SHORT NOTES ON GENERALIZED KUMMER THEORY

1. **Reference:** [Bos18, §4.10].

2. **Goal:** given a field K and a non-zero natural number n, characterize all Galois extensions of K whose Galois group is abelian with exponent $d \mid n$.

3. **Language:** by *abelian* extension we mean a Galois extension L/K with abelian Galois group; by *cyclic* extension we mean a Galois extension L/K with cyclic Galois group.

4. **Setting:**
    (1) Let K be a field and fix a separable closure K_s.
    (2) Let $n \in \mathbb{N}$ be a non-zero natural number.
    (3) Let $G := \mathrm{Gal}(K_s/K)$ be the absolute Galois group.
    (4) Let A be an abelian group endowed with the discrete topology and a continuous action of G on A via group automorphisms, which we will denote by $\sigma \cdot a =: \sigma(a)$.
    (5) For each intermediate field $K \subseteq L \subseteq K_s$ we denote
    $$A_L := \{a \in A \mid \sigma(a) = a \text{ for all } \sigma \in \mathrm{Gal}(K_s/L)\}.$$
    (6) Let $\wp \colon A \to A$ be a G-equivariant surjective homomorphism whose kernel, denoted μ_n, is a cyclic subgroup of order n of A_K.

    Continuity of the action of G on A ensures that for all $a \in A$ we have
    $$G(A/a) := \{\sigma \in G \mid \sigma(a) = a\} \hookrightarrow G.$$

Hence $G(A/a)$ is also closed in G and corresponds to an intermediate field $K \subseteq K_s^{G(A/a)} \subseteq K_s$ [Bos18, 4.2/3], let's denote it $K(a)$.

Lemma 1. *The intermediate field $K(a)$ is a finite extension of K.*

Proof. Let $\{L_i\}_{i \in I}$ be the direct system of all subfields of K_s which are finite field extensions of K. For each $i \in I$, let us denote by
$$f_i \colon G \to \mathrm{Gal}(L_i/K)$$
the restriction morphism. The topology in G is the coarsest one making all the f_i continuous. Since each $\mathrm{Gal}(L_i/K)$ is a finite group, endowed with the discrete topology, it follows that the topology on G should be the smallest topology in which all fibres of the morphisms f_i are open. But the fibres of

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all the f_i already form a basis for some topology on G, so the topology on G can be explicitly described in terms of this basis.

Since $G(A/a)$ is open and $\mathrm{id}_{K_s} \in G(A/a)$, there is some $i \in I$ such that

$$f_i^{-1}(f_i(\mathrm{id}_{K_s})) = \mathrm{Gal}(K_s/L_i) \subseteq G(A/a).$$

From Galois correspondence we deduce now that

$$K \subseteq K(a) \subseteq L_i,$$

hence $K(a)$ is also finite over K. $\square$

More generally, given a subset $\Delta \subseteq A$ we may consider the subgroup

$$G(A/\Delta) := \{\sigma \in G \mid \sigma(a) = a \text{ for all } a \in \Delta\} = \cap_{a \in \Delta} G(A/a),$$

which is then a closed subgroup but not necessarily an open subgroup. In any case we obtain an intermediate field $K \subseteq K_s^{G(A/\Delta)} \subseteq K_s$, which we will denote by $K(\Delta)$.

If L/K is Galois, then the action of G on A restricts to an action of G on A_L. Indeed, let $\tau \in G$, $\sigma \in \mathrm{Gal}(K_s/L)$ and $a \in A_L$. Since $\mathrm{Gal}(K_s/L) \trianglelefteq G$, there is some $\sigma' \in \mathrm{Gal}(K_s/L)$ such that

$$\sigma\tau(a) = \tau\sigma'(a) = \tau(a),$$

hence $\tau(a) \in A_L$. And by definition $\mathrm{Gal}(K_s/L)$ acts trivially on A_L, so we get an induced action of $G/\mathrm{Gal}(K_s/L)$ on A_L. Using again that L/K is Galois, we may identify this quotient group with $\mathrm{Gal}(L/K)$, obtaining an action of $\mathrm{Gal}(L/K)$ on A_L. We can then talk about the cohomology group $H^1(\mathrm{Gal}(L/K), A_L)$. A function $f \colon \mathrm{Gal}(L/K) \to A_L$ is called a *crossed homomorphism* if for all $\sigma, \tau \in \mathrm{Gal}(L/K)$ we have

$$f(\sigma\tau) = f(\sigma) + \sigma(f(\tau)).$$

A function $f \colon \mathrm{Gal}(L/K) \to A_L$ is called a *principal crossed homomorphism* if there exists some $a \in A_L$ such that for all $\sigma \in \mathrm{Gal}(L/K)$ we have

$$f(\sigma) = \sigma(a) - a.$$

Principal crossed homomorphisms form a subgroup of the group of crossed homomorphisms, and the quotient group is then our first cohomology group $H^1(\mathrm{Gal}(L/K), A_L)$.

We are ready now to state the main assumption on which we will rely:

Axiom 2. *For every cyclic extension L/K whose degree divides n we have*

$$H^1(\mathrm{Gal}(L/K), A_L) = 0.$$

References

[Bos18] Siegfried Bosch. *Algebra—from the viewpoint of Galois theory*. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser/Springer, Cham, german edition, 2018.

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