

# Various lecture notes

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## 1 Seminar on Condensed Mathematics - Talk 1: Condensed Sets - 21.10.19

Motivation: topological abelian groups do not form an abelian category.

**Example 1.1.**  $\mathbb{R}_{disc} \rightarrow \mathbb{R}$  is epi and mono, but not iso.

Another motivation is coherent duality:

**Theorem 1.2.** *Let  $f: X \rightarrow Y$  be a proper or quasi-projective morphism of Noetherian schemes of finite Krull dimension. Then there exists a right adjoint  $f^!$  to the derived direct image functor  $f_! = Rf_*: \mathcal{D}^b(\mathcal{QCoh}(X)) \rightarrow \mathcal{D}^b(\mathcal{QCoh}(Y))$ .*

At some point analytic rings will come up. We will then look at the category of solid modules, in which the 6-functor formalism works nicer than in the classical setting (e.g. when  $f_!$  is not defined in the classical setting,  $f_!$  takes non-discrete values in the condensed settings, which are "not there" in the classical setting).

**Definition 1.3.** Proétale site of a point, denoted  $*_{proét}$ , is the category of profinite sets with finite jointly surjective families of continuous maps as covers. A *condensed set* (resp. group, ring, ...) is a sheaf of sets (resp. groups, rings, ...) on  $*_{proét}$ . We denote by  $\text{Cond}(\mathcal{C})$  the category of condensed objects of a category  $\mathcal{C}$ .

**Definition 1.4.** A *condensed set* (resp. group, ring, ...) is a contravariant functor  $X$  from  $*_{\text{proét}}$  to the category of sets (resp. groups, rings, ...) such that

i)  $X(\emptyset) = *$ .

ii) For all profinite sets  $S_1$  and  $S_2$  the natural map

$$X(S_1 \sqcup S_2) \rightarrow X(S_1) \times X(S_2)$$

is an isomorphism.

iii) For any surjection of profinite sets  $f: S' \twoheadrightarrow S$  we get an induced<sup>1</sup> isomorphism

$$X(S) \rightarrow \{x \in X(S') \mid \pi_1^*(x) = \pi_2^*(x) \in X(S' \times_S S')\}$$

We will call  $X(*)$  the *underlying object* in  $\mathcal{C}$  of a condensed object.

*Remark 1.5.* We will use  $T$  for topological spaces vs.  $X, Y$  for condensed sets, as opposed to Scholze's mixing of those notations.

## 1.1 Recollections on sheaves on sites

Let  $F$  be a presheaf on a site, which is just a contravariant functor to whatever category in which our sheaves are gonna take values. If  $U = \cup_i U_i$  is an open cover, the topological sheaf axiom could be phrased as:  $F(U)$  is an equalizer of the diagram

$$\prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j).$$

Note that  $U_i \cap U_j$  is just the fiber product of the two inclusions.

**Definition 1.6** (Coverage). See definition 2.1 in nCat.

**Definition 1.7.**  $F$  a presheaf on  $\mathcal{C}$ . A collection  $(s_i) \in \prod_i F(U_i)$  for  $\{f_i: U_i \rightarrow U\}$  a covering is called a *matching family* if for all  $h: V \rightarrow U$  we have  $g^*(s_i) = h^*(s_j)$  for  $g$  and  $h$  in the diagram

$$\begin{array}{ccc} V & \xrightarrow{h} & U_j \\ \downarrow g & & \downarrow f_j \\ U_i & \xrightarrow{f_i} & U \end{array}$$

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<sup>1</sup>Since the pullback diagram is commutative, the image of  $X(f)$  is indeed induces a morphism as claimed.

**Definition 1.8.**  $F$  is a sheaf with respect to  $\{U_i \rightarrow U\}$  if for all matching families  $(s_i)$  there exists a unique  $s \in F(U)$  such that  $f_i^*(s) = s_i$ . We say that  $F$  is a *sheaf* if it is a sheaf for all covering families.

*Remark 1.9.* A sheaf of abelian groups is just a commutative group object in the category of sheaves of sets.

**Theorem 1.10.** *If  $\mathcal{C}$  is a site, then  $Ab(\mathcal{C})$  is an abelian category.*

**Definition 1.11.** An additive category is a category in which the hom-sets are endowed with an abelian group structure in a way that makes composition bilinear and such that finite biproducts exist.

Recall Grothendieck's axioms: AB1) Every morphism has a kernel and a cokernel. AB2) For every  $f: A \rightarrow B$ , the natural map  $\text{coim}(f) \rightarrow \text{im } f$  is an iso. AB3) All colimit exist. AB4) AB3) + arbitrary direct sums are exact. AB5) AB3) + arbitrary filtered colimits are exact. AB6) AB3) +  $J$  an index set,  $\forall j \in J$  a filtered category (think of directed set)  $I_j$ , functors  $M: I_j \rightarrow \mathcal{C}$ , then

$$\varinjlim_{(i_j \in I_j)_j} \prod_j M_{i_j} \rightarrow \prod_{j \in J} \varinjlim_{i_j \in I_j} M_{i_j}$$

**Theorem 1.12.**  $\mathcal{C}$  a site. Then  $Ab(\mathcal{C})$  satisfies AB3), AB4), AB5) and AB6).

In fact, our case is even nicer:

**Theorem 1.13.**  $\text{Cond}(Ab)$  in addition satisfies AB6) and AB4\*).

## 1.2 Compactly generated topological spaces

**Definition 1.14.** A topological space  $T$  is called *compactly generated* if any function  $f: T \rightarrow T'$  is continuous as soon as the composite  $S \rightarrow T \rightarrow T'$  is continuous for all maps  $S \rightarrow T$  where  $S$  is compact and Hausdorff. See also  $\text{nCat}$ .

The inclusion functor  $\mathcal{CG} \hookrightarrow \mathcal{Top}$  has a right adjoint  $(-)^{cg}$ . If  $T$  is any topological space, then the topology on  $T^{cg}$  is the finest topology on  $T$  such that  $\sqcup_{S \rightarrow T} S \rightarrow T$  is continuous, where  $S$  ranges over all compact Hausdorff spaces.

Let  $T$  be a topological space. We view  $T$  as a presheaf on  $*_{\text{proét}}$  by setting  $T(S) = \text{Hom}_{\mathcal{Top}}(S, T)$  for all profinite sets  $S$ . We denote this by  $\underline{T}$ . Claim:  $\underline{T}$  is a sheaf.

- i) The first condition  $\underline{T}(\emptyset) = *$  is true, because there is exactly one morphism from the empty set to any topological space.
- ii)  $\underline{T}(S_1 \sqcup S_2) = \underline{T}(S_1) \times \underline{T}(S_2)$  by universal property of disjoint union.
- iii) For any surjection  $S' \twoheadrightarrow S$  we get an isomorphism

$$\underline{T}(S) \rightarrow \{x \in \underline{T}(S') \mid \pi_1^*(x) = \pi_2^*(x) \in \underline{T}(S' \times_S S')\}$$

Since  $\mathcal{T}op \rightarrow \text{Cond}(\text{Set})$  preserves products, group objects are preserved, so it maps topological groups to condensed groups etc.

**Proposition 1.15.** *i) This functor is faithful and fully faithful when restricted to the full subcategory of compactly generated spaces.*

*ii) It admits a left adjoint  $X \mapsto X(*)_{top}$  where  $X(*)_{top}$  gets the quotient topology of  $\sqcup_{S \rightarrow X} S \rightarrow X(*)$  as above. The counit  $I(*)_{top} \rightarrow T$  agrees with  $T^{cg} \rightarrow T$ .*

Coming back to our original example:

**Example 1.16.**  $\mathbb{R}_{disc} \rightarrow \mathbb{R}$  can be seen in the condensed world as  $\underline{\mathbb{R}}_{disc} \rightarrow \underline{\mathbb{R}}$ , i.e. from locally constant functions to continuous functions. This is still a mono, but now it is not an epi. The cokernel  $Q$  can be described as  $Q(S) = \{S \rightarrow \mathbb{R} \text{ continuous}\} / \{S \rightarrow \mathbb{R} \text{ locally constant}\}$ . Note in particular that the underlying set of  $Q$  is just  $*$ , reflecting the fact that the cokernel was trivial in the classical setting.

## References