

Various lecture notes

Pedro Núñez

23rd October 2019

Contents

1	[CM] Talk 1 (Johan Commelin): Condensed Sets - 21.10.19	1
1.1	Introduction	1
1.2	Recollections on sheaves on sites	2
1.3	Compactly generated topological spaces	4
2	[LT] Lecture 1 - 22.10.19	5
2.1	Introduction and overview of the course	5
3	[WS] Kodaria 1 (Jin) - 23.10.19	7
3.1	Chow's theorem	7
3.2	Sheaves	8
3.3	A bit of Hodge theory	9

1 [CM] Talk 1 (Johan Commelin): Condensed Sets - 21.10.19

1.1 Introduction

One of the main motivations for condensed mathematics is that topological algebraic objects have usually poor categorical and functorial properties. For instance, topological abelian groups do not form an abelian category:

Example 1.1. $\mathbb{R}_{disc} \rightarrow \mathbb{R}$ is epi and mono, but not iso.

Another motivation is coherent duality:

Theorem 1.2. *Let $f: X \rightarrow Y$ be a proper or quasi-projective morphism of Noetherian schemes of finite Krull dimension. Then there exists a right adjoint $f^!$ to the derived direct image functor $f_! = Rf_*: \mathcal{D}^b(\mathcal{QCoh}(X)) \rightarrow \mathcal{D}^b(\mathcal{QCoh}(Y))$.*

At some point analytic rings will come up. We will then look at the category of solid modules, in which the 6-functor formalism works nicer than in the classical setting (e.g. when $f_!$ is not defined in the classical setting, $f_!$ takes non-discrete values in the condensed settings, which are "not there" in the classical setting).

Definition 1.3. Proétale site of a point, denoted $*_{proét}$, is the category of profinite sets with finite jointly surjective families of continuous maps as covers. A *condensed set* (resp. group, ring, ...) is a sheaf of sets (resp. groups, rings, ...) on $*_{proét}$. We denote by $\text{Cond}(\mathcal{C})$ the category of condensed objects of a category \mathcal{C} .

Definition 1.4. A *condensed set* (resp. group, ring, ...) is a contravariant functor X from $*_{proét}$ to the category of sets (resp. groups, rings, ...) such that

i) $X(\emptyset) = *$.

ii) For all profinite sets S_1 and S_2 the natural map

$$X(S_1 \sqcup S_2) \rightarrow X(S_1) \times X(S_2)$$

is an isomorphism.

iii) For any surjection of profinite sets $f: S' \twoheadrightarrow S$ we get an induced¹ isomorphism

$$X(S) \rightarrow \{x \in X(S') \mid \pi_1^*(x) = \pi_2^*(x) \in X(S' \times_S S')\}$$

We will call $X(*)$ the *underlying object* in \mathcal{C} of a condensed object.

Remark 1.5. We will use T for topological spaces vs. X, Y for condensed sets, as opposed to Scholze's mixing of those notations.

1.2 Recollections on sheaves on sites

Let F be a presheaf on a site, which is just a contravariant functor to whatever category in which our sheaves are gonna take values. If $U = \cup_i U_i$ is an open cover, the topological sheaf axiom could be phrased as: $F(U)$ is an equalizer of the diagram

$$\prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j).$$

Note that $U_i \cap U_j$ is just the fiber product of the two inclusions.

¹Since the pullback diagram is commutative, the image of $X(f)$ is indeed induces a morphism as claimed.

Definition 1.6 (Coverage). See definition 2.1 in nCat.

Definition 1.7. F a presheaf on \mathcal{C} . A collection $(s_i) \in \prod_i F(U_i)$ for $\{f_i: U_i \rightarrow U\}$ a covering is called a *matching family* if for all $h: V \rightarrow U$ we have $g^*(s_i) = h^*(s_j)$ for g and h in the diagram

$$\begin{array}{ccc} V & \xrightarrow{h} & U_j \\ \downarrow g & & \downarrow f_j \\ U_i & \xrightarrow{f_i} & U \end{array}$$

Definition 1.8. F is a sheaf with respect to $\{U_i \rightarrow U\}$ if for all matching families (s_i) there exists a unique $s \in F(U)$ such that $f_i^*(s) = s_i$. We say that F is a *sheaf* if it is a sheaf for all covering families.

Remark 1.9. A sheaf of abelian groups is just a commutative group object in the category of sheaves of sets.

Theorem 1.10. *If \mathcal{C} is a site, then $Ab(\mathcal{C})$ is an abelian category.*

Definition 1.11. An additive category is a category in which the hom-sets are endowed with an abelian group structure in a way that makes composition bilinear and such that finite biproducts exist.

Recall Grothendieck's axioms: AB1) Every morphism has a kernel and a cokernel. AB2) For every $f: A \rightarrow B$, the natural map $\text{coim}(f) \rightarrow \text{im } f$ is an iso. AB3) All colimit exist. AB4) AB3) + arbitrary direct sums are exact. AB5) AB3) + arbitrary filtered colimits are exact. AB6) AB3) + J an index set, $\forall j \in J$ a filtered category (think of directed set) I_j , functors $M: I_j \rightarrow \mathcal{C}$, then

$$\varinjlim_{(i_j \in I_j)_j} \prod_j M_{i_j} \rightarrow \prod_j \varinjlim_{i_j \in I_j} M_{i_j}$$

Theorem 1.12. \mathcal{C} a site. Then $Ab(\mathcal{C})$ satisfies AB3), AB4), AB5) and AB6).

In fact, our case is even nicer:

Theorem 1.13. $\text{Cond}(Ab)$ in addition satisfies AB6) and $AB4^*)$.

1.3 Compactly generated topological spaces

Definition 1.14. A topological space T is called *compactly generated* if any function $f: T \rightarrow T'$ is continuous as soon as the composite $S \rightarrow T \rightarrow T'$ is continuous for all maps $S \rightarrow T$ where S is compact and Hausdorff. See also \mathbf{nCat} .

The inclusion functor $\mathcal{CG} \hookrightarrow \mathcal{Top}$ has a right adjoint $(-)^{cg}$. If T is any topological space, then the topology on T^{cg} is the finest topology on T such that $\sqcup_{S \rightarrow T} S \rightarrow T$ is continuous, where S ranges over all compact Hausdorff spaces.

Let T be a topological space. We view T as a presheaf on $*_{pro\acute{e}t}$ by setting $T(S) = \text{Hom}_{\mathcal{Top}}(S, T)$ for all profinite sets S . We denote this by \underline{T} . Claim: \underline{T} is a sheaf.

- i) The first condition $\underline{T}(\emptyset) = *$ is true, because there is exactly one morphism from the empty set to any topological space.
- ii) $\underline{T}(S_1 \sqcup S_2) = \underline{T}(S_1) \times \underline{T}(S_2)$ by universal property of disjoint union.
- iii) For any surjection $S' \twoheadrightarrow S$ we get an isomorphism

$$\underline{T}(S) \rightarrow \{x \in \underline{T}(S') \mid \pi_1^*(x) = \pi_2^*(x) \in \underline{T}(S' \times_S S')\}$$

Since $\mathcal{Top} \rightarrow \text{Cond}(\text{Set})$ preserves products, group objects are preserved, so it maps topological groups to condensed groups etc.

Proposition 1.15. *i) This functor is faithful and fully faithful when restricted to the full subcategory of compactly generated spaces.*

- ii) *It admits a left adjoint $X \mapsto X(*)_{top}$ where $X(*)_{top}$ gets the quotient topology of $\sqcup_{S \rightarrow X} S \rightarrow X(*)$ as above. The counit $I(*)_{top} \rightarrow T$ agrees with $T^{cg} \rightarrow T$.*

Coming back to our original example:

Example 1.16. $\mathbb{R}_{disc} \rightarrow \mathbb{R}$ can be seen in the condensed world as $\mathbb{R}_{disc} \rightarrow \mathbb{R}$, i.e. from locally constant functions to continuous functions. This is still a mono, but now it is not an epi. The cokernel Q can be described as $Q(S) = \{S \rightarrow \mathbb{R} \text{ continuous}\} / \{S \rightarrow \mathbb{R} \text{ locally constant}\}$. Note in particular that the underlying set of Q is just $*$, reflecting the fact that the cokernel was trivial in the classical setting.

2 [LT] Lecture 1 - 22.10.19

2.1 Introduction and overview of the course

An *algebraic variety* is the solution set of a family of polynomial equations in \mathbb{C}^n . For example, if $f(x, y, z, t) = xy - tz$, then

$$V(f) = \{(x, y, z, t) \in \mathbb{C}^4 \mid xy - tz = 0\}$$

is an algebraic variety in \mathbb{C}^4 . Another example would be the parabola $\{y - x^2 = 0\} \subseteq \mathbb{C}^2$.

Let us focus on $V(f)$ and set $t = 1$. Then $X = V(f) \cap \{t = 1\} = \{(x, y, z) \in \mathbb{C}^3 \mid xy = z\}$ can be seen as a family of complex curves parametrized by the variable z . For $z = 0$, the complex curve X_0 has an *ordinary double point* at the origin:

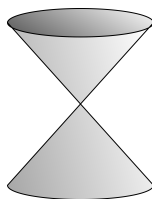


Figure 1: Topological picture of our ODP.

Singularities arise naturally while studying the topology of algebraic varieties, and ODP's are a particularly nice kind of singularities.

For $z \neq 0$ we get an equation which looks like $xy = 1$. In this case we have the following picture:

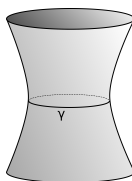


Figure 2: Topological picture of X_z .

As $z \rightarrow 0$, the central loop γ contracts to the ordinary double point. Note in particular that X_0 has trivial fundamental group (hence trivial 1-homology), whereas X_z does not.

We have a projection $\pi: X \rightarrow \mathbb{C}$, and Ehresmann's lemma tells us that for all disks $D \subseteq \mathbb{C}$ not containing 0 we have $\pi^{-1}(D) \cong D \times X_{z_0}$ for any $z_0 \in D$.

Question 2.1. Given an arbitrary nonsingular algebraic variety $X \subseteq \mathbb{C}^n$, can we find a map $\pi: X \rightarrow \mathbb{C}$ such that the fibers X_t are nonsingular for all but finitely many $t \in \mathbb{C}$ and such that the singular fibres have at worst ODP singularities?

Notice how we are missing information at infinity, e.g. $y = x^2$ versus $xy = 1$. The solution to this is to replace \mathbb{C}^n by \mathbb{CP}^n .

So let $X \subseteq \mathbb{P}^n$ be a nonsingular projective variety. Then we have:

Theorem 2.2. *There exists a family $(H_t)_{t \in \mathbb{CP}^1}$ of hyperplanes in \mathbb{CP}^n with $H_{[a,b]} = aH_0 + bH_\infty$ such that*

1. $X \subseteq \bigcup_{t \in \mathbb{CP}^1} H_t$.
2. $X_t = X \cap H_t$ is nonsingular except for finitely many critical values of t .
3. X_t has ODP singularities for each critical value t .

We call $(X_t)_{t \in \mathbb{CP}^1}$ a *Lefschetz pencil*. We get a rational map $X \dashrightarrow \mathbb{CP}^1$ sending $x \mapsto t$ whenever $x \in X_t$. If $x \in X_t \cap X_{t'}$ for $t \neq t'$, then $x \in H_0 \cap H_\infty$, so this rational map is not well-defined along $X \cap H_0 \cap H_\infty$. Blowing-up this subvariety of X we resolve the indeterminacy of the rational map and get a morphism $\tilde{X} \xrightarrow{\pi} \mathbb{CP}^1$ as we wanted.

As an application we obtain:

Theorem 2.3 (Lefschetz Hyperplane theorem). *$X \subseteq Y \subseteq \mathbb{CP}^N$ nonsingular varieties with X a hypersurface in the n -dimensional variety Y , then*

$$H_*(X) \rightarrow H_*(Y)$$

is an isomorphism for $ < n - 1$ and a surjection for $* = n - 1$.*

In particular, if $Y = \mathbb{CP}^n$, we have

$$H_*(\mathbb{CP}^n) = \begin{cases} \mathbb{Z} & \text{if } * \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

If $X \subseteq \mathbb{CP}^n$ is a nonsingular hypersurface, then its homology will be that of projective space on all degrees other than $n - 1$. Its $n - 1$ homology will depend on the variety, e.g. the ODP (trivial 1-homology) vs the ruled surface (with γ non trivial on 1-homology) from before.

Example 2.4. X elliptic curve in \mathbb{CP}^2 given by $y^2 = x(x - 1)(x - \lambda)$ for $\lambda \neq 0$. Let $L = \mathbb{CP}^1 \subseteq \mathbb{CP}^2$ and $P \in \mathbb{CP}^1 \setminus (X \cup L)$. We get $X \xrightarrow{\pi} \mathbb{CP}^1$ by projecting from P to L .

3 [WS] Kodaria 1 (Jin) - 23.10.19

3.1 Chow's theorem

Let $G_i = G_i(z_1, \dots, z_n)$ be homogeneous polynomials of degree d_i for $i \in \{1, \dots, k\}$. Let $V = V(G_1, \dots, G_k) = \{w \in \mathbb{C}^{n+1} \setminus \{0\} \mid G_i(w) = 0 \text{ for all } i \in \{1, \dots, k\}\} \subseteq \mathbb{CP}^n$. Assume $(\frac{\partial G_i}{\partial z_j}(w))$ is surjective at any $w \in V$.

$$\sum_{j=0}^n z_j \frac{\partial G_i}{\partial z_j} = d_i G(z_0, \dots, z_n),$$

if $\tilde{w} = (\tilde{z}_0, \dots, \tilde{z}_n) \in V$.

$$\sum_{j=0}^n \tilde{z}_j \frac{\partial G_i}{\partial z_j} \Big|_{\tilde{w}} = 0$$

$V \cap U_i$ for any $i \in \{0, \dots, n\}$, $U_i = \{[z_0 : \dots : z_n] \in \mathbb{CP}^n \mid z_i \neq 0\}$.

For $i = 0$, consider the chart (U_0, ϕ_0) with $\phi_0: U_0 \rightarrow \mathbb{C}^n$ given by $[z_0, \dots, z_n] \mapsto (\frac{z_1}{z_0}, \dots, \frac{z_n}{z_0})$. The inverse has a lift given by $\tilde{\psi}: \mathbb{C}^n \rightarrow \mathbb{C}^{n+1} \setminus \{0\}$ given by $(w_1, \dots, w_n) \mapsto (1, w_1, \dots, w_n)$.

$$\begin{array}{ccc} \mathbb{C}^n & & \\ \downarrow \tilde{\psi} & \searrow G \circ \tilde{\psi}_0 & \\ \mathbb{CP}^n & \xrightarrow{G} & \mathbb{C}^k \end{array}$$

$V \cap U_0 = G^{-1}(\{0\})$.

$G \circ \tilde{\psi}_0: (w_1, \dots, w_n) \mapsto (G_1(1, w_1, \dots, w_n), \dots, G_k(1, w_1, \dots, w_n))$.

$$\frac{\partial(G_i \circ \tilde{\psi}_0)}{\partial w_j} = \frac{\partial G_i}{\partial z_l} \frac{\partial(\tilde{\psi}_0)^l}{\partial w_j} \Big|_{(\tilde{w}_1, \dots, \tilde{w}_n)}$$

Call the LHS A_1 .

$$\frac{\partial G_i}{\partial z_l} \Big|_{\tilde{w}=(1, \tilde{w}_1, \dots, \tilde{w}_n)} \begin{pmatrix} 1 \\ \tilde{w}_1 \\ \vdots \\ \tilde{w}_n \end{pmatrix} = 0 \quad (1)$$

Note also that

$$\frac{\partial(\tilde{\psi}_0)^l}{\partial w_j} = \begin{pmatrix} 0 & \dots & 0 \\ 1 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 1 \end{pmatrix}.$$

Now

$$\left(\frac{\partial G_i}{\partial z_l}\right) = (a_{il}) = \begin{pmatrix} a_{10} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{k0} & \dots & a_{kn} \end{pmatrix}$$

$$A_1 = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{k1} & \dots & a_{kn} \end{pmatrix}$$

Since A is surjective and

$$A \begin{pmatrix} 1 \\ \tilde{w}_1 \\ \vdots \\ \tilde{w}_n \end{pmatrix} = 0,$$

hence A_1 is surjective.

Theorem 3.1 (Chow). *Every analytic closed subvariety $V \subseteq \mathbb{CP}^n$ is the zero locus of finite number of homogeneous polynomials.*

For this we will use as a black box:

Lemma 3.2 (Remmert-Stein). *$U \subseteq \mathbb{C}^n$ domain, S an analytic subvariety of U of $\dim = m$, W an analytic subvariety of $U \setminus S$ such that $\dim_p W > m$ for all regular points $p \in W$. Then \bar{W} is analytic.*

Now we can prove Chow's theorem. Suppose $\pi: \mathbb{C}^{n+1} \setminus \{0\} \rightarrow \mathbb{CP}^n$ has rank n everywhere. Then $\pi^{-1}(V)$ has dimension at least 1 everywhere in $\mathbb{C}^{n+1} \setminus \{0\}$ ($\pi^{-1}(V)$ is a cone missing the origin, so its closure is $\pi^{-1}(V) \cup \{0\}$). $S = \{0\}$, $W = \pi^{-1}(V)$. Then $V' = \bar{W} = \pi^{-1}(V) \cup \{0\}$ is an analytic variety of \mathbb{C}^{n+1} . Consider V' near 0. $V'_0 = U_\varepsilon(0) \cap V'$. $V'_0 = V(g_1, \dots, g_k)$ with g_i holomorphic on $U_\varepsilon(0)$. Expand g_i into a homogeneous polynomial $g_i = \sum_{n=1}^{\infty} g_{i,n}$. Then $g_i(tz) = \sum_{n=1}^{\infty} g_{i,n}(z)t^n$ for all $x \in \mathbb{C}^{n+1}$ and all $t \in \mathbb{C}$. If $z \in V'$, then $tz \in V'$ for all t . So $g_i(tz) \equiv 0$ implies $g_{i,n}(z) = 0$ for all i, n . So $V'_0 = V(\{g_{i,n}\})$. By Noetherianity, finitely many $g_{i,n}$ suffice. $V_0 = V(g^{(1)}, \dots, g^{(m)})$. Hence $V = V(\{g^{(1)}, \dots, g^{(m)}\})$ in \mathbb{CP}^n , and this finishes the proof.

3.2 Sheaves

For precise definitions and results in this subsection see Wikipedia, Stacks or nLab.

Definition of *sheaf* (of abelian groups) on a topological space X , *stalk* of a sheaf at a point $x \in X$, *germs* of a sheaf at a point... Note the similarities in terminology with plants.

Example 3.3. Constant sheaves $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$. Sheaf of smooth functions \mathcal{C}^∞ and its units \mathcal{C}^* . Sheaf of regular functions \mathcal{O} and units \mathcal{O}^* . Sheaf of meromorphic functions \mathcal{M} and \mathcal{M}^* .

Maps between sheaves, their kernels and their cokernels. Short exact sequences of sheaves.

Example 3.4. Let M be a complex manifold. The sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \rightarrow 0$$

is exact.

Definition of Čech cohomology of a sheaf $\mathcal{F} \in \text{Sh}(M)$ with respect to an open cover \mathcal{U} , which we denote by $H^p(\mathcal{U}, \mathcal{F})$ on degree p , and Čech cohomology of the sheaf \mathcal{F} as their direct limit over refinements, denoted $\check{H}^p(M, \mathcal{F})$.

Theorem 3.5 (Leray). *If \mathcal{U} is an acyclic cover, i.e. if there are no higher Čech cohomologies with respect to this cover, then the Čech complex associated to this cover computes Čech cohomology.*

Long exact sequence in Čech cohomology induced by a short exact sequence of sheaves.

3.3 A bit of Hodge theory

Decomposition of the tangent space at a point of a complex manifold, its tensor algebras, ∂ and $\bar{\partial}$ operators, Dolbeault cohomology groups, harmonic and Hodge decomposition.. See [GH78] or [Voi07].

References

- [GH78] P. Griffiths and J. Harris. *Principles of Algebraic Geometry*. John Wiley & Sons, 1978.
- [Voi07] Claire Voisin. *Hodge theory and complex algebraic geometry. I*, volume 76 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, english edition, 2007. Translated from the French by Leila Schneps.