Various lecture notes

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Contents

1 [CM] Talk 1 (Johan Commelin): Condensed Sets - 21.10.19

Motivation: topological abelian groups do not form an abelian category.

Example 1.1. $\mathbb{R}_{disc} \to \mathbb{R}$ is epi and mono, but not iso.

Another motivation is coherent duality:

Theorem 1.2. *Let $f: X \to Y$ be a proper or quasi-projective morphism of Noetherian schemes of finite Krull dimension. Then there exists a right adjoint $f^!$ to the derived direct image functor $f_! = Rf_*: \mathcal{D}^{\mathrm{b}}(\mathcal{QC}oh(X)) \to \mathcal{D}^{\mathrm{b}}(\mathcal{QC}oh(Y))$.*

At some point analytic rings will come up. We will then look at the category of solid modules, in which the 6-functor formalism works nicer than in the classical setting (e.g. when $f_!$ is not defined in the classical setting, $f_!$ takes non-discrete values in the condensed settings, which are "not there" in the classical setting).

Definition 1.3. Proétale site of a point, denoted $*_{pro\acute{e}t}$, is the category of profinite sets with finite jointly surjective families of continuous maps as covers. A *condensed set* (resp. group, ring, ...) is a sheaf of sets (resp. groups, rings, ...) on $*_{pro\acute{e}t}$. We denote by $\mathrm{Cond}(\mathcal{C})$ the category of condensed objects of a category $\mathcal{C}$.

Definition 1.4. A *condensed set* (resp. group, ring, ...) is a contravariant functor X from $*_{proét}$ to the category of sets (resp. groups, rings, ...) such that

i) $X(\varnothing) = *$.

ii) For all profinite sets S_1 and S_2 the natural map
$$X(S_1 \sqcup S_2) \to X(S_1) \times X(S_2)$$
is an isomorphism.

iii) For any surjection of profinite sets $f\colon S' \twoheadrightarrow S$ we get an induced[1] isomorphism
$$X(S) \to \{x \in X(S') \mid \pi_1^*(x) = \pi_2^*(x) \in X(S' \times_S S')\}$$

We will call $X(*)$ the *underlying object* in $\mathcal{C}$ of a condensed object.

Remark 1.5. We will use T for topological spaces vs. X, Y for condensed sets, as opposed to Scholze's mixing of those notations.

1.1 Recollections on sheaves on sites

Let F be a presheaf on a site, which is just a contravariant functor to whatever category in which our sheaves are gonna take values. If $U = \cup_i U_i$ is an open cover, the topological sheaf axiom could be phrased as: $F(U)$ is an equalizer of the diagram
$$\prod_i F(u_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j).$$

Note that $U_i \cap U_j$ is just the fiber product of the two inclusions.

Definition 1.6 (Coverage)**.** See definition 2.1 in nCat.

Definition 1.7. F a presheaf on $\mathcal{C}$. A collection $(s_i) \in \prod_i F(U_i)$ for $\{f_i\colon U_i \to U\}$ a covering is called a *matching family* if for all $h\colon V \to U$ we have $g^*(s_i) = h^*(s_j)$ for g and h in the diagram

$$\begin{tikzcd} V \arrow[r, "h"] \arrow[d, "g"] & U_j \arrow[d, "f_j"] \\ U_i \arrow[r, "f_i"] & U \end{tikzcd}$$

[1]Since the pullback diagram is commutative, the image of $X(f)$ is indeed induces a morphism as claimed.

Definition 1.8. F is a sheaf with respect to $\{U_i \to U\}$ if for all matching families (s_i) there exists a unique $s \in F(U)$ such that $f_i^*(s) = s_i$. We say that F is a *sheaf* if it is a sheaf for all covering families.

Remark 1.9. A sheaf of abelian groups is just a commutative group object in the category of sheaves of sets.

Theorem 1.10. *If* $\mathcal{C}$ *is a site, then* $\mathcal{A}b(\mathcal{C})$ *is an abelian category.*

Definition 1.11. An additive category is a category in which the hom-sets are endowed with an abelian group structure in a way that makes composition bilinear and such that finite biproducts exist.

Recall Grothendieck's axioms: AB1) Every morphism has a kernel and a cokernel. AB2) For every $f\colon A \to B$, the natural map $\operatorname{coim}(f) \to \operatorname{im} f$ is an iso. AB3) All colimit exist. AB4) AB3) + arbitrary direct sums are exact. AB5) AB3) + arbitrary filtered colimits are exact. AB6) AB3) + J an index set, $\forall j \in J$ a filtered category (think of directed set) I_j, functors $M\colon I_j \to \mathcal{C}$, then

$$\varinjlim_{(i_j \in I_j)_j} \prod_j M_{i_j} \to \prod_{j \in J} \varinjlim_{i_j \in I_j} M_{i_j}$$

Theorem 1.12. $\mathcal{C}$ *a site. Then* $\mathcal{A}b(\mathcal{C})$ *satisfies AB3), AB4), AB5) and AB6).*

In fact, our case is even nicer:

Theorem 1.13. $\mathrm{Cond}(\mathcal{A}b)$ *in addition satisfies AB6) and AB4*).*

1.2 Compactly generated topological spaces

Definition 1.14. A topological space T is called *compactly generated* if any function $f\colon T \to T'$ is continuous as soon as the composite $S \to T \to T'$ is continuous for all maps $S \to T$ where S is compact and Hausdorff. See also nCat.

The inclusion functor $\mathcal{CG} \hookrightarrow \mathcal{T}op$ has a right adjoint $(-)^{cg}$. If T is any topological space, then the topology on T^{cg} is the finest topology on T such that $\sqcup_{S \to T} S \to T$ is continuous, where S ranges over all compact Hausdorff spaces.

Let T be a topological space. We view T as a presheaf on $*_{pro\acute{e}t}$ by setting $T(S) = \mathrm{Hom}_{\mathcal{T}op}(S, T)$ for all profinite sets S. We denote this by $\underline{T}$. Claim: $\underline{T}$ is a sheaf.

i) The first condition $\underline{T}(\varnothing) = *$ is true, because there is exactly one morphism from the empty set to any topological space.

ii) $\underline{T}(S_1 \sqcup S_2) = \underline{T}(S_1) \times \underline{T}(S_2)$ by universal property of disjoint union.

iii) For any surjection $S' \twoheadrightarrow S$ we get an isomorphism

$$\underline{T}(S) \to \{x \in \underline{T}(S') \mid \pi_1^*(x) = \pi_2^*(x) \in \underline{T}(S' \times_S S')\}$$

Since $\mathcal{T}op \to \mathrm{Cond}(\mathcal{S}et)$ preserves products, group objects are preserved, so it maps topological groups to condensed groups etc.

Proposition 1.15. *i) This functor is faithful and fully faithful when restricted to the full subcategory of compactly generated spaces.*

ii) It admits a left adjoint $X \mapsto X(*)_{top}$ where $X(*)_{top}$ gets the quotient topology of $\sqcup_{S \to X} S \to X(*)$ as above. The counit $I(*)_{top} \to T$ agrees with $T^{cg} \to T$.

Coming back to our original example:

Example 1.16. $\mathbb{R}_{disc} \to \mathbb{R}$ can be seen in the condensed world as $\underline{\mathbb{R}_{disc}} \to \underline{\mathbb{R}}$, i.e. from locally constant functions to continuous functions. This is still a mono, but now it is not an epi. The cokernel Q can be described as $Q(S) = \{S \to \mathbb{R}$ continuous $\}/\{S \to \mathbb{R}$ locally constant $\}$. Note in particular that the underlying set of Q is just $*$, reflecting the fact that the cokernel was trivial in the classical setting.

References