# Various lecture notes

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# 1 [CM] Talk 1 (Johan Commelin): Condensed Sets - 21.10.19

Motivation: topological abelian groups do not form an abelian category.

**Example 1.1.** $\mathbb{R}_{disc} \to \mathbb{R}$ is epi and mono, but not iso.

Another motivation is coherent duality:

**Theorem 1.2.** *Let $f\colon X \to Y$ be a proper or quasi-projective morphism of Noetherian schemes of finite Krull dimension. Then there exists a right adjoint $f^!$ to the derived direct image functor $f_! = Rf_*\colon \mathcal{D}^{\mathrm{b}}(\mathcal{QC}oh(X)) \to \mathcal{D}^{\mathrm{b}}(\mathcal{QC}oh(Y))$.*

At some point analytic rings will come up. We will then look at the category of solid modules, in which the 6-functor formalism works nicer than in the classical setting (e.g. when $f_!$ is not defined in the classical setting, $f_!$ takes non-discrete values in the condensed settings, which are "not there" in the classical setting).

**Definition 1.3.** Proétale site of a point, denoted $*_{proét}$, is the category of profinite sets with finite jointly surjective families of continuous maps as covers. A *condensed set* (resp. group, ring, ...) is a sheaf of sets (resp.

groups, rings, ...) on $*_{pro\acute{e}t}$. We denote by Cond($\mathcal{C}$) the category of condensed objects of a category $\mathcal{C}$.

**Definition 1.4.** A *condensed set* (resp. group, ring, ...) is a contravariant functor $X$ from $*_{pro\acute{e}t}$ to the category of sets (resp. groups, rings, ...) such that

i) $X(\varnothing) = *$.

ii) For all profinite sets $S_1$ and $S_2$ the natural map

$$X(S_1 \sqcup S_2) \to X(S_1) \times X(S_2)$$

is an isomorphism.

iii) For any surjection of profinite sets $f\colon S' \twoheadrightarrow S$ we get an induced[1] isomorphism

$$X(S) \to \{x \in X(S') \mid \pi_1^*(x) = \pi_2^*(x) \in X(S' \times_S S')\}$$

We will call $X(*)$ the *underlying object* in $\mathcal{C}$ of a condensed object.

*Remark* 1.5. We will use $T$ for topological spaces vs. $X, Y$ for condensed sets, as opposed to Scholze's mixing of those notations.

## 1.1 Recollections on sheaves on sites

Let $F$ be a presheaf on a site, which is just a contravariant functor to whatever category in which our sheaves are gonna take values. If $U = \cup_i U_i$ is an open cover, the topological sheaf axiom could be phrased as: $F(U)$ is an equalizer of the diagram

$$\prod_i F(u_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j).$$

Note that $U_i \cap U_j$ is just the fiber product of the two inclusions.

**Definition 1.6** (Coverage)**.** See definition 2.1 in nCat.

**Definition 1.7.** $F$ a presheaf on $\mathcal{C}$. A collection $(s_i) \in \prod_i F(U_i)$ for $\{f_i\colon U_i \to U\}$ a covering is called a *matching family* if for all $h\colon V \to U$ we have $g^*(s_i) = h^*(s_j)$ for $g$ and $h$ in the diagram

[1] Since the pullback diagram is commutative, the image of $X(f)$ is indeed induces a morphism as claimed.

$$\begin{tikzcd} V \arrow[r, "h"] \arrow[d, "g"] & U_j \arrow[d, "f_j"] \\ U_i \arrow[r, "f_i"] & U \end{tikzcd}$$

**Definition 1.8.** $F$ is a sheaf with respect to $\{U_i \to U\}$ if for all matching families $(s_i)$ there exists a unique $s \in F(U)$ such that $f_i^*(s) = s_i$. We say that $F$ is a *sheaf* if it is a sheaf for all covering families.

*Remark* 1.9. A sheaf of abelian groups is just a commutative group object in the category of sheaves of sets.

**Theorem 1.10.** *If* $\mathcal{C}$ *is a site, then* $\mathcal{A}b(\mathcal{C})$ *is an abelian category.*

**Definition 1.11.** An additive category is a category in which the hom-sets are endowed with an abelian group structure in a way that makes composition bilinear and such that finite biproducts exist.

Recall Grothendieck's axioms: AB1) Every morphism has a kernel and a cokernel. AB2) For every $f \colon A \to B$, the natural map $\mathrm{coim}(f) \to \mathrm{im}\, f$ is an iso. AB3) All colimit exist. AB4) AB3) + arbitrary direct sums are exact. AB5) AB3) + arbitrary filtered colimits are exact. AB6) AB3) + $J$ an index set, $\forall j \in J$ a filtered category (think of directed set) $I_j$, functors $M \colon I_j \to \mathcal{C}$, then

$$\varinjlim_{(i_j \in I_j)_j} \prod_j M_{i_j} \to \prod_{j \in J} \varinjlim_{i_j \in I_j} M_{i_j}$$

**Theorem 1.12.** $\mathcal{C}$ *a site. Then* $\mathcal{A}b(\mathcal{C})$ *satisfies AB3), AB4), AB5) and AB6).*

In fact, our case is even nicer:

**Theorem 1.13.** $\mathrm{Cond}(\mathcal{A}b)$ *in addition satisfies AB6) and AB4*).*

## 1.2 Compactly generated topological spaces

**Definition 1.14.** A topological space $T$ is called *compactly generated* if any function $f \colon T \to T'$ is continuous as soon as the composite $S \to T \to T'$ is continuous for all maps $S \to T$ where $S$ is compact and Hausdorff. See also nCat.

The inclusion functor $\mathcal{CG} \hookrightarrow \mathcal{T}op$ has a right adjoint $(-)^{cg}$. If $T$ is any topological space, then the topology on $T^{cg}$ is the finest topology on $T$ such that $\sqcup_{S\to T} S \to T$ is continuous, where $S$ ranges over all compact Hausdorff spaces.

Let $T$ be a topological space. We view $T$ as a presheaf on $*_{pro\acute{e}t}$ by setting $T(S) = \mathrm{Hom}_{\mathcal{T}op}(S,T)$ for all profinite sets $S$. We denote this by $\underline{T}$. Claim: $\underline{T}$ is a sheaf.

i) The first condition $\underline{T}(\varnothing) = *$ is true, because there is exactly one morphism from the empty set to any topological space.

ii) $\underline{T}(S_1 \sqcup S_2) = \underline{T}(S_1) \times \underline{T}(S_2)$ by universal property of disjoint union.

iii) For any surjection $S' \twoheadrightarrow S$ we get an isomorphism

$$\underline{T}(S) \to \{x \in \underline{T}(S') \mid \pi_1^*(x) = \pi_2^*(x) \in \underline{T}(S' \times_S S')\}$$

Since $\mathcal{T}op \to \mathrm{Cond}(\mathcal{S}et)$ preserves products, group objects are preserved, so it maps topological groups to condensed groups etc.

**Proposition 1.15.** *i) This functor is faithful and fully faithful when restricted to the full subcategory of compactly generated spaces.*

*ii) It admits a left adjoint $X \mapsto X(*)_{top}$ where $X(*)_{top}$ gets the quotient topology of $\sqcup_{S\to X} S \to X(*)$ as above. The counit $I(*)_{top} \to T$ agrees with $T^{cg} \to T$.*

Coming back to our original example:

**Example 1.16.** $\mathbb{R}_{disc} \to \mathbb{R}$ can be seen in the condensed world as $\underline{\mathbb{R}_{disc}} \to \underline{\mathbb{R}}$, i.e. from locally constant functions to continuous functions. This is still a mono, but now it is not an epi. The cokernel $Q$ can be described as $Q(S) = \{S \to \mathbb{R} \text{ continuous }\}/\{S \to \mathbb{R} \text{ locally constant }\}$. Note in particular that the underlying set of $Q$ is just $*$, reflecting the fact that the cokernel was trivial in the classical setting.

# 2 [LT] Lecture 1 - 22.10.19

Today: big picture.

An *algebraic variety* is the solution set of a family of polynomial equations in $\mathbb{C}^n$. For example, if $f(x,y,z,t) = xy - tz$, then

$$\mathbb{V}(f) = \{(x,y,z,t) \in \mathbb{C}^4 \mid xy - tz = 0\}$$

is an algebraic variety in $\mathbb{C}^4$. Another example would be the parabola $\{y - x^2 = 0\} \subseteq \mathbb{C}^2$.

We can think of $\mathbb{V}(f)$ as a family of varieties parametrized by the variable $t$. For $t = 1$ we get the equation $xy - z = 0$ in $\mathbb{C}^3$. We can perform a change of coordinates $(x, y) \mapsto (x + iy, x - iy)$ to turn our equation into $x^2 + y^2 = z$. For $z = 0$, the variety $X_0 = \{x^2 + y^2 = 0\}$ has an *ordinary double point* at the origin [picture: cone] (a.k.a. node if we think of $X_0$ as a curve[2]. These are a particularly nice kind of singularities[3]. For $z \neq 0$ we get the equation $xy - 1 = 0$. This is a ruled surface $X_z$ [picture: chimeny of nuclear plant with a loop $\gamma$ at its base]. As $z \mapsto 0$, the central loop $\gamma$ contract to the ordinary double point. We have a projection $\pi \colon \mathbb{V}(f) \to \mathbb{C}$, and Ehresmann's lemma tells us that for all disk $D \subseteq \mathbb{C}$ not containing 0 we have $\pi^{-1}(D) \cong D \times X_{z_0}$ for any $z_0 \in D$.

Global picture: given an arbitrary nonsingular alg. variety $X \subseteq \mathbb{C}^n$, can we find a map $\pi \colon X \to \mathbb{C}$ such that the fibres $X_t$ are nonsingular for all but finitely many $t \in \mathbb{C}$ and such that the singular fibres have at worst ODP singularities?

Problem: we are missing information "at infinity", e.g. $y = x^2$ versus $xy = 1$. The solution is to replace $\mathbb{C}^n$ by $\mathbb{CP}^n$.

Let $X \subseteq \mathbb{P}^n$ be a nonsingular projective variety.

**Theorem 2.1.** *There exists a family $(H_t)_{t \in \mathbb{CP}^1}$ of hyperplanes in $\mathbb{CP}^n$ such that*

1. *$X \subseteq \cup_t H_t$.*
2. *$X_t = X \cap H_t$ is nonsingular except for finitely many "critical values" of $t$.*
3. *$X_t$ has ODP singularities for each critical value $t$.*

We call $(X_t)_t$ a *Lefschetz pencil.* We get a rational map $X \mapsto \mathbb{CP}^1$ sending $x \mapsto t$ whenever $x \in X_t$. This is not well-defined at $x \in \cap_t X_t$, but we can arrange for $(X_t)_t$ so that $\cap_t X_t = X_0 \cap X_\infty$. Blowing-up a suitable subvariety of $X$ we get maps $\tilde{X} \xrightarrow{\pi} \mathbb{CP}^1$ and $\tilde{X} \to X$ as we wanted. [picture A]

Some applications:

**Theorem 2.2** (Lefschetz Hyperplane theorem)**.** *$X \subseteq Y \subseteq \mathbb{CP}^N$ nonsingular varieties with $X$ a hypersurface in the $n$-dimensional variety $Y$, then*

$$H_*(X) \to H_*(Y)$$

[2]These are 1-dimensional complex varieties, so topologically they are surfaces.

[3]Singularities will appear naturally while studying the topology of algebraic varieties.

*is an isomorphism for* $* < n-1$ *and a surjection for* $* = n-1$.

In particular, if $Y = \mathbb{CP}^n$, we have

$$H_*(\mathbb{CP}^n) = \mathbb{Z} \text{ if } * \text{ is even or 0 otherwise.}$$

If $X \subseteq \mathbb{CP}^n$ is a nonsingular hypersurface, then its homology will be that of projective sapce on all digrees other than $n-1$. Its $n-1$ homology will depend on the variety, e.g. the ODP (trivial 1-homology) vs the ruled surface (with $\gamma$ non trivial on 1-homology) of before.

**Example 2.3** (Lefschetz pencil)**.** $X$ elliptic curve in $\mathbb{CP}^2$ given by $y^2 = x(x-1)(x-\lambda)$ for $\lambda \neq 0$. $L = \mathbb{CP}^1 \subseteq \mathbb{CP}^2$. $P \in \mathbb{CP}^1 \setminus (X \cup L)$. We get $X \xrightarrow{\pi} \mathbb{CP}^1$ once we choose a square root of $x(x-1)(x-\lambda)$. [Picture B]

# References