Various lecture notes

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Contents

1 About these notes

The purpose of these notes is to keep the material seen in lectures a bit organized and easily accesible from one single place, but they don't intend to be complete and they will surely be full of typos and mistakes[1].

Warnings will be marked with a dangerous bend symbol on the margin .

2 [CM] Talk 1 (Johan Commelin): Condensed Sets - 21.10.19

2.1 Introduction

One of the main motivations for condensed mathematics is that topological algebraic objects have usually poor categorical and functorial properties. For instance, topological abelian groups do not form an abelian category:

Example 2.1 $\mathbb{R}_{disc} \to \mathbb{R}$ is epi and mono, but not iso.

Another motivation is coherent duality:

Theorem 2.2 *Let $f\colon X \to Y$ be a proper or quasi-projective morphism of Noetherian schemes of finite Krull dimension. Then there exists a right adjoint $f^!$ to the derived direct image functor $f_! = Rf_*\colon \mathcal{D}^{\mathrm{b}}(\mathcal{QC}oh(X)) \to \mathcal{D}^{\mathrm{b}}(\mathcal{QC}oh(Y))$.*

At some point analytic rings will come up. We will then look at the category of solid modules, in which the 6-functor formalism works nicer than in the classical setting (e.g. when $f_!$ is not defined in the classical setting, $f_!$ takes non-discrete values in the condensed settings, which are "not there" in the classical setting).

Definition 2.3 Proétale site of a point, denoted $*_{prot}$, is the category of profinite sets with finite jointly surjective families of continuous maps as covers. A *condensed set* (resp. group, ring, ...) is a sheaf of sets (resp. groups, rings, ...) on $*_{prot}$. We denote by $\mathrm{Cond}(\mathcal{C})$ the category of condensed objects of a category $\mathcal{C}$.

[1]If you find any, please let me know! You can do this from GitHub or write me an email directly at pedro.nunez[at]math.uni-freiburg.de.

Definition 2.4 A *condensed set* (resp. group, ring, ...) is a contravariant functor X from $*_{prot}$ to the category of sets (resp. groups, rings, ...) such that

i) $X(\varnothing) = *$.

ii) For all profinite sets S_1 and S_2 the natural map

$$X(S_1 \sqcup S_2) \to X(S_1) \times X(S_2)$$

is an isomorphism.

iii) For any surjection of profinite sets $f\colon S' \twoheadrightarrow S$ we get an induced[2] isomorphism

$$X(S) \to \{x \in X(S') \mid \pi_1^*(x) = \pi_2^*(x) \in X(S' \times_S S')\}$$

We will call $X(*)$ the *underlying object* in $\mathcal{C}$ of a condensed object.

Remark 2.5. We will use T for topological spaces vs. X, Y for condensed sets, as opposed to Scholze's mixing of those notations.

2.2 Recollections on sheaves on sites

Let F be a presheaf on a site, which is just a contravariant functor to whatever category in which our sheaves are gonna take values. If $U = \cup_i U_i$ is an open cover, the topological sheaf axiom could be phrased as: $F(U)$ is an equalizer of the diagram

$$\prod_i F(u_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j).$$

Note that $U_i \cap U_j$ is just the fiber product of the two inclusions.

Definition 2.6 (Coverage) See definition 2.1 in nCat.

Definition 2.7 F a presheaf on $\mathcal{C}$. A collection $(s_i) \in \prod_i F(U_i)$ for $\{f_i\colon U_i \to U\}$ a covering is called a *matching family* if for all $h\colon V \to U$ we have $g^*(s_i) = h^*(s_j)$ for g and h in the diagram

[2] Since the pullback diagram is commutative, the image of $X(f)$ is indeed induces a morphism as claimed.

$$\begin{array}{ccc} V & \xrightarrow{h} & U_j \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f_j} \\ U_i & \xrightarrow{f_i} & U \end{array}$$

Definition 2.8 F is a sheaf with respect to $\{U_i \to U\}$ if for all matching families (s_i) there exists a unique $s \in F(U)$ such that $f_i^*(s) = s_i$. We say that F is a *sheaf* if it is a sheaf for all covering families.

Remark 2.9. A sheaf of abelian groups is just a commutative group object in the category of sheaves of sets.

Theorem 2.10 *If* $\mathcal{C}$ *is a site, then* $\mathcal{A}b(\mathcal{C})$ *is an abelian category.*

Definition 2.11 An additive category is a category in which the hom-sets are endowed with an abelian group structure in a way that makes composition bilinear and such that finite biproducts exist.

Recall Grothendieck's axioms: AB1) Every morphism has a kernel and a cokernel. AB2) For every $f: A \to B$, the natural map $\mathrm{coim}(f) \to \mathrm{im}\, f$ is an iso. AB3) All colimit exist. AB4) AB3) + arbitrary direct sums are exact. AB5) AB3) + arbitrary filtered colimits are exact. AB6) AB3) + J an index set, $\forall j \in J$ a filtered category (think of directed set) I_j, functors $M: I_j \to \mathcal{C}$, then

$$\varinjlim_{(i_j \in I_j)_j} \prod_j M_{i_j} \to \prod_{j \in J} \varinjlim_{i_j \in I_j} M_{i_j}$$

Theorem 2.12 $\mathcal{C}$ *a site. Then* $\mathcal{A}b(\mathcal{C})$ *satisfies AB3), AB4), AB5) and AB6).*

In fact, our case is even nicer:

Theorem 2.13 $\mathrm{Cond}(\mathcal{A}b)$ *in addition satisfies AB6) and AB4*).*

2.3 Compactly generated topological spaces

Definition 2.14 A topological space T is called *compactly generated* if any function $f: T \to T'$ is continuous as soon as the composite $S \to T \to T'$ is continuous for all maps $S \to T$ where S is compact and Hausdorff. See also nCat.

The inclusion functor $\mathcal{CG} \hookrightarrow \mathcal{T}op$ has a right adjoint $(-)^{cg}$. If T is any topological space, then the topology on T^{cg} is the finest topology on T such that $\sqcup_{S \to T} S \to T$ is continuous, where S ranges over all compact Hausdorff spaces.

Let T be a topological space. We view T as a presheaf on $*_{prot}$ by setting $T(S) = \mathrm{Hom}_{\mathcal{T}op}(S, T)$ for all profinite sets S. We denote this by $\underline{T}$. Claim: $\underline{T}$ is a sheaf.

i) The first condition $\underline{T}(\varnothing) = *$ is true, because there is exactly one morphism from the empty set to any topological space.

ii) $\underline{T}(S_1 \sqcup S_2) = \underline{T}(S_1) \times \underline{T}(S_2)$ by universal property of disjoint union.

iii) For any surjection $S' \twoheadrightarrow S$ we get an isomorphism

$$\underline{T}(S) \to \{x \in \underline{T}(S') \mid \pi_1^*(x) = \pi_2^*(x) \in \underline{T}(S' \times_S S')\}$$

Since $\mathcal{T}op \to \mathrm{Cond}(\mathcal{S}et)$ preserves products, group objects are preserved, so it maps topological groups to condensed groups etc.

Proposition 2.15 *i) This functor is faithful and fully faithful when restricted to the full subcategory of compactly generated spaces.*

ii) It admits a left adjoint $X \mapsto X(*)_{top}$ where $X(*)_{top}$ gets the quotient topology of $\sqcup_{S \to X} S \to X(*)$ as above. The counit $I(*)_{top} \to T$ agrees with $T^{cg} \to T$.

Coming back to our original example:

Example 2.16 $\mathbb{R}_{disc} \to \mathbb{R}$ can be seen in the condensed world as $\underline{\mathbb{R}_{disc}} \to \underline{\mathbb{R}}$, i.e. from locally constant functions to continuous functions. This is still a mono, but now it is not an epi. The cokernel Q can be described as $Q(S) = \{S \to \mathbb{R} \text{ continuous }\}/\{S \to \mathbb{R} \text{ locally constant }\}$. Note in particular that the underlying set of Q is just $*$, reflecting the fact that the cokernel was trivial in the classical setting.

3 [LT] Lecture 1 - 22.10.19

3.1 Introduction and overview of the course

An *algebraic variety* is the solution set of a family of polynomial equations in $\mathbb{C}^n$. For example, if $f(x, y, z, t) = xy - tz$, then

$$V(f) = \{(x, y, z, t) \in \mathbb{C}^4 \mid xy - tz = 0\}$$

is an algebraic variety in $\mathbb{C}^4$. Another example would be the parabola $\{y - x^2 = 0\} \subseteq \mathbb{C}^2$.

Let us focus on $V(f)$ and set $t = 1$. Then $X = V(f) \cap \{t = 1\} = \{(x, y, z) \in \mathbb{C}^3 \mid xy = z\}$ can be seen as a family of complex curves parametrized by the variable z. For $z = 0$, the complex curve X_0 has an *ordinary double point* at the origin:

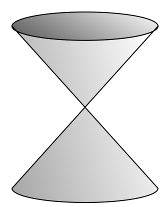

Figure 1: Topological picture of our ODP.

Singularities arise naturally while studying the topology of algebraic varieties, and ODP's are a particularly nice kind of singularities.

For $z \neq 0$ we get an equation which looks like $xy = 1$. In this case we have the following picture:

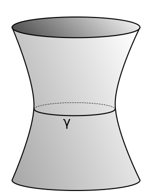

Figure 2: Topological picture of X_z.

As $z \to 0$, the central loop γ contracts to the ordinary double point. Note in particular that X_0 has trivial fundamental group (hence trivial 1-homology), whereas X_z does not.

We have a projection $\pi\colon X \to \mathbb{C}$, and Ehresmann's lemma tells us that for all disks $D \subseteq \mathbb{C}$ not containing 0 we have $\pi^{-1}(D) \cong D \times X_{z_0}$ for any $z_0 \in D$.

Question 3.1. Given an arbitrary nonsingular algebraic variety $X \subseteq \mathbb{C}^n$, can we find a map $\pi\colon X \to \mathbb{C}$ such that the fibers X_t are nonsingular for all but finitely many $t \in \mathbb{C}$ and such that the singular fibres have at worst ODP singularities?

Notice how we are missing information at infinity, e.g. $y = x^2$ versus $xy = 1$. The solution is to this is to replace $\mathbb{C}^n$ by $\mathbb{CP}^n$.

So let $X \subseteq \mathbb{P}^n$ be a nonsingular projective variety. Then we have:

Theorem 3.2 *There exists a family $(H_t)_{t\in\mathbb{CP}^1}$ of hyperplanes in $\mathbb{CP}^n$ with $H_{[a,b]} = aH_0 + bH_\infty$ such that*

1. *$X \subseteq \bigcup_{t\in\mathbb{CP}^1} H_t$.*
2. *$X_t = X \cap H_t$ is nonsingular except for finitely many critical values of t.*
3. *X_t has ODP singularities for each critical value t.*

We call $(X_t)_{t\in\mathbb{CP}^1}$ a *Lefschetz pencil*. We get a rational map $X \dashrightarrow \mathbb{CP}^1$ sending $x \mapsto t$ whenever $x \in X_t$. If $x \in X_t \cap X_{t'}$ for $t \neq t'$, then $x \in H_0 \cap H_\infty$, so this rational map is not well-defined along $X \cap H_0 \cap H_\infty$. Blowing-up this subvariety of X we resolve the indeterminacy of the rational map and get a morphism $\tilde{X} \xrightarrow{\pi} \mathbb{CP}^1$ as we wanted.

As an application we obtain:

Theorem 3.3 (Lefschetz Hyperplane theorem) *$X \subseteq Y \subseteq \mathbb{CP}^N$ nonsingular varieties with X a hypersurface in the n-dimensional variety Y, then*

$$H_*(X) \to H_*(Y)$$

is an isomorphism for $* < n - 1$ and a surjection for $* = n - 1$.

In particular, if $Y = \mathbb{CP}^n$, we have

$$H_*(\mathbb{CP}^n) = \begin{cases} \mathbb{Z} & \text{if * is even,} \\ 0 & \text{otherwise.} \end{cases}$$

If $X \subseteq \mathbb{CP}^n$ is a nonsingular hypersurface, then its homology will be that of projective sapce on all degrees other than $n-1$. Its $n-1$ homology will depend on the variety, e.g. the ODP (trivial 1-homology) vs the ruled surface (with γ non trivial on 1-homology) from before.

Example 3.4 X elliptic curve in $\mathbb{CP}^2$ given by $y^2 = x(x-1)(x-\lambda)$ for $\lambda \neq 0$. Let $L = \mathbb{CP}^1 \subseteq \mathbb{CP}^2$ and $P \in \mathbb{CP}^1 \setminus (X \cup L)$. We get $X \xrightarrow{\pi} \mathbb{CP}^1$ by projecting from P to L.

4 [WS] Kodaria 1 (Jin Li) - 23.10.19

4.1 Chow's theorem

Let $G_i = G_i(z_1, \ldots, z_n)$ be homogeneous polynomials of degree d_i for $i \in \{1, \ldots, k\}$. Let $V = V(G_1, \ldots, G_k) = \{w \in \mathbb{C}^{n+1} \setminus \{0\} \mid G_i(w) = 0 \text{ for all } i \in \{1, \ldots, k\}\} \subseteq \mathbb{CP}^n$. Assume $(\frac{\partial G_i}{\partial z_j}(w))_{i,j}$ is surjective at any $w \in V$. By Euler's theorem on homogeneous functions we have

$$\sum_{j=0}^{n} z_j \frac{\partial G_i}{\partial z_j} = d_i G_i(z_0, \ldots, z_n).$$

If $\tilde{w} = (\tilde{z}_0, \ldots, \tilde{z}_n) \in V$, then

$$\sum_{j=0}^{n} \tilde{z}_j \frac{\partial G_i}{\partial z_j}|_{\tilde{w}} = 0$$

$V \cap U_i$ for any $i \in \{0, \ldots, n\}$, $U_i = \{[z_0 : \ldots : z_n] \in \mathbb{CP}^n \mid z_i \neq 0\}$.

For $i = 0$, consider the chart (U_0, ϕ_0) with $\phi_0 \colon U_0 \to \mathbb{C}^n$ given by $[z_0, \ldots, z_n] \mapsto (\frac{z_1}{z_0}, \ldots, \frac{z_n}{z_0})$. The inverse has a lift given by $\tilde{\psi} \colon \mathbb{C}^n \to \mathbb{C}^{n+1} \setminus \{0\}$ given by $(w_1, \ldots, w_n) \mapsto (1, w_1, \ldots, w_n)$.

$$\begin{tikzcd}
\mathbb{C}^n \arrow[d, "\tilde{\psi}"] \arrow[dr, "G \circ \tilde{\psi_0}"] & \\
\mathbb{CP}^n \arrow[r, "G"] & \mathbb{C}^k
\end{tikzcd}$$

$V \cap U_0 = G^{-1}(\{0\})$.

$$G \circ \tilde{\psi}_0 \colon (w_1, \dots, w_n) \mapsto (G_1(1, w_1, \dots, w_n), \dots, G_k(1, w_1, \dots, w_n)).$$

$$\frac{\partial (G_i \circ \tilde{\psi}_0)}{\partial w_j} = \frac{\partial G_i}{\partial z_l} \frac{\partial (\tilde{\psi}_0)^l}{\partial w_j} |_{(\tilde{w_1}, \dots, \tilde{w_n})}$$

Call the LHS A_1.

$$\frac{\partial G_i}{\partial z_l} |_{\tilde{w}=(1,\tilde{w}_1,\dots,\tilde{w}_n)} \begin{pmatrix} 1 \\ \tilde{w}_1 \\ \vdots \\ \tilde{w}_n \end{pmatrix} = 0 \tag{1}$$

Note also that

$$\frac{\partial (\tilde{\psi}_0)^l}{\partial w_j} = \begin{pmatrix} 0 & \dots & 0 \\ 1 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 1 \end{pmatrix}.$$

Now

$$(\frac{\partial G_i}{\partial z_l}) = (a_{il}) = \begin{pmatrix} a_{10} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{k0} & \dots & a_{kn} \end{pmatrix}$$

$$A_1 = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{k1} & \dots & a_{kn} \end{pmatrix}$$

Since A is surjective and

$$A \begin{pmatrix} 1 \\ \tilde{w}_1 \\ \vdots \\ \tilde{w}_n \end{pmatrix} = 0,$$

hence A_1 is surjective.

Theorem 4.1 (Chow) *Every analytic closed subvariety $V \subseteq \mathbb{CP}^n$ is the zero locus of finite number of homogeneous polynomials.*

For this we will use as a black box:

Lemma 4.2 (Remmert-Stein) *Let $U \subseteq \mathbb{C}^n$ be a domain, S be an analytic subvariety of U of dimension m, and W be an analytic subvariety of $U \setminus S$ such that $\dim_p W > m$ for all regular points $p \in W$. Then $\bar{W}$ is analytic.*

Now we can prove Chow's theorem. Let $\pi\colon \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ be the projection. Then $\pi^{-1}(V)$ has dimension at least 1 everywhere in $\mathbb{C}^{n+1} \setminus \{0\}$. Moreover, $\pi^{-1}(V)$ is a cone missing the origin, so its closure is just $\pi^{-1}(V) \cup \{0\}$. Set $S = \{0\}$ and $W = \pi^{-1}(V)$. Then $V' = \bar{W} = \pi^{-1}(V) \cup \{0\}$ is an analytic variety of $\mathbb{C}^{n+1}$ by the Remmert-Stein theorem. In particular, near 0 we cam write

$$V_0' = U_\varepsilon(0) \cap V' = V(g_1, \ldots, g_k)$$

with g_i holomorphic on $U_\varepsilon(0)$. In particular each g_i is analytic, so we may write it as $g_i = \sum_{n=1}^\infty g_{i,n}$ where each $g_{i,n}$ is a homogeneous polynomial. Then $g_i(tz) = \sum_{n=1}^\infty g_{i,n}(z)t^n$ for all $x \in \mathbb{C}^{n+1}$ and all $t \in \mathbb{C}$. If $z \in V'$, then $tz \in V'$ for all t, because V' is a cone. So $g_i(tz) \equiv 0$ implies $g_{i,n}(z) = 0$ for all $i \in \{1, \ldots, k\}$ and all $n \in \mathbb{N}_{>0}$. Therefore $V_0' = V(\{g_{i,n}\})$. By Noetherianity, finitely many $g_{i,n}$ suffice, so we can write $V_0 = V(g^{(1)}, \ldots, g^{(m)})$ for some $g^{(i)} \in \{g_{i,n}\}$. Hence $V = V(g^{(1)}, \ldots, g^{(m)})$ in $\mathbb{CP}^n$ and this finishes the proof.

4.2 Sheaves

For precise definitions and results in this subsection see Wikipedia, Stacks or nLab.

Definition of *sheaf* (of abelian groups) on a topological space X, *stalk* of a sheaf at a point $x \in X$, *germs* of a sheaf at a point... Note the similarities in terminology with plants.

Example 4.3 Constant sheaves $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$. Sheaf of smooth functions $\mathcal{C}^\infty$ and its units $\mathcal{C}^*$. Sheaf of regular functions $\mathcal{O}$ and units $\mathcal{O}^*$. Sheaf of meromorphic functions $\mathcal{M}$ and $\mathcal{M}^*$.

Maps between sheaves, their kernels and their cokernels. Short exact sequences of sheaves.

Example 4.4 Let M be a complex manifold. The sequence

$$0 \to \mathbb{Z} \to \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \to 0$$

is exact.

Definition of Čech cohomology of a sheaf $\mathcal{F} \in \mathrm{Sh}(M)$ with respect to an open cover $\mathcal{U}$, which we denote by $H^p(\mathcal{U}, \mathcal{F})$ on degree p, and Čech cohomology of the sheaf F as their direct limit over refinements, denoted $\check{H}^p(M, \mathcal{F})$.

Theorem 4.5 (Leray) *If $\mathcal{U}$ is an acyclic cover, i.e. if there are no higher Čech cohomologies with espect to this cover, then the Čech complex associated to this cover computes Čech cohomology.*

Long exact sequence in Čech cohomology induced by a short exact sequence of sheaves.

4.3 A bit of Hodge theory

Decomposition of the tangent space at a point of a complex manifold, its tensor algebras, ∂ and $\bar{\partial}$ operators, Dolbeault cohomology groups, harmonic and Hodge decomposition. See [GH78] or [Voi07].

5 [LT] Lecture 2 - 24.10.19

Remark 5.1. Exercise sessions will be Thursday from 13h to 15h on SR318 (Starting next week).

As pointed out last week, we want to look at polynomials and their solutions sets. But polynomials are a bit too rigid. Instead, we look at polynomials as truncated power series, or more generally as *analytic functions*, which are functions which locally can be represented as power series. We will see that these are the same as holomorphic functions. In particular, every holomorphic function is C^∞.

$$\text{polynomials} \Rightarrow \text{convergent power series} \Rightarrow \text{analytic} \Leftrightarrow \\ \Leftrightarrow \text{holomorphic} \Rightarrow C^\infty \Rightarrow \text{continuous} \Rightarrow \text{abominations}$$

If we were analysts we would start at the bottom and then try to swim up. Instead we will start from the top and float downstream.

Notation 5.2. $\mathbb{E} = \mathbb{R}$ or $\mathbb{C}$. $z = (z_1, \ldots, z_n) \in \mathbb{E}^n$, $r \in \mathbb{R}_{\geqslant 0}$. Recall

$$|z| = \sqrt{2} z_1 \bar{z_1} + \cdots + z_n \bar{z_n},$$

$$\mathbb{D}(z, r) = \{w \in \mathbb{E}^n \mid |z - w| < r\}, \text{ and}$$
$$\bar{\mathbb{D}}(z, r) = \{w \in \mathbb{E}^n \mid |z - w| \leqslant r\}$$

called oepn and closed disks respectively. We call $\bar{\mathbb{D}}(z_1, r_1) \times \cdots \times \bar{\mathbb{D}}(z_n, r_n)$ an *open polydisk.*

5.1 Formal power series

Definition 5.3 Let $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$. A *formal power series* centered at a is an expression of the form

$$f(z) = f(z_1, \ldots, z_n) = \sum_{(r_1,\ldots,r_n)\in\mathbb{Z}^n_{\geqslant 0}} c_{r_1\cdots r_n}(z_1 - a_1)^{r_1} \cdots (z_n - a_n)^{r_n}$$

with $c_{r_1,\ldots,r_n} \in \mathbb{C}$.

Remark 5.4. We will restrict our attention to absolutely convergent series, so we do not need to order the indices in the sum to discuss convergence.

Definition 5.5 The series above *converges (uniformly) absolutely* on $X \subseteq \mathbb{C}^n$ if for all $z \in X$ the series of real numbers

$$\sum_{(r_1,\ldots,r_n)} |c_{r_1,\ldots,r_n}(z_1 - a_1)^{r_1} \cdots (z_n - a_n)^{r_n}|$$

converges (uniformly).

Recall that $\sum_n c_n z^n$ converges absolutely on $\mathbb{D}(0, R)$ where $R = \frac{1}{\limsup_{n\to\infty} |c_n|^{\frac{1}{n}}}$. It converges uniformly absolutely on each compact $K \subseteq \mathbb{D}(0, R)$.

Example 5.6 (Geometric series) The geometric series with ration $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ is defined as $\sum_{r_1,\ldots,r_n} z_1^{r_1} \cdots z_n^{r_n}$. It converges (uniformly) absolutely on (compact subsets of) $\mathbb{D}(0, 1)^n$ with sum equal

$$\prod_{k=1}^{n} \sum_{r_k \geqslant 0} z_k^{r_k} = \frac{1}{(1 - z_1) \cdots (1 - z_n)}$$

Lemma 5.7 (Abel) *Consider the series above, $w \in \mathbb{C}^n$ and $M \in \mathbb{R}_{>0}$. If $|c_r(w - a)^r| = |c_{r_1,\ldots,r_n}(w_1 - a_1)^{r_1} \cdots (w_n - a_n)^{r_n} < M$ for each $r \in \mathbb{Z}^n_{\geqslant 0}$, then $f(z)$ converges uniformly absolutely on each compact $K \subseteq D = \mathbb{D}(a_1, \rho_1) \times \mathbb{D}(a_n, \rho_n)$, where $\rho_i := |w_i - a_i|$.*

Proof. WLOG $\rho_k > 0$ for all $k \in \{1, \ldots, n\}$ (otherwise we'd have $D = \varnothing$). Let $K \subseteq D$. Then let $\delta_k := \max_{z\in K} \frac{|z_k - a_k|}{\rho_k} < 1$. Then for all $z \in K$ and for all $r \in \mathbb{Z}^n_{\geqslant 0}$ we have

$$|c_r(z - a)^r| \leqslant |c_r \rho^r| \leqslant M\delta^r.$$

Since all $\delta_k < 1$, by the previous example $\sum_r M\delta^r$ converges uniform absolutely on K. □

Definition 5.8 Uniform absolute convergence on compacts is also called *compact convergence.*

5.2 Analytic functions

Definition 5.9 Let $U \subseteq \mathbb{C}^n$ open.

i) $f: U \to \mathbb{C}$ is *analytic* at $a \in U$ if there exists an open neighbourhood $a \in V \subseteq U$ and c_r such that $f(z) = \sum_r c_r(z-a)^r$ converges compactly on V.

ii) $f: U \to \mathbb{C}$ is *analytic* on U if it is analytic at each point of U.

iii) $f: U \to \mathbb{C}^n$ is *analytic* on U if each component f_k is for all $k \in \{1, \dots, n\}$.

Exercise 5.10. Analytic at a implies continuous at a.

Exercise 5.11. If f, g are analytic, then so are $f+g$, $f-g$ and $g \circ f$ where defined.

Exercise 5.12. Let $U \subseteq C^n$ be an open subset, let $z \in U$ and $w \in \mathbb{C}^n$. Let $V = \{c \in \mathbb{C} \mid z + cw \in U\} \subseteq \mathbb{C}^n$.

i) V is open and $0 \in V$.

ii) For all $f: U \to \mathbb{C}$ analytic we have that $g(t) = f(z+tw)$ is analytic on V.

Theorem 5.13 (Identity theorem) *If $\varnothing \neq V \subseteq U \subseteq \mathbb{C}^n$ are open with U connected and $f: U \to \mathbb{C}$ is analytic with $f|_V = 0$, then $f = 0$.*

Proof. If $f(z) \neq 0$ for some $z \in U$, then by continuity of f we would have that f is nowhere zero on some open nbhd of z. Let $Z = \{w \in U \mid f \text{ vanishes in an open nbhd of } w\}$. Then Z is closed in U by what we just said. Also, $V \subseteq Z$ as V is open. Let $w \in Z$ and choose a polydisk $w \in D = \mathbb{D}(w_1, r_1) \times \cdots \times \mathbb{D}(w_n, r_n) \subseteq U$. If we show that $D \subseteq Z$, then every point of Z is in its interior and Z is therefore open.

So let $z \in D$ with $z \neq w$. Consider now $W = \{c \in \mathbb{C} \mid w + c(z-w) in \mathcal{U}\} \subseteq \mathbb{C}$, which is open by the previous exercise[3], and $g: t \mapsto f(w + t(z-w))$ is analytic on W. The identity theorem for single-variable analytic functions implies that $g = 0$ in a nbhd of w. Since D is convex, $[0,1] \subseteq W$. By the identity theorem in one variable, $g = 0$ on an open nbhd of $[0,1]$. Hence $g(1) = f(z) = 0$, so f vanishes on D and $D \subseteq Z$. □

[3] This step allows us to reduce our problem in several complex variables to a problem on a single complex variable.

5.3 Topology

Definition of topological space and examples (cofinite topology, Zariski topology). Continuous maps, homeomorphisms (isomorphism in the category of topological spaces). Example: graph of $f\colon X \to Y$ defined as $\Gamma_f = X \times_X Y$ maps homeomorphically onto X via the first projection. Subspace topology.

Connectedness, example: unit interval. Continuous image of connected is connected.

Hausdorffness, example: euclidean topology on $\mathbb{R}^n$. Non-example: real line with two origins[4]

Equivalently, X is Hd if and only if $\Delta \subseteq X \times X$ is closed. Hausdorffness is hereditary.

6 [FS] Matthias Paulsen - The construction problem for Hodge numbers - 25.10.19

In characteristic 0 this is j.w. with Stefan Schreieder. In positive characteristic this is j.w. with v. Dobbeen de Bruyn.

6.1 Overview over complex numbers

X smooth projective variety. Then we have Hodge theory, which allows us to decompose

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X),$$

where $H^{p,q}(X) \cong H^q(X, \Omega^p)$. The *Hodge numbers* are $h^{p,q}(X) = \dim_{\mathbb{C}} H^{p,q}(X)$. We usually arrange the Hodge numbers in the *Hodge diamond*

[4] These two examples show that Hausdorffness is not a local property, because the real line with two origins is locally the same as $\mathbb{R}$.

$$
\begin{array}{ccccccc}
 & & & h^{n,n} & & & \\
 & & h^{n,n-1} & & h^{n-1,n} & & \\
 & & & \cdots & & & \\
h^{n,0} & & & & & & h^{0,n} \\
 & & & \cdots & & & \\
 & & h^{1,0} & & h^{0,1} & & \\
 & & & h^{0,0} & & &
\end{array}
$$

Example 6.1 Let $X \subseteq \mathbb{P}^4$ be a hypersurface of degree 4. Then its Hodge diamond is

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 1 & & 0 & \\
5 & & 30 & & 30 & & 5 \\
 & 0 & & 1 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

We know:

i) $h^{p,q} = h^{q,p}$.

ii) $h^{0,0} = 1$.

iii) $h^{p,q} \geqslant h^{p-1,q-1}$ if $p+q \leqslant n$.

Question 6.2. Given $(h^{p,q})_{p,q}$ such that i), ii) and iii) hold. Does there exist X with $h^{p,q}(X) = h^{p,q}$ for all p, q?

1989 Partial results in dimensions 2 and 3.

2013 Kotschick and Schreieder determined the Hodge ring of Kaehler manifolds and showed that there are no linear relations besides of the previous three.

2015 Schreieder.

In any dimension n, a given row $k < n$ can be always achieved, except if $k = 2p$, in which case we need $h^{p,p} \geqslant O(p)$.

If we ignore the middle row, the outer Hodge numbers and the middle column, then everything else can be arbitraty.

Negative results: the previous question has negative answer, e.g. in dimension three, if we assume that $h^{1,1} = 1 =$ and $h^{0,2} \geqslant 1$, then $h^{3,0} < 12^6 h^{2,1}$.

Question 6.3 (Kollar). Are there any polynomial relations between the Hodge numbers, besides the ones induced by the symmetries above?

Question 6.4. Besides the "unexpected" inequalities, are there also number theoretic restrictions?

2019 Schreider and P.: modulo any integer $m \geqslant 1$, any Hodge diamond $(h^{p,q})_{p,q}$ satisfying the symmetries[5] above is realizable by a smooth projective variety X, i.e. such that

$$h^{p,q}(X) \equiv h^{p,q} (\text{mod } m).$$

[5] Note that condition iii) vanishes.

Corollary 6.5 *There are no polynomial relations and there are no "number theoretic" restrictions.*

The proof can be devided into two parts corresponding to the outer Hodge numbers and the remaining ones. The outer Hodge numbers are birational invariants. The first part is to produce a variety which has the right outer Hodge numbers. And then all the inner ones can be obtain by repeated blow-ups.

6.2 Constructions

We will have 3 building blocks:

- Products (and then use Kuenneth's formula).
- Hypersurfaces to reduce the dimension again (apply Lefschetz hyperplane theorem, picture [A]).
- Blow-ups of subvarieties (see picture [B]).

We start then with a curve, in which the problem is completely solvable:

$$\begin{array}{ccc} & 1 & \\ g & & g \\ & 1 & \end{array}$$

Starting from this we can build up our Hodge diamonds modulo m.

6.3 Positive characteristic

We don't have complex conjugation, but we still have Serre duality imposing its 180 degrees rotation on the Hodge diamonds. Is this the only restriction?

Example 6.6 (Serre) Construction of a surface with Hodge diamond

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 1 & \\ ? & & ? & & ? \\ & 0 & & 1 & \\ & & 1 & & \end{array}$$

It seems that Matthias is up to something cool using this example!

7 [CM] Talk 2 (Pedro Núñez) - Condensed Abelian Groups - 28.10.19

See the full script on GitHub.

7.1 Recollections from the previous talk

Recall $*_{prot}$ and $\mathcal{A} = \mathrm{Sh}(*_{prot}, \mathcal{A}b)$.

Equivalent description of $\mathcal{A}$ from last talk.

7.2 A nicer description of our category

Definition 7.1 Extremally disconnected.

Note that $\mathcal{ED} \subsetneq \mathcal{CH}$.

Fact: $S \in \mathcal{CH}$ is in $\mathcal{ED}$ if and only if every surjection $S' \twoheadrightarrow S$ from a compact Hausdorff space admits a section. Using Stone-Čech compactification β, this implies that every $S \in \mathcal{CH}$ admits a surjection from an extremally disconnected set

$$\exists \tilde{S} = \beta(S_{disc}) \twoheadrightarrow S$$

Lemma 7.2 *$\mathcal{A} = \mathrm{Sh}(\mathcal{ED}, \mathcal{A}b)$.*

Corollary 7.3 $\mathcal{A} = \{\mathcal{F} \in \mathrm{Fun}(\mathcal{ED}^{op}, \mathcal{A}b) \mid i) \wedge ii)\}$.

Corollary 7.4 *(Co)limits exist in $\mathcal{A}$ and can be constructed pointwise.*

7.3 Abelianity and compact-projective generation

Recall definition of Grothendieck category.

Theorem 7.5 *$\mathcal{A}$ is abelian with (AB6) and (AB4*) and it is generated by compact projective objects.*

Corollary 7.6 *$\mathcal{A}$ has enough injectives and projectives.*

— **BREAK** —

7.4 Closed symmetric monoidal structure

Briefly outline monoidal categories.

Proposition 7.7 *$\mathcal{A}$ is symmetric monoidal and the functor $\mathbb{Z}[-]\colon \mathrm{Cond}(\mathcal{S}et) \to \mathcal{A}$ is symmetric monoidal.*

Proposition 7.8 *For all condesned set $\mathcal{X}$, the condensed abelian group $\mathbb{Z}[\mathcal{X}]$ is flat, i.e. the functor $\mathbb{Z}[\mathcal{X}] \otimes (-)$ is exact.*

Proposition 7.9 *For all $\mathcal{F} \in \mathcal{A}$ the functor $\mathcal{F} \otimes (-)$ has a right adjoint $[\mathcal{F}, -]$, i.e. $\mathcal{A}$ is closed symmetric monoidal.*

7.5 Derived category

Brief description.

Remark 7.10. Triangulated structure and the problem it carries.

Basic derived functors

$$RF\colon \mathcal{D}^+(\mathcal{A}) = \mathcal{K}^+(\mathcal{I}) \to \mathcal{D}(\mathcal{B}).$$

Example 7.11 $\mathrm{Hom}(-, \mathcal{F})$ and $\mathrm{Hom}^\bullet(-, \mathcal{F}^\bullet)$.

Extension to the whole $\mathcal{D}$ using Spaltenstein's resolutions. Formula

$$\mathrm{Hom}_{\mathcal{D}}(\mathcal{F}^\bullet, \mathcal{G}^\bullet[i]) = \mathrm{Ext}^i(\mathcal{F}^\bullet, \mathcal{G}^\bullet).$$

Closed symmetric monoidal structure.

Example 7.12 $\mathbb{Z}[\mathcal{X}] \otimes^L (-) = \mathbb{Z}[\mathcal{X}] \otimes (-)$ is just degree-wise tensor product, because this functor is already exact so it need not be derived.

Proposition 7.13 *Compact generation.*

Mention Brown representability.

8 [LT] Lecture 3 - 29.10.19

Goals for today:

- **Prop:** f analytic $\Rightarrow f$ complex differentiable (holomorphic).
- **Cor:** f analytic $\Rightarrow f \in \mathcal{C}^\infty$.
- Implicit and inverse function theorems for analytic functions.

8.1 Complex differentiable functions

Recall: $U \subseteq \mathbb{R}^n$ open, $f\colon U \to \mathbb{R}^m$ is *differentiable* at $a \in U$ if $\exists\mathbb{R}$-linear map $df_a\colon \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - df_a(h)}{|h|} = 0.$$

With respect to the standard bases, df_a is given by the *Jacobian* matrix $J_f(a) = (\frac{\partial f_i}{\partial f_j})_{i,j}$.

We say that $f\colon U \to \mathbb{R}^m$ is $\mathcal{C}^\infty$ or *smooth* if all partial derivatives exist (to all orders) and are continuous.

Recall also that $f \in \mathcal{C}^1$ implies f differentiable.

Definition 8.1 $X \subseteq \mathbb{R}^n$ not necessarily open, $f\colon X \to \mathbb{R}^m$ is *differentiable* or $\mathcal{C}^\infty$ if for all $x \in X$ we can find an open nbhd $x \in U \subseteq \mathbb{R}^n$ and a differentiable or $\mathcal{C}^\infty$ function $F\colon U \to \mathbb{R}^m$ such that $F|_{X \cap U} = f|_{X \cap U}$.

Definition 8.2 Let $U \subseteq \mathbb{C}^n$ open. We say that $f\colon U \to \mathbb{C}$ is *complex differentiable* at $a \in U$ if it is continous and

i) if $n = 1$, then the limit $f'(a) := \lim_{z \to a} \frac{f(z)-f(a)}{z-a}$ exists.

ii) if $n > 1$, then for all $1 \leqslant k \leqslant n$ and for all $z_1, \dots, \hat{z_k}, \dots, z_n$ the function

$$f_k(z) := f(z_1, \dots, z_{k-1}, z, z_{k+1}, \dots, z_n)$$

is complex differentiable as in 1).

Example 8.3 $\operatorname{Re}, \operatorname{Im} \colon \mathbb{C} \to \mathbb{R}$ are $\mathcal{C}^\infty$, not complex differentiable.

If f is complex differentiable, then $\frac{\partial f}{\partial x_i}, \frac{\partial f}{\partial y_i}$ exist and are continuous, i.e. $f \in \mathcal{C}^1$.

Proposition 8.4 *Let $U \subseteq \mathbb{C}^n$ open and $f \colon U \to \mathbb{C}$ is analytic. Then f is complex differentiable and f' is analytic.*

Proof. $f(z_1, \dots, z_n) = \sum_r c_r (z-a)^r$ locally near $a \in U$. Continuity is OK, we need to show that

$$\sum c_r (\zeta_1 - a_1)^{r_1} \cdots (\zeta_{n-1} - a_{n-1})^{r_{n-1}} (z_n - a_n)^{r_n}$$

is complex differentiable for all $(\zeta_1, \dots, \zeta_n)$ with respect to z_n. This is a power series in a single variable, hence complex differentiable. $\square$

Corollary 8.5 *f analytic implies f smooth.*

Proof. f analytic implies f' analytic, hence f'' analytic, and so on. $\square$

8.2 Implicit function theorem

Notation 8.6. We denote by $\mathbb{C}[[z_1, \dots, z_n]]$ the ring of formal power series in $z_1, \dots, z_n$ centered at 0 (with Cauchy product as multiplication) and by $\mathbb{C}\{z_1, \dots, z_n\}$ the subring of all power series convergent in a nbd of 0.

Lemma 8.7 [6] *Let $f(z_0, \dots, z_n) = \sum_r c_r z^r$ with $c_{0,\dots,0} \neq 0$. Then $\exists! g \in \mathbb{C}[[z_1, \dots, z_n]]$ such that $fg = 1$.*

Proof. WLOG $c_{0,\dots,0} = 1$. Let $z = (z_1, \dots, z_{n-1})$ and $w = z_n$. Let $f = \sum_k a_k(z) w^k$ with $a_k(z) \in \mathbb{C}[[z]]$. Let $g = \sum_k b_k(z) w^k$ with $b_k(z) \in \mathbb{C}[[z]]$.

[6] Typical proof with power series: do the naive thing inducting on the degree and the number of variables.

Then $fg = \sum_{l=0}^{\infty}(\sum_{k=0}^{l} a_k b_{l-k})w^l$. Call $\delta_{0,l} = \sum_{k=0}^{l} a_k b_{l-k}$. $l = 0$ implies $1 = a_0 b_0$, hence $b_0 = a_0^{-1}$. For $l > 0$ we get $0 = \sum_{k=0}^{l} a_k b_{l-k} a_k b_{l-k}$, hence

$$b_l = -a_0^{-1} \sum_{k=1}^{l} a_k b_{l-k}.$$

□

Theorem 8.8 (Formal implicit function theorem) [7] *If $f \in \mathbb{C}[[z,w]]$, $f(0) = 0$ and $\frac{\partial f}{\partial w} \neq 0$, then $\exists! u \in \mathbb{C}[[z,w]]$ and $\exists! r \in \mathbb{C}[[z]]$ such that $uf = w - r$ and $u(0) \neq 0$.*

Proof. 1) Consider the $\mathbb{C}$-linear maps

$$R\colon \mathbb{C}[[z,w]] \to \mathbb{C}[[z]]$$
$$p = \sum_r c_r(z) w^r \mapsto c_0(z)$$

$$H\colon \mathbb{C}[[z,w]] \to \mathbb{C}[[z,w]]$$
$$p = \sum_r c_r w^r \mapsto \frac{p - R(p)}{w} = \sum_{r>0} c_r w^{r-1}$$

2) $\forall p$ we have $p = wH(p) + R(p)$.

3) $H(f)$ is a unit by the previous lemma and $\frac{\partial f}{\partial w} \neq 0$.

4) Suffices to find u s.t.

$$0 = H(w - uf) = H(w) - H(uf)$$

$$0 = 1 - H(uf) \quad (*)$$

Indeed, $(*)$ and 2) imply that $R(w - uf) = w - uf$, hence $r := w - uf \in \mathbb{C}[[z]]$ $(w = uf + (w - uf) = uf + r)$.

[7] Near 0 we have $f(z,w) = 0$ iff $0 = u(z,w)f(z,w)$ iff $0 = w - r(z)$, i.e. the vanishing locus of f is the graph of r near 0, hence this is indeed an implicit function theorem.

5)

$$uf \stackrel{2)}{=} u(wH(f) + R(f))$$
$$= uwH(f) + uR(f)$$
$$(*) \Leftrightarrow 0 = 1 - H(uf)$$
$$0 = 1 - H(uwH(f) + uR(f))$$
$$0 = 1 - (\frac{uwH(f) - R(uwH(f))}{w}) - H(uR(f))$$
$$(**) 0 = 1 - uH(f) - H(uR(f))$$

6) $H(f)$ unit $\Rightarrow$ suffices to find $v = uH(f)$. $\mu := -R(f)H(f)^{-1}$ s.t. $u = vH(f)^{-1}$ satisfies $(**)$.

7)

$$(**) \Leftrightarrow 0 = 1 - v + H(\frac{-uR(f)}{H(f)} H(f))$$
$$(***) 0 = 1 - v + H(\mu v)$$

8) $M\colon \mathbb{C}[[z,w]] \to \mathbb{C}[[z,w]]$ defined as $p \mapsto H(\mu p)$. Note: if z^k divides p, then z^{k+1} divides $M(p)$.

9)

$$0 = 1 - v + M(v)$$
$$(***') v = 1 + M(v)$$

$\mathbb{C}$-linearity of M implies that $v = 1 + M(v) = 1 + M(1 + M(v))$. Hence for all $k \geqslant 0$ we have

$$v = 1 + M(1) + \cdots + M^k(1) + M^k(v).$$

10) z^k divides $M^k(1)$, z^{k+1} divides $M^{k+1}(v)$. Hence $\sum_k M^k(1)$ is convergent as a formal power series.

11) If v satisfying $(***')$, then $v = \sum_k M^k(1)$. Hence v is unique and thus so are u, r.

12) $v = \sum_k M^k(1)$ satisfies $(***')$. $v = 1 + M(1) + \cdots + M^k(1) + W_k$, hence $z^{k+!}$ divides W_k by 10). $v-1-M(v) = 1+M(1)+\cdots+M^k(1)+W_k - 1 - M(1) - \cdots - M^k(1) - M^{k+1}(1) - M(W_k)$, hence for all k we have that z^{k+1} divides $v-1-M(v)$. Therefore $v-1-M(v)=0$.

$\square$

Theorem 8.9 (Convergent implicit function theorem) *With notation as above, if $f \in \mathbb{C}\{z, w\}$, then $u \in \mathbb{C}\{z, w\}$ and $r \in \mathbb{C}\{z\}$.*

Proof. See Samuel-Zariski. $\square$

Exercise 8.10. Formulate and deduce the analytic inverse and implicit function theorem.

8.3 More basic topology

Product spaces: definition and universal property. Example: $\mathbb{R}^n \cong \mathbb{R} \times \mathbb{R}^{n-1}$ with the usual topology. Non example: $\mathbb{C}^{n+m} \ncong \mathbb{C}^n \times \mathbb{C}^m$ with the Zariski topology.

Quotient spaces: definition and universal property. Examples: $S^1 = [0, 1]/1 \sim= \mathbb{R}/\mathbb{Z}$ and $\mathbb{T}^2$ gluing the boundary of the square $[0, 1]^2$ appropriately.

Quasi-compactness: definition. Non-example: $\mathbb{R}$. Example: $[a, b] \subseteq \mathbb{R}$. In $\mathbb{R}^n$ quasi-compact iff closed bounded. More facts: union of qc is qc, continuous image of qc is qc, closed subset of qc is qc.

9 [WS] Kodaira 2 (Vera) - Introduction to Hodge Manifolds - 30.10.19

Outline for today:

1) Recollections.

2) Hodge Manifolds.

3) Outlook: Kodaira's Embedding Theorem.

9.1 Recollections

9.1.1 Cohomology Theories

1. Recall the *Dolbeault-Cohomology* defined as

$$H^{p,q}(X) := \frac{\ker(\bar{\partial}\colon \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p,q+1}(X))}{\operatorname{im}(\bar{\partial}\colon \mathcal{A}^{p,q-1}(X) \to \mathcal{A}^{p,q}(X))}.$$

2. Recall the *de Rham cohomology* defined as

$$H^k_{dR}(X,\mathbb{C}) = \frac{\ker(d\colon \mathcal{A}^k_{\mathbb{C}}(X) \to \mathcal{A}^{k+1}_{\mathbb{C}}(X))}{\operatorname{im}(d\colon \mathcal{A}^{k-1}_{\mathbb{C}}(X) \to \mathcal{A}^k_{\mathbb{C}}(X))},$$

where $\mathcal{A}^k_{\mathbb{C}} = \Gamma(X, \Lambda^k TX^* \otimes \mathbb{C})$.

Theorem 9.1 *On a compact Kaehler manifold* $(X, I, <\cdot,\cdot>)$ *we have*

$$H^k_{dR}(X,\mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}(X).$$

On the other hand, we have

Theorem 9.2 (de Rham Theorem)

$$H^k_{dR}(X,\mathbb{C}) \cong H^k(X,\mathbb{C}).$$

3. Integral cohomology.

Definition 9.3 Let X be a compact complex manifold. A closed differential form φ on X is called *integral* if its cohomology class $[\varphi] \in H^k_{dR}(X,\mathbb{C})$ is in the image of the mapping

$$H^k(X,\mathbb{Z}) \to H^k(X,\mathbb{C}) \cong H^k_{dR}(X,\mathbb{C}).$$

Today we will be interested in integral $(1,1)$-classes, i.e.

$$\begin{tikzcd} & H^2(X,\mathbb{Z}) \arrow[dl, "j_*"'] \arrow[d] \\ H^{1,1}(X) \subseteq H^2_{dR}(X,\mathbb{C}) \arrow[r, "\cong"] & H^2(X,\mathbb{C}) \end{tikzcd}$$

We denote $\tilde{H}^2_{dR}(X,\mathbb{Z}) := j_* H^2(X,\mathbb{Z})$.

9.1.2 Holomorphic line bundles and first Chern class

Recall the exponential sequence

$$0 \to \mathbb{Z} \to \mathcal{O} \xrightarrow{\exp} \mathcal{O}^\times \to 0$$

of sheaves. Last week we have seen that this sequence is exact and thus it induces a long exact sequence of cohomology groups

$$\cdots \to \check{H}^1(X, \mathcal{O}^\times) \xrightarrow{\delta} \check{H}^2(X, \mathbb{Z}) \to \cdots$$

Fact 1: $\check{H}^1(X, \mathcal{O}^\times)$ classifies holomorphic line bundles up to holomorphic isomorphism.

Fact 2: In the sequence above, δ is "almost" c_1.

$$\begin{tikzcd} H^1(X, \mathcal{O}^\times) \arrow[r, "\delta"] \arrow[dr, "c_1"'] & H^2(X, \mathbb{Z}) \arrow[d] \\ & H^2_{dR}(X, \mathbb{R}) \end{tikzcd}$$

i.e. holomorphic line bundles have "integral Chern classes".

Remark 9.4. Indeed, the Lefschetz $(1,1)$-theorem states that on compact Kaehler manifolds X we have

$$c_1(H^1(X, \mathcal{O}^\times)) = \tilde{H}^2(X, \mathbb{Z}) \cap H^{1,1}(X).$$

In the proof of the previous commutative diagram, an explicit formula for the $(1,1)$-form representing $c_1(E, h)$, namely

$$\frac{1}{2\pi i}\partial\bar{\partial}\log(h_\alpha),$$

where h_α is the metric on U_α.

9.1.3 Kaehler manifolds

Definition 9.5 Given a complex manifold (M, I, g) where the complex structure I is compatible with the metric g, we associate a $(1,1)$-form

$$w(\cdot,\cdot) = g(I\cdot,\cdot)$$

called the *Kaehler form.*

We say that M is Kaehler, if it admits some metric such that the associated Kaehler form is closed ($dw = 0$).

In that case, $w \in \mathcal{A}^{1,1}(M)$ defines a cohomology class $[w] \in H^{1,1}(M)$, called the *Kaehler class.*

Example 9.6 (The Fubini-Study metric on $\mathbb{CP}^n$) In homogeneous coordinates, the corresponding Kaehler form is given as

$$w_a = i\partial\bar{\partial}\log(|\zeta_a|^2 + 1)$$

on $U_a = \{(z^0 : \ldots : z^n) \mid z^a \neq 0\}$. The map $\zeta_a \colon U_a \to \mathbb{C}^n$ sends $(z^0 : \ldots : z^n) \mapsto (\frac{z^0}{z^a}, \ldots, \hat{1}, \ldots, \frac{z^n}{z^a})$.

From this we can also deduce the same for any projective complex manifold using the following:

Lemma 9.7 *Let (X, I, g) be a Kaehler manifold with Kaehler form w and let M be a complex submanifold. Then g induces a Kaehler manifold on M, thus M is a Kaehler manifold.*

9.2 Hodge Manifolds

Definition 9.8 Let (X, h) be a Kaehler manifold with Kaehler metric h and let w be the associated Kaehler form. If w is integral, it is called a *Hodge form* on X and h is called a *Hodge metric.* A Kaehler manifold is called *Hodge manifold* if it admits a Hodge metric.

Example 9.9 (Complex projective space) Let $\mathbb{CP}^n$ be endowed with the Fubini-Study metric g_{FS} and the associated Kaehler form w_{FS}. Then w_{FS} is a Hodge form. Indeed, let $E \to \mathbb{CP}^n$ be the tautological line bundle. By Fact 2, $c_1(E) \in \tilde{H}^2(X, \mathbb{Z})$. And we have

$$\left[\frac{1}{2\pi} w_{FS}\right] = -c_1(E).$$

Thus $\mathbb{CP}^n$ is a Hodge manifold.

Example 9.10 (Projective complex manifolds) Let X be a projective complex manifold, i.e. $X \subseteq \mathbb{CP}^n$ as closed complex submanifold. The restriction of the Hodge form on $\mathbb{CP}^n$ is a Hodge form on X.

Fundamental fact: all projective complex manifolds are Hodge.

Example 9.11 (Compact connected Riemann surfaces) Let X be a compact connected Riemann surface. The claim is that X is a Hodge manifold. We have $H^2(X, \mathbb{C}) \cong H_0(X, \mathbb{C}) \cong \mathbb{C}$. Moreover, $H^2(X, \mathbb{C}) \cong H^{1,1}(\mathbb{C})$. Let $\tilde{w}$ be the Kaehler form associated to some metric on X. Then $[\tilde{w}] \in H^{1,1}(\mathbb{C})$ generates $H^2_{dR}(X, \mathbb{C})$. Let $c := \int_X \tilde{w}$. Then $w := \frac{1}{c}\tilde{w}$ is an integral positive form on X of type $(1,1)$.

9.3 Outlook

We will soon prove the converse of our fundamental fact, namely:

Theorem 9.12 (Kodaira's Embedding) *Every Hodge manifold admits a closed immersion into $\mathbb{CP}^n$ for n sufficiently large.*

Combining this with Chow's theorem seen last time, we obtain as a corollary that Hodge manifolds are always algebraic.

10 [LT] Lecture 4 - 31.10.19

10.1 Key theorems

Theorem 10.1 (Liouville) *$f \colon \mathbb{C}^n \to \mathbb{C}$ bounded. Then f constant.*

Proof. Let $z, w \in \mathbb{C}^n$. We claim that $f(z) = f(w)$. Consider $h \colon c \mapsto z + c(w - z)$ from $\mathbb{C} \to \mathbb{C}^n$. This is analytic. Hence $g := f \circ h$ is analytic. $\mathrm{im}(g) \subseteq \mathrm{im}(f)$, so g is also bounded. By the 1-dimensional Liouville theorem, g is constant. But then $f(z) = g(0) = g(1) = f(w)$. □

Theorem 10.2 (Maximum modulus principle) *$U \subseteq \mathbb{C}^n$ open connected, $f \colon U \to \mathbb{C}$ analytic, $a \in U$. If $|f(a)| \geqslant |f(z)|$ for each z in an open nbd $a \in V \subseteq U$, then f is constant on U.*

Proof. Identity theorem implies that if f is constant on V, then f is constant on U. So WLOG $V = D$ a polydisk centered at a. Consider $z_1 \mapsto f(z_1, a_2, \ldots, a_n)$ from $D(a, r)$ to $\mathbb{C}$. This is analytic and its modulus attains a maximum at $z_1 = a_1$. By the single variable maximum modulus principle, this map is constant on $D(a_1, r_1)$.

Suppose $(z_1, \ldots, z_k) \mapsto f(z_1, \ldots, z_k, a_{k+1}, \ldots, a_n)$ is constant on $D(a_1, r_!) \times \cdots \times D(a_k, r_k)$ for some $1 \leqslant k < n$. Then $\forall (w_1, \ldots, w_k, z_{k+1}, a_{k+2}, \ldots, a_n)$, $z_{k+1} \mapsto f(w_1, \ldots, w_k, z_{k+1}, a_{k+2}, \ldots, a_n)$ is analytic, its modulus has a maximum at $z_{k+1} = a_{k+1}$. This implies by the single variable case that it is constant. By induction, we win. □

Theorem 10.3 (Open mapping theorem) *$U \subseteq \mathbb{C}^n$ connected open, $f \colon U \to \mathbb{C}$ nonconstant, analytic. If $V \subseteq U$ is open, then $f(V) \subseteq \mathbb{C}$ is open as well.*

Proof. Let $z \in V$ and $w = f(z)$. It suffices to construct an open nbd of w contained in $f(V)$. There exists an open polydisk D centered at z containedin V, and V is a union of such polydisks. Hence WLOG $V = D$.

f nonconstant implies that $\exists z' \in D$ with $f(z) \neq f(z')$. The set $W = \{c \in \mathbb{C} \mid z + c(z' - z) \in D\} \subseteq \mathbb{C}$ is open. The composite $f \circ h \colon W \to \mathbb{C}$ is analytic and nonconstant. By the single variable open mapping theorem, $f \circ h$ is open. Hence $\operatorname{im}(f \circ h) \subseteq \mathbb{C}$ is open, and $\operatorname{im}(f \circ h) \subseteq f(V)$. □

10.2 Cauchy-Riemann equations

Proposition 10.4 *$U \subseteq \mathbb{C}^n$ open, $f \colon U \to \mathbb{C}$ complex differentiable. If $f = u + iv$ with $u, v \colon U \to \mathbb{R}$ and we use coordinates $z_k = x_k + iy_k = \operatorname{Re}(z_k) + i \operatorname{Im}(z_k)$, then*

$$(CR_k) \quad \frac{\partial u}{\partial x_k} = \frac{\partial v}{\partial y_k} \text{ and } \frac{\partial u}{\partial y_k} = -\frac{\partial v}{\partial x_k}$$

Proof. Let $e_k := (0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 on the k^{th} coordinate.

$$\begin{aligned}\frac{f(a + he_k) - f(a)}{h} &= \frac{(u(a + he_k) + iv(a + he_k)) - (u(a) + iv(a))}{h} \\ &\overset{(*)}{=} \frac{u(a + he_k) - u(a)}{h} + i\frac{v(a + he_k) - v(a)}{h} \\ &\overset{h_{\text{real}}}{\Rightarrow} (*) \to \frac{\partial u}{\partial x_k}(a) + i\frac{\partial v}{\partial x_k}(a)(**) \\ &\qquad \overset{h_{\text{imaginary}}}{\Rightarrow} (*) \to -i\frac{\partial u}{\partial y_k}(a) + \frac{\partial v}{\partial y_k}(a)(***)\end{aligned}$$

f complex differentiable implies $(**) = (***)$. Now compare real and imaginary part and we are done. □

Corollary 10.5 *$f \colon U \to \mathbb{R}$ complex differentiable imlpies f constant.*

Definition 10.6 Let $f\colon U \to \mathbb{C}$ be a function. Let

$$\frac{\partial f}{\partial x_k} = \frac{\partial u}{\partial x_k} + i\frac{\partial v}{\partial x_k} \text{ and } \frac{\partial f}{\partial y_k} = \frac{\partial u}{\partial y_k} + i\frac{\partial v}{\partial y_k}.$$

Then we define the *Wirtinger operators* as

$$\frac{\partial f}{\partial z_k} := \frac{1}{2}\left(\frac{\partial f}{\partial x_k} - i\frac{\partial f}{\partial y_k}\right) \text{ and } \frac{\partial f}{\partial \bar{z}_k} := \frac{1}{2}\left(\frac{\partial f}{\partial x_k} + i\frac{\partial f}{\partial y_k}\right).$$

Proposition 10.7 *For $U\colon \mathbb{C}^n$ open and $f\colon U \to \mathbb{C}$ with partial derivatives, the following are equivalent:*

1) *f satisfies (CR_k) for each k.*
2) $\frac{\partial f}{\partial x_k} = -i\frac{\partial f}{\partial y_k}$ *for all k.*
3) $\frac{\partial f}{\partial \bar{z}_k} = 0$ *for all k.*
4) $\frac{\partial f}{\partial z_k} = \frac{\partial f}{\partial x_k} = \frac{\partial u}{\partial x_k} + i\frac{\partial v}{\partial x_k} = \frac{\partial f}{\partial (iy_k)}$ *for all k.*
5) *Via $\mathbb{C}^2 \cong \mathbb{R}^{2n}$ sending $(z_1, \dots, z_n) \mapsto (x_1, y_1, \dots, x_n, y_n)$, for all $a \in U$, the Jacobian matrix $J_f^{\mathbb{R}}(a)$ is composed of 2×2 blocks of the form*

$$\begin{pmatrix} \frac{\partial u}{\partial x_k} & -\frac{\partial v}{\partial x_k} \\ \frac{\partial v}{\partial x_k} & \frac{\partial u}{\partial x_k} \end{pmatrix}(a).$$

6) *For all a, $J_f^{\mathbb{R}}(a)\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is $\mathbb{C}$-linear.*

Proof. For equivalences between 1), 2), 3) and 4), expand and compare real and imaginary parts. For 1) $\Leftrightarrow$ 5) use the definition of the Jacobian matrix. For 5) $\Leftrightarrow$ 6) observe that $\mathbb{C} = \{a + ib \mid a, b \in \mathbb{R}\} \xrightarrow{\cong} \{\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \mid a, b \in \mathbb{R}\}$ is compactible with sums and product. □

Remark 10.8. If f is $\mathcal{C}^1$ and satisfies any of these conditions, then f is complex differentiable and

$$\frac{\partial f}{\partial z_k}(a) = \lim_{h \to 0} \frac{f(a + he_k) - f(a)}{h}.$$

Proposition 10.9 *Let $U\colon \mathbb{C}^n$ open and $f\colon U \to \mathbb{C}^n$ complex differentiable. Then for all $a \in U$*

$$0 \leqslant \left|\det(J_f^{\mathbb{C}}(a))\right|^2 = \det(J_f^{\mathbb{R}}(a)),$$

where $J_f^{\mathbb{C}}(a) = \left(\frac{\partial f_k}{\partial z_l}\right)_{k,l}$. In particular, f preserves orientations.

Idea. Rewrite $J_f^{\mathbb{R}}(a)$ in the basis $(z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n)$. The resulting matrix is the matrix of derivatives of $(f_1, \dots, f_n, \bar{f}_1, \dots, \bar{f}_n)$ with respect to $(z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n)$. Then

$$J_f^{\mathbb{R}}(a) = \begin{pmatrix} J_f^{\mathbb{C}}(a) & 0 \\ 0 & \overline{J_f^{\mathbb{C}}(a)} \end{pmatrix}.$$

□

10.3 Complex differentiable implies analytic

Theorem 10.10 (Cauchy integral formula) *$U\colon \mathbb{C}^n$ open, $f\colon U \to \mathbb{C}$ complex differentiable, $\bar{D} = \bar{D}_1 \times \bar{D}_n$ closed polydisk in U, $z \in D$ (which is the interior of $\bar{D}$. Then*

$$f(z_1, \dots, z_n) = \left(\frac{1}{2\pi i}\right)^n \int_{\partial D_n} \cdots \int_{\partial D_1} \frac{f(w_1, \dots, w_n)}{(w_1 - z_1) \cdots (w_n - z_n)} dw_1 \cdots dw_n.$$

Proof. Use the single variable version, the fact that complex differentiable means copmlex differentiable separately in each variable and the induct on n. □

Corollary 10.11 *If f is complex differentiable, then f is analytic.*

Corollary 10.12 *If f is complex differentiable, then $f \in \mathcal{C}^\infty$.*

Proof. Mimic single variable proof. □

10.4 Topology: gluing constructions

Definition and construction of pushouts.

10.5 Complex projective space

Definition as a quotient. Compact and connected because it is the image of the sphere under the quotient map. Hausdorff for the exercise session.

11 [CM] Talk 3 (Nicola) - Cohomology - 04.11.19

11.1 Singular cohomology

Recollection of singular cohomology of a topological space X with coefficients in some abelian group A, denoted $H^p_{sing}(X, A)$.

11.2 Čech cohomology

Recollection of Čech cohomology of a sheaf of abelian groups $\mathcal{F}$ on a topological space X, denoted $\check{H}^p(X, \mathcal{F})$. The Čech complex associated to an open cover $\mathcal{U}$ will be denoted $C^\bullet(\mathcal{U})$ or $C^\bullet(\mathcal{U}, \mathcal{F})$ is we want to make explicit the sheaf $\mathcal{F}$.

11.3 Sheaf cohomology

Recollection of sheaf cohomology of a sheaf of abelian groups $\mathcal{F}$ on a topological space X, denoted $H^p_{Sh}(X, \mathcal{F})$.

11.4 Comparison

Let $U \subseteq X$ be an open subset and $\mathcal{U}$ an open cover on X. We can restrict this open cover to an open cover $\mathcal{U}|_U$ of U. Then $C^p(\mathcal{U})(U) := C^p(\mathcal{U}|_U)$ defines a new sheaf of abelian groups on X.

Fact: $0 \to \mathcal{F} \to C^0(\mathcal{U}) \to C^1(\mathcal{U}) \to \cdots$ is a resolution of $\mathcal{F}$. Hence we obtain a map $\varinjlim_{\mathcal{U}} H^q(\mathcal{U}, \mathcal{F}) \to H^q_{Sh}(X, \mathcal{F})$.

Facts:

1) Čech cohomology and sheaf cohomology always agree on degrees zero and one.

2) X locally contractible implies that singular cohomology with coefficients on A agrees with sheaf cohomology of the constant sheaf A.

3) X paracompact and Hausdorff implies that Čech cohomology and sheaf cohomology agree.

Remark 11.1. Using hypercovers instead of Čech, this is always true (on any site with fibre products). See [Sta19, Tag 01H0].

Example 11.2 If X is a CW complex, then all three cohomologies agree.

Example 11.3 If X is profinite, then we have

$$\check{H}^i(X,\mathbb{Z}) = H^i_{Sh}(X,\mathbb{Z}) = 0 = H^i_{sing}(X,\mathbb{Z})$$

for all $i > 0$, but $H^0_{Sh}(X,\mathbb{Z}) = C(X,\mathbb{Z}) \neq H^0_{sing}(X,\mathbb{Z}) = \mathrm{Hom}_{\mathcal{S}et}(X,\mathbb{Z})$ unless X is finite.

11.5 Hypercovers

Recollection of simplicial sets and simplicial objects on a category. Convention: we also consider the empty set ("-1-simplices").

Definition 11.4 A *hypercover* of $X \in \mathrm{Ob}(\mathcal{C})$ is a simplicial object in $\mathcal{C}$ such that $X_{-1} = X$ with some nice properties.

Definition 11.5 $H^\bullet_{cond}(S,\mathcal{F}) := H^\bullet_{HC}(S_{cond},\mathcal{F}) = H^\bullet_{HC}(S_{cond},\mathcal{F}) = \varinjlim_{S_\bullet \to S} H^\bullet(S_\bullet,\mathcal{F})$ for $S \in \mathcal{CH}aus$ and $\mathcal{F} \in \mathcal{A}b(S_{cond})$, where S_{cond} denotes the proetale site of compact Hausdorff spaces over S.

The RHS inside the direct limit is defined as follows. Start from $S_\bullet \to S$. Then get a diagram $\mathcal{F} \to \pi_{0,*}\pi_0^{-1}\mathcal{F} \rightrightarrows \cdots$. With the alternating sum of the face maps get $0 \to \pi_{0,*}\pi_0^{-1}\mathcal{F} \to \cdots$. From this complex we take global sections and get a complex of groups. Then we take cohomology of this complex. Finally we take the direct limit over all simplicial resolutions.

By [?, Theorems 2.3, 3.10-11], for all $S \in \mathcal{CH}aus$ there exists a hypercover $S_\bullet \to S$ by extremally disconnected S_n which is natural in S with proper surjective face maps such that for all $\mathcal{F} \in \mathcal{A}b(S)$ we have

$$H^\bullet(S_\bullet,\mathcal{F}) = H^\bullet_{cond}(S,\mathcal{F})$$

The reason is that $0 \to \pi_{0,*}\pi_0^{-1}\mathcal{F} \to \pi_{1,*}\pi_1^{-1}\mathcal{F} \to \cdots$ is a soft resolution of $\mathcal{F}$.

In particular, for $\mathcal{F} = \mathbb{Z}$, we have

$$\Gamma(S, \pi_{n,*}\pi_n^{-1}\mathbb{Z}) = \Gamma(S_n, \mathbb{Z}).$$

Hence we have

$$H^\bullet_{cond}(S, \mathbb{Z}) = H^\bullet(0 \to \Gamma(S_0, \mathbb{Z}) \to \cdots).$$

Theorem 11.6 *$H^\bullet_{cond}(S, \mathbb{Z}) = H^\bullet_{Sh}(S, \mathbb{Z})$.*

Proof. If S is profinite, then we can write S as an inverse limit $S = \varprojlim_j S_j$ of finite discrete spaces S_j. Then

$$H^i_{Sh}(S, \mathbb{Z}) = \begin{cases} 0 & \text{if } i > 0 \\ C(S, \mathbb{Z}) & \text{if } i = 0 \end{cases}.$$

$C(S, \mathbb{Z}) = \Gamma(S, \mathbb{Z}) = \varinjlim_j \Gamma(S_j, \mathbb{Z}) \overset{\text{by def (?)}}{=} H^0_{cond}(S, \mathbb{Z})$.

Now WTS $H^i_{cond}(S, \mathbb{Z}) = 0$ for all $i > 0$. Claim: it is sufficient to show that for all $S' \twoheadrightarrow S$ of profinite sets, then

$$0 \to \Gamma(S, \mathbb{Z}) \to \Gamma(S', \mathbb{Z}) \to \Gamma(S' \times_S S') \to \cdots$$

is exact.

So let $S' \twoheadrightarrow S = \lim(S'_j \twoheadrightarrow S_j)$ of finite sets. This implies splitting, so $0 \to \Gamma(S_j, \mathbb{Z}) \to \Gamma(S'_j, \mathbb{Z}) \to \cdots$ is split exact and $\varinjlim_j \Gamma = \Gamma \varprojlim_j$, so $\varinjlim_j(\cdots)$ is still split exact.

For a general compact Hausdorff space use some compatibilities between derived functors. $\square$

Theorem 11.7 $H^i_{cond}(S, \mathbb{R}) = \begin{cases} 0 & \textit{if } i > 0 \\ C(S, \mathbb{R}) & \textit{if } i = 0 \end{cases}$.

Proof. I.e. want to show that $0 \to C(S, \mathbb{R}) \to C(S_0, \mathbb{R}) \to \cdots$ is exact. $C(S, \mathbb{R}) \overset{?}{=} C(S, \mathbb{R})$. Moreover we will show that if $df = 0$, then $f = dg$ with g such that $||g|| \leqslant (i + 2 + \varepsilon)||f||$.

Let S and an HC S_i finite. Then HC splits: $\pi_n \colon S_n \to S$ and $\exists \psi_n \colon S \to S_n$ such that $\pi_n \circ \psi_n = \mathrm{id}_S$ and $\psi_n \circ \pi_n = r_n \colon S_n \to S_n$.

Claim: $r_\bullet \overset{h}{\simeq} \mathrm{id}_{S_\bullet}$, i.e. there exists $h \colon \Delta^1 \times S_\bullet \to S_\bullet$ with $h|_{\{0\} \times S_\bullet} = \mathrm{id}$ and $h|_{\{1\} \times S_\bullet} = r_\bullet$. See notes for an argument.

Now we use the Dold-Kan correspondence: there is an equivalence between the category of simplicial abelian groups and the bounded below cochain complexes of abelian groups which preserves homotopy equivalence. We have $S \overset{h}{\simeq} S_\bullet$, so $0 \to C(S,\mathbb{R}) \to C(S_0,\mathbb{R}) \to \cdots$ is chain homotopy equivalent to $0 \to C(S,\mathbb{R}) \to C(S,\mathbb{R}) \to \cdots$. Since one is exact, the other one is also exact. This proves the statement for finite spaces.

For profinite sets we use limits.

For the general case, assume $f \in C(S,\mathbb{R})$, $f \neq 0$, $s \in S$. $f_s := f|_{S_i \times_S \{s\}} \overset{!}{=} dg_s$ with $g_s \in C(S_{i-1} \times_S \{s\}, \mathbb{R})$. □

12 [LT] Lecture 5 - 5.11.19

New chapter: Manifolds and vector bundles!

12.1 Manifolds

Definition 12.1 Manifold of dimension n: locally homeomorphic to $\mathbb{R}^n$, Hausdorff and second countable.

Example 12.2 $\mathbb{R}^n$.

Remark 12.3. Voisin omits Hausdorffness and second countability, even though Hausdorffness is definitely needed.

Our main objects of study are affine/projective varieties over $\mathbb{C}$. These satisfy the definition of manifold we have given.

There is a more general notion of manifold with boundary, but we will not need it for the moment.

Definition 12.4 Real n-dimensional chart (U,φ) on a topological manifold X [picture].

Complex n-dimensional chart (U,φ) on a topological manifold X (can only exist on even dimensional manifolds).

Example 12.5 $X \subseteq \mathbb{R}^n$ open implies that (X, id_X) is a chart.

Example 12.6 S^1 with charts given by (the inverse of) the exponential.

Example 12.7 Real projective n-space (recall we defined it via the usual quotient map) with the usual charts $x_i \neq 0$ is a manifold[8].

Exercise 12.8. Let H be a hyperplane on real n-projective space and let U be its complement. Find a chart from U to $\mathbb{R}^n$.

Example 12.9 We consider the situation of the implicit function theorem (Forster, Analysis 2). Let $U_1 \subseteq \mathbb{R}^n$ open and $U_2 \subseteq \mathbb{R}^m$ open. Let

$$\begin{aligned} F\colon U_1 \times U_2 &\to \mathbb{R}^m \\ (x, y) &\mapsto F(x, y) \end{aligned}$$

be smooth. Let $(a, b) \in U_1 \times U_2$ be such that $F(a, b) = 0$ and $\frac{\partial F}{\partial y}(a, b)$ is invertible. Then there are open nhds $V_1 \subseteq U_1$ of a and $V_2 \subseteq U_2$ of b such that there is a smooth function $g\colon V_1 \to V_2$ such that $F(x, g(x)) = 0$ for all $x \in V_1$. Moreover, if $(x, y) \in V_1 \times V_2$ such that $F(x, y) = 0$, then $y = g(x)$.

Let $X = \{(x, y) \mid F(x, y) = 0\} \subseteq U_1 \times U_2$ and $U = X \cap (V_1 \times V_2)$. Let $\varphi\colon U \to V_1$ be the projection. Then (U, φ) is a chart. The inverse is given by $x \mapsto (x, g(x))$. E.g. $F(x, y) = x^2 + y^2 - 1$...

Definition 12.10 A *smooth atlas* $(U_i, \varphi_i)_{i \in I}$ on X is a family of real n-dimensional charts on X satisfying the following condition:

1) $(U_i)_{i \in I}$ is an open cover of X.

2) For every $i, j \in I$, the transition map

$$\varphi_{ij}\colon \varphi_j(U_i \cap U_j) \xrightarrow{\varphi_j^{-1}} U_i \cap U_j \xrightarrow{\varphi_i} \varphi_i(U_i \cap U_j)$$

is smooth.

Two smooth atlases $(U_i, \varphi_i)_{i \in I}$ and $(U'_i, \varphi'_i)_{i \in I'}$ are equivalent if their union is an atlas.

A *smooth manifold* is a topological manifold together with an equivalence class of smooth atlases.

[8] Second contability follows from the projection being an open map. For instance, the identity on $\mathbb{R}$ from the euclidean topology to the cofinal topology is not open and its image is not second countable.

Definition 12.11 A *holomorphic atlas* is defined analogously using complex charts and holomorphic transition maps. A *complex manifold* is a topological manifold together with an equivalence class of holomorphic atlases.

A complex manifold of (complex) dimension 1 is called a *Riemann surface.*

Example 12.12 $U \subseteq \mathbb{R}^n$ open, the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, Moebius strip, Klein bottle, real projective space [pictures of the gluing].

Example 12.13 Compute the formula for the transition maps on real projective space.

Lemma 12.14 *Let $U \subseteq \mathbb{R}^N$ be an open subset and $F\colon U \to \mathbb{R}^m$ smooth. Let $X = \{x \in \mathbb{R}^N \mid F(x) = 0\}$. Assume that the Jacobian $(\frac{\partial F_i}{\partial x_j})_{i,j}$ has maximal rank m at all $x \in X$ (in particular, assume that $N \geqslant m$). Then X carries a canonical structure of smooth manifold.*

Proof. Let $x \in X$. After reordering of the coordinates of $\mathbb{R}^n$, we may choose a nbhd of x of the form $U_1 \times U_2 \subseteq \mathbb{R}^{N-m} \times \mathbb{R}^m$ satisfying the assumption of the implicit function theorem. This gives a chart near x. The transition maps are smooth. $\square$

Remark 12.15. The same lemma works in the holomorphic setting.

Example 12.16 $X = \{(x, y) \in \mathbb{C}^2 \mid y^2 = x^3+x^2\}$. We apply the lemma with $F(x, y) = y^2 - x^3 - x^2$, $N = 2$ and $m = 1$. The Jacobian is $(-3x^2 - 2x, 2y)$, which has rank 1 at all points of X except for $(0, 0)$, which is called a *singular point*.

Example 12.17 Every complex manifold of dimension n also defines a smooth manifold of dimension $2n$ by identifying $\mathbb{C}^n \cong \mathbb{R}^{2n}$.

Definition 12.18 Let X, Y be smooth (resp. complex) manifolds. Let $f\colon X \to Y$ be a continuous map. We say that f is *smooth* (resp. *holomorphic*) if for every $x \in X$ there exists a chart (U, φ) with $x \in U$ and a chart (V, ψ) with $f(x) = y \in V$ such that $f(U) \subseteq V$ and $\varphi(U) \xrightarrow{\varphi^{-1}} U \xrightarrow{f|_U} V \xrightarrow{\psi} \psi(V)$ is smooth (resp. holomorphic).

Exercise 12.19. $U \subseteq \mathbb{C}$ open, $f\colon U \to \mathbb{C}$ holomorphic viewed as a holomorphic map of complex manifolds. f non constant. Let $z_0 \in U$. Show that there are charts (U_1, φ_1) around z_0 with $\varphi_1(z_0) = 0$ and (U_2, φ_2) around $f(z_0)$ with $\varphi_2(f(z_0)) = 0$ such that $f(U_1) \subseteq U_2$ and the induced map from $\varphi_1(U_1)$ to $\varphi_2(U_2)$ is given by $z \mapsto z^d$.

13 [WS] Kodaira 3 (Leonardo) - Kodaira Vanishing Theorem - 6.11.19

Let X be a compact complex manifold and E a holomorphic vector bundle[9] on X.

Let $\mathcal{A}^{p,q}(E)$ denote the sheaf of E-valued (p,q)-forms, i.e. $\mathcal{A}^{p,q}(E) = \mathcal{A}^{p,q} \otimes E$. There is no analogue of d on $\mathcal{A}^{p,q}$, but

$$\bar{\partial}\colon \mathcal{A}^{p,q}(E) \to \mathcal{A}^{p,q+1}(E)$$

is well defined. Why? We can take a holomorphic frame $\{e_\alpha\}$ for E, and if we are given $\eta = \sum \eta_\alpha \otimes e_\alpha$ with $\eta_\alpha \in \mathcal{A}^{p,q}$, then $\bar{\partial}\eta = \sum \bar{\partial}\eta_\alpha \otimes e_\alpha$.

$$0 \to \Omega^p(E) \to \mathcal{A}^{p,0}(E) \xrightarrow{\bar{\partial}_E} \mathcal{A}^{p,1}(E) \xrightarrow{\bar{\partial}_E} \cdots$$

where $\Omega^p(E)$ are the p-holomorphic forms on E. This way we obtain *generalized Dolbeault cohomology*:

$$(H^{p,q}(X,E) :=) H^q(X, \Omega^p \otimes E) := \frac{\ker(\mathcal{A}^{p,q}(E) \xrightarrow{\bar{\partial}} \mathcal{A}^{p,q+1}(E))}{\operatorname{im}(\mathcal{A}^{p,q-1}(E) \to \mathcal{A}^{p,q}(E))}$$

13.1 Vector bundles and metrics

Recall that an *Hermitian metric* of X is

$$\sum_{i,j} h_{i,j} dz_i \otimes d\bar{z}_j,$$

] where $\{z_i\}$ is a local holomorphic coordinate system and $h_{i,j}$ is a positive definite hermitian form. This induces a metric on every $\mathcal{A}^{p,q}(X)$ and a volume form

$$\mathrm{vol} = \sqrt{\det(h_{i,j})} dz_1 \wedge d\bar{z}_1 \wedge \cdots$$

We define the *Hodge $*$-operator* $\bar{*}\colon \mathcal{A}^{p,q}(X) \to \mathcal{A}^{n-p,n-q}(X)$ by

$$\alpha \wedge \bar{*}\beta = (\alpha, \beta)\, \mathrm{vol} \in \mathcal{A}^{n,n}(X).$$

[9] Which we sometimes identify with its sheaf of smooth sections. If we want to consider holomorphic sections, then we will use the notation $\mathcal{O}(E)$.

An *Hermitian metric* h_E on E is a hermitian metric on every fiber E_x of E varying smoothly. Such an h_E induces $E \xrightarrow{\sim} \bar{E}^*$ sending $\sigma \mapsto h_E(\sigma, -)$. Now we have the skew-linear map

$$\begin{aligned} \bar{*}_E \colon \mathcal{A}^{p,q}(E) &\to \mathcal{A}^{n-p,n-q}(E^*) \\ \alpha \otimes e &\mapsto \bar{*}(\alpha) \otimes h_E(-, e) \end{aligned}$$

And the maps

$$\begin{aligned} \Lambda \colon \mathcal{A}^{p,q}(E) \otimes \mathcal{A}^{p',q'}(E^*) &\to \mathcal{A}^{p+p',q+q'}(X) \\ (\alpha \otimes e) \otimes (\beta \otimes f) &\mapsto f(e)(\alpha \wedge \beta) \end{aligned}$$

Let $\{e_\alpha\}$ be a unitary frame of E, i.e. $h_E(e_\alpha, e_\beta) = \delta_{\alpha\beta}$.[10] If we have $\eta = \sum \eta_\alpha \otimes e_\alpha$, then $\bar{*}_E(\eta) = \sum \bar{*}\eta_\alpha \otimes e^*_\alpha$, where $\{e^*_\alpha\}$ is the dual unitary frame on E^*. We have $\bar{*}_{E^*}\bar{*}_E = (-1)^{r(2n-r)}$ on $\mathcal{A}^r(E)$.

$\bar{\partial}_E \colon \mathcal{A}^{p,q}(E) \to \mathcal{A}^{p,q+1}(E)$ yields $\bar{\partial}^*_E$ adjoint map, with description $\bar{\partial}^*_E = -\bar{*}_{E^*}\bar{\partial}_{E^*}\bar{*}_E$.

We now consider the laplacian

$$\Delta_{\bar{\partial}_E} = \bar{\partial}^*_E\bar{\partial}_E + \bar{\partial}_E\bar{\partial}^*_E \colon \mathcal{A}^{p,q}(E) \to \mathcal{A}^{p,q}(E)$$

and denote $\mathcal{H}^{p,q}(E) = \{\sigma \in \mathcal{A}^{p,q}(E) \mid \Delta_{\bar{\partial}_E}(\sigma) = 0\}$.

Theorem 13.1 *$\mathcal{H}^{p,q}(E) \cong H^{p,q}(X, E) = H^q(X, \Omega^p \otimes E)$.*

$\bar{*}_E\Delta_{\bar{\partial}_E} = \Delta_{\bar{\partial}_E}\bar{*}_E \Rightarrow \bar{*}_E \colon \mathcal{H}^{p,q}(E) \xrightarrow{\sim} \mathcal{H}^{n-p,n-q}(E^*)$. *The first thing is isomorphic to $H^q(X, \Omega^p \otimes E)$ and the second thing to $H^{n-q}(X, \Omega^{n-p} \otimes E^*)$, and this is known as Kodaira-Serre duality.*

For $p = 0$, $H^q(X, \mathcal{O}(E)) \cong H^{n-q}(X, K_X \otimes E^*)$ where K_X denotes the canonical bundle.

13.2 Connections

Need replacement for $d \colon \mathcal{A}^r(X) \to \mathcal{A}^{r+1}(X)$.

Definition 13.2 Let $E \to X$ be a vector bundle. A *connection* is $\nabla \colon \mathcal{A}^0(E) \to \mathcal{A}^1(E)$ such that $\nabla(f\xi) = df \cdot \xi + f\nabla\xi$ for all $f \in \mathcal{C}^\infty(X)$ and all $xi \in \mathcal{A}^0(E)$.

[10] We can choose either a unitary frame or a holomorphic frame, but not both at the same time. Hence we work rather with smooth sections and consider such a unitary frame.

$\mathcal{A}^1(E) = \mathcal{A}^{1,0}(E) \oplus \mathcal{A}^{0,1}(E)$. $\nabla = \nabla' + \nabla''$. $\nabla' \colon \mathcal{A}^0(E) \to \mathcal{A}^{1,0}(E)$ and $\nabla'' \colon \mathcal{A}^0(E) \to \mathcal{A}^{0,1}(E)$.

Definition 13.3 We say that ∇ is compatible with the complex structure if $\nabla'' = \bar{\partial}_E$, or equivalently if $\nabla''\sigma = 0$ for every holomorphic section σ.

Definition 13.4 ∇ is compatible with the metric on E if for all $\xi, \eta \in \mathcal{A}^0(E)$ we have

$$d\langle \xi, \eta \rangle = \langle \nabla \xi, \eta \rangle + \langle \xi, \nabla \eta \rangle.$$

Theorem 13.5 *There exists a unique connection ∇ on an holomorphic vector bundle E with hermitian metric h_E such that it is compatible with the complex structure and with the metric. It is called the Chern connection.*

We can extend $\nabla \colon \mathcal{A}^r(E) \to \mathcal{A}^{r+1}(E)$ by asking that

$$\nabla(\eta \otimes e) = d\eta \otimes e + (-1)^r \eta \wedge \nabla(e)$$

for $\eta \in \mathcal{A}^r(X)$ and $e \in A^0(E)$.

Definition 13.6 $\nabla^2 \colon \mathcal{A}^0(E) \to \mathcal{A}^2(E)$ is called the *curvature*.

∇^2 is linear over $\mathcal{C}^\infty(X)$. For all $f \in \mathcal{C}^\infty$ and $\sigma \in \mathcal{A}^0(E)$ we have

$$\nabla^2(f\sigma) = \nabla(df \otimes \sigma + f\nabla\sigma) = -df \wedge \nabla\sigma + df \wedge \nabla\sigma + f\nabla^2\sigma.$$

Let E a holomorphic line bundle over X with a Hermitian metric h_E and ∇ the corresponding Chern connection. Let $\{\sigma\}$ be a holomorphic section. Then $\nabla^2\sigma = \Theta \wedge \sigma$, where Θ is a global $(1,1)$-form.

Recall that $c_1(E) = [\frac{i}{2\pi}\Theta] \in H^{1,1}(X)$, where

$$\Theta = \bar{\partial}\partial \log h_E(\sigma, \sigma)$$

Question 13.7. Given a holomorphic line bundle, when can we choose metrics h on X (Kaehler), h_E on E, such that $w = \frac{i}{2\pi}\Theta$, where w a Kaehler form?

Definition 13.8 When this happens, we call the line bundle *positive*.

Proposition 13.9 *If $\exists$ metric h_E on E such that $w \in [\frac{1}{2\pi}\Theta]$, then there exists a metric h'_E such that $w = \frac{1}{2\pi}\Theta'$.*

Example 13.10 Let $X = \mathbb{P}^n$ and consider a hyperplane line bundle $E(= \mathcal{O}(1))$ on $\mathbb{P}^n$. Then E is positive. Indeed, E^* is the tautological line bundle on $\mathbb{P}^n$. Define h_{E^*} by setting

$$|(z_0, \dots, z_n)|^2 = \sum z_i \bar{z}_i \Rightarrow \Theta = [...]$$

So E^* is negative and therefore E is positive.

Theorem 13.11 (Kodaira Vanishing) *Let $E > 0$ (i.e. let E be a positive line bundle). Then $H^q(X, \Omega^p \otimes E) = 0$ for every $p + q > n$, which by Kodaira-Serre duality is equivalent to saying that for every $E < 0$ we have $H^q(X, \Omega^p \otimes E) = 0$ for all $p + q < 0$.*

Proof. Let $L = w \wedge (-) \colon \mathcal{A}^{p,q}(E) \to \mathcal{A}^{p+1,q+1}(E)$ and consider its adjoint operator L^*. In a unitary frame $\{e_\alpha\}$ of E, for $\eta = \sum \eta_\alpha \otimes e_\alpha$, then $L(\eta) = \sum w \wedge \eta_\alpha \otimes e_\alpha$ and $L^*(\eta) = \sum L^*(\eta_\alpha) \otimes e_\alpha$, where L^* adjoint on $\mathcal{A}^{p,q}(X)$. So $[L, L^*]$ is multiplicative on $\mathcal{A}^{p,q}(E)$ by the scalar $(p + q - n)$. [Recall: $[L^*, \bar{\partial}] = -i\partial^*$ on $\mathcal{A}^{p,q}(X)$] Here $[L^*, \bar{\partial}] = -i(\nabla')^*$ on $\mathcal{A}^{p,q}(E)$.

To prove the Kodaira vanishing theorem we only need two inequalities which are called the *Nakano inequalities*: for every harmonic form $\xi \in \mathcal{H}^{p,q}(E)$ we have

1) $\frac{i}{2\pi}\langle \Theta \wedge L_E^*(\xi), \xi\rangle \leqslant 0.$

2) $\frac{i}{2\pi}\langle L_E^*(\Theta \wedge \xi), \xi\rangle \geqslant 0.$

Now if $E > 0$, then from the difference 2) $-$ 1) we deduce

$$\frac{i}{2\pi}\langle L_E^*(\Theta \wedge \xi - \Theta \wedge L_E^* \xi, \xi\rangle \geqslant 0.$$

Since E is positive, we can choose a metric on X such that $w = \frac{i}{2\pi}\Theta$. Hence the previous expression becomes $\langle [L_E^*, L_E](\xi), \xi\rangle \geqslant 0$. By assumption $\xi \in \mathcal{H}^{p,q}(E)$, hence this expression is in turn equivalent to $-(p+q-n)\langle \xi, \xi\rangle \geqslant 0$. So if $p + q > n$, then $\xi = 0$, thus $H^q(X, \Omega^p \otimes E) = \mathcal{H}^{p,q}(E) = 0$. $\square$

Let us now prove the Nakano inequalities, which were used in the previous proof. $\Theta \wedge (-) = \nabla^2 = \nabla^1 \bar{\partial}_E + \bar{\partial}_E \nabla'$ $(\nabla = \nabla' + \bar{\partial}_E)$. $\Delta_{\bar{\partial}_E}(\xi) = 0 \Leftrightarrow \bar{\partial}_E(\xi) = \bar{\partial}_E^*(\xi) = 0$.

$$\begin{aligned} i\langle (\nabla')^*\xi, (\nabla')^*\xi\rangle &= -\langle [L_E^*, \bar{\partial}_E]\xi, (\nabla')^*(\xi)\rangle = \langle \bar{\partial}_E L_E^* \xi, (\nabla'))^*\xi\rangle = \\ &= \langle L_E^*(\xi), \bar{\partial}_E^*(\nabla')^*(\xi) = \langle L_E^*(\xi), [\bar{\partial}_E^*(\nabla')^* + (\nabla')^*\bar{\partial}_E^*](\xi)\rangle = \\ &= \langle \Theta \wedge L_E^*(\xi), \xi\rangle \end{aligned}$$

This shows the first Nakano inequality.

Remark 13.12. As a special case, if E is a positive line bundle, then

$$H^q(X, K_X \otimes E) = 0$$

for all $q > 0$.

14 [LT] Lecture 6 - 7.11.19

Why are we considering real manifolds at all? (We are interested in smooth projective varieties and complex manifolds are much nicer). It turns out that there are things we can do only with real manifolds. This is also where second countability will come into play. Consider:

Example 14.1

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & \text{if } x \geqslant 0 \\ 0 & \text{if } x \leqslant 0 \end{cases}$$

Because of the identity principle, holomorphic functions are too rigid (zero on an open implies zero everywhere). On the other hand, smooth functions can wiggle around. This will be very useful for the following reason:

Theorem 14.2 (Partition of Unity) *Let X be a smooth manifold and $\{U_\alpha\}_{\alpha \in I}$ an open cover. There are smooth functions $f_n \colon X \to \mathbb{R}$ for $n \in \mathbb{N}$ with compact support taking values in $[0, 1]$ such that*

i) for every $n \in \mathbb{N}$ there is some $\alpha \in I$ such that $\mathrm{Supp}(f_n) \subseteq U_\alpha$,

ii) for every $x \in X$ there are only finitely many $n \in \mathbb{N}$ such that $f_n(x) \neq 0$, and

iii) $\sum_{n=1}^{\infty} f_n(x) = 1$.

Example 14.3 On $X = \mathbb{R}$, with $U = \mathbb{R}$, the single function $f_1 = 1$ does not work, because it does not have compact support. Instead, choose smooth positive functions g_n for $n \in \mathbb{Z}$ such that g_n has constant value 1 between n and $n + 1$ and constant value 0 outside of the interval $[n - 1, n + 2]$. [picture] This way we get compact supports and local finiteness. We can sum them up at each point to obtain a smooth function $\sum_{n \in \mathbb{Z}} g_n$. We can divide each term of the sum by $g > 0$ to obtain new smooth functions to obtain the desired result.

Remark 14.4. We used second contability of $\mathbb{R}$ to divide it into countably many intervals.

For a full proof of the previous result, see Warner Theorem 1.11.

Lemma 14.5 (Warner, Lemma 1.9) *Let X be a topological space which is locally compact, Hausdorff and second countable. Then every open cover has a refinement which is countable, locally finite and consisting only of open sets with compact closure.*

Proof. Let $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ be a fixed open cover. As X is second countable, there is a sequence $(V_i)_{i\in\mathbb{N}}$ of open sets such that every open set is a union of such V_i. By local compactness and Hausdorffness we can further assume that all $\bar{V}_i$ are compact, because throwing away those which do not have compact closure still forms a basis. For every $x \in X$ there is an index $i(x) \in \mathbb{N}$ such that $x \in V_{i(x)}$. The family $\{V_{i(x)}\}_{x\in X}$ is a countable open cover of X and all $\bar{V}_i$ are compact. We construct a sequence $G_1 \subseteq G_2 \subseteq \cdots$ of open subsets with compact closure such that $\cup_{i=1}^{\infty} G_i = X$ and $\bar{G}_i \subseteq G_{i+1}$:

$$G_1 \subseteq \bar{G}_1 \subseteq G_2 \subseteq \bar{G}_2 \subseteq \cdots$$

Put $G_1 = V_1$. Let j_2 be the smallest index bigger than 1 such that $\bar{G}_1 \subseteq \cup_{i=1}^{j_2} =: G_2$, which exists by compactness of $\bar{G}_1$. Let j_3 be the smallest index bigger than j_2 such that... etc. Then

$$\cup_{i=1}^{\infty} G_i = \cup_{n=1}^{\infty} \cup_{j=1}^{j_n} V_j = \cup_{i=1}^{\infty} V_i = X.$$

The set $\bar{G}_i \setminus G_{i-1}$ is compact and contained in the open set $G_{i+1} \setminus \bar{G}_{i-2}$. For each $i \geqslant 3$ choose a finite subcover of the open cover $\{U_\alpha \cap (G_{i+1} \setminus \bar{G}_{i-2})\}_\alpha$ of $\bar{G}_i \setminus G_{i-1}$. Also choose a finite open subcover of $\{U_\alpha \cap G_\xi\}$ of $\bar{G}_2$. This is finally the refinement we wanted. $\square$

Now we can give the idea of the proof of Thm 1.19. A partition of unity subordinated to a refinement of $\{U_\alpha\}$ is also a partition of unity subordinated to the original open cover. We replace our cover by the refinement in the lemma. Wlog the cover is countable, locally finite and $\bar{U}_\alpha$ is compact. This looks like the cover of $\mathbb{R}$ given in the previous exmaple, and then we find suitable smooth functions, etc.

14.1 Vector bundles

Our goal is to define the tangent bundle of a manifold.

Example 14.6 Let $X = S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. The tangent line at $(x_0, y_0) \in S^1$ has the parametrization $t \mapsto (x_0, y_0) + t(y_0, -x_0)$. The tangent space is given by

$$T_{(x_0,y_0)}S^1 = \{t(y_0, -x_0) \mid t \in \mathbb{R}\}.$$

Together they form the tangent bundle $TS^1 = \{(v, w) \in S^1 \times \mathbb{R}^2 \mid v \perp w\}$. This comes with a projection $p\colon TS^1 \to S^1$. The fibre of p in TS^1 has the structure of a vector space.

Definition 14.7 Let X be a topological space. A *real vector bundle* of rank r over X consists of a continuous map $\pi\colon V \to X$ together with the structure of a real vector space of dimension r on each $V_x = \pi^{-1}(x)$ such that there exists an open cover $\{U_i\}$ of X and local trivializations

$$\begin{tikzcd} \pi^{-1}(U_i) \arrow[rr, "\sim"] \arrow[dr, "\pi|_{\pi^{-1}(U_i)}"'] & & U_i \times \mathbb{R}^r \arrow[dl] \\ & U_i & \end{tikzcd}$$

such that for every $x \in U_i$ the induced map $V_x \to \{x\} \times \mathbb{R}^r \to \mathbb{R}^r$ is an isomorphism of vector spaces.

Let $f\colon X \to Y$ be continuous and let $\pi\colon V \to X$ and $\xi\colon W \to Y$ two real vector bundles. A morphism of vector bundles over f is a continuous map $F\colon V \to W$ such that the following diagram commutes

$$\begin{tikzcd} V \arrow[r, "F"] \arrow[d, "\pi"] & W \arrow[d, "\xi"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

and such that the induced maps $V_x \to W_{f(x)}$ is $\mathbb{R}$-linear.

If we replace $\mathbb{R}$ by $\mathbb{C}$ in this definition we get *complex vector bundles*, and a vector bundle of rank 1 is called a *line bundle*.

Remark 14.8. Every complex vector bundle defines a real vector bundle.

Exercise 14.9. Let $\pi\colon V \to X$ be a vector bundle of rank r. Show that it is trivial if and only if it there are r continuous section such that on each point their images form a basis of the fibre.

15 [FS] Harry Schmidt - Uniformizations of polynomial dynamical systems - 8.11.19

Let K be a number field and K_v a completion at a place $v \in M_K := \{ \text{places of } K\}$. We denote by $\bar{K}_v$ the algebraic closure of K_v.

Let $f \in K[z]$ be a monic polynomial of degree $d \geqslant 2$. We define $f^{0,n} = f(f^{0,-1})$ and $f^{0,0} = z$.

Remark 15.1. Let $f\colon \mathbb{P}^1 \to \mathbb{P}^1$. If $\mu\colon \mathbb{P}^1 \to \mathbb{P}^1$ is a Moebius transformation, then $\mu \circ f \circ \mu^{-1}$ has the same dynamics as f.

Theorem 15.2 (Botall, Jones, S.) *Let $\alpha \in K$. Define*

$$\mathcal{L}_\alpha^{(n)} = \{\beta \in \bar{K} \mid f^{0,n}(\beta) = f^{0,n}(\alpha)\}.$$

There exists $\beta \in \mathcal{L}_\alpha^{(n)}$ such that

$$(*) \quad [K(\beta) : K] \underset{f,\alpha,K}{>>} d^{n/5}$$

.

Example 15.3 (P. Ingram) $f = z^d$, $\mathcal{L}_\alpha^{(n)} = \alpha\mu_{d^n}$, where $\mu_{d^n} = \{\xi \mid \xi^{d^n} = 1\}$. There exists $\Phi \in \frac{1}{z} + \cdots \in K[[\frac{1}{z}]]$ with the property $\Phi(f(z)) = \Phi(z)^d$.

Theorem 15.4 (Ingram, de Marco et al) *For each place $v \in M^K$ there exists $\delta_v \geqslant 1$ such that Φ converges on*

$$D_\infty^{\delta_v} = \{z \in \bar{K}_v \mid |z|_v > \delta_v\},$$

and $\delta_v = 1$ for almost all v.

Idea: let n be minimal such that $f^{0,n}(\alpha) = f^{0,n}(\beta)$. Then $\Phi(\beta) = \xi\Phi(\alpha)$ for a root of unity ξ of order $N \geqslant 2^n$. If v is a finite place, then $[K_v(\xi) : K_v] \underset{K,\varepsilon}{} N^{1-\varepsilon}$.

If v is infinite, then we write $\xi = \exp(2\pi i \frac{m}{N})$ for $m \in \mathbb{Z}$ and $(m, N) = 1$. Observe that $\beta = \Phi^{-1}(\exp(2\pi i \frac{m}{N})\Phi(\alpha))$. We study the function $g(\tau) = \Phi^{-1}(\exp(2\pi i \tau)\Phi(\psi))$, $\mathrm{im}(\tau) \geqslant 0$ and $g(\tau)$ is definable in $\mathbb{R}_{an,exp}$.

Theorem 15.5 (Serre, Tate, Ihora '60) *Suppose $C \subseteq \mathbb{G}_m^2$ is an irreducible curve with generic point (X_e, Y_e) and that*

$$|\mathcal{C} \cap \mu_\infty^2| = \infty \Rightarrow \exists n, m \geqslant 0, , X_e^n Y_e^n = 1.$$

Conjecture 15.6 *Suppose $\mathcal{C}$ is an invertible curve in $\mathbb{A}^2$. Then if $|\mathcal{C} \cap (\mathcal{L}_\alpha^\infty)^2| = \infty$, $\mathcal{C}$ is a component of a curve defined by $f^{0,n}(X) - f^{0,n}(Y) = 0$ or $\mathcal{C}$ is of the form $\beta \times \mathbb{A}^1$, $\mathbb{A}^1 \times \beta$ for $\beta \in \mathcal{L}_\alpha^\infty$.*

Here $\mathcal{L}_\alpha^\infty = \cup_{n\in\mathbb{N}} \mathcal{L}_\alpha^{(n)}$.

Remark 15.7. If we had $(*)$ for all primitive solutions of $f^{0,n}(X) = f^{0,n}(\alpha)$, then we could use the Pola-Zimmer strategy. We do have a Galois-bound if $|f^{0,n}(\alpha)| \to \infty$ for v a finite place. To apply the strategy we would also need $|f(\alpha)|_v \to \infty$ at infinite places.

Theorem 15.8 *Suppose $|f^{0,n}(\alpha)|_v \to \infty$ at a finite place $v \in M_K$. Then the conjecture is true.*

Idea: use the following elementary

Lemma 15.9 *Let $(\xi_1, \xi_2) \in \mathbb{G}_m^2(\bar{\mathbb{Q}})$ be of order N. There exists ξ_N a primitive N-th root of 1 and ξ_e, ξ_e' two e-th roots of 1 such that*

$$(\xi_1, \xi_2) = (\xi_e \xi_N^{K_1}, \xi_e' \xi_N^{K_2}),$$

where $e \cdot \max\{|K_1|, |K_2|\} \leqslant N^{3/4}$.

Proof. The idea is to apply the pigeon principle. Let $P \in K[X, Y]$ be an irreducible polynomial such thath $P(\xi_1, \xi_2) = 0$. Then $Q(x) = P(\xi_e X^{K_1}, \xi_e' X^{K_2})$ vanishes at $X = \xi_N$. We have $Q(X) \in K[X, \frac{1}{X}]$ and $\deg Q \leqslant 2 \deg P \frac{N^{3/4}}{e}$. Assuming $Q \neq 0$, the number of zeroes of Q is $<<$ than $\deg P N^{3/4}$. But [...] $\square$

In our situation we consider

$$Q(x) = P(\Phi^{-1}(\xi_e X^{K_1} \Phi(\alpha)), \Phi^{-1}(\xi_e' X^{K_2} \Phi(\alpha))),$$

which defines a v-adic analytic function. Then we can use some theory for v-adic functions.

16 [CM] Talk 4 (Oliver Braunling) - Locally compact abelian groups - 11.11.19

Goal of this lecture: there is a "classical" approach to locally compact abelian groups giving an exact category called LCA and a bounded derived category $\mathcal{D}^{\mathrm{b}}(\mathrm{LCA})$, and we will show that $\mathcal{D}^{\mathrm{b}}(\mathrm{LCA}) \hookrightarrow \mathcal{D}^{\mathrm{b}}(\mathrm{Cond}(\mathcal{A}b))$ is a fully faithful functor.

Definition 16.1 A topological group is called *locally compact* if it posseses a compact neighbourhood of the identity element.

Remark 16.2. Compact in the previous definition is french compact, i.e. Hausdorff and quasi-compact. In particular, locally compact abelian groups are Hausdorff.

Theorem 16.3 (Theorem 4.1) *i) Let A be a locally compact abelian group. Then there is an integer n and an iso $A \cong \mathbb{R}^n \times A'$, where A' admits a compact open subgroup. Hence we can write*[11]

$$C \hookrightarrow A' \twoheadrightarrow A'/C$$

with C compact and A'/C discrete.

ii) Pontryagin duality induces an equivalence of categories

$$\begin{aligned} \mathrm{LCA}^{op} &\xrightarrow{\sim} \mathrm{LCA} \\ A &\mapsto \mathrm{Hom}_{\mathcal{T}op\mathcal{A}b}(A, \Pi) \end{aligned}$$

where the hom-space is considered with the compact open topology and $\mathbb{T} := U(1) = \mathbb{R}/\mathbb{Z}$. We denote the hom-space above by $A^\vee$.

iii) Pontryagin duality exchanges compact and discrete groups.

Remark 16.4. i) A product $\prod_{i\in I} \mathbb{R}$ is locally compact if and only if $|I| < \infty$.

ii) Open implies clopen for topological groups.

iii) $H \subseteq G$ open if and only if G/H is discrete.

Proposition 16.5 (Proposition 4.2) *Let A, B be Hausdorff topological abelian groups and A, B compactly generated. Then there is a natural isomorphism of condensed abelian groups*

$$\underline{\mathrm{Hom}}(\underline{A}, \underline{B}) \xrightarrow{\sim} \underline{\mathrm{Hom}_{\mathcal{T}op\mathcal{A}b}(A, B)}$$

[11] During this talk, $\hookrightarrow$ means injective and closed and $\twoheadrightarrow$ means surjective and open.

Proof. Recall that we have this functor $\underline{(-)}\colon \mathcal{T}op\mathcal{C} \to \mathrm{Cond}(\mathcal{C})$ sending $T \mapsto \underline{T}$, where $\underline{T}(S) = \mathrm{Hom}_{\mathcal{T}op}(S, T)$. Recall Proposition 1.7. in the script: this functor is faithful, and restricted to the category of compactly generated Hausdorff objects it is also full. Recall also that if S is profinite and M is a condensed abelian group, then

$$M(S) = \mathrm{Hom}_{\mathrm{Cond}(\mathcal{A}b)}(\mathbb{Z}[S], M).$$

Then

$$\underline{\mathrm{Hom}}(\underline{A}, \underline{B})(S) = \mathrm{Hom}_{\mathrm{Cond}(\mathcal{A}b)}(\mathbb{Z}[S], \underline{\mathrm{Hom}}(\underline{A}, \underline{B})) \overset{adj.}{\cong} \mathrm{Hom}_{\mathrm{Cond}(\mathcal{A}b)}(\mathbb{Z}[S] \otimes \underline{A}, \underline{B})$$

and

$$\underline{\mathrm{Hom}(A, B)}(S) = \mathrm{Hom}_{\mathcal{T}op}(S, \mathrm{Hom}_{\mathcal{T}op\mathcal{A}b}(A, B)) \overset{adj.}{\cong} \mathrm{Hom}_{\mathcal{T}op}(S \times A, B) = \mathrm{Hom}_{\mathrm{Cond}(Set)}(\underline{S \times A}, B)$$

Check carefully what is meant by each hom-set.

□

Theorem 16.6 (Theorem 4.3) *Suppose $A = \prod_I \mathbb{T}$ be a condensed abelian group, where I is any index set.*

i) For any discrete abelian group M,

$$R\,\underline{\mathrm{Hom}}(A, M) = \bigoplus_I M[-1].$$

ii) $R\,\underline{\mathrm{Hom}}(A, \mathbb{R}) = 0$.

First some preparation:

Theorem 16.7 (Theorem 4.5, Eilenberg-MacLane, Breen, Deligne) *There is a functorial resolution of an abelian group A of the form*

$$(*) \quad \cdots \to \oplus_{j=1}^{n_i} \mathbb{Z}[A^{r_{i,j}}] \to \cdots \mathbb{Z}[A^3] \oplus \mathbb{Z}[A^2] \to \mathbb{Z}[A^2] \to \mathbb{Z}[A] \to A \to$$

for finite $n_i, r_{i,j}$.

Corollary 16.8 (Corollary 4.8) *For condensed abelian groups A, M and any extremally disconnected set S there is a spectral sequence*

$$E_1^{i_1,i_2} = \prod_{j=1}^{n_{i_1}} H^{i_2}_{Cond}(A^{r_{i_1,i_2}} \times S, M) \Rightarrow \underline{\mathrm{Ext}}^{i_1+i_2}(A, M)(S),$$

where $\underline{\mathrm{Ext}}^i(X, Y) = R\,\underline{\mathrm{Hom}}(X, Y[i])$.

Proof. Recall $R\,\underline{\mathrm{Hom}}(A,M)(S) = R\,\mathrm{Hom}(A\otimes\mathbb{Z}[S],M)$, and for finite direct sumes we have $\mathbb{Z}[A^i]\otimes\mathbb{Z}[S]\cong\mathbb{Z}[A^i\times S]$. We use the resolution of Theorem 4.5 and the previous iso to compute $R\,\mathrm{Hom}(A\otimes\mathbb{Z}[S],M)$ as

$$\cdots\to\bigoplus\mathrm{Hom}(\mathbb{Z}[A^{r_{i,j}}\times S],M)\to\cdots,$$

then do this resolution for each term and apply some horse-shoe lemma on projective resolutions on the LHS of the previous homs to conclude with the spectral sequence of the double complex. $\square$

Remark 16.9. The previous corollary is functorial in A, M and in S.

Now we prove the theorem (Theorem 4.3).

Proof of *i*) first. If I finite, we can assume that I is a singleton. Let's just assume that $A=\mathbb{T}$. We need to compute $\mathbb{R}\,\mathrm{Hom}(\mathbb{T},M)$.. We have a triangle $\mathbb{Z}\to\mathbb{R}\to\mathbb{T}\xrightarrow{+1}$, which rotated yields a distinguished triangle

$$\mathbb{R}\to\mathbb{T}\to\mathbb{Z}[1]\xrightarrow{1}.$$

So the claim is equivalent to the claim $\mathbb{R}\,\mathrm{Hom}(\mathbb{R},M)=0$ for any discrete group M, because we have a distinguished triangle

$$\mathbb{R}\,\underline{\mathrm{Hom}}(\mathbb{Z}[1],M)\to R\,\mathrm{Hom}(\mathbb{T},M)\to R\,\mathrm{Hom}(\mathbb{R},M)\xrightarrow{+1}$$

It suffices to show that $0\to\mathbb{R}$ induces an iso after applying $R\,\mathrm{Hom}(-,M)$, for which in turn it is sufficient to know that the pullback $0\times\mathrm{id}\colon S\to\mathbb{R}^r\times S$ induces an iso in all cohomology groups

$$H^i(\mathbb{R}^r\times S,M))\to H^i(S,M).$$

Now we use the result that $R\Gamma(\mathbb{R}^r\times S,M)=\mathbb{R}\varprojlim_r R\Gamma([-n,n]^r\times S,M)$, and since $[-n,n]\times S$ is compact, by the last talk this is classical sheaf cohomology, where we can use homotopy invariance. Then use functoriality in the previous lemma to conclude.

Proof of *ii*) now. For this we need a lemma (Proposition 4.17): consider a functorial resolution $F(A)_\bullet$ of an abelian group such taht $F(A)_i\cong\oplus_{fin}\mathbb{Z}[A^{(fin)}]$. Then for any natural number $n\geqslant 1$, the map $(-\cdot n)\colon F(*)\to F(A)$ and the map $[n]\colon F(A)\to F(A)$ induced from multiplication on A are homotopic by a functorial homotopy $h_\bullet\colon F(A)_\bullet\to F(A)_{\bullet+1}$. By theorem 3.3 from lat lecture, $R\,\mathrm{Hom}(A,\mathbb{R})(S)$ can be computed by the complex

$$0\to\oplus_{j=1}^{n_0}C(A^{r_{0,j}}\times S,\mathbb{R})\to\oplus_{j=1}^{n_1}C(A^{r_{1,j}}\times S,\mathbb{R})\to\cdots$$

of Banach spaces (i.e. differentials are continuous etc). Assume $f \in \oplus_{j=1}^{n_i} C(A^{r_{i,j}} \times S, \mathbb{R})$ satisfies $df = 0$. Then $2f = [2]^*(f) + d(h_{i-1}^*(f))$. Hence

$$f = \frac{1}{2}[2]^*(f) + d(\frac{1}{2}h_{i-1}^*(f)).$$

Now we iterate this idea! After n iterations we obtain

$$f = \frac{1}{2^n}[2^n]^*(f) + d(\frac{1}{2}h_{i-1}^*(f) + \frac{1}{4}h_{i-1}^*([2]^*(f)) + \ldots).$$

Note that $[2^n]^*(f)$ stays bounded as $n \to \infty$.

Corollary 16.10 $R\underline{\mathrm{Hom}}(\mathbb{R}, \mathbb{R}) = \mathbb{R}$ *using again the triangle* $\mathbb{Z} \hookrightarrow \mathbb{R} \twoheadrightarrow \mathbb{T}$.

Recall the goal theorem: $\mathcal{D}^{\mathrm{b}}(\mathrm{LCA}) \hookrightarrow \mathcal{D}^{\mathrm{b}}(\mathrm{Cond}(\mathcal{A}b))$ is fully faithful. LCA is a quasi-abelian category. There are two approaches to define its bounded derived category. One of them by J.P. Schneider using these quasi-abelian categories. Another one was by Hoffmann-Spitzweck, which is using exact categories of Quillen. They showed that these are a replacement for quasi-abelian category. We may not have kernels and cokernels, so we cannot talk about cohomology objects! How do we define the derived category? Here are the steps:

1. Define the category of complexes.
2. Go to the homotopy category.
3. Define the derived category as the Verdier quotient of the homotopy category by acyclic complexes, where a complex is called acyclic if each differential factors as $d\colon C^n \twoheadrightarrow Z^{n+1}A \hookrightarrow A^{n+1}$ and $Z^nA \hookrightarrow A^n \twoheadrightarrow Z^{n+1}A$.

Now the proof idea goes as follows. We first set up a functor from $\mathcal{D}^{\mathrm{b}}(\mathrm{LCA})$ to $\mathcal{D}^{\mathrm{b}}(\mathrm{Cond}(\mathcal{A}b))$ (this we skip). For the full faithfulness statement we need to show that $\mathbb{R}\,\mathrm{Hom}_{\mathcal{D}^{\mathrm{b}}(\mathrm{LCA})}(A, B) \to \mathbb{R}\,\mathrm{Hom}_{\mathcal{D}^{\mathrm{b}}(\mathrm{Cond}(\mathcal{A}b))}(\psi A, \psi B)$ is an isomorphism of abelian groups. We can reduce to the case of $A, B \in \mathrm{LCA}$. By the structure theorem for LCA groups, any LCA group has the shape $\mathbb{R} \times A'$ with $C \hookrightarrow A' \twoheadrightarrow D$ with C compact and D discrete. D is a $\mathbb{Z}$-module, hence has a projective resolution $\oplus\mathbb{Z} \to \oplus\mathbb{Z} \twoheadrightarrow D$. Again arguing with triangles we reduce to direct sums of $\mathbb{Z}$, then reduce to $\mathbb{Z}$. For the compact use triangles with tori.

17 [LT] Lecture 7 - 12.11.19

$\pi\colon V \to X$ smooth vector bundle. The data of the transition maps is then equivalent to the data of a smooth map $U_i \cap U_j \xrightarrow{\varphi_{i,j}} \mathrm{GL}_r(\mathbb{R})$. However, not every such map will do, because transition maps have a certain compatibility on triple intersections. More precisely, they satisfy the *cocycle condition*

$$\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$$

over $U_i \cap U_j \cap U_k$.

Exercise 17.1. The cocycle condition implies that $\varphi_{ii} = \mathrm{id}$.

Proposition 17.2 *Let X be a topological space. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover and consider continuous maps $U_i \cap U_j \to \mathrm{GL}_r(\mathbb{R})$ such that the cocycle conditions is satisfied. Then there is a real vector bundle $\pi\colon V \to X$ of rank r with transition functions $\tau_{ij}\colon U_i \cap U_j \times \mathbb{R}^r \to U_i \times U_j \times \mathbb{R}^r$ given by $(x, v) \mapsto (x, \varphi_{ij}(x)(v))$ together with local trivializations on $\mathcal{U}$. Moreover, this data is unique up to unique isomorphism.*

Proof. We glue $V_i = U_i \times \mathbb{R}^r$ along the open subsets $U_i \cap U_j \times \mathbb{R}^r \subseteq V_i$ along τ_{ij}. This is well-defined because of the cocycle condition. We get a topological space V. By construction there is a continuous map $\pi\colon V \to X$. The fibre above x is $\{x\} \times \mathbb{R}^r$. This defines a vector space structure. We get local trivializations on $\pi^{-1}(U_i) \cong V_i \xrightarrow{\sim} U_i \times \mathbb{R}^r$. □

Example 17.3 $X = S^1 \subseteq \mathbb{R}^2$. Cover the circle by $U_i = \{(x, y) \in \mathbb{R}^2 \mid x \neq (-1)^i 1\}$ for $i \in \{1, 2\}$. The trivial bundle $S^1 \times \mathbb{R}$ has the transition map $U_1 \cap U_2 \to \mathrm{GL}_1(\mathbb{R}) = \mathbb{R}^\times$ given by $p \mapsto 1$ for all $p \in S^1$. The Moebius bundle on the other hand has the transition map

$$p \mapsto \begin{cases} 1 & \text{if } y > 0, \\ -1 & \text{if } y < 0. \end{cases}$$

Lemma 17.4 *Let X be a topological space and $\pi\colon V \to X$ a vector bundle of rank r. Then there is a canonical bundle $\xi\colon V^* \to X$, called the dual bundle of V, such that $\xi^{-1}(x) = V_x^* = \mathrm{Hom}_{\mathbb{R}}(V_x, \mathbb{R})$.*

Proof. We will specify the gluing data. We use local trivializations on the cover U_i of X and transition maps $\varphi_{ij}\colon U_i \cap U_j \to \mathrm{GL}_r(\mathbb{R})$ for V. Put $\psi_{ij} = (\varphi_{ij}^{-1})^{tr}$, where the inverse and the transpose are taken pointwise on X. □

Exercise 17.5. Weite out all the details.

The same method works for other constructions, e.g. $\oplus$, $\otimes$, exterior powers...

Definition 17.6 i) Let X be a smooth manifold. A smooth real or complex vector bundle is a smooth map $\pi\colon V \to X$ of smooth manifolds together with the structure of real or complex topological vector bundle such that the trivializations are smooth.

ii) Let X be a complex manifold. A holomorphic vector bundle is a holomorphic map of complex manifolds $\pi\colon V \to X$ together with the structure of a complex vector bundle such that the trivializations are holomorphic.

17.1 Tangent bundles

Let X be a smooth manifold of dimension n and $x \in X$. We want to define the tangent space of X in x. Tangent vectors are "directions".

Example 17.7 If $X \subseteq \mathbb{R}^n$ is open, then the tangent space at each point is $\mathbb{R}^n$. Every $v \in \mathbb{R}^n \setminus \{0\}$ defines a straight line $t \mapsto x + tv$ through x. If (U, φ) is a chart around x, we could use preimages of straight lines. But the result depends on the chart.

Definition 17.8 Let X be a smooth manifold and $x \in X$. A *smooth path* through x is a smooth map $\gamma\colon (-\varepsilon, \varepsilon) \to X$ for some $\varepsilon > 0$ such that $\gamma(0) = x$. We say that two paths (through x) are equivalent if they have the same tangent vector in $\mathbb{R}^n$ for some chart. A *tangent vector* in X at x is an equivalence class of paths through x. Let T_xX denote the set of tangent vectors.

Remark 17.9. By the chain rule, the previous equivalence relation is chart independent.

There is a more elegant approach. Recall that a path defines a directional derivative:

Definition 17.10 Let X be a smooth manifold and $x \in X$. Let γ be a smooth path through x. Let $f\colon U \to \mathbb{R}$ be a smooth function on an open nbhd U of x. We put

$$\partial_\gamma f = \lim_{h \to 0} \frac{f(\gamma(h)) - f(x)}{h} = (f \circ \gamma)'(0),$$

and call it the *derivative of f in the direction of γ*.

Lemma 17.11 *Two paths define the same tangent vector at x if and only if for all smooth $f\colon U \to \mathbb{R}$ the directional derivatives agree.*

Proof. Let γ_1, γ_2 be equivalent paths. We choose a chart (V, φ) around x. Up to some shrinking we may assume $V = U$. We may replace X by $\varphi(V)$, f by $f \circ \varphi^{-1}$ and γ_i by $\varphi \circ \gamma_i$. This does not change the directional derivatives.

$$\partial_{\gamma_i} f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right) \begin{pmatrix} \gamma'_{i_1}(0) \\ \vdots \\ \gamma'_{i_n}(0) \end{pmatrix}$$

Hence it only depends on the equivalence class of the paths. Conversely, assume γ_1, γ_2 define the same directional derivatives. Choose a chart (U, φ) around x. Consider the coordinate functions $x_i\colon \mathbb{R}^n \to \mathbb{R}$. Then consider $f_i = x_i \circ \varphi\colon U \to \mathbb{R}$. The tangent vectors are then given on each component by the directional derivatives of the f_i. $\square$

18 [WS] Kodaira 4 (Yuhang): Quadratic transformations

Recall: let X be a complex manifold and let $E \to X$ be a vector bundle on X. Denote the associated connection by D. Then the curvature Θ is defined as

$$\Theta = D^2 = D \circ D\colon \mathcal{A}^0(E) \to \mathcal{A}^2(E).$$

Proposition 18.1 *Given a Hermitian holomorphic line bundle $E \to X$ with Hermitian metric h and the induced connection D_h, we have*

1. *Θ is a $(1,1)$-type.*
2. *$\bar{\partial}\Theta = 0$.*
3. *If given a locally finite cover $\{U_i\}$ of X which is a holomorphic trivialization, with holomorphic frame f_i on U_i, then*

$$\Theta(f_i) = \bar{\partial}\partial \log(h_i).$$

Lemma 18.2 *Given two Hermitian line bundles E, F over X with hermitian metrics h_E, h_F, then the associated curvature satisfies the following*

$$\Theta_{E\otimes F} = \Theta_E + \Theta_F.$$

Proof. Consider cover $\{U_i\}$ of X, then locally on U_i we have

$$\Theta_{E\otimes F} = \bar{\partial}\partial \log(h_{E,i}h_{F,i}) = \bar{\partial}\partial \log(h_{E,i}) + \bar{\partial} \log(h_{E,i}) = \Theta_E + \Theta_F.$$

□

18.1 Quadratic transformations

Let X be a complex manifold and $p \in X$. Let U be a coordinate nbhd of p in X with local coordinates $z_1, \ldots, z_n$ such that $p = (0, \ldots, 0)$ with respect to this coordinates. Consider the product $U \times \mathbb{P}^{n-1}$, where we use $[t_1, \ldots, t_n]$ as homogeneous coordinates for $\mathbb{P}^{n-1}$. Then let

$$W = \{((z_1, \ldots, z_n), [t_1, \ldots, t_n]) \in U \times \mathbb{P}^{n-1} \mid t_i z_j = t_j z_i \text{ for all } i, j\}.$$

1. There is a holomorphic projection $\pi\colon W \to U$ sending $(z, t) \mapsto z$.
2. $S := \pi^{-1}(0) \cong \mathbb{P}^{n-1}$.
3. $\pi|_{W\setminus S}$ is biholomorphic.

Definition 18.3 The *quadratic transformation* $\tilde{X} = Q_p(X)$ of X at p is defined as

$$\tilde{X} = W \bigcup_{U\setminus\{p\}} (X \setminus \{p\})$$

with projection $\pi_p\colon \tilde{X} \to X$.

We set $\tilde{U}_i \subseteq W$ as the set given by $t_i \neq 0$, the local coordinates of $\tilde{U}$ being given by

$$z(i)_j = \begin{cases} \frac{z_j}{z_i} = \frac{t_j}{t_i} & \text{if } j \neq i, \\ z_i & \text{if } i = j. \end{cases}$$

and $\pi_{\tilde{U}_i}$ being given by

$$(z(i)_1, \ldots, z(i)_n) \mapsto (z(i)_1 z(i)_i, \ldots, z(i)_i, \ldots, z(i)_n z(i)_i).$$

Example 18.4 Consider $V = \{x^2 = y^2\} \subseteq \mathbb{C}^2$.

1. $\pi_0^{-1}(V \setminus \{0\}) \subseteq Q_0(\mathbb{C}^2)$ is given by the equation

$$\{((z_1, z_2), [t_1, t_2]) \in \mathbb{C}^2 \times \mathbb{P}^1 \mid z_1 t_2 = z_2 t_1, z_1^2 = z_2^2, (z_1, z_2) \neq (0,0)\}$$

2. Consider the affine chart with $t_1 = 1$. Then we have equations

$$\begin{aligned} \{(z_1, z_2, t_2) \in \mathbb{C}^3 \mid z_1 t_2 = z_2, z_1^2 = z_0^2, (z_1, z_2) \neq (0,0)\} = \\ = \{(z_1, z_1 t_2, t_2) \in \mathbb{C}^3 \mid t_2^2 = 1, z_1 \neq 0\} = \\ = \{(z_1, z_1, 1) \in \mathbb{C}^3 \mid z_1 \neq 0\} \cup \{(z_1, -z_1, -1) \in \mathbb{C}^3 \mid z_1 \neq 0\} \end{aligned}$$

We take the closure to obtain the final equations

$$\{(z_1, z_1, 1) \in \mathbb{C}^3\} \cup \{(z_1, -z_1, -1) \in \mathbb{C}^3\}$$

18.2 Divisors

Definition 18.5 Let X be a complex manifold. An *analytic subvariety* of X is a closed subset $Y \subseteq X$ such that for any point $x \in X$ there exists an open nbhd $x \in U \subseteq X$ such that $Y \cap U$ is the zero locus of finitely many holomorphic functions $f_1, \dots, f_r \in \mathcal{O}_X(U)$. An analytic subvariety Y is called *irreducible* if it cannot be written as $Y = Y_1 \cup Y_2$ for some proper analytic subvarieties $Y_i \subseteq Y$.

Definition 18.6 An *analytic hypersurface* is an analytic subvariety $Y \subseteq X$ of codimension 1.

Definition 18.7 A *Weil divisor* D on X is a formal linear combination

$$D = \sum a_i [Y_i]$$

with $a_i \in \mathbb{Z}$ and $Y_i \subseteq X$ irreducible analytic hypersurfaces. The divisor group is denoted by $\mathrm{Div}(X)$.

Facts:

1. Weil divisors are the same as Cartier divisors on a complex manifolds.
2. Up to linear equivalence and if X is projective, divisors are also the same as elements of the Picard group.

Let X be a compact complex manifold and $\pi_p \colon \tilde{X} \to X$ a quadratic transformation. Let $S = \pi_p^{-1}(p)$ be the exceptional divisor, which locally in W is given by $z_1 = \ldots = z_n = 0$. The line bundle associated to S is denoted by $L(S) \to \tilde{X}$. We denote the hyperplane bundle $H \to S$ and its given by $(t_i = 0)$.

$$\begin{aligned} \sigma \colon W &\to \mathbb{P}^{n-1} \\ (z,t) &\mapsto t \end{aligned}$$

Proposition 18.8 $L|_W = \sigma^* H^\vee$.

Proof. In $\mathbb{P}^{n-1}$, the hyperplane is given by $[t_j/t_i = 0]$ in $V_i \subseteq \mathbb{P}^{n-1}$. Over $V_i \cap V_j$ we have

$$h_{ij} = \frac{t_i}{t_j}\left(\frac{t_i}{t_j}\right)^{-1} = \frac{t_j}{t_i}.$$

$\sigma^* H$ has the same transition in $(U \times (V_i \cap V_j)) \subseteq W$. For $L|_W$ in $((U \times V_i)) \subseteq W$, S is given by $[z_i = 0]$, thus the transition function is

$$g_{ij} = \frac{z_i}{z_j}.$$

Since $g_{ij} = g_{ij}^{-1}$, $L|_W = \sigma^* H^\vee$. □

Lemma 18.9 *Denoting the canonical line bundles by $K_{(-)}$, we have*

$$K_{\tilde{X}} = \pi_p K_X \otimes L_p^{n-1}.$$

Proof. Since π_p on $\tilde{X} \backslash W$ is biholomorphic and $L_p|_{X \backslash W}$ is trivial, the pullback of the canonical is the canonical over this open. On W, observe that K_X on U is trivial. We only need to show that $K_{\tilde{X}}|_W = L_p^{n-1}|_W$. Consider $f = dz_1 \wedge \cdots \wedge dz_n$ on $U \subseteq X$. Then $\pi_p^* f$ on $\tilde{U}_i$ is given by

$$f_i = z_i^{n-1} dz(i)_1 \wedge \cdots \wedge dz(i)_n.$$

□

Let $L_p \to \tilde{X}$ be the line bundle associated to the exceptional divisor S. If we blow up 2 points $p \neq q$, we get $Q_p(Q_q(X)) \cong Q_q(Q_p(X))$. We denote the projection $\pi_{p,q}$ and $L_{p,q}$ the dual of the line bundle associated to the divisor $\pi_{p,q}^{-1}(\{p,q\})$.

Proposition 18.10 *Let $E \to X$ be a positive holomorphic line bundle. Then there exists an integer μ_0 such that if $\mu > \mu_0$, then for any points $p, q \in X$ with $p \neq q$ we have that*

1. $\pi_p^* E^\mu \otimes L_p^\vee \otimes K_{Q_p(X)}^\vee$,
2. $\pi_p^* E^\mu \otimes (L_p^\vee)^2 \otimes K_{Q_p(X)}^\vee$, *and*
3. $\pi_{p,q}^* E^\mu \otimes L_{p,q}^\vee \otimes K_{Q_p Q_q(X)}^\vee$

are all positive holomorphic line bundles.

[Recall the definition of positive line bundle and the fact that the hyperplane bundle is positive.]

19 [LT] Lecture 8 - 14.11.19

Definition 19.1 Let X be a smooth (resp. complex) manifold, $x \in X$ a point and $U, V \subseteq X$ open nbhds of x in X. Let $f\colon U \to \mathbb{R}$ (resp. $\mathbb{C}$) and $g\colon V \to \mathbb{R}$ (resp. $\mathbb{C}$) be two smooth (resp. holomorphic) functions. We say that f and g are equivalent at x if there is a nbhd $W \subseteq U \cap V$ of x such that $f|_W = g|_W$. A *germ of a smooth (resp. holomorphic) function* at x is an equivalence clas for this relation. We use $\mathcal{A}_x$ (resp. $\mathcal{O}_x$) for the local rings of germs of smooth (resp. holomorphic) functions at x.

Definition 19.2 Let X be a smooth manifold and $x \in X$. A *derivation at* x is an $\mathbb{R}$-linear map $D\colon \mathcal{A} \to \mathbb{R}$ such that

$$\text{(Leibniz)} \quad \forall [f], [g] \in \mathcal{A}_x, , D(fg) = D(f)g(x) + f(x)D(g).$$

We will denote the $\mathbb{R}$-vector space of derivations at x by $\mathrm{Der}_{\mathbb{R}}(\mathcal{A}_x, \mathbb{R})$.

Lemma 19.3 *Let X be a smooth manifold and $x \in X$. Let $\gamma\colon (-\varepsilon, \varepsilon) \to X$ be a path through x. The map $\partial_\gamma\colon \mathcal{A}_x \to \mathbb{R}$ sending $f \mapsto \partial_\gamma f$ is a well-defined $\mathbb{R}$-linear derivation.*

Proof. $\mathbb{R}$-linearity is okay, so let us check the Leibniz rule.

$$\partial_\gamma(fg) = ((fg) \circ \gamma)'(0) = ((f \circ \gamma)(g \circ \gamma))'(0),$$

from which we can conclude applying the product rule. □

Proposition 19.4 *Let X be a smooth manifold of dimension n and $p \in X$.*

1. $\mathrm{Der}_{\mathbb{R}}(\mathcal{A}_p, \mathbb{R})$ *is an $\mathbb{R}$-vector space of dimension n.*

2. $T_pX \cong \mathrm{Der}_{\mathbb{R}}(\mathcal{A}_p, \mathbb{R})$.

Proof. WLOG $X = U \subseteq \mathbb{R}^n$ open. Let $x_1, \dots, x_n \colon U \to \mathbb{R}$ be the coordinate functions and define paths $\gamma_k(t) = p + te_k$ for $1 \leqslant k \leqslant n$ and small values of $t > 0$. Then ∂_{γ_k} is by definition the partial derivative $\frac{\partial}{\partial x_k} =: \partial_k$. The claim is that $\partial_1, \dots, \partial_n$ is a basis for $\mathrm{Der}_{\mathbb{R}}(\mathcal{A}_p, \mathbb{R})$. First we show linear independence. Suppose that $0 = \sum_k \lambda_k \partial_k$. Let $v = \sum_k \lambda_k e_k$. Then $D_v = \sum_k \lambda_k \partial_k = 0$, so the tangent vector $\gamma(t) = p + tv$ is equivalent to zero. This implies that $v = 0$, so $\lambda_1 = \ldots = \lambda_n = 0$. Now we show that $\partial_1, \dots, \partial_n$ generate $\mathrm{Der}_{\mathbb{R}}(\mathcal{A}_p, \mathbb{R})$. Let $\partial \colon \mathcal{A}_p \to \mathbb{R}$ be a derivation and let $\lambda_k := \partial x_k$ for $1 \leqslant k \leqslant n$. The claim is that $\partial = \sum_k \lambda_k \partial_k$. Let $f \colon U \to \mathbb{R}$ smooth and consider the expansion

$$(*) \quad f(y) = f(p) + \sum_k (\partial_k f)(y_k - p_k) + \sum_{k,l} g_{kl}(p)(y_k - p_k)(y_l - p_l)$$

for g_{kl} smooth. Apply ∂ and $\sum_k \lambda_k \partial_k$ to $(*)$ to get

$$\partial f = 0 + \Big(\sum_k (\partial_k f)\partial y_k\Big)\Big|_{y=p}$$

and similarly with $\sum_k \lambda_k \partial_k$. We find that $\partial f = \sum_k \lambda_k \partial_k f$. $\square$

Corollary 19.5 $T_xX \cong \mathrm{Der}_{\mathbb{R}}(\mathcal{A}_x, \mathbb{R}) \cong \mathrm{Hom}_{\mathbb{R}}(\mathfrak{m}_x/\mathfrak{m}_x^2, \mathbb{R})$.

Definition 19.6 Let X be a smooth manifold of dimension n.

1. $TX := \coprod_{x \in X} T_xX \xrightarrow{\pi} X$ is called the *tangent bundle*. For coordinate chart (U, φ) we have

$$\begin{aligned} \pi^{-1}(U) &\xrightarrow{\sim} U \times \mathbb{R}^n \\ \sum_k \lambda_k(x)\partial_k &\mapsto (x, \lambda_1(x), \dots, \lambda_n(x)) \end{aligned}$$

   We give TX the topology obtained from the $\pi^{-1}(U)$ by gluing, so that it becomes a rank n vector bundle over X and thus a smooth manifold of dimension $2n$.

2. T^*X denotes the dual vector bundle of TX, called the *cotangent bundle*.

3. A *vector field* on X over $U \subseteq X$ open is a smooth section $\xi \colon U \to TX$ of π.

4. A *differential form* on X over $U \subseteq X$ oepn is a smooth section of T^*X.

5. A *differential form* of degree r on X over $U \subseteq X$ open is a smooth section of $\Lambda^r T^*X$.

Exercise 19.7. $U \subseteq \mathbb{R}^{n+1}$ oepn, $F \colon U \to \mathbb{R}$ smooth function, $X = \mathbb{V}(F) := \{x \in U \mid F(x) = 0\}$. If the gradient of F is nonzero on X, then give an explicit identification of T_xX with the space of tangent directions to X in $\mathbb{R}^{n+1}$.

Remark 19.8. $T_x^*X \cong \mathfrak{m}_x/\mathfrak{m}_x^2$. If $f \colon U \to \mathbb{R}$ smooth, then we have a differential form $df \colon y \mapsto [f - f(y)]$. Show (exercise) that df is smooth. [*Hint:* in a coordinate chart (U, φ), $d\varphi_k(x) \in T_x^*X$ is dual to ∂_k for $1 \leqslant k \leqslant n$.]

19.1 Functoriality

Let $\psi \colon X \to Y$ be a smooth map between smooth manifolds and let $x \in X$. Then we get an induced map

$$\begin{aligned} d\psi_x \colon T_xX &\longrightarrow T_{\psi(x)}Y \\ \partial &\longmapsto [f \mapsto \partial(f \circ \psi)] \end{aligned}$$

which is $\mathbb{R}$-linear.

Definition 19.9 Let $f \colon X \to Y$ be a smooth map. Then the *pushforward along* f is the map

$$df \colon TX \xrightarrow{\coprod df_x} TY,$$

which is a morphism of smooth vector bundles (exercise).

Proposition 19.10 *1.* $d(\mathrm{id}_X) = \mathrm{id}_{TX}$.

2. If $f \colon X \to Y$, $g \colon Y \to Z$ are smooth maps, then $dg \circ df = d(g \circ f)$.

Remark 19.11. In other words, $X \mapsto TX$, $f \mapsto df$ is a functor.

Corollary 19.12 *If $f \colon X \to Y$ is a diffeomorphism, then $df \colon TX \to TY$ is a diffeomorphism of smooth vector bundles.*

19.2 Immersions and embeddings

Definition 19.13 We say that a map $f\colon X \to Y$ is an *immersion* if it is smooth and for all $x \in X$ the differenrial df_x is injective. We say that a map $f\colon X \to Y$ is a *submersion* if it is smooth and for all $x \in X$ the differential df_x is surjective.

Definition 19.14 We say that X is a *submanifold* of Y if there exists an injective immersion $X \hookrightarrow Y$.

Definition 19.15 We say that $f\colon X \to Y$ is a *embedding* if it is an immersion that is a diffeomorphism onto $f(X)$.

Example 19.16 $(x, y) \mapsto x$ from $\mathbb{R}^2$ to $\mathbb{R}$ is a submersion but not an immersion.

Example 19.17 $x \mapsto (x, 0)$ from $\mathbb{R}$ to $\mathbb{R}^2$ is an immersion, but not a submersion. In fact, it is an embedding.

Exercise 19.18. $S^1 \hookrightarrow \mathbb{R}^2$ is an embedding.

Example 19.19 $X = Y = \mathbb{R}$ and $f(x) = x^3$ is injective, but it is neither an immersion nor a submersion.

Example 19.20 $X = (\frac{-\pi}{4}, \frac{3\pi}{4})$, $Y = \mathbb{R}^2$ and $f\colon X \to Y$ sending $x \mapsto \cos(x)\cos(2x), \sin(x)\cos(2x))$. Then f is injective but it is not an embedding[12] (it already fails to be a homeomorphism onto its image).

19.3 Tangent bunles to complex manifolds

To translate the previous constructions to the holomorphic setting, replace $\gamma\colon (-\varepsilon, \varepsilon) \to X$ by $\gamma\colon \mathbb{D}(0, \varepsilon) \to X$ holomorphic. Equivalence classes of such are $\mathbb{C}$-linear derivations

$$\mathrm{Der}_{\mathbb{C}}(\mathcal{O}_x, \mathbb{C}) \cong (\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee.$$

We get the *holomorphic tangent bundle* $TX \to X$ of rank $\dim_{\mathbb{C}}(X)$ as before. Let $X_{\mathbb{R}}$ denote the real manifold underlying X. Then we have $(TX)_{\mathbb{R}} \cong T(X_{\mathbb{R}})$ as smooth bundles, but this is not a $\mathbb{C}$-linear isomorphism.

Remark 19.21. Let X be a complex manifold. Then $TX \ncong T(X_{\mathbb{R}}) \otimes_{\mathbb{R}} \mathbb{C}$.

[12] This example shows that being an embedding is not local on the source.

20 [CM] Solid abelian groups - 18.11.19

Definition 20.1 Let S be a profinite set, given as $\varprojlim_i S_i$. Then we define

$$\mathbb{Z}[S]^{\blacksquare} = \varprojlim_i \mathbb{Z}[S_i],$$

called the free solid abelian group on S.

A condensed abelian group is called solid if... A complex of condensed abelian groups is called solid if...

Later we will show that $C \in \mathcal{D}(\mathrm{Cond}(\mathcal{A}b))$ is solid if and only if $H^i(C)$ is solid for all i.

Notation 20.2. Denote $\mathcal{S}olid \subseteq \mathrm{Cond}(\mathcal{A}b)$ the full subcategory of solid condensed abelian groups.

Remark 20.3. $\mathbb{Z}[S]^{\blacksquare} = \varprojlim_i \mathbb{Z}[S_i] = \varprojlim_i \hom(C(S_i, \mathbb{Z}), \mathbb{Z}) = \hom(\varinjlim_i C(S_i, \mathbb{Z}), \mathbb{Z}) = \hom(C(S, \mathbb{Z}), \mathbb{Z})$.

Theorem 20.4 (Theorem 5.4) *S profinite. Then $C(S, \mathbb{Z})$ is a free abelian group.*

Sketch. We can find a continuous injection $S \hookrightarrow \prod_I \{0,1\}$. We can take $I = \{Z \subseteq S \mid Z \text{ clopen }\}$. Injectivity then is a consequence of total disconnectedness. The data of a continuous function $\varphi \colon S \to \mathbb{Z}$ is equivalent to a decomposition $S = \coprod_{finite} S_i$ where S_i is clopen. So a generating system of $C(S, \mathbb{Z})$ is given by... a basis is given by... (see Scholze's notes). □

Corollary 20.5 $\mathbb{Z}[S]^{\blacksquare} = \prod_I \mathbb{Z}$.

Proof. $\mathbb{Z}[S]^{\blacksquare} = \hom(C(S, \mathbb{Z}), \mathbb{Z}) = \prod_I \hom(\mathbb{Z}, \mathbb{Z}) = \prod_I \mathbb{Z}$. □

Proposition 20.6 (Proposition 5.7) *If S is profinite, then $\mathbb{Z}[S]^{\blacksquare}$ is solid in $\mathcal{D}(\mathrm{Cond}(\mathcal{A}b))$ and $\mathrm{Cond}(\mathcal{A}b)$.*

Proof. Have to check: $R\operatorname{Hom}(\mathbb{Z}[T], \mathbb{Z}[T]^{\blacksquare}) \cong R\operatorname{Hom}(\mathbb{Z}[T]^{\blacksquare}, \mathbb{Z}[S]^{\blacksquare})$, T profinite. Reduce $\mathbb{Z}[S]^{\blacksquare} = \mathbb{Z}$. From the previous lecture we know $R\operatorname{Hom}(\mathbb{Z}[T], \mathbb{Z}) = R\Gamma(T, \mathbb{Z}) = C(T, \mathbb{Z})[0]$. We have a SES $0 \to \prod_J \mathbb{Z} \to \prod_J \mathbb{R} \to \prod_J \mathbb{R}/\mathbb{Z} \to 0$, where $\mathbb{Z}[T]^{\blacksquare} = \prod_J \mathbb{Z}$. Apply $R\operatorname{Hom}(-, \mathbb{Z})$. We get a triangle

$$R\operatorname{Hom}(\prod \mathbb{R}/\mathbb{Z}, \mathbb{Z}) \to R\operatorname{Hom}(\prod \mathbb{R}, \mathbb{Z}) \to R\operatorname{Hom}(\prod \mathbb{Z}, \mathbb{Z}) \xrightarrow{+1} \cdots$$

Note $R\operatorname{Hom}(\prod \mathbb{R}, \mathbb{Z}) = R\operatorname{Hom}_{\mathbb{R}}(\prod \mathbb{R}, R\operatorname{hom}(\mathbb{R}, \mathbb{Z})) = 0$. This follows from Theorem 4.3. $R\operatorname{Hom}(\mathbb{R}/\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}[-1]$, so $R\operatorname{Hom}(\mathbb{R}, \mathbb{Z}) = 0$. Thus $R\operatorname{Hom}(\prod \mathbb{Z}, \mathbb{Z}) \cong R\operatorname{Hom}(\prod \mathbb{R}/\mathbb{Z}, \mathbb{Z})[-1]$. By theorem 4.3. this is $\bigoplus_J \mathbb{Z}[-1][1] = \bigoplus_J \mathbb{Z}$. By 5.4 this is precisely $C(T, \mathbb{Z})$. $\square$

So far we have a notion of solid and we have the notion of free solid abelian groups. The question is: can we solidify every condensed abelina group? Turns out we can:

Theorem 20.7 (Theorem 5.8) *i) $\mathcal{S}olid \subseteq \operatorname{Cond}(\mathcal{A}b)$ is an abelian subcategory with all limits, colimits and extensions, projective generators $\prod_I \mathbb{Z}$ coming from free solid abelian groups and with a left adjoint $L = (-)^{\blacksquare} \colon \operatorname{Cond}(\mathcal{A}b) \to \mathcal{S}olid$ to the inclusion $\mathcal{S}olid \subseteq \operatorname{Cond}(\mathcal{A}b)$ and it extends the functor $\mathbb{Z}[S] \mapsto \mathbb{Z}[S]^{\blacksquare}$.*

ii) $\mathcal{D}(\mathcal{S}olid) \subseteq \mathcal{D}(\operatorname{Cond}(\mathcal{A}b))$ is fully faithful and has essential image precisely the solid objects in $\mathcal{D}(\operatorname{Cond}(\mathcal{A}b))$. We have the same description as before, i.e. a complex is solid iff all its cohomologies are solid. We have moreover a left adjoint to this inclusion $\mathcal{D}(\mathcal{S}olid) \subseteq \mathcal{D}(\operatorname{Cond}(\mathcal{A}b))$, denoted $C \mapsto C^{\blacksquare, L}$, which is actually the left derived functor of L.

Lemma 20.8 (Lemma 5.9) *Let $\mathcal{A}$ be an abelian category with colimits. Let $\mathcal{A}_0 \subseteq \mathcal{A}$ be compact projective generators. Let $F \colon \mathcal{A}_0 \to \mathcal{A}$ be a functor together with a natural transformation $\operatorname{id} \to F$ such that: For objects Y, Z direct sums of images of F, $f \colon Y \to Z$ with kernel K, we hvae that*

$$R\operatorname{Hom}(\mathcal{F}(X), K) \xrightarrow{\sim} R\operatorname{Hom}(X, K).$$

Let $\mathcal{A}_F \subseteq \mathcal{A}$ be the full subcategory of all objects $Y \in \mathcal{A}$ such that for all $X \in \mathcal{A}_0$ we have

$$\operatorname{Hom}(F(X), Y) \xrightarrow{\sim} \operatorname{Hom}(X, Y).$$

Let $\mathcal{D}_F(\mathcal{A}) \subseteq \mathcal{D}(\mathcal{A})$ be the full subcategory of all $C \in \mathcal{D}(\mathcal{A})$ such that for $X \in \mathcal{A}_0$ we have

$$R\operatorname{Hom}(F(X), C) \xrightarrow{\sim} R\operatorname{Hom}(X, C).$$

Then: $\mathcal{A}_F \subseteq \mathcal{A}$ is an abelian category with colimits, limits and extensions. The images $F(X)$ for $X \in \mathcal{A}_0$ are compact, projective generatros of $\mathcal{A}_F$. Have a left adjoint $L \colon \mathcal{A} \to \mathcal{A}_F$ to the inclusion that extends our functor $F \colon \mathcal{A}_0 \to \mathcal{A}_F$. Similar for the derived category: we get a left adjoint to *the inclusion $\mathcal{D}_F(\mathcal{A}) \subseteq \mathcal{D}(\mathcal{A})$ which is the left derived functor of $\mathcal{A} \to \mathcal{A}_F$. Moreover, $\mathcal{D}(\mathcal{A}_F) \to \mathcal{D}(\mathcal{A})$ is fully faithful and has essential image $\mathcal{D}_F(\mathcal{A})$. Furthermore, $C \in D(\mathcal{A})$ lies in $\mathcal{D}_F(\mathcal{A})$ if and only if all $H^i(C) \in \mathcal{A}_F$.*

Lemma 20.9 (Lemma 5.10) *Let $\mathcal{A}$ be an abelian category with colimits and $\mathcal{A}_0 \subseteq \mathcal{A}$ be compact projective generators. Let $F\colon \mathcal{A}_0 \to \mathcal{A}$ be a functor together with a natural transformation $\mathrm{id} \to F$ such that for all $X \in \mathcal{A}_0$ and all complexes $C = \cdots \to C_1 \to C_0 \to 0 \to \cdots$ with C_i is direct sum of images of F we have*

$$R\operatorname{Hom}(F(X), C) \xrightarrow{\sim} R\operatorname{Hom}(X, C).$$

Then we have for all $X \in \mathcal{A}_0$ and all $f\colon Y \to Z$ wtih kernel K and Y, Z direct sums of images of F that

$$R\operatorname{Hom}(F(X), K) \xrightarrow{\sim} R\operatorname{Hom}(X, K).$$

Proof. Let Y, Z, f and K as in the statement. Take a resolution

$$\cdots \to \oplus_{ij} X_{ij} \to \cdots \to K$$

with $X_{ij} \in \mathcal{A}_0$. Denote this projective resolution of K by B. We get a complex $C = (\cdots \to \oplus F(X_{ij}) \to \cdots \to 0)$. By assumption $R\operatorname{Hom}(F(X_{ij}), Y) = R\operatorname{Hom}(X_{ij}, Y)$. Then by a spectral sequence argument we have $R\operatorname{Hom}(B, Y) = R\operatorname{Hom}(C, Y)$. The same for Z. Thus, $B \to C \to Y$ factors through $B \to K \subseteq Y$, so composing with $Y \to Z$ we get 0. Hene $B \xrightarrow{\sim} K$ (factors over C) and B is a retract of C. The condition $R\operatorname{Hom}(F(X), C) \xrightarrow{\sim} R\operatorname{Hom}(X, C)$ passes to B, thus to K. □

20.1 Proof of theorem 5.8

We take $\mathcal{A}_0$ to be the full subcategory of objects $\mathbb{Z}[S]$ for S extremally disconnected and $\mathcal{A} = \mathrm{Cond}(\mathcal{A}b)$. The functor $F\colon \mathcal{A}_0 \to \mathcal{A}$ is $\mathbb{Z}[S] \mapsto \mathbb{Z}[S]^{\blacksquare}$. This is indeed a functor because $\mathbb{Z}[S]^{\blacksquare} = \hom(C(S,\mathbb{Z}),\mathbb{Z}) = \hom(\hom(\mathbb{Z}[S],\mathbb{Z}),\mathbb{Z})$ is functorial on $\mathbb{Z}[S]$. We check the assumptions of lemma 5.10. Let $C = (\cdots \to C_1 \to C_0 \to 0)$ be a complex with C_i a direct sum of F, i.e. $C_i = \oplus \prod \mathbb{Z}$. We want to show that $R\operatorname{Hom}(\mathbb{Z}[S]^{\blacksquare}, C) = R\operatorname{Hom}(\mathbb{Z}[S], C)$ for S extremally disconnected. Let $\mathcal{M}(S,\mathbb{Z}) = \hom(C(S,\mathbb{Z}),\mathbb{Z}) = \mathbb{Z}[S]^{\blacksquare}$, $\mathcal{M}(S,\mathbb{R}) = \hom(C(S,\mathbb{Z}),\mathbb{R}) = \prod \mathbb{R}$ and $\mathcal{M}(S,\mathbb{R}/\mathbb{Z}) = \hom(C(S,\mathbb{Z}),\mathbb{R}/\mathbb{Z}) = \prod \mathbb{R}/\mathbb{Z}$. We have a SES

$$0 \to \mathcal{M}(S,\mathbb{Z}) \to \mathcal{M}(S,\mathbb{R}) \to \mathcal{M}(S,\mathbb{R}/\mathbb{Z}) \to 0.$$

The claim is that $R\hom(\mathcal{M}(S,\mathbb{R}/\mathbb{Z}),C)(S') = R\Gamma(S\times S',C)[-1]$. Let us see first why this claim implies that we are in the assumptions of lemma 5.10. Let $S=\{*\}$. Then $R\hom(\mathbb{R}/\mathbb{Z},C)(S') = R\Gamma(S',C)[-1] = R\operatorname{Hom}(\mathbb{Z}[S'],C)[-1]=$ $R\hom(\mathbb{Z},C)(S')[-1] = R\hom(\mathbb{Z}[1],C)(S')$. This comes from the SES

$$0\to\mathbb{Z}\to\mathbb{R}\to\mathbb{R}/\mathbb{Z}\to 0$$

which gives the triangle $R\hom(\mathbb{R}/\mathbb{Z},C)\to R\hom(\mathbb{R},C)\to R\hom(\mathbb{Z},C)\to \cdots$. Since $R\hom(\mathbb{R},C)=0$ we get $R\hom(\mathbb{R}/\mathbb{Z},C)\cong R\hom(\mathbb{Z},C)[-1]$. Thus $R\hom(\mathcal{M}(S,\mathbb{R}),C) = R\hom(\mathcal{M}(S,\mathbb{R}),R\hom(\mathbb{R},C))=0$. We have a distinguished triangle

$$R\hom(\mathcal{M}(S,\mathbb{R}/\mathbb{Z}),C)\to R\hom(\mathcal{M}(S,\mathbb{R}),C)\to R\hom(\mathcal{M}(S,\mathbb{Z}),C)$$

The middle term is again zero, so

$$R\hom(\mathcal{M}(S,\mathbb{Z}),C) = R\hom(\mathcal{M}(S,\mathbb{R}/\mathbb{Z}),C)[1] \overset{claim}{=} R\Gamma(S\times S',C) = R\hom(\mathbb{Z}[S],C)(S')$$

For the claim: if X is a compact Hausdorff topological space, then $\mathbb{Z}[X]$ is pseudo coherent, i.e. the ext-groups commute with filtered colimits. In particular, since $\mathcal{M}(S,\mathbb{R}/\mathbb{Z}) = \prod\mathbb{R}/\mathbb{Z}$, we get that $\mathbb{Z}[\prod\mathbb{R}/\mathbb{Z}]$ is pseudo-coherent. Because of the resolution

$$\cdots\to\mathbb{Z}[\prod\mathbb{R}/\mathbb{Z}]\to\prod\mathbb{R}/\mathbb{Z}\to 0$$

we also have that $\prod\mathbb{R}/\mathbb{Z}$ is pseudo coherent. Also, if $\mathbb{Z}[S]$ is pseudo coherent, then so is $\mathbb{Z}[S]\otimes\prod\mathbb{R}/\mathbb{Z}$. Claim for $C=M[0]$, $M=\oplus\prod\mathbb{Z}$.

$$\begin{aligned} R\hom(\mathcal{M}(S,\mathbb{R}/\mathbb{Z}),\oplus\prod\mathbb{Z}) &= \oplus R\hom(\mathcal{M}(S,\mathbb{R}/\mathbb{Z}),\prod\mathbb{Z}) = \\ &\oplus\prod R\hom(\mathcal{M}(S,\mathbb{R}/\mathbb{Z}),\mathbb{Z}) \overset{LLM4.3}{=} C(S\times S',\mathbb{Z})[-1]\end{aligned}$$

Hence the claim for C concentrated in one degree, which formally implies the claim for C bounded (with triangles and induction on the length). In general: C can be written as the homotopy colimit of the truncations $C_{\leqslant n}$ (see [Sta19, Tag 093W]). Homotopy colimit means that we have a triangle

$$\oplus C_{\leqslant n}\xrightarrow{1-\oplus f_n}\oplus C_{\leqslant n}\to C$$

Apply $R\hom(\mathcal{M}(S,\mathbb{R}/\mathbb{Z}),-)$ to this triangle. Then use pseudo coherence to take the direct sum out.

21 [LT] Lecture 9 - 19.11.19

21.1 Fundamental examples

Proposition 21.1 *Let $f\colon X \to Y$ be a smooth map of smooth manifolds. Then $\Gamma_f = \{(x, f(x)) \mid x \in X\} \subseteq X \times Y$ is a closed embedded submanifold.*

Proof. It is closed by Hausdorffness. Let $\varphi\colon X \to X \times Y$ sending $x \mapsto (x, f(x))$ (which is smooth). Let $\pi\colon X \times Y \to X$ be the projection onto the first factor (also smooth). Then $\pi \circ \varphi = \mathrm{id}_X$, so $d(\pi \circ \varphi) = (d\pi) \circ (d\varphi) = \mathrm{id}_{TX}$, so $d\varphi_x$ is injective for all $x \in X$. Thus φ is an immersion. $\square$

Corollary 21.2 *$X \subseteq \mathbb{CP}^n$ vanishing locus of a famliy of homogeneous polynomials. If the Jacobian matrix has maximal rank at each point of X, then X is an embedded submanifold of $\mathbb{CP}^n$.*

Proof. Use implicit function theorem and previous proposition. $\square$

Theorem 21.3 (Constant rank theorem) *Let $f\colon X \to Y$ be a smooth map of smooth manifolds and let $m = \dim X$, $n = \dim Y$, $x \in X$ and $y = f(x)$. If f is of constant rank r, then we can find charts (U, φ) at x and (V, ψ) at y such that $f(U) \subseteq V$ and for all $t \in \varphi(U)$ we have $(\psi \circ f \circ \varphi^{-1})(t_1, \ldots, t_r, 0, \ldots, 0)$. [Here the rank of f at x, denoted $\mathrm{rk}_x(f) = \mathrm{rk}(df_x)$.]*

Proof. WLOG $X = W \subseteq \mathbb{R}^m$, $Y = W' \subseteq \mathbb{R}^n$ open and $x = 0$, $y = 0$. Then df_0 is the Jacobian matrix. Define $\varphi\colon W \to \mathbb{R}^m$ by

$$\varphi(x) = (f_1(x), \ldots, f_r(x), x_{r+1}, \ldots, x_m)$$

for all $x \in W$ (we may choose charts so that the first r coordinates are linearly indepenent). $\varphi(0) = 0$ and

$$d\varphi_0 = \begin{pmatrix} \left(\frac{\partial f_k}{\partial x_l}\right) & * \\ 0 & I_{m-r} \end{pmatrix}.$$

Since $d\varphi_0$ is invertible, by the inverse function theorem we can find $0 \in W_1 \subseteq W$ open and $0 \in U_1 \subseteq \mathbb{R}^m$ open such thath

$$\varphi|_{W_1}\colon W_1 \to U_1$$

is a diffeomorphism. On U_1 we have $(f\circ\varphi^{-1})(x) = (x_1, \dots, x_r, \tilde{f}_{r+1}(x), \dots, \tilde{f}_n(x))$ for some smooth functions $\tilde{f}_k$ with $r < k \leqslant n$. We have

$$((df) \circ (d\varphi^{-1}))_0 = d(f \circ \varphi^{-1})_0 = \begin{pmatrix} I_r & 0 \\ * & \left(\frac{\partial \tilde{f}_k}{\partial x_l}\right) \end{pmatrix}.$$

$d\varphi^{-1}$ is invertible, so $\operatorname{rk}(d(f \circ \varphi^{-1})) = \operatorname{rk}(df) = r$ on U_1. Hence $\left(\frac{\partial \tilde{f}_k}{\partial x_l}\right) = 0$, so $\tilde{f}_k$ is independent of $x_{r+1}, \dots, x_n$. $T(y_1, \dots, y_n) := (y_1, \dots, y_r, y_{r+1} + \tilde{f}_{r+1}(y_1, \dots, y_r), \dots, y_n + \tilde{f}_n(y_1, \dots, y_n))$. We have $T(0) = 0$ and

$$dT_y = \begin{pmatrix} I_r & 0 \\ * & I_{n-r} \end{pmatrix}.$$

Hence T is a diffeomorphism from a nbhd of $0 \in \tilde{V} \subseteq \mathbb{R}^n$ to a nbhd of $0 \in V \subseteq W'$. Choose $\tilde{U} \subseteq U$ open such that $(f \circ \varphi^{-1})(\tilde{U}) \subseteq V$. Let $U := \varphi^{-1}(\tilde{U})$ and $\psi = T^{-1}$. Consider

$$\tilde{U} \xrightarrow{\varphi^{-1}} U \xrightarrow{f} V \xrightarrow{\psi} \tilde{V}.$$

For $x \in \tilde{U}$ we get $(x_1, \dots, x_r, 0, \dots, 0)$ after applying the previous composition. □

Special cases:

1. f submersion.
2. f immersion.

In the first case, if $f\colon X \to Y$ is a submersion, locally on X and on Y we have

$$\begin{aligned} \mathbb{R}^{n+r} &\longrightarrow \mathbb{R}^n \\ (t_1, \dots, t_{n+r}) &\longmapsto (t_1, \dots, t_n) \end{aligned}$$

Proposition 21.4 *If $f\colon X \to Y$ is a smooth map of smooth manifolds and $y \in Y$ is such that for all $x \in f^{-1}\{y\}$ the map df_x is surjective, then $f^{-1}\{y\} \subseteq X$ is an embedded submanifold.*

Proof. Exercise! □

21.2 Ehresmann's theorem

Definition 21.5 A smooth map $f\colon X \to Y$ is a *trivial fibration* if there exists a smooth manifold F and a diffeomorphism $\varphi\colon X \to Y \times F$ such that

$$\begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr, "f"'] & & Y \times F \arrow[dl, "\pi"] \\ & Y & \end{tikzcd}$$

We say that f is a *locally trivial fibration* if for all $y \in Y$ there exists an open $y \in U \subseteq Y$ such that $f^{-1}(U) \to U$ is a trivial fibration.

Exercise 21.6. $\mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ sending $(x, y) \mapsto x$ or $\mathbb{T}^2 \to S^1$ projecting onto one of its factors.

Exercise 21.7. The Mobius strip is a locally trivial fibration which is not globally trivial.

Theorem 21.8 (Ehresmann) *Let $f\colon X \to Y$ be a smooth, surjective, submersive and proper map of smooth manifolds. Then f is a locally trivial fibration.*

Idea. Locally on X we have trivializations, so we get some horizontal lines. We extend them across other neighbourhoods in a way that is compatible with the corresponding trivializations to obtain trivializations locally on Y. □

21.3 Proper maps

Definition 21.9 Let $f\colon X \to Y$ be a function and X, Y topological spaces. We say that f is *proper* if for all $K \subseteq Y$ quasi-compact we have that $f^{-1}(K)$ is quasi-compact.

Example 21.10 $X \to *$ is proper if and only if X is quasi-compact.

Example 21.11 $\mathbb{R} \to \mathbb{R}/\mathbb{Z} = S^1$ is not proper.

Proposition 21.12 *Let $f\colon X \to Y$ continuous.*

1. *If f is closed with quasi-compact fibres, then f is proper.*

2. *If f is proper and Y is a compactly generated Hausdorff space, then f is closed.*

3. *If X is quasi-compact and Y is Hausdorff, then f is closed and proper.*

Proof. For 1), let $K \subseteq Y$ be q.c. The claim is that $f^{-1}(K)$ is q.c. Let U_α be an open cover of $f^{-1}(K)$ and consider the fibre $f^{-1}\{k\}$ for $k \in K$. We can find a finite subcover U_{α_i} of the fibre. Since f is closed, $f(X \setminus \cup_i U_{\alpha_i})$ is also closed. Hence $V_k := Y \setminus f(X \setminus \cup_i U_{\alpha_i})$ is open. By construction $k \in V_k$ for each $k \in K$, so V_k is an open cover of K. We can find a finite subcover by q.c. But then $f^{-1}(K) \subseteq f^{-1}(V_{k_1} \cup \cdots V_{k_n}) \subseteq \cup_j U_{\alpha_{i_j}}$, so this is the desired finite subcover.

For 2), recall that compact generation means that we can test closedness by intersecting with all compact subspaces. Let $Z \subseteq X$ be closed and $K \subseteq Y$ be quasi-compact. Claim: $f(Z) \cap K$ is closed. We know that $f^{-1}(K)$ is q.c., so $f^{-1}(K) \cap Z$ is q.c. Continuous image of q.c. is q.c., so $f(f^{-1}(K) \cap Z) = K \cap f(Z)$ q.c. Since Y is Hausdorff, $K \cap f(Z)$ is closed.

For 3), if Z is closed in a q.c. space, then it is q.c. Hence $f(Z)$ is q.c. inside a Hd space, so it is again closed. Properness is left as an exercise. $\square$

If $f\colon X \to Y$ is proper and $j\colon U \subseteq X$ is an open subset, then there is no reason to expct the composition $U \to Y$ is proper. From this we can also build counterexamples to the Ehresmann theorem, e.g. remove a bunch of points from a covering space.

21.4 Flows

Let X be a smooth manifold.

Definition 21.13 A *global flow* on X is a smooth map

$$\mathbb{R} \times X \to X$$

such that for all $x \in X$ and for all times $s, t \in \mathbb{R}$ we have

i) $\Phi(0, x) = x$.

ii) $\Phi(s, \Phi(t, x)) = \Phi(s + t, x)$.

Example 21.14 Let $X = \mathbb{R}$ and $L\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ sending $(s, t) \mapsto s + t$.

Example 21.15 Consider $\Phi\colon \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}^2$ sending

$$\left(t, \begin{pmatrix} p \\ q \end{pmatrix}\right) \mapsto e^{-t/2} \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}.$$

Definition 21.16 Let Φ be a flow on X. The *velocity field* of Φ is the vector field

$$\begin{aligned} \vec{\Phi}\colon X &\longrightarrow TX \\ x &\longmapsto \frac{d}{dt}|_0 \Phi(t,x) \end{aligned}$$

We will see that all vector fields are velocoty fields of flows.

Definition 21.17 Let $\Phi\colon \mathbb{R}\times X \to X$ be a flow and $x, y \in X$. We will say that $x \sim y$ if and only if there exists some $t \in \mathbb{R}$ such that $\Phi(t,x) = y$.

22 [WS] Kodaira 5 - 20.11.19

Theorem 22.1 *If X is a Hodge manifold, then there exists an embedding into $\mathbb{CP}^m$ for $m \in \mathbb{N}$.*

Recall that a Hodge manifold is a Kaehler manifold such that the Kaehler form is positive and integral.

Last week we have seen that there exists $\mu_0 \in \mathbb{N}$ such that

i) $\pi_p^* E^\mu \otimes L_p^* \otimes K^*_{Q_p(X)}$,

ii) $\pi_p E^\mu \otimes (L_p^*)^2 \otimes K^*_{Q_p(X)}$, and

iii) $\pi_{p,q} E^\mu \otimes L_{p,q} \otimes K^*_{Q_{p,q}(X)}$

are positive. We get

$$X \longrightarrow \mathbb{P}(\Gamma(F))$$

for $F = E^\mu$.

For $p \in X$ consider $\mathfrak{m}_p \subseteq \mathcal{O}_X$. We consider the exact sequence

$$0 \to \mathfrak{m}_p^2 \to \mathcal{O}_X \to \mathcal{O}_X/\mathfrak{m}_p^2 \to 0$$

and the exact sequence

$$0 \to \mathfrak{m}_{p,q} \to \mathcal{O}_X \to \mathcal{O}_X/\mathfrak{m}_{p,q} \to 0,$$

where $\mathfrak{m}_{p,q} = \{f \in \mathcal{O}_X \mid f(p) = f(q) = 0\}$. Tensor both short exact sequences with $\mathcal{O}_X(F)$. We have

$$((\mathcal{O}_X/\mathfrak{m}_p^2) \otimes \mathcal{O}_X(F))_x = \begin{cases} \mathcal{O}_p/\mathfrak{m}_p^2 \otimes F_p & \text{if } x = p \\ 0 & \text{if } x \neq p \end{cases}$$

and

$$((\mathcal{O}_X/\mathfrak{m}_{p,q}) \otimes \mathcal{O}_X(F))_x = \begin{cases} F_p & \text{if } x = p \\ F_q & \text{if } x = q \\ 0 & \text{if } p \neq x \neq q \end{cases}$$

Lemma 22.2 $\mathcal{O}_p/\mathfrak{m}_p^2 \cong \mathbb{C} \oplus T_p^*(X)$ *via* $f \mapsto f(p) + df(p)$.

Proof. Choose local coordinates around p. Then

$$f(z) = f(p) + \sum_{|\alpha| \geqslant 1} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} f}{\partial z^\alpha}(p)(z-p)^\alpha,$$

whence

$$[f]_p = [f(p) + \sum_{|\alpha|=1} \frac{\partial f}{\partial z^\alpha}(p)(z-p)^\alpha$$

□

Consider $\Gamma(F) \xrightarrow{r} \mathcal{O}_X/\mathfrak{m}_p^2 \otimes \mathcal{O}_X(F)$. For a local fram $f \in \Gamma(U, F)$ and $\xi \in \Gamma(F)$ we have

$$r(\xi) = (\xi(f)(p), d\xi(f)(p)) \otimes f(p).$$

Similarly we denote $\Gamma(F) \xrightarrow{s} F_p \oplus F_q$.

Lemma 22.3 *If s and r are surjective for all $p, q \in X$, then there exists an embedding $X \hookrightarrow \mathbb{CP}^m$ where $m+1$ is the dimension of $\Gamma(F)$.*

Proof. Choose a basis $\{\varphi_0, \ldots, \varphi_m\}$ of $\Gamma(F)$ and let $p \in X$ and $f \in \Gamma(U, F)$ a local frame. We define

$$\begin{aligned} \Phi_p \colon X &\longrightarrow \mathbb{CP}^m \\ x &\longmapsto [\varphi_0(f)(x), \ldots, \varphi_m(f)(x)] \end{aligned}$$

This is well defined by surjectivity of r, i.e. we can find some i such that $\varphi_i(f)(p) \neq 0$. Let $\tilde{f} = cf$ be a different local frame, where c is a non-vanishing holomorphic function. Then $\varphi_i(\tilde{f}) = c\varphi_i(f)$, so $[\varphi_0(\tilde{f})(x), \ldots] = [\varphi_0(f)(x), \ldots]$. For a basis $\{\tilde{\varphi}_i = \sum_j c_{ij}\varphi_j\}$ of $\Gamma(F)$, we have the following:

$$\begin{tikzcd} & \mathbb{CP}^m \arrow[d, "C=[c_{ij}]"] \\ X \arrow[ur, "\Phi_\varphi"] \arrow[r, "\Phi_{\tilde\varphi}"] & \mathbb{CP}^m \end{tikzcd}$$

Since C is non-singular, the map on the right is biholomorphic. To prove that Φ_φ is an immersion at $p \in X$, we choose $\xi_0, \ldots, \xi_n \in \Gamma(F)$ (for $n = \dim(X)$) such that $\xi_0(f)(p) = 1$, $\xi_i(f)(p) = 0$ and $d\xi_i(f)(p) = dz^j$ for $f \in \Gamma(U, F)$ a local frame and for z a local basis of X around p. We complete $\xi_0, \ldots, \xi_n$ to a basis of $\Gamma(F)$, call this basis $\tilde\varphi$, and we consider $\Phi_{\tilde\varphi}$. We choose homogeneous coordinates $(1, x_1, \ldots, x_n, \ldots)$ in $\mathbb{CP}^m$. Then

$$\begin{aligned}\left(\det(\frac{\partial(x_1,\ldots,x_n)}{\partial(z_1,\ldots,z_n)})dz^1 \wedge \ldots \wedge dz^n\right)|_p = \\ = d\left(\frac{\xi_1(f)}{\xi_0(f)}\right)|_p \wedge \ldots \wedge d\left(\frac{\xi_n(f)}{\xi_0(f)}\right)|_p = dz^1 \wedge \ldots \wedge dz^n.\end{aligned}$$

Since $\Phi_{\tilde\varphi}$ is an immersion, Φ_φ is an immersion. Let $p, q \in X$. Since s is surjective, we find $\xi_0, \xi_1 \in \Gamma(F)$ such that $\xi_0(p) \neq 0 \neq \xi_1(q)$, $\xi_0(p) \neq 0 \neq \xi_1(q)$, $\xi_1(p) = 0 = \xi_0(q)$. We get a complete basis $\tilde\varphi$. $\Phi_{\tilde\varphi}(p) \neq \Phi_{\tilde\varphi}(q)$. □

Lemma 22.4 *Let X be a Hodge manifold. r, s are surjective for all points $p, q \in X$.*

Proof. Let $p \in X$. Consider the blowup $\pi_p \colon Q_p(X) \to X$ and denote $\tilde F = \pi_p^*(F)$ the pullback. Denote also the exceptional fibre by $S = \pi_p^{-1}(\{p\})$ and $J_S = \{f \in \mathcal{O}_{Q_p(X)} \mid f|_S = 0\}$. Consider the exact sequence

$$0 \to \mathcal{O}(\tilde F) \otimes J_S^2 \to \mathcal{O}(\tilde F) \to \mathcal{O}(\tilde F) \otimes \mathcal{O}_{Q_p(X)}/J_S^2 \to 0$$

and the exact sequence

$$0 \to \mathcal{O}(F) \otimes \mathfrak{m}_p^2 \to \mathcal{O}(F) \to \mathcal{O}(F) \to \mathcal{O}(F) \otimes \mathcal{O}_{Q_p(X)}/\mathfrak{m}_p^2 \to 0.$$

Now consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & \Gamma(\mathcal{O}(\tilde F) \otimes J_S^2) \arrow[r] \arrow[d, shift left, "\alpha"] & \Gamma(\mathcal{O}(\tilde F)) \arrow[r] \arrow[d, shift left, "\beta"] & \Gamma(\mathcal{O}(\tilde F) \otimes \mathcal{O}_{Q_p(X)}/J_S^2) \\ 0 \arrow[r] & \Gamma(\mathcal{O}(F) \otimes \mathfrak{m}_p^2) \arrow[r] \arrow[u, shift left, "\pi_p^*"] & \Gamma(\mathcal{O}(F)) \arrow[r] \arrow[u, shift left, "\pi_p^*"] & \Gamma(\mathcal{O}(F) \otimes \mathcal{O}_{Q_p(X)}/\mathfrak{m}_p^2) \arrow[u, "\pi_p^*"'] \end{tikzcd}$$

Let $\xi \in \Gamma(\mathcal{O}(\tilde{F}))$, $\tilde{\beta}(\xi)(x) = \Pi_p \xi(\pi_p^{-1}(x))$ for all $x \in X \setminus \{p\}$, where $\Pi_p \colon \tilde{F} \to F$. For $n = 1$, nothing happens, so $\beta = \mathrm{id}$. For $n > 1$ we have $\tilde{\beta}(\xi) \in \Gamma(X \setminus \{p\}, F)$. By Hartog's theorem, there exists a unique holomorphic extension $\tilde{\tilde{\beta}}(\xi)$ to $\Gamma(X, F)$. Denote it by $\beta(\xi)$. Then $\pi_p^{-1} = \beta$. We have $\alpha = \beta|_{\Gamma(\mathcal{O}(\tilde{F}) \otimes J_S)}$. We are left to show that $H^1(\mathcal{O}(\tilde{F}) \otimes J_S^2) = 0$. J_S is the sheaf of a line bundle. $(z^i, t) = S = \{t = 0\}$. If L is the line bundle associated to S, then J_S is the sheaf $L^\vee$. $H^1(\mathcal{O}(\tilde{F}) \otimes J_S^2) = H^1(\mathcal{O}(\tilde{F}) \otimes (L^\vee)^2)$.

Recall that $F = E^\mu$ for $\mu \geqslant \mu_0$ such that

$$\pi_p^* E^\mu \otimes (L^\vee)^2 \otimes K_{Q_p(X)}^\vee > 0.$$

By Kodaira-vanishing we have $H^1(\mathcal{O}(\tilde{F}) \otimes (L^\vee)^2) = 0$, hence we can complete the exact sequence above to

$$0 \to \Gamma(\mathcal{O}(\tilde{F}) \otimes J_S^2) \to \Gamma(\mathcal{O}(\tilde{F})) \to \Gamma(\mathcal{O}(\tilde{F}) \otimes \mathcal{O}/J_S^2) \to 0.$$

This implies that r is surjective for all $p, q \in X$.

It remains to show that s is surjective for all $p, q \in X$. Let $\pi_{p,q} \colon \tilde{X} = Q_{p,q}(X) \to X$ be the blowup at these two points and $\pi_{p,q}^* F =: \tilde{F}$. Let $J_S = \{f \in \mathcal{O}_{\tilde{X}} \mid f|_{\pi_{p,q}^{-1}(\{p,q\})} = 0\}$. We get a "horizontal ladder" commutative diagram as in the previous case, in which we again we get a zero on the right of the first row. This means that s is surjective. $\square$

References

[GH78] P. Griffiths and J. Harris. *Principles of Algebraic Geometry*. John Wiley & Sons, 1978.

[Sta19] The Stacks project authors. The stacks project. `https://stacks.math.columbia.edu`, 2019.

[Voi07] Claire Voisin. *Hodge theory and complex algebraic geometry. I*, volume 76 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, english edition, 2007. Translated from the French by Leila Schneps.