Various lecture notes

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1 About these notes

The purpose of these notes is to keep the material seen in lectures a bit organized and easily accesible from one single place, but they don't intend to be complete and they will surely be full of typos and mistakes[1].

If interesting questions occur during the lectures they will be reflected here in blue color. If I want to add anything which was not said in the lecture, I will use green color instead.

Warnings will be marked with a dangerous bend symbol on the margin .

[1] If you find any, please let me know! You can do this from GitHub or write me an email directly at pedro.nunez[at]math.uni-freiburg.de.

2 [CM] Talk 1 (Johan Commelin): Condensed Sets - 21.10.19

2.1 Introduction

One of the main motivations for condensed mathematics is that topological algebraic objects have usually poor categorical and functorial properties. For instance, topological abelian groups do not form an abelian category:

Example 2.1. $\mathbb{R}_{disc} \to \mathbb{R}$ is epi and mono, but not iso.

Another motivation is coherent duality:

Theorem 2.2. *Let $f\colon X \to Y$ be a proper or quasi-projective morphism of Noetherian schemes of finite Krull dimension. Then there exists a right adjoint $f^!$ to the derived direct image functor $f_! = Rf_*\colon \mathcal{D}^{\mathrm{b}}(\mathcal{QC}oh(X)) \to \mathcal{D}^{\mathrm{b}}(\mathcal{QC}oh(Y))$.*

At some point analytic rings will come up. We will then look at the category of solid modules, in which the 6-functor formalism works nicer than in the classical setting (e.g. when $f_!$ is not defined in the classical setting, $f_!$ takes non-discrete values in the condensed settings, which are "not there" in the classical setting).

Definition 2.3. Proétale site of a point, denoted $*_{pro\acute{e}t}$, is the category of profinite sets with finite jointly surjective families of continuous maps as covers. A *condensed set* (resp. group, ring, ...) is a sheaf of sets (resp. groups, rings, ...) on $*_{pro\acute{e}t}$. We denote by $\mathrm{Cond}(\mathcal{C})$ the category of condensed objects of a category $\mathcal{C}$.

Definition 2.4. A *condensed set* (resp. group, ring, ...) is a contravariant functor X from $*_{pro\acute{e}t}$ to the category of sets (resp. groups, rings, ...) such that

i) $X(\varnothing) = *$.

ii) For all profinite sets S_1 and S_2 the natural map

$$X(S_1 \sqcup S_2) \to X(S_1) \times X(S_2)$$

is an isomorphism.

iii) For any surjection of profinite sets $f\colon S' \twoheadrightarrow S$ we get an induced[2] isomorphism

$$X(S) \to \{x \in X(S') \mid \pi_1^*(x) = \pi_2^*(x) \in X(S' \times_S S')\}$$

We will call $X(*)$ the *underlying object* in $\mathcal{C}$ of a condensed object.

Remark 2.5. We will use T for topological spaces vs. X, Y for condensed sets, as opposed to Scholze's mixing of those notations.

2.2 Recollections on sheaves on sites

Let F be a presheaf on a site, which is just a contravariant functor to whatever category in which our sheaves are gonna take values. If $U = \cup_i U_i$ is an open cover, the topological sheaf axiom could be phrased as: $F(U)$ is an equalizer of the diagram

$$\prod_i F(u_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j).$$

Note that $U_i \cap U_j$ is just the fiber product of the two inclusions.

Definition 2.6 (Coverage)**.** See definition 2.1 in nCat.

Definition 2.7. F a presheaf on $\mathcal{C}$. A collection $(s_i) \in \prod_i F(U_i)$ for $\{f_i\colon U_i \to U\}$ a covering is called a *matching family* if for all $h\colon V \to U$ we have $g^*(s_i) = h^*(s_j)$ for g and h in the diagram

$$\begin{tikzcd} V \arrow[r, "h"] \arrow[d, "g"] & U_j \arrow[d, "f_j"] \\ U_i \arrow[r, "f_i"] & U \end{tikzcd}$$

Definition 2.8. F is a sheaf with respect to $\{U_i \to U\}$ if for all matching families (s_i) there exists a unique $s \in F(U)$ such that $f_i^*(s) = s_i$. We say that F is a *sheaf* if it is a sheaf for all covering families.

Remark 2.9. A sheaf of abelian groups is just a commutative group object in the category of sheaves of sets.

Theorem 2.10. *If $\mathcal{C}$ is a site, then $\mathcal{Ab}(\mathcal{C})$ is an abelian category.*

[2]Since the pullback diagram is commutative, the image of $X(f)$ is indeed induces a morphism as claimed.

Definition 2.11. An additive category is a category in which the hom-sets are endowed with an abelian group structure in a way that makes composition bilinear and such that finite biproducts exist.

Recall Grothendieck's axioms: AB1) Every morphism has a kernel and a cokernel. AB2) For every $f\colon A \to B$, the natural map $\mathrm{coim}(f) \to \mathrm{im}\, f$ is an iso. AB3) All colimit exist. AB4) AB3) + arbitrary direct sums are exact. AB5) AB3) + arbitrary filtered colimits are exact. AB6) AB3) + J an index set, $\forall j \in J$ a filtered category (think of directed set) I_j, functors $M\colon I_j \to \mathcal{C}$, then

$$\varinjlim_{(i_j \in I_j)_j} \prod_j M_{i_j} \to \prod_{j \in J} \varinjlim_{i_j \in I_j} M_{i_j}$$

Theorem 2.12. $\mathcal{C}$ *a site. Then* $\mathcal{Ab}(\mathcal{C})$ *satisfies AB3), AB4), AB5) and AB6).*

In fact, our case is even nicer:

Theorem 2.13. $\mathrm{Cond}(\mathcal{Ab})$ *in addition satisfies AB6) and AB4*).*

2.3 Compactly generated topological spaces

Definition 2.14. A topological space T is called *compactly generated* if any function $f\colon T \to T'$ is continuous as soon as the composite $S \to T \to T'$ is continuous for all maps $S \to T$ where S is compact and Hausdorff. See also nCat.

The inclusion functor $\mathcal{CG} \hookrightarrow \mathcal{T}op$ has a right adjoint $(-)^{cg}$. If T is any topological space, then the topology on T^{cg} is the finest topology on T such that $\sqcup_{S \to T} S \to T$ is continuous, where S ranges over all compact Hausdorff spaces.

Let T be a topological space. We view T as a presheaf on $*_{pro\acute{e}t}$ by setting $T(S) = \mathrm{Hom}_{\mathcal{T}op}(S, T)$ for all profinite sets S. We denote this by $\underline{T}$. Claim: $\underline{T}$ is a sheaf.

i) The first condition $\underline{T}(\varnothing) = *$ is true, because there is exactly one morphism from the empty set to any topological space.

ii) $\underline{T}(S_1 \sqcup S_2) = \underline{T}(S_1) \times \underline{T}(S_2)$ by universal property of disjoint union.

iii) For any surjection $S' \twoheadrightarrow S$ we get an isomorphism

$$\underline{T}(S) \to \{x \in \underline{T}(S') \mid \pi_1^*(x) = \pi_2^*(x) \in \underline{T}(S' \times_S S')\}$$

Since $\mathcal{T}op \to \mathrm{Cond}(\mathcal{S}et)$ preserves products, group objects are preserved, so it maps topological groups to condensed groups etc.

Proposition 2.15. *i) This functor is faithful and fully faithful when restricted to the full subcategory of compactly generated spaces.*

ii) It admits a left adjoint $X \mapsto X(*)_{top}$ where $X(*)_{top}$ gets the quotient topology of $\sqcup_{S \to X} S \to X(*)$ as above. The counit $I(*)_{top} \to T$ agrees with $T^{cg} \to T$.

Coming back to our original example:

Example 2.16. $\mathbb{R}_{disc} \to \mathbb{R}$ can be seen in the condensed world as $\underline{\mathbb{R}_{disc}} \to \underline{\mathbb{R}}$, i.e. from locally constant functions to continuous functions. This is still a mono, but now it is not an epi. The cokernel Q can be described as $Q(S) = \{S \to \mathbb{R} \text{ continuous }\}/\{S \to \mathbb{R} \text{ locally constant }\}$. Note in particular that the underlying set of Q is just $*$, reflecting the fact that the cokernel was trivial in the classical setting.

3 [LT] Lecture 1 - 22.10.19

3.1 Introduction and overview of the course

An *algebraic variety* is the solution set of a family of polynomial equations in $\mathbb{C}^n$. For example, if $f(x, y, z, t) = xy - tz$, then

$$V(f) = \{(x, y, z, t) \in \mathbb{C}^4 \mid xy - tz = 0\}$$

is an algebraic variety in $\mathbb{C}^4$. Another example would be the parabola $\{y - x^2 = 0\} \subseteq \mathbb{C}^2$.

Let us focus on $V(f)$ and set $t = 1$. Then $X = V(f) \cap \{t = 1\} = \{(x, y, z) \in \mathbb{C}^3 \mid xy = z\}$ can be seen as a family of complex curves parametrized by the variable z. For $z = 0$, the complex curve X_0 has an *ordinary double point* at the origin:

Singularities arise naturally while studying the topology of algebraic varieties, and ODP's are a particularly nice kind of singularities.

For $z \neq 0$ we get an equation which looks like $xy = 1$. In this case we have the following picture:

As $z \to 0$, the central loop γ contracts to the ordinary double point. Note in particular that X_0 has trivial fundamental group (hence trivial 1-homology), whereas X_z does not.

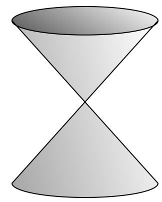

Figure 1: Topological picture of our ODP.

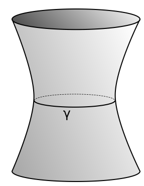

Figure 2: Topological picture of X_z.

We have a projection $\pi\colon X \to \mathbb{C}$, and Ehresmann's lemma tells us that for all disks $D \subseteq \mathbb{C}$ not containing 0 we have $\pi^{-1}(D) \cong D \times X_{z_0}$ for any $z_0 \in D$.

Question 3.1. Given an arbitrary nonsingular algebraic variety $X \subseteq \mathbb{C}^n$, can we find a map $\pi\colon X \to \mathbb{C}$ such that the fibers X_t are nonsingular for all but finitely many $t \in \mathbb{C}$ and such that the singular fibres have at worst ODP singularities?

Notice how we are missing information at infinity, e.g. $y = x^2$ versus $xy = 1$. The solution is to this is to replace $\mathbb{C}^n$ by $\mathbb{CP}^n$.

So let $X \subseteq \mathbb{P}^n$ be a nonsingular projective variety. Then we have:

Theorem 3.2. *There exists a family $(H_t)_{t\in\mathbb{CP}^1}$ of hyperplanes in $\mathbb{CP}^n$ with $H_{[a,b]} = aH_0 + bH_\infty$ such that*

1. $X \subseteq \bigcup_{t\in\mathbb{CP}^1} H_t$.
2. $X_t = X \cap H_t$ *is nonsingular except for finitely many critical values of* t.
3. X_t *has ODP singularities for each critical value* t.

We call $(X_t)_{t\in\mathbb{CP}^1}$ a *Lefschetz pencil*. We get a rational map $X \dashrightarrow \mathbb{CP}^1$ sending $x \mapsto t$ whenever $x \in X_t$. If $x \in X_t \cap X_{t'}$ for $t \neq t'$, then $x \in H_0 \cap H_\infty$, so this rational map is not well-defined along $X \cap H_0 \cap H_\infty$. Blowing-up this subvariety of X we resolve the indeterminacy of the rational map and get a morphism $\tilde{X} \xrightarrow{\pi} \mathbb{CP}^1$ as we wanted.

As an application we obtain:

Theorem 3.3 (Lefschetz Hyperplane theorem)**.** *$X \subseteq Y \subseteq \mathbb{CP}^N$ nonsingular varieties with X a hypersurface in the n-dimensional variety Y, then*

$$H_*(X) \to H_*(Y)$$

is an isomorphism for $* < n-1$ and a surjection for $* = n-1$.

In particular, if $Y = \mathbb{CP}^n$, we have

$$H_*(\mathbb{CP}^n) = \begin{cases} \mathbb{Z} & \text{if * is even,} \\ 0 & \text{otherwise.} \end{cases}$$

If $X \subseteq \mathbb{CP}^n$ is a nonsingular hypersurface, then its homology will be that of projective sapce on all degrees other than $n-1$. Its $n-1$ homology will depend on the variety, e.g. the ODP (trivial 1-homology) vs the ruled surface (with γ non trivial on 1-homology) from before.

Example 3.4. X elliptic curve in $\mathbb{CP}^2$ given by $y^2 = x(x-1)(x-\lambda)$ for $\lambda \neq 0$. Let $L = \mathbb{CP}^1 \subseteq \mathbb{CP}^2$ and $P \in \mathbb{CP}^1 \setminus (X \cup L)$. We get $X \xrightarrow{\pi} \mathbb{CP}^1$ by projecting from P to L.

4 [WS] Kodaria 1 (Jin Li) - 23.10.19

4.1 Chow's theorem

Let $G_i = G_i(z_1, \dots, z_n)$ be homogeneous polynomials of degree d_i for $i \in \{1, \dots, k\}$. Let $V = V(G_1, \dots, G_k) = \{w \in \mathbb{C}^{n+1} \setminus \{0\} \mid G_i(w) = 0 \text{ for all } i \in \{1, \dots, k\}\} \subseteq \mathbb{CP}^n$. Assume $(\frac{\partial G_i}{\partial z_j}(w))$ is surjective at any $w \in V$.

$$\sum_{j=0}^{n} z_j \frac{\partial G_i}{\partial z_j} = d_i G(z_0, \dots, z_n),$$

if $\tilde{w} = (\tilde{z}_0, \dots, \tilde{z}_n) \in V$.

$$\sum_{j=0}^{n} \tilde{z}_j \frac{\partial G_i}{\partial z_j}|_{\tilde{w}} = 0$$

$V \cap U_i$ for any $i \in \{0, \dots, n\}$, $U_i = \{[z_0 : \dots : z_n] \in \mathbb{CP}^n \mid z_i \neq 0\}$.

For $i = 0$, consider the chart (U_0, ϕ_0) with $\phi_0 \colon U_0 \to \mathbb{C}^n$ given by $[z_0, \dots, z_n] \mapsto (\frac{z_1}{z_0}, \dots, \frac{z_n}{z_0})$. The inverse has a lift given by $\tilde{\psi} \colon \mathbb{C}^n \to \mathbb{C}^{n+1} \setminus \{0\}$ given by $(w_1, \dots, w_n) \mapsto (1, w_1, \dots, w_n)$.

$$
\begin{tikzcd}
\mathbb{C}^n \arrow[d, "\tilde{\psi}"] \arrow[dr, "G\circ\tilde{\psi}_0"] & \\
\mathbb{CP}^n \arrow[r, "G"] & \mathbb{C}^k
\end{tikzcd}
$$

$V \cap U_0 = G^{-1}(\{0\})$.

$G \circ \tilde{\psi_0} \colon (w_1, \ldots, w_n) \mapsto (G_1(1, w_1, \ldots, w_n), \ldots, G_k(1, w_1, \ldots, w_n))$.

$$\frac{\partial(G_i \circ \tilde{\psi_0})}{\partial w_j} = \frac{\partial G_i}{\partial z_l}\frac{\partial(\tilde{\psi_0})^l}{\partial w_j}|_{(\tilde{w_1}, \ldots, \tilde{w_n})}$$

Call the LHS A_1.

$$\frac{\partial G_i}{\partial z_l}|_{\tilde{w}=(1,\tilde{w}_1,\ldots,\tilde{w}_n)} \begin{pmatrix} 1 \\ \tilde{w}_1 \\ \vdots \\ \tilde{w}_n \end{pmatrix} = 0 \tag{1}$$

Note also that

$$\frac{\partial(\tilde{\psi_0})^l}{\partial w_j} = \begin{pmatrix} 0 & \ldots & 0 \\ 1 & \ldots & 0 \\ \vdots & & \vdots \\ 0 & \ldots & 1 \end{pmatrix}.$$

Now

$$(\frac{\partial G_i}{\partial z_l}) = (a_{il}) = \begin{pmatrix} a_{10} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{k0} & \ldots & a_{kn} \end{pmatrix}$$

$$A_1 = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{k1} & \ldots & a_{kn} \end{pmatrix}$$

Since A is surjective and

$$A \begin{pmatrix} 1 \\ \tilde{w}_1 \\ \vdots \\ \tilde{w}_n \end{pmatrix} = 0,$$

hence A_1 is surjective.

Theorem 4.1 (Chow)**.** *Every analytic closed subvariety $V \subseteq \mathbb{CP}^n$ is the zero locus of finite number of homogeneous polynomials.*

For this we will use as a black box:

Lemma 4.2 (Remmert-Stein)**.** *$U \subseteq \mathbb{C}^n$ domain, S an analytic subvariety of U of $\dim = m$, W an analytic subvariety of $U \setminus S$ such that $\dim_p W > m$ for all regular points $p \in W$. Then $\bar{W}$ is analytic.*

Now we can prove Chow's theorem. Suppose $\pi \colon \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ has rank n everywhere. Then $\pi^{-1}(V)$ has dimension at least 1 everywhere in $\mathbb{C}^{n+1} \setminus \{0\}$ ($\pi^{-1}(V)$ is a cone missing the origin, so its closure is $\pi^{-1}(V) \cup \{0\}$). $S = \{0\}$, $W = \pi^{-1}(V)$. Then $V' = \bar{W} = \pi^{-1}(V) \cup \{0\}$ is an analytic variety of $\mathbb{C}^{n+1}$. Consider V' near 0. $V_0' = U_\varepsilon(0) \cap V'$. $V_0' = V(g_1, \ldots, g_k)$ with g_i holomorphic on $U_\varepsilon(0)$. Expand g_i into a homogeneous polynomial $g_i = \sum_{n=1}^{\infty} g_{i,n}$. Then $g_i(tz) = \sum_{n=1}^{\infty} g_{i,n}(z) t^n$ for all $x \in \mathbb{C}^{n+1}$ and all $t \in \mathbb{C}$. If $z \in V'$, then $tz \in V'$ for all t. So $g_i(tz) \equiv 0$ implies $g_{i,n}(z) = 0$ for all i, n. So $V_0' = V(\{g_{i,n}\})$. By Noetherianity, finitely many $g_{i,n}$ suffice. $V_0 = V(g^{(1)}, \ldots, g^{(m)})$. Hence $V = V(\{g^{(1)}, \ldots, g^{(m)}\})$ in $\mathbb{CP}^n$, and this finishes the proof.

4.2 Sheaves

For precise definitions and results in this subsection see Wikipedia, Stacks or nLab.

Definition of *sheaf* (of abelian groups) on a topological space X, *stalk* of a sheaf at a point $x \in X$, *germs* of a sheaf at a point... Note the similarities in terminology with plants.

Example 4.3. Constant sheaves $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$. Sheaf of smooth functions $\mathcal{C}^\infty$ and its units $\mathcal{C}^*$. Sheaf of regular functions $\mathcal{O}$ and units $\mathcal{O}^*$. Sheaf of meromorphic functions $\mathcal{M}$ and $\mathcal{M}^*$.

Maps between sheaves, their kernels and their cokernels. Short exact sequences of sheaves.

Example 4.4. Let M be a complex manifold. The sequence

$$0 \to \mathbb{Z} \to \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \to 0$$

is exact.

Definition of Čech cohomology of a sheaf $\mathcal{F} \in \mathrm{Sh}(M)$ with respect to an open cover $\mathcal{U}$, which we denote by $H^p(\mathcal{U}, \mathcal{F})$ on degree p, and Čech cohomology of the sheaf F as their direct limit over refinements, denoted $\check{H}^p(M, \mathcal{F})$.

Theorem 4.5 (Leray)**.** *If $\mathcal{U}$ is an acyclic cover, i.e. if there are no higher Čech cohomologies with espect to this cover, then the Čech complex associated to this cover computes Čech cohomology.*

Long exact sequence in Čech cohomology induced by a short exact sequence of sheaves.

4.3 A bit of Hodge theory

Decomposition of the tangent space at a point of a complex manifold, its tensor algebras, ∂ and $\bar{\partial}$ operators, Dolbeault cohomology groups, harmonic and Hodge decomposition.. See [GH78] or [Voi07].

5 [LT] Lecture 2 - 24.10.19

Remark 5.1. Exercise sessions will be Thursday from 13h to 15h on SR318 (Starting next week).

As pointed out last week, we want to look at polynomials and their solutions sets. But polynomials are a bit too rigid. Instead, we look at polynomials as truncated power series, or more generally as *analytic functions*, which are functions which locally can be represented as power series. We will see that these are the same as holomorphic functions. In particular, every holomorphic function is C^∞.

$$\begin{aligned} &\text{polynomials} \Rightarrow \text{convergent power series} \Rightarrow \text{analytic} \Leftrightarrow \\ &\qquad \Leftrightarrow \text{holomorphic} \Rightarrow C^\infty \Rightarrow \text{continuous} \Rightarrow \text{abominations} \end{aligned}$$

If we were analysts we would start at the bottom and then try to swim up. Instead we will start from the top and float downstream.

Notation 5.2. $\mathbb{E} = \mathbb{R}$ or $\mathbb{C}$. $z = (z_1, \dots, z_n) \in \mathbb{E}^n$, $r \in \mathbb{R}_{\geqslant 0}$. Recall

$$|z| = \sqrt{2} z_1 \bar{z_1} + \cdots + z_n \bar{z_n},$$

$$\mathbb{D}(z,r) = \{w \in \mathbb{E}^n \mid |z-w| < r\}, \text{ and}$$

$$\bar{\mathbb{D}}(z,r) = \{w \in \mathbb{E}^n \mid |z-w| \leqslant r\}$$

called oepn and closed disks respectively. We call $\bar{\mathbb{D}}(z_1, r_1) \times \cdots \times \bar{\mathbb{D}}(z_n, r_n)$ an *open polydisk.*

5.1 Formal power series

Definition 5.3. Let $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$. A *formal power series* centered at a is an expression of the form

$$f(z) = f(z_1, \ldots, z_n) = \sum_{(r_1,\ldots,r_n)\in\mathbb{Z}^n_{\geqslant 0}} c_{r_1\cdots r_n}(z_1 - a_1)^{r_1}\cdots(z_n - a_n)^{r_n}$$

with $c_{r_1,\ldots,r_n} \in \mathbb{C}$.

Remark 5.4. We will restrict our attention to absolutely convergent series, so we do not need to order the indices in the sum to discuss convergence.

Definition 5.5. The series above *converges (uniformly) absolutely* on $X \subseteq \mathbb{C}^n$ if for all $z \in X$ the series of real numbers

$$\sum_{(r_1,\ldots,r_n)} |c_{r_1,\ldots,r_n}(z_1 - a_1)^{r_1}\cdots(z_n - a_n)^{r_n}|$$

converges (uniformly).

Recall that $\sum_n c_n z^n$ converges absolutely on $\mathbb{D}(0, R)$ where $R = \frac{1}{\operatorname{limsup}_{n\to\infty}|c_n|^{\frac{1}{n}}}$. It converges uniformly absolutely on each compact $K \subseteq \mathbb{D}(0, R)$.

Example 5.6 (Geometric series)**.** The geometric series with ration $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ is defined as $\sum_{r_1,\ldots,r_n} z_1^{r_1}\cdots z_n^{r_n}$. It converges (uniformly) absolutely on (compact subsets of) $\mathbb{D}(0, 1)^n$ with sum equal

$$\prod_{k=1}^{n}\sum_{r_k\geqslant 0} z_k^{r_k} = \frac{1}{(1 - z_1)\cdots(1 - z_n)}$$

Lemma 5.7 (Abel)**.** *Consider the series above, $w \in \mathbb{C}^n$ and $M \in \mathbb{R}_{>0}$. If $|c_r(w-a)^r| = |c_{r_1,\ldots,r_n}(w_1-a_1)^{r_1}\cdots(w_n-a_n)^{r_n} < M$ for each $r \in \mathbb{Z}^n_{\geqslant 0}$, then $f(z)$ converges uniformly absolutely on each compact $K \subseteq D = \mathbb{D}(a_1, \rho_1) \times \mathbb{D}(a_n, \rho_n)$, where $\rho_i := |w_i - a_i|$.*

Proof. WLOG $\rho_k > 0$ for all $k \in \{1, \ldots, n\}$ (otherwise we'd have $D = \varnothing$). Let $K \subseteq D$. Then let $\delta_k := \max_{z\in K}\frac{|z_k - a_k|}{\rho_k} < 1$. Then for all $z \in K$ and for all $r \in \mathbb{Z}^n_{\geqslant 0}$ we have

$$|c_r(z - a)^r| \leqslant |c_r\rho^r| \leqslant M\delta^r.$$

Since all $\delta_k < 1$, by the previous example $\sum_r M\delta^r$ converges uniform absolutely on K. $\square$

Definition 5.8. Uniform absolute convergence on compacts is also called *compact convergence.*

5.2 Analytic functions

Definition 5.9. Let $U \subseteq \mathbb{C}^n$ open.

i) $f: U \to \mathbb{C}$ is *analytic* at $a \in U$ if there exists an open neighbourhood $a \in V \subseteq U$ and c_r such that $f(z) = \sum_r c_r(z-a)^r$ converges compactly on V.

ii) $f: U \to \mathbb{C}$ is *analytic* on U if it is analytic at each point of U.

iii) $f: U \to \mathbb{C}^n$ is *analytic* on U if each component f_k is for all $k \in \{1, \dots, n\}$.

Exercise 5.10. Analytic at a implies continuous at a.

Exercise 5.11. If f, g are analytic, then so are $f + g$, $f - g$ and $g \circ f$ where defined.

Exercise 5.12. Let $U \subseteq C^n$ be an open subset, let $z \in U$ and $w \in \mathbb{C}^n$. Let $V = \{c \in \mathbb{C} \mid z + cw \in U\} \subseteq \mathbb{C}^n$.

i) V is open and $0 \in V$.

ii) For all $f: U \to \mathbb{C}$ analytic we have that $g(t) = f(z + tw)$ is analytic on V.

Theorem 5.13 (Identity theorem)**.** *If $\varnothing \neq V \subseteq U \subseteq \mathbb{C}^n$ are open with U connected and $f: U \to \mathbb{C}$ is analytic with $f|_V = 0$, then $f = 0$.*

Proof. If $f(z) \neq 0$ for some $z \in U$, then by continuity of f we would have that f is nowhere zero on some open nbhd of z. Let $Z = \{w \in U \mid f \text{ vanishes in an open nbhd of } w\}$. Then Z is closed in U by what we just said. Also, $V \subseteq Z$ as V is open. Let $w \in Z$ and choose a polydisk $w \in D = \mathbb{D}(w_1, r_1) \times \cdots \times \mathbb{D}(w_n, r_n) \subseteq U$. If we show that $D \subseteq Z$, then every point of Z is in its interior and Z is therefore open.

So let $z \in D$ with $z \neq w$. Consider now $W = \{c \in \mathbb{C} \mid w + c(z-w) in \mathcal{U}\} \subseteq \mathbb{C}$, which is open by the previous exercise[3], and $g: t \mapsto f(w + t(z-w))$ is analytic on W. The identity theorem for single-variable analytic functions implies that $g = 0$ in a nbhd of w. Since D is convex, $[0, 1] \subseteq W$. By the identity theorem in one variable, $g = 0$ on an open nbhd of $[0, 1]$. Hence $g(1) = f(z) = 0$, so f vanishes on D and $D \subseteq Z$. □

[3]This step allows us to reduce our problem in several complex variables to a problem on a single complex variable.

5.3 Topology

Definition of topological space and examples (cofinite topology, Zariski topology). Continuous maps, homeomorphisms (isomorphism in the category of topological spaces). Example: graph of $f\colon X \to Y$ defined as $\Gamma_f = X \times_X Y$ maps homeomorphically onto X via the first projection. Subspace topology.

Connectedness, example: unit interval. Continuous image of connected is connected.

Hausdorffness, example: euclidean topology on $\mathbb{R}^n$. Non-example: real line with two origins.

Remark 5.14. These two examples show that Hausdorffness is not a local property, because the real line with two origins is locally the same as $\mathbb{R}$.

Equivalently, X is Hd if and only if $\Delta \subseteq X \times X$ is closed. Hausdorffness is hereditary.

6 [FS] Matthias Paulsen - The construction problem for Hodge numbers - 25.10.19

In characteristic 0 this is j.w. with Stefan Schreieder. In positive characteristic this is j.w. with v. Dobbeen de Bruyn.

6.1 Overview over complex numbers

X smooth projective variety. Then we have Hodge theory, which allows us to decompose

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X),$$

where $H^{p,q}(X) \cong H^q(X, \Omega^p)$. The *Hodge numbers* are $h^{p,q}(X) = \dim_{\mathbb{C}} H^{p,q}(X)$. We usually arrange the Hodge numbers in the *Hodge diamond*

$$
\begin{array}{ccccccc}
 & & & h^{n,n} & & & \\
 & & h^{n,n-1} & & h^{n-1,n} & & \\
 & & & \cdots & & & \\
h^{n,0} & & & & & & h^{0,n} \\
 & & & \cdots & & & \\
 & & h^{1,0} & & h^{0,1} & & \\
 & & & h^{0,0} & & &
\end{array}
$$

Example 6.1. Let $X \subseteq \mathbb{P}^4$ be a hypersurface of degree 4. Then its Hodge diamond is

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 1 & & 0 & \\
5 & & 30 & & 30 & & 5 \\
 & 0 & & 1 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

We know:

i) $h^{p,q} = h^{q,p}$.

ii) $h^{0,0} = 1$.

iii) $h^{p,q} \geqslant h^{p-1,q-1}$ if $p + q \leqslant n$.

Question 6.2. Given $(h^{p,q})_{p,q}$ such that $i)$, $ii)$ and $iii)$ hold. Does there exist X with $h^{p,q}(X) = h^{p,q}$ for all p, q?

1989 Partial results in dimensions 2 and 3.

2013 Kotschick and Schreieder determined the Hodge ring of Kaehler manifolds and showed that there are no linear relations besides of the previous three.

2015 Schreieder.

In any dimension n, a given row $k < n$ can be always achieved, except if $k = 2p$, in which case we need $h^{p,p} \geqslant O(p)$.

If we ignore the middle row, the outer Hodge numbers and the middle column, then everything else can be arbitraty.

Negative results: the previous question has negative answer, e.g. in dimension three, if we assume that $h^{1,1} = 1 =$ and $h^{0,2} \geqslant 1$, then $h^{3,0} < 12^6 h^{2,1}$.

Question 6.3 (Kollar)**.** Are there any polynomial relations between the Hodge numbers, besides the ones induced by the symmetries above?

Question 6.4. Besides the "unexpected" inequalities, are there also number theoretic restrictions?

2019 Schreider and P.: modulo any integer $m \geqslant 1$, any Hodge diamond $(h^{p,q})_{p,q}$ satisfying the symmetries[4] above is realizable by a smooth projective variety X, i.e. such that

$$h^{p,q}(X) \equiv h^{p,q} (\text{mod } m).$$

Corollary 6.5. *There are no polynomial relations and there are no "number theoretic" restrictions.*

The proof can be devided into two parts corresponding to the outer Hodge numbers and the remaining ones. The outer Hodge numbers are birational invariants. The first part is to produce a variety which has the right outer Hodge numbers. And then all the inner ones can be obtain by repeated blow-ups.

[4]Note that condition $iii)$ vanishes.

6.2 Constructions

We will have 3 building blocks:

- Products (and then use Kuenneth's formula).
- Hypersurfaces to reduce the dimension again (apply Lefschetz hyperplane theorem, picture [A]).
- Blow-ups of subvarieties (see picture [B]).

We start then with a curve, in which the problem is completely solvable:

$$\begin{array}{ccc} & 1 & \\ g & & g \\ & 1 & \end{array}$$

Starting from this we can build up our Hodge diamonds modulo m. [*Some sketch of proof here is omitted, see future paper I guess.*]

6.3 Positive characteristic

We don't have complex conjugation, but we still have Serre duality imposing its 180 degrees rotation on the Hodge diamonds. Is this the only restriction?

Example 6.6 (Serre)**.** Construction of a surface with Hodge diamond

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 1 & \\ ? & & ? & & ? \\ & 0 & & 1 & \\ & & 1 & & \end{array}$$

It seems that Matthias is up to something cool using this example!

7 [CM] Talk 2 (Pedro Núñez) - Condensed Abelian Groups - 28.10.19

See the full script on GitHub.

7.1 Recollections from the previous talk

Recall $*_{pro\acute{e}t}$ and $\mathcal{A} = \mathrm{Sh}(*_{pro\acute{e}t}, \mathcal{A}b)$.

Equivalent description of $\mathcal{A}$ from last talk.

7.2 A nicer description of our category

Definition 7.1. Extremally disconnected.

Note that $\mathcal{ED} \subsetneq \mathcal{CH}$.

Fact: $S \in \mathcal{CH}$ is in $\mathcal{ED}$ if and only if every surjection $S' \twoheadrightarrow S$ from a compact Hausdorff space admits a section. Using Stone-Čech compactification β, this implies that every $S \in \mathcal{CH}$ admits a surjection from an extremally disconnected set

$$\exists \tilde{S} = \beta(S_{disc}) \twoheadrightarrow S$$

Lemma 7.2. $\mathcal{A} = \mathrm{Sh}(\mathcal{ED}, \mathcal{A}b)$.

Corollary 7.3. $\mathcal{A} = \{\mathcal{F} \in \mathrm{Fun}(\mathcal{ED}^{op}, \mathcal{A}b) \mid i) \wedge ii)\}$.

Corollary 7.4. *(Co)limits exist in $\mathcal{A}$ and can be constructed pointwise.*

7.3 Abelianity and compact-projective generation

Recall definition of Grothendieck category.

Theorem 7.5. *$\mathcal{A}$ is abelian with (AB6) and (AB4*) and it is generated by compact projective objects.*

Corollary 7.6. *$\mathcal{A}$ has enough injectives and projectives.*

— BREAK —

7.4 Closed symmetric monoidal structure

Briefly outline monoidal categories.

Proposition 7.7. *$\mathcal{A}$ is symmetric monoidal and the functor $\mathbb{Z}[-]\colon \mathrm{Cond}(\mathcal{S}et) \to \mathcal{A}$ is symmetric monoidal.*

Proposition 7.8. *For all condesned set $\mathcal{X}$, the condensed abelian group $\mathbb{Z}[\mathcal{X}]$ is flat, i.e. the functor $\mathbb{Z}[\mathcal{X}] \otimes (-)$ is exact.*

Proposition 7.9. *For all $\mathcal{F} \in \mathcal{A}$ the functor $\mathcal{F} \otimes (-)$ has a right adjoint $[\mathcal{F}, -]$, i.e. $\mathcal{A}$ is closed symmetric monoidal.*

7.5 Derived category

Brief description.

Remark 7.10. Triangulated structure and the problem it carries.

Basic derived functors

$$RF \colon \mathcal{D}^+(\mathcal{A}) = \mathcal{K}^+(\mathcal{I}) \to \mathcal{D}(\mathcal{B}).$$

Example 7.11. $\mathrm{Hom}(-, \mathcal{F})$ and $\mathrm{Hom}^\bullet(-, \mathcal{F}^\bullet)$.

Extension to the whole $\mathcal{D}$ using Spaltenstein's resolutions. Formula

$$\mathrm{Hom}_{\mathcal{D}}(\mathcal{F}^\bullet, \mathcal{G}^\bullet[i]) = \mathrm{Ext}^i(\mathcal{F}^\bullet, \mathcal{G}^\bullet).$$

Closed symmetric monoidal structure.

Example 7.12. $\mathbb{Z}[\mathcal{X}] \otimes^L (-) = \mathbb{Z}[\mathcal{X}] \otimes (-)$ is just degree-wise tensor product, because this functor is already exact so it need not be derived.

Proposition 7.13. *Compact generation.*

Mention Brown representability.

8 [LT] Lecture 3 - 29.10.19

Goals for today:

- **Prop:** f analytic $\Rightarrow f$ complex differentiable (holomorphic).
- **Cor:** f analytic $\Rightarrow f \in \mathcal{C}^\infty$.
- Implicit and inverse function theorems for analytic functions.

8.1 Complex differentiable functions

Recall: $U \subseteq \mathbb{R}^n$ open, $f\colon U \to \mathbb{R}^m$ is *differentiable* at $a \in U$ if $\exists\mathbb{R}$-linear map $df_a\colon \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{h\to 0} \frac{f(a+h) - f(a) - df_a(h)}{|h|} = 0.$$

With respect to the standard bases, df_a is given by the *Jacobian* matrix $J_f(a) = (\frac{\partial f_i}{\partial f_j})_{i,j}$.

We say that $f\colon U \to \mathbb{R}^m$ is $\mathcal{C}^\infty$ or *smooth* if all partial derivatives exist (to all orders) and are continuous.

Recall also that $f \in \mathcal{C}^1$ implies f differentiable.

Definition 8.1. $X \subseteq \mathbb{R}^n$ not necessarily open, $f\colon X \to \mathbb{R}^m$ is *differentiable* or $\mathcal{C}^\infty$ if for all $x \in X$ we can find an open nbhd $x \in U \subseteq \mathbb{R}^n$ and a differentiable or $\mathcal{C}^\infty$ function $F\colon U \to \mathbb{R}^m$ such that $F|_{X\cap U} = f|_{X\cap U}$.

Definition 8.2. Let $U \subseteq \mathbb{C}^n$ open. We say that $f\colon U \to \mathbb{C}$ is *complex differentiable* at $a \in U$ if it is continous and

i) if $n = 1$, then the limit $f'(a) := \lim_{z\to a} \frac{f(z)-f(a)}{z-a}$ exists.

ii) if $n > 1$, then for all $1 \leqslant k \leqslant n$ and for all $z_1, \ldots, \hat{z_k}, \ldots, z_n$ the function
$$f_k(z) := f(z_1, \ldots, z_{k-1}, z, z_{k+1}, \ldots, z_n)$$
is complex differentiable as in 1).

Example 8.3. $\mathrm{Re}, \mathrm{Im}\colon \mathbb{C} \to \mathbb{R}$ are $\mathcal{C}^\infty$, not complex differentiable.

If f is complex differentiable, then $\frac{\partial f}{\partial x_i}, \frac{\partial f}{\partial y_i}$ exist and are continuous, i.e. $f \in \mathcal{C}^1$.

Proposition 8.4. *Let $U \subseteq \mathbb{C}^n$ open and $f\colon U \to \mathbb{C}$ is analytic. Then f is complex differentiable and f' is analytic.*

Proof. $f(z_1, \ldots, z_n) = \sum_r c_r(z-a)^r$ locally near $a \in U$. Continuity is OK, we need to show that

$$\sum c_r(\zeta_1 - a_1)^{r_1} \cdots (\zeta_{n-1} - a_{n-1})^{r_{n-1}}(z_n - a_n)^{r_n}$$

is complex differentiable for all $(\zeta_1, \ldots, \zeta_n)$ with respect to z_n. This is a power series in a single variable, hence complex differentiable. □

Corollary 8.5. *f analytic implies f smooth.*

Proof. f analytic implies f' analytic, hence f'' analytic, and so on. □

8.2 Implicit function theorem

Notation 8.6. We denote by $\mathbb{C}[[z_1, \dots, z_n]]$ the ring of formal power series in $z_1, \dots, z_n$ centered at 0 (with Cauchy product as multiplication) and by $\mathbb{C}\{z_1, \dots, z_n\}$ the subring of all power series convergent in a nbd of 0.

Lemma 8.7. [5] *Let $f(z_0, \dots, z_n) = \sum_r c_r z^r$ with $c_{0,\dots,0} \neq 0$. Then $\exists! g \in \mathbb{C}[[z_1, \dots, z_n]]$ such that $fg = 1$.*

Proof. WLOG $c_{0,\dots,0} = 1$. Let $z = (z_1, \dots, z_{n-1})$ and $w = z_n$. Let $f = \sum_k a_k(z) w^k$ with $a_k(z) \in \mathbb{C}[[z]]$. Let $g = \sum_k b_k(z) w^k$ with $b_k(z) \in \mathbb{C}[[z]]$. Then $fg = \sum_{l=0}^{\infty} (\sum_{k=0}^{l} a_k b_{l-k}) w^l$. Call $\delta_{0,l} = \sum_{k=0}^{l} a_k b_{l-k}$. $l = 0$ implies $1 = a_0 b_0$, hence $b_0 = a_0^{-1}$. For $l > 0$ we get $0 = \sum_{k=0}^{l} a_k b_{l-k} a_k b_{l-k}$, hence

$$b_l = -a_0^{-1} \sum_{k=1}^{l} a_k b_{l-k}.$$

□

Theorem 8.8 (Formal implicit function theorem)**.** [6] *If $f \in \mathbb{C}[[z, w]]$, $f(0) = 0$ and $\frac{\partial f}{\partial w} \neq 0$, then $\exists! u \in \mathbb{C}[[z, w]]$ and $\exists! r \in \mathbb{C}[[z]]$ such that $uf = w - r$ and $u(0) \neq 0$.*

Proof. 1) Consider the $\mathbb{C}$-linear maps

$$\begin{aligned} R\colon \mathbb{C}[[z, w]] &\to \mathbb{C}[[z]] \\ p = \sum_r c_r(z) w^r &\mapsto c_0(z) \end{aligned}$$

$$\begin{aligned} H\colon \mathbb{C}[[z, w]] &\to \mathbb{C}[[z, w]] \\ p = \sum_r c_r w^r &\mapsto \frac{p - R(p)}{w} = \sum_{r>0} c_r w^{r-1} \end{aligned}$$

2) $\forall p$ we have $p = wH(p) + R(p)$.

3) $H(f)$ is a unit by the previous lemma and $\frac{\partial f}{\partial w} \neq 0$.

[5] Typical proof with power series: do the naive thing inducting on the degree and the number of variables.

[6] Near 0 we have $f(z, w) = 0$ iff $0 = u(z, w) f(z, w)$ iff $0 = w - r(z)$, i.e. the vanishing locus of f is the graph of r near 0, hence this is indeed an implicit function theorem.

4) Suffices to find u s.t.

$$0 = H(w - uf) = H(w) - H(uf)$$

$$0 = 1 - H(uf) \quad (*)$$

Indeed, $(*)$ and 2) imply that $R(w-uf) = w-uf$, hence $r := w-uf \in \mathbb{C}[[z]]$ $(w = uf + (w - uf) = uf + r)$.

5)

$$uf \overset{2)}{=} u(wH(f) + R(f))$$

$$= uwH(f) + uR(f)$$

$$(*) \Leftrightarrow 0 = 1 - H(uf)$$

$$0 = 1 - H(uwH(f) + uR(f))$$

$$0 = 1 - (\frac{uwH(f) - R(uwH(f))}{w}) - H(uR(f))$$

$$(**)0 = 1 - uH(f) - H(uR(f))$$

6) $H(f)$ unit $\Rightarrow$ suffices to find $v = uH(f)$. $\mu := -R(f)H(f)^{-1}$ s.t. $u = vH(f)^{-1}$ satisfies $(**)$.

7)

$$(**) \Leftrightarrow 0 = 1 - v + H(\frac{-uR(f)}{H(f)}H(f))$$

$$(***)0 = 1 - v + H(\mu v)$$

8) $M \colon \mathbb{C}[[z,w]] \to \mathbb{C}[[z,w]]$ defined as $p \mapsto H(\mu p)$. Note: if z^k divides p, then z^{k+1} divides $M(p)$.

9)

$$0 = 1 - v + M(v)$$

$$(***')v = 1 + M(v)$$

$\mathbb{C}$-linearity of M implies that $v = 1+M(v) = 1+M(1+M(v))$. Hence for all $k \geqslant 0$ we have

$$v = 1 + M(1) + \cdots + M^k(1) + M^k(v).$$

10) z^k divides $M^k(1)$, z^{k+1} divides $M^{k+1}(v)$. Hence $\sum_k M^k(1)$ is convergent as a formal power series.

11) If v satisfying $(***')$, then $v = \sum_k M^k(1)$. Hence v is unique and thus so are u, r.

12) $v = \sum_k M^k(1)$ satisfies $(***')$. $v = 1 + M(1) + \cdots + M^k(1) + W_k$, hence $z^{k+!}$ divides W_k by 10). $v - 1 - M(v) = 1 + M(1) + \cdots + M^k(1) + W_k - 1 - M(1) - \cdots - M^k(1) - M^{k+1}(1) - M(W_k)$, hence for all k we have that z^{k+1} divides $v - 1 - M(v)$. Therefore $v - 1 - M(v) = 0$.

□

Theorem 8.9 (Convergent implicit function theorem)**.** *With notation as above, if $f \in \mathbb{C}\{z, w\}$, then $u \in \mathbb{C}\{z, w\}$ and $r \in \mathbb{C}\{z\}$.*

Proof. See Samuel-Zariski. □

Exercise 8.10. Formulate and deduce the analytic inverse and implicit function theorem.

8.3 More basic topology

Product spaces: definition and universal property. Example: $\mathbb{R}^n \cong \mathbb{R} \times \mathbb{R}^{n-1}$ with the usual topology. Non example: $\mathbb{C}^{n+m} \ncong \mathbb{C}^n \times \mathbb{C}^m$ with the Zariski topology.

Quotient spaces: definition and universal property. Examples: $S^1 = [0,1]/1 \sim= \mathbb{R}/\mathbb{Z}$ and $\mathbb{T}^2$ gluing the boundary of the square $[0,1]^2$ appropriately.

Quasi-compactness: definition. Non-example: $\mathbb{R}$. Example: $[a,b] \subseteq \mathbb{R}$. In $\mathbb{R}^n$ quasi-compact iff closed bounded. More facts: union of qc is qc, continuous image of qc is qc, closed subset of qc is qc.

9 [WS] Kodaira 2 (Vera) - Introduction to Hodge Manifolds - 30.10.19

Outline for today:

1) Recollections.

2) Hodge Manifolds.

3) Outlook: Kodaira's Embedding Theorem.

9.1 Recollections

9.1.1 Cohomology Theories

1. Recall the *Dolbeault-Cohomology* defined as

$$H^{p,q}(X) := \frac{\ker(\bar{\partial}\colon \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p,q+1}(X))}{\operatorname{im}(\bar{\partial}\colon \mathcal{A}^{p,q-1}(X) \to \mathcal{A}^{p,q}(X))}.$$

2. Recall the *de Rham cohomology* defined as

$$H^k_{dR}(X, \mathbb{C}) = \frac{\ker(d\colon \mathcal{A}^k_{\mathbb{C}}(X) \to \mathcal{A}^{k+1}_{\mathbb{C}}(X))}{\operatorname{im}(d\colon \mathcal{A}^{k-1}_{\mathbb{C}}(X) \to \mathcal{A}^k_{\mathbb{C}}(X))},$$

where $\mathcal{A}^k_{\mathbb{C}} = \Gamma(X, \Lambda^k TX^* \otimes \mathbb{C})$.

Theorem 9.1. *On a compact Kaehler manifold* $(X, I, < \cdot, \cdot >)$ *we have*

$$H^k_{dR}(X, \mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}(X).$$

On the other hand, we have

Theorem 9.2 (de Rham Theorem)**.**

$$H^k_{dR}(X, \mathbb{C}) \cong H^k(X, \mathbb{C}).$$

3. Integral cohomology.

Definition 9.3. Let X be a compact complex manifold. A closed differential form φ on X is called *integral* if its cohomology class $[\varphi] \in H^k_{dR}(X, \mathbb{C})$ is in the image of the mapping

$$H^k(X, \mathbb{Z}) \to H^k(X, \mathbb{C}) \cong H^k_{dR}(X, \mathbb{C}).$$

Today we will be interested in integral $(1,1)$-classes, i.e.

$$\begin{tikzcd} & H^2(X,\mathbb{Z}) \arrow[dl, "j_*"'] \arrow[d] \\ H^{1,1}(X) \subseteq H^2_{dR}(X,\mathbb{C}) \arrow[r, "\cong"] & H^2(X,\mathbb{C}) \end{tikzcd}$$

We denote $\tilde{H}^2_{dR}(X, \mathbb{Z}) := j_* H^2(X, \mathbb{Z})$.

9.1.2 Holomorphic line bundles and first Chern class

Recall the exponential sequence

$$0 \to \mathbb{Z} \to \mathcal{O} \xrightarrow{\exp} \mathcal{O}^\times \to 0$$

of sheaves. Last week we have seen that this sequence is exact and thus it induces a long exact sequence of cohomology groups

$$\cdots \to \check{H}^1(X, \mathcal{O}^\times) \xrightarrow{\delta} \check{H}^2(X, \mathbb{Z}) \to \cdots$$

Fact 1: $\check{H}^1(X, \mathcal{O}^\times)$ classifies holomorphic line bundles up to holomorphic isomorphism.

Fact 2: In the sequence above, δ is "almost" c_1.

$$\begin{tikzcd} H^1(X, \mathcal{O}^\times) \arrow[r, "\delta"] \arrow[dr, "c_1"'] & H^2(X, \mathbb{Z}) \arrow[d] \\ & H^2_{dR}(X, \mathbb{R}) \end{tikzcd}$$

i.e. holomorphic line bundles have "integral Chern classes".

Remark 9.4. Indeed, the Lefschetz $(1,1)$-theorem states that on compact Kaehler manifolds X we have

$$c_1(H^1(X, \mathcal{O}^\times)) = \tilde{H}^2(X, \mathbb{Z}) \cap H^{1,1}(X).$$

In the proof of the previous commutative diagram, an explicit formula for the $(1,1)$-form representing $c_1(E, h)$, namely

$$\frac{1}{2\pi i} \partial\bar{\partial} \log(h_\alpha),$$

where h_α is the metric on U_α.

9.1.3 Kaehler manifolds

Definition 9.5. Given a complex manifold (M, I, g) where the complex structure I is compatible with the metric g, we associate a $(1,1)$-form

$$w(\cdot, \cdot) = g(I\cdot, \cdot)$$

called the *Kaehler form.*

We say that M is Kaehler, if it admits some metric such that the associated Kaehler form is closed ($dw = 0$).

In that case, $w \in \mathcal{A}^{1,1}(M)$ defines a cohomology class $[w] \in H^{1,1}(M)$, called the *Kaehler class.*

Example 9.6 (The Fubini-Study metric on $\mathbb{CP}^n$)**.** In homogeneous coordinates, the corresponding Kaehler form is given as

$$w_a = i\partial\bar{\partial}\log(|\zeta_a|^2 + 1)$$

on $U_a = \{(z^0 : \ldots : z^n) \mid z^a \neq 0\}$. The map $\zeta_a \colon U_a \to \mathbb{C}^n$ sends $(z^0 : \ldots : z^n) \mapsto (\frac{z^0}{z^a}, \ldots, \hat{1}, \ldots, \frac{z^n}{z^a})$.

From this we can also deduce the same for any projective complex manifold using the following:

Lemma 9.7. *Let (X, I, g) be a Kaehler manifold with Kaehler form w and let M be a complex submanifold. Then g induces a Kaehler manifold on M, thus M is a Kaehler manifold.*

9.2 Hodge Manifolds

Definition 9.8. Let (X, h) be a Kaehler manifold with Kaehler metric h and let w be the associated Kaehler form. If w is integral, it is called a *Hodge form* on X and h is called a *Hodge metric.* A Kaehler manifold is called *Hodge manifold* if it admits a Hodge metric.

Example 9.9 (Complex projective space)**.** Let $\mathbb{CP}^n$ be endowed with the Fubini-Study metric g_{FS} and the associated Kaehler form w_{FS}. Then w_{FS} is a Hodge form. Indeed, let $E \to \mathbb{CP}^n$ be the tautological line bundle. By Fact 2, $c_1(E) \in \tilde{H}^2(X, \mathbb{Z})$. And we have

$$\left[\frac{1}{2\pi} w_{FS}\right] = -c_1(E).$$

Thus $\mathbb{CP}^n$ is a Hodge manifold.

Example 9.10 (Projective complex manifolds)**.** Let X be a projective complex manifold, i.e. $X \subseteq \mathbb{CP}^n$ as closed complex submanifold. The restriction of the Hodge form on $\mathbb{CP}^n$ is a Hodge form on X.

Fundamental fact: all projective complex manifolds are Hodge.

Example 9.11 (Compact connected Riemann surfaces)**.** Let X be a compact connected Riemann surface. The claim is that X is a Hodge manifold. We have $H^2(X, \mathbb{C}) \cong H_0(X, \mathbb{C}) \cong \mathbb{C}$. Moreover, $H^2(X, \mathbb{C}) \cong H^{1,1}(\mathbb{C})$. Let $\tilde{w}$ be the Kaehler form associated to some metric on X. Then $[\tilde{w}] \in H^{1,1}(\mathbb{C})$ generates $H^2_{dR}(X, \mathbb{C})$. Let $c := \int_X \tilde{w}$. Then $w := \frac{1}{c}\tilde{w}$ is an integral positive form on X of type $(1, 1)$.

9.3 Outlook

We will soon prove the converse of our fundamental fact, namely:

Theorem 9.12 (Kodaira's Embedding)**.** *Every Hodge manifold admits a closed immersion into* $\mathbb{CP}^n$ *for* n *sufficiently large.*

Combining this with Chow's theorem seen last time, we obtain as a corollary that Hodge manifolds are always algebraic.

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