SHORT NOTES ON GENERALIZED KUMMER THEORY

1. **Reference:** [Bos18, §4.10].
2. **Setting:**
   (1) Let K be a field and fix a separable closure K_s.
   (2) Let $n \in \mathbb{N}$ be a non-zero natural number.
   (3) Let $G := \mathrm{Gal}(K_s/K)$ be the absolute Galois group.
   (4) Let A be an abelian group endowed with the discrete topology and a continuous action of G on A via group automorphisms, which we will denote by $\sigma \cdot a =: \sigma(a)$.
   (5) For each intermediate field $K \subseteq L \subseteq K_s$ we denote
   $$A_L := \{a \in A \mid \sigma(a) = a \text{ for all } \sigma \in \mathrm{Gal}(K_s/L)\}.$$
   (6) Let $\wp\colon A \to A$ be a G-equivariant surjective homomorphism whose kernel, denoted μ_n, is a cyclic subgroup of order n of A_K.

Continuity of the action of G on A ensures that for all $a \in A$ we have
$$G(A/a) := \{\sigma \in G \mid \sigma(a) = a\} \overset{\circ}{\hookrightarrow} G.$$
Hence $G(A/a)$ is also closed in G and corresponds to an intermediate field $K \subseteq K_s^{G(A/a)} \subseteq K_s$ [Bos18, 4.2/3], let's denote it $K(a)$.

Lemma 1. *The intermediate field $K(a)$ is a finite extension of K.*

Proof. Let $\{L_i\}_{i\in I}$ be the direct system of all subfields of K_s which are finite field extensions of K. For each $i \in I$, let us denote by
$$f_i\colon G \to \mathrm{Gal}(L_i/K)$$
the restriction morphism. The topology in G is the coarsest one making all the f_i continuous. Since each $\mathrm{Gal}(L_i/K)$ is a finite group, endowed with the discrete topology, it follows that the topology on G should be the smallest topology in which all fibres of the morphisms f_i are open. But the fibres of all the f_i already form a basis for some topology on G, so the topology on G can be explicitly described in terms of this basis.

Since $G(A/a)$ is open and $\mathrm{id}_{K_s} \in G(A/a)$, there is some $i \in I$ such that
$$f_i^{-1}(f_i(\mathrm{id}_{K_s})) = \mathrm{Gal}(K_s/L_i) \subseteq G(A/a).$$
From Galois correspondence we deduce now that
$$K \subseteq K(a) \subseteq L_i,$$

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hence $K(a)$ is also finite over K. $\square$

More generally, given a subset $\Delta \subseteq A$ we may consider the subgroup

$$G(A/\Delta) := \{\sigma \in G \mid \sigma(a) = a \text{ for all } a \in \Delta\} = \Cap_{a \in \Delta} G(A/a),$$

which is then a closed subgroup but not necessarily an open subgroup. In any case we obtain an intermediate field $K \subseteq K_s^{G(A/\Delta)} \subseteq K_s$, which we will denote by $K(\Delta)$.

References

[Bos18] Siegfried Bosch. *Algebra—from the viewpoint of Galois theory*. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser/Springer, Cham, german edition, 2018.

Pedro Núñez
Albert-Ludwigs-Universität Freiburg, Mathematisches Institut
Ernst-Zermelo-Strasse 1, 79104 Freiburg im Breisgau (Germany)
Email address: pedro.nunez@math.uni-freiburg.de
Homepage: *https://home.mathematik.uni-freiburg.de/nunez*

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