

# SHORT NOTES ON GENERALIZED KUMMER THEORY

## 1. PRELIMINARIES

**Goal:** given a field  $K$  and a non-zero natural number  $n$ , characterize all Galois extensions of  $K$  whose Galois group is abelian with exponent  $d \mid n$ .

**Language:** by *abelian* extension we mean a Galois extension  $L/K$  with abelian Galois group; by *cyclic* extension we mean a Galois extension  $L/K$  with cyclic Galois group.

**Reference:** [Bos18, §4.10].

## 2. SETTING

- (1) Let  $K$  be a field and fix a separable closure  $K_s$ .
- (2) Let  $n \in \mathbb{N}$  be a non-zero natural number.
- (3) Let  $G := \text{Gal}(K_s/K)$  be the absolute Galois group.
- (4) Let  $A$  be an abelian group endowed with the discrete topology and a continuous action of  $G$  on  $A$  via group automorphisms, which we will denote by  $\sigma \cdot a =: \sigma(a)$ .
- (5) For each intermediate field  $K \subseteq L \subseteq K_s$  we denote

$$A_L := \{a \in A \mid \sigma(a) = a \text{ for all } \sigma \in \text{Gal}(K_s/L)\}.$$

- (6) Let  $\wp: A \rightarrow A$  be a  $G$ -equivariant surjective homomorphism whose kernel, denoted  $\mu_n$ , is a cyclic subgroup of order  $n$  of  $A_K$ .

Continuity of the action of  $G$  on  $A$  ensures that for all  $a \in A$  we have

$$G(A/a) := \{\sigma \in G \mid \sigma(a) = a\} \hookrightarrow G.$$

Hence  $G(A/a)$  is also closed in  $G$  and corresponds to an intermediate field  $K \subseteq K_s^{G(A/a)} \subseteq K_s$  [Bos18, 4.2/3], let's denote it  $K(a)$ .

**Lemma 1.** *The intermediate field  $K(a)$  is a finite extension of  $K$ .*

*Proof.* Let  $\{L_i\}_{i \in I}$  be the direct system of all subfields of  $K_s$  which are finite field extensions of  $K$ . For each  $i \in I$ , let us denote by

$$f_i: G \rightarrow \text{Gal}(L_i/K)$$

the restriction morphism. The topology in  $G$  is the coarsest one making all the  $f_i$  continuous. Since each  $\text{Gal}(L_i/K)$  is a finite group, endowed with the discrete topology, it follows that the topology on  $G$  should be the smallest

topology in which all fibres of the morphisms  $f_i$  are open. But the fibres of all the  $f_i$  already form a basis for some topology on  $G$ , so the topology on  $G$  can be explicitly described in terms of this basis.

Since  $G(A/a)$  is open and  $\text{id}_{K_s} \in G(A/a)$ , there is some  $i \in I$  such that

$$f_i^{-1}(f_i(\text{id}_{K_s})) = \text{Gal}(K_s/L_i) \subseteq G(A/a).$$

From Galois correspondence we deduce now that

$$K \subseteq K(a) \subseteq L_i,$$

hence  $K(a)$  is also finite over  $K$ . □

More generally, given a subset  $\Delta \subseteq A$  we may consider the subgroup

$$G(A/\Delta) := \{\sigma \in G \mid \sigma(a) = a \text{ for all } a \in \Delta\} = \bigcap_{a \in \Delta} G(A/a),$$

which is then a closed subgroup but not necessarily an open subgroup. In any case we obtain an intermediate field  $K \subseteq K_s^{G(A/\Delta)} \subseteq K_s$ , which we will denote by  $K(\Delta)$ .

If  $L/K$  is Galois, then the action of  $G$  on  $A$  restricts to an action of  $G$  on  $A_L$ . Indeed, let  $\tau \in G$ ,  $\sigma \in \text{Gal}(K_s/L)$  and  $a \in A_L$ . Since  $\text{Gal}(K_s/L) \trianglelefteq G$ , there is some  $\sigma' \in \text{Gal}(K_s/L)$  such that

$$\sigma\tau(a) = \tau\sigma'(a) = \tau(a),$$

hence  $\tau(a) \in A_L$ . And by definition  $\text{Gal}(K_s/L)$  acts trivially on  $A_L$ , so we get an induced action of  $G/\text{Gal}(K_s/L)$  on  $A_L$ . Using again that  $L/K$  is Galois, we may identify this quotient group with  $\text{Gal}(L/K)$ , obtaining an action of  $\text{Gal}(L/K)$  on  $A_L$ . We can then talk about the cohomology group  $H^1(\text{Gal}(L/K), A_L)$ . A function  $f: \text{Gal}(L/K) \rightarrow A_L$  is called a *crossed homomorphism* if for all  $\sigma, \tau \in \text{Gal}(L/K)$  we have

$$f(\sigma\tau) = f(\sigma) + \sigma(f(\tau)).$$

A function  $f: \text{Gal}(L/K) \rightarrow A_L$  is called a *principal crossed homomorphism* if there exists some  $a \in A_L$  such that for all  $\sigma \in \text{Gal}(L/K)$  we have

$$f(\sigma) = \sigma(a) - a.$$

Principal crossed homomorphisms form a subgroup of the group of crossed homomorphisms, and the quotient group is then our first cohomology group  $H^1(\text{Gal}(L/K), A_L)$ .

We are ready now to state the main assumption on which we will rely:

**Axiom 2.** *For every cyclic extension  $L/K$  whose degree divides  $n$  we have*

$$H^1(\text{Gal}(L/K), A_L) = 0.$$

## 3. THE PAIRING ASSOCIATED TO A SUBGROUP

Let  $C \subseteq A_K$  be a subgroup and consider  $\wp^{-1}(C) \subseteq A$ . By  $G$ -equivariance of  $\wp$  and our assumption that  $C \subseteq A_K$ , any  $\sigma \in G$  restricts to a homomorphism  $\sigma: \wp^{-1}(C) \rightarrow \wp^{-1}(C)$ . If  $\sigma(a) = 0$  for  $a \in \wp^{-1}(C)$ , then

$$\wp(\sigma(a)) = \sigma(\wp(a)) = \wp(a) = 0,$$

because  $\wp(a) \in C \subseteq A_K$ . Therefore  $a \in \mu_n \subseteq A_K$ , and this implies in turn that  $\sigma(a) = a = 0$ . So the restriction of  $\sigma$  is an injective homomorphism  $\wp^{-1}(C) \rightarrow \wp^{-1}(C)$ . For  $a \in \wp^{-1}(C)$  we have

$$\sigma(a) - a \in \mu_n$$

again by  $G$ -equivariance of  $\wp$  and our assumption that  $C \subseteq A_K$ . So if  $\sigma(a) \in \wp^{-1}(C)$ , then

$$\wp(\sigma(a)) = \wp(a) \in C$$

and  $a \in \wp^{-1}(C)$  as well, showing that the restriction of  $\sigma$  is also surjective. Hence  $\sigma$  restricts to a bijection  $\wp^{-1}(C) \rightarrow \wp^{-1}(C)$ . We obtain in this manner a group homomorphism

$$G \rightarrow \text{Aut}(\wp^{-1}(C)).$$

The kernel of this group homomorphism is  $G(A/\wp^{-1}(C))$  by definition. It is therefore a normal subgroup of  $G$ , which means in turn that  $K(\wp^{-1}(C))/K$  is a Galois extension with Galois group  $G_C \cong G/G(A/\wp^{-1}(C))$ . In particular, we also obtain an induced action of the Galois group  $G_C$  on  $\wp^{-1}(C)$ .

We define now a pairing

$$\begin{aligned} G_C \times C &\longrightarrow \mu_n \\ (\sigma, c) &\longmapsto \sigma(a) - a, \text{ for } a \in \wp^{-1}(c). \end{aligned}$$

To check that it is well-defined, pick some other  $a' \in \wp^{-1}(c)$ . This element will differ from the previous  $a$  by some  $b \in \mu_n$ , hence

$$\sigma(a') - a' = \sigma(a) + \sigma(b) - a - b = \sigma(a) - a.$$

All good then. Assume from now on that  $\wp(A_K) \subseteq C$ . We factor then the previous pairing into the pairing that we are interested in:

$$\begin{aligned} \langle \cdot, \cdot \rangle: G_C \times C/\wp(A_K) &\longrightarrow \mu_n \\ (\sigma, \bar{c}) &\longmapsto \sigma(a) - a, \text{ for } a \in \wp^{-1}(c). \end{aligned}$$

**Proposition 3.** *The pairing  $\langle \cdot, \cdot \rangle$  is non-degenerate.*

*Proof.* We have to show that the induced morphisms

$$\varphi_1: G_C \rightarrow \text{Hom}(C/\wp(A_K), \mu_n) \quad \text{and} \quad \varphi_2: C/\wp(A_K) \rightarrow \text{Hom}(G_C, \mu_n)$$

are injective.

Suppose that  $\sigma \in G_C$  is such that  $\langle \sigma, \bar{c} \rangle = 0$  for all  $\bar{c} \in C/\wp(A_K)$ . In particular, if  $\sigma' \in G$  is a preimage of  $\sigma$ , then  $\sigma(a) = a$  for all  $a \in \wp^{-1}(C)$ . This means precisely that  $\sigma' \in G(A/\wp^{-1}(C))$ , hence  $\sigma = 1_{G_C}$ .

Suppose now that  $c \in C$  is such that  $\langle \sigma, \bar{c} \rangle = 0$  for all  $\sigma \in G_C$ . We want to show that  $c \in \wp(A_K)$ , so let  $a \in \wp^{-1}(c)$ . For all  $\sigma' \in G$  we have  $\sigma'(a) = a$ , which means that  $a \in A_K$  and therefore  $\bar{c} = 0$ .  $\square$

**Proposition 4.**  $K(\wp^{-1}(C))/K$  is finite if and only if  $(C : \wp(A_K))$  is finite.

*Proof.* Suppose first that  $[K(\wp^{-1}(C)) : K]$  is finite. Then its Galois group  $G_C$  would be finite as well, so  $\text{Hom}(G_C, \mu_n)$  is finite. But  $\varphi_2$  is injective by Proposition 3, so  $C/\wp(A_K)$  must be finite as well.

Conversely, suppose that  $C/\wp(A_K)$  is finite. Again, this implies that  $\text{Hom}(C/\wp(A_K), \mu_n)$  is finite, so injectivity of  $\varphi_1$  shows that  $[K(\wp^{-1}(C)) : K]$  is finite as well.  $\square$

**Lemma 5.** Let  $n \in \mathbb{N}$  be a non-zero natural number and let  $H$  be a finite abelian group with exponent  $d \mid n$ . Then there exists an isomorphism  $H \cong \text{Hom}(H, \mathbb{Z}/n\mathbb{Z})$ .

*Proof.* By the structure theorem for finitely generated abelian groups it suffices to show the result for  $H = \mathbb{Z}/d\mathbb{Z}$ . We first reduce the result to the case  $d = n$ . There is a unique cyclic subgroup  $H_d \subseteq \mathbb{Z}/n\mathbb{Z}$  of order  $d$ . Every homomorphism  $\mathbb{Z}/d\mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z}$  factors then through  $H_d$ , so the canonical map

$$\text{Hom}(\mathbb{Z}/d\mathbb{Z}, H_d) \hookrightarrow \text{Hom}(\mathbb{Z}/d\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$$

is an isomorphism. Since  $H_d \cong \mathbb{Z}/d\mathbb{Z}$ , it suffices to show that there is an isomorphism

$$\mathbb{Z}/d\mathbb{Z} \rightarrow \text{Hom}(\mathbb{Z}/d\mathbb{Z}, \mathbb{Z}/d\mathbb{Z}),$$

i.e. it suffices to show the case  $d = n$ .

In this case we consider the surjective homomorphism

$$\mathbb{Z} \rightarrow \text{Hom}(\mathbb{Z}/d\mathbb{Z}, \mathbb{Z}/d\mathbb{Z})$$

$$1 \mapsto \text{id}.$$

Its kernel is  $d\mathbb{Z}$ , so passing to the quotient yields the desired isomorphism.  $\square$

**Proposition 6.** If  $K(\wp^{-1}(C))/K$  or  $(C : \wp(A_K))$  are finite, then  $\varphi_1$  and  $\varphi_2$  from Proposition 3 are isomorphisms and

$$[K(\wp^{-1}(C)) : K] = (C : \wp(A_K)).$$

*Proof.* By Proposition 4, if either of the two is finite, so is the other one. By Lemma 5 we have isomorphisms

$$C/\wp(A_K) \cong \text{Hom}(C/\wp(A_K), \mu_n) \quad \text{and} \quad G_C \cong \text{Hom}(G_C, \mu_n).$$

We have

$$\begin{aligned}
 [K(\wp^{-1}(C)) : K] &= |G_C| \\
 &\leq |\operatorname{Hom}(C/\wp(A_K), \mu_n)| \\
 &= |C/\wp(A_K)| \\
 &\leq |\operatorname{Hom}(G_C, \mu_n)| \\
 &= |G_C| \\
 &= [K(\wp^{-1}(C)) : K].
 \end{aligned}$$

Therefore  $[K(\wp^{-1}(C)) : K] = (C : \wp(A_K))$  and  $\varphi_1$  and  $\varphi_2$  are isomorphisms.  $\square$

**Proposition 7.** *Even if  $[K(\wp^{-1}(C)) : K]$  and  $(C : \wp(A_K))$  are not finite,  $\varphi_1$  is still an isomorphism and  $\varphi_2$  induces an isomorphism*

$$C/\wp(A_K) \cong \operatorname{Hom}_{\text{cont}}(G_C, \mu_n)$$

*onto the subgroup of continuous homomorphisms.*

*Proof.* Consider the directed system  $\{C_i\}_{i \in I}$  of all subgroups  $C_i$  of  $C$  containing  $\wp(A_K)$  and such that  $(C_i : \wp(A_K))$  is finite. Since any finite intermediate extension  $K \subseteq L \subseteq K(\wp^{-1}(C))$  is contained in some  $K(\wp^{-1}(C_i))$ , we can write

$$G_C \cong \varprojlim_{i \in I} G_{C_i}.$$

Consider now for each  $i \in I$  the commutative diagram

$$\begin{array}{ccc}
 G_C & \xleftarrow{\varphi_1} & \operatorname{Hom}(C/\wp(A_K), \mu_n) \\
 \downarrow & & \downarrow \\
 G_{C_i} & \xrightarrow[\varphi_{1,i}]{\cong} & \operatorname{Hom}(C_i/\wp(A_K), \mu_n)
 \end{array}$$

in which both horizontal arrows are the restrictions. Let  $f \in \operatorname{Hom}(C/\wp(A_K), \mu_n)$  and consider for each  $i \in I$  its restriction  $f|_{C_i/\wp(A_K)}$ , which comes from a unique  $\sigma_i \in G_{C_i}$ . This yields a family of automorphisms  $(\sigma_i)_{i \in I}$ . We claim that this is a compatible family, giving therefore an element in  $G_C$  which maps to  $f$  and proving surjectivity of  $\varphi_1$ . To show this, let  $i \leq j$  and consider the diagram

$$\begin{array}{ccc}
 G_{C_j} & \xrightarrow{\varphi_{1,j}} & \operatorname{Hom}(C_j/\wp(A_K), \mu_n) \\
 \downarrow & & \downarrow \\
 G_{C_i} & \xrightarrow[\varphi_{1,i}]{} & \operatorname{Hom}(C_i/\wp(A_K), \mu_n)
 \end{array}$$

in which both vertical arrows are the restrictions. We want to check that it commutes. The isomorphism  $G/G(A/\wp^{-1}(C_k)) \rightarrow G_{C_k}$  is given by restriction of automorphisms for all  $k \in I$ . So every  $\tau \in G_{C_j}$  can be written as  $\sigma|_{K(\wp^{-1}(C_j))}$  for some  $\sigma \in G$ . For  $c \in C_i \subseteq C_j$  and  $a \in \wp^{-1}(c)$  we have then

$$\varphi_{1,i}(\tau|_{K(\wp^{-1}(C_i))})(\bar{c}) = \sigma(a) - a = \varphi_{1,j}(\tau)|_{C_i/\wp(A_K)}(\bar{c}),$$

showing commutativity of the diagram and thus finishing the proof of bijectivity of  $\varphi_1$ .

Before discussing the assertion about  $\varphi_2$ , we claim that every continuous homomorphism  $g: G_C \rightarrow \mu_n$  comes from some homomorphism  $g_i: G_{C_i} \rightarrow \mu_n$  via the restriction  $f_i: G_C \rightarrow G_{C_i}$ . Indeed, let  $\xi \in \mu_n$  be a generator. Given such  $g$  and given some  $k\xi \in \mu_n$  in the image of  $g$ , say  $k\xi = g(\sigma)$ , the preimage  $g^{-1}(k\xi)$  is open in  $G_C$ . Since the fibers of the restrictions form a basis for the topology on  $G_C$ , there exists some  $i_k \in I$  such that  $f_{i_k}^{-1}f_{i_k}(\sigma) \subseteq g^{-1}(g(\sigma))$ . Let now  $i = \max_k \{i_k\}$  and define

$$\begin{aligned} g_i: G_{C_i} &\longrightarrow \mu_n \\ \sigma|_{K(\wp^{-1}(C_i))} &\longmapsto g(\sigma). \end{aligned}$$

If  $\sigma|_{K(\wp^{-1}(C_i))} = \tau|_{K(\wp^{-1}(C_i))}$  and  $g(\sigma) = k\xi$ , then we have  $i_k \leq i$  and therefore

$$\tau \in f_i^{-1}f_i(\sigma) \subseteq f_{i_k}^{-1}f_{i_k}(\sigma) \subseteq g^{-1}g(\sigma),$$

showing that  $g_i$  is well-defined. And by construction  $g = g_i \circ f_i$ , proving the claim.

Moving on to the assertion about  $\varphi_2$ , suppose  $g \in \text{Hom}(G_C, \mu_n)$  is in the image of  $\varphi_2$ , say  $g = \varphi_2(\bar{c})$ . Then it is continuous, because the formula we used to define it involves only the continuous action of  $G$  on  $A$  and the continuous group operations in  $A$ . But we can also check this directly: by homogeneity it suffices to show that

$$g^{-1}(0) = \{\sigma \in G_C \mid \sigma(a) - a = 0 \text{ for } a \in \wp^{-1}(c)\}$$

is closed in  $G_C$ , which is true because its preimage under the quotient map is the closed subgroup  $G(A/\wp^{-1}(c))$  of  $G$ . Conversely, suppose  $g \in \text{Hom}(G_C, \mu_n)$  is a continuous homomorphism. Then by our previous claim we may find some  $i \in I$  such that  $g = g_i \circ f_i$  for some  $g_i: G_{C_i} \rightarrow \mu_n$ , where  $f_i: G_C \rightarrow G_{C_i}$  denotes the restriction. This means that in the commutative square

$$\begin{array}{ccc} C_i/\wp(A_K) & \xrightarrow{\varphi_{2,i}} & \text{Hom}(G_{C_i}, \mu_n) \\ \downarrow & & \downarrow \\ C/\wp(A_K) & \xrightarrow{\varphi_2} & \text{Hom}(G_C, \mu_n) \end{array}$$

our  $g$  lies in the image of the right vertical arrow. By Proposition 6 the top horizontal arrow is an isomorphism. Hence  $g$  also lies in the image of  $\varphi_2$ .  $\square$

## REFERENCES

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