition, 2018.

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