

# VARIOUS LECTURE NOTES

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## CONTENTS

### 1. ABOUT THESE NOTES

The purpose of these notes is to keep the material seen in lectures a bit organized and easily accessible from one single place, but they don't intend to be complete and they will surely be full of typos and mistakes<sup>1</sup>.

Warnings will be marked with a **dangerous bend** symbol on the margin .



### 2. [CM] TALK 1 (JOHAN) - CONDENSED SETS - 21.10.19

**2.1. Introduction.** One of the main motivations for condensed mathematics is that topological algebraic objects have usually poor categorical and functorial properties. For instance, topological abelian groups do not form an abelian category:

**Example 2.1**  $\mathbb{R}_{disc} \rightarrow \mathbb{R}$  is epi and mono, but not iso.

Another motivation is coherent duality:

**Theorem 2.2** *Let  $f: X \rightarrow Y$  be a proper or quasi-projective morphism of Noetherian schemes of finite Krull dimension. Then there exists a right adjoint  $f^!$  to the derived direct image functor  $f_! = Rf_*: \mathbf{D}^b(\mathbf{QCoh}(X)) \rightarrow \mathbf{D}^b(\mathbf{QCoh}(Y))$ .*

At some point analytic rings will come up. We will then look at the category of solid modules, in which the 6-functor formalism works nicer than in the classical setting (e.g. when  $f_!$  is not defined in the classical setting,  $f_!$  takes non-discrete values in the condensed settings, which are "not there" in the classical setting).

**Definition 2.3** Proétale site of a point, denoted  $*_{prot}$ , is the category of profinite sets with finite jointly surjective families of continuous maps as covers. A *condensed set* (resp. group, ring, ...) is a sheaf of sets (resp. groups, rings, ...) on  $*_{prot}$ . We denote by  $\mathbf{Cond}(\mathbf{C})$  the category of condensed objects of a category  $\mathbf{C}$ .

**Definition 2.4** A *condensed set* (resp. group, ring, ...) is a contravariant functor  $X$  from  $*_{prot}$  to the category of sets (resp. groups, rings, ...) such that

- i)  $X(\emptyset) = *$ .

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*Date:* 31st January 2020.

<sup>1</sup>If you find any, please let me know! You can do this from [GitHub](#) or more directly with an email at [pedro.nunez@math.uni-freiburg.de](mailto:pedro.nunez@math.uni-freiburg.de)

ii) For all profinite sets  $S_1$  and  $S_2$  the natural map

$$X(S_1 \sqcup S_2) \rightarrow X(S_1) \times X(S_2)$$

is an isomorphism.

iii) For any surjection of profinite sets  $f: S' \twoheadrightarrow S$  we get an induced<sup>2</sup> isomorphism

$$X(S) \rightarrow \{x \in X(S') \mid \pi_1^*(x) = \pi_2^*(x) \in X(S' \times_S S')\}$$

We will call  $X(*)$  the *underlying object* in  $\mathbf{C}$  of a condensed object.

*Remark 2.5.* We will use  $T$  for topological spaces vs.  $X, Y$  for condensed sets, as opposed to Scholze's mixing of those notations.

**2.2. Recollections on sheaves on sites.** Let  $F$  be a presheaf on a site, which is just a contravariant functor to whatever category in which our sheaves are gonna take values. If  $U = \cup_i U_i$  is an open cover, the topological sheaf axiom could be phrased as:  $F(U)$  is an equalizer of the diagram

$$\prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j).$$

Note that  $U_i \cap U_j$  is just the fiber product of the two inclusions.

**Definition 2.6** (Coverage) See definition 2.1 in **nCat**.

**Definition 2.7**  $F$  a presheaf on  $\mathbf{C}$ . A collection  $(s_i) \in \prod_i F(U_i)$  for  $\{f_i: U_i \rightarrow U\}$  a covering is called a *matching family* if for all  $h: V \rightarrow U$  we have  $g^*(s_i) = h^*(s_j)$  for  $g$  and  $h$  in the diagram

$$\begin{array}{ccc} V & \xrightarrow{h} & U_j \\ \downarrow g & & \downarrow f_j \\ U_i & \xrightarrow{f_i} & U \end{array}$$

**Definition 2.8**  $F$  is a sheaf with respect to  $\{U_i \rightarrow U\}$  if for all matching families  $(s_i)$  there exists a unique  $s \in F(U)$  such that  $f_i^*(s) = s_i$ . We say that  $F$  is a *sheaf* if it is a sheaf for all covering families.

*Remark 2.9.* A sheaf of abelian groups is just a commutative group object in the category of sheaves of sets.

**Theorem 2.10** If  $\mathbf{C}$  is a site, then  $\mathbf{Ab}(\mathbf{C})$  is an abelian category.

**Definition 2.11** An additive category is a category in which the hom-sets are endowed with an abelian group structure in a way that makes composition bilinear and such that finite biproducts exist.

Recall Grothendieck's axioms: AB1) Every morphism has a kernel and a cokernel. AB2) For every  $f: A \rightarrow B$ , the natural map  $\text{coim}(f) \rightarrow \text{im } f$  is an iso. AB3) All colimit exist. AB4) AB3) + arbitrary direct sums are exact. AB5) AB3) + arbitrary filtered colimits are exact. AB6) AB3) +  $J$  an index set,  $\forall j \in J$  a filtered category (think of directed set)  $I_j$ , functors  $M: I_j \rightarrow \mathbf{C}$ , then

$$\varinjlim_{(i_j \in I_j)_j} \prod_j M_{i_j} \rightarrow \prod_j \varinjlim_{i_j \in I_j} M_{i_j}$$

**Theorem 2.12**  $\mathbf{C}$  a site. Then  $\mathbf{Ab}(\mathbf{C})$  satisfies AB3), AB4), AB5) and AB6).

<sup>2</sup>Since the pullback diagram is commutative, the image of  $X(f)$  is indeed induces a morphism as claimed.

In fact, our case is even nicer:

**Theorem 2.13**  $\text{Cond}(\mathbf{Ab})$  in addition satisfies AB6) and AB4\*).

### 2.3. Compactly generated topological spaces.

**Definition 2.14** A topological space  $T$  is called *compactly generated* if any function  $f: T \rightarrow T'$  is continuous as soon as the composite  $S \rightarrow T \rightarrow T'$  is continuous for all maps  $S \rightarrow T$  where  $S$  is compact and Hausdorff. See also [nCat](#).

The inclusion functor  $\mathbf{CG} \hookrightarrow \mathbf{Top}$  has a right adjoint  $(-)^{cg}$ . If  $T$  is any topological space, then the topology on  $T^{cg}$  is the finest topology on  $T$  such that  $\sqcup_{S \rightarrow T} S \rightarrow T$  is continuous, where  $S$  ranges over all compact Hausdorff spaces.

Let  $T$  be a topological space. We view  $T$  as a presheaf on  $*_{prot}$  by setting  $T(S) = \text{Hom}_{\mathbf{Top}}(S, T)$  for all profinite sets  $S$ . We denote this by  $\underline{T}$ . Claim:  $\underline{T}$  is a sheaf.

- i) The first condition  $\underline{T}(\emptyset) = *$  is true, because there is exactly one morphism from the empty set to any topological space.
- ii)  $\underline{T}(S_1 \sqcup S_2) = \underline{T}(S_1) \times \underline{T}(S_2)$  by universal property of disjoint union.
- iii) For any surjection  $S' \twoheadrightarrow S$  we get an isomorphism

$$\underline{T}(S) \rightarrow \{x \in \underline{T}(S') \mid \pi_1^*(x) = \pi_2^*(x) \in \underline{T}(S' \times_S S')\}$$

Since  $\mathbf{Top} \rightarrow \text{Cond}(\mathbf{Set})$  preserves products, group objects are preserved, so it maps topological groups to condensed groups etc.

**Proposition 2.15** i) This functor is faithful and fully faithful when restricted to the full subcategory of compactly generated spaces.

- ii) It admits a left adjoint  $X \mapsto X(*)_{top}$  where  $X(*)_{top}$  gets the quotient topology of  $\sqcup_{S \rightarrow X} S \rightarrow X(*)$  as above. The counit  $I(*)_{top} \rightarrow T$  agrees with  $T^{cg} \rightarrow T$ .

Coming back to our original example:

**Example 2.16**  $\mathbb{R}_{disc} \rightarrow \mathbb{R}$  can be seen in the condensed world as  $\underline{\mathbb{R}_{disc}} \rightarrow \underline{\mathbb{R}}$ , i.e. from locally constant functions to continuous functions. This is still a mono, but now it is not an epi. The cokernel  $Q$  can be described as  $Q(S) = \{S \rightarrow \mathbb{R} \text{ continuous}\} / \{S \rightarrow \mathbb{R} \text{ locally constant}\}$ . Note in particular that the underlying set of  $Q$  is just  $*$ , reflecting the fact that the cokernel was trivial in the classical setting.

## 3. [LT] LECTURE 1 - 22.10.19

**3.1. Introduction and overview of the course.** An *algebraic variety* is the solution set of a family of polynomial equations in  $\mathbb{C}^n$ . For example, if  $f(x, y, z, t) = xy - tz$ , then

$$V(f) = \{(x, y, z, t) \in \mathbb{C}^4 \mid xy - tz = 0\}$$

is an algebraic variety in  $\mathbb{C}^4$ . Another example would be the parabola  $\{y - x^2 = 0\} \subseteq \mathbb{C}^2$ .

Let us focus on  $V(f)$  and set  $t = 1$ . Then  $X = V(f) \cap \{t = 1\} = \{(x, y, z) \in \mathbb{C}^3 \mid xy = z\}$  can be seen as a family of complex curves parametrized by the variable  $z$ . For  $z = 0$ , the complex curve  $X_0$  has an *ordinary double point* at the origin:

Singularities arise naturally while studying the topology of algebraic varieties, and ODP's are a particularly nice kind of singularities.

For  $z \neq 0$  we get an equation which looks like  $xy = 1$ . In this case we have the following picture:

As  $z \rightarrow 0$ , the central loop  $\gamma$  contracts to the ordinary double point. Note in particular that  $X_0$  has trivial fundamental group (hence trivial 1-homology), whereas  $X_z$  does not.

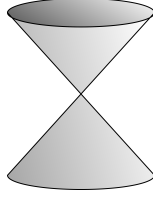
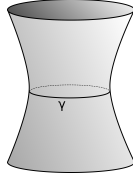


FIGURE 1. Topological picture of our ODP.

FIGURE 2. Topological picture of  $X_z$ .

We have a projection  $\pi: X \rightarrow \mathbb{C}$ , and Ehresmann's lemma tells us that for all disks  $D \subseteq \mathbb{C}$  not containing 0 we have  $\pi^{-1}(D) \cong D \times X_{z_0}$  for any  $z_0 \in D$ .

**Question 3.1.** Given an arbitrary nonsingular algebraic variety  $X \subseteq \mathbb{C}^n$ , can we find a map  $\pi: X \rightarrow \mathbb{C}$  such that the fibers  $X_t$  are nonsingular for all but finitely many  $t \in \mathbb{C}$  and such that the singular fibres have at worst ODP singularities?

Notice how we are missing information at infinity, e.g.  $y = x^2$  versus  $xy = 1$ . The solution to this is to replace  $\mathbb{C}^n$  by  $\mathbb{CP}^n$ .

So let  $X \subseteq \mathbb{P}^n$  be a nonsingular projective variety. Then we have:

**Theorem 3.2** *There exists a family  $(H_t)_{t \in \mathbb{CP}^1}$  of hyperplanes in  $\mathbb{CP}^n$  with  $H_{[a,b]} = aH_0 + bH_\infty$  such that*

- (1)  $X \subseteq \bigcup_{t \in \mathbb{CP}^1} H_t$ .
- (2)  $X_t = X \cap H_t$  is nonsingular except for finitely many critical values of  $t$ .
- (3)  $X_t$  has ODP singularities for each critical value  $t$ .

We call  $(X_t)_{t \in \mathbb{CP}^1}$  a *Lefschetz pencil*. We get a rational map  $X \dashrightarrow \mathbb{CP}^1$  sending  $x \mapsto t$  whenever  $x \in X_t$ . If  $x \in X_t \cap X_{t'}$  for  $t \neq t'$ , then  $x \in H_0 \cap H_\infty$ , so this rational map is not well-defined along  $X \cap H_0 \cap H_\infty$ . Blowing-up this subvariety of  $X$  we resolve the indeterminacy of the rational map and get a morphism  $\tilde{X} \xrightarrow{\pi} \mathbb{CP}^1$  as we wanted.

As an application we obtain:

**Theorem 3.3 (Lefschetz Hyperplane theorem)**  *$X \subseteq Y \subseteq \mathbb{CP}^N$  nonsingular varieties with  $X$  a hypersurface in the  $n$ -dimensional variety  $Y$ , then*

$$H_*(X) \rightarrow H_*(Y)$$

*is an isomorphism for  $*$  <  $n - 1$  and a surjection for  $*$  =  $n - 1$ .*

In particular, if  $Y = \mathbb{CP}^n$ , we have

$$H_*(\mathbb{CP}^n) = \begin{cases} \mathbb{Z} & \text{if } * \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

If  $X \subseteq \mathbb{CP}^n$  is a nonsingular hypersurface, then its homology will be that of projective space on all degrees other than  $n - 1$ . Its  $n - 1$  homology will depend on the variety, e.g. the ODP (trivial 1-homology) vs the ruled surface (with  $\gamma$  non trivial on 1-homology) from before.

**Example 3.4**  $X$  elliptic curve in  $\mathbb{CP}^2$  given by  $y^2 = x(x - 1)(x - \lambda)$  for  $\lambda \neq 0$ . Let  $L = \mathbb{CP}^1 \subseteq \mathbb{CP}^2$  and  $P \in \mathbb{CP}^1 \setminus (X \cup L)$ . We get  $X \xrightarrow{\pi} \mathbb{CP}^1$  by projecting from  $P$  to  $L$ .

#### 4. [WS] KODARIA 1 (JIN) - 23.10.19

**4.1. Chow's theorem.** Let  $G_i = G_i(z_1, \dots, z_n)$  be homogeneous polynomials of degree  $d_i$  for  $i \in \{1, \dots, k\}$ . Let  $V = V(G_1, \dots, G_k) = \{w \in \mathbb{C}^{n+1} \setminus \{0\} \mid G_i(w) = 0 \text{ for all } i \in \{1, \dots, k\}\} \subseteq \mathbb{CP}^n$ . Assume  $(\frac{\partial G_i}{\partial z_j}(w))_{i,j}$  is surjective at any  $w \in V$ . By Euler's theorem on homogeneous functions we have

$$\sum_{j=0}^n z_j \frac{\partial G_i}{\partial z_j} = d_i G_i(z_0, \dots, z_n).$$

If  $\tilde{w} = (\tilde{z}_0, \dots, \tilde{z}_n) \in V$ , then

$$\sum_{j=0}^n \tilde{z}_j \frac{\partial G_i}{\partial z_j} \Big|_{\tilde{w}} = 0$$

$V \cap U_i$  for any  $i \in \{0, \dots, n\}$ ,  $U_i = \{[z_0 : \dots : z_n] \in \mathbb{CP}^n \mid z_i \neq 0\}$ .

For  $i = 0$ , consider the chart  $(U_0, \phi_0)$  with  $\phi_0: U_0 \rightarrow \mathbb{C}^n$  given by  $[z_0, \dots, z_n] \mapsto (\frac{z_1}{z_0}, \dots, \frac{z_n}{z_0})$ . The inverse has a lift given by  $\tilde{\psi}: \mathbb{C}^n \rightarrow \mathbb{C}^{n+1} \setminus \{0\}$  given by  $(w_1, \dots, w_n) \mapsto (1, w_1, \dots, w_n)$ .

$$\begin{array}{ccc} \mathbb{C}^n & & \\ \downarrow \tilde{\psi} & \searrow G \circ \tilde{\psi}_0 & \\ \mathbb{CP}^n & \xrightarrow{G} & \mathbb{C}^k \end{array}$$

$V \cap U_0 = G^{-1}(\{0\})$ .

$G \circ \tilde{\psi}_0: (w_1, \dots, w_n) \mapsto (G_1(1, w_1, \dots, w_n), \dots, G_k(1, w_1, \dots, w_n))$ .

$$\frac{\partial (G_i \circ \tilde{\psi}_0)}{\partial w_j} = \frac{\partial G_i}{\partial z_l} \frac{\partial (\tilde{\psi}_0)^l}{\partial w_j} \Big|_{(\tilde{w}_1, \dots, \tilde{w}_n)}$$

Call the LHS  $A_1$ .

$$\frac{\partial G_i}{\partial z_l} \Big|_{\tilde{w}=(1, \tilde{w}_1, \dots, \tilde{w}_n)} \begin{pmatrix} 1 \\ \tilde{w}_1 \\ \vdots \\ \tilde{w}_n \end{pmatrix} = 0 \quad (1)$$

Note also that

$$\frac{\partial (\tilde{\psi}_0)^l}{\partial w_j} = \begin{pmatrix} 0 & \dots & 0 \\ 1 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 1 \end{pmatrix}.$$

Now

$$\left( \frac{\partial G_i}{\partial z_l} \right) = (a_{il}) = \begin{pmatrix} a_{10} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{k0} & \dots & a_{kn} \end{pmatrix}$$

$$A_1 = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{k1} & \dots & a_{kn} \end{pmatrix}$$

Since  $A$  is surjective and

$$A \begin{pmatrix} 1 \\ \tilde{w}_1 \\ \vdots \\ \tilde{w}_n \end{pmatrix} = 0,$$

hence  $A_1$  is surjective.

**Theorem 4.1 (Chow)** *Every analytic closed subvariety  $V \subseteq \mathbb{CP}^n$  is the zero locus of finite number of homogeneous polynomials.*

For this we will use as a black box:

**Lemma 4.2 (Riemann-Roch)** *Let  $U \subseteq \mathbb{C}^n$  be a domain,  $S$  be an analytic subvariety of  $U$  of dimension  $m$ , and  $W$  be an analytic subvariety of  $U \setminus S$  such that  $\dim_p W > m$  for all regular points  $p \in W$ . Then  $\bar{W}$  is analytic.*

Now we can prove Chow's theorem. Let  $\pi: \mathbb{C}^{n+1} \setminus \{0\} \rightarrow \mathbb{CP}^n$  be the projection. Then  $\pi^{-1}(V)$  has dimension at least 1 everywhere in  $\mathbb{C}^{n+1} \setminus \{0\}$ . Moreover,  $\pi^{-1}(V)$  is a cone missing the origin, so its closure is just  $\pi^{-1}(V) \cup \{0\}$ . Set  $S = \{0\}$  and  $W = \pi^{-1}(V)$ . Then  $V' = \bar{W} = \pi^{-1}(V) \cup \{0\}$  is an analytic variety of  $\mathbb{C}^{n+1}$  by the Riemann-Roch theorem. In particular, near 0 we can write

$$V'_0 = U_\varepsilon(0) \cap V' = V(g_1, \dots, g_k)$$

with  $g_i$  holomorphic on  $U_\varepsilon(0)$ . In particular each  $g_i$  is analytic, so we may write it as  $g_i = \sum_{n=1}^{\infty} g_{i,n}$  where each  $g_{i,n}$  is a homogeneous polynomial. Then  $g_i(tz) = \sum_{n=1}^{\infty} g_{i,n}(z)t^n$  for all  $x \in \mathbb{C}^{n+1}$  and all  $t \in \mathbb{C}$ . If  $z \in V'$ , then  $tz \in V'$  for all  $t$ , because  $V'$  is a cone. So  $g_i(tz) \equiv 0$  implies  $g_{i,n}(z) = 0$  for all  $i \in \{1, \dots, k\}$  and all  $n \in \mathbb{N}_{>0}$ . Therefore  $V'_0 = V(\{g_{i,n}\})$ . By Noetherianity, finitely many  $g_{i,n}$  suffice, so we can write  $V_0 = V(g^{(1)}, \dots, g^{(m)})$  for some  $g^{(i)} \in \{g_{i,n}\}$ . Hence  $V = V(g^{(1)}, \dots, g^{(m)})$  in  $\mathbb{CP}^n$  and this finishes the proof.

**4.2. Sheaves.** For precise definitions and results in this subsection see Wikipedia, Stacks or nLab.

Definition of *sheaf* (of abelian groups) on a topological space  $X$ , *stalk* of a sheaf at a point  $x \in X$ , *germs* of a sheaf at a point... Note the similarities in terminology with plants.

**Example 4.3** Constant sheaves  $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ . Sheaf of smooth functions  $C^\infty$  and its units  $C^*$ . Sheaf of regular functions  $\mathcal{O}$  and units  $\mathcal{O}^*$ . Sheaf of meromorphic functions  $\mathcal{M}$  and  $\mathcal{M}^*$ .

Maps between sheaves, their kernels and their cokernels. Short exact sequences of sheaves.

**Example 4.4** Let  $M$  be a complex manifold. The sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \rightarrow 0$$

is exact.

Definition of Čech cohomology of a sheaf  $\mathcal{F} \in \mathbf{Sh}(M)$  with respect to an open cover  $\mathcal{U}$ , which we denote by  $H^p(\mathcal{U}, \mathcal{F})$  on degree  $p$ , and Čech cohomology of the sheaf  $\mathcal{F}$  as their direct limit over refinements, denoted  $\check{H}^p(M, \mathcal{F})$ .

**Theorem 4.5 (Leray)** If  $\mathcal{U}$  is an acyclic cover, i.e. if there are no higher Čech cohomologies with respect to this cover, then the Čech complex associated to this cover computes Čech cohomology.

Long exact sequence in Čech cohomology induced by a short exact sequence of sheaves.

**4.3. A bit of Hodge theory.** Decomposition of the tangent space at a point of a complex manifold, its tensor algebras,  $\partial$  and  $\bar{\partial}$  operators, Dolbeault cohomology groups, harmonic and Hodge decomposition. See [?] or [?].

## 5. [LT] LECTURE 2 - 24.10.19

**Remark 5.1.** Exercise sessions will be Thursday from 13h to 15h on SR318 (Starting next week).

As pointed out last week, we want to look at polynomials and their solutions sets. But polynomials are a bit too rigid. Instead, we look at polynomials as truncated power series, or more generally as *analytic functions*, which are functions which locally can be represented as power series. We will see that these are the same as holomorphic functions. In particular, every holomorphic function is  $C^\infty$ .

$$\begin{aligned} \text{polynomials} &\Rightarrow \text{convergent power series} \Rightarrow \text{analytic} \Leftrightarrow \\ &\Leftrightarrow \text{holomorphic} \Rightarrow C^\infty \Rightarrow \text{continuous} \Rightarrow \text{abominations} \end{aligned}$$

If we were analysts we would start at the bottom and then try to swim up. Instead we will start from the top and float downstream.

**Notation 5.2.**  $\mathbb{E} = \mathbb{R}$  or  $\mathbb{C}$ .  $z = (z_1, \dots, z_n) \in \mathbb{E}^n$ ,  $r \in \mathbb{R}_{\geq 0}$ . Recall

$$|z| = \sqrt{2z_1\bar{z}_1 + \dots + z_n\bar{z}_n},$$

$$\mathbb{D}(z, r) = \{w \in \mathbb{E}^n \mid |z - w| < r\}, \text{ and}$$

$$\bar{\mathbb{D}}(z, r) = \{w \in \mathbb{E}^n \mid |z - w| \leq r\}$$

called open and closed disks respectively. We call  $\bar{\mathbb{D}}(z_1, r_1) \times \dots \times \bar{\mathbb{D}}(z_n, r_n)$  an *open polydisk*.

### 5.1. Formal power series.

**Definition 5.3** Let  $a = (a_1, \dots, a_n) \in \mathbb{C}^n$ . A *formal power series* centered at  $a$  is an expression of the form

$$f(z) = f(z_1, \dots, z_n) = \sum_{(r_1, \dots, r_n) \in \mathbb{Z}_{\geq 0}^n} c_{r_1 \dots r_n} (z_1 - a_1)^{r_1} \dots (z_n - a_n)^{r_n}$$

with  $c_{r_1, \dots, r_n} \in \mathbb{C}$ .

**Remark 5.4.** We will restrict our attention to absolutely convergent series, so we do not need to order the indices in the sum to discuss convergence.

**Definition 5.5** The series above *converges (uniformly) absolutely* on  $X \subseteq \mathbb{C}^n$  if for all  $z \in X$  the series of real numbers

$$\sum_{(r_1, \dots, r_n)} |c_{r_1, \dots, r_n} (z_1 - a_1)^{r_1} \dots (z_n - a_n)^{r_n}|$$

converges (uniformly).

Recall that  $\sum_n c_n z^n$  converges absolutely on  $\mathbb{D}(0, R)$  where  $R = \frac{1}{\limsup_{n \rightarrow \infty} |c_n|^{\frac{1}{n}}}$ . It converges uniformly absolutely on each compact  $K \subseteq \mathbb{D}(0, R)$ .

**Example 5.6 (Geometric series)** The geometric series with ration  $z = (z_1, \dots, z_n) \in \mathbb{C}^n$  is defined as  $\sum_{r_1, \dots, r_n} z_1^{r_1} \cdots z_n^{r_n}$ . It converges (uniformly) absolutely on (compact subsets of)  $\mathbb{D}(0, 1)^n$  with sum equal

$$\prod_{k=1}^n \sum_{r_k \geq 0} z_k^{r_k} = \frac{1}{(1 - z_1) \cdots (1 - z_n)}$$

**Lemma 5.7 (Abel)** Consider the series above,  $w \in \mathbb{C}^n$  and  $M \in \mathbb{R}_{>0}$ . If  $|c_r(w - a)^r| = |c_{r_1, \dots, r_n}(w_1 - a_1)^{r_1} \cdots (w_n - a_n)^{r_n}| < M$  for each  $r \in \mathbb{Z}_{\geq 0}^n$ , then  $f(z)$  converges uniformly absolutely on each compact  $K \subseteq D = \mathbb{D}(a_1, \rho_1) \times \mathbb{D}(a_n, \rho_n)$ , where  $\rho_i := |w_i - a_i|$ .

*Proof.* WLOG  $\rho_k > 0$  for all  $k \in \{1, \dots, n\}$  (otherwise we'd have  $D = \emptyset$ ). Let  $K \subseteq D$ . Then let  $\delta_k := \max_{z \in K} \frac{|z_k - a_k|}{\rho_k} < 1$ . Then for all  $z \in K$  and for all  $r \in \mathbb{Z}_{\geq 0}^n$  we have

$$|c_r(z - a)^r| \leq |c_r \rho^r| \leq M \delta^r.$$

Since all  $\delta_k < 1$ , by the previous example  $\sum_r M \delta^r$  converges uniform absolutely on  $K$ .  $\square$

**Definition 5.8** Uniform absolute convergence on compacts is also called *compact convergence*.

## 5.2. Analytic functions.

**Definition 5.9** Let  $U \subseteq \mathbb{C}^n$  open.

- i)  $f : U \rightarrow \mathbb{C}$  is *analytic* at  $a \in U$  if there exists an open neighbourhood  $a \in V \subseteq U$  and  $c_r$  such that  $f(z) = \sum_r c_r (z - a)^r$  converges compactly on  $V$ .
- ii)  $f : U \rightarrow \mathbb{C}$  is *analytic* on  $U$  if it is analytic at each point of  $U$ .
- iii)  $f : U \rightarrow \mathbb{C}^n$  is *analytic* on  $U$  if each component  $f_k$  is for all  $k \in \{1, \dots, n\}$ .

*Exercise 5.10.* Analytic at  $a$  implies continuous at  $a$ .

*Exercise 5.11.* If  $f, g$  are analytic, then so are  $f + g, f - g$  and  $g \circ f$  where defined.

*Exercise 5.12.* Let  $U \subseteq \mathbb{C}^n$  be an open subset, let  $z \in U$  and  $w \in \mathbb{C}^n$ . Let  $V = \{c \in \mathbb{C} \mid z + cw \in U\} \subseteq \mathbb{C}$ .

- i)  $V$  is open and  $0 \in V$ .
- ii) For all  $f : U \rightarrow \mathbb{C}$  analytic we have that  $g(t) = f(z + tw)$  is analytic on  $V$ .

**Theorem 5.13 (Identity theorem)** If  $\emptyset \neq V \subseteq U \subseteq \mathbb{C}^n$  are open with  $U$  connected and  $f : U \rightarrow \mathbb{C}$  is analytic with  $f|_V = 0$ , then  $f = 0$ .

*Proof.* If  $f(z) \neq 0$  for some  $z \in U$ , then by continuity of  $f$  we would have that  $f$  is nowhere zero on some open nbhd of  $z$ . Let  $Z = \{w \in U \mid f \text{ vanishes in an open nbhd of } w\}$ . Then  $Z$  is closed in  $U$  by what we just said. Also,  $V \subseteq Z$  as  $V$  is open. Let  $w \in Z$  and choose a polydisk  $w \in D = \mathbb{D}(w_1, r_1) \times \cdots \times \mathbb{D}(w_n, r_n) \subseteq U$ . If we show that  $D \subseteq Z$ , then every point of  $Z$  is in its interior and  $Z$  is therefore open.

So let  $z \in D$  with  $z \neq w$ . Consider now  $W = \{c \in \mathbb{C} \mid w + c(z - w) \in U\} \subseteq \mathbb{C}$ , which is open by the previous exercise<sup>3</sup>, and  $g : t \mapsto f(w + t(z - w))$  is analytic on  $W$ . The identity theorem for single-variable analytic functions implies that  $g = 0$  in a nbhd of  $w$ . Since  $D$

<sup>3</sup>This step allows us to reduce our problem in several complex variables to a problem on a single complex variable.



is convex,  $[0, 1] \subseteq W$ . By the identity theorem in one variable,  $g = 0$  on an open nbhd of  $[0, 1]$ . Hence  $g(1) = f(z) = 0$ , so  $f$  vanishes on  $D$  and  $D \subseteq Z$ .  $\square$

**5.3. Topology.** Definition of topological space and examples (cofinite topology, Zariski topology). Continuous maps, homeomorphisms (isomorphism in the category of topological spaces). Example: graph of  $f: X \rightarrow Y$  defined as  $\Gamma_f = X \times_X Y$  maps homeomorphically onto  $X$  via the first projection. Subspace topology.

Connectedness, example: unit interval. Continuous image of connected is connected.

Hausdorffness, example: euclidean topology on  $\mathbb{R}^n$ . Non-example: real line with two origins<sup>4</sup>

Equivalently,  $X$  is Hd if and only if  $\Delta \subseteq X \times X$  is closed. Hausdorffness is hereditary.

#### 6. [FS] MATTHIAS PAULSEN - THE CONSTRUCTION PROBLEM FOR HODGE NUMBERS - 25.10.19

In characteristic 0 this is j.w. with Stefan Schreieder. In positive characteristic this is j.w. with v. Dobbeen de Bruyn.

**6.1. Overview over complex numbers.**  $X$  smooth projective variety. Then we have Hodge theory, which allows us to decompose

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X),$$

where  $H^{p,q}(X) \cong H^q(X, \Omega^p)$ . The *Hodge numbers* are  $h^{p,q}(X) = \dim_{\mathbb{C}} H^{p,q}(X)$ . We usually arrange the Hodge numbers in the *Hodge diamond*

$$\begin{array}{ccccc} & & h^{n,n} & & \\ & & \vdots & & \\ & h^{n,n-1} & & h^{n-1,n} & \\ & \vdots & & \vdots & \\ & h^{n,0} & & h^{0,n} & \\ & \vdots & & \vdots & \\ & h^{1,0} & & h^{0,1} & \\ & & h^{0,0} & & \end{array}$$

**Example 6.1** Let  $X \subseteq \mathbb{P}^4$  be a hypersurface of degree 4. Then its Hodge diamond is

<sup>4</sup>These two examples show that Hausdorffness is not a local property, because the real line with two origins is locally the same as  $\mathbb{R}$ .

$$\begin{array}{ccccccc}
& & & & 1 & & \\
& & & & & & \\
& & & 0 & & 0 & \\
& & & & & & \\
& & 0 & & 1 & & 0 \\
& & & & & & \\
5 & & 30 & & 30 & & 5 \\
& & & & & & \\
& & 0 & & 1 & & 0 \\
& & & & & & \\
& & 0 & & 0 & & \\
& & & & 1 & & 
\end{array}$$

We know:

- i)  $h^{p,q} = h^{q,p}$ .
- ii)  $h^{0,0} = 1$ .
- iii)  $h^{p,q} \geq h^{p-1,q-1}$  if  $p + q \leq n$ .

*Question 6.2.* Given  $(h^{p,q})_{p,q}$  such that i), ii) and iii) hold. Does there exist  $X$  with  $h^{p,q}(X) = h^{p,q}$  for all  $p, q$ ?

1989 Partial results in dimensions 2 and 3.

2013 Kotschick and Schreieder determined the Hodge ring of Kaehler manifolds and showed that there are no linear relations besides of the previous three.

2015 Schreieder.

In any dimension  $n$ , a given row  $k < n$  can be always achieved, except if  $k = 2p$ , in which case we need  $h^{p,p} \geq O(p)$ .

If we ignore the middle row, the outer Hodge numbers and the middle column, then everything else can be arbitrary.

Negative results: the previous question has negative answer, e.g. in dimension three, if we assume that  $h^{1,1} = 1$  and  $h^{0,2} \geq 1$ , then  $h^{3,0} < 12^6 h^{2,1}$ .

*Question 6.3* (Kollar). Are there any polynomial relations between the Hodge numbers, besides the ones induced by the symmetries above?

*Question 6.4.* Besides the "unexpected" inequalities, are there also number theoretic restrictions?

2019 Schreieder and P.: modulo any integer  $m \geq 1$ , any Hodge diamond  $(h^{p,q})_{p,q}$  satisfying the symmetries<sup>5</sup> above is realizable by a smooth projective variety  $X$ , i.e. such that

$$h^{p,q}(X) \equiv h^{p,q} \pmod{m}.$$

**Corollary 6.5** *There are no polynomial relations and there are no "number theoretic" restrictions.*

<sup>5</sup>Note that condition iii) vanishes.

The proof can be divided into two parts corresponding to the outer Hodge numbers and the remaining ones. The outer Hodge numbers are birational invariants. The first part is to produce a variety which has the right outer Hodge numbers. And then all the inner ones can be obtained by repeated blow-ups.

**6.2. Constructions.** We will have 3 building blocks:

- Products (and then use Kuenneth's formula).
- Hypersurfaces to reduce the dimension again (apply Lefschetz hyperplane theorem, picture [A]).
- Blow-ups of subvarieties (see picture [B]).

We start then with a curve, in which the problem is completely solvable:

$$\begin{array}{ccc}
 & 1 & \\
 & & \\
 g & & g \\
 & & \\
 & 1 & 
 \end{array}$$

Starting from this we can build up our Hodge diamonds modulo  $m$ .

**6.3. Positive characteristic.** We don't have complex conjugation, but we still have Serre duality imposing its 180 degrees rotation on the Hodge diamonds. Is this the only restriction?

**Example 6.6 (Serre)** Construction of a surface with Hodge diamond

$$\begin{array}{ccccc}
 & & 1 & & \\
 & & & & \\
 & 0 & & 1 & \\
 & & & & \\
 ? & & ? & & ? \\
 & & & & \\
 & 0 & & 1 & \\
 & & & & \\
 & & 1 & & 
 \end{array}$$

It seems that Matthias is up to something cool using this example!

## 7. [CM] TALK 2 (PEDRO) - CONDENSED ABELIAN GROUPS - 28.10.19

See the full script on [GitHub](#).

**7.1. Recollections from the previous talk.** Recall  $*_{prot}$  and  $A = \text{Sh}(*_{prot}, \text{Ab})$ .  
Equivalent description of  $A$  from last talk.

### 7.2. A nicer description of our category.

**Definition 7.1** Extremally disconnected.

Note that  $\mathbf{ED} \subsetneq \mathbf{CHaus}$ .

**Fact:**  $S \in \mathbf{CHaus}$  is in  $\mathbf{ED}$  if and only if every surjection  $S' \twoheadrightarrow S$  from a compact Hausdorff space admits a section. Using Stone-Čech compactification  $\beta$ , this implies that every  $S \in \mathbf{CHaus}$  admits a surjection from an extremally disconnected set

$$\exists \tilde{S} = \beta(S_{disc}) \twoheadrightarrow S$$

**Lemma 7.2**  $\mathbf{A} = \mathbf{Sh}(\mathbf{ED}, \mathbf{Ab})$ .

**Corollary 7.3**  $\mathbf{A} = \{\mathcal{F} \in \mathbf{Fun}(\mathbf{ED}^{op}, \mathbf{Ab}) \mid i) \wedge ii)\}$ .

**Corollary 7.4** (Co)limits exist in  $\mathbf{A}$  and can be constructed pointwise.

**7.3. Abelianity and compact-projective generation.** Recall definition of Grothendieck category.

**Theorem 7.5**  $\mathbf{A}$  is abelian with  $(AB6)$  and  $(AB4^*)$  and it is generated by compact projective objects.

**Corollary 7.6**  $\mathbf{A}$  has enough injectives and projectives.

— BREAK —

**7.4. Closed symmetric monoidal structure.** Briefly outline monoidal categories.

**Proposition 7.7**  $\mathbf{A}$  is symmetric monoidal and the functor  $\mathbb{Z}[-]: \mathbf{Cond}(\mathbf{Set}) \rightarrow \mathbf{A}$  is symmetric monoidal.

**Proposition 7.8** For all condensed set  $X$ , the condensed abelian group  $\mathbb{Z}[X]$  is flat, i.e. the functor  $\mathbb{Z}[X] \otimes (-)$  is exact.

**Proposition 7.9** For all  $\mathcal{F} \in \mathbf{A}$  the functor  $\mathcal{F} \otimes (-)$  has a right adjoint  $[\mathcal{F}, -]$ , i.e.  $\mathbf{A}$  is closed symmetric monoidal.

**7.5. Derived category.** Brief description.

*Remark 7.10.* Triangulated structure and the problem it carries.

Basic derived functors

$$RF: \mathbf{D}^+(\mathbf{A}) = \mathbf{K}^+(I) \rightarrow \mathbf{D}(\mathcal{B}).$$

**Example 7.11**  $\mathbf{Hom}(-, \mathcal{F})$  and  $\mathbf{Hom}^\bullet(-, \mathcal{F}^\bullet)$ .

Extension to the whole  $\mathbf{D}$  using Spaltenstein's resolutions. Formula

$$\mathbf{Hom}_{\mathbf{D}}(\mathcal{F}^\bullet, \mathcal{G}^\bullet[i]) = \mathbf{Ext}^i(\mathcal{F}^\bullet, \mathcal{G}^\bullet).$$

Closed symmetric monoidal structure.

**Example 7.12**  $\mathbb{Z}[X] \otimes^L (-) = \mathbb{Z}[X] \otimes (-)$  is just degree-wise tensor product, because this functor is already exact so it need not be derived.

**Proposition 7.13** Compact generation.

Mention Brown representability.

## 8. [LT] LECTURE 3 - 29.10.19

**Goals for today:**

- **Prop:**  $f$  analytic  $\Rightarrow f$  complex differentiable (holomorphic).
- **Cor:**  $f$  analytic  $\Rightarrow f \in C^\infty$ .
- Implicit and inverse function theorems for analytic functions.

**8.1. Complex differentiable functions.** Recall:  $U \subseteq \mathbb{R}^n$  open,  $f: U \rightarrow \mathbb{R}^m$  is *differentiable* at  $a \in U$  if  $\exists \mathbb{R}$ -linear map  $df_a: \mathbb{R}^n \rightarrow \mathbb{R}^m$  such that

$$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a) - df_a(h)}{|h|} = 0.$$

With respect to the standard bases,  $df_a$  is given by the *Jacobian matrix*  $J_f(a) = (\frac{\partial f_i}{\partial f_j})_{i,j}$ .

We say that  $f: U \rightarrow \mathbb{R}^m$  is  $C^\infty$  or *smooth* if all partial derivatives exist (to all orders) and are continuous.

Recall also that  $f \in C^1$  implies  $f$  differentiable.

**Definition 8.1**  $X \subseteq \mathbb{R}^n$  not necessarily open,  $f: X \rightarrow \mathbb{R}^m$  is *differentiable* or  $C^\infty$  if for all  $x \in X$  we can find an open nbhd  $x \in U \subseteq \mathbb{R}^n$  and a differentiable or  $C^\infty$  function  $F: U \rightarrow \mathbb{R}^m$  such that  $F|_{X \cap U} = f|_{X \cap U}$ .

**Definition 8.2** Let  $U \subseteq \mathbb{C}^n$  open. We say that  $f: U \rightarrow \mathbb{C}$  is *complex differentiable* at  $a \in U$  if it is continuous and

- if  $n = 1$ , then the limit  $f'(a) := \lim_{z \rightarrow a} \frac{f(z) - f(a)}{z - a}$  exists.
- if  $n > 1$ , then for all  $1 \leq k \leq n$  and for all  $z_1, \dots, \hat{z}_k, \dots, z_n$  the function

$$f_k(z) := f(z_1, \dots, z_{k-1}, z, z_{k+1}, \dots, z_n)$$

is complex differentiable as in 1).

**Example 8.3**  $\text{Re}, \text{Im}: \mathbb{C} \rightarrow \mathbb{R}$  are  $C^\infty$ , not complex differentiable.

If  $f$  is complex differentiable, then  $\frac{\partial f}{\partial x_i}, \frac{\partial f}{\partial y_i}$  exist and are continuous, i.e.  $f \in C^1$ .

**Proposition 8.4** Let  $U \subseteq \mathbb{C}^n$  open and  $f: U \rightarrow \mathbb{C}$  is analytic. Then  $f$  is complex differentiable and  $f'$  is analytic.

*Proof.*  $f(z_1, \dots, z_n) = \sum_r c_r (z - a)^r$  locally near  $a \in U$ . Continuity is OK, we need to show that

$$\sum c_r (\zeta_1 - a_1)^{r_1} \dots (\zeta_{n-1} - a_{n-1})^{r_{n-1}} (z_n - a_n)^{r_n}$$

is complex differentiable for all  $(\zeta_1, \dots, \zeta_n)$  with respect to  $z_n$ . This is a power series in a single variable, hence complex differentiable.  $\square$

**Corollary 8.5**  $f$  analytic implies  $f$  smooth.

*Proof.*  $f$  analytic implies  $f'$  analytic, hence  $f''$  analytic, and so on.  $\square$

**8.2. Implicit function theorem.**

**Notation 8.6.** We denote by  $\mathbb{C}[[z_1, \dots, z_n]]$  the ring of formal power series in  $z_1, \dots, z_n$  centered at 0 (with Cauchy product as multiplication) and by  $\mathbb{C}\{z_1, \dots, z_n\}$  the subring of all power series convergent in a nbd of 0.

**Lemma 8.7** <sup>6</sup> Let  $f(z_0, \dots, z_n) = \sum_r c_r z^r$  with  $c_{0, \dots, 0} \neq 0$ . Then  $\exists! g \in \mathbb{C}[[z_1, \dots, z_n]]$  such that  $fg = 1$ .

<sup>6</sup>Typical proof with power series: do the naive thing inducting on the degree and the number of variables.

*Proof.* WLOG  $c_{0,\dots,0} = 1$ . Let  $z = (z_1, \dots, z_{n-1})$  and  $w = z_n$ . Let  $f = \sum_k a_k(z)w^k$  with  $a_k(z) \in \mathbb{C}[[z]]$ . Let  $g = \sum_k b_k(z)w^k$  with  $b_k(z) \in \mathbb{C}[[z]]$ . Then  $fg = \sum_{l=0}^{\infty} (\sum_{k=0}^l a_k b_{l-k})w^l$ . Call  $\delta_{0,l} = \sum_{k=0}^l a_k b_{l-k}$ .  $l = 0$  implies  $1 = a_0 b_0$ , hence  $b_0 = a_0^{-1}$ . For  $l > 0$  we get  $0 = \sum_{k=0}^l a_k b_{l-k} a_k b_{l-k}$ , hence

$$b_l = -a_0^{-1} \sum_{k=1}^l a_k b_{l-k}.$$

□

**Theorem 8.8 (Formal implicit function theorem)**<sup>7</sup> If  $f \in \mathbb{C}[[z, w]]$ ,  $f(0) = 0$  and  $\frac{\partial f}{\partial w} \neq 0$ , then  $\exists! u \in \mathbb{C}[[z, w]]$  and  $\exists! r \in \mathbb{C}[[z]]$  such that  $uf = w - r$  and  $u(0) \neq 0$ .

*Proof.* 1) Consider the  $\mathbb{C}$ -linear maps

$$R: \mathbb{C}[[z, w]] \rightarrow \mathbb{C}[[z]]$$

$$p = \sum_r c_r(z)w^r \mapsto c_0(z)$$

$$H: \mathbb{C}[[z, w]] \rightarrow \mathbb{C}[[z, w]]$$

$$p = \sum_r c_r w^r \mapsto \frac{p - R(p)}{w} = \sum_{r>0} c_r w^{r-1}$$

2)  $\forall p$  we have  $p = wH(p) + R(p)$ .

3)  $H(f)$  is a unit by the previous lemma and  $\frac{\partial f}{\partial w} \neq 0$ .

4) Suffices to find  $u$  s.t.

$$0 = H(w - uf) = H(w) - H(uf)$$

$$0 = 1 - H(uf) \quad (*)$$

Indeed,  $(*)$  and 2) imply that  $R(w - uf) = w - uf$ , hence  $r := w - uf \in \mathbb{C}[[z]]$  ( $w = uf + (w - uf) = uf + r$ ).

5)

$$uf \stackrel{2)}{=} u(wH(f) + R(f))$$

$$= uwH(f) + uR(f)$$

$$(*) \Leftrightarrow 0 = 1 - H(uf)$$

$$0 = 1 - H(uwH(f) + uR(f))$$

$$0 = 1 - \left( \frac{uwH(f) - R(uwH(f))}{w} \right) - H(uR(f))$$

$$(**) 0 = 1 - uH(f) - H(uR(f))$$

6)  $H(f)$  unit  $\Rightarrow$  suffices to find  $v = uH(f)$ .  $\mu := -R(f)H(f)^{-1}$  s.t.  $u = vH(f)^{-1}$  satisfies  $(**)$ .

7)

$$(**) \Leftrightarrow 0 = 1 - v + H\left(\frac{-uR(f)}{H(f)}\right)H(f)$$

$$(***) 0 = 1 - v + H(\mu v)$$

<sup>7</sup>Near 0 we have  $f(z, w) = 0$  iff  $0 = u(z, w)f(z, w)$  iff  $0 = w - r(z)$ , i.e. the vanishing locus of  $f$  is the graph of  $r$  near 0, hence this is indeed an implicit function theorem.

- 8)  $M: \mathbb{C}[[z, w]] \rightarrow \mathbb{C}[[z, w]]$  defined as  $p \mapsto H(\mu p)$ . Note: if  $z^k$  divides  $p$ , then  $z^{k+1}$  divides  $M(p)$ .

9)

$$0 = 1 - v + M(v)$$

$$(***)v = 1 + M(v)$$

$\mathbb{C}$ -linearity of  $M$  implies that  $v = 1 + M(v) = 1 + M(1 + M(v))$ . Hence for all  $k \geq 0$  we have

$$v = 1 + M(1) + \cdots + M^k(1) + M^k(v).$$

- 10)  $z^k$  divides  $M^k(1)$ ,  $z^{k+1}$  divides  $M^{k+1}(v)$ . Hence  $\sum_k M^k(1)$  is convergent as a formal power series.
- 11) If  $v$  satisfying  $(***)$ , then  $v = \sum_k M^k(1)$ . Hence  $v$  is unique and thus so are  $u, r$ .
- 12)  $v = \sum_k M^k(1)$  satisfies  $(***)$ .  $v = 1 + M(1) + \cdots + M^k(1) + W_k$ , hence  $z^{k+1}$  divides  $W_k$  by 10).  $v - 1 - M(v) = 1 + M(1) + \cdots + M^k(1) + W_k - 1 - M(1) - \cdots - M^k(1) - M^{k+1}(1) - M(W_k)$ , hence for all  $k$  we have that  $z^{k+1}$  divides  $v - 1 - M(v)$ . Therefore  $v - 1 - M(v) = 0$ .

□

**Theorem 8.9 (Convergent implicit function theorem)** With notation as above, if  $f \in \mathbb{C}\{z, w\}$ , then  $u \in \mathbb{C}\{z, w\}$  and  $r \in \mathbb{C}\{z\}$ .

*Proof.* See Samuel-Zariski.

□

**Exercise 8.10.** Formulate and deduce the analytic inverse and implicit function theorem.

**8.3. More basic topology.** Product spaces: definition and universal property. Example:  $\mathbb{R}^n \cong \mathbb{R} \times \mathbb{R}^{n-1}$  with the usual topology. Non example:  $\mathbb{C}^{n+m} \not\cong \mathbb{C}^n \times \mathbb{C}^m$  with the Zariski topology.

Quotient spaces: definition and universal property. Examples:  $S^1 = [0, 1]/1 \sim \mathbb{R}/\mathbb{Z}$  and  $\mathbb{T}^2$  gluing the boundary of the square  $[0, 1]^2$  appropriately.

Quasi-compactness: definition. Non-example:  $\mathbb{R}$ . Example:  $[a, b] \subseteq \mathbb{R}$ . In  $\mathbb{R}^n$  quasi-compact iff closed bounded. More facts: union of qc is qc, continuous image of qc is qc, closed subset of qc is qc.

## 9. [WS] KODAIRA 2 (VERA) - INTRODUCTION TO HODGE MANIFOLDS - 30.10.19

Outline for today:

- 1) Recollections.
- 2) Hodge Manifolds.
- 3) Outlook: Kodaira's Embedding Theorem.

### 9.1. Recollections.

9.1.1. *Cohomology Theories.* 1. Recall the *Dolbeault-Cohomology* defined as

$$H^{p,q}(X) := \frac{\ker(\bar{\partial}: \mathcal{A}^{p,q}(X) \rightarrow \mathcal{A}^{p,q+1}(X))}{\text{im}(\bar{\partial}: \mathcal{A}^{p,q-1}(X) \rightarrow \mathcal{A}^{p,q}(X))}.$$

2. Recall the *de Rham cohomology* defined as

$$H_{dR}^k(X, \mathbb{C}) = \frac{\ker(d: \mathcal{A}_{\mathbb{C}}^k(X) \rightarrow \mathcal{A}_{\mathbb{C}}^{k+1}(X))}{\text{im}(d: \mathcal{A}_{\mathbb{C}}^{k-1}(X) \rightarrow \mathcal{A}_{\mathbb{C}}^k(X))},$$

where  $\mathcal{A}_{\mathbb{C}}^k = \Gamma(X, \wedge^k TX^* \otimes \mathbb{C})$ .

**Theorem 9.1** On a compact Kaehler manifold  $(X, I, < \cdot, \cdot >)$  we have

$$H_{dR}^k(X, \mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}(X).$$

On the other hand, we have

**Theorem 9.2** (de Rham Theorem)

$$H_{dR}^k(X, \mathbb{C}) \cong H^k(X, \mathbb{C}).$$

3. Integral cohomology.

**Definition 9.3** Let  $X$  be a compact complex manifold. A closed differential form  $\varphi$  on  $X$  is called *integral* if its cohomology class  $[\varphi] \in H_{dR}^k(X, \mathbb{C})$  is in the image of the mapping

$$H^k(X, \mathbb{Z}) \rightarrow H^k(X, \mathbb{C}) \cong H_{dR}^k(X, \mathbb{C}).$$

Today we will be interested in integral  $(1, 1)$ -classes, i.e.

$$\begin{array}{ccc} & & H^2(X, \mathbb{Z}) \\ & \swarrow j_* & \downarrow \\ H^{1,1}(X) \subseteq H_{dR}^2(X, \mathbb{C}) & \xrightarrow{\cong} & H^2(X, \mathbb{C}) \end{array}$$

We denote  $\tilde{H}_{dR}^2(X, \mathbb{Z}) := j_* H^2(X, \mathbb{Z})$ .

9.1.2. *Holomorphic line bundles and first Chern class.* Recall the exponential sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O} \xrightarrow{\exp} \mathcal{O}^\times \rightarrow 0$$

of sheaves. Last week we have seen that this sequence is exact and thus it induces a long exact sequence of cohomology groups

$$\cdots \rightarrow \check{H}^1(X, \mathcal{O}^\times) \xrightarrow{\delta} \check{H}^2(X, \mathbb{Z}) \rightarrow \cdots$$

**Fact 1:**  $\check{H}^1(X, \mathcal{O}^\times)$  classifies holomorphic line bundles up to holomorphic isomorphism.

**Fact 2:** In the sequence above,  $\delta$  is “almost”  $c_1$ .

$$\begin{array}{ccc} H^1(X, \mathcal{O}^\times) & \xrightarrow{\delta} & H^2(X, \mathbb{Z}) \\ & \searrow c_1 & \downarrow \\ & & H_{dR}^2(X, \mathbb{R}) \end{array}$$

i.e. holomorphic line bundles have “integral Chern classes”.

*Remark 9.4.* Indeed, the Lefschetz  $(1, 1)$ -theorem states that on compact Kaehler manifolds  $X$  we have

$$c_1(H^1(X, \mathcal{O}^\times)) = \tilde{H}^2(X, \mathbb{Z}) \cap H^{1,1}(X).$$

In the proof of the previous commutative diagram, an explicit formula for the  $(1, 1)$ -form representing  $c_1(E, h)$ , namely

$$\frac{1}{2\pi i} \partial \bar{\partial} \log(h_\alpha),$$

where  $h_\alpha$  is the metric on  $U_\alpha$ .



### 9.1.3. Kaehler manifolds.

**Definition 9.5** Given a complex manifold  $(M, I, g)$  where the complex structure  $I$  is compatible with the metric  $g$ , we associate a  $(1, 1)$ -form

$$w(\cdot, \cdot) = g(I, \cdot)$$

called the *Kaehler form*.

We say that  $M$  is Kaehler, if it admits some metric such that the associated Kaehler form is closed ( $dw = 0$ ).

In that case,  $w \in \mathcal{A}^{1,1}(M)$  defines a cohomology class  $[w] \in H^{1,1}(M)$ , called the *Kaehler class*.

**Example 9.6 (The Fubini-Study metric on  $\mathbb{CP}^n$ )** In homogeneous coordinates, the corresponding Kaehler form is given as

$$w_a = i\partial\bar{\partial} \log(|\zeta_a|^2 + 1)$$

on  $U_a = \{(z^0 : \dots : z^n) \mid z^a \neq 0\}$ . The map  $\zeta_a : U_a \rightarrow \mathbb{C}^n$  sends  $(z^0 : \dots : z^n) \mapsto (\frac{z^0}{z^a}, \dots, \hat{1}, \dots, \frac{z^n}{z^a})$ .

From this we can also deduce the same for any projective complex manifold using the following:

**Lemma 9.7** Let  $(X, I, g)$  be a Kaehler manifold with Kaehler form  $w$  and let  $M$  be a complex submanifold. Then  $g$  induces a Kaehler manifold on  $M$ , thus  $M$  is a Kaehler manifold.

### 9.2. Hodge Manifolds.

**Definition 9.8** Let  $(X, h)$  be a Kaehler manifold with Kaehler metric  $h$  and let  $w$  be the associated Kaehler form. If  $w$  is integral, it is called a *Hodge form* on  $X$  and  $h$  is called a *Hodge metric*. A Kaehler manifold is called *Hodge manifold* if it admits a Hodge metric.

**Example 9.9 (Complex projective space)** Let  $\mathbb{CP}^n$  be endowed with the Fubini-Study metric  $g_{FS}$  and the associated Kaehler form  $w_{FS}$ . Then  $w_{FS}$  is a Hodge form. Indeed, let  $E \rightarrow \mathbb{CP}^n$  be the tautological line bundle. By Fact 2,  $c_1(E) \in \tilde{H}^2(X, \mathbb{Z})$ . And we have

$$\left[ \frac{1}{2\pi} w_{FS} \right] = -c_1(E).$$

Thus  $\mathbb{CP}^n$  is a Hodge manifold.

**Example 9.10 (Projective complex manifolds)** Let  $X$  be a projective complex manifold, i.e.  $X \subseteq \mathbb{CP}^n$  as closed complex submanifold. The restriction of the Hodge form on  $\mathbb{CP}^n$  is a Hodge form on  $X$ .

**Fundamental fact:** all projective complex manifolds are Hodge.

**Example 9.11 (Compact connected Riemann surfaces)** Let  $X$  be a compact connected Riemann surface. The claim is that  $X$  is a Hodge manifold. We have  $H^2(X, \mathbb{C}) \cong H_0(X, \mathbb{C}) \cong \mathbb{C}$ . Moreover,  $H^2(X, \mathbb{C}) \cong H^{1,1}(\mathbb{C})$ . Let  $\tilde{w}$  be the Kaehler form associated to some metric on  $X$ . Then  $[\tilde{w}] \in H^{1,1}(\mathbb{C})$  generates  $H_{dR}^2(X, \mathbb{C})$ . Let  $c := \int_X \tilde{w}$ . Then  $w := \frac{1}{c} \tilde{w}$  is an integral positive form on  $X$  of type  $(1, 1)$ .

**9.3. Outlook.** We will soon prove the converse of our fundamental fact, namely:

**Theorem 9.12 (Kodaira's Embedding)** Every Hodge manifold admits a closed immersion into  $\mathbb{CP}^n$  for  $n$  sufficiently large.

Combining this with Chow's theorem seen last time, we obtain as a corollary that Hodge manifolds are always algebraic.

## 10. [LT] LECTURE 4 - 31.10.19

## 10.1. Key theorems.

**Theorem 10.1 (Liouville)**  $f: \mathbb{C}^n \rightarrow \mathbb{C}$  bounded. Then  $f$  constant.

*Proof.* Let  $z, w \in \mathbb{C}^n$ . We claim that  $f(z) = f(w)$ . Consider  $h: c \mapsto z + c(w - z)$  from  $\mathbb{C} \rightarrow \mathbb{C}^n$ . This is analytic. Hence  $g := f \circ h$  is analytic.  $\text{im}(g) \subseteq \text{im}(f)$ , so  $g$  is also bounded. By the 1-dimensional Liouville theorem,  $g$  is constant. But then  $f(z) = g(0) = g(1) = f(w)$ .  $\square$

**Theorem 10.2 (Maximum modulus principle)**  $U \subseteq \mathbb{C}^n$  open connected,  $f: U \rightarrow \mathbb{C}$  analytic,  $a \in U$ . If  $|f(a)| \geq |f(z)|$  for each  $z$  in an open nbd  $a \in V \subseteq U$ , then  $f$  is constant on  $U$ .

*Proof.* Identity theorem implies that if  $f$  is constant on  $V$ , then  $f$  is constant on  $U$ . So WLOG  $V = D$  a polydisk centered at  $a$ . Consider  $z_1 \mapsto f(z_1, a_2, \dots, a_n)$  from  $D(a, r)$  to  $\mathbb{C}$ . This is analytic and its modulus attains a maximum at  $z_1 = a_1$ . By the single variable maximum modulus principle, this map is constant on  $D(a_1, r_1)$ .

Suppose  $(z_1, \dots, z_k) \mapsto f(z_1, \dots, z_k, a_{k+1}, \dots, a_n)$  is constant on  $D(a_1, r_1) \times \dots \times D(a_k, r_k)$  for some  $1 \leq k < n$ . Then  $\forall (w_1, \dots, w_k, z_{k+1}, a_{k+2}, \dots, a_n)$ ,  $z_{k+1} \mapsto f(w_1, \dots, w_k, z_{k+1}, a_{k+2}, \dots, a_n)$  is analytic, its modulus has a maximum at  $z_{k+1} = a_{k+1}$ . This implies by the single variable case that it is constant. By induction, we win.  $\square$

**Theorem 10.3 (Open mapping theorem)**  $U \subseteq \mathbb{C}^n$  connected open,  $f: U \rightarrow \mathbb{C}$  nonconstant, analytic. If  $V \subseteq U$  is open, then  $f(V) \subseteq \mathbb{C}$  is open as well.

*Proof.* Let  $z \in V$  and  $w = f(z)$ . It suffices to construct an open nbd of  $w$  contained in  $f(V)$ . There exists an open polydisk  $D$  centered at  $z$  contained in  $V$ , and  $V$  is a union of such polydisks. Hence WLOG  $V = D$ .

$f$  nonconstant implies that  $\exists z' \in D$  with  $f(z) \neq f(z')$ . The set  $W = \{c \in \mathbb{C} \mid z + c(z' - z) \in D\} \subseteq \mathbb{C}$  is open. The composite  $f \circ h: W \rightarrow \mathbb{C}$  is analytic and nonconstant. By the single variable open mapping theorem,  $f \circ h$  is open. Hence  $\text{im}(f \circ h) \subseteq \mathbb{C}$  is open, and  $\text{im}(f \circ h) \subseteq f(V)$ .  $\square$

## 10.2. Cauchy-Riemann equations.

**Proposition 10.4**  $U \subseteq \mathbb{C}^n$  open,  $f: U \rightarrow \mathbb{C}$  complex differentiable. If  $f = u + iv$  with  $u, v: U \rightarrow \mathbb{R}$  and we use coordinates  $z_k = x_k + iy_k = \text{Re}(z_k) + i \text{Im}(z_k)$ , then

$$(CR_k) \quad \frac{\partial u}{\partial x_k} = \frac{\partial v}{\partial y_k} \text{ and } \frac{\partial u}{\partial y_k} = -\frac{\partial v}{\partial x_k}$$

*Proof.* Let  $e_k := (0, \dots, 0, 1, 0, \dots, 0)$  with 1 on the  $k^{\text{th}}$  coordinate.

$$\begin{aligned} \frac{f(a + he_k) - f(a)}{h} &= \frac{(u(a + he_k) + iv(a + he_k)) - (u(a) + iv(a))}{h} \\ &\stackrel{(*)}{=} \frac{u(a + he_k) - u(a)}{h} + i \frac{v(a + he_k) - v(a)}{h} \\ &\stackrel{h_{\text{real}}}{\Rightarrow} (*) \rightarrow \frac{\partial u}{\partial x_k}(a) + i \frac{\partial v}{\partial x_k}(a) (**) \\ &\stackrel{h_{\text{imaginary}}}{\Rightarrow} (*) \rightarrow -i \frac{\partial u}{\partial y_k}(a) + \frac{\partial v}{\partial y_k}(a) (***) \end{aligned}$$

$f$  complex differentiable implies  $(**) = (***)$ . Now compare real and imaginary part and we are done.  $\square$

**Corollary 10.5**  $f: U \rightarrow \mathbb{R}$  complex differentiable implies  $f$  constant.

**Definition 10.6** Let  $f: U \rightarrow \mathbb{C}$  be a function. Let

$$\frac{\partial f}{\partial x_k} = \frac{\partial u}{\partial x_k} + i \frac{\partial v}{\partial x_k} \text{ and } \frac{\partial f}{\partial y_k} = \frac{\partial u}{\partial y_k} + i \frac{\partial v}{\partial y_k}.$$

Then we define the Wirtinger operators as

$$\frac{\partial f}{\partial z_k} := \frac{1}{2} \left( \frac{\partial f}{\partial x_k} - i \frac{\partial f}{\partial y_k} \right) \text{ and } \frac{\partial f}{\partial \bar{z}_k} := \frac{1}{2} \left( \frac{\partial f}{\partial x_k} + i \frac{\partial f}{\partial y_k} \right).$$

**Proposition 10.7** For  $U: \mathbb{C}^n$  open and  $f: U \rightarrow \mathbb{C}$  with partial derivatives, the following are equivalent:

- 1)  $f$  satisfies  $(CR_k)$  for each  $k$ .
- 2)  $\frac{\partial f}{\partial x_k} = -i \frac{\partial f}{\partial y_k}$  for all  $k$ .
- 3)  $\frac{\partial f}{\partial \bar{z}_k} = 0$  for all  $k$ .
- 4)  $\frac{\partial f}{\partial z_k} = \frac{\partial f}{\partial x_k} = \frac{\partial u}{\partial x_k} + i \frac{\partial v}{\partial x_k} = \frac{\partial f}{\partial(iy_k)}$  for all  $k$ .
- 5) Via  $\mathbb{C}^2 \cong \mathbb{R}^{2n}$  sending  $(z_1, \dots, z_n) \mapsto (x_1, y_1, \dots, x_n, y_n)$ , for all  $a \in U$ , the Jacobian matrix  $J_f^{\mathbb{R}}(a)$  is composed of  $2 \times 2$  blocks of the form

$$\begin{pmatrix} \frac{\partial u}{\partial x_k} & -\frac{\partial v}{\partial x_k} \\ \frac{\partial v}{\partial x_k} & \frac{\partial u}{\partial x_k} \end{pmatrix} (a).$$

- 6) For all  $a$ ,  $J_f^{\mathbb{R}}(a): \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$  is  $\mathbb{C}$ -linear.

*Proof.* For equivalences between 1), 2), 3) and 4), expand and compare real and imaginary parts. For 1)  $\Leftrightarrow$  5) use the definition of the Jacobian matrix. For 5)  $\Leftrightarrow$  6) observe that  $\mathbb{C} = \{a+ib \mid a, b \in \mathbb{R}\} \xrightarrow{\cong} \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \mid a, b \in \mathbb{R} \right\}$  is compatible with sums and product.  $\square$

**Remark 10.8.** If  $f$  is  $C^1$  and satisfies any of these conditions, then  $f$  is complex differentiable and

$$\frac{\partial f}{\partial z_k}(a) = \lim_{h \rightarrow 0} \frac{f(a + he_k) - f(a)}{h}.$$

**Proposition 10.9** Let  $U: \mathbb{C}^n$  open and  $f: U \rightarrow \mathbb{C}^n$  complex differentiable. Then for all  $a \in U$

$$0 \leq \left| \det(J_f^{\mathbb{C}}(a)) \right|^2 = \det(J_f^{\mathbb{R}}(a)),$$

where  $J_f^{\mathbb{C}}(a) = \left( \frac{\partial f_k}{\partial z_l} \right)_{k,l}$ . In particular,  $f$  preserves orientations.

*Idea.* Rewrite  $J_f^{\mathbb{R}}(a)$  in the basis  $(z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n)$ . The resulting matrix is the matrix of derivatives of  $(f_1, \dots, f_n, \bar{f}_1, \dots, \bar{f}_n)$  with respect to  $(z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n)$ . Then

$$J_f^{\mathbb{R}}(a) = \begin{pmatrix} J_f^{\mathbb{C}}(a) & 0 \\ 0 & \overline{J_f^{\mathbb{C}}(a)} \end{pmatrix}.$$

$\square$

### 10.3. Complex differentiable implies analytic.

**Theorem 10.10 (Cauchy integral formula)**  $U: \mathbb{C}^n$  open,  $f: U \rightarrow \mathbb{C}$  complex differentiable,  $\bar{D} = \bar{D}_1 \times \bar{D}_n$  closed polydisk in  $U$ ,  $z \in D$  (which is the interior of  $\bar{D}$ ). Then

$$f(z_1, \dots, z_n) = \left( \frac{1}{2\pi i} \right)^n \int_{\partial D_n} \cdots \int_{\partial D_1} \frac{f(w_1, \dots, w_n)}{(w_1 - z_1) \cdots (w_n - z_n)} dw_1 \cdots dw_n.$$

*Proof.* Use the single variable version, the fact that complex differentiable means complex differentiable separately in each variable and the induct on  $n$ .  $\square$

**Corollary 10.11** If  $f$  is complex differentiable, then  $f$  is analytic.

**Corollary 10.12** If  $f$  is complex differentiable, then  $f \in C^\infty$ .

*Proof.* Mimic single variable proof.  $\square$

### 10.4. Topology: gluing constructions. Definition and construction of pushouts.

**10.5. Complex projective space.** Definition as a quotient. Compact and connected because it is the image of the sphere under the quotient map. Hausdorff for the exercise session.

## 11. [CM] TALK 3 (NICOLA) - COHOMOLOGY - 04.11.19

**11.1. Singular cohomology.** Recollection of singular cohomology of a topological space  $X$  with coefficients in some abelian group  $A$ , denoted  $H_{sing}^p(X, A)$ .

**11.2. Čech cohomology.** Recollection of Čech cohomology of a sheaf of abelian groups  $\mathcal{F}$  on a topological space  $X$ , denoted  $\check{H}^p(X, \mathcal{F})$ . The Čech complex associated to an open cover  $\mathcal{U}$  will be denoted  $C^\bullet(\mathcal{U})$  or  $C^\bullet(\mathcal{U}, \mathcal{F})$  if we want to make explicit the sheaf  $\mathcal{F}$ .

**11.3. Sheaf cohomology.** Recollection of sheaf cohomology of a sheaf of abelian groups  $\mathcal{F}$  on a topological space  $X$ , denoted  $H_{Sh}^p(X, \mathcal{F})$ .

**11.4. Comparison.** Let  $U \subseteq X$  be an open subset and  $\mathcal{U}$  an open cover on  $X$ . We can restrict this open cover to an open cover  $\mathcal{U}|_U$  of  $U$ . Then  $C^p(\mathcal{U})(U) := C^p(\mathcal{U}|_U)$  defines a new sheaf of abelian groups on  $X$ .

**Fact:**  $0 \rightarrow \mathcal{F} \rightarrow C^0(\mathcal{U}) \rightarrow C^1(\mathcal{U}) \rightarrow \cdots$  is a resolution of  $\mathcal{F}$ . Hence we obtain a map  $\varinjlim_{\mathcal{U}} H^q(\mathcal{U}, \mathcal{F}) \rightarrow H_{Sh}^q(X, \mathcal{F})$ .

**Facts:**

- 1) Čech cohomology and sheaf cohomology always agree on degrees zero and one.
- 2)  $X$  locally contractible implies that singular cohomology with coefficients on  $A$  agrees with sheaf cohomology of the constant sheaf  $A$ .
- 3)  $X$  paracompact and Hausdorff implies that Čech cohomology and sheaf cohomology agree.

*Remark 11.1.* Using hypercovers instead of Čech, this is always true (on any site with fibre products). See [?, Tag 01H0].

**Example 11.2** If  $X$  is a CW complex, then all three cohomologies agree.

**Example 11.3** If  $X$  is profinite, then we have

$$\check{H}^i(X, \mathbb{Z}) = H_{Sh}^i(X, \mathbb{Z}) = 0 = H_{sing}^i(X, \mathbb{Z})$$

for all  $i > 0$ , but  $H_{Sh}^0(X, \mathbb{Z}) = C(X, \mathbb{Z}) \neq H_{sing}^0(X, \mathbb{Z}) = \text{Hom}_{\text{Set}}(X, \mathbb{Z})$  unless  $X$  is finite.

**11.5. Hypercovers.** Recollection of simplicial sets and simplicial objects on a category. Convention: we also consider the empty set (“−1-simplices”).

**Definition 11.4** A *hypercov*er of  $X \in \text{Ob}(\mathbf{C})$  is a simplicial object in  $\mathbf{C}$  such that  $X_{-1} = X$  with some nice properties.

**Definition 11.5**  $H_{cond}^\bullet(S, \mathcal{F}) := H_{HC}^\bullet(S_{cond}, \mathcal{F}) = H_{HC}^\bullet(S_{cond}, \mathcal{F}) = \varinjlim_{S_\bullet \rightarrow S} H^\bullet(S_\bullet, \mathcal{F})$  for  $S \in \mathbf{CHaus}$  and  $\mathcal{F} \in \mathbf{Ab}(S_{cond})$ , where  $S_{cond}$  denotes the proetale site of compact Hausdorff spaces over  $S$ .

The RHS inside the direct limit is defined as follows. Start from  $S_\bullet \rightarrow S$ . Then get a diagram  $\mathcal{F} \rightarrow \pi_{0,*}\pi_0^{-1}\mathcal{F} \rightrightarrows \dots$ . With the alternating sum of the face maps get  $0 \rightarrow \pi_{0,*}\pi_0^{-1}\mathcal{F} \rightarrow \dots$ . From this complex we take global sections and get a complex of groups. Then we take cohomology of this complex. Finally we take the direct limit over all simplicial resolutions.

By [?, Theorems 2.3, 3.10-11], for all  $S \in \mathbf{CHaus}$  there exists a hypercover  $S_\bullet \rightarrow S$  by extremally disconnected  $S_n$  which is natural in  $S$  with proper surjective face maps such that for all  $\mathcal{F} \in \mathbf{Ab}(S)$  we have

$$H^\bullet(S_\bullet, \mathcal{F}) = H_{cond}^\bullet(S, \mathcal{F})$$

The reason is that  $0 \rightarrow \pi_{0,*}\pi_0^{-1}\mathcal{F} \rightarrow \pi_{1,*}\pi_1^{-1}\mathcal{F} \rightarrow \dots$  is a soft resolution of  $\mathcal{F}$ .

In particular, for  $\mathcal{F} = \mathbb{Z}$ , we have

$$\Gamma(S, \pi_{n,*}\pi_n^{-1}\mathbb{Z}) = \Gamma(S_n, \mathbb{Z}).$$

Hence we have

$$H_{cond}^\bullet(S, \mathbb{Z}) = H^\bullet(0 \rightarrow \Gamma(S_0, \mathbb{Z}) \rightarrow \dots).$$

**Theorem 11.6**  $H_{cond}^\bullet(S, \mathbb{Z}) = H_{Sh}^\bullet(S, \mathbb{Z})$ .

*Proof.* If  $S$  is profinite, then we can write  $S$  as an inverse limit  $S = \varprojlim_j S_j$  of finite discrete spaces  $S_j$ . Then

$$H_{Sh}^i(S, \mathbb{Z}) = \begin{cases} 0 & \text{if } i > 0 \\ C(S, \mathbb{Z}) & \text{if } i = 0 \end{cases}.$$

$$C(S, \mathbb{Z}) = \Gamma(S, \mathbb{Z}) = \varinjlim_j \Gamma(S_j, \mathbb{Z}) \stackrel{\text{by def (?)}}{=} H_{cond}^0(S, \mathbb{Z}).$$

Now WTS  $H_{cond}^i(S, \mathbb{Z}) = 0$  for all  $i > 0$ . Claim: it is sufficient to show that for all  $S' \twoheadrightarrow S$  of profinite sets, then

$$0 \rightarrow \Gamma(S, \mathbb{Z}) \rightarrow \Gamma(S', \mathbb{Z}) \rightarrow \Gamma(S' \times_S S') \rightarrow \dots$$

is exact.

So let  $S' \twoheadrightarrow S = \varinjlim_j S_j$  of finite sets. This implies splitting, so  $0 \rightarrow \Gamma(S_j, \mathbb{Z}) \rightarrow \Gamma(S'_j, \mathbb{Z}) \rightarrow \dots$  is split exact and  $\varinjlim_j \Gamma = \Gamma \varinjlim_j$ , so  $\varinjlim_j (\dots)$  is still split exact.

For a general compact Hausdorff space use some compatibilities between derived functors.  $\square$

**Theorem 11.7**  $H_{cond}^i(S, \mathbb{R}) = \begin{cases} 0 & \text{if } i > 0 \\ C(S, \mathbb{R}) & \text{if } i = 0 \end{cases}.$

*Proof.* I.e. want to show that  $0 \rightarrow C(S, \mathbb{R}) \rightarrow C(S_0, \mathbb{R}) \rightarrow \dots$  is exact.  $C(S, \mathbb{R}) \stackrel{?}{=} C(S, \mathbb{R})$ . Moreover we will show that if  $df = 0$ , then  $f = dg$  with  $g$  such that  $\|g\| \leq (i + 2 + \varepsilon)\|f\|$ .

Let  $S$  and an HC  $S_i$  finite. Then HC splits:  $\pi_n: S_n \rightarrow S$  and  $\exists \psi_n: S \rightarrow S_n$  such that  $\pi_n \circ \psi_n = \text{id}_S$  and  $\psi_n \circ \pi_n = r_n: S_n \rightarrow S_n$ .

Claim:  $r_\bullet \xrightarrow{h} \text{id}_{S_\bullet}$ , i.e. there exists  $h: \Delta^1 \times S_\bullet \rightarrow S_\bullet$  with  $h|_{\{0\} \times S_\bullet} = \text{id}$  and  $h|_{\{1\} \times S_\bullet} = r_\bullet$ . See notes for an argument.

Now we use the Dold-Kan correspondence: there is an equivalence between the category of simplicial abelian groups and the bounded below cochain complexes of abelian groups which preserves homotopy equivalence. We have  $S \xrightarrow{h} S_\bullet$ , so  $0 \rightarrow C(S, \mathbb{R}) \rightarrow C(S_0, \mathbb{R}) \rightarrow \dots$  is chain homotopy equivalent to  $0 \rightarrow C(S, \mathbb{R}) \rightarrow C(S, \mathbb{R}) \rightarrow \dots$ . Since one is exact, the other one is also exact. This proves the statement for finite spaces.

For profinite sets we use limits.

For the general case, assume  $f \in C(S, \mathbb{R})$ ,  $f \neq 0$ ,  $s \in S$ .  $f_s := f|_{S_i \times_S \{s\}} \stackrel{!}{=} dg_s$  with  $g_s \in C(S_{i-1} \times_S \{s\}, \mathbb{R})$ .  $\square$

## 12. [LT] LECTURE 5 - 05.11.19

New chapter: Manifolds and vector bundles!

### 12.1. Manifolds.

**Definition 12.1** Manifold of dimension  $n$ : locally homeomorphic to  $\mathbb{R}^n$ , Hausdorff and second countable.

**Example 12.2**  $\mathbb{R}^n$ .

*Remark 12.3.* Voisin omits Hausdorffness and second countability, even though Hausdorffness is definitely needed.

Our main objects of study are affine/projective varieties over  $\mathbb{C}$ . These satisfy the definition of manifold we have given.

There is a more general notion of manifold with boundary, but we will not need it for the moment.

**Definition 12.4** Real  $n$ -dimensional chart  $(U, \varphi)$  on a topological manifold  $X$  [picture].

Complex  $n$ -dimensional chart  $(U, \varphi)$  on a topological manifold  $X$  (can only exist on even dimensional manifolds).

**Example 12.5**  $X \subseteq \mathbb{R}^n$  open implies that  $(X, \text{id}_X)$  is a chart.

**Example 12.6**  $S^1$  with charts given by (the inverse of) the exponential.

**Example 12.7** Real projective  $n$ -space (recall we defined it via the usual quotient map) with the usual charts  $x_i \neq 0$  is a manifold<sup>8</sup>.

*Exercise 12.8.* Let  $H$  be a hyperplane on real  $n$ -projective space and let  $U$  be its complement. Find a chart from  $U$  to  $\mathbb{R}^n$ .

**Example 12.9** We consider the situation of the implicit function theorem (Forster, Analysis 2). Let  $U_1 \subseteq \mathbb{R}^n$  open and  $U_2 \subseteq \mathbb{R}^m$  open. Let

$$\begin{aligned} F: U_1 \times U_2 &\rightarrow \mathbb{R}^m \\ (x, y) &\mapsto F(x, y) \end{aligned}$$

<sup>8</sup>Second countability follows from the projection being an open map. For instance, the identity on  $\mathbb{R}$  from the euclidean topology to the cofinal topology is not open and its image is not second countable.

be smooth. Let  $(a, b) \in U_1 \times U_2$  be such that  $F(a, b) = 0$  and  $\frac{\partial F}{\partial y}(a, b)$  is invertible. Then there are open nhds  $V_1 \subseteq U_1$  of  $a$  and  $V_2 \subseteq U_2$  of  $b$  such that there is a smooth function  $g: V_1 \rightarrow V_2$  such that  $F(x, g(x)) = 0$  for all  $x \in V_1$ . Moreover, if  $(x, y) \in V_1 \times V_2$  such that  $F(x, y) = 0$ , then  $y = g(x)$ .

Let  $X = \{(x, y) \mid F(x, y) = 0\} \subseteq U_1 \times U_2$  and  $U = X \cap (V_1 \times V_2)$ . Let  $\varphi: U \rightarrow V_1$  be the projection. Then  $(U, \varphi)$  is a chart. The inverse is given by  $x \mapsto (x, g(x))$ . E.g.  $F(x, y) = x^2 + y^2 - 1$ ...

**Definition 12.10** A *smooth atlas*  $(U_i, \varphi_i)_{i \in I}$  on  $X$  is a family of real  $n$ -dimensional charts on  $X$  satisfying the following condition:

- 1)  $(U_i)_{i \in I}$  is an open cover of  $X$ .
- 2) For every  $i, j \in I$ , the transition map

$$\varphi_{ij}: \varphi_j(U_i \cap U_j) \xrightarrow{\varphi_j^{-1}} U_i \cap U_j \xrightarrow{\varphi_i} \varphi_i(U_i \cap U_j)$$

is smooth.

Two smooth atlases  $(U_i, \varphi_i)_{i \in I}$  and  $(U'_i, \varphi'_i)_{i \in I'}$  are equivalent if their union is an atlas.

A *smooth manifold* is a topological manifold together with an equivalence class of smooth atlases.

**Definition 12.11** A *holomorphic atlas* is defined analogously using complex charts and holomorphic transition maps. A *complex manifold* is a topological manifold together with an equivalence class of holomorphic atlases.

A complex manifold of (complex) dimension 1 is called a *Riemann surface*.

**Example 12.12**  $U \subseteq \mathbb{R}^n$  open, the torus  $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$ , Moebius strip, Klein bottle, real projective space [pictures of the gluing].

**Example 12.13** Compute the formula for the transition maps on real projective space.

**Lemma 12.14** Let  $U \subseteq \mathbb{R}^N$  be an open subset and  $F: U \rightarrow \mathbb{R}^m$  smooth. Let  $X = \{x \in \mathbb{R}^N \mid F(x) = 0\}$ . Assume that the Jacobian  $(\frac{\partial F_i}{\partial x_j})_{i,j}$  has maximal rank  $m$  at all  $x \in X$  (in particular, assume that  $N \geq m$ ). Then  $X$  carries a canonical structure of smooth manifold.

*Proof.* Let  $x \in X$ . After reordering of the coordinates of  $\mathbb{R}^n$ , we may choose a nbhd of  $x$  of the form  $U_1 \times U_2 \subseteq \mathbb{R}^{N-m} \times \mathbb{R}^m$  satisfying the assumption of the implicit function theorem. This gives a chart near  $x$ . The transition maps are smooth.  $\square$

*Remark 12.15.* The same lemma works in the holomorphic setting.

**Example 12.16**  $X = \{(x, y) \in \mathbb{C}^2 \mid y^2 = x^3 + x^2\}$ . We apply the lemma with  $F(x, y) = y^2 - x^3 - x^2$ ,  $N = 2$  and  $m = 1$ . The Jacobian is  $(-3x^2 - 2x, 2y)$ , which has rank 1 at all points of  $X$  except for  $(0, 0)$ , which is called a *singular point*.

**Example 12.17** Every complex manifold of dimension  $n$  also defines a smooth manifold of dimension  $2n$  by identifying  $\mathbb{C}^n \cong \mathbb{R}^{2n}$ .

**Definition 12.18** Let  $X, Y$  be smooth (resp. complex) manifolds. Let  $f: X \rightarrow Y$  be a continuous map. We say that  $f$  is *smooth* (resp. *holomorphic*) if for every  $x \in X$  there exists a chart  $(U, \varphi)$  with  $x \in U$  and a chart  $(V, \psi)$  with  $f(x) = y \in V$  such that  $f(U) \subseteq V$  and  $\varphi(U) \xrightarrow{\varphi^{-1}} U \xrightarrow{f|_U} V \xrightarrow{\psi} \psi(V)$  is smooth (resp. holomorphic).

*Exercise 12.19.*  $U \subseteq \mathbb{C}$  open,  $f: U \rightarrow \mathbb{C}$  holomorphic viewed as a holomorphic map of complex manifolds.  $f$  non constant. Let  $z_0 \in U$ . Show that there are charts  $(U_1, \varphi_1)$  around  $z_0$  with  $\varphi_1(z_0) = 0$  and  $(U_2, \varphi_2)$  around  $f(z_0)$  with  $\varphi_2(f(z_0)) = 0$  such that  $f(U_1) \subseteq U_2$  and the induced map from  $\varphi_1(U_1)$  to  $\varphi_2(U_2)$  is given by  $z \mapsto z^d$ .

13. [WS] KODAIRA 3 (LEONARDO) - KODAIRA VANISHING THEOREM - 06.11.19

Let  $X$  be a compact complex manifold and  $E$  a holomorphic vector bundle<sup>9</sup> on  $X$ .

Let  $\mathcal{A}^{p,q}(E)$  denote the sheaf of  $E$ -valued  $(p, q)$ -forms, i.e.  $\mathcal{A}^{p,q}(E) = \mathcal{A}^{p,q} \otimes E$ . There is no analogue of  $d$  on  $\mathcal{A}^{p,q}$ , but

$$\bar{\partial}: \mathcal{A}^{p,q}(E) \rightarrow \mathcal{A}^{p,q+1}(E)$$

is well defined. Why? We can take a holomorphic frame  $\{e_\alpha\}$  for  $E$ , and if we are given  $\eta = \sum \eta_\alpha \otimes e_\alpha$  with  $\eta_\alpha \in \mathcal{A}^{p,q}$ , then  $\bar{\partial}\eta = \sum \bar{\partial}\eta_\alpha \otimes e_\alpha$ .

$$0 \rightarrow \Omega^p(E) \rightarrow \mathcal{A}^{p,0}(E) \xrightarrow{\bar{\partial}_E} \mathcal{A}^{p,1}(E) \xrightarrow{\bar{\partial}_E} \dots$$

where  $\Omega^p(E)$  are the  $p$ -holomorphic forms on  $E$ . This way we obtain *generalized Dolbeault cohomology*:

$$(H^{p,q}(X, E) :=) H^q(X, \Omega^p \otimes E) := \frac{\ker(\mathcal{A}^{p,q}(E) \xrightarrow{\bar{\partial}} \mathcal{A}^{p,q+1}(E))}{\text{im}(\mathcal{A}^{p,q-1}(E) \rightarrow \mathcal{A}^{p,q}(E))}$$

13.1. **Vector bundles and metrics.** Recall that an *Hermitian metric* of  $X$  is

$$\sum_{i,j} h_{i,j} dz_i \otimes d\bar{z}_j,$$

] where  $\{z_i\}$  is a local holomorphic coordinate system and  $h_{i,j}$  is a positive definite hermitian form. This induces a metric on every  $\mathcal{A}^{p,q}(X)$  and a volume form

$$\text{vol} = \sqrt{\det(h_{i,j})} dz_1 \wedge d\bar{z}_1 \wedge \dots$$

We define the *Hodge \*-operator*  $\bar{*}: \mathcal{A}^{p,q}(X) \rightarrow \mathcal{A}^{n-p,n-q}(X)$  by

$$\alpha \wedge \bar{*}\beta = (\alpha, \beta) \text{vol} \in \mathcal{A}^{n,n}(X).$$

An *Hermitian metric*  $h_E$  on  $E$  is a hermitian metric on every fiber  $E_x$  of  $E$  varying smoothly. Such an  $h_E$  induces  $E \xrightarrow{\sim} \bar{E}^*$  sending  $\sigma \mapsto h_E(\sigma, -)$ . Now we have the skew-linear map

$$\begin{aligned} \bar{*}_E: \mathcal{A}^{p,q}(E) &\rightarrow \mathcal{A}^{n-p,n-q}(E^*) \\ \alpha \otimes e &\mapsto \bar{*}(\alpha) \otimes h_E(-, e) \end{aligned}$$

And the maps

$$\begin{aligned} \Lambda: \mathcal{A}^{p,q}(E) \otimes \mathcal{A}^{p',q'}(E^*) &\rightarrow \mathcal{A}^{p+p',q+q'}(X) \\ (\alpha \otimes e) \otimes (\beta \otimes f) &\mapsto f(e)(\alpha \wedge \beta) \end{aligned}$$

Let  $\{e_\alpha\}$  be a unitary frame of  $E$ , i.e.  $h_E(e_\alpha, e_\beta) = \delta_{\alpha\beta}$ .<sup>10</sup> If we have  $\eta = \sum \eta_\alpha \otimes e_\alpha$ , then  $\bar{*}_E(\eta) = \sum \bar{*}\eta_\alpha \otimes e_\alpha^*$ , where  $\{e_\alpha^*\}$  is the dual unitary frame on  $E^*$ . We have  $\bar{*}_{E^*}\bar{*}_E = (-1)^{r(2n-r)}$  on  $\mathcal{A}^r(E)$ .

$\bar{\partial}_E: \mathcal{A}^{p,q}(E) \rightarrow \mathcal{A}^{p,q+1}(E)$  yields  $\bar{\partial}_E^*$  adjoint map, with description  $\bar{\partial}_E^* = -\bar{*}_{E^*}\bar{\partial}_E\bar{*}_E$ .

<sup>9</sup>Which we sometimes identify with its sheaf of smooth sections. If we want to consider holomorphic sections, then we will use the notation  $\mathcal{O}(E)$ .

<sup>10</sup>We can choose either a unitary frame or a holomorphic frame, but not both at the same time. Hence we work rather with smooth sections and consider such a unitary frame.



We now consider the laplacian

$$\Delta_{\bar{\partial}_E} = \bar{\partial}_E^* \bar{\partial}_E + \bar{\partial}_E \bar{\partial}_E^*: \mathcal{A}^{p,q}(E) \rightarrow \mathcal{A}^{p,q}(E)$$

and denote  $\mathcal{H}^{p,q}(E) = \{\sigma \in \mathcal{A}^{p,q}(E) \mid \Delta_{\bar{\partial}_E}(\sigma) = 0\}$ .

**Theorem 13.1**  $\mathcal{H}^{p,q}(E) \cong H^{p,q}(X, E) = H^q(X, \Omega^p \otimes E)$ .

$\bar{*}_E \Delta_{\bar{\partial}_E} = \Delta_{\bar{\partial}_E} \bar{*}_E \Rightarrow \bar{*}_E: \mathcal{H}^{p,q}(E) \xrightarrow{\sim} \mathcal{H}^{n-p, n-q}(E^*)$ . The first thing is isomorphic to  $H^q(X, \Omega^p \otimes E)$  and the second thing to  $H^{n-q}(X, \Omega^{n-p} \otimes E^*)$ , and this is known as Kodaira-Serre duality.

For  $p = 0$ ,  $H^q(X, \mathcal{O}(E)) \cong H^{n-q}(X, K_X \otimes E^*)$  where  $K_X$  denotes the canonical bundle.

**13.2. Connections.** Need replacement for  $d: \mathcal{A}^r(X) \rightarrow \mathcal{A}^{r+1}(X)$ .

**Definition 13.2** Let  $E \rightarrow X$  be a vector bundle. A *connection* is  $\nabla: \mathcal{A}^0(E) \rightarrow \mathcal{A}^1(E)$  such that  $\nabla(f\xi) = df \cdot \xi + f\nabla\xi$  for all  $f \in C^\infty(X)$  and all  $\xi \in \mathcal{A}^0(E)$ .

$\mathcal{A}^1(E) = \mathcal{A}^{1,0}(E) \oplus \mathcal{A}^{0,1}(E)$ .  $\nabla = \nabla' + \nabla''$ .  $\nabla': \mathcal{A}^0(E) \rightarrow \mathcal{A}^{1,0}(E)$  and  $\nabla'': \mathcal{A}^0(E) \rightarrow \mathcal{A}^{0,1}(E)$ .

**Definition 13.3** We say that  $\nabla$  is compatible with the complex structure if  $\nabla'' = \bar{\partial}_E$ , or equivalently if  $\nabla''\sigma = 0$  for every holomorphic section  $\sigma$ .

**Definition 13.4**  $\nabla$  is compatible with the metric on  $E$  if for all  $\xi, \eta \in \mathcal{A}^0(E)$  we have

$$d\langle \xi, \eta \rangle = \langle \nabla \xi, \eta \rangle + \langle \xi, \nabla \eta \rangle.$$

**Theorem 13.5** There exists a unique connection  $\nabla$  on an holomorphic vector bundle  $E$  with hermitian metric  $h_E$  such that it is compatible with the complex structure and with the metric. It is called the Chern connection.

We can extend  $\nabla: \mathcal{A}^r(E) \rightarrow \mathcal{A}^{r+1}(E)$  by asking that

$$\nabla(\eta \otimes e) = d\eta \otimes e + (-1)^r \eta \wedge \nabla(e)$$

for  $\eta \in \mathcal{A}^r(X)$  and  $e \in A^0(E)$ .

**Definition 13.6**  $\nabla^2: \mathcal{A}^0(E) \rightarrow \mathcal{A}^2(E)$  is called the *curvature*.

$\nabla^2$  is linear over  $C^\infty(X)$ . For all  $f \in C^\infty(X)$  and  $\sigma \in \mathcal{A}^0(E)$  we have

$$\nabla^2(f\sigma) = \nabla(df \otimes \sigma + f\nabla\sigma) = -df \wedge \nabla\sigma + df \wedge \nabla\sigma + f\nabla^2\sigma.$$

Let  $E$  a holomorphic line bundle over  $X$  with a Hermitian metric  $h_E$  and  $\nabla$  the corresponding Chern connection. Let  $\{\sigma\}$  be a holomorphic section. Then  $\nabla^2\sigma = \Theta \wedge \sigma$ , where  $\Theta$  is a global  $(1, 1)$ -form.

Recall that  $c_1(E) = [\frac{i}{2\pi}\Theta] \in H^{1,1}(X)$ , where

$$\Theta = \bar{\partial}\partial \log h_E(\sigma, \sigma)$$

**Question 13.7.** Given a holomorphic line bundle, when can we choose metrics  $h$  on  $X$  (Kaehler),  $h_E$  on  $E$ , such that  $w = \frac{i}{2\pi}\Theta$ , where  $w$  a Kaehler form?

**Definition 13.8** When this happens, we call the line bundle *positive*.

**Proposition 13.9** If  $\exists$  metric  $h_E$  on  $E$  such that  $w \in [\frac{1}{2\pi}\Theta]$ , then there exists a metric  $h'_E$  such that  $w = \frac{1}{2\pi}\Theta'$ .

**Example 13.10** Let  $X = \mathbb{P}^n$  and consider a hyperplane line bundle  $E (= \mathcal{O}(1))$  on  $\mathbb{P}^n$ . Then  $E$  is positive. Indeed,  $E^*$  is the tautological line bundle on  $\mathbb{P}^n$ . Define  $h_{E^*}$  by setting

$$|(z_0, \dots, z_n)|^2 = \sum z_i \bar{z}_i \Rightarrow \Theta = [\dots]$$

So  $E^*$  is negative and therefore  $E$  is positive.

**Theorem 13.11 (Kodaira Vanishing)** Let  $E > 0$  (i.e. let  $E$  be a positive line bundle). Then  $H^q(X, \Omega^p \otimes E) = 0$  for every  $p + q > n$ , which by Kodaira-Serre duality is equivalent to saying that for every  $E < 0$  we have  $H^q(X, \Omega^p \otimes E) = 0$  for all  $p + q < 0$ .

*Proof.* Let  $L = w \wedge (-): \mathcal{A}^{p,q}(E) \rightarrow \mathcal{A}^{p+1,q+1}(E)$  and consider its adjoint operator  $L^*$ . In a unitary frame  $\{e_\alpha\}$  of  $E$ , for  $\eta = \sum \eta_\alpha \otimes e_\alpha$ , then  $L(\eta) = \sum w \wedge \eta_\alpha \otimes e_\alpha$  and  $L^*(\eta) = \sum L^*(\eta_\alpha) \otimes e_\alpha$ , where  $L^*$  adjoint on  $\mathcal{A}^{p,q}(X)$ . So  $[L, L^*]$  is multiplicative on  $\mathcal{A}^{p,q}(E)$  by the scalar  $(p + q - n)$ . [Recall:  $[L^*, \bar{\partial}] = -i\partial^*$  on  $\mathcal{A}^{p,q}(X)$ ] Here  $[L^*, \bar{\partial}] = -i(\nabla')^*$  on  $\mathcal{A}^{p,q}(E)$ .

To prove the Kodaira vanishing theorem we only need two inequalities which are called the *Nakano inequalities*: for every harmonic form  $\xi \in \mathcal{H}^{p,q}(E)$  we have

- 1)  $\frac{i}{2\pi} \langle \Theta \wedge L_E^*(\xi), \xi \rangle \leq 0$ .
- 2)  $\frac{i}{2\pi} \langle L_E^*(\Theta \wedge \xi), \xi \rangle \geq 0$ .

Now if  $E > 0$ , then from the difference 2) - 1) we deduce

$$\frac{i}{2\pi} \langle L_E^*(\Theta \wedge \xi - \Theta \wedge L_E^* \xi), \xi \rangle \geq 0.$$

Since  $E$  is positive, we can choose a metric on  $X$  such that  $w = \frac{i}{2\pi} \Theta$ . Hence the previous expression becomes  $\langle [L_E^*, L_E](\xi), \xi \rangle \geq 0$ . By assumption  $\xi \in \mathcal{H}^{p,q}(E)$ , hence this expression is in turn equivalent to  $-(p + q - n) \langle \xi, \xi \rangle \geq 0$ . So if  $p + q > n$ , then  $\xi = 0$ , thus  $H^q(X, \Omega^p \otimes E) = \mathcal{H}^{p,q}(E) = 0$ .  $\square$

Let us now prove the Nakano inequalities, which were used in the previous proof.  $\Theta \wedge (-) = \nabla^2 = \nabla^1 \bar{\partial}_E + \bar{\partial}_E \nabla'$  ( $\nabla = \nabla' + \bar{\partial}_E$ ).  $\Delta_{\bar{\partial}_E}(\xi) = 0 \Leftrightarrow \bar{\partial}_E(\xi) = \bar{\partial}_E^*(\xi) = 0$ .

$$\begin{aligned} i \langle (\nabla')^* \xi, (\nabla')^* \xi \rangle &= -\langle [L_E^*, \bar{\partial}_E] \xi, (\nabla')^*(\xi) \rangle = \langle \bar{\partial}_E L_E^* \xi, (\nabla')^* \xi \rangle = \\ &= \langle L_E^*(\xi), \bar{\partial}_E^*(\nabla')^*(\xi) \rangle = \langle L_E^*(\xi), [\bar{\partial}_E^*(\nabla')^* + (\nabla')^* \bar{\partial}_E^*](\xi) \rangle = \\ &= \langle \Theta \wedge L_E^*(\xi), \xi \rangle \end{aligned}$$

This shows the first Nakano inequality.

*Remark 13.12.* As a special case, if  $E$  is a positive line bundle, then

$$H^q(X, K_X \otimes E) = 0$$

for all  $q > 0$ .

#### 14. [LT] LECTURE 6 - 07.11.19

Why are we considering real manifolds at all? (We are interested in smooth projective varieties and complex manifolds are much nicer). It turns out that there are things we can do only with real manifolds. This is also where second countability will come into play. Consider:

**Example 14.1**

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & \text{if } x \geq 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

Because of the identity principle, holomorphic functions are too rigid (zero on an open implies zero everywhere). On the other hand, smooth functions can wiggle around. This will be very useful for the following reason:

**Theorem 14.2 (Partition of Unity)** *Let  $X$  be a smooth manifold and  $\{U_\alpha\}_{\alpha \in I}$  an open cover. There are smooth functions  $f_n: X \rightarrow \mathbb{R}$  for  $n \in \mathbb{N}$  with compact support taking values in  $[0, 1]$  such that*

- i) *for every  $n \in \mathbb{N}$  there is some  $\alpha \in I$  such that  $\text{Supp}(f_n) \subseteq U_\alpha$ ,*
- ii) *for every  $x \in X$  there are only finitely many  $n \in \mathbb{N}$  such that  $f_n(x) \neq 0$ , and*
- iii)  *$\sum_{n=1}^{\infty} f_n(x) = 1$ .*

**Example 14.3** On  $X = \mathbb{R}$ , with  $U = \mathbb{R}$ , the single function  $f_1 = 1$  does not work, because it does not have compact support. Instead, choose smooth positive functions  $g_n$  for  $n \in \mathbb{Z}$  such that  $g_n$  has constant value 1 between  $n$  and  $n+1$  and constant value 0 outside of the interval  $[n-1, n+2]$ . [picture] This way we get compact supports and local finiteness. We can sum them up at each point to obtain a smooth function  $\sum_{n \in \mathbb{Z}} g_n$ . We can divide each term of the sum by  $g > 0$  to obtain new smooth functions to obtain the desired result.

*Remark 14.4.* We used second countability of  $\mathbb{R}$  to divide it into countably many intervals.

For a full proof of the previous result, see Warner Theorem 1.11.

**Lemma 14.5 (Warner, Lemma 1.9)** *Let  $X$  be a topological space which is locally compact, Hausdorff and second countable. Then every open cover has a refinement which is countable, locally finite and consisting only of open sets with compact closure.*

*Proof.* Let  $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$  be a fixed open cover. As  $X$  is second countable, there is a sequence  $(V_i)_{i \in \mathbb{N}}$  of open sets such that every open set is a union of such  $V_i$ . By local compactness and Hausdorffness we can further assume that all  $\bar{V}_i$  are compact, because throwing away those which do not have compact closure still forms a basis. For every  $x \in X$  there is an index  $i(x) \in \mathbb{N}$  such that  $x \in V_{i(x)}$ . The family  $\{V_{i(x)}\}_{x \in X}$  is a countable open cover of  $X$  and all  $\bar{V}_i$  are compact. We construct a sequence  $G_1 \subseteq G_2 \subseteq \dots$  of open subsets with compact closure such that  $\bigcup_{i=1}^{\infty} G_i = X$  and  $\bar{G}_i \subseteq G_{i+1}$ :

$$G_1 \subseteq \bar{G}_1 \subseteq G_2 \subseteq \bar{G}_2 \subseteq \dots$$

Put  $G_1 = V_1$ . Let  $j_2$  be the smallest index bigger than 1 such that  $\bar{G}_1 \subseteq \bigcup_{i=1}^{j_2} V_i =: G_2$ , which exists by compactness of  $\bar{G}_1$ . Let  $j_3$  be the smallest index bigger than  $j_2$  such that... etc. Then

$$\bigcup_{i=1}^{\infty} G_i = \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{j_n} V_j = \bigcup_{i=1}^{\infty} V_i = X.$$

The set  $\bar{G}_i \setminus G_{i-1}$  is compact and contained in the open set  $G_{i+1} \setminus \bar{G}_{i-2}$ . For each  $i \geq 3$  choose a finite subcover of the open cover  $\{U_\alpha \cap (G_{i+1} \setminus \bar{G}_{i-2})\}_\alpha$  of  $\bar{G}_i \setminus G_{i-1}$ . Also choose a finite open subcover of  $\{U_\alpha \cap G_\xi\}$  of  $\bar{G}_2$ . This is finally the refinement we wanted.  $\square$

Now we can give the idea of the proof of Thm 1.19. A partition of unity subordinated to a refinement of  $\{U_\alpha\}$  is also a partition of unity subordinated to the original open cover. We replace our cover by the refinement in the lemma. Wlog the cover is countable, locally finite and  $\bar{U}_\alpha$  is compact. This looks like the cover of  $\mathbb{R}$  given in the previous example, and then we find suitable smooth functions, etc.

**14.1. Vector bundles.** Our goal is to define the tangent bundle of a manifold.

**Example 14.6** Let  $X = S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ . The tangent line at  $(x_0, y_0) \in S^1$  has the parametrization  $t \mapsto (x_0, y_0) + t(y_0, -x_0)$ . The tangent space is given by

$$T_{(x_0, y_0)}S^1 = \{t(y_0, -x_0) \mid t \in \mathbb{R}\}.$$

Together they form the tangent bundle  $TS^1 = \{(v, w) \in S^1 \times \mathbb{R}^2 \mid v \perp w\}$ . This comes with a projection  $p: TS^1 \rightarrow S^1$ . The fibre of  $p$  in  $TS^1$  has the structure of a vector space.

**Definition 14.7** Let  $X$  be a topological space. A *real vector bundle* of rank  $r$  over  $X$  consists of a continuous map  $\pi: V \rightarrow X$  together with the structure of a real vector space of dimension  $r$  on each  $V_x = \pi^{-1}(x)$  such that there exists an open cover  $\{U_i\}$  of  $X$  and local trivializations

$$\begin{array}{ccc} \pi^{-1}(U_i) & \xrightarrow{\sim} & U_i \times \mathbb{R}^r \\ & \searrow \pi|_{\pi^{-1}(U_i)} \swarrow & \\ & U_i & \end{array}$$

such that for every  $x \in U_i$  the induced map  $V_x \rightarrow \{x\} \times \mathbb{R}^r \rightarrow \mathbb{R}^r$  is an isomorphism of vector spaces.

Let  $f: X \rightarrow Y$  be continuous and let  $\pi: V \rightarrow X$  and  $\xi: W \rightarrow Y$  two real vector bundles. A morphism of vector bundles over  $f$  is a continuous map  $F: V \rightarrow W$  such that the following diagram commutes

$$\begin{array}{ccc} V & \xrightarrow{F} & W \\ \downarrow \pi & & \downarrow \xi \\ X & \xrightarrow{f} & Y \end{array}$$

and such that the induced maps  $V_x \rightarrow W_{f(x)}$  is  $\mathbb{R}$ -linear.

If we replace  $\mathbb{R}$  by  $\mathbb{C}$  in this definition we get *complex vector bundles*, and a vector bundle of rank 1 is called a *line bundle*.

*Remark 14.8.* Every complex vector bundle defines a real vector bundle.

*Exercise 14.9.* Let  $\pi: V \rightarrow X$  be a vector bundle of rank  $r$ . Show that it is trivial if and only if there are  $r$  continuous section such that on each point their images form a basis of the fibre.

15. [FS] HARRY SCHMIDT - UNIFORMIZATIONS OF POLYNOMIAL DYNAMICAL SYSTEMS - 08.11.19

Let  $K$  be a number field and  $K_v$  a completion at a place  $v \in M_K := \{\text{places of } K\}$ . We denote by  $\bar{K}_v$  the algebraic closure of  $K_v$ .

Let  $f \in K[z]$  be a monic polynomial of degree  $d \geq 2$ . We define  $f^{0,n} = f(f^{0,-1})$  and  $f^{0,0} = z$ .

*Remark 15.1.* Let  $f: \mathbb{P}^1 \rightarrow \mathbb{P}^1$ . If  $\mu: \mathbb{P}^1 \rightarrow \mathbb{P}^1$  is a Moebius transformation, then  $\mu \circ f \circ \mu^{-1}$  has the same dynamics as  $f$ .

**Theorem 15.2** (Botall, Jones, S.) Let  $\alpha \in K$ . Define

$$\mathcal{L}_\alpha^{(n)} = \{\beta \in \bar{K} \mid f^{0,n}(\beta) = f^{0,n}(\alpha)\}.$$

There exists  $\beta \in \mathcal{L}_\alpha^{(n)}$  such that

$$(*) \quad [K(\beta) : K]_{f, \alpha, K} \gg d^{n/5}$$

**Example 15.3** (P. Ingram)  $f = z^d$ ,  $\mathcal{L}_\alpha^{(n)} = \alpha \mu_{d^n}$ , where  $\mu_{d^n} = \{\xi \mid \xi^{d^n} = 1\}$ . There exists  $\Phi \in \frac{1}{z} + \dots \in K[[\frac{1}{z}]]$  with the property  $\Phi(f(z)) = \Phi(z)^d$ .

**Theorem 15.4** (Ingram, de Marco et al) *For each place  $v \in M^K$  there exists  $\delta_v \geq 1$  such that  $\Phi$  converges on*

$$D_\infty^{\delta_v} = \{z \in \bar{K}_v \mid |z|_v > \delta_v\},$$

and  $\delta_v = 1$  for almost all  $v$ .

**Idea:** let  $n$  be minimal such that  $f^{0,n}(\alpha) = f^{0,n}(\beta)$ . Then  $\Phi(\beta) = \xi \Phi(\alpha)$  for a root of unity  $\xi$  of order  $N \geq 2^n$ . If  $v$  is a finite place, then  $[K_v(\xi) : K_v] N^{1-\varepsilon}$ .

If  $v$  is infinite, then we write  $\xi = \exp(2\pi i \frac{m}{N})$  for  $m \in \mathbb{Z}$  and  $(m, N) = 1$ . Observe that  $\beta = \Phi^{-1}(\exp(2\pi i \frac{m}{N}) \Phi(\alpha))$ . We study the function  $g(\tau) = \Phi^{-1}(\exp(2\pi i \tau) \Phi(\psi))$ ,  $\text{im}(\tau) \geq 0$  and  $g(\tau)$  is definable in  $\mathbb{R}_{an, exp}$ .

**Theorem 15.5** (Serre, Tate, Ihara '60) *Suppose  $C \subseteq \mathbb{G}_m^2$  is an irreducible curve with generic point  $(X_e, Y_e)$  and that*

$$|C \cap \mu_\infty^2| = \infty \Rightarrow \exists n, m \geq 0, X_e^n Y_e^n = 1.$$

**Conjecture 15.6** *Suppose  $C$  is an invertible curve in  $\mathbb{A}^2$ . Then if  $|C \cap (\mathcal{L}_\alpha^\infty)^2| = \infty$ ,  $C$  is a component of a curve defined by  $f^{0,n}(X) - f^{0,n}(Y) = 0$  or  $C$  is of the form  $\beta \times \mathbb{A}^1, \mathbb{A}^1 \times \beta$  for  $\beta \in \mathcal{L}_\alpha^\infty$ .*

Here  $\mathcal{L}_\alpha^\infty = \cup_{n \in \mathbb{N}} \mathcal{L}_\alpha^{(n)}$ .

**Remark 15.7.** If we had  $(*)$  for all primitive solutions of  $f^{0,n}(X) = f^{0,n}(\alpha)$ , then we could use the Pila-Zimmer strategy. We do have a Galois-bound if  $|f^{0,n}(\alpha)| \rightarrow \infty$  for  $v$  a finite place. To apply the strategy we would also need  $|f(\alpha)|_v \rightarrow \infty$  at infinite places.

**Theorem 15.8** *Suppose  $|f^{0,n}(\alpha)|_v \rightarrow \infty$  at a finite place  $v \in M_K$ . Then the conjecture is true.*

**Idea:** use the following elementary

**Lemma 15.9** *Let  $(\xi_1, \xi_2) \in \mathbb{G}_m^2(\bar{\mathbb{Q}})$  be of order  $N$ . There exists  $\xi_N$  a primitive  $N$ -th root of 1 and  $\xi_e, \xi'_e$  two  $e$ -th roots of 1 such that*

$$(\xi_1, \xi_2) = (\xi_e \xi_N^{K_1}, \xi'_e \xi_N^{K_2}),$$

where  $e \cdot \max\{|K_1|, |K_2|\} \leq N^{3/4}$ .

**Proof.** The idea is to apply the pigeon principle. Let  $P \in K[X, Y]$  be an irreducible polynomial such that  $P(\xi_1, \xi_2) = 0$ . Then  $Q(x) = P(\xi_e X^{K_1}, \xi'_e X^{K_2})$  vanishes at  $X = \xi_N$ . We have  $Q(X) \in K[X, \frac{1}{X}]$  and  $\deg Q \leq 2 \deg P \frac{N^{3/4}}{e}$ . Assuming  $Q \neq 0$ , the number of zeroes of  $Q$  is  $< \deg Q$  than  $P N^{3/4}$ . But [...]

In our situation we consider

$$Q(x) = P(\Phi^{-1}(\xi_e X^{K_1} \Phi(\alpha)), \Phi^{-1}(\xi'_e X^{K_2} \Phi(\alpha))),$$

which defines a  $v$ -adic analytic function. Then we can use some theory for  $v$ -adic functions.

## 16. [CM] TALK 4 (OLIVER) - LOCALLY COMPACT ABELIAN GROUPS - 11.11.19

Goal of this lecture: there is a “classical” approach to locally compact abelian groups giving an exact category called LCA and a bounded derived category  $D^b(\text{LCA})$ , and we will show that  $D^b(\text{LCA}) \hookrightarrow D^b(\text{Cond}(\text{Ab}))$  is a fully faithful functor.

**Definition 16.1** A topological group is called *locally compact* if it possesses a compact neighbourhood of the identity element.

*Remark 16.2.* Compact in the previous definition is french compact, i.e. Hausdorff and quasi-compact. In particular, locally compact abelian groups are Hausdorff.

**Theorem 16.3 (Theorem 4.1)** i) Let  $A$  be a locally compact abelian group. Then there is an integer  $n$  and an iso  $A \cong \mathbb{R}^n \times A'$ , where  $A'$  admits a compact open subgroup. Hence we can write<sup>11</sup>

$$C \hookrightarrow A' \twoheadrightarrow A'/C$$

with  $C$  compact and  $A'/C$  discrete.

ii) Pontryagin duality induces an equivalence of categories

$$\text{LCA}^{op} \xrightarrow{\sim} \text{LCA}$$

$$A \mapsto \text{Hom}_{\text{TopAb}}(A, \mathbb{T})$$

where the hom-space is considered with the compact open topology and  $\mathbb{T} := U(1) = \mathbb{R}/\mathbb{Z}$ . We denote the hom-space above by  $A^\vee$ .

iii) Pontryagin duality exchanges compact and discrete groups.

*Remark 16.4.* i) A product  $\prod_{i \in I} \mathbb{R}$  is locally compact if and only if  $|I| < \infty$ .

ii) Open implies clopen for topological groups.

iii)  $H \subseteq G$  open if and only if  $G/H$  is discrete.

**Proposition 16.5 (Proposition 4.2)** Let  $A, B$  be Hausdorff topological abelian groups and  $A, B$  compactly generated. Then there is a natural isomorphism of condensed abelian groups

$$\underline{\text{Hom}}(\underline{A}, \underline{B}) \xrightarrow{\sim} \underline{\text{Hom}}_{\text{TopAb}}(A, B)$$

*Proof.* Recall that we have this functor  $(-): \text{TopC} \rightarrow \text{Cond}(\text{C})$  sending  $T \mapsto \underline{T}$ , where  $\underline{T}(S) = \text{Hom}_{\text{Top}}(S, T)$ . Recall Proposition 1.7. in the script: this functor is faithful, and restricted to the category of compactly generated Hausdorff objects it is also full. Recall also that if  $S$  is profinite and  $M$  is a condensed abelian group, then

$$M(S) = \text{Hom}_{\text{Cond}(\text{Ab})}(\mathbb{Z}[S], M).$$

Then

$$\underline{\text{Hom}}(\underline{A}, \underline{B})(S) = \text{Hom}_{\text{Cond}(\text{Ab})}(\mathbb{Z}[S], \underline{\text{Hom}}(\underline{A}, \underline{B})) \xrightarrow{adj.} \text{Hom}_{\text{Cond}(\text{Ab})}(\mathbb{Z}[S] \otimes \underline{A}, \underline{B})$$

Check carefully what is meant by each hom-set.

and

$$\underline{\text{Hom}}(\underline{A}, \underline{B})(S) = \text{Hom}_{\text{Top}}(S, \text{Hom}_{\text{TopAb}}(A, B)) \xrightarrow{adj.} \text{Hom}_{\text{Top}}(S \times A, B) = \text{Hom}_{\text{Cond}(\text{Set})}(\underline{S \times A}, \underline{B})$$

□

**Theorem 16.6 (Theorem 4.3)** Suppose  $A = \prod_I \mathbb{T}$  be a condensed abelian group, where  $I$  is any index set.

<sup>11</sup>During this talk,  $\hookrightarrow$  means injective and closed and  $\twoheadrightarrow$  means surjective and open.

i) For any discrete abelian group  $M$ ,

$$R\text{Hom}(A, M) = \bigoplus_I M[-1].$$

ii)  $R\text{Hom}(A, \mathbb{R}) = 0$ .

First some preparation:

**Theorem 16.7** (Theorem 4.5, Eilenberg-MacLane, Breen, Deligne) *There is a functorial resolution of an abelian group  $A$  of the form*

$$(*) \quad \cdots \rightarrow \bigoplus_{j=1}^{n_i} \mathbb{Z}[A^{r_{i,j}}] \rightarrow \cdots \mathbb{Z}[A^3] \oplus \mathbb{Z}[A^2] \rightarrow \mathbb{Z}[A^2] \rightarrow \mathbb{Z}[A] \rightarrow A \rightarrow$$

for finite  $n_i, r_{i,j}$ .

**Corollary 16.8** (Corollary 4.8) *For condensed abelian groups  $A, M$  and any extremally disconnected set  $S$  there is a spectral sequence*

$$E_1^{i_1, i_2} = \prod_{j=1}^{n_{i_1}} H_{\text{Cond}}^{i_2}(A^{r_{i_1, j}} \times S, M) \Rightarrow \underline{\text{Ext}}^{i_1 + i_2}(A, M)(S),$$

where  $\underline{\text{Ext}}^i(X, Y) = R\text{Hom}(X, Y[i])$ .

*Proof.* Recall  $R\text{Hom}(A, M)(S) = R\text{Hom}(A \otimes \mathbb{Z}[S], M)$ , and for finite direct sumes we have  $\mathbb{Z}[A^i] \otimes \mathbb{Z}[S] \cong \mathbb{Z}[A^i \times S]$ . We use the resolution of Theorem 4.5 and the previous iso to compute  $R\text{Hom}(A \otimes \mathbb{Z}[S], M)$  as

$$\cdots \rightarrow \bigoplus \text{Hom}(\mathbb{Z}[A^{r_{i,j}} \times S], M) \rightarrow \cdots,$$

then do this resolution for each term and apply some horse-shoe lemma on projective resolutions on the LHS of the previous homs to conclude with the spectral sequence of the double complex.  $\square$

*Remark 16.9.* The previous corollary is functorial in  $A, M$  and in  $S$ .

Now we prove the theorem (Theorem 4.3).

Proof of i) first. If  $I$  finite, we can assume that  $I$  is a singleton. Let's just assume that  $A = \mathbb{T}$ . We need to compute  $R\text{Hom}(\mathbb{T}, M)$ . We have a triangle  $\mathbb{Z} \rightarrow \mathbb{R} \rightarrow \mathbb{T} \xrightarrow{+1}$ , which rotated yields a distinguished triangle

$$\mathbb{R} \rightarrow \mathbb{T} \rightarrow \mathbb{Z}[1] \xrightarrow{1}.$$

So the claim is equivalent to the claim  $R\text{Hom}(\mathbb{R}, M) = 0$  for any discrete group  $M$ , because we have a distinguished triangle

$$R\text{Hom}(\mathbb{Z}[1], M) \rightarrow R\text{Hom}(\mathbb{T}, M) \rightarrow R\text{Hom}(\mathbb{R}, M) \xrightarrow{+1}$$

It suffices to show that  $0 \rightarrow \mathbb{R}$  induces an iso after applying  $R\text{Hom}(-, M)$ , for which in turn it is sufficient to know that the pullback  $0 \times \text{id}: S \rightarrow \mathbb{R}^r \times S$  induces an iso in all cohomology groups

$$H^i(\mathbb{R}^r \times S, M) \rightarrow H^i(S, M).$$

Now we use the result that  $R\Gamma(\mathbb{R}^r \times S, M) = \mathbb{R} \varprojlim_r R\Gamma([-n, n]^r \times S, M)$ , and since  $[-n, n]^r \times S$  is compact, by the last talk this is classical sheaf cohomology, where we can use homotopy invariance. Then use functoriality in the previous lemma to conclude.

Proof of ii) now. For this we need a lemma (Proposition 4.17): consider a functorial resolution  $F(A)_\bullet$  of an abelian group such that  $F(A)_i \cong \bigoplus_{fin} \mathbb{Z}[A^{(fin)}]$ . Then for any

natural number  $n \geq 1$ , the map  $(- \cdot n): F(*) \rightarrow F(A)$  and the map  $[n]: F(A) \rightarrow F(A)$  induced from multiplication on  $A$  are homotopic by a functorial homotopy  $h_\bullet: F(A)_\bullet \rightarrow F(A)_{\bullet+1}$ . By theorem 3.3 from last lecture,  $R\text{Hom}(A, \mathbb{R})(S)$  can be computed by the complex

$$0 \rightarrow \bigoplus_{j=1}^{n_0} C(A^{r_0,j} \times S, \mathbb{R}) \rightarrow \bigoplus_{j=1}^{n_1} C(A^{r_1,j} \times S, \mathbb{R}) \rightarrow \dots$$

of Banach spaces (i.e. differentials are continuous etc). Assume  $f \in \bigoplus_{j=1}^{n_i} C(A^{r_i,j} \times S, \mathbb{R})$  satisfies  $df = 0$ . Then  $2f = [2]^*(f) + d(h_{i-1}^*(f))$ . Hence

$$f = \frac{1}{2}[2]^*(f) + d\left(\frac{1}{2}h_{i-1}^*(f)\right).$$

Now we iterate this idea! After  $n$  iterations we obtain

$$f = \frac{1}{2^n}[2^n]^*(f) + d\left(\frac{1}{2}h_{i-1}^*(f) + \frac{1}{4}h_{i-1}^*([2]^*(f)) + \dots\right).$$

Note that  $[2^n]^*(f)$  stays bounded as  $n \rightarrow \infty$ .

**Corollary 16.10**  $R\text{Hom}(\mathbb{R}, \mathbb{R}) = \mathbb{R}$  using again the triangle  $\mathbb{Z} \hookrightarrow \mathbb{R} \rightarrow \mathbb{T}$ .

Recall the goal theorem:  $\mathbf{D}^b(\text{LCA}) \hookrightarrow \mathbf{D}^b(\text{Cond}(\mathbf{Ab}))$  is fully faithful. LCA is a quasi-abelian category. There are two approaches to define its bounded derived category. One of them by J.P. Schneider using these quasi-abelian categories. Another one was by Hoffmann-Spitzweck, which is using exact categories of Quillen. They showed that these are a replacement for quasi-abelian category. We may not have kernels and cokernels, so we cannot talk about cohomology objects! How do we define the derived category? Here are the steps:

- (1) Define the category of complexes.
- (2) Go to the homotopy category.
- (3) Define the derived category as the Verdier quotient of the homotopy category by acyclic complexes, where a complex is called acyclic if each differential factors as  $d: C^n \rightarrow Z^{n+1}A \hookrightarrow A^{n+1}$  and  $Z^nA \hookrightarrow A^n \rightarrow Z^{n+1}A$ .

Now the proof idea goes as follows. We first set up a functor from  $\mathbf{D}^b(\text{LCA})$  to  $\mathbf{D}^b(\text{Cond}(\mathbf{Ab}))$  (this we skip). For the full faithfulness statement we need to show that  $\mathbb{R}\text{Hom}_{\mathbf{D}^b(\text{LCA})}(A, B) \rightarrow \mathbb{R}\text{Hom}_{\mathbf{D}^b(\text{Cond}(\mathbf{Ab}))}(\psi A, \psi B)$  is an isomorphism of abelian groups. We can reduce to the case of  $A, B \in \text{LCA}$ . By the structure theorem for LCA groups, any LCA group has the shape  $\mathbb{R} \times A'$  with  $C \hookrightarrow A' \rightarrow D$  with  $C$  compact and  $D$  discrete.  $D$  is a  $\mathbb{Z}$ -module, hence has a projective resolution  $\bigoplus \mathbb{Z} \rightarrow \bigoplus \mathbb{Z} \rightarrow D$ . Again arguing with triangles we reduce to direct sums of  $\mathbb{Z}$ , then reduce to  $\mathbb{Z}$ . For the compact use triangles with tori.

## 17. [LT] LECTURE 7 - 12.11.19

$\pi: V \rightarrow X$  smooth vector bundle. The data of the transition maps is then equivalent to the data of a smooth map  $U_i \cap U_j \xrightarrow{\varphi_{i,j}} \text{GL}_r(\mathbb{R})$ . However, not every such map will do, because transition maps have a certain compatibility on triple intersections. More precisely, they satisfy the *cocycle condition*

$$\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$$

over  $U_i \cap U_j \cap U_k$ .

*Exercise 17.1.* The cocycle condition implies that  $\varphi_{ii} = \text{id}$ .



**Proposition 17.2** Let  $X$  be a topological space. Let  $\mathcal{U} = \{U_i\}_{i \in I}$  be an open cover and consider continuous maps  $U_i \cap U_j \rightarrow \text{GL}_r(\mathbb{R})$  such that the cocycle condition is satisfied. Then there is a real vector bundle  $\pi: V \rightarrow X$  of rank  $r$  with transition functions  $\tau_{ij}: U_i \cap U_j \times \mathbb{R}^r \rightarrow U_i \times U_j \times \mathbb{R}^r$  given by  $(x, v) \mapsto (x, \varphi_{ij}(x)(v))$  together with local trivializations on  $\mathcal{U}$ . Moreover, this data is unique up to unique isomorphism.

*Proof.* We glue  $V_i = U_i \times \mathbb{R}^r$  along the open subsets  $U_i \cap U_j \times \mathbb{R}^r \subseteq V_i$  along  $\tau_{ij}$ . This is well-defined because of the cocycle condition. We get a topological space  $V$ . By construction there is a continuous map  $\pi: V \rightarrow X$ . The fibre above  $x$  is  $\{x\} \times \mathbb{R}^r$ . This defines a vector space structure. We get local trivializations on  $\pi^{-1}(U_i) \cong V_i \xrightarrow{\sim} U_i \times \mathbb{R}^r$ .  $\square$

**Example 17.3**  $X = S^1 \subseteq \mathbb{R}^2$ . Cover the circle by  $U_i = \{(x, y) \in \mathbb{R}^2 \mid x \neq (-1)^i 1\}$  for  $i \in \{1, 2\}$ . The trivial bundle  $S^1 \times \mathbb{R}$  has the transition map  $U_1 \cap U_2 \rightarrow \text{GL}_1(\mathbb{R}) = \mathbb{R}^\times$  given by  $p \mapsto 1$  for all  $p \in S^1$ . The Moebius bundle on the other hand has the transition map

$$p \mapsto \begin{cases} 1 & \text{if } y > 0, \\ -1 & \text{if } y < 0. \end{cases}$$

**Lemma 17.4** Let  $X$  be a topological space and  $\pi: V \rightarrow X$  a vector bundle of rank  $r$ . Then there is a canonical bundle  $\xi: V^* \rightarrow X$ , called the dual bundle of  $V$ , such that  $\xi^{-1}(x) = V_x^* = \text{Hom}_{\mathbb{R}}(V_x, \mathbb{R})$ .

*Proof.* We will specify the gluing data. We use local trivializations on the cover  $U_i$  of  $X$  and transition maps  $\varphi_{ij}: U_i \cap U_j \rightarrow \text{GL}_r(\mathbb{R})$  for  $V$ . Put  $\psi_{ij} = (\varphi_{ij}^{-1})^{tr}$ , where the inverse and the transpose are taken pointwise on  $X$ .  $\square$

*Exercise 17.5.* Write out all the details.

The same method works for other constructions, e.g.  $\oplus$ ,  $\otimes$ , exterior powers...

**Definition 17.6** i) Let  $X$  be a smooth manifold. A smooth real or complex vector bundle is a smooth map  $\pi: V \rightarrow X$  of smooth manifolds together with the structure of real or complex topological vector bundle such that the trivializations are smooth.  
ii) Let  $X$  be a complex manifold. A holomorphic vector bundle is a holomorphic map of complex manifolds  $\pi: V \rightarrow X$  together with the structure of a complex vector bundle such that the trivializations are holomorphic.

**17.1. Tangent bundles.** Let  $X$  be a smooth manifold of dimension  $n$  and  $x \in X$ . We want to define the tangent space of  $X$  in  $x$ . Tangent vectors are “directions”.

**Example 17.7** If  $X \subseteq \mathbb{R}^n$  is open, then the tangent space at each point is  $\mathbb{R}^n$ . Every  $v \in \mathbb{R}^n \setminus \{0\}$  defines a straight line  $t \mapsto x + tv$  through  $x$ . If  $(U, \varphi)$  is a chart around  $x$ , we could use preimages of straight lines. But the result depends on the chart.

**Definition 17.8** Let  $X$  be a smooth manifold and  $x \in X$ . A *smooth path* through  $x$  is a smooth map  $\gamma: (-\varepsilon, \varepsilon) \rightarrow X$  for some  $\varepsilon > 0$  such that  $\gamma(0) = x$ . We say that two paths (through  $x$ ) are equivalent if they have the same tangent vector in  $\mathbb{R}^n$  for some chart. A *tangent vector* in  $X$  at  $x$  is an equivalence class of paths through  $x$ . Let  $T_x X$  denote the set of tangent vectors.

**Remark 17.9.** By the chain rule, the previous equivalence relation is chart independent.

There is a more elegant approach. Recall that a path defines a directional derivative:

**Definition 17.10** Let  $X$  be a smooth manifold and  $x \in X$ . Let  $\gamma$  be a smooth path through  $x$ . Let  $f: U \rightarrow \mathbb{R}$  be a smooth function on an open nbhd  $U$  of  $x$ . We put

$$\partial_\gamma f = \lim_{h \rightarrow 0} \frac{f(\gamma(h)) - f(x)}{h} = (f \circ \gamma)'(0),$$

and call it the *derivative of  $f$  in the direction of  $\gamma$* .

**Lemma 17.11** Two paths define the same tangent vector at  $x$  if and only if for all smooth  $f: U \rightarrow \mathbb{R}$  the directional derivatives agree.

*Proof.* Let  $\gamma_1, \gamma_2$  be equivalent paths. We choose a chart  $(V, \varphi)$  around  $x$ . Up to some shrinking we may assume  $V = U$ . We may replace  $X$  by  $\varphi(V)$ ,  $f$  by  $f \circ \varphi^{-1}$  and  $\gamma_i$  by  $\varphi \circ \gamma_i$ . This does not change the directional derivatives.

$$\partial_{\gamma_i} f = \left( \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right) \begin{pmatrix} \gamma'_{i_1}(0) \\ \vdots \\ \gamma'_{i_n}(0) \end{pmatrix}$$

Hence it only depends on the equivalence class of the paths. Conversely, assume  $\gamma_1, \gamma_2$  define the same directional derivatives. Choose a chart  $(U, \varphi)$  around  $x$ . Consider the coordinate functions  $x_i: \mathbb{R}^n \rightarrow \mathbb{R}$ . Then consider  $f_i = x_i \circ \varphi: U \rightarrow \mathbb{R}$ . The tangent vectors are then given on each component by the directional derivatives of the  $f_i$ .  $\square$

#### 18. [WS] KODAIRA 4 (YUHAN): QUADRATIC TRANSFORMATIONS

Recall: let  $X$  be a complex manifold and let  $E \rightarrow X$  be a vector bundle on  $X$ . Denote the associated connection by  $D$ . Then the curvature  $\Theta$  is defined as

$$\Theta = D^2 = D \circ D: \mathcal{A}^0(E) \rightarrow \mathcal{A}^2(E).$$

**Proposition 18.1** Given a Hermitian holomorphic line bundle  $E \rightarrow X$  with Hermitian metric  $h$  and the induced connection  $D_h$ , we have

- (1)  $\Theta$  is a  $(1, 1)$ -type.
- (2)  $\bar{\partial}\Theta = 0$ .
- (3) If given a locally finite cover  $\{U_i\}$  of  $X$  which is a holomorphic trivialization, with holomorphic frame  $f_i$  on  $U_i$ , then

$$\Theta(f_i) = \bar{\partial}\partial \log(h_i).$$

**Lemma 18.2** Given two Hermitian line bundles  $E, F$  over  $X$  with hermitian metrics  $h_E, h_F$ , then the associated curvature satisfies the following

$$\Theta_{E \otimes F} = \Theta_E + \Theta_F.$$

*Proof.* Consider cover  $\{U_i\}$  of  $X$ , then locally on  $U_i$  we have

$$\Theta_{E \otimes F} = \bar{\partial}\partial \log(h_{E,i} h_{F,i}) = \bar{\partial}\partial \log(h_{E,i}) + \bar{\partial}\partial \log(h_{F,i}) = \Theta_E + \Theta_F.$$

$\square$

**18.1. Quadratic transformations.** Let  $X$  be a complex manifold and  $p \in X$ . Let  $U$  be a coordinate nbhd of  $p$  in  $X$  with local coordinates  $z_1, \dots, z_n$  such that  $p = (0, \dots, 0)$  with respect to these coordinates. Consider the product  $U \times \mathbb{P}^{n-1}$ , where we use  $[t_1, \dots, t_n]$  as homogeneous coordinates for  $\mathbb{P}^{n-1}$ . Then let

$$W = \{((z_1, \dots, z_n), [t_1, \dots, t_n]) \in U \times \mathbb{P}^{n-1} \mid t_i z_j = t_j z_i \text{ for all } i, j\}.$$

- (1) There is a holomorphic projection  $\pi: W \rightarrow U$  sending  $(z, t) \mapsto z$ .
- (2)  $S := \pi^{-1}(0) \cong \mathbb{P}^{n-1}$ .
- (3)  $\pi|_{W \setminus S}$  is biholomorphic.

**Definition 18.3** The *quadratic transformation*  $\tilde{X} = Q_p(X)$  of  $X$  at  $p$  is defined as

$$\tilde{X} = W \bigcup_{U \setminus \{p\}} (X \setminus \{p\})$$

with projection  $\pi_p: \tilde{X} \rightarrow X$ .

We set  $\tilde{U}_i \subseteq W$  as the set given by  $t_i \neq 0$ , the local coordinates of  $\tilde{U}$  being given by

$$z(i)_j = \begin{cases} \frac{z_j}{z_i} = \frac{t_j}{t_i} & \text{if } j \neq i, \\ z_i & \text{if } i = j. \end{cases}$$

and  $\pi_{\tilde{U}_i}$  being given by

$$(z(i)_1, \dots, z(i)_n) \mapsto (z(i)_1 z(i)_i, \dots, z(i)_i, \dots, z(i)_n z(i)_i).$$

**Example 18.4** Consider  $V = \{x^2 = y^2\} \subseteq \mathbb{C}^2$ .

- (1)  $\pi_0^{-1}(V \setminus \{0\}) \subseteq Q_0(\mathbb{C}^2)$  is given by the equation

$$\{((z_1, z_2), [t_1, t_2]) \in \mathbb{C}^2 \times \mathbb{P}^1 \mid z_1 t_2 = z_2 t_1, z_1^2 = z_2^2, (z_1, z_2) \neq (0, 0)\}$$

- (2) Consider the affine chart with  $t_1 = 1$ . Then we have equations

$$\begin{aligned} \{(z_1, z_2, t_2) \in \mathbb{C}^3 \mid z_1 t_2 = z_2, z_1^2 = z_2^2, (z_1, z_2) \neq (0, 0)\} &= \\ &= \{(z_1, z_1 t_2, t_2) \in \mathbb{C}^3 \mid t_2^2 = 1, z_1 \neq 0\} = \\ &= \{(z_1, z_1, 1) \in \mathbb{C}^3 \mid z_1 \neq 0\} \cup \{(z_1, -z_1, -1) \in \mathbb{C}^3 \mid z_1 \neq 0\} \end{aligned}$$

We take the closure to obtain the final equations

$$\{(z_1, z_1, 1) \in \mathbb{C}^3\} \cup \{(z_1, -z_1, -1) \in \mathbb{C}^3\}$$

## 18.2. Divisors.

**Definition 18.5** Let  $X$  be a complex manifold. An *analytic subvariety* of  $X$  is a closed subset  $Y \subseteq X$  such that for any point  $x \in X$  there exists an open nbhd  $x \in U \subseteq X$  such that  $Y \cap U$  is the zero locus of finitely many holomorphic functions  $f_1, \dots, f_r \in \mathcal{O}_X(U)$ . An analytic subvariety  $Y$  is called *irreducible* if it cannot be written as  $Y = Y_1 \cup Y_2$  for some proper analytic subvarieties  $Y_i \subseteq Y$ .

**Definition 18.6** An *analytic hypersurface* is an analytic subvariety  $Y \subseteq X$  of codimension 1.

**Definition 18.7** A *Weil divisor*  $D$  on  $X$  is a formal linear combination

$$D = \sum a_i [Y_i]$$

with  $a_i \in \mathbb{Z}$  and  $Y_i \subseteq X$  irreducible analytic hypersurfaces. The divisor group is denoted by  $\text{Div}(X)$ .

**Facts:**

- (1) Weil divisors are the same as Cartier divisors on a complex manifolds.
- (2) Up to linear equivalence and if  $X$  is projective, divisors are also the same as elements of the Picard group.

Let  $X$  be a compact complex manifold and  $\pi_p: \tilde{X} \rightarrow X$  a quadratic transformation. Let  $S = \pi_p^{-1}(p)$  be the exceptional divisor, which locally in  $W$  is given by  $z_1 = \dots = z_n = 0$ . The line bundle associated to  $S$  is denoted by  $L(S) \rightarrow \tilde{X}$ . We denote the hyperplane bundle  $H \rightarrow S$  and its given by  $(t_i = 0)$ .

$$\begin{aligned} \sigma: W &\rightarrow \mathbb{P}^{n-1} \\ (z, t) &\mapsto t \end{aligned}$$

**Proposition 18.8**  $L|_W = \sigma^*H^\vee$ .

*Proof.* In  $\mathbb{P}^{n-1}$ , the hyperplane is given by  $[t_j/t_i = 0]$  in  $V_i \subseteq \mathbb{P}^{n-1}$ . Over  $V_i \cap V_j$  we have

$$h_{ij} = \frac{t_i}{t_j} \left( \frac{t_i}{t_j} \right)^{-1} = \frac{t_j}{t_i}.$$

$\sigma^*H$  has the same transition in  $(U \times (V_i \cap V_j)) \subseteq W$ . For  $L|_W$  in  $((U \times V_i)) \subseteq W$ ,  $S$  is given by  $[z_i = 0]$ , thus the transition function is

$$g_{ij} = \frac{z_i}{z_j}.$$

Since  $g_{ij} = g_{ij}^{-1}$ ,  $L|_W = \sigma^*H^\vee$ . □

**Lemma 18.9** Denoting the canonical line bundles by  $K(-)$ , we have

$$K_{\tilde{X}} = \pi_p^*K_X \otimes L_p^{n-1}.$$

*Proof.* Since  $\pi_p$  on  $\tilde{X} \setminus W$  is biholomorphic and  $L_p|_{X \setminus W}$  is trivial, the pullback of the canonical is the canonical over this open. On  $W$ , observe that  $K_X$  on  $U$  is trivial. We only need to show that  $K_{\tilde{X}}|_W = L_p^{n-1}|_W$ . Consider  $f = dz_1 \wedge \dots \wedge dz_n$  on  $U \subseteq X$ . Then  $\pi_p^*f$  on  $\tilde{U}_i$  is given by

$$f_i = z_i^{n-1} dz(i)_1 \wedge \dots \wedge dz(i)_n.$$

□

Let  $L_p \rightarrow \tilde{X}$  be the line bundle associated to the exceptional divisor  $S$ . If we blow up 2 points  $p \neq q$ , we get  $Q_p(Q_q(X)) \cong Q_q(Q_p(X))$ . We denote the projection  $\pi_{p,q}$  and  $L_{p,q}$  the dual of the line bundle associated to the divisor  $\pi_{p,q}^{-1}(\{p, q\})$ .

**Proposition 18.10** Let  $E \rightarrow X$  be a positive holomorphic line bundle. Then there exists an integer  $\mu_0$  such that if  $\mu > \mu_0$ , then for any points  $p, q \in X$  with  $p \neq q$  we have that

- (1)  $\pi_p^*E^\mu \otimes L_p^\vee \otimes K_{Q_p(X)}^\vee$ ,
- (2)  $\pi_p^*E^\mu \otimes (L_p^\vee)^2 \otimes K_{Q_p(X)}^\vee$ , and
- (3)  $\pi_{p,q}^*E^\mu \otimes L_{p,q}^\vee \otimes K_{Q_p Q_q(X)}^\vee$

are all positive holomorphic line bundles.

[Recall the definition of positive line bundle and the fact that the hyperplane bundle is positive.]

## 19. [LT] LECTURE 8 - 14.11.19

**Definition 19.1** Let  $X$  be a smooth (resp. complex) manifold,  $x \in X$  a point and  $U, V \subseteq X$  open nbhds of  $x$  in  $X$ . Let  $f: U \rightarrow \mathbb{R}$  (resp.  $\mathbb{C}$ ) and  $g: V \rightarrow \mathbb{R}$  (resp.  $\mathbb{C}$ ) be two smooth (resp. holomorphic) functions. We say that  $f$  and  $g$  are equivalent at  $x$  if there is a nbhd  $W \subseteq U \cap V$  of  $x$  such that  $f|_W = g|_W$ . A *germ of a smooth (resp. holomorphic) function* at  $x$  is an equivalence class for this relation. We use  $\mathcal{A}_x$  (resp.  $\mathcal{O}_x$ ) for the local rings of germs of smooth (resp. holomorphic) functions at  $x$ .

**Definition 19.2** Let  $X$  be a smooth manifold and  $x \in X$ . A *derivation at  $x$*  is an  $\mathbb{R}$ -linear map  $D: \mathcal{A}_x \rightarrow \mathbb{R}$  such that

$$(\text{Leibniz}) \quad \forall [f], [g] \in \mathcal{A}_x, D(fg) = D(f)g(x) + f(x)D(g).$$

We will denote the  $\mathbb{R}$ -vector space of derivations at  $x$  by  $\text{Der}_{\mathbb{R}}(\mathcal{A}_x, \mathbb{R})$ .

**Lemma 19.3** Let  $X$  be a smooth manifold and  $x \in X$ . Let  $\gamma: (-\varepsilon, \varepsilon) \rightarrow X$  be a path through  $x$ . The map  $\partial_\gamma: \mathcal{A}_x \rightarrow \mathbb{R}$  sending  $f \mapsto \partial_\gamma f$  is a well-defined  $\mathbb{R}$ -linear derivation.

*Proof.*  $\mathbb{R}$ -linearity is okay, so let us check the Leibniz rule.

$$\partial_\gamma(fg) = ((fg) \circ \gamma)'(0) = ((f \circ \gamma)(g \circ \gamma))'(0),$$

from which we can conclude applying the product rule.  $\square$

**Proposition 19.4** Let  $X$  be a smooth manifold of dimension  $n$  and  $p \in X$ .

- (1)  $\text{Der}_{\mathbb{R}}(\mathcal{A}_p, \mathbb{R})$  is an  $\mathbb{R}$ -vector space of dimension  $n$ .
- (2)  $T_p X \cong \text{Der}_{\mathbb{R}}(\mathcal{A}_p, \mathbb{R})$ .

*Proof.* WLOG  $X = U \subseteq \mathbb{R}^n$  open. Let  $x_1, \dots, x_n: U \rightarrow \mathbb{R}$  be the coordinate functions and define paths  $\gamma_k(t) = p + te_k$  for  $1 \leq k \leq n$  and small values of  $t > 0$ . Then  $\partial_{\gamma_k}$  is by definition the partial derivative  $\frac{\partial}{\partial x_k} =: \partial_k$ . The claim is that  $\partial_1, \dots, \partial_n$  is a basis for  $\text{Der}_{\mathbb{R}}(\mathcal{A}_p, \mathbb{R})$ . First we show linear independence. Suppose that  $0 = \sum_k \lambda_k \partial_k$ . Let  $v = \sum_k \lambda_k e_k$ . Then  $D_v = \sum_k \lambda_k \partial_k = 0$ , so the tangent vector  $\gamma(t) = p + tv$  is equivalent to zero. This implies that  $v = 0$ , so  $\lambda_1 = \dots = \lambda_n = 0$ . Now we show that  $\partial_1, \dots, \partial_n$  generate  $\text{Der}_{\mathbb{R}}(\mathcal{A}_p, \mathbb{R})$ . Let  $\partial: \mathcal{A}_p \rightarrow \mathbb{R}$  be a derivation and let  $\lambda_k := \partial x_k$  for  $1 \leq k \leq n$ . The claim is that  $\partial = \sum_k \lambda_k \partial_k$ . Let  $f: U \rightarrow \mathbb{R}$  smooth and consider the expansion

$$(*) \quad f(y) = f(p) + \sum_k (\partial_k f)(y_k - p_k) + \sum_{k,l} g_{kl}(p)(y_k - p_k)(y_l - p_l)$$

for  $g_{kl}$  smooth. Apply  $\partial$  and  $\sum_k \lambda_k \partial_k$  to  $(*)$  to get

$$\partial f = 0 + \left( \sum_k (\partial_k f) \partial y_k \right) \Big|_{y=p}$$

and similarly with  $\sum_k \lambda_k \partial_k$ . We find that  $\partial f = \sum_k \lambda_k \partial_k f$ .  $\square$

**Corollary 19.5**  $T_x X \cong \text{Der}_{\mathbb{R}}(\mathcal{A}_x, \mathbb{R}) \cong \text{Hom}_{\mathbb{R}}(\mathfrak{m}_x / \mathfrak{m}_x^2, \mathbb{R})$ .

**Definition 19.6** Let  $X$  be a smooth manifold of dimension  $n$ .

- (1)  $TX := \coprod_{x \in X} T_x X \xrightarrow{\pi} X$  is called the *tangent bundle*. For coordinate chart  $(U, \varphi)$  we have

$$\begin{aligned} \pi^{-1}(U) &\xrightarrow{\sim} U \times \mathbb{R}^n \\ \sum_k \lambda_k(x) \partial_k &\mapsto (x, \lambda_1(x), \dots, \lambda_n(x)) \end{aligned}$$

We give  $TX$  the topology obtained from the  $\pi^{-1}(U)$  by gluing, so that it becomes a rank  $n$  vector bundle over  $X$  and thus a smooth manifold of dimension  $2n$ .

- (2)  $T^*X$  denotes the dual vector bundle of  $TX$ , called the *cotangent bundle*.
- (3) A *vector field* on  $X$  over  $U \subseteq X$  open is a smooth section  $\xi: U \rightarrow TX$  of  $\pi$ .
- (4) A *differential form* on  $X$  over  $U \subseteq X$  open is a smooth section of  $T^*X$ .
- (5) A *differential form* of degree  $r$  on  $X$  over  $U \subseteq X$  open is a smooth section of  $\Lambda^r T^*X$ .

**Exercise 19.7.**  $U \subseteq \mathbb{R}^{n+1}$  open,  $F: U \rightarrow \mathbb{R}$  smooth function,  $X = \mathbb{V}(F) := \{x \in U \mid F(x) = 0\}$ . If the gradient of  $F$  is nonzero on  $X$ , then give an explicit identification of  $T_x X$  with the space of tangent directions to  $X$  in  $\mathbb{R}^{n+1}$ .

**Remark 19.8.**  $T_x^*X \cong \mathfrak{m}_x/\mathfrak{m}_x^2$ . If  $f: U \rightarrow \mathbb{R}$  smooth, then we have a differential form  $df: y \mapsto [f - f(y)]$ . Show (exercise) that  $df$  is smooth. [Hint: in a coordinate chart  $(U, \varphi)$ ,  $d\varphi_k(x) \in T_x^*X$  is dual to  $\partial_k$  for  $1 \leq k \leq n$ .]

**19.1. Functoriality.** Let  $\psi: X \rightarrow Y$  be a smooth map between smooth manifolds and let  $x \in X$ . Then we get an induced map

$$\begin{aligned} d\psi_x: T_x X &\longrightarrow T_{\psi(x)} Y \\ \partial &\longmapsto [f \mapsto \partial(f \circ \psi)] \end{aligned}$$

which is  $\mathbb{R}$ -linear.

**Definition 19.9** Let  $f: X \rightarrow Y$  be a smooth map. Then the *pushforward along  $f$*  is the map

$$df: TX \xrightarrow{\sqcup df_x} TY,$$

which is a morphism of smooth vector bundles (exercise).

**Proposition 19.10** (1)  $d(\text{id}_X) = \text{id}_{TX}$ .

(2) If  $f: X \rightarrow Y$ ,  $g: Y \rightarrow Z$  are smooth maps, then  $dg \circ df = d(g \circ f)$ .

**Remark 19.11.** In other words,  $X \mapsto TX$ ,  $f \mapsto df$  is a functor.

**Corollary 19.12** If  $f: X \rightarrow Y$  is a diffeomorphism, then  $df: TX \rightarrow TY$  is a diffeomorphism of smooth vector bundles.

## 19.2. Immersions and embeddings.

**Definition 19.13** We say that a map  $f: X \rightarrow Y$  is an *immersion* if it is smooth and for all  $x \in X$  the differential  $df_x$  is injective. We say that a map  $f: X \rightarrow Y$  is a *submersion* if it is smooth and for all  $x \in X$  the differential  $df_x$  is surjective.

**Definition 19.14** We say that  $X$  is a *submanifold* of  $Y$  if there exists an injective immersion  $X \hookrightarrow Y$ .

**Definition 19.15** We say that  $f: X \rightarrow Y$  is a *embedding* if it is an immersion that is a diffeomorphism onto  $f(X)$ .

**Example 19.16**  $(x, y) \mapsto x$  from  $\mathbb{R}^2$  to  $\mathbb{R}$  is a submersion but not an immersion.

**Example 19.17**  $x \mapsto (x, 0)$  from  $\mathbb{R}$  to  $\mathbb{R}^2$  is an immersion, but not a submersion. In fact, it is an embedding.

**Exercise 19.18.**  $S^1 \hookrightarrow \mathbb{R}^2$  is an embedding.

**Example 19.19**  $X = Y = \mathbb{R}$  and  $f(x) = x^3$  is injective, but it is neither an immersion nor a submersion.

**Example 19.20**  $X = (-\frac{\pi}{4}, \frac{3\pi}{4})$ ,  $Y = \mathbb{R}^2$  and  $f: X \rightarrow Y$  sending  $x \mapsto (\cos(x) \cos(2x), \sin(x) \cos(2x))$ . Then  $f$  is injective but it is not an embedding<sup>12</sup> (it already fails to be a homeomorphism onto its image).

**19.3. Tangent bunles to complex manifolds.** To translate the previous constructions to the holomorphic setting, replace  $\gamma: (-\varepsilon, \varepsilon) \rightarrow X$  by  $\gamma: \mathbb{D}(0, \varepsilon) \rightarrow X$  holomorphic. Equivalence classes of such are  $\mathbb{C}$ -linear derivations

$$\mathrm{Der}_{\mathbb{C}}(\mathcal{O}_x, \mathbb{C}) \cong (\mathfrak{m}_x / \mathfrak{m}_x^2)^{\vee}.$$

We get the *holomorphic tangent bundle*  $TX \rightarrow X$  of rank  $\dim_{\mathbb{C}}(X)$  as before. Let  $X_{\mathbb{R}}$  denote the real manifold underlying  $X$ . Then we have  $(TX)_{\mathbb{R}} \cong T(X_{\mathbb{R}})$  as smooth bundles, but this is not a  $\mathbb{C}$ -linear isomorphism.

*Remark 19.21.* Let  $X$  be a complex manifold. Then  $TX \not\cong T(X_{\mathbb{R}}) \otimes_{\mathbb{R}} \mathbb{C}$ .

20. [CM] TALK 5 (DARIO) - SOLID ABELIAN GROUPS - 18.11.19

**Definition 20.1** Let  $S$  be a profinite set, given as  $\varprojlim_i S_i$ . Then we define

$$\mathbb{Z}[S]^{\blacksquare} = \varprojlim_i \mathbb{Z}[S_i],$$

called the free solid abelian group on  $S$ .

A condensed abelian group is called solid if... A complex of condensed abelian groups is called solid if...

Later we will show that  $C \in \mathbf{D}(\mathrm{Cond}(\mathbf{Ab}))$  is solid if and only if  $H^i(C)$  is solid for all  $i$ .

*Notation 20.2.* Denote  $\mathbf{Solid} \subseteq \mathrm{Cond}(\mathbf{Ab})$  the full subcategory of solid condensed abelian groups.

*Remark 20.3.*  $\mathbb{Z}[S]^{\blacksquare} = \varprojlim_i \mathbb{Z}[S_i] = \varprojlim_i \mathcal{H}om(C(S_i, \mathbb{Z}), \mathbb{Z}) = \mathcal{H}om(\varinjlim_i C(S_i, \mathbb{Z}), \mathbb{Z}) = \mathcal{H}om(C(S, \mathbb{Z}), \mathbb{Z})$ .

**Theorem 20.4** (Theorem 5.4)  $S$  profinite. Then  $C(S, \mathbb{Z})$  is a free abelian group.

*Sketch.* We can find a continuous injection  $S \hookrightarrow \prod_I \{0, 1\}$ . We can take  $I = \{Z \subseteq S \mid Z \text{ clopen}\}$ . Injectivity then is a consequence of total disconnectedness. The data of a continuous function  $\varphi: S \rightarrow \mathbb{Z}$  is equivalent to a decomposition  $S = \bigsqcup_{finite} S_i$  where  $S_i$  is clopen. So a generating system of  $C(S, \mathbb{Z})$  is given by... a basis is given by... (see Scholze's notes).  $\square$

**Corollary 20.5**  $\mathbb{Z}[S]^{\blacksquare} = \prod_I \mathbb{Z}$ .

*Proof.*  $\mathbb{Z}[S]^{\blacksquare} = \mathcal{H}om(C(S, \mathbb{Z}), \mathbb{Z}) = \prod_I \mathcal{H}om(\mathbb{Z}, \mathbb{Z}) = \prod_I \mathbb{Z}$ .  $\square$

**Proposition 20.6** (Proposition 5.7) If  $S$  is profinite, then  $\mathbb{Z}[S]^{\blacksquare}$  is solid in  $\mathbf{D}(\mathrm{Cond}(\mathbf{Ab}))$  and  $\mathrm{Cond}(\mathbf{Ab})$ .

<sup>12</sup>This example shows that being an embedding is not local on the source.

*Proof.* Have to check:  $R\mathrm{Hom}(\mathbb{Z}[T], \mathbb{Z}[T]^\blacksquare) \cong R\mathrm{Hom}(\mathbb{Z}[T]^\blacksquare, \mathbb{Z}[S]^\blacksquare)$ ,  $T$  profinite. Reduce  $\mathbb{Z}[S]^\blacksquare = \mathbb{Z}$ . From the previous lecture we know  $R\mathrm{Hom}(\mathbb{Z}[T], \mathbb{Z}) = R\Gamma(T, \mathbb{Z}) = C(T, \mathbb{Z})[0]$ . We have a SES  $0 \rightarrow \prod_J \mathbb{Z} \rightarrow \prod_J \mathbb{R} \rightarrow \prod_J \mathbb{R}/\mathbb{Z} \rightarrow 0$ , where  $\mathbb{Z}[T]^\blacksquare = \prod_J \mathbb{Z}$ . Apply  $R\mathrm{Hom}(-, \mathbb{Z})$ . We get a triangle

$$R\mathrm{Hom}(\prod_J \mathbb{R}/\mathbb{Z}, \mathbb{Z}) \rightarrow R\mathrm{Hom}(\prod_J \mathbb{R}, \mathbb{Z}) \rightarrow R\mathrm{Hom}(\prod_J \mathbb{Z}, \mathbb{Z}) \xrightarrow{+1} \dots$$

Note  $R\mathrm{Hom}(\prod_J \mathbb{R}, \mathbb{Z}) = R\mathrm{Hom}_{\mathbb{R}}(\prod_J \mathbb{R}, R\mathcal{H}om(\mathbb{R}, \mathbb{Z})) = 0$ . This follows from Theorem 4.3.  $R\mathrm{Hom}(\mathbb{R}/\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}[-1]$ , so  $R\mathrm{Hom}(\mathbb{R}, \mathbb{Z}) = 0$ . Thus  $R\mathrm{Hom}(\prod_J \mathbb{Z}, \mathbb{Z}) \cong R\mathrm{Hom}(\prod_J \mathbb{R}/\mathbb{Z}, \mathbb{Z})[-1]$ . By theorem 4.3. this is  $\bigoplus_J \mathbb{Z}[-1][1] = \bigoplus_J \mathbb{Z}$ . By 5.4 this is precisely  $C(T, \mathbb{Z})$ .  $\square$

So far we have a notion of solid and we have the notion of free solid abelian groups. The question is: can we solidify every condensed abelian group? Turns out we can:

**Theorem 20.7 (Theorem 5.8)** i) **Solid**  $\subseteq \mathrm{Cond}(\mathbf{Ab})$  is an abelian subcategory with all limits, colimits and extensions, projective generators  $\prod_I \mathbb{Z}$  coming from free solid abelian groups and with a left adjoint  $L = (-)^\blacksquare : \mathrm{Cond}(\mathbf{Ab}) \rightarrow \mathbf{Solid}$  to the inclusion **Solid**  $\subseteq \mathrm{Cond}(\mathbf{Ab})$  and it extends the functor  $\mathbb{Z}[S] \mapsto \mathbb{Z}[S]^\blacksquare$ .  
 ii) **D(Solid)**  $\subseteq \mathbf{D}(\mathrm{Cond}(\mathbf{Ab}))$  is fully faithful and has essential image precisely the solid objects in  $\mathbf{D}(\mathrm{Cond}(\mathbf{Ab}))$ . We have the same description as before, i.e. a complex is solid iff all its cohomologies are solid. We have moreover a left adjoint to this inclusion **D(Solid)**  $\subseteq \mathbf{D}(\mathrm{Cond}(\mathbf{Ab}))$ , denoted  $C \mapsto C^{\blacksquare, L}$ , which is actually the left derived functor of  $L$ .

**Lemma 20.8 (Lemma 5.9)** Let  $\mathbf{A}$  be an abelian category with colimits. Let  $\mathbf{A}_0 \subseteq \mathbf{A}$  be compact projective generators. Let  $F : \mathbf{A}_0 \rightarrow \mathbf{A}$  be a functor together with a natural transformation  $\mathrm{id} \rightarrow F$  such that: For objects  $Y, Z$  direct sums of images of  $F$ ,  $f : Y \rightarrow Z$  with kernel  $K$ , we have that

$$R\mathrm{Hom}(\mathcal{F}(X), K) \xrightarrow{\sim} R\mathrm{Hom}(X, K).$$

Let  $\mathbf{A}_F \subseteq \mathbf{A}$  be the full subcategory of all objects  $Y \in \mathbf{A}$  such that for all  $X \in \mathbf{A}_0$  we have

$$\mathrm{Hom}(F(X), Y) \xrightarrow{\sim} \mathrm{Hom}(X, Y).$$

Let  $\mathbf{D}_F(\mathbf{A}) \subseteq \mathbf{D}(\mathbf{A})$  be the full subcategory of all  $C \in \mathbf{D}(\mathbf{A})$  such that for  $X \in \mathbf{A}_0$  we have

$$R\mathrm{Hom}(F(X), C) \xrightarrow{\sim} R\mathrm{Hom}(X, C).$$

Then:  $\mathbf{A}_F \subseteq \mathbf{A}$  is an abelian category with colimits, limits and extensions. The images  $F(X)$  for  $X \in \mathbf{A}_0$  are compact, projective generators of  $\mathbf{A}_F$ . Have a left adjoint  $L : \mathbf{A} \rightarrow \mathbf{A}_F$  to the inclusion that extends our functor  $F : \mathbf{A}_0 \rightarrow \mathbf{A}_F$ . Similar for the derived category: we get a left adjoint to the inclusion  $\mathbf{D}_F(\mathbf{A}) \subseteq \mathbf{D}(\mathbf{A})$  which is the left derived functor of  $\mathbf{A} \rightarrow \mathbf{A}_F$ . Moreover,  $\mathbf{D}(\mathbf{A}_F) \rightarrow \mathbf{D}(\mathbf{A})$  is fully faithful and has essential image  $\mathbf{D}_F(\mathbf{A})$ . Furthermore,  $C \in \mathbf{D}(\mathbf{A})$  lies in  $\mathbf{D}_F(\mathbf{A})$  if and only if all  $H^i(C) \in \mathbf{A}_F$ .

**Lemma 20.9 (Lemma 5.10)** Let  $\mathbf{A}$  be an abelian category with colimits and  $\mathbf{A}_0 \subseteq \mathbf{A}$  be compact projective generators. Let  $F : \mathbf{A}_0 \rightarrow \mathbf{A}$  be a functor together with a natural transformation  $\mathrm{id} \rightarrow F$  such that for all  $X \in \mathbf{A}_0$  and all complexes  $C = \dots \rightarrow C_1 \rightarrow C_0 \rightarrow 0 \rightarrow \dots$  with  $C_i$  is direct sum of images of  $F$  we have

$$R\mathrm{Hom}(F(X), C) \xrightarrow{\sim} R\mathrm{Hom}(X, C).$$

Then we have for all  $X \in \mathbf{A}_0$  and all  $f : Y \rightarrow Z$  with kernel  $K$  and  $Y, Z$  direct sums of images of  $F$  that

$$R\mathrm{Hom}(F(X), K) \xrightarrow{\sim} R\mathrm{Hom}(X, K).$$



*Proof.* Let  $Y, Z, f$  and  $K$  as in the statement. Take a resolution

$$\cdots \rightarrow \oplus_{ij} X_{ij} \rightarrow \cdots \rightarrow K$$

with  $X_{ij} \in \mathbf{A}_0$ . Denote this projective resolution of  $K$  by  $B$ . We get a complex  $C = (\cdots \rightarrow \oplus F(X_{ij}) \rightarrow \cdots \rightarrow 0)$ . By assumption  $R\mathrm{Hom}(F(X_{ij}), Y) = R\mathrm{Hom}(X_{ij}, Y)$ . Then by a spectral sequence argument we have  $R\mathrm{Hom}(B, Y) = R\mathrm{Hom}(C, Y)$ . The same for  $Z$ . Thus,  $B \rightarrow C \rightarrow Y$  factors through  $B \rightarrow K \subseteq Y$ , so composing with  $Y \rightarrow Z$  we get 0. Hence  $B \xrightarrow{\sim} K$  (factors over  $C$ ) and  $B$  is a retract of  $C$ . The condition  $R\mathrm{Hom}(F(X), C) \xrightarrow{\sim} R\mathrm{Hom}(X, C)$  passes to  $B$ , thus to  $K$ .  $\square$

**20.1. Proof of theorem 5.8.** We take  $\mathbf{A}_0$  to be the full subcategory of objects  $\mathbb{Z}[S]$  for  $S$  extremally disconnected and  $\mathbf{A} = \mathrm{Cond}(\mathbf{Ab})$ . The functor  $F: \mathbf{A}_0 \rightarrow \mathbf{A}$  is  $\mathbb{Z}[S] \mapsto \mathbb{Z}[S]^\blacksquare$ . This is indeed a functor because  $\mathbb{Z}[S]^\blacksquare = \mathcal{H}om(C(S, \mathbb{Z}), \mathbb{Z}) = \mathcal{H}om(\mathcal{H}om(\mathbb{Z}[S], \mathbb{Z}), \mathbb{Z})$  is functorial on  $\mathbb{Z}[S]$ . We check the assumptions of lemma 5.10. Let  $C = (\cdots \rightarrow C_1 \rightarrow C_0 \rightarrow 0)$  be a complex with  $C_i$  a direct sum of  $F$ , i.e.  $C_i = \oplus \prod \mathbb{Z}$ . We want to show that  $R\mathrm{Hom}(\mathbb{Z}[S]^\blacksquare, C) = R\mathrm{Hom}(\mathbb{Z}[S], C)$  for  $S$  extremally disconnected. Let  $\mathcal{M}(S, \mathbb{Z}) = \mathcal{H}om(C(S, \mathbb{Z}), \mathbb{Z}) = \mathbb{Z}[S]^\blacksquare$ ,  $\mathcal{M}(S, \mathbb{R}) = \mathcal{H}om(C(S, \mathbb{Z}), \mathbb{R}) = \prod \mathbb{R}$  and  $\mathcal{M}(S, \mathbb{R}/\mathbb{Z}) = \mathcal{H}om(C(S, \mathbb{Z}), \mathbb{R}/\mathbb{Z}) = \prod \mathbb{R}/\mathbb{Z}$ . We have a SES

$$0 \rightarrow \mathcal{M}(S, \mathbb{Z}) \rightarrow \mathcal{M}(S, \mathbb{R}) \rightarrow \mathcal{M}(S, \mathbb{R}/\mathbb{Z}) \rightarrow 0.$$

The claim is that  $R\mathcal{H}om(\mathcal{M}(S, \mathbb{R}/\mathbb{Z}), C)(S') = R\Gamma(S \times S', C)[-1]$ . Let us see first why this claim implies that we are in the assumptions of lemma 5.10. Let  $S = \{*\}$ . Then  $R\mathcal{H}om(\mathbb{R}/\mathbb{Z}, C)(S') = R\Gamma(S', C)[-1] = R\mathrm{Hom}(\mathbb{Z}[S'], C)[-1] = R\mathcal{H}om(\mathbb{Z}, C)(S')[-1] = R\mathcal{H}om(\mathbb{Z}[1], C)(S')$ . This comes from the SES

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z} \rightarrow 0$$

which gives the triangle  $R\mathcal{H}om(\mathbb{R}/\mathbb{Z}, C) \rightarrow R\mathcal{H}om(\mathbb{R}, C) \rightarrow R\mathcal{H}om(\mathbb{Z}, C) \rightarrow \cdots$ . Since  $R\mathcal{H}om(\mathbb{R}, C) = 0$  we get  $R\mathcal{H}om(\mathbb{R}/\mathbb{Z}, C) \cong R\mathcal{H}om(\mathbb{Z}, C)[-1]$ . Thus  $R\mathcal{H}om(\mathcal{M}(S, \mathbb{R}), C) = R\mathcal{H}om(\mathcal{M}(S, \mathbb{R}), R\mathcal{H}om(\mathbb{R}, C)) = 0$ . We have a distinguished triangle

$$R\mathcal{H}om(\mathcal{M}(S, \mathbb{R}/\mathbb{Z}), C) \rightarrow R\mathcal{H}om(\mathcal{M}(S, \mathbb{R}), C) \rightarrow R\mathcal{H}om(\mathcal{M}(S, \mathbb{Z}), C)$$

The middle term is again zero, so

$$R\mathcal{H}om(\mathcal{M}(S, \mathbb{Z}), C) = R\mathcal{H}om(\mathcal{M}(S, \mathbb{R}/\mathbb{Z}), C)[1] \stackrel{\text{claim}}{=} R\Gamma(S \times S', C) = R\mathcal{H}om(\mathbb{Z}[S], C)(S')$$

For the claim: if  $X$  is a compact Hausdorff topological space, then  $\mathbb{Z}[X]$  is pseudo coherent, i.e. the ext-groups commute with filtered colimits. In particular, since  $\mathcal{M}(S, \mathbb{R}/\mathbb{Z}) = \prod \mathbb{R}/\mathbb{Z}$ , we get that  $\mathbb{Z}[\prod \mathbb{R}/\mathbb{Z}]$  is pseudo-coherent. Because of the resolution

$$\cdots \rightarrow \mathbb{Z}[\prod \mathbb{R}/\mathbb{Z}] \rightarrow \prod \mathbb{R}/\mathbb{Z} \rightarrow 0$$

we also have that  $\prod \mathbb{R}/\mathbb{Z}$  is pseudo coherent. Also, if  $\mathbb{Z}[S]$  is pseudo coherent, then so is  $\mathbb{Z}[S] \otimes \prod \mathbb{R}/\mathbb{Z}$ . Claim for  $C = M[0]$ ,  $M = \oplus \prod \mathbb{Z}$ .

$$\begin{aligned} R\mathcal{H}om(\mathcal{M}(S, \mathbb{R}/\mathbb{Z}), \oplus \prod \mathbb{Z}) &= \oplus R\mathcal{H}om(\mathcal{M}(S, \mathbb{R}/\mathbb{Z}), \prod \mathbb{Z}) = \\ &= \oplus \prod R\mathcal{H}om(\mathcal{M}(S, \mathbb{R}/\mathbb{Z}), \mathbb{Z}) \stackrel{LLM^{4.3}}{=} C(S \times S', \mathbb{Z})[-1] \end{aligned}$$

Hence the claim for  $C$  concentrated in one degree, which formally implies the claim for  $C$  bounded (with triangles and induction on the length). In general:  $C$  can be written as the

homotopy colimit of the truncations  $C_{\leq n}$  (see [?, Tag 093W]). Homotopy colimit means that we have a triangle

$$\oplus C_{\leq n} \xrightarrow{1-\oplus f_n} \oplus C_{\leq n} \rightarrow C$$

Apply  $R\mathcal{H}om(\mathcal{M}(S, \mathbb{R}/\mathbb{Z}), -)$  to this triangle. Then use pseudo coherence to take the direct sum out.

## 21. [LT] LECTURE 9 - 19.11.19

### 21.1. Fundamental examples.

**Proposition 21.1** *Let  $f: X \rightarrow Y$  be a smooth map of smooth manifolds. Then  $\Gamma_f = \{(x, f(x)) \mid x \in X\} \subseteq X \times Y$  is a closed embedded submanifold.*

*Proof.* It is closed by Hausdorffness. Let  $\varphi: X \rightarrow X \times Y$  sending  $x \mapsto (x, f(x))$  (which is smooth). Let  $\pi: X \times Y \rightarrow X$  be the projection onto the first factor (also smooth). Then  $\pi \circ \varphi = \text{id}_X$ , so  $d(\pi \circ \varphi) = (d\pi) \circ (d\varphi) = \text{id}_{TX}$ , so  $d\varphi_x$  is injective for all  $x \in X$ . Thus  $\varphi$  is an immersion.  $\square$

**Corollary 21.2**  *$X \subseteq \mathbb{CP}^n$  vanishing locus of a family of homogeneous polynomials. If the Jacobian matrix has maximal rank at each point of  $X$ , then  $X$  is an embedded submanifold of  $\mathbb{CP}^n$ .*

*Proof.* Use implicit function theorem and previous proposition.  $\square$

**Theorem 21.3 (Constant rank theorem)** *Let  $f: X \rightarrow Y$  be a smooth map of smooth manifolds and let  $m = \dim X$ ,  $n = \dim Y$ ,  $x \in X$  and  $y = f(x)$ . If  $f$  is of constant rank  $r$ , then we can find charts  $(U, \varphi)$  at  $x$  and  $(V, \psi)$  at  $y$  such that  $f(U) \subseteq V$  and for all  $t \in \varphi(U)$  we have  $(\psi \circ f \circ \varphi^{-1})(t_1, \dots, t_r, 0, \dots, 0)$ . [Here the rank of  $f$  at  $x$ , denoted  $\text{rk}_x(f) = \text{rk}(df_x)$ .]*

*Proof.* WLOG  $X = W \subseteq \mathbb{R}^m$ ,  $Y = W' \subseteq \mathbb{R}^n$  open and  $x = 0$ ,  $y = 0$ . Then  $df_0$  is the Jacobian matrix. Define  $\varphi: W \rightarrow \mathbb{R}^m$  by

$$\varphi(x) = (f_1(x), \dots, f_r(x), x_{r+1}, \dots, x_m)$$

for all  $x \in W$  (we may choose charts so that the first  $r$  coordinates are linearly independent).  $\varphi(0) = 0$  and

$$d\varphi_0 = \begin{pmatrix} \left( \frac{\partial f_k}{\partial x_l} \right) & * \\ 0 & I_{m-r} \end{pmatrix}.$$

Since  $d\varphi_0$  is invertible, by the inverse function theorem we can find  $0 \in W_1 \subseteq W$  open and  $0 \in U_1 \subseteq \mathbb{R}^m$  open such that

$$\varphi|_{W_1}: W_1 \rightarrow U_1$$

is a diffeomorphism. On  $U_1$  we have  $(f \circ \varphi^{-1})(x) = (x_1, \dots, x_r, \tilde{f}_{r+1}(x), \dots, \tilde{f}_n(x))$  for some smooth functions  $\tilde{f}_k$  with  $r < k \leq n$ . We have

$$((df) \circ (d\varphi^{-1}))_0 = d(f \circ \varphi^{-1})_0 = \begin{pmatrix} I_r & 0 \\ * & \left( \frac{\partial \tilde{f}_k}{\partial x_l} \right) \end{pmatrix}.$$

$d\varphi^{-1}$  is invertible, so  $\text{rk}(d(f \circ \varphi^{-1})) = \text{rk}(df) = r$  on  $U_1$ . Hence  $\left( \frac{\partial \tilde{f}_k}{\partial x_l} \right) = 0$ , so  $\tilde{f}_k$  is independent of  $x_{r+1}, \dots, x_n$ .  $T(y_1, \dots, y_n) := (y_1, \dots, y_r, y_{r+1} + \tilde{f}_{r+1}(y_1, \dots, y_r), \dots, y_n +$

$\tilde{f}_n(y_1, \dots, y_n)$ ). We have  $T(0) = 0$  and

$$dT_y = \begin{pmatrix} I_r & 0 \\ * & I_{n-r} \end{pmatrix}.$$

Hence  $T$  is a diffeomorphism from a nbhd of  $0 \in \tilde{V} \subseteq \mathbb{R}^n$  to a nbhd of  $0 \in V \subseteq W'$ . Choose  $\tilde{U} \subseteq U$  open such that  $(f \circ \varphi^{-1})(\tilde{U}) \subseteq V$ . Let  $U := \varphi^{-1}(\tilde{U})$  and  $\psi = T^{-1}$ . Consider

$$\tilde{U} \xrightarrow{\varphi^{-1}} U \xrightarrow{f} V \xrightarrow{\psi} \tilde{V}.$$

For  $x \in \tilde{U}$  we get  $(x_1, \dots, x_r, 0, \dots, 0)$  after applying the previous composition.  $\square$

Special cases:

- (1)  $f$  submersion.
- (2)  $f$  immersion.

In the first case, if  $f: X \rightarrow Y$  is a submersion, locally on  $X$  and on  $Y$  we have

$$\begin{aligned} \mathbb{R}^{n+r} &\longrightarrow \mathbb{R}^n \\ (t_1, \dots, t_{n+r}) &\longmapsto (t_1, \dots, t_n) \end{aligned}$$

**Proposition 21.4** If  $f: X \rightarrow Y$  is a smooth map of smooth manifolds and  $y \in Y$  is such that for all  $x \in f^{-1}\{y\}$  the map  $df_x$  is surjective, then  $f^{-1}\{y\} \subseteq X$  is an embedded submanifold.

*Proof.* Exercise!  $\square$

## 21.2. Ehresmann's theorem.

**Definition 21.5** A smooth map  $f: X \rightarrow Y$  is a *trivial fibration* if there exists a smooth manifold  $F$  and a diffeomorphism  $\varphi: X \rightarrow Y \times F$  such that

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \times F \\ & \searrow f & \swarrow \pi \\ & Y & \end{array}$$

We say that  $f$  is a *locally trivial fibration* if for all  $y \in Y$  there exists an open  $U \subseteq Y$  such that  $f^{-1}(U) \rightarrow U$  is a trivial fibration.

*Exercise 21.6.*  $\mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  sending  $(x, y) \mapsto x$  or  $\mathbb{T}^2 \rightarrow S^1$  projecting onto one of its factors.

*Exercise 21.7.* The Möbius strip is a locally trivial fibration which is not globally trivial.

**Theorem 21.8 (Ehresmann)** Let  $f: X \rightarrow Y$  be a smooth, surjective, submersive and proper map of smooth manifolds. Then  $f$  is a locally trivial fibration.

*Idea.* Locally on  $X$  we have trivializations, so we get some horizontal lines. We extend them across other neighbourhoods in a way that is compatible with the corresponding trivializations to obtain trivializations locally on  $Y$ .  $\square$

## 21.3. Proper maps.

**Definition 21.9** Let  $f: X \rightarrow Y$  be a function and  $X, Y$  topological spaces. We say that  $f$  is *proper* if for all  $K \subseteq Y$  quasi-compact we have that  $f^{-1}(K)$  is quasi-compact.

**Example 21.10**  $X \rightarrow *$  is proper if and only if  $X$  is quasi-compact.

**Example 21.11**  $\mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z} = S^1$  is not proper.

**Proposition 21.12** Let  $f: X \rightarrow Y$  continuous.

- (1) If  $f$  is closed with quasi-compact fibres, then  $f$  is proper.
- (2) If  $f$  is proper and  $Y$  is a compactly generated Hausdorff space, then  $f$  is closed.
- (3) If  $X$  is quasi-compact and  $Y$  is Hausdorff, then  $f$  is closed and proper.

*Proof.* For 1), let  $K \subseteq Y$  be q.c. The claim is that  $f^{-1}(K)$  is q.c. Let  $U_\alpha$  be an open cover of  $f^{-1}(K)$  and consider the fibre  $f^{-1}\{k\}$  for  $k \in K$ . We can find a finite subcover  $U_{\alpha_i}$  of the fibre. Since  $f$  is closed,  $f(X \setminus \cup_i U_{\alpha_i})$  is also closed. Hence  $V_k := Y \setminus f(X \setminus \cup_i U_{\alpha_i})$  is open. By construction  $k \in V_k$  for each  $k \in K$ , so  $V_k$  is an open cover of  $K$ . We can find a finite subcover by q.c. But then  $f^{-1}(K) \subseteq f^{-1}(V_{k_1} \cup \dots \cup V_{k_n}) \subseteq \cup_j U_{\alpha_{ij}}$ , so this is the desired finite subcover.

For 2), recall that compact generation means that we can test closedness by intersecting with all compact subspaces. Let  $Z \subseteq X$  be closed and  $K \subseteq Y$  be quasi-compact. Claim:  $f(Z) \cap K$  is closed. We know that  $f^{-1}(K)$  is q.c., so  $f^{-1}(K) \cap Z$  is q.c. Continuous image of q.c. is q.c., so  $f(f^{-1}(K) \cap Z) = K \cap f(Z)$  q.c. Since  $Y$  is Hausdorff,  $K \cap f(Z)$  is closed.

For 3), if  $Z$  is closed in a q.c. space, then it is q.c. Hence  $f(Z)$  is q.c. inside a Hd space, so it is again closed. Properness is left as an exercise.  $\square$



If  $f: X \rightarrow Y$  is proper and  $j: U \subseteq X$  is an open subset, then there is no reason to expect the composition  $U \rightarrow Y$  is proper. From this we can also build counterexamples to the Ehresmann theorem, e.g. remove a bunch of points from a covering space.

**21.4. Flows.** Let  $X$  be a smooth manifold.

**Definition 21.13** A *global flow* on  $X$  is a smooth map

$$\mathbb{R} \times X \rightarrow X$$

such that for all  $x \in X$  and for all times  $s, t \in \mathbb{R}$  we have

- i)  $\Phi(0, x) = x$ .
- ii)  $\Phi(s, \Phi(t, x)) = \Phi(s + t, x)$ .

**Example 21.14** Let  $X = \mathbb{R}$  and  $L: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  sending  $(s, t) \mapsto s + t$ .

**Example 21.15** Consider  $\Phi: \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$  sending

$$\left(t, \begin{pmatrix} p \\ q \end{pmatrix}\right) \mapsto e^{-t/2} \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}.$$

**Definition 21.16** Let  $\Phi$  be a flow on  $X$ . The *velocity field* of  $\Phi$  is the vector field

$$\begin{aligned} \vec{\Phi}: X &\longrightarrow TX \\ x &\longmapsto \frac{d}{dt}\bigg|_0 \Phi(t, x) \end{aligned}$$

We will see that all vector fields are velocity fields of flows.

**Definition 21.17** Let  $\Phi: \mathbb{R} \times X \rightarrow X$  be a flow and  $x, y \in X$ . We will say that  $x \sim y$  if and only if there exists some  $t \in \mathbb{R}$  such that  $\Phi(t, x) = y$ .

22. [WS] KODAIRA 5 - 20.11.19

**Theorem 22.1** If  $X$  is a Hodge manifold, then there exists an embedding into  $\mathbb{CP}^m$  for  $m \in \mathbb{N}$ .

Recall that a Hodge manifold is a Kaehler manifold such that the Kaehler form is positive and integral.

Last week we have seen that there exists  $\mu_0 \in \mathbb{N}$  such that

- i)  $\pi_p^* E^\mu \otimes L_p^* \otimes K_{Q_p(X)}^*$ ,
- ii)  $\pi_p E^\mu \otimes (L_p^*)^2 \otimes K_{Q_p(X)}^*$ , and
- iii)  $\pi_{p,q} E^\mu \otimes L_{p,q} \otimes K_{Q_{p,q}(X)}^*$

are positive. We get

$$X \longrightarrow \mathbb{P}(\Gamma(F))$$

for  $F = E^\mu$ .

For  $p \in X$  consider  $\mathfrak{m}_p \subseteq \mathcal{O}_X$ . We consider the exact sequence

$$0 \rightarrow \mathfrak{m}_p^2 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X/\mathfrak{m}_p^2 \rightarrow 0$$

and the exact sequence

$$0 \rightarrow \mathfrak{m}_{p,q} \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X/\mathfrak{m}_{p,q} \rightarrow 0,$$

where  $\mathfrak{m}_{p,q} = \{f \in \mathcal{O}_X \mid f(p) = f(q) = 0\}$ . Tensor both short exact sequences with  $\mathcal{O}_X(F)$ . We have

$$((\mathcal{O}_X/\mathfrak{m}_p^2) \otimes \mathcal{O}_X(F))_x = \begin{cases} \mathcal{O}_p/\mathfrak{m}_p^2 \otimes F_p & \text{if } x = p \\ 0 & \text{if } x \neq p \end{cases}$$

and

$$((\mathcal{O}_X/\mathfrak{m}_{p,q}) \otimes \mathcal{O}_X(F))_x = \begin{cases} F_p & \text{if } x = p \\ F_q & \text{if } x = q \\ 0 & \text{if } p \neq x \neq q \end{cases}$$

**Lemma 22.2**  $\mathcal{O}_p/\mathfrak{m}_p^2 \cong \mathbb{C} \oplus T_p^*(X)$  via  $f \mapsto f(p) + df(p)$ .

*Proof.* Choose local coordinates around  $p$ . Then

$$f(z) = f(p) + \sum_{|\alpha| \geq 1} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} f}{\partial z^\alpha}(p)(z-p)^\alpha,$$

whence

$$[f]_p = [f(p) + \sum_{|\alpha|=1} \frac{\partial f}{\partial z^\alpha}(p)(z-p)^\alpha]$$

□

Consider  $\Gamma(F) \xrightarrow{r} \mathcal{O}_X/\mathfrak{m}_p^2 \otimes \mathcal{O}_X(F)$ . For a local frame  $f \in \Gamma(U, F)$  and  $\xi \in \Gamma(F)$  we have

$$r(\xi) = (\xi(f)(p), d\xi(f)(p)) \otimes f(p).$$

Similarly we denote  $\Gamma(F) \xrightarrow{s} F_p \oplus F_q$ .

**Lemma 22.3** *If  $s$  and  $r$  are surjective for all  $p, q \in X$ , then there exists an embedding  $X \hookrightarrow \mathbb{CP}^m$  where  $m+1$  is the dimension of  $\Gamma(F)$ .*

*Proof.* Choose a basis  $\{\varphi_0, \dots, \varphi_m\}$  of  $\Gamma(F)$  and let  $p \in X$  and  $f \in \Gamma(U, F)$  a local frame. We define

$$\begin{aligned} \Phi_p : X &\longrightarrow \mathbb{CP}^m \\ x &\longmapsto [\varphi_0(f)(x), \dots, \varphi_m(f)(x)] \end{aligned}$$

This is well defined by surjectivity of  $r$ , i.e. we can find some  $i$  such that  $\varphi_i(f)(p) \neq 0$ . Let  $\tilde{f} = cf$  be a different local frame, where  $c$  is a non-vanishing holomorphic function. Then  $\varphi_i(\tilde{f}) = c\varphi_i(f)$ , so  $[\varphi_0(\tilde{f})(x), \dots] = [\varphi_0(f)(x), \dots]$ . For a basis  $\{\tilde{\varphi}_i = \sum_j c_{ij}\varphi_j\}$  of  $\Gamma(F)$ , we have the following:

$$\begin{array}{ccc} & & \mathbb{CP}^m \\ & \nearrow \Phi_\varphi & \downarrow C=[c_{ij}] \\ X & \xrightarrow{\Phi_{\tilde{\varphi}}} & \mathbb{CP}^m \end{array}$$

Since  $C$  is non-singular, the map on the right is biholomorphic. To prove that  $\Phi_\varphi$  is an immersion at  $p \in X$ , we choose  $\xi_0, \dots, \xi_n \in \Gamma(F)$  (for  $n = \dim(X)$ ) such that  $\xi_0(f)(p) = 1$ ,  $\xi_i(f)(p) = 0$  and  $d\xi_i(f)(p) = dz^i$  for  $f \in \Gamma(U, F)$  a local frame and for  $z$  a local basis of  $X$  around  $p$ . We complete  $\xi_0, \dots, \xi_n$  to a basis of  $\Gamma(F)$ , call this basis  $\tilde{\varphi}$ , and we consider  $\Phi_{\tilde{\varphi}}$ . We choose homogeneous coordinates  $(1, x_1, \dots, x_n, \dots)$  in  $\mathbb{CP}^m$ . Then

$$\begin{aligned} \left( \det \left( \frac{\partial(x_1, \dots, x_n)}{\partial(z_1, \dots, z_n)} \right) dz^1 \wedge \dots \wedge dz^n \right) |_p &= \\ &= d \left( \frac{\xi_1(f)}{\xi_0(f)} \right) |_p \wedge \dots \wedge d \left( \frac{\xi_n(f)}{\xi_0(f)} \right) |_p = dz^1 \wedge \dots \wedge dz^n. \end{aligned}$$

Since  $\Phi_{\tilde{\varphi}}$  is an immersion,  $\Phi_\varphi$  is an immersion. Let  $p, q \in X$ . Since  $s$  is surjective, we find  $\xi_0, \xi_1 \in \Gamma(F)$  such that  $\xi_0(p) \neq 0 \neq \xi_1(q)$ ,  $\xi_0(p) \neq 0 \neq \xi_1(q)$ ,  $\xi_1(p) = 0 = \xi_0(q)$ . We get a complete basis  $\tilde{\varphi}$ .  $\Phi_{\tilde{\varphi}}(p) \neq \Phi_{\tilde{\varphi}}(q)$ .  $\square$

**Lemma 22.4** *Let  $X$  be a Hodge manifold.  $r, s$  are surjective for all points  $p, q \in X$ .*

*Proof.* Let  $p \in X$ . Consider the blowup  $\pi_p: Q_p(X) \rightarrow X$  and denote  $\tilde{F} = \pi_p^*(F)$  the pullback. Denote also the exceptional fibre by  $S = \pi_p^{-1}(\{p\})$  and  $J_S = \{f \in \mathcal{O}_{Q_p(X)} \mid f|_S = 0\}$ . Consider the exact sequence

$$0 \rightarrow \mathcal{O}(\tilde{F}) \otimes J_S^2 \rightarrow \mathcal{O}(\tilde{F}) \rightarrow \mathcal{O}(\tilde{F}) \otimes \mathcal{O}_{Q_p(X)}/J_S^2 \rightarrow 0$$

and the exact sequence

$$0 \rightarrow \mathcal{O}(F) \otimes \mathfrak{m}_p^2 \rightarrow \mathcal{O}(F) \rightarrow \mathcal{O}(F) \rightarrow \mathcal{O}(F) \otimes \mathcal{O}_{Q_p(X)}/\mathfrak{m}_p^2 \rightarrow 0.$$

Now consider the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Gamma(\mathcal{O}(\tilde{F}) \otimes J_S^2) & \longrightarrow & \Gamma(\mathcal{O}(\tilde{F})) & \longrightarrow & \Gamma(\mathcal{O}(\tilde{F}) \otimes \mathcal{O}_{Q_p(X)}/J_S^2) \\ & & \pi_p^* \uparrow \alpha & & \pi_p^* \uparrow \beta & & \pi_p^* \uparrow \\ 0 & \longrightarrow & \Gamma(\mathcal{O}(F) \otimes \mathfrak{m}_p^2) & \longrightarrow & \Gamma(\mathcal{O}(F)) & \longrightarrow & \Gamma(\mathcal{O}(F) \otimes \mathcal{O}_{Q_p(X)}/\mathfrak{m}_p^2) \end{array}$$

Let  $\xi \in \Gamma(\mathcal{O}(\tilde{F}))$ ,  $\tilde{\beta}(\xi)(x) = \Pi_p \xi(\pi_p^{-1}(x))$  for all  $x \in X \setminus \{p\}$ , where  $\Pi_p: \tilde{F} \rightarrow F$ . For  $n = 1$ , nothing happens, so  $\beta = \text{id}$ . For  $n > 1$  we have  $\tilde{\beta}(\xi) \in \Gamma(X \setminus \{p\}, F)$ . By Hartog's theorem, there exists a unique holomorphic extension  $\tilde{\beta}(\xi)$  to  $\Gamma(X, F)$ . Denote it by  $\beta(\xi)$ . Then  $\pi_p^{-1} = \beta$ . We have  $\alpha = \beta|_{\Gamma(\mathcal{O}(\tilde{F}) \otimes J_S)}$ . We are left to show that  $H^1(\mathcal{O}(\tilde{F}) \otimes J_S^2) = 0$ .  $J_S$  is the sheaf of a line bundle.  $(z^i, t) = S = \{t = 0\}$ . If  $L$  is the line bundle associated to  $S$ , then  $J_S$  is the sheaf  $L^\vee$ .  $H^1(\mathcal{O}(\tilde{F}) \otimes J_S^2) = H^1(\mathcal{O}(\tilde{F}) \otimes (L^\vee)^2)$ .

Recall that  $F = E^\mu$  for  $\mu \geq \mu_0$  such that

$$\pi_p^* E^\mu \otimes (L^\vee)^2 \otimes K_{Q_p(X)}^\vee > 0.$$

By Kodaira-vanishing we have  $H^1(\mathcal{O}(\tilde{F}) \otimes (L^\vee)^2) = 0$ , hence we can complete the exact sequence above to

$$0 \rightarrow \Gamma(\mathcal{O}(\tilde{F}) \otimes J_S^2) \rightarrow \Gamma(\mathcal{O}(\tilde{F})) \rightarrow \Gamma(\mathcal{O}(\tilde{F}) \otimes \mathcal{O}/J_S^2) \rightarrow 0.$$

This implies that  $r$  is surjective for all  $p, q \in X$ .

It remains to show that  $s$  is surjective for all  $p, q \in X$ . Let  $\pi_{p,q}: \tilde{X} = Q_{p,q}(X) \rightarrow X$  be the blowup at these two points and  $\pi_{p,q}^*F =: \tilde{F}$ . Let  $J_S = \{f \in \mathcal{O}_{\tilde{X}} \mid f|_{\pi_{p,q}^{-1}(\{p,q\})} = 0\}$ . We get a “horizontal ladder” commutative diagram as in the previous case, in which we again get a zero on the right of the first row. This means that  $s$  is surjective.  $\square$

### 23. [LT] LECTURE 10 - 21.11.19

**23.1. Flows, vector fields and integral curves.** Let  $X$  be a smooth manifold.

**Definition 23.1** Let  $\gamma: \mathbb{R} \rightarrow X$  be smooth. The *velocity vector*  $\frac{d\gamma}{dt}|_{t=s}$  of  $\gamma$  at  $s \in \mathbb{R}$  is

$$\frac{d\gamma}{dt}|_{t=s} := d\gamma(\vec{L}(s)) = [\gamma L_s] = [t \mapsto \gamma(s+t)] \in T_{\gamma(s)}X,$$

where  $L: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  sends  $(s, t) \mapsto s + t$  and  $L_s: \mathbb{R} \rightarrow \mathbb{R}$  sends  $t \mapsto t + s$ , and  $\vec{L}$  denotes the velocity vector field of  $L$ .

**Definition 23.2** Let  $\xi: X \rightarrow TX$  be a smooth vector field. An *integral curve* of  $\xi$  starting at  $p \in X$  is a smooth path  $\gamma: J \rightarrow X$  defined on some open interval  $0 \in J \subseteq \mathbb{R}$  such that

- (1) For all  $s \in J$  we have  $\frac{d\gamma}{dt}|_{t=s} = \xi(\gamma(s))$ .
- (2)  $\gamma(0) = p$ .

**Example 23.3**  $X = \mathbb{R}^2$  and  $\xi = \frac{\partial}{\partial x}$ ,  $p = (a, b)$  and  $\gamma(t) = (t + a, b)$ .

**Example 23.4**  $X = \mathbb{R}^2$ ,  $\xi = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$ .  $\gamma(t) = (x(t), y(t))$ ,  $p = (a, b)$ . We need to solve the differential equation  $\gamma'(t) = \xi(\gamma(t))$  and  $\gamma(0) = (a, b)$ . The first condition is equivalent to

$$x'(t) \frac{\partial}{\partial x}|_{\gamma(t)} + y'(t) \frac{\partial}{\partial y}|_{\gamma(t)} = x(t) \frac{\partial}{\partial x} + y(t) \frac{\partial}{\partial y}$$

and the second one is equivalent to  $x(0) = a$  and  $y(0) = b$ . Hence

$$\gamma(t) = (ae^t, be^t).$$

**Remark 23.5.** There are vector fields that are not velocity vector fields of global flows.



**Example 23.6**  $X = \mathbb{R}_{<0} \times \mathbb{R}$ ,  $\xi = \frac{\partial}{\partial x}$ . Integral curve  $\gamma(t) = (t + a, b)$  but  $\gamma$  is only defined for  $t < -a$ .

**Definition 23.7** A *flow domain*  $D$  is an open  $D \subseteq \mathbb{R} \times X$  such that

- (1) For all  $p \in X$  the set  $D_p := \{t \in \mathbb{R} \mid (t, p) \in D\}$  is an open interval.
- (2) For all  $p \in X$  we have  $0 \in D_p$ .

**Definition 23.8** A *local flow*  $\Phi: D \rightarrow X$  is a smooth map such that

- (1)  $D$  is a flow domain.
- (2) For all  $p \in X$  we have  $\Phi(0, p) = p$ .
- (3) For all  $p \in X$ , all  $s \in D_p$  and all  $t \in D_{\Phi(s, p)}$  we have

$$t + s \in D_p \Rightarrow \Phi(t, \Phi(s, p)) = \Phi(t + s, p).$$

**Remark 23.9.** Local flows have velocity vector fields defined just as for global flows.

**Definition 23.10** A local flow  $\Phi: D \rightarrow X$  is *maximal* if there is no flow domain  $E$  strictly containing  $D$  together with an extension  $\bar{\Phi}$  of the flow  $\Phi$  to  $E$ .

*Exercise 23.11.* If  $\Phi: D \rightarrow X$  is a local flow, then  $\vec{\Phi}$  is a smooth vector field and for all  $p \in X$ , and for all  $p \in X$ ,  $\Phi^{(p)} := \Phi(-, p)$  is an integral curve of  $\vec{\Phi}$  starting at  $p$ .

**Theorem 23.12 (ODE existence, uniqueness and smoothness)** Let  $U \subseteq \mathbb{R}^n$  be an open set and  $\xi: U \rightarrow \mathbb{R}^n$  be a smooth map. Let  $(t_0, x_0) \in \mathbb{R} \times U$ . For each  $0 < \varepsilon < 1$  there exists  $x_0 \in U_0 \subseteq U$  open neighbourhood and a smooth map

$$\Phi: (t_0 - \varepsilon, t_0 + \varepsilon) \times U_0 \rightarrow U$$

such that for all  $x \in U_0$ ,  $\gamma(t) := \Phi(t, x)$  is the unique solution to the initial value problem given by  $\gamma'_k(t) = \xi_k(\gamma(t))$  and  $\gamma_k(t_0) = x_k$  on the interval  $(t_0 - \varepsilon, t_0 + \varepsilon)$ .

For a proof, see Lee's *Introduction to smooth manifolds*, appendix D.

**Lemma 23.13** Let  $\xi$  be a smooth vector field on  $X$ , and  $\gamma_0, \gamma_1: J \rightarrow X$  integral curves of  $\xi$  on an open interval  $J \subseteq \mathbb{R}$ . Let  $S := \{t \in J \mid \gamma_0(t) = \gamma_1(t)\}$ . If  $S \neq \emptyset$ , then  $S = J$ , i.e.  $\gamma_0 = \gamma_1$ .

*Proof.* Let  $t_0 \in S$ . Since  $X$  is Hausdorff, the set  $S$  is closed. For all  $t_1 \in S$ ,  $\gamma_0$  and  $\gamma_1$  are solutions of the same ODE with initial condition  $\gamma_0(t_1) = p = \gamma_1(t_1)$ . By the ODE theorem, the two integral curves agree on an open neighbourhood of  $t_1$ . Hence  $S$  is also closed in  $J$ . Since  $J$  is connected, we conclude  $S = J$ .  $\square$

**Theorem 23.14 (Flow theorem)** Let  $\xi$  be a smooth vector field on  $X$ . Then there exists a unique maximal local flow  $\Phi: D \rightarrow X$  such that  $\vec{\Phi} = \xi$ .

**Lemma 23.15 (Escape lemma)** Let  $K \subseteq X$  be quasi-compact.  $\Phi: D \rightarrow X$  maximal local flow on  $X$  such that for some  $p \in X$  we have  $D_p = (a_p, b_p) \subseteq \mathbb{R}$  with  $b_p < \infty$ . Then  $\exists \varepsilon > 0$  such that  $\Phi(t, p) \notin K$  for  $t > b_p - \varepsilon$ .

*Proof.* Let  $K$  be q.c. Then  $\exists \varepsilon > 0$  such that  $[-\varepsilon, \varepsilon] \times K \subseteq D \cap (\mathbb{R} \times K)$ . Indeed, choose an open cover  $\{(-\delta_\alpha, \delta_\alpha) \times U_\alpha\}_{\alpha \in A}$  of  $\{0\} \times K$  by subsets of  $D$  with  $\delta_\alpha > 0$  and  $U_\alpha \subseteq X$  open. Choose a finite subcover corresponding to a finite subset  $A' \subseteq A$  and let  $0 < \varepsilon < \min_{\alpha \in A'} \delta_\alpha$ . If  $T \in (b_p - \varepsilon, b_p)$  has  $\Phi(T, p) \in K$ , then let  $\Phi(t, p) := \Phi(t - T, \Phi(T, p))$  for  $t \in [T, T + \varepsilon]$ . This extends  $\Phi$  to a larger flow domain, so  $\Phi$  was not maximal. This contradicts our assumption.  $\square$

Now we can prove Ehresmann's fibration theorem!

**Theorem 23.16 (Ehresmann's fibration)** Let  $f: X \rightarrow Y$  be a proper submersion of smooth manifolds. Then  $f$  is a locally trivial fibration.

*Proof.* WLOG  $Y = \mathbb{R}^n$  (being proper/submersive is local on the target). Submersive maps are open maps (exercise) and last time we showed that proper maps are closed, so  $f(X) \subseteq Y$  is open and closed. Since  $\mathbb{R}^n$  is connected,  $f(X) = \mathbb{R}^n$  or  $X = \emptyset$  (which a priori was not ruled out from our definition of manifolds). WLOG  $X \neq \emptyset$ . This reduces the claim to the following special case.

**Lemma 23.17** Let  $f: X \rightarrow \mathbb{R}^n$  surjective proper submersion. Then  $f$  is a trivial fibration,

*Proof.* Constant rank theorem: for all  $p \in X$  there exist charts  $\varphi_p: U_p \rightarrow V_p$  for  $U_p \subseteq X$  and  $V_p \subseteq \mathbb{R}^{n+k}$  opens such that



$$\begin{array}{ccc}
U_p & \xrightarrow{f} & \mathbb{R}^n \\
\downarrow \varphi_p & & \parallel \\
V_p \subseteq \mathbb{R}^{n+m} & \xrightarrow{\pi} & \mathbb{R}^n
\end{array}$$

$$(x_1, \dots, x_{n+}) \longmapsto (x_1, \dots, x_n)$$

Choose a partition of unity  $\{\psi_k\}_{k \geq 0}$  subordinate to the cover  $\{U_p\}_{p \in X}$ . For all  $k \geq 0$ , fix  $p \in X$  such that  $\text{Supp}(\psi_k) \subseteq U_p$  and set  $\varphi_k := \varphi_p$ . For all  $1 \leq i \leq n$ , let  $\xi_i: X \rightarrow TX$  be the smooth vector field given by

$$\begin{aligned}
\xi_i(q) &= \sum_k \psi_k(q) d(\varphi_k^{-1}) \frac{\partial}{\partial x_i} (\varphi_k(q)) = \\
&= \sum_k \psi_k(q) [t \mapsto \varphi_k^{-1}(\varphi_k(q) + t e_i)]
\end{aligned}$$

$f|_{U_k} = \pi \circ \varphi_k$ . Goal:  $df \xi_i(q) = \frac{\partial}{\partial x_i} (f(q))$ .

$$\begin{aligned}
df \xi_i(q) &= \sum_k \psi_k(q) d(f \varphi_k^{-1}) \frac{\partial}{\partial x_i} (\varphi_k(q)) = \\
&= \sum_k \psi_k(q) d\pi_k \frac{\partial}{\partial x_i} (\varphi_k(q)) = \sum_k \psi_k(q) \frac{\partial}{\partial x_i} (f(q)) = \\
&= \frac{\partial}{\partial x_i} (f(q))
\end{aligned}$$

This shows that  $\xi_i$  lifts  $\frac{\partial}{\partial x_i}$ . Fix  $1 \leq i \leq n$ . Then  $\bar{\gamma}_i(t) := u + t e_i$  unique solution to the ODE  $\bar{\gamma}'_i(t) = e_i$  and  $\bar{\gamma}_i(0) = u$ .  $\Phi_i: D_i \rightarrow X$  is locally given by  $\xi_i$ .  $D_{i,q} := \{t \in \mathbb{R} \mid (t, q) \in D_i\}$ .  $\gamma_i(t) := \Phi_i(t, q)$ . We have a commutative diagram

$$\begin{array}{ccccc}
D_{i,q} & \xrightarrow{\gamma_i} & X & \xrightarrow{f} & \mathbb{R}^n \\
& \searrow \gamma'_i & \downarrow \xi_i & & \downarrow \frac{\partial}{\partial x_i} \\
& & TX & \xrightarrow{df} & T\mathbb{R}^n
\end{array}$$

$df(\gamma'_i) = \frac{d}{dt}(f \circ \gamma_i)$ ,  $(f \circ \gamma_i)(0) = f(q)$ . By ODE uniqueness,  $f(\Phi_i(t, q)) = f(\gamma_i(t)) = f(q) + t e_i$ .

Claim:  $D_i = \mathbb{R} \times X$ , i.e.  $\Phi_i$  is global.  $f(\Phi_i(t, q)) = f(q) + t e_i$ , so for all  $a, b \in \mathbb{R}$  the image  $f(\Phi_i((a, b), q))$  is contained in some q.c.  $K \subseteq \mathbb{R}^n$ . Hence  $\Phi_i((a, b), q) \subseteq f^{-1}(K)$  q.c. (by properness). By the escape lemma,  $D_{i,q} = \mathbb{R}$ .

Knowing that the flow is global we can define an explicit diffeomorphism as follows:

$$\begin{aligned}
\tau: \mathbb{R}^n \times f^{-1}(0) &\longrightarrow X \\
(t, q) &\longmapsto ((\Phi_1)_{t_1} \circ (\Phi_2)_{t_2} \circ \dots \circ (\Phi_n)_{t_n})(q)
\end{aligned}$$

The inverse of this diffeomorphism is given by the formula

$$\tau^{-1}(q) = (f(q), ((\Phi_n)_{-f_n(q)} \circ \dots \circ (\Phi_1)_{-f_1(q)})(q)).$$

□

□

## 24. [FS] ERWAN ROUSSEAU - AUTOMORPHISMS OF FOLIATIONS - 22.11.19

Joint work with F. Lo Bianco, J. Pereira and F. Tauzet.

- Example 24.1** (1) Let  $X$  be projective over  $\mathbb{C}$  and  $\pi: X \rightarrow B$ , a foliation  $\mathcal{F}$  onto a projective curve  $B$  with  $g(B) \geq 2$ . Let  $\text{Aut}(X, \mathcal{F}) \subseteq \text{Aut}(X)$  be the subgroup of automorphisms preserving  $\mathcal{F}$ . We have a map  $\text{Aut}(X, \mathcal{F}) \rightarrow \text{Aut}(B)$ . The second one is finite because  $B$  is a hyperbolic projective curve. Hence  $\exists$  finite index subgroup in  $\text{Aut}(X, \mathcal{F})$  which leaves each leaf invariant. Thus we have a finite transverse action.
- (2) Let  $\pi: X \rightarrow \mathbb{P}^1$  a foliation,  $\kappa(X) \geq 0$ , then  $\text{Aut}(X, \mathcal{F})$  has a finite transverse action. Corollary of Viehweg-Zuo:  $\pi$  has at least 3 singular fibres.  $\text{Aut}(X, \mathcal{F}) \rightarrow \text{Aut}(\mathbb{P}^1, p_1 + \dots + p_s)$ , where  $p_i$  correspond to the singular fibres, with  $s \geq 3$ , and  $K_{\mathbb{P}^1} + \sum(p_i) > 0$  (log version of general type). Hence  $\text{Aut}(\mathbb{P}^1, p_1 + \dots + p_s)$  is finite.
- (3) Lo Bianco:

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \downarrow \pi & & \downarrow \pi \\ B & \xrightarrow{f_B} & B \end{array}$$

such that  $\kappa(X) \geq 0$  and  $\exists L \rightarrow B$  big line bundle preserved by  $f_B$ . Hence  $f_B$  is terzian.

*Question 24.2.* Can we generalize these statements to more general fibrations?

$g(B) \geq 2 \Rightarrow$  *transverse hyperbolic foliations*.  $(X, \mathcal{F})$  codimension 1, holomorphic.  $\mathcal{F}$  is defined by local first integrals  $F_i: U_i \rightarrow \mathbb{D}$ , which are uniquely define modulo  $\text{Aut}(\mathbb{D})$ , i.e. we can glue them with relations of the form  $F_j = \psi_{ij} \circ F_i$ .

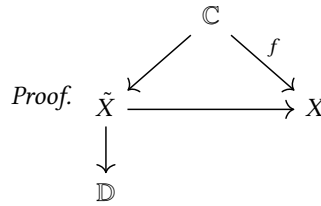
Associated to a foliation, we have two natural objects to consider. There is a monodromy action, so we pass to the universal cover. We get the *developing map*  $\pi: \tilde{X} \rightarrow b b D$ . The second natural object is the monodromy  $\rho: \pi_1(X) \rightarrow \text{PSL}_2(\mathbb{R})$ , which is the obstruction to the analytic continuation, i.e. when we go along a path the result differs by an element of  $\text{PSL}_2(\mathbb{R})$ .

**Theorem 24.3** *Let  $X$  be a projective manifold and  $\mathcal{F}$  a transversally hyperbolic  $\mathcal{F}$ . Then  $\text{Aut}(X, \mathcal{F})$  is transversally finite (i.e. there is a finite index subgroup which preserves every fibre).*

*Proof.* We have 2 cases analyzing the monodromy. Let  $G = \rho(\pi_1(X))$  be the image of the monodromy. First case would be that it is a compact lattice. And the second case is that it is dense (in the euclidean topology). In the first case we have a foliation  $\pi: X \rightarrow C = \mathbb{D}/G$  (hyperbolic up to an etale cover of  $X$ ). In the second case, use a result of Calette-Simpson ( $X$  projective): there exists a morphism  $\psi: X \rightarrow Y = \mathbb{D}^N/\Gamma$  such that  $\mathcal{F} = \psi^* \mathcal{F}_i$ , where  $\mathcal{F}_i$  is called Hilbert modular foliation.  $\psi: X \rightarrow Z = \psi(X) \subseteq Y = \mathbb{D}^N/\Gamma$ . By a result of Burenbarbe-Cadael:  $Z$  is of general type. We pass to the Stein factorization  $\pi: X \rightarrow B$  (such that  $\psi: X \xrightarrow{\pi} B \rightarrow Z$ ). To conclude,  $\pi$  is  $\text{Aut}(X, \mathcal{F})$  invariant  $\Leftrightarrow \mathcal{F}_i$  is purely transcendental, i.e. the leaves do not contain algebraic curves.  $\square$

From the proof one sees that we can deduce the following:

**Theorem 24.4** *Let  $(X, \mathcal{F})$  be transversally hyperbolic. Then any entire curve  $f: \mathbb{C} \rightarrow X$  is not Zariski dense.*



Hence  $f$  has to be tangent to  $\mathcal{F}$ . If the leaf is algebraic, then we are good. If not, we use argument as in the previous theorem.  $\square$

In the language of Campana, a manifold with the previous property is called *non-special*.

Recall: Lang has a beautiful dictionary for general type stuff. Let  $X$  be projective  $/\mathbb{C}$ . Define  $\text{Exc}(X) = \overline{\cup f(\mathbb{C})}$  where  $f: \mathbb{C} \rightarrow X$  are entire curves and the closure is taken in the Zariski topology. His conjecture is

$$X \text{ is of general type} \Leftrightarrow \text{Exc}(X) \neq X.$$

There are also connections to arithmetic:  $X$  of general type should correspond to the fact that  $X$  is pseudo-mordellic, i.e. that the set of rational points is not dense.

Now extend to the non-special case:  $X = \mathbb{P}^1 \times C_{g \geq 2}$  ( $\text{Exc}(X) = X$ ). By Faltings, the rational points can only show up in finitely many fibres. We can always lift an entire curve  $f: \mathbb{C} \rightarrow X$  to the projectivized tangent bundle  $\mathbb{P}(T_X) \rightarrow X$  by taking its derivative  $f_{[1]}: \mathbb{C} \rightarrow \mathbb{P}(T_X)$ . Set

$$\text{Exc}_1(X) = \overline{\cup f_{[1]}(\mathbb{C})}$$

with the closure taken on the Zariski topology.

**Conjecture 24.5**  $X$  non-special iff  $\text{Exc}_1(X) \neq X$ .

**Question 24.6.** If we have a projective manifold  $X$  and  $\mathcal{F}$  a foliation such that any entire curve  $f: \mathbb{C} \rightarrow X$  is tangent to  $\mathcal{F}$ , is any entire curve non-Zariski dense?

Both statements can be false if  $X$  is not Kaehler!



**Example 24.7** (Oeljeklaus-Toma manifolds) Let  $K$  be a number field,  $\sigma_1, \dots, \sigma_s$  real embeddings,  $\sigma_{s+1}, \dots, \sigma_{s+2t}$  complex embeddings. Assume  $t > 0$ .  $a \in \mathcal{O}_K$  act on  $\mathbb{H}^s \times \mathbb{C}^t$  as a translation by  $(\sigma_1(a), \dots, \sigma_{s+t}(a))$ .  $u \in \mathcal{O}_K^\times$  act by multiplication. The amalgamated product of  $U$  and  $\mathcal{O}_K$  acts on  $\mathbb{H}^s \times \mathbb{C}^t$ , so  $\exists U$  such that  $X(K, U)$  is a compact complex manifold.  $\mathcal{O}_K^{*,+}/U$ . We get automorphisms of  $X(K, U)$ , and then we can apply Dirichlet's theorem.

## 25. [CM] TALK 6 (VINCENT) - SOLID ABELIAN GROUPS II - 25.11.19

**Goal:** Corollaries of the big theorem of last time, solidified tensor product and some computations.

Reference for details of the next proof:  $\beta S$  as a topological space.

**Corollary 25.1** Compact projective objects in Solid are precisely the condensed abelian groups

$$\prod_I \mathbb{Z}$$

for arbitrary indexing set  $I$ .

*Proof.* We know  $S = \beta S_0$ ,  $|S_0| \gg 0$ . We get  $\mathbb{Z}[S]^\blacksquare \cong \prod_I \mathbb{Z}$ , and  $|I| > |S_0|$ .

1. The first claim is that  $I$  can be taken arbitrarily large. For this we bound it below by the cardinality of  $S_0$  and then show that  $S_0$  can be taken arbitrarily large. Clopen subsets forming a basis of  $S = \beta S_0$  are *Stone sets*

$$\begin{aligned} \hat{A} &\xrightarrow{1:1} A \subseteq S_0 \quad \hat{p} \text{ for each } p \in S_0 \\ \hat{A} \subseteq \hat{B} &\Leftrightarrow A \subseteq B \end{aligned}$$

Basis of  $C(S, \mathbb{Z})$  is given by minimal words in  $e_\mu$  for  $\mu \subseteq S$  clopen. The functions  $e_\mu$  are part of this basis (not completely obvious), so we get our bound.

2. If  $I$  is given,  $I_0 \rightarrow I, \mathbb{Z}^I \rightarrow \mathbb{Z}^{I_0}$  admits a retract, so  $\mathbb{Z}^I$  is projective and compact.

The other direction is omitted.  $\square$

**Corollary 25.2**  $\mathbf{D}(\mathbf{Solid})$  is compactly generated and compact objects are bounded complexes with terms  $\prod_I \mathbb{Z}$ . We have an equivalence of categories  $\mathbf{D}(\mathbf{Solid})^w \xrightarrow{\sim} \mathbf{D}^b(\mathbb{Z})^{op}$  sending  $C \mapsto R\mathbf{Hom}(C, \mathbb{Z})$  from left to right and  $C' \mapsto R\mathbf{Hom}(C', \mathbb{Z})$ .

*Proof.* Key ingredient in this proof: Stacks Project, tag 07LT and tag 07VI. The idea is then first get  $\mathbf{D}(\mathbf{Solid})^w \cong (\mathbf{D}(\mathbf{Solid})^w)^\vee$  sending  $C \mapsto R\mathbf{Hom}(C, \mathbb{Z})$ . Then get  $(\mathbf{D}(\mathbf{Solid})^w)^\vee \cong \mathbf{D}^b(\mathbb{Z})^{op}$  sending  $R\mathbf{Hom}(C, \mathbb{Z}) \mapsto R\mathbf{Hom}(C, \mathbb{Z})$ .  $\square$

*Remark 25.3.*  $\cdots \rightarrow 0 \rightarrow \prod_I \mathbb{Z} \rightarrow 0 \rightarrow \cdots \mapsto R\mathbf{Hom}(\prod_I \mathbb{Z}, \mathbb{Z}) \xrightarrow{p.34} \cdots \rightarrow 0 \rightarrow \oplus_I \mathbb{Z} \rightarrow 0 \rightarrow \cdots$ .

**Corollary 25.4** The derived solidification of  $\mathbb{R}$  is trivial, i.e.  $\mathbb{R}^{L\blacksquare} = 0$ .

*Proof.*  $\mathbf{Hom}_{\mathbf{D}(\mathbf{Solid})}(\mathbb{R}^{L\blacksquare}, C) = 0$  for all solid  $C$ . Indeed, it is isomorphic by adjunction to  $\mathbf{Hom}_{\mathbf{D}(\mathbf{Cond})}(\mathbb{R}, C) = H^0(R\mathbf{Hom}(\mathbb{R}, C)) = 0$ , where we use equation 6.2 to conclude (i.e.  $R\mathbf{Hom}(\mathbb{R}, C) = 0$  for  $C$  is positive degrees with  $C_i = \oplus \prod$ ).  $\square$

**Corollary 25.5** Let  $C \in \mathbf{D}(\mathbf{Solid})$ . Then for all profinite  $S$  we have  $R\mathbf{Hom}(\mathbb{Z}[S]^\blacksquare, C) \cong R\mathbf{Hom}(\mathbb{Z}[S], C)$ .

*Proof.* With  $C$  as in the previous corollary, by equation 6.1, we have

$$\begin{aligned} R\mathbf{Hom}(\mathbb{Z}[S]^\blacksquare, C)(S') &\cong R\Gamma(S \times S', C) \cong R\mathbf{Hom}(\mathbb{Z}[S \times S'], C) \\ R\mathbf{Hom}(\mathbb{Z}[S] \otimes \mathbb{Z}[S'], C) &\cong R\mathbf{Hom}(\mathbb{Z}[S], C)(S') \end{aligned}$$

$\square$

**25.1. Solidified tensor products.** The motivation is that  $\mathbb{Z}_p \otimes \mathbb{R}$  is in no good way a topological group. And  $\mathbb{Z}_p \hat{\otimes} \mathbb{R}$  is tricky, because there are many ways. We want to define  $(-) \otimes^\blacksquare (-)$  and  $(-)^{L\blacksquare}(-)$ .

**Theorem 25.6** (1) There is a unique way to endow  $\mathbf{Solid}$  with a symmetric monoidal structure such that  $(-)^\blacksquare$  is a symmetric monoidal structure.

(2) Same on  $\mathbf{D}(\mathbf{Solid})$  such that  $(-)^{L\blacksquare}$  is symmetric monoidal.

Furthermore,  $(-) \otimes^{L\blacksquare} (-)$  is the left derived of  $(-) \otimes^\blacksquare (-)$ .

*Proof.* Assume  $(-)^\blacksquare$  or  $(-)^{L\blacksquare}$  is symmetric monoidal. Then  $M \otimes^\blacksquare N = (M \otimes N)^\blacksquare$  and  $C \otimes^{L\blacksquare} C' = (C \otimes^{L\blacksquare} (-)C')^{L\blacksquare}$ . This shows uniqueness. For the existence it suffices to show

$$(M \otimes^\blacksquare N) \cong (M \otimes N)^\blacksquare \xrightarrow{\sim} (M^\blacksquare \otimes N^\blacksquare)^\blacksquare = M^\blacksquare \otimes^\blacksquare N^\blacksquare, \quad (*)$$

meaning that associator, unit, and all those coherence conditions follow from the condensed abelian group world. To show the previous isomorphism, we consider the factorization

$$\begin{array}{ccc} (M \otimes N)^{\blacksquare} & \xrightarrow{\quad} & (M^{\blacksquare} \otimes N^{\blacksquare})^{\blacksquare} \\ & \searrow \sim & \nearrow \sim \\ & (M^{\blacksquare} \otimes N)^{\blacksquare} & \end{array}$$

Both tensoring and solidification commutes with colimits, so it will suffice to show this on projective generators. Consider the diagram

$$\begin{array}{ccc} P.Gen \times P.Gen & \xrightarrow{\quad} & Cond \times Cond \\ \downarrow \scriptstyle (M \otimes N)^{\blacksquare} & \searrow \scriptstyle \text{Solid} & \\ (M^{\blacksquare} \otimes N)^{\blacksquare} & & \end{array}$$

and then use abstract nonsense (Kan extensions). We have

$$(\mathbb{Z}[S] \otimes \mathbb{Z}[T])^{\blacksquare} \cong (\mathbb{Z}[S]^{\blacksquare} \otimes \mathbb{Z}[T])^{\blacksquare},$$

and for  $A \in \mathbf{Solid}$  we have

$$\mathrm{Hom}((\mathbb{Z}[S]^{\blacksquare} \otimes \mathbb{Z}[T]), A) \cong \underline{\mathrm{Hom}}(\mathbb{Z}[S]^{\blacksquare}, A)(T).$$

The LHS is iso. to  $\mathrm{Hom}((\mathbb{Z}[S] \otimes \mathbb{Z}[T])^{\blacksquare}, A)$  and the RHS to  $\underline{\mathrm{Hom}}(\mathbb{Z}[S], A)(T)$ . Erase all squares except the one on  $\underline{\mathrm{Hom}}(\mathbb{Z}[S]^{\blacksquare}, A)(T)$  to finish.

To show that  $(-) \otimes^{L\blacksquare} (-)$  is the left derived functor of  $(-) \otimes^{\blacksquare} (-)$ , we reduce again to projective generators. But if  $\mathcal{M} = \mathbb{Z}[S]^{\blacksquare}$  and  $\mathcal{M}' = \mathbb{Z}[T]^{\blacksquare}$ , then

$$(*) \quad \mathcal{M} \otimes^{L\blacksquare} \mathcal{M}' \cong \mathcal{M} \otimes^{\blacksquare} \mathcal{M}'[0].$$

□

## 25.2. Calculations.

**Proposition 25.7** Let  $\mathcal{M} = \mathbb{Z}^I \in \mathbf{Solid}$  and  $\mathcal{M}' = \mathbb{Z}^J \in \mathbf{Solid}$ . Then  $\mathcal{M} \otimes^{L\blacksquare} \mathcal{M}' \cong \prod_{I \times J} \mathbb{Z}$ .

Using this we can start computing.

**Example 25.8**  $\mathbb{Z}[[U]] \otimes^{L\blacksquare} \mathbb{Z}[[T]] \cong \prod_{\mathbb{N}} \mathbb{Z} \otimes^{L\blacksquare} \prod_{\mathbb{N}} \mathbb{Z} \cong \prod_{\mathbb{N} \times \mathbb{N}} \mathbb{Z} \cong \mathbb{Z}[[U, T]]$ .

**Example 25.9** Recall the analytic completion:

$$(*) \quad 0 \rightarrow \mathbb{Z}[[U]] \xrightarrow{(u-p)} \mathbb{Z}[[U]] \rightarrow \mathbb{Z}_p \rightarrow 0.$$

where we consider it as a sequence in condensed abelian groups. The reason is that for LCA we have:  $A \rightarrow B$  implies  $\underline{A} \rightarrow \underline{B}$ . Then express the previous SES defining each object as a limit and use commutativity of limits with  $\underline{\quad}$ . Hence the functor  $\underline{\quad}$  yields a SES.

Thus we have a projective resolution  $(*)$  of  $\mathbb{Z}_p$  and we can compute tensor products

$$\mathbb{Z}_p \otimes^{L\blacksquare} \mathbb{Z}[[U]] \cong \mathbb{Z}_p[[U]]$$

and

$$\mathbb{Z}_p \otimes^{L\blacksquare} \mathbb{Z}_p = \begin{cases} 0 & \text{if } l \neq p \\ \mathbb{Z}_p & \text{if } l = p \end{cases}$$

If  $X$  is any of the above, then  $X \otimes^{L\blacksquare} \mathbb{R} = (X \otimes^L \mathbb{R})^{L\blacksquare} = (X^{L\blacksquare} \otimes^L \mathbb{R}^{L\blacksquare})^{L\blacksquare} = 0$ .

**Example 25.10** Let  $X$  be a CW complex. Then  $\mathbb{Z}[X]^{L\blacksquare} \cong H_*(X)$ .

*Proof.* Note that the RHS commutes with colimits. The LHS is a composition of stuff that commutes with colimits, so it also commutes with colimits. Thus we can assume that  $X$  is a finite CW complex, in particular compact and Hausdorff. Thus  $\mathbb{Z}[X]$  is pseudo coherent in  $\text{Cond}$  and  $\mathbb{Z}[X]^{L\blacksquare}$  is pseudo coherent in  $\mathbf{D}(\text{Solid})$ . Also  $H_*(X) \cong C_*(X)$  is pseudo coherent. On such objects,  $R\text{Hom}(-, \mathbb{Z})$  takes discrete values in  $\mathbf{D}(\mathbb{Z})^{\geq 0}$ . It is fully faithful.  $\square$

26. [LT] LECTURE 11 - 26.11.19

**Theorem 26.1 (Flow theorem)** *Let  $\xi$  be a smooth vector field on  $X$ . There exists a unique maximal flow  $\Phi$  on  $X$  such that*

$$\frac{d\Phi}{dt} = \xi.$$

*Proof.* By the ODE theorem, there exists an integral curve of  $\xi$  starting at each point  $p \in X$ . Define

$$\text{intcurve}(\xi, p) = \{ \text{integral curves of } \xi \text{ starting at } p \}$$

and  $A(\xi, p) = \{ \text{dom}(\gamma) \mid \gamma \in \text{intcurve}(\xi, p) \}$ . Set also  $D_p = \cup_{J \in A(\xi, p)} J \subseteq \mathbb{R}$  and

$$\begin{aligned} \Phi_p: D_p &\rightarrow X \\ t &\mapsto \gamma(t) \end{aligned}$$

where  $\gamma: J \rightarrow X$  is in  $\text{intcurve}(\xi, p)$ . By the ODE theorem,  $\Phi_p$  is well-defined. Define

$$D(\xi) := \{(t, p) \in \mathbb{R} \times X \mid t \in D_p\}$$

and

$$\begin{aligned} \Phi: D(\xi) &\rightarrow X \\ (t, p) &\mapsto \Phi_p(t) \end{aligned}$$

It remains to show:

- (1) For all  $p \in X$  we have  $\Phi(0, p) = p$ .
- (2) For all  $p \in X$  and all  $s, t \in \mathbb{R}$  we have  $\Phi(t, \Phi(s, p)) = \Phi(t + s, p)$  when both sides are defined.
- (3)  $D(\xi) \subseteq \mathbb{R} \times X$  is open.
- (4)  $\Phi: D(\xi) \rightarrow X$  is smooth.

Point 1) is okay by definition. For 2), let  $s \in D_p$ ,  $q := \Phi(s, p)$ ,  $\gamma(t) := \Phi(t + s, p)$  where defined. Chain rule implies that  $\gamma \in \text{intcurve}(\xi, q)$ , so  $\gamma = \Phi_q$  and  $\Phi(t + s, p) = \Phi(t, \Phi(s, p))$ .

For 3) and 4), let  $W := \{(t, p) \in D(\xi) \mid \text{there exists an open interval } 0, t \in J \subseteq \mathbb{R} \text{ and there exists an open neighbourhood } p \in U \subseteq X \text{ such that } \Phi \text{ is defined and smooth on } J \times U\}$ . Since  $W \subseteq \mathbb{R} \times X$  is open and  $\Phi|_W$  is smooth, it suffices to show that  $W = D(\xi)$ . Suppose  $(t_0, p_0) \in D(\xi) \setminus W$ . We have two cases:  $t_0 > 0$  and  $t_0 < 0$ . Both are proved by the same method, so let us do  $t_0 > 0$ . Let  $\tau := \sup\{t \in \mathbb{R} \mid (t, p_0) \in W\}$ . By the ODE theorem,  $\Phi$  is defined and smooth in an open neighbourhood of  $(0, p_0)$  by (\*), where

$$(*) \quad \text{double check taht } (0, p) \in W \forall p \in X.$$

Hence  $\tau > 0$ .  $q_0 = \Phi_{p_0}(\tau)$ .

By the ODE theorem, there exists  $\varepsilon > 0$  and an open nbhd  $q_0 \in U_0$  such that  $\Phi: (-\varepsilon, \varepsilon) \times U_0 \rightarrow X$  is defined and smooth. Let  $t_1 < \tau$  with  $t_1 + \varepsilon > \tau$ ,  $\Phi_{p_0}(t_1) \in U_0$ .  $t_1 < \tau$  implies... proof erased from the board (change of lecturer), sorry!  $\square$

**26.1. Some elements of Morse theory.** Morse theory is a method that we have to understand the topology of manifolds. We do this via a Morse function  $X \rightarrow \mathbb{R}$ . [Picture in all the books and on Wikipedia with the torus sitting vertically on a plane. The function is given by the height. Ehresmann applies, but above the topology of the fibre is different from in the one in the middle. The reason are some special points in which Ehresmann does not apply.] See book on the references! (*Topology from the differential viewpoint* by John Milnor).

**Definition 26.2** Let  $X$  be a smooth manifold and  $f: X \rightarrow \mathbb{R}$  smooth. A point  $x \in X$  is called *critical* if  $f$  does not have full rank at this point, i.e. if  $df_x = 0$ . A value  $t \in \mathbb{R}$  is called a *critical value* of  $f$  if there is a critical point in  $f^{-1}(t)$ . In the non-critical points, the rank of  $df_x$  is maximal, i.e. rank 1, and  $f$  is a submersion near  $X$ .

**Theorem 26.3 (Sard)** Let  $f: X \rightarrow \mathbb{R}^n$  smooth map. Let  $\Sigma \subseteq X$  be the set of points  $x \in X$  such that  $df_x$  is not surjective. Then  $f(\Sigma) \subseteq \mathbb{R}^n$  has (Lebesgue) measure 0.

*Proof.* See [Hir94, pp. 68-72] or [Mil97, pp. 16-19].  $\square$

**Example 26.4 (Torus)** There are 4 critical values. The bottom is a minimum of  $h$ , the top is a maximum of  $h$ , the other two are saddle points. By calculus we can distinguish these cases by looking at the second derivatives (or at the Hessian matrix).

**Definition 26.5** Let  $f: X \rightarrow \mathbb{R}$  smooth,  $x \in X$  a critical point. We define the bilinear form

$$\text{Hess}_x(f): T_x X \times T_x X \rightarrow \mathbb{R}$$

by  $\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) \mapsto \frac{\partial^2 f}{\partial x_i \partial x_j}(x)$ , where  $x_1, \dots, x_n$  are local coordinates of  $X$  near  $x$ .

*Exercise 26.6.* This is well defined, i.e. independent of the choice of local coordinates.

Note that  $\text{Hess}_x(f)$  is symmetric ( $f$  is even  $C^\infty$ ). Symmetric bilinear forms are classified: there is a basis of the vector space such that the representing matrix has the following shape

$$\begin{pmatrix} -I_r & 0 & 0 \\ 0 & I_s & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We call  $r + s$  the *rank* and  $r$  the *index*.

**Definition 26.7** We say that  $x \in X$  is a *non-degenerate* critical point if it is a critical point and  $\text{Hess}_x(f)$  is non-degenerate, i.e. invertible, i.e.  $n = r + s$ .

**Example 26.8**  $X = \mathbb{R}^n$ ,  $f(x_1, \dots, x_n) = -x_1^2 - x_2^2 - \dots - x_r^2 + x_{r+1}^2 + \dots + x_{r+s}^2$ . Let  $\partial_i$  denote  $\frac{\partial}{\partial x_i}$ .

$$df = \frac{\partial f}{\partial x_1} \partial_1 + \dots + \frac{\partial f}{\partial x_n} \partial_n = 0$$

$$\text{grad}(f) = (-2x_1, \dots)$$

Critical points:  $\{x \in \mathbb{R}^n \mid x_1 = \dots = x_{r+s} = 0\}$ . Hessian matrix is diagonal with first some -2's and then some 2's. It is non-degenerate if and only if there is a single critical point (at the origin).

**Proposition 26.9 (Morse lemma)** *Let  $P \in X$  be a non-degenerate critical point of a smooth function  $f$ . Then there is an open neighbourhood  $U$  of  $P$  and coordinate functions  $x_1, \dots, x_n$  on  $X$  such that*

$$f(x) = f(P) - \sum_{i=1}^r x_i^2 + \sum_{i=r+1}^n x_i^2$$

where  $r$  is the index of  $f$  at  $P$ .

*Proof.* WLOG  $X \subseteq \mathbb{R}^n$  and  $P = 0$ . Let  $x_1, \dots, x_n$  be the standard coordinates on  $\mathbb{R}^n$ .

$$f(x_1, \dots, x_n) = f(0) + \sum_{i=1}^n \left( \frac{\partial f}{\partial x_i} \right)(0) x_i + \sum_{i,j} x_i x_j g_{ij}(x_1, \dots, x_n)$$

for smooth functions  $g_{ij}: X \rightarrow \mathbb{R}$ . The linear term goes away, because 0 is a critical point. The remaining term is a bilinear map  $x^t G x$  with  $G = (g_{ij})_{i,j}$ . WLOG  $g_{ij}$  is symmetric (replace  $g_{ij}$  by  $\frac{1}{2}(g_{ij} + g_{ji})$ ). Applying the product rule for derivatives we have

$$\frac{\partial f}{\partial x_k} = \sum_j x_j g_{kj} + \sum_i x_i g_{ik} + \sum_{i,j} x_i x_j \frac{\partial g_{ij}}{\partial x_k}$$

The previous two terms can be grouped as  $2 \sum_j x_j g_{kj}$ . Now

$$\frac{\partial^2 f}{\partial x_l \partial x_k} = 2g_{kl} + 2 \sum_j x_j \frac{\partial g_{kj}}{\partial x_l} + 2 \sum_j x_j \frac{\partial g_{lj}}{\partial x_k} + \sum_{i,j} x_i x_j \frac{\partial g_{ij}}{\partial x_l \partial x_k}$$

Hence  $\text{Hess}_0(f) = (2g_{kl}(0))$ . They form a non-degenerate matrix. Let  $A \subseteq \text{GL}_n(\mathbb{R})$  be the matrix that transforms this matrix into normal form as above, i.e.

$$A^t(g_{ij}(0))A = \begin{pmatrix} -I_r & 0 \\ 0 & I_{n-r} \end{pmatrix}$$

We make a change of coordinates  $x' = A^{-1}x$ . WLOG  $f(0) = 0$ , so that

$$f(x') = x'^t A^t(g_{ij})Ax' = \sum_{i,j} x'_i x'_j g'_{ij}(x')$$

such that  $(g'_{ij}(0)) = A^t(g_{ij})A$ . [To be continued on thursday!] □

**Corollary 26.10** *Non-degenerate critical points are isolated.*

*Proof.* Compute in the previous coordinates. □

27. [WS] DERIVED CATS 1 (IVAN) - INTRODUCTION TO COHERENT SHEAVES - 27.11.19

As a motivation we have:

**Theorem 27.1 (Swan's theorem)** *Let  $M$  be a smooth real manifold. Then the additive functor*

$$\text{Vec}(M) \rightarrow \text{Mod}(C^\infty(M))$$

*sending  $E$  to its global sections induces an equivalence of categories onto the full subcategory of finitely generated projective  $C^\infty(M)$ -modules.*

*Remark 27.2.* (1) Given any vector bundle  $E$ , there exists  $F$  such that  $E \oplus F$  is trivial.

Hence  $\Gamma(M, E)$  is a direct summand of a free module, thus projective.

(2) Any vector bundle  $E$  admits a finite trivializing cover  $\{U_i\}$ . Using partitions of unity we can extend local frames to a finite number of global sections generating  $\Gamma(M, E)$  over  $C^\infty(M)$ .



**Moral 1:** The category of vector bundles is somehow weird to consider, meaning that it is not an abelian category.

What goes wrong explicitly? Consider  $M = \mathbb{R}$  and  $E = \mathbb{R} \times \mathbb{R}$ . Consider the endomorphism

$$\begin{aligned} \varphi: \mathbb{R} \times \mathbb{R} &\rightarrow \mathbb{R} \times \mathbb{R} \\ (x, \lambda) &\mapsto (x, x\lambda) \end{aligned}$$

The kernel is not a vector bundle, because the fibres have different dimensions.

*Remark 27.3.* What if we take  $X$  to be a compact complex manifold? By the maximum principle,  $\mathcal{O}_X(X) \cong \mathbb{C}$ . It is not true that the category of holomorphic vector bundles on  $X$  is equivalent to finite dimensional  $\mathbb{C}$ -vector spaces.

**Moral 2:** When things get more rigid, we cannot read everything from the global sections. We need to take into account also local sections.

The right functor to consider in general is  $\mathbf{Vec}(X) \rightarrow \mathbf{Sh}(X)$ , associating to a vector bundle  $E$  its sheaf of sections.

From now on we will work with  $X$  connected complex manifold/irreducible algebraic variety over  $\mathbb{C}$ .  $\mathcal{O}_X$  will denote the sheaf of holomorphic or regular functions on  $X$ .

*Remark 27.4.* We can multiply local sections of  $E$  by local functions. So we actually have sheaves of  $\mathcal{O}_X$ -modules.

**Definition 27.5** Sheaves of  $\mathcal{O}_X$ -modules.

*Remark 27.6.* If  $E = X \times \mathbb{C}^m$ , then  $\mathcal{E} = \mathcal{O}_X^m$ . If  $E$  is any vector bundle, then  $\mathcal{E}$  is a locally free sheaf.

**Definition 27.7** Locally free sheaves of  $\mathcal{O}_X$ -modules.

Hence our functor  $\mathbf{Vec}(X) \rightarrow \mathbf{Sh}(X)$  takes values in the full subcategory of locally free sheaves.

**Theorem 27.8** The functor  $\mathbf{Vec}(X) \rightarrow \{ \text{locally free sheaves} \}$  is an equivalence of categories.

*Proof.* Fix a trivializing cover  $U_i$ . We want to show that vector bundles trivial on  $\{U_i\}$  are the same as locally free sheaves free on  $\{U_i\}$ . We show that both these categories are equivalent to the category of transition functions.  $\square$

*Remark 27.9.* More explicitly we can write

$$E := \coprod_{x \in X} \mathcal{E}_x / \mathfrak{m}_x \mathcal{E}_x.$$

So we have a fully faithful functor  $\mathbf{Vec}(X) \rightarrow \mathbf{Mod}(X)$  into an abelian category. The problem is that the abelian category  $\mathbf{Mod}(X)$  is too big. This leads to the following definition:

**Definition 27.10** An  $\mathcal{O}_X$ -module  $\mathcal{F}$  is called *coherent* if locally it is the cokernel of two vector bundles, i.e. for all  $x \in X$  there is an open neighbourhood  $x \in U \subseteq X$  and an exact sequence

$$\mathcal{O}_U^{\oplus m} \rightarrow \mathcal{O}_U^{\oplus n} \rightarrow \mathcal{F}|_U \rightarrow 0.$$

This is a natural generalization of vector bundles. We will see that these sheaves do form an abelian category.

- Example 27.11** (1) Any locally free sheaf is coherent.  
 (2)  $X = \mathbb{C}^2$ ,  $0 \in \mathbb{C}^2$  the origin. Let  $\mathcal{I}_0 \subseteq \mathcal{O}_X$  be the subsheaf of all functions vanishing at 0. We have an exact sequence

$$0 \rightarrow \mathcal{O}_X \xrightarrow{\begin{pmatrix} y \\ -x \end{pmatrix}} \mathcal{O}_X^{\oplus 2} \xrightarrow{(x,y)} \mathcal{I}_0 \rightarrow 0.$$

Hence  $\mathcal{I}_0$  is a coherent sheaf. But  $\mathcal{I}_0$  is not locally free, because the fibres are  $\mathbb{C}$  if  $x \neq 0$  and  $\mathbb{C}^2$  if  $x = 0$ .

- (3)  $X = \mathbb{C}^2$  and  $0 \in \mathbb{C}^2$ . We define the skyscraper sheaf with value  $\mathbb{C}$  at 0 by

$$\delta_0(U) = \begin{cases} \mathbb{C} & \text{if } 0 \in U \\ 0 & \text{otherwise} \end{cases}$$

$\delta_0$  is not locally free, but we have a short exact sequence

$$0 \rightarrow \mathcal{I}_0 \rightarrow \mathcal{O}_X \rightarrow \delta_0 \rightarrow 0$$

and thus an exact sequence

$$\mathcal{O}_X^{\oplus 2} \rightarrow \mathcal{O}_X \rightarrow \delta_0 \rightarrow 0.$$

**Facts:**

- (1)  $\mathcal{F}$  coherent has always locally a resolution by locally free sheaves.  
 (2)  $i: Z \hookrightarrow X$  smooth (closed) submanifold. We have  $\mathcal{I}_Z$  and  $i_*\mathcal{O}_Z$  coherent, and we have a short exact sequence

$$0 \rightarrow \mathcal{I}_Z \rightarrow \mathcal{O}_X \rightarrow i_*\mathcal{O}_Z \rightarrow 0.$$

### 27.1. Why care about coherent sheaves?

- (1) They are singular vector bundles in a way, so they are useful to compactify moduli spaces of vector bundles.  
 (2) To completely characterize projective varieties with Gabriel's theorem. So  $\mathbf{Coh}(X)$  is not very manageable.

Instead we will consider  $\mathbf{D}^b(X) = \mathbf{D}^b(\mathbf{Coh}(X))$  in the following lectures. The goal will be Bondal-Orlov's recognition theorem: smooth projective varieties with ample or anti-ample canonical bundles are completely determined by their bounded derived categories.

In order to let the derived category machinery work, we need to start with an abelian category. Before giving the proof, we need a characterization of coherent sheaves.

**27.2. Characterization of coherent sheaves.** An  $\mathcal{O}_X$ -module  $\mathcal{F}$  is coherent if and only if the following two conditions hold:

- (1) Locally we have an exact sequence  $\mathcal{O}_U^{\oplus n} \rightarrow \mathcal{F}|_U \rightarrow 0$ , i.e.  $\mathcal{F}$  is of *finite type*.  
 (2) For all  $U \subseteq X$ , any  $\mathcal{O}_U^{\oplus m} \rightarrow \mathcal{F}|_U$  has finite type kernel.

When is  $\mathcal{O}_X$  coherent with respect to this second definition? In the algebraic setting, this follows from noetherianity arguments. In the analytic setting, this is a deep theorem due to Oka.

### 27.3. The category of coherent sheaves is abelian.

**Theorem 27.12**  $\mathbf{Coh}(X)$  is abelian.

*Proof.* Since  $\mathbf{Coh}(X)$  sits inside the abelian category  $\mathbf{Mod}(X)$ , we only need to check that given  $\mathcal{F}, \mathcal{G} \in \mathbf{Coh}(X)$ , the following three properties hold:

- (1)  $\mathcal{F} \oplus \mathcal{G} \in \mathbf{Coh}(X)$ .
- (2)  $\ker(\varphi) \in \mathbf{Coh}(X)$  for all  $\varphi: \mathcal{F} \rightarrow \mathcal{G}$ .
- (3)  $\operatorname{coker}(\varphi) \in \mathbf{Coh}(X)$ .

Condition 1) is left to the interested reader.

For 2), condition *b*) in our characterization is trivial. To check *a*), consider  $U \subseteq X$  and  $\mathcal{O}_U^{\oplus n} \rightarrow \mathcal{F}|_U \rightarrow 0$  exact. We have a diagram

$$\begin{array}{ccccc} \ker(\tilde{\varphi}) & \longrightarrow & \mathcal{O}_U^{\oplus n} & & \\ \downarrow & & \downarrow & \searrow \tilde{\varphi} & \\ \ker(\varphi|_U) & \longrightarrow & \mathcal{F}|_U & \xrightarrow{\varphi} & \mathcal{G}|_U \\ & & \downarrow & & \\ & & 0 & & \end{array}$$

Since  $\ker(\tilde{\varphi})$  is of finite type, we have  $\ker(\tilde{\varphi}) \twoheadrightarrow \ker(\varphi|_U)$ , so  $\ker(\varphi)$  is also finite type.

For 3), since  $\mathcal{G} \twoheadrightarrow \operatorname{coker}(\varphi)$  is a quotient, *a*) is trivial. So let us check *b*). Let  $\tilde{\psi}: \mathcal{O}_U^{\oplus n} \rightarrow \operatorname{coker}(\varphi|_U)$  be a map. We need to check that  $\ker(\tilde{\psi})$  is of finite type. Consider the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_U^{\oplus m} & \longrightarrow & \mathcal{O}_U^{\oplus n} \oplus \mathcal{O}_U^{\oplus n} & \longrightarrow & \mathcal{O}_U^{\oplus n} \longrightarrow 0 \\ & & \downarrow \Phi & & \downarrow \psi & \swarrow & \downarrow \tilde{\psi} \\ 0 & \longrightarrow & \operatorname{im}(\varphi|_U) & \longrightarrow & \mathcal{G}|_U & \longrightarrow & \operatorname{coker}(\varphi|_U) \longrightarrow 0 \end{array}$$

in which both rows are exact. Apply the snake lemma to get  $\ker(\psi) \rightarrow \ker(\tilde{\psi}) \rightarrow 0 = \operatorname{coker}(\Phi)$ . Since  $\ker(\psi)$  is of finite type, so is  $\ker(\tilde{\psi})$ .  $\square$

### 28. [LT] LECTURE 12 - 28.11.19

The next step in our proof is to make a non-linear change of coordinates to get rid of the  $g_{ij}$  of the last time by completing the square. We have  $g_{11}(0) = \pm 1$ . Consider for instance  $g_{11}(0) = 1$ . Then  $g_{11}$  is positive on a nbhd of 0. We replace  $X$  by this neighbourhood, so that  $g_{11}$  is positive on  $X$  and we can take roots. We put

$$y_1 = \sqrt{g_{11}}x_1 + \sqrt{g_{11}}^{-1}g_{12}x_2 + \cdots + \sqrt{g_{11}}^{-1}g_{1n}x_n$$

and hence  $y_1^2 = g_{11}x_1^2 + 2g_{12}x_1x_2 + \cdots + 2g_{1n}x_1x_n + \text{rest (without } x_1)$ . In the new coordinates  $y_1, x_2, \dots, x_n$  we have

$$f = y^2 + \sum_{i,j>1} x_i x_j h_{ij}$$

with new functions  $h_{ij}$ . By induction, we find new coordinates  $y_2, \dots, y_n$  such that  $f$  has the desired form.

The case  $g_{11}(0) = -1$  is similar. In this case  $g_{11}$  is negative on a neighbourhood of 0, so we replace again  $X$  by this neighbourhood. The coordinate that we have to do then is

$$y_1 = \sqrt{-g_{11}}x_1 + \sqrt{-g_{11}}^{-1}g_{12}x_2 + \cdots + \sqrt{-g_{11}}^{-1}g_{1n}x_n,$$

so that  $-y_1^2 = g_{11}x_1^2 + 2g_{12}x_1x_2 + \dots$ , and then finish the argument as before. This concludes the proof.

### 28.1. Same story in the holomorphic setting.

**Definition 28.1** Let  $X$  be a complex manifold and  $f: X \rightarrow \mathbb{C}$  holomorphic. A point  $x \in X$  is called *critical point* if  $df_x = 0$ . A value  $t \in \mathbb{C}$  is called a *critical value* if there is a critical point in  $X$  mapping to  $t$  under  $f$ .

*Remark 28.2.* In a non-critical point,  $\text{rk}_{\mathbb{C}}(df_x) = 1$ , so  $f$  is a submersion of smooth manifolds. If in addition  $f$  is proper, then we can apply Ehresmann to get a local fibration of smooth manifolds away from the critical values.

**Definition 28.3** Let  $X$  be a complex manifold and  $f: X \rightarrow \mathbb{C}$  holomorphic. Let  $x \in X$  be a critical point. We define the  $\mathbb{C}$ -bilinear form

$$\begin{aligned} \text{Hess}_x(f): T_x X \times T_x X &\rightarrow \mathbb{C} \\ (\partial_{z_i}, \partial_{z_j}) &\mapsto \frac{\partial^2 f}{\partial z_j \partial z_i}(x) \end{aligned}$$

with  $z_1, \dots, z_n$  local coordinates near  $x$ .

As in the real case,  $\text{Hess}_x(f)$  is well-defined and symmetric. There is a basis of  $T_x X$  such that the bilinear form is represented by the diagonal matrix

$$\begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix}$$

**Definition 28.4** We say that  $x \in X$  is a *non-degenerate critical point* if it is a critical point and the bilinear form given by the Hessian is non-degenerate. In this case, we also say that the fibre  $X_{f(x)}$  has an *ordinary double point*.

**Example 28.5** Let  $X = \mathbb{C}^n$ ,  $f(x_1, \dots, x_n) = \sum_{i=1}^s x_i^2$  with  $s \leq n$ . Then  $df = (2x_1, \dots, 2x_s, 0, \dots, 0)$ . Critical points are  $\{x \in \mathbb{C}^n \mid x_1 = \dots = x_s = 0\}$ . In the Hessian matrix at  $x$  written as above we get  $I_s = 2\text{id}$ . It is non-degenerate if and only if  $s = n$ , i.e. if and only if 0 is the only critical point. The fibre is then  $X_0 = \{\sum x_i^2 = 0\}$ .

$f: X \rightarrow \mathbb{C}$  smooth,  $p \in X$  non-degenerate critical point. Then  $\exists U$  open nbhd of  $p$  and local coordinates  $x_1, \dots, x_n$  on  $U$  such that

$$f(x) = f(p) - \sum_{i=1}^r x_i^2 + \sum_{i=r+1}^n x_i^2$$

**Proposition 28.6 (Morse lemma)** Let  $p \in X$  be non-degenerate critical point of a holomorphic function  $f: X \rightarrow \mathbb{C}$ . Then there is an open nbhd  $U$  of  $p$  and local holomorphic coordinates  $x_1, \dots, x_n$  such that

$$f(x) = f(p) + \sum_{i=1}^n x_i^2$$

*Proof.* Same argument as in the smooth case. □

### 28.2. Topology of level sets. [Torus picture]

Let  $f: X \rightarrow \mathbb{R}$  be smooth. Our aim is to understand the set  $X_{\leq M} = f^{-1}((-\infty, M])$  or more generally the sets  $X_{[M_1, M_2]}$  and  $X_M$  for  $M \in \mathbb{R}$  and  $[M_1, M_2] \subseteq \mathbb{R}$ .

**Lemma 28.7** *If  $M$  is not a critical value with non-empty fibre, then  $X_{\leq M}$  is a manifold with boundary, i.e. every point has a nbhd homeomorphic to an open set in  $\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$ .*

*Proof.* The preimage of  $(-\infty, M)$  is open in  $X$ , hence a manifold. Consider  $x_0 \in X_M$ , which is not a critical point by assumption. By assumption,  $f$  is a submersion near  $x_0$ . By the constant rank theorem, there are local coordinates near  $x_0$  such that  $f$  is the projection to the first coordinate. This is the coordinate chart for the manifold with boundary.  $\square$

**Definition 28.8** Let  $X$  be a smooth manifold. A function  $f: X \rightarrow \mathbb{R}$  is called a *Morse function* if all critical points are non-degenerate and if it is exhaustie, i.e. if for every  $M \in \mathbb{R}$  the closed subset  $X_{\leq M}$  is compact.

In particular,  $f$  is proper. Hence, if  $[M_1, M_2]$  does not contain a critical value,  $f$  defines a trivial fibration

$$X_{[M_1, M_2]} \cong [M_1, M_2] \times K$$

for the compact fibre  $K$ .

**Definition 28.9** Cells, cell complexes, finite cell complexes.

*Exercise 28.10.*  $X$  and  $Y$  finite cell complexes. Show that  $X \times Y$  also has the structure of a finite cell complex.

**Definition 28.11** Homotopy relative to a subspace. Homotopy equivalence and homotopy type.

**Definition 28.12**  $X$  topological space,  $A \subseteq X$  closed subset,  $i: A \rightarrow X$  the inclusion. We say that  $A$  is a *deformation retract* of  $X$  if there is a continuous map  $r: X \rightarrow A$  such that  $r \circ i = \text{id}_A$  and a homotopy relative to  $A$  from  $i \circ r$  to  $\text{id}_X$ .

Hence this is stronger than homotopy equivalence.

**Theorem 28.13**  $X$  a smooth manifold,  $f: X \rightarrow \mathbb{R}$  a Morse function and  $M \in \mathbb{R}$ . Then  $X_{\leq M}$  has the homotopy type of a finite cell complex.

29. [FS] CHRISTIAN GLEISSNER - KLEIN'S QUARTIC, FERMAT'S CUBIC AND RIGID MANIFOLDS WITH KODAIRA DIMENSION 1 - 29.11.19

Joint work with Ingrid Bauer. We work over  $\mathbb{C}$ .

**Definition 29.1**  $X$  compact  $\mathbb{C}$ -mfd.

- a)  $X$  is (locally) rigid iff for all deformations  $f: (X, X) \rightarrow (S, 0)$  there exists  $0 \in U \subseteq S$  open such that for all  $t \in U$  we have  $X_t \cong X$ , where  $\cong$  means biholomorphic.
- b)  $X$  is infinitesimally rigid iff  $H^1(T_X) = 0$ .

The (locally) means that we have defined locally rigidity, but we only say rigid. We have  $b) \Rightarrow a)$ , and in 2016 Bauer and Pignatelli proved that  $a)$  does not imply  $b)$ .

**Example 29.2**  $\mathbb{P}^1 \times \mathbb{P}^1 = X$  has  $H^1(T_X) = 0$  but  $X$  is deformation equivalent to the Hirzebruch surface  $\mathbb{F}_2$ .

**Curves:** only  $\mathbb{P}^1$  is rigid, so for  $\kappa(X) \neq -\infty$  we know that  $X$  is not rigid.

**Surfaces:** if  $X$  is rigid, then  $\kappa(X) = -\infty$  or  $\kappa(X) = 2$ .

**Dimension at least 3:**

- a) Beauville '83: rigid CY mfds  $X_n$  with  $n = 3, 4, 6$  ( $\kappa(X_n) = 0$ ).
- b) Catanese Bauer '16: for all  $n \geq 4$  there is a rigid  $n$ -fold  $X_n$  with  $\kappa(X_n) = 0$ .
- c) Catanese Bauer '16: for all  $n \geq 3$  and all  $2 \leq \kappa \leq n$  there exists  $X_{n,\kappa}$  an  $n$ -fold which is rigid and has  $\kappa(X_{n,\kappa}) = \kappa$ .

**Theorem 29.3** (-,Bauer) *For all  $n \geq 3$  there exists a rigid  $X_n$  with  $\kappa(X_n) = 1$ .*

**29.1. Idea of the construction.** Start with  $Y_n$  an  $n$ -fold with  $\kappa(Y_n) = 1$ , e.g.  $Y_n = E^{n-1} \times C$  for  $E$  elliptic curve and  $C$  hyperbolic curve. Take a finite subgroup  $G \subseteq \text{Aut}(Y_n)$  such that  $H^1(T_{Y_n})^G = 0$ . Then consider  $X_n = Y_n/G$ .

*Remark 29.4.* If codimension of fixed locus is at least 2, then  $H^1(T_{Y_n})^G = H^1(T_{X_n})$ .

*Problem 29.5.* a)  $X_n$  may be singular. We want  $\kappa(X_n) = 1$ , so we only allow canonical singularities.

b) We need to compare  $H^1(T_{\hat{X}_n})$  with  $H^1(T_{X_n})$ .

Consider the Leray spectral sequence:

$$E_2^{p,q} = H^p(R^q f_* T_{\hat{X}_n}) \Rightarrow H^\bullet(T_{\hat{X}_n})$$

The low terms yield the 5-term spectral sequence

$$0 \rightarrow E_2^{1,0} \rightarrow H^1(T_{\hat{X}_n}) \rightarrow E_2^{0,1} \rightarrow E_2^{2,0} \rightarrow H^2(T_{\hat{X}_n})$$

We want  $f_* T_{\hat{X}_n} = T_{X_n}$  and  $R^1 f_* T_{\hat{X}_n} = 0$ , for which we need to choose our resolution carefully.

**29.2. The quotient  $X_n$ .**  $C = \{x_0^3 x_1 + x_1^3 x_2 + x_2^3 x_0 = 0\} \subseteq \mathbb{P}^2$  with  $g(C) = 3$  (Klein's quartic). Then  $\text{Aut}(C) \cong \text{PSL}(2, \mathbb{F}_7)$ , which has order 168 (maximum for  $g = 3$  curves). We choose  $\lambda^7 = 1$  and

$$G = \left\langle \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \lambda^4 & 0 & 0 \\ 0 & \lambda^2 & 0 \\ 0 & 0 & \lambda \end{pmatrix} \right\rangle$$

Call  $S$  resp.  $T$  the first resp. second matrix. Then  $\text{ord}(S) = 3$  and  $\text{ord}(T) = 7$  and  $G \cong \mathbb{Z}/7\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ . This group acts on  $E = \mathbb{C}/\mathbb{Z}[\omega]$  by  $f_S(z) = \omega z$  and  $f_T(z) = z + \frac{1+3\omega}{7}$ . We take  $Y_n = E^{n-1} \times C$  with  $G \rightarrow Y_n$  via

$$S^a T^b(z_1, \dots, z_{n-1}, Y) = (f_{S^a T^b}(z_1), \dots, f_{S^a T^b}(z_{n-1}), S^a T^b y)$$

and set  $X_n = Y_n/G$ .

**Proposition 29.6**  $X_n$  has  $3^{n-1}$  singularities of type  $\frac{1}{3}(1, \dots, 1)$  and  $3^{n-1}$  singularities of type  $\frac{1}{3}(1, \dots, 1, 2)$  (all isolated). This implies that they are canonical, and in particular  $\kappa(X_n) = 1$ .

**29.3. Verification that  $H^1(T_{Y_n})^G = 0$ .**

*Remark 29.7.*  $H^0(\Omega_E^{\otimes q})^G = \langle dz^{\otimes q} \rangle^G = 0$  if  $q = 1$  or  $q = 2$ .

**Lemma 29.8**  $H^0(\Omega_C^{\otimes q})^G = 0$  for  $q = 1$  and  $q = 2$ .

**Corollary 29.9**  $H^1(T_{Y_n})^G = 0$ .

*Proof.* Remark plus lemma plus Kuenneth plus Serre duality. □

#### 29.4. The resolution $f: \hat{X}_n \rightarrow X_n$ .

*Remark 29.10.* To construct  $f$  such that  $f_*T_{\hat{X}_n} = T_{X_n}$  and  $R^1f_*T_{\hat{X}_n} = 0$  is a local problem, since the singular locus is isolated.

Hence we can use toric geometry. We resolve our singularity locally with some  $U$ . They glue to a toric variety  $U_\Sigma$ , where  $\Sigma$  is the corresponding fan. For any toric variety we have an Euler sequence like the one on projective space:

$$0 \rightarrow \mathcal{O}_{U_\Sigma} \rightarrow \bigoplus_{i=1}^3 \mathcal{O}(D_i) \oplus \mathcal{O}(E) \rightarrow T_{U_\Sigma} \rightarrow 0$$

Toric, quotient and canonical singularities are all rational. Hence  $R^i f_* \mathcal{O}_{U_\Sigma} = 0$  for  $i \geq 1$ . Thus

$$R^1 f_* T_{U_\Sigma} \cong \bigoplus R^1 f_* \mathcal{O}(D_i) \oplus R^1 f_* \mathcal{O}(E)$$

For example for the right one we use the standard short exact sequence

$$0 \rightarrow \mathcal{O}_{U_\Sigma} \rightarrow \mathcal{O}_{U_\Sigma}(E) \rightarrow \mathcal{O}_E(E) \rightarrow 0$$

Which implies that

$$R^1 f_* \mathcal{O}_{U_\Sigma}(E) = R^1 f_* \mathcal{O}_E(E) = H^1(E, \mathcal{O}_E(E)) = 0$$

because  $\mathcal{O}_E(E) = \mathcal{O}_{\mathbb{P}^2}(9)$ . Similarly for the other one, so  $R^1 f_* T_{U_\Sigma} = 0$ . We are done if we show that  $f_* T_{U_\Sigma}$  is reflexive, because then  $f_* T_{U_\Sigma} = T_U$ . This follows by a result of Greb, Kebekus and Peternell, which show that  $f_* \Omega_{U_\Sigma}^2$  is reflexive.

### 30. [CM] TALK 7 (JOHAN) - ANALYTIC RINGS - 02.12.19

**30.1. What we have seen so far and where we are going.** In the last two talks we looked at things called *solid abelian groups*. To every extremally disconnected set  $S$  we attached the object  $\mathbb{Z}[S]^\blacksquare$ . Dario showed us that using this functor we can define the subcategory **Solid**  $\subseteq \text{Cond}(\text{Ab})$ , and we got a long list of good categorical properties. In the next talk, Vincent told us about the tensor product and how to complete it in a decent way. The goal for this talk is to look at what the ingredients were in the last two talks to get this category and this nice properties and axiomatize it. This will lead first to preanalytic rings and then to analytic rings. The previous construction will then be an example of an analytic ring  $\mathbb{Z}^\blacksquare$ .

#### 30.2. Preanalysis.

**Definition 30.1** A *pre-analytic ring*  $\mathcal{A}$  consists of the data:

- i) A condensed ring  $\underline{\mathcal{A}}$ .
- ii) A functor

$$\{ \text{extremally disconnected sets} \} \rightarrow \mathbf{Mod}_{\underline{\mathcal{A}}}^{\text{cond}}$$

sending  $S \mapsto \mathcal{A}[S]$ .

- iii) A natural transformation  $S \rightarrow \mathcal{A}[S]$ .

with the condition that the functor  $S \mapsto \mathcal{A}[S]$  takes finite disjoint unions to products.

**Definition 30.2** A morphism  $\mathcal{A} \rightarrow \mathcal{B}$  is a morphism of condensed rings  $\underline{\mathcal{A}} \rightarrow \underline{\mathcal{B}}$  together with a natural transformation  $\mathcal{A}[S] \rightarrow \mathcal{B}[S]$  of  $\underline{\mathcal{A}}$ -modules.

**Example 30.3** i)  $\mathbb{Z}^\blacksquare$  is given by  $\underline{\mathbb{Z}}^\blacksquare = \mathbb{Z}$  with the functor  $S \mapsto \mathbb{Z}[S]^\blacksquare$  (nat. transf. left to reader).

- ii)  $\mathbb{Z}_p, \blacksquare$  is given by  $\mathbb{Z}_p$  with the functor  $S \mapsto \lim_i \mathbb{Z}_p[S_i]$ .
- iii) If  $A$  is a condensed ring, then  $A, \blacksquare$  is given by  $\underline{A} = A$  and  $S \mapsto \lim_i A[S_i]$ . Remark:  $A, \blacksquare[S]$  are the  $A$ -valued measures on  $S$ .
- iv) If  $A$  is a discrete ring, then  $(A, \mathbb{Z}), \blacksquare$  is given by  $\underline{A}$  and  $S \mapsto \mathbb{Z}, \blacksquare[S] \otimes_{\mathbb{Z}} A$ .
- v) If  $A$  is a condensed ring we can take  $A$  itself and  $S \mapsto A[S]$ , but this is not well-behaved.

### 30.3. Analysis.

**Theorem 30.4** Let  $\mathcal{A}$  be a pre-analytic ring. TFAE:

- i) For any complex

$$C = \cdots \rightarrow C_i \rightarrow \cdots \rightarrow C_1 \rightarrow C_0 \rightarrow 0$$

of  $\underline{\mathcal{A}}$ -modules such that all  $C_i$  are direct sums of  $\mathcal{A}[T]$  for varying  $T$ , the map

$$R\underline{\mathrm{Hom}}_{\underline{\mathcal{A}}}(\mathcal{A}[S], C) \rightarrow R\underline{\mathrm{Hom}}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}}[S], C)$$

is an isomorphism in  $\mathrm{Cond}(\mathbf{Ab})$  for all  $S$ .

- ii) For all  $I, J$  and extremally disconnected families  $(S_i)_{i \in I}$  and  $(S'_j)_{j \in J}$  and all maps  $f: \bigoplus_{i \in I} \mathcal{A}[S_i] \rightarrow \bigoplus_{j \in J} \mathcal{A}[S'_j]$  with kernel  $K$ , the map

$$R\underline{\mathrm{Hom}}_{\underline{\mathcal{A}}}(\mathcal{A}[S], K) \rightarrow R\underline{\mathrm{Hom}}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}}[S], K)$$

is an isomorphism for all  $S$ .

**Definition 30.5** An analytic ring  $\mathcal{A}$  is a pre-analytic ring which satisfies any of the previous equivalent conditions.

*Remark 30.6.* The analogous of property i) above was our original definition of solid abelian groups.

**Example 30.7**  $\mathbb{Z}, \blacksquare$  is analytic.

**Definition 30.8** Let  $\mathcal{A}$  be an analytic ring. We define with  $\mathbf{Mod}_{\mathcal{A}}^{cond}$  as the full subcategory of  $\mathbf{Mod}_{\underline{\mathcal{A}}}^{cond}$  of objects  $\mathcal{M}$  for which the map

$$\mathrm{Hom}_{\underline{\mathcal{A}}}(\mathcal{A}[S], \mathcal{M}) \rightarrow \mathcal{M}(S)$$

is an isomorphism for all  $S$ .

**Proposition 30.9**  $\mathbf{Mod}_{\mathcal{A}}^{cond} \subseteq \mathbf{Mod}_{\underline{\mathcal{A}}}^{cond}$  is stable under limits, colimits and extensions. The  $\mathcal{A}[S]$  form a family of compact projective generators. The inclusion functor admits a left adjoint

$$M \mapsto M \otimes_{\underline{\mathcal{A}}} \mathcal{A}$$

If  $\underline{\mathcal{A}}$  is commutative, then there is a unique symmetric monoidal tensor product  $\otimes_{\mathcal{A}}$  on  $\mathbf{Mod}_{\mathcal{A}}^{cond}$  such that the functor

$$M \mapsto M \otimes_{\mathcal{A}} \mathcal{A}$$

is symmetric monoidal.

The functor  $\mathbf{D}(\mathbf{Mod}_{\mathcal{A}}^{cond}) \rightarrow \mathbf{D}(\mathbf{Mod}_{\underline{\mathcal{A}}}^{cond})$  is fully faithful and its essential image is stable under limits, colimits and it is given by those  $C$  in  $\mathbf{D}(\mathbf{Mod}_{\underline{\mathcal{A}}}^{cond})$  such that

$$R\underline{\mathrm{Hom}}_{\underline{\mathcal{A}}}(\mathcal{A}[S], C) \rightarrow R\underline{\mathrm{Hom}}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}}[S], C)$$

is an isomorphism for all  $S$ . In that case, the derived internal homs also agree.

We also have  $C \in \mathbf{D}(\mathbf{Mod}_{\mathcal{A}}^{cond})$  if and only if all  $H^i(C)$  are in  $\mathbf{Mod}_{\mathcal{A}}^{cond}$ .



The functor admits a left adjoint which is the left derived of  $(-) \otimes_{\underline{\mathcal{A}}} \mathcal{A}$

$$C \mapsto C \otimes_{\underline{\mathcal{A}}}^L \mathcal{A}.$$

If  $\underline{\mathcal{A}}$  is commutative, then  $\exists! \otimes_{\underline{\mathcal{A}}}^L$  on  $\mathbf{D}(\mathbf{Mod}_{\underline{\mathcal{A}}}^{cond})$  such that the functor  $C \mapsto C \otimes_{\underline{\mathcal{A}}}^L \mathcal{A}$  is symmetric monoidal.

#### 30.4. New names for old things.

$$\mathbf{Solid} \leftrightarrow \mathbf{Mod}_{\mathbb{Z}_{\blacksquare}}^{cond}$$

$$(M \mapsto M^{\blacksquare}) \leftrightarrow (M \mapsto M \otimes_{\mathbb{Z}} \mathbb{Z}_{\blacksquare})$$

$$\otimes^{\blacksquare} \leftrightarrow \otimes_{\mathbb{Z}_{\blacksquare}}$$

$$(C \mapsto C^{L\blacksquare}) \leftrightarrow (C \mapsto C \otimes_{\mathbb{Z}}^L \mathbb{Z}_{\blacksquare})$$

$$\otimes^{L\blacksquare} \leftrightarrow \otimes_{\mathbb{Z}_{\blacksquare}}^L$$

#### 30.5. Some more results.

**Proposition 30.10** *Let  $f: \mathcal{A} \rightarrow \mathcal{B}$  be a morphism. The functor*

$$\mathbf{Mod}_{\underline{\mathcal{A}}}^{cond} \xrightarrow{\otimes_{\underline{\mathcal{A}}} \mathcal{B}} \mathbf{Mod}_{\underline{\mathcal{B}}}^{cond} \xrightarrow{\otimes_{\underline{\mathcal{B}}} \mathcal{B}} \mathbf{Mod}_{\mathcal{B}}^{cond}$$

*factors over  $\mathbf{Mod}_{\underline{\mathcal{A}}}^{cond}$  via a functor denoted  $M \otimes_{\mathcal{A}} \mathcal{B}$ .*

*Analogously for the derived version  $C \mapsto C \otimes_{\mathcal{A}}^L \mathcal{B}$ .*

**Proposition 30.11** *i)  $\mathbb{Z}_{p,\blacksquare}$  is analytic.  
ii) For any discrete  $A$ ,  $(A, \mathbb{Z}_{\blacksquare})$  is also analytic.*

*Proof.* Denote both analytic rings with  $\mathcal{A}$  for the moment. Take a complex  $C$  as in the definition above. We need to check that

$$R\mathbf{Hom}_{\underline{\mathcal{A}}}(\mathcal{A}[S], C) \rightarrow R\mathbf{Hom}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}}[S], C)$$

is an isomorphism for all  $S$ . Note that  $C$  is in  $\mathbf{D}(\mathbf{Solid})$ . We also have that  $\mathcal{A}[S] = \mathbb{Z}_{\blacksquare}[S] \otimes_{\mathbb{Z}_{\blacksquare}}^L \mathcal{A}$ . Now we get a long chain of equalities:

$$\begin{aligned} R\mathbf{Hom}_{\underline{\mathcal{A}}}(\mathcal{A}[S], C) &= R\mathbf{Hom}_{\underline{\mathcal{A}}}(\mathbb{Z}_{\blacksquare}[S] \otimes_{\mathbb{Z}_{\blacksquare}}^L \mathcal{A}, C) = \\ &= R\mathbf{Hom}(\mathbb{Z}_{\blacksquare}[S], C) = R\mathbf{Hom}(\mathbb{Z}[S], C) = R\mathbf{Hom}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}}[S], C) \end{aligned}$$

□

**Example 30.12** If we take  $(\mathbb{Q}_p, \mathbb{Z}_p)_{\blacksquare}$ , then  $(\mathbb{Q}_p, \mathbb{Z}_p)_{\blacksquare}[S]$  are the measures  $\mathcal{M}(S, \mathbb{Z}_p)[\frac{1}{p}]$  with  $p$  inverted, which is equal to  $\mathcal{M}^b(S, \mathbb{Q}_p)$ , where  $b$  stands for bounded.

## 31. [LT] LECTURE 13 - 03.12.19

Recall that a deformation retract of  $X$  onto  $A \subseteq X$  is a homotopy between the identity on  $X$  and a retraction  $r: X \rightarrow A$  (composed with the inclusion) relative to  $A$ .

**Example 31.1** Let  $f: X \rightarrow \mathbb{R}^n$  be a trivial fibration. Then  $f^{-1}(y_0)$  is a deformation retract of  $X$ .

**Proposition 31.2** Let  $f: X \rightarrow \mathbb{R}$  be a Morse function and  $\lambda \in \mathbb{R}$  a critical value. Let  $\lambda < M \in \mathbb{R}$  such that  $f$  does not have critical values in  $(\lambda, M]$ . Then for all  $0 < \varepsilon < M - \lambda$  the level set  $X_{\leq \lambda + \varepsilon}$  is a deformation retract of  $X_{\leq M}$ .

*Proof.* Apply the Ehresmann lemma to the restriction of  $f$  to  $X_{[\lambda + \varepsilon, M]}$  (for which we make this interval slightly bigger without introducing any critical values). Then there is a diffeomorphism

$$X_{[\lambda + \varepsilon, M]} \cong X_{\lambda + \varepsilon} \times [\lambda + \varepsilon, M].$$

The retraction of  $[\lambda + \varepsilon, M]$  to  $\{\lambda + \varepsilon\}$  induces a deformation on the boundary(?) (boundary? borders?) [...]

Now we want to study critical points. Let  $X = \mathbb{R}^2$  and  $f: X \rightarrow \mathbb{R}$  defined by  $(x_1, x_2) \mapsto -x_1^2 + x_2^2$ . The critical value is 0 and the critical point is  $(0, 0)$ . The fibre at 0 is  $X_0 = f^{-1}(0) = \{(x_1, x_2) \mid x_2^2 - x_1^2 = 0\}$ , which is the union of two lines. The preimage of some  $\alpha \in \mathbb{R}_{\neq 0}$  is

$$X_\alpha = \{(x_1, x_2) \mid (x_2 - x_1)(x_2 + x_1) = \alpha\}.$$

Setting the first factor to  $a$  and the second to  $b$ , we have equations of the form  $ab = 1$ ,  $ab = -1$ , and so on. The level sets are hyperbolas which are asymptotically coming closer to the fibre  $X_0$ . Nord and south correspond to  $\alpha > 0$ , whereas east and west correspond to  $\alpha < 0$ .

We want to get from  $X_{<-\varepsilon}$  to  $X_{<\varepsilon}$ . We glue on  $[-\sqrt{\varepsilon}, \sqrt{\varepsilon}] \subseteq \mathbb{R}$  ( $x_1$  axis). [Picture]

More generally, let  $X = \mathbb{R}^n$  and  $f(x_1, \dots, x_n) = -\sum_{i=1}^r x_i^2 + \sum_{i=r+1}^n x_i^2$ . The picture is the same but now the axes are higher dimensional. We want to compare  $X_{\leq -\varepsilon}$  to  $X_{\leq \varepsilon}$ . The interval is replaced by  $B^r(\varepsilon) = \{x \mid \sum_{i=1}^r x_i^2 \leq \varepsilon, x_{r+j} = 0\}$ . We have  $B^r(\varepsilon) \subseteq X_{\leq \varepsilon}$ .

Its intersection with  $X_{\leq -\varepsilon}$  is

$$B^r(\varepsilon) \cap X_{\leq -\varepsilon} = \partial B^r(\varepsilon) = S^{r-1}$$

**Lemma 31.3** With notation as above,  $X_{\leq \varepsilon}$  deformation retracts to  $X_{\leq -\varepsilon} \cup B^r(\varepsilon)$ .

*Proof.* We need to define a deformation from  $X_{\leq \varepsilon}$  to the subset (which is clear from the picture). Voisin [Voi07a, Prop.1.12] and Milnor [Mil63, p.15-19] write this down. We use a third method, inspired by the differential geometric approach to Morse theory. Let  $G: \mathbb{R}^n \rightarrow \mathbb{R}^n$  be the negative gradient field

$$G(x_1, \dots, x_n) = -(\partial_1 f, \partial_2 f, \dots, \partial_n f) = (2x_1, \dots, 2x_r, -2x_{r+1}, \dots, -2x_n)$$

Let  $\Phi$  be its flow. Going along the flow reduces the value of  $f$ . They flow in the direction of steepest descent. 0 is a fixed point. Most flow lines tend to infinity as  $t$  goes to  $\pm\infty$ . In particular, they connect a point on the level set  $X_\varepsilon$  to a point on the level set  $X_{-\varepsilon}$  after a finite amount of time. The flow is faster away from the origin, so we can do this whole deformation in finite time. On the exceptional set  $\{(x_1, \dots, x_r, 0, \dots, 0)\}$  the point 0 is repelling, and on  $\{(0, \dots, 0, x_{r+1}, \dots, x_n)\}$  it is attracting. The repelling set (also called the *unstable manifold*) is where we find  $B^r(\varepsilon)$ . We proceed as follows. Follow  $\Phi$  for a finite amount of time until the points on  $X_\varepsilon$  with  $|x| > R$  until they have reached  $X_{-\varepsilon}$ .

Then move along the flow  $(x, t) \mapsto (x_1, \dots, x_r, tx_{r+1}, \dots, tx_n)$ . In both cases we leave points stationary once they reached  $X_{\leq -\varepsilon} \cup B^r(\varepsilon)$ .  $\square$

**Theorem 31.4** Let  $X$  be a smooth manifold and  $f: X \rightarrow \mathbb{R}$  be a Morse function and  $\lambda \in \mathbb{R}$  a critical value. Let  $x_1, \dots, x_n$  be the critical points for the values  $r_1, \dots, r_n$ , called their Morse indices. Then there is  $\varepsilon > 0$  small enough such that there is a deformation retraction from  $X_{\leq \lambda + \varepsilon}$  to  $X_{\leq \lambda - \varepsilon} \cup (\cup_{i=1}^m B^{r_i})$  with cells  $B^{r_1}, \dots, B^{r_n}$  of dimensions  $r_1, \dots, r_n$  glued to  $X_{\leq \lambda - \varepsilon}$  along their boundaries such that their boundaries remain disjoint.

*Proof.* The idea is to use the negative gradient flow away from the critical points. The values  $\varepsilon$  can be chosen such that the exceptional balls are disjoint. Within the exceptional ball we use the deformation constructed in the lemma. However, the gradient is not a well-defined tangent vector. The invariant object is

$$df = (\partial_1 f)dx_1 + \dots + (\partial_n f)dx_n$$

In the proof of the lemma, we use an identification of  $T^*X$  and  $TX$ . More precisely, we choose a scalar product. Globally, such an object is called a Riemann metric. They exist (partition of unity). Hence the proof is complete.  $\square$

*Remark 31.5.* To ensure finiteness we may want to take  $X$  compact. Otherwise infinitely many balls, but no big deal in any case.

[Picture, contrasting the local picture above with our Morse picture of a torus]

## 32. [WS] DERIVED CATS 2 (LUCA) - INTRODUCTION TO DERIVED CATEGORIES - 04.12.19

**32.1. Overview.** Last week we introduced the category of coherent sheaves on any algebraic variety  $X$ . We showed that both  $\mathbf{Coh}(X)$  and  $\mathbf{Mod}(X)$  are abelian categories.

Today: given any abelian category  $\mathbf{A}$  we construct its derived category  $\mathbf{D}(\mathbf{A})$ , which roughly speaking is the category of cochain complexes of objects in  $\mathbf{A}$  in which we have inverted the class of all quasi-isomorphisms.

**Definition 32.1** A morphism  $I^\bullet \rightarrow J^\bullet$  in  $\mathbf{C}(\mathbf{A})$  is a quasi-isomorphism if all the maps induced in cohomology  $H^i(I^\bullet) \rightarrow H^i(J^\bullet)$  are isomorphisms.

Motivation: given  $\mathcal{F} \in \mathbf{Coh}(X)$  we can encode all cohomology groups  $H^i(X, \mathcal{F})$  into a single complex of coherent sheaves.

### 32.2. Construction of sheaf cohomology.

*Fact 32.2.* (1)  $\mathbf{Mod}(X)$  contains special objects called *injective objects*, which have nice algebraic properties.

(2)  $\mathcal{F} \in \mathbf{Mod}(X)$  admits an *injective resolution*, i.e. an exact sequence

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{I}^0 \rightarrow \mathcal{I}^1 \rightarrow \dots$$

in which all  $\mathcal{I}^i$  are injective sheaves. In other words, a quasi-isomorphism

$$\mathcal{F}[0] \rightarrow (\dots \rightarrow 0 \rightarrow \mathcal{I}^0 \rightarrow \mathcal{I}^1 \rightarrow \dots)$$

in which  $\mathcal{F}[0] = \mathcal{F}$  denotes the sheaf  $\mathcal{F}$  seen as a complex concentrated on degree zero.

**Definition 32.3** The cohomology groups  $H^i(X, \mathcal{F})$  are defined as follows:

- (1) Take an injective resolution  $\mathcal{F} \rightarrow \mathcal{I}^\bullet$ .
- (2) Apply the global section functor  $\Gamma(X, -)$  to the complex  $\mathcal{I}^\bullet$ .
- (3) Take cohomology on degree  $i$  of the resulting complex.

*Remark 32.4.* The functor  $\Gamma(X, -)$  is left exact, so  $H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$  as expected. But it is not exact, and this is the reason why we get interesting cohomology groups.

*Fact 32.5.* Let  $\mathcal{I}^\bullet, \mathcal{J}^\bullet$  be two injective resolutions of  $\mathcal{F}$ . Then the identity on  $\mathcal{F}$  induces a morphism  $\mathcal{I}^\bullet \rightarrow \mathcal{J}^\bullet$  which is uniquely defined up to homotopy. In particular, since the roles are symmetric,  $\mathcal{I}^\bullet$  and  $\mathcal{J}^\bullet$  are quasi-isomorphic via these maps. This implies that the cohomology groups are well-defined.

In practice we use other kinds of resolutions such as flasque resolutions, which are more meaningful geometrically.

*Fact 32.6.* If we have an injective resolution  $\mathcal{I}^\bullet$  and a flasque resolution  $\mathcal{L}^\bullet$  of our sheaf  $\mathcal{F}$ , then we have a quasi-isomorphism  $\mathcal{L}^\bullet \rightarrow \mathcal{I}^\bullet$  which is unique up to homotopy. Hence flasque resolutions compute the right cohomology groups.

Summarizing:

- (1) The injective resolution of  $\mathcal{F}$  is a more intrinsic cohomological invariant.
- (2) Flasque resolutions have more geometric nature than injective resolutions.

Let  $\mathcal{L}_1^\bullet$  and  $\mathcal{L}_2^\bullet$  be two flasque resolutions of  $\mathcal{F}$ . Unfortunately, we cannot find a quasi-isomorphism  $\mathcal{L}_1^\bullet \rightarrow \mathcal{L}_2^\bullet$ . Instead, we can find an injective resolution  $\mathcal{I}^\bullet$  of  $\mathcal{F}$  and a diagram of the form

$$\begin{array}{ccc} \mathcal{L}_1^\bullet & & \mathcal{L}_2^\bullet \\ & \searrow \text{qis} & \swarrow \text{qis} \\ & \mathcal{I}^\bullet & \end{array}$$

This will give isomorphisms between the cohomology groups of each flasque resolution, but they do not come from a morphism between the complexes. Therefore we want to invert quasi-isomorphisms.

**32.3. From abelian category to derived category.** Let  $\mathbf{A}$  be an abelian category. Its *derived category*  $\mathbf{D}(\mathbf{A})$  is constructed in two steps:

- (1) First we pass from cochain complexes to the homotopy category  $\mathbf{K}(\mathbf{A})$ , which is the quotient of  $\mathbf{K}(\mathbf{A})$  by chain-homotopies. Hence injective resolutions are unique (up to unique isomorphism) in  $\mathbf{K}(\mathbf{A})$ .
- (2) From  $\mathbf{K}(\mathbf{A}) \rightarrow \mathbf{D}(\mathbf{A})$  we pass by inverting quasi-isomorphisms. This is analogous to inverting non-zero integers to pass from  $\mathbb{Z}$  to  $\mathbb{Q}$ . The key property of the set  $S$  of quasi-isomorphisms in  $\mathbf{K}(\mathbf{A})$  which allows us to do this is the following. For all  $s \in S$  quasi-isomorphism and all diagrams in  $\mathbf{K}(\mathbf{A})$  of the form

$$\begin{array}{ccc} \mathcal{F}^\bullet & \xrightarrow{f} & (\mathcal{F}'')^\bullet \\ \downarrow s & & \\ (\mathcal{F}')^\bullet & & \end{array}$$

can be completed into a diagram<sup>13</sup>

$$\begin{array}{ccc} \mathcal{F}^\bullet & \xrightarrow{f} & (\mathcal{F}'')^\bullet \\ \downarrow s & & \downarrow t \\ (\mathcal{F}')^\bullet & \xrightarrow{g} & \mathcal{G}^\bullet \end{array}$$

<sup>13</sup>This cannot be taken to be a pushout diagram (see [?, Exercise 2.19]), but we can take it to be a *homotopy pushout*? (suggested by Sebastian)

in which  $t \in S$ . This allows us to define morphisms in  $D(A)$  by *calculus of fractions*: a morphism in  $D(A)$  is represented by a diagram

$$\begin{array}{ccc} \mathcal{F}^\bullet & & \mathcal{G}^\bullet \\ & \searrow f & \swarrow t \\ & \mathcal{H}^\bullet & \end{array}$$

which can be thought of as  $t^{-1}f$ .

**32.4. Triangles in  $D(A)$ .** In an abelian category  $A$ , every short exact sequence gives rise to a long exact sequence in cohomology. Indeed, from the short exact sequence  $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$  of objects in  $A$ , we get the short exact sequence  $0 \rightarrow (\mathcal{F}')^\bullet \rightarrow \mathcal{F}^\bullet \rightarrow (\mathcal{F}'')^\bullet \rightarrow 0$  of complexes. This yields then a long exact cohomology sequence in  $A$ .

*Question 32.7.* Given more generally a morphism  $f: \mathcal{F}^\bullet \rightarrow \mathcal{G}^\bullet$  in  $C(A)$ , can we put all maps  $H^i(f)$  into a long exact sequence?

**Definition 32.8** The *cone of  $f$*  is a complex  $C(f)^\bullet$  which on degree  $n$  is given by  $\mathcal{F}^{n+1} \oplus \mathcal{G}^n$  and whose differential is given by

$$d^n = \begin{pmatrix} -d_{\mathcal{F}^\bullet}^{n+1} & 0 \\ f_{n+1}^n & d_{\mathcal{G}^\bullet}^n \end{pmatrix}$$

*Fact 32.9.* (1) We have a short exact sequence

$$0 \rightarrow \mathcal{G}^\bullet \rightarrow C(f)^\bullet \rightarrow \mathcal{F}^\bullet[1] \rightarrow 0$$

where  $(\mathcal{F}^\bullet[1])^m = \mathcal{F}^{m+1}$  and  $d_{\mathcal{F}^\bullet[1]} = -d_{\mathcal{F}^\bullet}$ . This induces a long exact sequence in cohomology

$$\dots \rightarrow H^i(\mathcal{G}^\bullet) \rightarrow H^i(C(f)^\bullet) \rightarrow H^{i+1}(\mathcal{F}^\bullet) \xrightarrow{H^{i+1}(f)} H^{i+1}(\mathcal{G}^\bullet) \rightarrow \dots$$

(2) Given a short exact sequence

$$0 \rightarrow \mathcal{F}^\bullet \xrightarrow{f} \mathcal{G}^\bullet \xrightarrow{g} \mathcal{H}^\bullet \rightarrow 0$$

we get two quasi-isomorphisms  $C(f)^\bullet \rightarrow \mathcal{H}^\bullet$  and  $\mathcal{F}^\bullet[1] \rightarrow C(g)^\bullet$ , which in the derived category will become isomorphisms.

(3) We also get

$$\dots \rightarrow C(f)^\bullet[-1] \rightarrow \mathcal{F}^\bullet \rightarrow \mathcal{G}^\bullet \rightarrow C(f)^\bullet \rightarrow \mathcal{F}^\bullet[1] \xrightarrow{f[1]} \dots$$

in which the composition of any two consecutive morphisms is null-homotopic, hence 0 in the derived category.

*Remark 32.10.* In the previous situation we get a long exact sequence

$$\dots \rightarrow H^i(\mathcal{F}^\bullet) \rightarrow H^i(\mathcal{G}^\bullet) \rightarrow H^i(C(f)^\bullet) \rightarrow H^{i+1}(\mathcal{F}^\bullet) \rightarrow \dots$$

**Definition 32.11** A *triangle* in  $D(A)$  is a sequence of morphisms

$$(\mathcal{L}')^\bullet \rightarrow \mathcal{L}^\bullet \rightarrow (\mathcal{L}'')^\bullet \rightarrow (\mathcal{L}')^\bullet[1]$$

isomorphic (in  $D(A)$ ) to a sequence of the form

$$\mathcal{F}^\bullet \xrightarrow{f} \mathcal{G}^\bullet \rightarrow C(f) \rightarrow \mathcal{F}^\bullet[1],$$

where a morphism of triangles is defined as a commutative diagram between them in the straightforward way.

- Fact 32.12.* (1)  $(\text{id}, 0, 0)$  is a triangle.  
 (2) Every morphism  $\mathcal{F}^\bullet \rightarrow \mathcal{G}^\bullet$  is part of a triangle.  
 (3) Triangles are stable under rotation (watch out for the sign!).  
 (4) Any triangle gives rise to a long exact cohomology sequence.

*Fact 32.13.* We can always complete a morphism of triangles when one of the maps in the commutative diagram is missing.

**32.5. Derived functors.** Let  $\mathbf{A} = \mathbf{Coh}(X)$ . The naive extension of  $\Gamma(X, -)$  to  $\mathbf{D}(X)$  does not work (for example, it does not map zero objects to zero objects, so it would not preserve isomorphisms). But we can define a best approximation for  $\Gamma(X, -)$  on  $\mathbf{D}^+(\mathbf{Coh}(X))$ .

**Definition 32.14** The *right derived functor*  $\mathbf{R}\Gamma(X, -): \mathbf{D}^+(\mathbf{Mod}(X)) \rightarrow \mathbf{D}^+(\mathbb{C})$  is defined as follows:

- (1) For  $\mathcal{F} \in \mathbf{D}^+(\mathbf{Mod}(X))$ , take an injective resolution  $\mathcal{F}^\bullet \xrightarrow{\sim} \mathcal{I}^\bullet$ .
- (2) Apply  $\Gamma(X, -)$  to  $\mathcal{I}^\bullet$  componentwise. The resulting complex is by definition  $\mathbf{R}\Gamma(X, \mathcal{F}^\bullet)$ .

*Remark 32.15.* This is again well-defined and we may use flasque resolutions to compute.

*Remark 32.16.* The entire construction can be dualized replacing injective objects with projective objects, left exact functors with right exact functors, right derived functors with left derived functors and bounded below by bounded above.

**32.6. Useful formulas.** Functors on  $\mathbf{Coh}(X)$ :

- (1)  $\Gamma(X, -)$  left exact.
- (2)  $\text{Hom}(-, -)$  left exact in both variables.
- (3)  $\mathcal{H}om(-, -)$  left exact in both variables.
- (4)  $\otimes$  right exact.

Compatibilities (see [?, Chapter 3]).

### 33. [LT] LECTURE 14 - 05.12.19

**33.1. Vanishing spheres.**  $X$  complex manifold,  $f: X \rightarrow \mathbb{C}$  holomorphic,  $g = \text{Re}(f): X \rightarrow \mathbb{R}$ . Let  $x \in X$  be non-critical for  $f$ . Then  $x$  is non-critical for  $g$ .

**Lemma 33.1**  $X$  a complex manifold of dimension  $n$ ,  $f: X \rightarrow \mathbb{C}$  holomorphic,  $x \in X$  a non-degenerate critical point of  $f$ . Then  $x$  is also a non-degenerate critical point with respect to  $g$  of Morse index  $n$ .

*Proof.* We use local coordinates from the holomorphic Morse lemma.

$$f(z_1, \dots, z_n) = \sum_{j=1}^n z_j^2$$

for  $z_j = x_j + iy_j$ . Then

$$g(x_1, y_1, \dots, x_n, y_n) = \sum_{j=1}^n (x_j^2 - y_j^2).$$

□

We concentrate on the local behaviour. Let  $f: X \rightarrow V \subseteq \mathbb{C}$  be holomorphic, with  $0 \in \Delta \subseteq V$  a small enough open disk around 0.

**Theorem 33.2** Let  $f: X \rightarrow V$  be holomorphic with  $\dim X = n$ . Let  $x_0 \in X$  with  $f(x_0) = 0$  an ordinary double point. Let  $t \in \Delta^* := \Delta \setminus \{0\}$ . Then  $X_\Delta = f^{-1}(\Delta)$  deformation retracts to  $X_t \cup B_t^n$ , where  $B_t^n$  is an  $n$ -dimensional real ball (cell) glued to  $X_t$  along the vanishing sphere  $S_t^{n-1}$ .

*Proof.* Idea:

- (1)  $X_\Delta$  retracts to  $X_{\Delta, g \leq \varepsilon}$ .
- (2)  $X_{\Delta, g \leq -\varepsilon}$  retracts to  $X_{g=-\varepsilon}$ .
- (3) Morse theory tells us that we get from one to the other by gluing an  $n$ -cell.
- (4)  $X_{g=-\varepsilon} \sim X_{-\varepsilon}$ , all fibres are diffeomorphic away from the critical value by Ehresmann.

□

Near  $x_0$  we consider an open set  $\mathcal{U} \subseteq X$  homeomorphic to a ball in  $\mathbb{C}^n$ , local coordinates  $z_1, \dots, z_n$  on  $\mathcal{U}$  centered at  $x_0$  such that  $f(z_1, \dots, z_n) = \sum_{i=1}^n z_i^2$ . For  $t = se^{i\theta} \in \Delta^*$  we denote  $\mathcal{U}_t = \mathcal{U} \cap f^{-1}(t)$ ,

$$S_t^{n-1} = \{z \in \mathcal{U} \mid z_j = \sqrt{se^{i\theta/2}} x_j, x_j \in \mathbb{R}, \sum_{j=1}^n x_j^2 = 1\} \subseteq \mathcal{U}_t$$

and

$$B_t^n = \{z \in \mathcal{U} \mid z_j = \sqrt{se^{i\theta/2}} x_j, x_j \in \mathbb{R}, \sum_{j=1}^n x_j^2 \leq 1\} \subseteq \{z \in \mathcal{U} \mid |f(z)| \leq s\}.$$

We cannot make  $\mathcal{U}$  larger, but we can make  $\Delta$  smaller. Once  $\mathcal{U}$  is big enough compared to  $\Delta$ , the these sets do not depend on  $\mathcal{U}$ .  $S_t^{n-1}$  is a sphere and  $B_t^n$  is a ball.

**Definition 33.3**  $S_t^{n-1}$  is the *vanishing sphere* of the family  $(\mathcal{U}_\tau)_{\tau \in \Delta}$  and  $B_t^n$  is called the *cone of the vanishing sphere*. Its boundary is  $S_t^{n-1}$ .

Note that  $\lim_{t \rightarrow 0} S_t^{n-1} = \{0\} \subseteq \mathcal{U}_0$ .

*Remark 33.4.*  $S_t^{n-1}$  depends on the choice of coordinates. Once we have introduced homology, we will see that the homology class of  $S_t^{n-1}$  (the *vanishing cycle*) is independent of the choice of coordinates.

**Example 33.5** Let  $t = -\varepsilon \in \Delta^*$ . Then  $-\varepsilon = \varepsilon e^{\pi i}$ .

$$\begin{aligned} B_{-\varepsilon}^n &= \{z \in \mathcal{U} \mid z_j = \sqrt{\varepsilon} e^{\pi i/2} x_j, x_j \in \mathbb{R}, \sum_{j=1}^n x_j^2 \leq 1\} = \\ &= B_g^n (= \{(x_1, \dots, x_m) \mid x_{n+1} = x_{n+2} = \dots = x_m = 0, \sum_j x_j^2 \leq \varepsilon\}) \\ &= \{(x_1, y_1, \dots, x_n, y_n) \mid x_1 = x_2 = \dots = x_n = 0, \sum_j y_j^2 \leq \varepsilon\}. \end{aligned}$$

*Proof of the theorem above.*  $f: X \rightarrow V$ , restrict to  $X_\Delta = f^{-1}(\Delta) \rightarrow \Delta$ . WLOG  $t = -\varepsilon$ . We use the Ehresmann flow wrt  $g$ . It exists as long as we avoid 0. It induces a retract from  $X_{\Delta, g \leq -\varepsilon}$  to  $X_{\Delta, g=-\varepsilon}$  and from  $X_\Delta$  to  $X_{\Delta, g \leq \varepsilon}$ . We use the deformation of real Morse theory from  $X_{\Delta, -\varepsilon \leq g \leq \varepsilon}$  to  $X_{\Delta, g=-\varepsilon} \cup B_{-\varepsilon}^n$  via the vanishing sphere. The level set  $X_{\Delta, g=-\varepsilon}$  is the preimage under  $f$  of  $\Delta_{-\varepsilon} = \{z \in \Delta \mid \operatorname{Re}(z) = -\varepsilon\}$ . We use the Ehresmann flow to deform  $X_{\Delta, g=-\varepsilon}$  to  $X_{-\varepsilon}$ . □

### 33.2. Lefschetz pencils.

**Problem 33.6.**  $X$  compact  $\mathbb{C}$ -manifold. Then all  $f: X \rightarrow \mathbb{R}$  holomorphic are constant.

The solution will be to restrict our attention to algebraic varieties inside projective space and define an algebraic analogue of Morse functions.  $X \rightarrow \mathbb{C}$  replaced by  $X \dashrightarrow \mathbb{CP}^1$ .

**33.3. Recollections on algebraic varieties.** Definition, singular and regular points, dimension, irreducibility (we consider  $\emptyset$  irreducible).

**Definition 33.7** Let  $X \subseteq \mathbb{C}^N$  be an algebraic variety and  $x \in X$ .

- (1)  $T_x X \subseteq T_x \mathbb{C}^N$  is the subset of vectors  $v \in T_x \mathbb{C}^N$  such that  $df_y(v) = 0$  for all  $y \in X$  and all  $f \in \mathbb{C}[z_1, \dots, z_N]$  vanishing on  $X$ .
- (2)  $\hat{T}_x X \subseteq \mathbb{P}^N$  *projective completion* of  $T_x X$  is defined as  $\mathbb{P}(T_x X \oplus \mathbb{C})$ .

Let the projective space  $\mathbb{P}$  be written as  $\mathbb{P}(V)$  for some vector space  $V$ . Then the *dual* projective space is defined as  $\mathbb{P}(V^\vee)$ , denoted  $\mathbb{P}(V)^\vee = \mathbb{P}^\vee$ . Note that  $\xi \in \mathbb{P}^\vee$  corresponds to a class of linear forms  $[\xi: V \rightarrow \mathbb{C}]$  which in turn corresponds to a set of classes of vectors  $\{[v] \in \mathbb{P} \mid \xi(v) = 0\} =: H_\xi$ , which is a hyperplane in  $\mathbb{P}$ . On the other hand, a point  $p \in \mathbb{P}$  corresponds to a hyperplane  $H_p = \{H \subseteq \mathbb{P} \mid p \in H\}$  or equivalently to the hyperplane of classes  $\{[\xi] \in \mathbb{P}^\vee \mid \xi(p) = 0\}$ .

We have a canonical isomorphism  $\mathbb{P} \cong (\mathbb{P}^\vee)^\vee$ .

*Incidence variety:*  $\Phi = \{(x, \xi) \in \mathbb{P} \times \mathbb{P}^\vee \mid x \in H_\xi\} \subseteq \mathbb{P} \times \mathbb{P}^\vee = \{(x, \xi) \in \mathbb{P} \times \mathbb{P}^\vee \mid \xi \in H_x\}$ . This is a projective variety (via the Segre embedding  $\mathbb{P}^n \times \mathbb{P}^m \rightarrow \mathbb{P}^{(n+1)(m+1)-1}$ ).

**Remark 33.8.**  $(\mathbb{P} \times \mathbb{P}^\vee)^\vee \cong \mathbb{P}^\vee \times (\mathbb{P}^\vee)^\vee \cong \mathbb{P}^\vee \times \mathbb{P} \cong \mathbb{P} \times \mathbb{P}^\vee$ . Hence we may also regard  $\Phi$  as a subvariety of  $\mathbb{P}^\vee \times \mathbb{P}$ .

**Exercise 33.9.** We have two projections from  $\Phi$  to  $\mathbb{P}$  and to  $\mathbb{P}^\vee$ . Show that both are Zariski-locally trivial fibrations with fibres  $\mathbb{P}^{N-1}$ .

34. [FS] AMOS TURCHET - SPECIAL VS WEAKLY-SPECIAL MANIFOLDS - 06.12.19

Joint work in progress with E. Rousseau, J. Wang.

**34.1. Introduction. Main question:** characterize varieties  $X$  over  $k$  such that  $X(L)$  is Zariski dense for some finite field extension  $k \subseteq L$ .

**Definition 34.1**  $X$  over  $k$  projective non-singular (geometrically integral). We say that  $X(k)$  is *potentially dense* if there exists a finite field extension  $k \subseteq L$  such that  $X(L)$  is Zariski dense.

The question is then whether we can characterize this property in terms of geometry.

**Example 34.2** If  $\dim(X_k) = 1$ , we have a trichotomy distinguishing cases according to the genus. Assume  $k$  is a number field.

- (1) If  $g(X) = 0$ , then  $X(k)$  is either empty or Zariski-dense. For example, consider  $x^2 + y^2 + 1 = 0$  has no  $\mathbb{Q}$ -rational points. On the other hand, after a degree two extension, we get Zariski denseness ( $\mathbb{Q}(i)$ -rational points are Zariski dense). Hence  $X(k)$  is potentially dense. We can actually prove that it is always Zariski dense after a degree two extension.
- (2) If  $g(X) = 1$ , then by the Mordell-Weil theorem,  $X(k)$  is a finitely generated group, hence isomorphic to  $G \oplus \mathbb{Z}^r$  for  $G$  a finite group. There exists a finite extension  $k \subseteq L$  such that  $X(L)$  has a point of  $\infty$ -order, i.e.  $X(L) = G \oplus \mathbb{Z}^r$  with  $r \geq 1$ . So  $X(k)$  is potentially dense.



- (3) If  $g(X) \geq 2$ , then Faltings theorem tells us that  $X(L)$  is finite for all finite field extension  $k \subseteq L$ . In particular,  $X(k)$  is not potentially dense.

**34.2. Higher dimensions.**  $X/k$  general type.

**Conjecture 34.3 (Lang-(Bombieri))**  $X$  general type implies  $X(k)$  is not potentially dense. More precisely, there exists a closed subvariety  $Z \subseteq X$  with  $Z \neq X$  such that for all  $k \subseteq L$  we have  $\overline{X(L)}_{\geq 1} \subseteq Z$ , where  $\overline{X(L)}_{\geq 1}$  is the positive dimensional irreducible components of  $\overline{X(L)}$ .

If we believe in this conjecture, then we have the following:

- (1)  $X(k)$  is potentially dense implies that  $X$  is not of general type.
- (2)  $X \rightarrow Y$  dominant with  $\dim Y > 0$ . If  $X(k)$  is potentially dense, then  $Y$  is not of general type.

**Question 34.4 (J. Harris).** If  $X/k$  has no dominant maps  $X \rightarrow Y$  general type, is  $X(k)$  potentially dense?

**Example 34.5 (CT-S-SD '97)**  $D$  curve,  $g(D) = 1$ ,  $\rho: D \rightarrow D$  fixed point free involution.  $C$  curve,  $g(C) \geq 2$  hyperelliptic and  $\sigma: C \rightarrow C$  a hyperelliptic involution. Now we take the product  $Y = D \times_k C$  and  $\varphi: Y \rightarrow Y$  fixed point free involution. Let  $X = Y / \langle \varphi \rangle$ . Then

- (1)  $X$  is smooth geometrically integral with  $\kappa(X) = 1$ .
- (2) There is no dominant map  $X \rightarrow Z$  with  $Z$  general type of dimension  $\dim Z > 0$ .
- (3) For all finite field extensions  $k \subseteq L$ ,  $X(L)$  is not Zariski dense.

So what goes wrong with this variety? There are two explanations:

- (1)  $Y \rightarrow X$  is an étale cover. But  $Y$  also has a map to  $C$  with  $C$  of general type and positive dimension. This is relevant due to the Chevalley-Weil theorem.
- (2)  $X \rightarrow C / \langle \sigma \rangle \cong \mathbb{P}^1$  has  $n \geq 6$  double fibers, say over the points  $Q_1, \dots, Q_n \in \mathbb{P}^1$ .  $X_{Q_i}$  the fibre over  $Q_i$  is a double fibre. On  $\mathbb{P}^1$  we consider the divisor  $\Delta = \frac{1}{2}(Q_1 + \dots + Q_n)$ . Then  $(\mathbb{P}^1, \Delta)$  is of general type as a Campana orbifold, meaning that  $K_{\mathbb{P}^1} + \Delta$  is big.

More generally (Campana), we consider a morphism  $f: X \rightarrow (C, \Delta_f)$  in which  $\Delta_f = \sum (1 - \frac{1}{m_Q})Q$ , with  $m_Q$  the minimum among multiplicities of irreducible components of the fibre over  $Q$ .

**34.3. Two conjectures.**  $X/k$  with  $k$  a number field and  $X$  projective and nonsingular (geometrically integral). We have the following two conjectures:

- (1)  $X(k)$  is potentially dense if and only if there is no étale cover  $Y \rightarrow X$  such that  $Y$  dominates a variety of general type.
- (2) There exists no fibration  $f: X \rightarrow Z$  such that  $(Z, \Delta_f)$  is of (log) general type (meaning that  $K_Z + \Delta_f$ ).

Conjecture (1) is due to Abramovich-Colliot-Thelene and conjecture (2) is due to Campana.

**Definition 34.6** Campana called the  $X$  with property in (1) *weakly special* and the ones with property in (2) *special*.

**Remark 34.7.** (1) Special implies weakly special.  
(2) Weakly special implies special if  $\dim X \leq 2$ .

In dimension 3 we already have examples of things that are weakly special but not special.

**Example 34.8** (Bogomolov-Tschinkel, early 2000's)  $X \rightarrow S$  with  $S$  a surface and  $\kappa(X) = 1$ .

**Example 34.9** (Rousseau-T-Wang) Families of examples  $X_m \rightarrow S$ . The main feature here is that  $X_m$  is simply connected. Hence there are no étale covers whatsoever, so we only need to check that  $X_m$  does not map to anything of general type. But this very fibration  $X_m \rightarrow S$  yields a divisor  $\Delta$  on  $S$  such that  $(S, \Delta)$  is of general type.

The idea would be to show that the rational points of  $X_m$  (which is weakly special) are not potentially dense. But unfortunately, there is no control on  $X_m(L)$  for  $k \subseteq L$ . But we get analogue statements when  $k = K(C)$  (for  $K = \bar{K}$  of characteristic zero) or for entire curves  $\mathbb{C} \rightarrow X$ .

**Analogy:** if  $k$  is a number field, then its behaviour is close to that of a function field of a curve (on an alg. closed field of characteristic zero).

Assume  $X$  of general type. We have conjectures on the left: Lang ( $\overline{X(L)} \neq X$ ), Vojta (height bounds). On the right, we can think of a variety  $X$  over  $k(C)$  as a fibration  $\mathcal{X} \rightarrow C$ . A rational point would then be a section of this fibration. Height bound means then that there is  $A > 0$  constant such that  $\deg_H \sigma(C) \leq A(2g(C) - 2)$  outside of an exceptional set.

The idea is that special varieties are not the opposite of general type varieties, but rather of non-special varieties.

**Example 34.10**  $k$  a number field.  $f: C \times \mathbb{P}^1 \rightarrow C$  with  $g(C) \geq 2$  is not special but not of general type either.  $Z$  (the exceptional set) is  $\bigcup \overline{X(L)}_{\geq 1}$ .  $P \in C(L)$  finite.  $f^{-1}(P) \cong \mathbb{P}^1 \subseteq Z$ , which means that  $Z = X = C \times \mathbb{P}^1$ . So this is a bit of a problem (because  $X$  is not general type).

**Theorem 34.11** (Rousseau-T-Wang) We constructed  $X_m \rightarrow (S, \Delta)$  weakly special but not special. Then

- (1) We get height bounds for  $\sigma: C \rightarrow X_m \rightarrow (S, \Delta)$ .
- (2) Every entire curve  $f: \mathbb{C} \rightarrow X$  is degenerate, meaning that  $\overline{f(\mathbb{C})} \neq X$ .

35. [CM] TALK 8 (TANUJ) - SIX FUNCTOR FORMALISM - 09.12.19

Last time we defined *preanalytic rings* as the data

$$\mathcal{A} = (\underline{\mathcal{A}}, S \mapsto \mathcal{A}[S], S \rightarrow \mathcal{A}[S])$$

where  $\underline{\mathcal{A}}$  is a condensed ring, the second thing is a module in  $\mathbf{Mod}_{\underline{\mathcal{A}}}^{cond}$  and the third thing is a natural transformation.

**Example 35.1** A discrete ring. Then  $A_{\blacksquare} = (A, S \mapsto \lim A[S_i], \dots)$ .  $(A, \mathbb{Z})_{\blacksquare} = (A, S \mapsto \mathbb{Z}_{\blacksquare}[S] \otimes A)$ .

Recall also that a *analytic ring* is a preanalytic ring  $\mathcal{A}$  for which any complex

$$C := (\dots \rightarrow C_i \rightarrow \dots \rightarrow C_0 \rightarrow 0$$

in which the terms are direct sums of objects of the form  $\mathcal{A}[T]$  has the property that

$$\mathbf{R} \mathrm{Hom}_{\underline{\mathcal{A}}}(\mathcal{A}[S], C) \rightarrow \mathbf{R} \mathrm{Hom}_{\underline{\mathcal{A}}}(\mathcal{A}[S], C)$$

is an isomorphism.

We also saw that there exists a nice subcategory  $\mathbf{Mod}_{\underline{\mathcal{A}}}^{cond} \subseteq \mathbf{Mod}_{\underline{\mathcal{A}}}^{cond}$  which was explicitly described as the full subcategory of objects satisfying...

Let  $A$  be a finitely generated  $\mathbb{Z}$ -algebra. We will be looking at  $A = \mathbb{Z}[T]$ . The functor

$$j_*: D(A_{\blacksquare}) \rightarrow D((A, \mathbb{Z})_{\blacksquare})$$

admits a left adjoint  $j^*$  which in turn has a left adjoint  $j_!$ .

**Theorem 35.2** (1)  $A_{\blacksquare}$  is an analytic ring.

(2)  $j_*$  is a full inclusion  $D(A_{\blacksquare}) \subseteq D((A, \mathbb{Z})_{\blacksquare})$ .

(3)  $j^*: D((A, \mathbb{Z})_{\blacksquare}) \rightarrow D(A_{\blacksquare})$  admits a left adjoint  $j_!$ . Moreover, it satisfies

$$j_! j^*(M) = M \otimes_{(A, \mathbb{Z})_{\blacksquare}} j_! A.$$

**Goal 1:**  $A_{\blacksquare}$  is an analytic ring.

We immediately specialize to  $A = \mathbb{Z}[T]$ . Define  $A_{\infty} := \mathbb{Z}((T^{-1}))$  which is the Laurent power series in  $T^{-1}$ .

*Observation 35.3.* We have a short exact sequence

$$0 \rightarrow \mathbb{Z}[[U]] \otimes A \xrightarrow{(-) \cdot (UT^{-1})} \mathbb{Z}[[U]] \otimes A \xrightarrow{U \mapsto T^{-1}} A_{\infty} \rightarrow 0.$$

*Remark 35.4.* The first two guys in the previous short exact sequence are compact projective, hence  $A_{\infty}$  is compact as well.

*Observation 35.5.* If  $C \in D(\text{Cond}(A))$  whose terms are products of direct sums of  $A$ .

$$\mathbf{R}\underline{\text{Hom}}_A(A_{\infty}, C) = 0.$$

*Remark 35.6.* Through a series of reductions we can assume that  $C = A$  (this uses compactness of  $A_{\infty}$ ).

We can compute from the previous short exact sequence

$$\mathbf{R}\text{Hom}_{\mathbb{Z}}(\mathbb{Z}[[U]], A) \xrightarrow{UT^{-1}} \mathbf{R}\text{Hom}(\mathbb{Z}[[U]], A)$$

same as

$$\frac{A[U^{-1}]}{A} \xrightarrow{UT^{-1}} \frac{A[U^{-1}]}{A}.$$

Why is this true? We have a pairing  $\mathbb{Z}[[T]] \times \mathbb{Z}[T] \rightarrow \mathbb{Z}$ . Now  $\mathbb{Z}[[T]] \cong \prod_{\mathbb{N}} \mathbb{Z}$  and  $\mathbb{Z}[T] \cong \oplus_{\mathbb{N}} \mathbb{Z}$ . We use here the isomorphisms seen in previous lectures

$$\text{Hom}(\prod \mathbb{Z}, \mathbb{Z}) \cong \oplus \mathbb{Z} \quad \text{and} \quad \text{Hom}(\oplus \mathbb{Z}, \mathbb{Z}) \cong \prod \mathbb{Z}.$$

*Observation 35.7.* For any indexing set  $I$ , for the injective map

$$(\prod_I \mathbb{Z}) \otimes A \rightarrow \prod_I A$$

the cokernel is naturally an  $A_{\infty}$ -module.

Let us see now that  $A_{\blacksquare}$  is an analytic ring. We take any  $C$  of the form mentioned above. We need to show that for any  $S$  we have an isomorphism

$$\mathbf{R}\underline{\text{Hom}}(A[S], C) \xrightarrow{\sim} \mathbf{R}\underline{\text{Hom}}(A_{\blacksquare}[S], C).$$

Since  $C \in D((A, \mathbb{Z})_{\blacksquare})$  and  $(A, \mathbb{Z})_{\blacksquare}$  is an analytic ring, we have an isomorphism

$$\mathbf{R}\underline{\text{Hom}}(A[S], C) \xrightarrow{\sim} \mathbf{R}\underline{\text{Hom}}((A, \mathbb{Z})_{\blacksquare}, C).$$

We have  $(A, \mathbb{Z})_{\blacksquare}[S] = \mathbb{Z}_{\blacksquare}[S] \otimes A \cong (\prod_I \mathbb{Z}) \otimes A$  and  $A_{\blacksquare}[S] = \prod_I A$ . By the previous observation of being an  $A_{\infty}$ -module and since  $\mathbf{R}\underline{\text{Hom}}(A_{\infty}, C) = 0$  we have the desired isomorphism.

The goal now is defining  $j_!$ , which should be the left adjoint to  $j^*$ . The pullback  $j^*$  is defined as

$$j^*(M) = M \otimes_{(A, \mathbb{Z})_{\blacksquare}}^L A_{\blacksquare}.$$

First we define an endofunctor  $D((A, \mathbb{Z})_{\blacksquare}) \rightarrow D((A, \mathbb{Z})_{\blacksquare})$  by sending

$$M \mapsto M \otimes_{A \otimes_{\mathbb{Z}} \mathbb{Z}_{\blacksquare}} A_{\infty} / A[-1].$$

We can then find a dashed functor  $j_!$  yielding a factorization as in the commutative diagram below

$$\begin{array}{ccc} & D((A, \mathbb{Z})_{\blacksquare}) & \\ j_! \swarrow & \uparrow & \\ D(A_{\blacksquare}) & & D((A, \mathbb{Z})_{\blacksquare}) \\ & j^* \nwarrow & \end{array}$$

Assuming for the moment that we have the desired adjunction, we get that our functor is fully faithful for free, because

$$j^* j_* = \text{id} + j_! j^*.$$

$j_!$  is a fully faithful functor and left adjoint to  $j^*$ . We will show that  $j_! \dashv j^*$  by this iso...

For all  $M, N \in D((A, \mathbb{Z})_{\blacksquare})$ , it suffices to show

$$\mathbf{R}\underline{\text{Hom}}_A(j_! M, N) \stackrel{(1)}{\cong} \mathbf{R}\text{Hom}(j_! M, N \otimes_{(A, \mathbb{Z})_{\blacksquare}}^L A_{\blacksquare}) \stackrel{(2)}{\cong} \mathbf{R}\text{Hom}(M, N \otimes_{(A, \mathbb{Z})_{\blacksquare}}^L A_{\blacksquare}).$$

Isomorphism (2) follows from the cone of  $j_! M \rightarrow M$  being an  $A_{\infty}$ -modules. Isomorphism (1) follows from:

- (1)  $j^*$  has kernel an  $A_{\infty}$ -module.
- (2) The cone of  $N \rightarrow N \otimes_{(A, \mathbb{Z})_{\blacksquare}}^L A_{\blacksquare}$  is an  $A_{\infty}$ -module.

This implies that  $j_!$  is an adjoint as claimed.

What about  $j^!$ ? This is given by the theorem below (Theorem 8.2. in Scholze's notes).

What about other f.g.  $\mathbb{Z}$ -algebras? This is done in the appendix to lecture VIII.

**Theorem 35.8** *Let  $A$  be a finitely generated  $\mathbb{Z}$ -algebra. Let  $f: \text{Spec } A \rightarrow \text{Spec } \mathbb{Z}$ . We can form a left adjoint*

$$f_!: D(A_{\blacksquare}) \rightarrow D((A, \mathbb{Z})_{\blacksquare}) \rightarrow D(\mathbb{Z}_{\blacksquare}).$$

*Then  $f_!$  commutes with all direct sums, preserves compact objects and we have a projection formula*

$$f_!((M \otimes_{\mathbb{Z}_{\blacksquare}}^L A_{\blacksquare}) \otimes_{A_{\blacksquare}}^L N) \cong M \otimes_{\mathbb{Z}_{\blacksquare}}^L f_! M.$$

*We can also form a right adjoint*

$$f^!: D(\mathbb{Z}_{\blacksquare}) \rightarrow D(A_{\blacksquare})$$

*with  $f^! M = (M \otimes_{\mathbb{Z}_{\blacksquare}}^L A_{\blacksquare}) \otimes_{A_{\blacksquare}}^L f^! \mathbb{Z}$ .*

### 36. [LT] LECTURE 15 - 10.12.19

**Goal:** generalize the duality of  $(\mathbb{P}, \mathbb{P}^{\vee})$  seen last time to  $(X, X^{\vee})$  for  $X \subseteq \mathbb{P}$  a projective variety.

**Definition 36.1** The hyperplane  $\xi \in \mathbb{P}^{\vee}$  is *tangent to  $X$  at  $x \in X$*  if  $\hat{T}_x X \subseteq H_{\xi}$  or, equivalently,  $x \in H_{\xi}$  and  $T_x X \subseteq T_x H_{\xi}$ .

**36.1. Motivational example.** Let  $X$  be a nonsingular hypersurface in  $\mathbb{P}$ , i.e.  $X = V(f)$  for  $f$  a homogeneous nonconstant polynomial and  $X = X_{reg}$ . For all  $x \in X$  we have a hyperplane  $\hat{T}_x X \subseteq \mathbb{P}$ . We can define the generalization of the Gauss map:

$$\begin{aligned} \varphi: X &\rightarrow \mathbb{P}^\vee \\ x &\mapsto [\hat{T}_x X] \end{aligned}$$

In suitable coordinate charts,  $\varphi$  is given by  $\varphi(x) = df_x$ . Hence  $\varphi$  is an algebraic map. Now define

$$X^\vee = \varphi(X) = \{\xi \in \mathbb{P}^\vee \mid \xi \text{ tangent to } X \text{ at some } x \in X\}.$$

For all  $\xi \in X^\vee$  we have  $\dim_\xi(X^\vee) = N - 1$ . Intuition: we can move a tangent hyperplane ( $\hat{T}_x X$ ) in any direction along  $X$  and  $\dim(X) = N - 1$ .

*Remark 36.2.* Even if  $X$  is smooth,  $X^\vee$  may have singularities. For example, if  $\xi$  is tangent to  $X$  at  $x$  and  $y$  with  $x \neq y$ , then we have some singularities ( $X^\vee$  looks like a quotient of  $X$  by an equivalence relation near  $\varphi(x) = \varphi(y)$ ).

In particular, we do not have a definition of  $(X^\vee)^\vee$  yet.

**36.2. Dual varieties in general. Idea:**  $X \subseteq \mathbb{P}$  projective variety,  $x \in X$  and  $\xi \in \mathbb{P}^\vee$ . We say that  $\xi$  is *tangent to  $X$  at  $x$*  if

- (1)  $x \in X_{reg}$  and  $\xi$  is tangent to  $X_{reg}$  at  $x$ , or
- (2)  $x \in X_{sing}$  and  $\xi$  is a limit of hyperplanes tangent to  $X_{reg}$ .

$$X^\vee = \{\xi \in \mathbb{P}^\vee \mid \xi \text{ tangent to } X \text{ at some } x \in X\}.$$

Consider

$$\Psi_X^\circ = \{(x, \xi) \in \mathbb{P} \times \mathbb{P}^\vee \mid x \in X_{reg} \text{ and } \xi \text{ tangent to } X_{reg} \text{ at } x\} \subseteq \Psi_X,$$

where  $\Psi_X$  is the Zariski closure of  $\Psi_X^\circ$ . Since  $X_{reg} \subseteq X$  is open and tangency is a closed condition on  $X_{reg}$ , we see that  $\Psi_X^\circ$  is a quasi-projective variety. We have projections

$$\begin{array}{ccc} & \Psi_X & \\ p_1 \swarrow & & \searrow p_2 \\ \mathbb{P} & & \mathbb{P}^\vee \end{array}$$

**Definition 36.3** The *dual variety* of  $X$  is defined as  $X^\vee = p_2(\Psi_X) \subseteq \mathbb{P}^\vee$ .

- Remark 36.4.*
- (1)  $X^\vee \subseteq \mathbb{P}^\vee$  is Zariski closed, so  $X^\vee$  is a projective variety.
  - (2)  $X$  nonsingular implies that  $\Psi_X^\circ = \Psi_X$  is already closed.
  - (3)  $X$  nonsingular implies that  $p_1$  is a Zariski locally trivial fibration with fibres isomorphic to projective spaces of dimension  $N - 1 - \dim(X)$ .
  - (4)  $X$  nonsingular and irreducible, then  $\Psi_X$  is nonsingular and irreducible. Hence  $X^\vee$  is irreducible.
  - (5)  $X$  nonsingular hypersurface implies that  $p_1$  is an isomorphism and  $p_2$  is the Gauss map defined before.

**Goal:** imitate Morse function by choosing  $L = \mathbb{P}^1 \hookrightarrow \mathbb{P}^\vee$  such that  $L \cap (X^\vee)_{sing} = \emptyset$  and  $L$  is transverse to  $X^\vee$ .

*Notation 36.5.*  $\Phi_X = \{(x, \xi) \in \mathbb{P} \times \mathbb{P}^\vee \mid x \in X \cap H_\xi\} = X \times_{\mathbb{P}} \Phi \subseteq X \times \mathbb{P}^\vee$ .

*Remark 36.6.* We have a commutative diagram with morphisms whose fibres are as indicated below:

$$\begin{array}{ccccccc}
& & & & \mathbb{P}^\vee & & \\
& & & & \uparrow p_2, \mathbb{P}^N & & \\
& & & \mathbb{P}^{N-1} & \nearrow & & \\
\Psi_X & \hookrightarrow & \Phi_X & \hookrightarrow & \Phi & \hookrightarrow & \mathbb{P} \times \mathbb{P}^\vee \\
& \searrow \mathbb{P}^{N-1-\dim(X)} & & \searrow \mathbb{P}^{N-1} & \searrow \mathbb{P}^{N-1} & \searrow p_1, \mathbb{P}^N & \\
& & & X & \hookrightarrow & \mathbb{P} & 
\end{array}$$

**Remark 36.7.**  $X$  nonsingular irreducible implies that  $\Phi_X$  is nonsingular and irreducible, since  $p_1: \Phi_X \rightarrow X$  is a locally trivial fibration with fibres  $\mathbb{P}^{N-1}$ .

The fibers of  $p_2: \Phi_X \rightarrow \mathbb{P}^\vee$  are of the form  $p_2^{-1}(\xi) = X \cap H_\xi$ .

**Notation 36.8.** Define  $Y$  as the following pullback:

$$\begin{array}{ccc}
Y & \hookrightarrow & \Phi_X \\
\downarrow & & \downarrow p_2 \\
\mathbb{P}^1 = L & \xrightarrow{i} & \mathbb{P}^\vee
\end{array}$$

**Goal:** find a suitable inclusion  $i$  such that  $Y \rightarrow L$  behaves like a Morse function.

**36.3. Critical points of  $P_2$ .** Assume from now on that  $X$  is nonsingular.

**Lemma 36.9** (1) The critical points of  $p_2: \Phi_X \rightarrow \mathbb{P}^\vee$  are the points of the form  $(x, \xi)$  with  $\xi$  tangent to  $X$  at  $x$ , i.e. critical points of  $p_2$  are  $\Psi_X$ .  
(2)  $(x, \xi)$  critical point of  $p_2$  implies that

$$(dp_2)_{(x, \xi)}(T_{(x, \xi)}\Phi_X) = T_\xi(X^\vee).$$

*Proof.* For (1), let  $(x, \xi) \in \Phi_X$ . If  $\xi$  is not tangent to  $X$  at  $x$ , then each infinitesimal path through  $\xi$  consists of hyperplanes intersecting  $X$  near  $x$ . Hence  $(dp_2)_{(x, \xi)}$  is surjective. Conversely, if  $\xi$  is tangent to  $X$  at  $x$ , then the infinitesimal path through  $\xi$  in the direction normal to  $X$  consists of hyperplanes that do not intersect with  $X$  near  $x$  (except for  $\xi$  itself). This normal direction in  $T_\xi(\mathbb{P}^\vee)$  is not in the image of  $(dp_2)_{(x, \xi)}$ .

For (2), use similar arguments. The image of  $(dp_2)_{(x, \xi)}$  consists of infinitesimal paths through  $\xi$  consisting of hyperplanes intersecting  $X$  near  $x$ . If  $\xi$  is tangent to  $X$  at  $x$ , then such paths are the ones along which  $\xi$  remains tangent to  $X$ , i.e. elements of  $T_\xi(X^\vee)$ .  $\square$

**Theorem 36.10** Let  $X$  be a nonsingular irreducible projective variety  $\subseteq \mathbb{P}$  and  $x \in \mathbb{P}$ .

- (1)  $x \in X \Leftrightarrow H_x$  is tangent to  $X^\vee$ .
- (2)  $(X^\vee)^\vee = X$ .

*Proof.* Note that (1)  $\Leftrightarrow$  (2). Via the identification  $(\mathbb{P}^\vee)^\vee = \mathbb{P}$  we have

$$\Psi_{X^\vee} = \{(x, \xi) \in \mathbb{P} \times \mathbb{P}^\vee \mid H_x \text{ tangent to } X^\vee \text{ at } \xi\}$$

Condition (2) is equivalent to  $\Psi_{X^\vee} = \Psi_X$ . Let us check this.

We have  $x \in X \Leftrightarrow \exists \xi$  such that  $(x, \xi) \in \Psi_X$ .  $H_x$  tangent to  $X^\vee$  iff  $\exists \xi$  such that  $(x, \xi) \in \Psi_{X^\vee}$ . Both  $\Psi_X$  and  $\Psi_{X^\vee}$  are closed and irreducible subvarieties of the same dimension, so it suffices to show the inclusion  $\Psi_X \subseteq \Psi_{X^\vee}$ .

Let  $\Omega = \{(x, \xi) \in \Psi_X \mid \xi \in (X^\vee)_{\text{reg}}\} \subseteq \Psi_X$ . This can also be described as the set

$$\{(x, \xi) \mid \xi \text{ tangent to } X \text{ at } x \text{ and only at } x\}$$

We have  $\emptyset \neq \Omega \subseteq \Psi_X$  is a Zariski-open ( $(X^\vee)_{\text{reg}} \subseteq X^\vee$  is dense open and  $p_2: \Psi_X \rightarrow X^\vee$  is dominant). Hence  $\bar{\Omega} = \Psi_X$ , so it suffices to show that  $\Omega \subseteq \Psi_{X^\vee}$ .  $\{x\} \times H_x \subseteq \Phi_X$ .

$T_\xi(H_x) \stackrel{(*)}{\subseteq} (dp_2)(T_{(x,\xi)}\Phi_X$  by the lemma, and also  $(dp_2)(T_{(x,\xi)}\Psi_X \subseteq (dp_2)(T_{(x,\xi)}\Phi_X)$  and  $(dp_2)(T_{(x,\xi)}\Psi_X) \stackrel{(**)}{\subseteq} T_\xi(X^\vee)$ .

$(x, \xi) \in \Omega$  implies  $(**)$  is an equality.  $(*)$  is also an equality. Hence  $T_\xi(X^\vee) \subseteq T_\xi(H_x)$  and therefore  $H_x$  is tangent to  $X^\vee$  at  $\xi$ , i.e.  $(x, \xi) \in \Psi_{X^\vee}$ .  $\square$

37. [WS] DERIVED CATS 3 (SEVERIN) - THE DERIVED CATEGORY OF PROJECTIVE SPACE AND SEMIORTHOGONAL DECOMPOSITIONS - 11.12.19

**Main result:**

$$\mathbf{D}^b(\mathbb{P}^n) \cong \langle \mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n) \rangle \cong \mathbf{D}^b(A),$$

where  $A$  is a finite dimensional  $\mathbb{C}$ -algebra, namely

$$A \cong \text{End}(\mathcal{O} \oplus \mathcal{O}(1) \oplus \dots \oplus \mathcal{O}(n)).$$

Recall that from an abelian category  $\mathbf{A}$  we constructed first its homotopy category  $\mathbf{K}(\mathbf{A})$  and then its derived category  $\mathbf{D}(\mathbf{A})$ . For  $A, B \in \mathbf{A}$ , we have  $\text{Hom}_{\mathbf{D}(\mathbf{A})}(A, B[i]) \cong \text{Hom}_{\mathbf{D}(\mathbf{A})}(A, I^\bullet[i])$ , where  $I^\bullet$  is an injective resolution of  $B$ . Hence such a morphism is determined by  $f \in \text{Hom}(A, I^i)$  such that  $df = 0$ , hence  $f$  is nullhomotopic. This implies that

$$\text{Hom}_{\mathbf{D}(\mathbf{A})}(A, B[i]) \cong \text{Ext}^i(A, B).$$

**Definition 37.1** An object  $E \in \mathbf{D}(\mathbf{A})$  is called *exceptional* if

$$\text{Hom}_{\mathbf{D}(\mathbf{A})}(E, E[i]) = \begin{cases} \mathbb{C} & \text{if } i = 0 \\ 0 & \text{if } i \neq 0 \end{cases}$$

**Example 37.2** (1) By Kodaira vanishing, the line bundle  $\mathcal{O}(j)$  on  $\mathbb{P}^n$  is exceptional, because:

$$\text{Ext}^\bullet(\mathcal{O}(j), \mathcal{O}(j)) \cong H^\bullet(\mathcal{O}) = \begin{cases} \mathbb{C} & \text{if } \bullet = 0 \\ 0 & \text{otherwise.} \end{cases}$$

- (2) By the same argument, any line bundle on a Fano variety is exceptional.
- (3) Let  $x \in X$  be a point in a smooth projective variety. Let  $\tilde{X}$  be the blowup and  $E$  be the exceptional divisor. Then  $\mathcal{O}_E(j)$  is an exceptional object on  $\tilde{X}$ , seen as a sheaf on  $\tilde{X}$  via the pushforward along the inclusion.
- (4)  $\mathcal{O}_P$  is not exceptional, because

$$\text{Ext}^1(\mathcal{O}_P, \mathcal{O}_P) \cong T_P X.$$

- (5) A line bundle on a curve of genus  $\geq 1$  is not exceptional.
- (6) The structure sheaf on a K3 surface is not exceptional.

**Definition 37.3**  $E_1, \dots, E_n$  is an *exceptional collection* if each  $E_i$  is exceptional and  $\text{RHom}_{\mathbf{D}}^\bullet(E_j, E_i) = 0$  whenever  $j > i$ .

**Example 37.4** Again by Kodaira vanishing,  $\mathcal{O}, \dots, \mathcal{O}(n)$  on  $\mathbb{P}^n$  are an exceptional collection.

**Definition 37.5** An exceptional collection is *full* if  $E_i$  generate  $\mathbf{D}(\mathbf{A})$  as a triangulated category.

**Theorem 37.6 (Beilinson)**  $\mathcal{O}, \dots, \mathcal{O}(n)$  on  $\mathbb{P}^n$  is full.

*Sketch of the proof.*  $\mathcal{O}, \dots, \mathcal{O}(n)$ .

- (1) These sheaves generate skyscraper sheaves  $\mathcal{O}_{\{P\}}$  where  $\{l_1 = \dots = l_n = 0\}$ .

- (2) Exceptional objects  $\langle \mathcal{O}(j) \rangle \hookrightarrow \mathbf{D}^b(X)$  yield admissible subcategories, meaning that the inclusion functor has left and right adjoints.
- (3)  $\langle \mathcal{O}, \dots, \mathcal{O}(n) \rangle^\perp = 0$ , where  $(-)^\perp$  means the orthogonal complement, which is the full subcategory of objects  $F \in \mathbf{D}^b(X)$  such that  $\mathbf{RHom}(-, F)$  vanishes on  $\langle \mathcal{O}, \dots, \mathcal{O}(n) \rangle$ .

□

Now we generalize this:

**Definition 37.7**  $\mathbf{D}(X) \cong \langle C_1, \dots, C_n \rangle$  is called a *semiorthogonal decomposition* if all subcategories  $C_i$  are admissible and  $C_j \subseteq (C_i)^\perp$  whenever  $j < i$ .

*Remark 37.8.*  $E$  is exceptional if and only if  $\mathbf{D}^b(\mathbb{C}) \rightarrow \mathbf{D}^b(X)$  given by  $V^\bullet \mapsto V^\bullet \otimes E$  has adjoint  $\mathbf{RHom}(E, -)$ .

**Example 37.9**  $X \subseteq \mathbb{P}^n$  degree  $d \leq n$  hypersurface, then

$$\mathbf{D}^b(X) = \langle C, \mathcal{O}_X, \dots, \mathcal{O}_X(n-d) \rangle$$

for some admissible subcategory  $C$ .

38. [LT] LECTURE 16 - 12.12.19

Recall our setting:

$$\begin{array}{ccc} & \Phi = \{(x, \xi) \in \mathbb{P} \times \mathbb{P}^\vee \mid x \in H_\xi\} & \\ p_1 \swarrow & & \searrow p_2 \\ \mathbb{P} & & \mathbb{P}^\vee \end{array}$$

38.1. **Trivialization of  $p_1$  over  $U = \{x \in \mathbb{P} \mid x_0 \neq 0\}$ .**

$$U \times (\mathbb{P}^{N-1})^\vee \rightarrow p_1^{-1}(U)$$

$$(x, z) \mapsto (x, [-\sum_{k=1}^N x_k z_k : x_0 z_1 : \dots : x_0 z_N])$$

where  $z = [z_1 : \dots : z_N] \in \mathbb{P}^{N-1}$  is identified with the hyperplane  $V(\sum_{k=1}^N z_k w_k) \subseteq \mathbb{P}^{N-1}$ .

**Lemma 38.1 (Lemma A)** Let  $(a, \alpha) \in \Phi_X$ . The following are equivalent:

- (1)  $(a, \alpha) \in \Psi_X$ .
- (2)  $\hat{T}_a X \subseteq H_\alpha$ .
- (3)  $(dp_2)_{(T_{(a, \alpha)} \Phi_X)} \neq T_\alpha(\mathbb{P}^\vee)$ .
- (4)  $(a, \alpha)$  critical point of  $p_2: \Phi_X \rightarrow \mathbb{P}^\vee$ .
- (5)  $(dp_2)_{(T_{(a, \alpha)} \Phi_X)} = T_\alpha H_a$ .

*Proof.* Unwrapping the definitions we have

$$(1) \Leftrightarrow (2) \quad \text{and} \quad (3) \Leftrightarrow (4).$$

Choose coordinates on  $\mathbb{P}$  and  $\mathbb{P}^\vee$  such that  $a = [1 : 0 : \dots : 0]$  and  $H_\alpha = V(x_N)$ . Using the trivialization of  $p_1$  given above in this coordinate system,  $p_2$  looks like

$$\mathbb{A}^N \times \mathbb{A}^{N-1} \ni (x, u) \mapsto (-\sum_{k=1}^{N-1} x_k u_k - x_N, u_1, \dots, u_{N-1})$$

in a neighbourhood of  $(a, \alpha)$ .



For (1)  $\Leftrightarrow$  (3): the restriction of  $p_2$  to  $(\mathbb{A}^N \cap X) \times \mathbb{A}^{N-1}$  is not submersive if and only if  $dx_N|_{T_0(\mathbb{A}^N \cap X)} = 0$ , if and only if  $\hat{T}_\alpha X \subseteq H_\alpha$ .

For (3)  $\Leftrightarrow$  (5) we have  $\{a\} \times H_\alpha \subseteq \Phi_X$ . Hence  $T_\alpha H_\alpha = dp_2(T_{(a,\alpha)}(\{a\} \times H_\alpha)) \subseteq dp_2(T_{(a,\alpha)}\Phi_X)$ .  $\dim(T_\alpha H_\alpha) = N - 1$ . Hence  $T_\alpha H_\alpha = dp_2(T_{(a,\alpha)}\Phi_X) \Leftrightarrow dp_2(T_{(a,\alpha)}\Phi_X) < N$ .  $\square$

### 38.2. Transversality.

**Definition 38.2**  $N$  smooth manifold,  $L, M \subseteq N$  submanifolds. We say that  $L$  is *transverse* to  $M$  if for all  $p \in L \cap M$  we have

$$T_p L + T_p M = T_p N.$$

**Example 38.3** Coordinate hyperplanes in  $\mathbb{C}^N$ .

**Example 38.4 (Non-example)**  $H \subseteq \mathbb{P}$  hyperplane tangent to  $X \subseteq \mathbb{P}$ .

More generally,  $f: X \rightarrow Z$  and  $g: Y \rightarrow Z$  are *transverse* if for all  $x \in X$  and all  $y \in Y$  with  $z = f(x) = g(y)$  we have  $\text{im}(df_x) + \text{im}(dg_y) = T_z Z$ .

**Definition 38.5** A *projective line* is a subvariety  $L \subseteq \mathbb{P}$  cut out by  $N - 1$  linear homogeneous polynomials.

**Example 38.6**  $L = V(x_0, \dots, x_{N-2}) \cong \mathbb{P}^1$ .

**Definition 38.7** Let  $L \subseteq \mathbb{P}$  be a projective line and  $X \subseteq \mathbb{P}$  be a projective variety. Then  $L$  is *transverse* to  $X$  if

- (1)  $L \cap X_{\text{sing}} = \emptyset$ , and
- (2)  $L$  is transverse to  $X_{\text{reg}}$ .

**Remark 38.8.** Let  $L$  be transverse to  $X$ . Then  $\dim(X) = N - 1$  at each  $x \in L \cap X$ .

**Henceforth:**  $X \subseteq \mathbb{P}$  irreducible nonsingular projective variety,  $X \neq \mathbb{P}$ .

**Lemma 38.9** Let  $\xi \in \mathbb{P}^\vee \setminus X^\vee$ .

- (1)  $\mathbb{P}(\xi) = \{L \subseteq \mathbb{P}^\vee \mid L \text{ projective line and } \xi \in L\}$  is a projective space of dimension  $N - 1$ .
- (2)  $\{L \in \mathbb{P}(\xi) \mid L \text{ transverse to } X^\vee\} \subseteq \mathbb{P}(\xi)$  is open and dense.

*Proof.* (1) is an exercise. For (2), consider

$$\begin{aligned} \varphi: X^\vee &\rightarrow \mathbb{P}(\xi) \\ \eta &\mapsto \overline{\xi\eta} \end{aligned}$$

In other words,  $\eta$  is mapped to the projective line in  $\mathbb{P}^\vee$  spanned by  $\xi$  and  $\eta$ .

Facts:  $\varphi$  algebraic and  $X^\vee$  projective implies that  $\varphi(X^\vee) \subseteq \mathbb{P}(\xi)$  is closed, so  $\mathbb{P}(\xi) \setminus \varphi(X^\vee)$  is open.  $\varphi$  algebraic implies that  $\dim(\varphi(X^\vee)) \leq \dim(X^\vee)$ .

If  $\dim(X^\vee) < N - 1$ , then  $\dim(\varphi(X^\vee)) < N - 1$ . Hence  $\mathbb{P}(\xi) \setminus \varphi(X^\vee) \neq \emptyset$ .

If  $\dim(X^\vee) = N - 1$ , then let  $\eta \in (X^\vee)_{\text{reg}}$ .  $\dim(\hat{T}_\eta(X^\vee)) = N - 1$ .  $\eta$  critical point of  $\varphi$  iff  $\overline{\xi\eta} \subseteq \hat{T}_\eta(X^\vee)$  iff  $\overline{\xi\eta}$  not transverse to  $X^\vee$  at  $\eta$ .

Therefore

$$\{L \subseteq \mathbb{P}(\xi) \mid L \text{ not transverse to } X^\vee\} = \varphi((X^\vee)_{\text{sing}}) \cup \text{critical}(\varphi|_{(X^\vee)_{\text{reg}}}) \subsetneq \mathbb{P}(\xi).$$

$\square$

### 38.3. Construction of Lefschetz pencil.

**Definition 38.10** A *pencil of hyperplanes* is a projective line in  $\mathbb{P}^\vee$ .

Choose  $\xi_0 \in \mathbb{P}^\vee \setminus X^\vee$ .  $\xi_0$  classifies a hyperplane  $H_{\xi_0}$  transverse to  $X$  by definition of  $X^\vee$ .

Let  $L \subseteq \mathbb{P}^\vee$  be any projective line through  $\xi_0$  transverse to  $X^\vee$  [such exist by the lemma]. Let  $A = L^\vee = \{x \in \mathbb{P} \mid L \subseteq H_x\} \subseteq \mathbb{P}$  is the *axis* of the pencil  $L$ .

*Remark 38.11.*  $\text{codim}_{\mathbb{P}}(L^\vee) = 2$ . If  $L = \overline{\xi_0 \eta}$  for some  $\eta$ , then  $L^\vee$  is given by the hyperplanes in  $\mathbb{P}^\vee$  containing  $\xi_0$  and  $\eta$ .

*Remark 38.12.*  $L = \{\eta \in \mathbb{P}^\vee \mid A \subseteq H_\eta\}$  [hence the name axis].

*Notation 38.13.*  $Y = X \cap A$ ,  $\tilde{X} = \{(x, \xi) \in X \times L \mid x \in H_\xi\}$ . We have projections  $p_1: \tilde{X} \rightarrow X$  and  $p_2: \tilde{X} \rightarrow L$ , given by the restriction of the projections from  $\mathbb{P} \times \mathbb{P}^\vee$ . Moreover,  $\tilde{X} = p_1^{-1}(X) \cap p_2^{-1}(L) \subseteq \mathbb{P} \times \mathbb{P}^\vee$ .  $\tilde{Y} = p_1^{-1}(Y) = Y \times L \subseteq \tilde{X}$  closed subspace.

*Remark 38.14.*  $p_1: \tilde{X} \setminus \tilde{Y} \rightarrow X \setminus Y$  is an isomorphism with inverse given by  $x \mapsto (x, \xi)$ , where  $H_\xi$  is the hyperplane spanned by  $A$  and  $x$ .

*Remark 38.15.* For all  $\xi \in L$  we have  $p_2^{-1}(\xi) = X \cap H_\xi$  hyperplane section. We call  $p_2: \tilde{X} \rightarrow L$  a *Lefschetz fibration* and  $L$  a *Lefschetz pencil* on  $X$ .

**Proposition 38.16 (Proposition 1)**  $A$  is transverse to  $X$ . In particular,  $Y = X \cap A$  is of codimension 2 in  $X$ .

*Proof.* Suppose  $A$  is transverse to  $X$  at  $a \in A \cap X$ . Then  $\exists$  hyperplane  $H \subseteq \mathbb{P}$  such that  $\hat{T}_a X \cup A \subseteq H$ .

$$\begin{aligned} & \xRightarrow{A \text{ axis of } L} \exists \alpha \in L \text{ s.t. } H = H_\alpha \\ & \Rightarrow (a, \alpha) \in \Psi_X \\ & \Rightarrow \alpha \in X^\vee \end{aligned}$$

$L$  transverse to  $X^\vee$  implies that  $X^\vee$  is non singular of dimension  $N - 1$  at  $\alpha$ .

$$\xRightarrow{\text{Last time}} T_\alpha(X^\vee) = H_\alpha$$

$a \in A \Rightarrow L \subseteq H_a \Rightarrow \hat{T}_a X + L = H_a$ . Contradiction:  $L$  transverse to  $X^\vee$ .  $\square$

**Proposition 38.17 (Proposition 2)**  $p_2: \Phi_X \rightarrow \mathbb{P}^\vee$  is transverse to  $L$ . In particular  $\tilde{X} = p_2^{-1}(L)$  is non singular.

*Proof.* Let  $(a, \alpha) \in \tilde{X}$ . If  $p_2$  is submersive at  $(a, \alpha)$ , there is nothing left to do. Suppose  $(a, \alpha)$  is a critical point of  $p_2$ . By Lemma A,  $(a, \alpha) \in \Psi_X$ , hence  $(dp_2)(T_{(a, \alpha)}\Phi_X) = T_\alpha H_a \supseteq T_\alpha(X^\vee)$ .

Since  $L$  is transverse to  $X^\vee$  at  $\alpha$ ,  $(dp_2)(T_{(a, \alpha)}\Phi_X) + T_\alpha L \supseteq T_\alpha(X^\vee) + T_\alpha L = T_\alpha(\mathbb{P}^\vee)$ . Hence  $p_2$  is transverse to  $L$  at  $(a, \alpha)$ .  $\square$

**Proposition 38.18 (Proposition 3)** Critical points of  $p_2: \tilde{X} \rightarrow L$  do not intersect  $\tilde{Y}$ .

*Proof.* Let  $(a, \alpha) \in \tilde{X}$  be a critical point of  $p_2$ . By definition of axis  $A$ ,  $p_2: \tilde{Y} = Y \times L \rightarrow L$  is the projection.

By proposition 1,  $Y$  is nonsingular. Hence  $p_2$  is submersive along  $Y \times L$ .  $\square$

39. [FS] PIETER BELMANS - DEFORMATIONS OF HILBERT SCHEMES OF POINTS VIA DERIVED CATEGORIES - 13.12.19

Let  $k = \mathbb{C}$ ,  $X$  a smooth projective variety of dimension  $\geq 2$ . Whenever  $\dim X = 2$  we will denote  $X = S$ .

The ingredients for deformation theory of varieties are:

- (1)  $H^1(X, \mathcal{T}_X)$  parametrizes first order deformations, i.e. deformations of  $X$  over the dual numbers.
- (2)  $H^0(X, \mathcal{T}_X)$  parametrizes infinitesimal automorphisms of  $X$ , which we can regard as the Lie algebra of  $\text{Aut}(X)$ .
- (3)  $H^2(X, \mathcal{T}_X)$  gives obstructions to higher order liftings.

The talk of today's talk is to see how to compute dimensions of those vector spaces.

**39.1. Classification of surfaces.** The first step is to look at the Kodaira dimension  $\kappa(S) \in \{-\infty, 0, 1, 2\}$ .

$\kappa$	$H^0(\mathcal{T}_S)$	$H^1(\mathcal{T}_S)$	$H^2(\mathcal{T}_S)$
2	0	$\geq 0$	$\geq 0$
0 (only K3)	0	20	0
$-\infty$ (only $\mathbb{P}^2$ )	8	0	0

The number 0 in the middle column of  $\mathbb{P}^2$  tells us that  $\mathbb{P}^2$  is *rigid*.

**39.2. Hilbert schemes.** These are moduli spaces of ideal sheaves of colength  $n$ , i.e.  $\mathcal{I} \subseteq \mathcal{O}_S$  s.t. support of  $\mathcal{O}_S/\mathcal{I}$  is finite of length  $n$ , i.e. moduli of stable sheaves with Chern character  $(1, 0, -n)$ .

$$\text{Hilb}^n(S) \rightarrow \text{Sym}^n(S) := S^n / \text{Sym}_n$$

The quotient on the right is singular of dimension  $2n$ . This is a crepant resolution called *Hilbert-Chow* resolution.

**Example 39.1** If we take  $S$  a K3 surface, then  $\text{Hilb}^n(S)$  is a hyperkaehler variety (Beauville). [There exist 2 infinite families and 2 sporadic examples of hyperkaehler varieties, so it would be interesting to find more examples.]

**Question 39.2.** How does the deformation theory of  $S$  influence the deformation theory of  $\text{Hilb}^n(S)$ ?

Let  $\mathbb{H} = \text{Hilb}^n(S)$ .

$\kappa$	$H^0(\mathcal{T}_{\mathbb{H}})$	$H^1(\mathcal{T}_{\mathbb{H}})$	$H^2(\mathcal{T}_{\mathbb{H}})$
2 (general type)	(*)	(*)	(*)
0 (K3)	0	21	0
$-\infty$ ( $\mathbb{P}^2$ )	8	10	0

(\*) Fantucchi: deformations are isomorphic.

**Remark 39.3.** These numbers are independent of  $n$ .

**Theorem 39.4 (Hitchin)** If  $H^1(S, \mathcal{O}_S) = 0$ , then there exists a SES

$$0 \rightarrow H^1(S, \mathcal{T}_S) \rightarrow H^1(\mathbb{H}, \mathcal{T}_{\mathbb{H}}) \rightarrow H^0(S, \omega_S^\vee) \rightarrow 0$$

This SES explains some of the previous numbers in the table above.

**Remark 39.5 (Left column).** (1) Boisiine:  $\text{Aut}(S) \hookrightarrow \text{Aut}(\text{Hilb}^n(S))$  for all  $S$  s.t. this is an iso on the identity components, i.e.  $H^0(S, \mathcal{T}_S) \cong H^0(\mathbb{H}, \mathcal{T}_{\mathbb{H}})$ .

- (2) If  $S$  is a K3. Beauville constructed (in the 1980's) for a quartic surface an automorphism of  $\mathrm{Hilb}^2(S)$  which does not come from  $S$  itself. This shows that the previous inclusion can be strict.
- (3) B-Oberdieck-Rennemo:  $S$  (weak) Fano or of general type. The result is: the previous inclusion is an isomorphism unless  $n = 2$  and  $S = C_1 \times C_2$  is the product of two curves. The point is that the automorphism swapping the  $C_1$ 's in  $C_1 \times C_2 \times C_1 \times C_2 = S \times S$  can be lifted to  $\mathrm{Hilb}^2(C_1 \times C_2)$ .

**Question 39.6.** (1) Can we explain these extra deformations?

- (2) What about other moduli spaces of sheaves? Because every result mentioned so far depends very much on the Hilbert-Chow resolution.
- (3) What can we say about higher dimensions?

**Goal:** reprove Hitchin in a way that is easier to generalize (using derived categories).

### 39.3. Hochschild cohomology.

$$HH^i(X) := \mathrm{Ext}_{X \times X}^i(\Delta_* \mathcal{O}_X, \Delta_* \mathcal{O}_X) \cong \mathrm{Hom}_{\mathrm{D}^b(X \times X)}(\Delta_* \mathcal{O}_X, \Delta_* \mathcal{O}_X[i])$$

We have:

**Theorem 39.7** (Hochschild-Kostant-Rosenberg decomposition)

$$HH^i(X) \cong \bigoplus_{p+q=i} H^p(X, \Lambda^q \mathcal{T}_X)$$

And also:

**Theorem 39.8** (Lowen-Van der Bergh)  $HH^2(X)$  controls deformation theory of  $\mathrm{Coh}(X)$ , as an abelian category.

$$HH^2(X) \cong H^2(X, \mathcal{O}_X) \oplus H^1(X, \mathcal{T}_X) \oplus H^0(X, \Lambda^2 \mathcal{T}_X)$$

We will assume  $H^2(X, \mathcal{O}_X) = 0$ . The part on  $H^1(X, \mathcal{T}_X)$  controls geometric deformations, whereas  $H^0(X, \Lambda^2 \mathcal{T}_X)$  controls noncommutative deformations, which are related to Poisson geometry.

**Goal:** compare  $H^i(\mathcal{T}_{\mathbb{H}})$  to  $HH^{i+1}(S)$ .

**39.4. Comparison.** On  $S \times \mathbb{H}$  there is a universal ideal sheaf, which we will denote by  $\mathcal{I}$ , which exists because the Hilbert scheme of points is a *fine* moduli space. This gives an instance of a Fourier-Mukai transform

$$\begin{aligned} \Phi_{\mathcal{I}} : \mathrm{D}^b(S) &\rightarrow \mathrm{D}^b(\mathbb{H}) \\ \mathcal{E} &\mapsto \mathrm{R}\gamma_{\mathbb{H},*}(\gamma_S^* \mathcal{E} \otimes^{\mathrm{L}} \mathcal{I}) \end{aligned}$$

**Theorem 39.9** (Krug-Soma) If  $H^1(S, \mathcal{O}_S) = H^2(S, \mathcal{O}_S) = 0$ , then  $\Phi_{\mathcal{I}}$  is fully faithful.

This implies that the functor  $(-)*\mathcal{I} : \mathrm{D}^b(S \times S) \rightarrow \mathrm{D}^b(S \times \mathbb{H})$  sending  $\Delta_* \mathcal{O}_S$  to  $\mathcal{I}$  is fully faithful.

From now on: joint work with Theo Rouedschloleus, Lie Fu.

$$HH^i(S) = \mathrm{Hom}_{S \times S}(\Delta_* \mathcal{O}_S, \Delta_* \mathcal{O}_S[i]) = \mathrm{Ext}_{S \times \mathbb{H}}^i(\mathcal{I}, \mathcal{I})$$

We try to compute this ext group with the relative local to global spectral sequence.

$$E_2^{ij} = H^i(\mathbb{H}, \mathcal{E} \mathrm{xt}_{\gamma_{\mathbb{H}}}^j(\mathcal{I}, \mathcal{I})) \Rightarrow \mathrm{Ext}_{S \times \mathbb{H}}^{i+j}(\mathcal{I}, \mathcal{I})$$

where  $\mathcal{E} \mathrm{xt}_{\gamma_{\mathbb{H}}}$  is the  $j^{\mathrm{th}}$ -derived functor of  $\gamma_{\mathbb{H},*} \circ \mathcal{H} \mathrm{om}(\mathcal{I}, -)$ . Here  $\mathbb{H}$  could be another moduli space of sheaves.

- (1)  $j = 0$  get  $\mathcal{O}_{\mathbb{H}}$ .
- (2)  $j = 1$  get  $\mathcal{T}_{\mathbb{H}}$ .
- (3)  $j = 2$  get 0.
- (4)  $j \geq 3$  get 0.

*Exercise 39.10.* Write own  $E_2$  and use it to reprove Hitchins.

**39.5. Higher dimensions.** Usually,  $\text{Hilb}^n(X)$  is very singular, except when  $n = 2$  or  $n = 3$ . We will work in the case  $n = 2$ .

**Theorem 39.11** (B-Fu-Roued...)

$$\Phi_{\mathcal{J}}: \mathbf{D}^b(X) \rightarrow \mathbf{D}^b(\text{Hilb}^2(X))$$

is fully faithful if  $H^i(X, \mathcal{O}_X) = 0$  for all  $i \geq 1$ .

**Theorem 39.12** (B-Fu-Roued...) Given the previous result, the previous spectral sequence degenerates at  $E_2$ . Moreover, a partial description of what happens on  $E_2$  can be given.

**Corollary 39.13** Deformation theory of  $X$  is equal to the deformation theory of  $\text{Hilb}^2(X)$ .

**Example 39.14**  $\mathbb{P}^2$  is rigid but  $\text{Hilb}^2(\mathbb{P}^2)$  is not. On the other hand,  $\mathbb{P}^3$  is rigid and  $\text{Hilb}^2(\mathbb{P}^3)$  as well. Both of them have non-commutative deformations.

#### 40. [LT] LECTURE 17 - 17.12.19

*Recall:*  $f: X \rightarrow Z$  is transverse to  $g: Y \rightarrow Z$  if  $\text{im}(df_x) + \text{im}(dg_y) = \dim(T_z Z)$  for each  $x \in X$  and each  $y \in Y$  such that  $f(x) = z = g(y)$ .

**Proposition 40.1**  $f: X \rightarrow Y$  smooth map,  $i: Z \hookrightarrow Y$  smooth submanifold of codimension  $r$ . If  $f$  is transverse to  $i$ , then  $f^{-1}(Z) \subseteq X$  is a submanifold of codimension  $r$  as well.

*Proof.* The condition is local at each  $x \in f^{-1}(Z)$ . Let  $x \in f^{-1}(Z)$  and  $y = f(x) \in Z$ . Constant rank theorem implies that  $Z = V(g_1, \dots, g_r)$  with  $g_k$  independent smooth functions  $Z \rightarrow \mathbb{R}$ . Hence  $f^{-1}(Z) = V(g_1 \circ f, \dots, g_r \circ f)$ .  $g = (g_1, \dots, g_r): Z \rightarrow \mathbb{R}^r$  submersion.  $d(g \circ f)_x = dg_y \circ df_x$ .  $d(g \circ f)_x(T_x X) = dg_y(\text{im}(df_x))$ . This is equal to  $dg_y(\text{im}(df_x) + \ker(dg_y))$ , which in turn is equal to  $dg_y(\text{im}(df_x) + T_y Z)$  and equal to  $dg_y(T_y Y)$ , which by submersivity of  $g$  is equal to  $\mathbb{R}^r$ . Hence  $g \circ f$  is submersive at  $x$ . Thus  $f^{-1}(Z) = (g \circ f)^{-1}(0)$  is a submanifold of codimension  $r$ .  $\square$

**Proposition 40.2** (Proposition 4)  $p_2: \text{critpt}(p_2: \tilde{X} \rightarrow L) \xrightarrow{\sim} L \cap X^\vee$ , i.e., each singular fiber of  $p_2: \tilde{X} \rightarrow L$  has a unique singularity.

*Proof.* Similar to proofs from the end of the previous lecture.  $\square$

**Proposition 40.3** (Proposition 5) Each critical point  $(a, \alpha)$  of  $p_2: \tilde{X} \rightarrow L$  is non degenerate.

*Proof.* When proving proposition 4 we find that  $(a, \alpha)$  is also a critical point of  $p_2: \Phi_X \rightarrow \mathbb{P}^\vee$ . Hence  $T_a X \subseteq T_a H_\alpha$ .  $a \notin A$  implies that we can choose coordinates  $[z_0, \dots, z_N]$  on  $\mathbb{P}$  such that

- (1)  $a = [1, 0, \dots, 0]$ ,
- (2)  $A = V(z_0, z_N)$ ,
- (3)  $H_\alpha = V(z_N)$ , and
- (4)  $\Phi = \{([z_0, \dots, z_N], [\xi_0, \dots, \xi_N]) \mid \sum_k z_k \xi_k = 0\}$ .

Here  $\xi_i$  are a dual basis to the  $z_i$ . We want to describe  $p_2: \Phi_X \rightarrow \mathbb{P}^\vee$  in a neighbourhood of  $(a, \alpha)$ . Let  $U := \{x \in X \mid x = [1 : x_1 : \dots : x_N]\}$ . Last time: trivialization of  $p_1$  over  $U$ .  $\varphi: U \times \mathbb{P}^{N-1} \rightarrow p_1^{-1}(U)$  was given by

$$(x, [z_1, \dots, z_N]) \mapsto (x, [-\sum_{k \geq 0} x_k z_k : z_1 : \dots : z_N])$$

Let  $t_1, \dots, t_d$  be local holomorphic coordinates on  $X$  near  $a$ . Then we have functions  $x_k(t) = x_k(t_1, \dots, t_d)$  such that each point in a neighbourhood of  $a$  in  $X$  can be written in the form  $[1 : x_1(t) : \dots : x_N(t)]$ . Hence we get local coordinates on  $\Phi_X$  near  $(a, \alpha)$

$$(t_1, \dots, t_d, z_1, \dots, z_{N-1}) \mapsto \varphi([1 : x_1(t) : \dots : x_N(t)], [z_1 : \dots : z_{N-1}])$$

The map  $p_2: \Phi_X \rightarrow \mathbb{P}^\vee$  takes the form

$$(t_1, \dots, t_d, z_1, \dots, z_{N-1}) \mapsto (-\sum_{k \geq 0} x_k(t) z_k, z_1, \dots, z_{N-1}) =: (g(t, z), z_1, \dots, z_{N-1})$$

Now compute the Jacobian matrix of  $p_2$  in these coordinates:

$$\begin{pmatrix} \frac{\partial g}{\partial t_1} & \cdots & \frac{\partial g}{\partial t_d} & * & \cdots & * \\ & & 0 & & I & \end{pmatrix}$$

$\Psi = \text{critpt}(p_2: \Phi_X \rightarrow \mathbb{P}^\vee) = \{p \in \Phi_X \mid \text{the above Jacobian does not have full rank}\}$ . Hence  $\Psi = V(\frac{\partial g}{\partial t_1}, \dots, \frac{\partial g}{\partial t_d})$  locally. Hence  $p_2|_{\Psi_X}$  has rank  $N-1$ . Thus the Jacobian of  $\frac{\partial g}{\partial t_k}$  with the Jacobian of  $p_2$  has rank  $N+d-1$ .

$$\begin{pmatrix} \frac{\partial^2 g}{\partial t_1^2} & \cdots & \frac{\partial^2 g}{\partial t_1 \partial t_d} & & \\ \vdots & & \vdots & & * \\ \frac{\partial^2 g}{\partial t_1 \partial t_d} & \cdots & \frac{\partial^2 g}{\partial t_d^2} & & \\ 0 & \cdots & 0 & I & \end{pmatrix}$$

Hence the Hessian has full rank  $d$ , so  $f$  has non-degenerate critical point at  $(a, \alpha)$ .  $\square$

**40.1. Singular homology.** References: Spanier and Hatcher. Definition and computations for points and contractible spaces.

41. [WS] DERIVED CATS 4 (PEDRO) - SERRE DUALITY ON THE DERIVED CATEGORY - 18.12.19

See script on [Github](#).

42. [LT] LECTURE 18 - 19.12.19

Homotopy invariance of singular homology. Excisive couples (from Spanier). Skeleta relative one lower skeleta. Long exact sequences, which can be arranged into a spectral sequence. Excision. Eilenberg-Steenrod axioms.

43. [FS] SIMON FELTEN - SMOOTHING (TOROIDAL) NORMAL CROSSING SPACES - 20.12.19

Joint work with M. Filip, H. Ruddit.

### 43.1. What does this mean?

**Definition 43.1** A complex space  $X$  is a *normal crossing space* if it is locally isomorphic to  $\{x_1 \cdots x_r = 0\} \in \mathbb{C}^n$  for  $r \leq n$ . It is *simple* if its irreducible components are smooth.

**Example 43.2** The crossing of two lines is SNC. The nodal curve is NC but not SNC. The cusp is not NC.

**Example 43.3**  $X = \{x_0 \cdots x_3 = 0\} \subseteq \mathbb{P}^3$ .

$$\begin{array}{ccc} & X & \xrightarrow{\quad} \mathcal{X} \\ \text{Definition 43.4 (Smoothing)} & \downarrow & \downarrow \\ & \{0\} & \xrightarrow{\quad} \Delta = \{z \in \mathbb{C} \mid |z| < 1\} \end{array}$$

**Example 43.5**  $\mathcal{X} = \{sx_0x_1x_2x_3 - t(x_0^4 + \cdots x_3^4)\} \subseteq \mathbb{P}^3 \times \mathbb{P}^1$  mapping to  $\mathbb{P}^1$ .

*Question 43.6.* Which compact nc spaces have a smoothing?

**43.2. Spectral sequences.** First we start with the classical case. Let  $X$  be a compact Kaehler manifold. We have the Hodge decomposition:

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^q(X, \Omega_X^p)$$

This is not well behaved under deformations, because holomorphicity isn't. A key substitute is the Hodge filtration, and related to it the degeneracy of the Hodge-de Rham spectral sequence

$$E_1^{p,q} = H^1(X, \Omega_X^p) \Rightarrow \mathbb{H}^{p+q}(X, \Omega_X^\bullet) \cong H^{p+q}(X, \mathbb{C})$$

This sequence degenerates at  $E_1$ .

**Algebraic proof:** Faltings in 1985, Deligne-Illusie in 1987.

Now we look at the open case. Consider  $D \subseteq X$  an nc divisor. Then  $j: U = X \setminus D \hookrightarrow X$  and we can consider *logarithmic differential forms*  $\Omega_X^\bullet(\log D) \subseteq j_*\Omega_U^\bullet$ .

**Example 43.7**  $\{z_1 = 0\} \subseteq \mathbb{C}^2$ . Then the log differentials are  $\mathcal{O}_{\mathbb{C}^2} \frac{dz_1}{z_1} \oplus \mathcal{O}_{\mathbb{C}^2} dz_2$ .

In this situation we also have the Hodge-de Rham spectral sequence.

$$E_1^{p,q} = H^q(X, \Omega_X^p(\log D)) \Rightarrow \mathbb{H}^{p+q}(X, \Omega_X^\bullet(\log D)) \cong H^{p+q}(U, \mathbb{C})$$

Finally we come to the toroidal case. A *toroidal pair* is a pair  $(X, D)$  where  $D$  is a divisor such that etale locally  $D \subseteq X$  is isomorphic to  $D_\Sigma \subseteq X_\Sigma$ .

$j: V \subseteq X$  locus where  $X$  is regular, then  $D|_V \subseteq V$  is nc.  $\Omega_X^\bullet(\log D) = j_*\Omega_V^\bullet(\log D|_V)$  is the de Rham complex.

**Theorem 43.8 (F-Filip-Ruddet, Danilov's conjecture)** *The Hodge-de Rham spectral sequence  $E_1^{p,q} = H^q(X, \Omega_X^p(\log D)) \Rightarrow \mathbb{H}^{p+q}(X, \Omega_X^\bullet(\log D))$  degenerates at  $E_1$ .*

History:

- (1) Danilov:  $D = \emptyset$ .
- (2) Kato-Illusie-Nakayama:  $D$  is locally the entire toric bdy.
- (3) Steenbrink: related result in the orbifold case.

**43.3. Smoothing.** We first recall *semistable reduction*. In the analytic case we have

$$\begin{array}{c} \mathcal{Y} \\ \downarrow \pi \\ \Delta \end{array}$$

an algebraic degeneration, flat and proper, such that  $\mathcal{Y}_t$  is smooth for  $t \neq 0$ . Now we can consider the base change

$$\begin{array}{ccccc} \mathcal{X} = \text{Bl}(\tilde{\mathcal{Y}}) & \longrightarrow & \tilde{\mathcal{Y}} & \longrightarrow & \mathcal{Y} \\ & \searrow f & \downarrow \tilde{\pi} & & \downarrow \pi \\ & & \Delta & \xrightarrow{z \mapsto z^n} & \Delta \end{array}$$

such that  $\mathcal{X}$  is smooth,  $f^{-1}(0) \subseteq \mathcal{X}$  is snc divisor.  $f$  is called a *semistable degeneration*.

**Theorem 43.9 (Friedmann 1983)** *X compact SNC space.  $X = f^{-1}(0)$  for some semistable  $f: \mathcal{X} \rightarrow \Delta$ . Then  $\mathcal{T}_X^1 = \mathcal{E}xt^1(\Omega_X^1, \mathcal{O}_X) \cong \mathcal{O}_D$  for  $D = X_{\text{sing}}$ . ( $X$  is  $d$ -semistable).*

*If  $X$  is an SNC surface, compact and Kaehler,  $d$ -semistable,  $K_X = 0$ ,  $H^1(X, \mathcal{O}_X) = 0$ . Then  $\exists f: \mathcal{X} \rightarrow \Delta$  semistable with  $f^{-1}(0) \cong X$ .*

**Theorem 43.10 (Kawamata-Namikawa, 1994)** *X compact SNC space, Kaehler comp.,  $\dim X = d$ .  $K_X = 0$ ,  $H^{d-1}(X, \mathcal{O}_X) = 0$ .  $H^{d-2}(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0$ ,  $d$ -semistable. Then  $\exists$  semistable  $f: \mathcal{X} \rightarrow \Delta$  with  $f^{-1}(0) \cong X$ .*

$d$ -semistability is closely related to  $\mathcal{X}$  being smooth.

**Theorem 43.11 (F-Filip-Ruddet)** *X proper algebraic NC space. Assume that  $-K_X$  is effective.  $\mathcal{T}_X^1$  is globally generated,  $D = X_{\text{sing}}$  is projective. Then  $\exists$  smoothing.*

#### 43.4. Examples.

**Example 43.12**  $\mathbb{P}^3$  is a smooth Fano variety.  $D \subseteq \mathbb{P}^3$  smooth anticanonical divisor. We get  $X$  by gluing two copies of  $\mathbb{P}^3$  along this anticanonical divisor, i.e.

$$X = \mathbb{P}^3 \sqcup_D \mathbb{P}^3$$

Then  $\mathcal{T}_X^1 \not\cong \mathcal{O}_D$ .

**Remark 43.13.**  $\mathcal{X} \rightarrow \Delta$  is semistable outside of a locus  $Z$ .

**Example 43.14**  $X = \{xy = 0\} \subseteq \mathbb{C}^3$ .  $D \cong \mathbb{C}$ .  $t \in \mathcal{T}_X^1$ . Then  $Z = \{s = 0\} = \{x\}$ . Smoothing:

$$\begin{array}{ccc} X & \hookrightarrow & \mathcal{X} = \{xy - ts = 0\} \subseteq \mathbb{C}^4 \\ \downarrow & & \downarrow \\ \{0\} & \longrightarrow & \mathbb{C}_s \end{array}$$

44. [LT] LECTURE 19 - 07.01.20

Recall that on the algebraic geometry side we had a variety  $X \subseteq \mathbb{P}^n$  and a fibration

$$\begin{array}{ccc} X & \longleftarrow & \tilde{X} \subseteq \mathbb{P}^n \times (\mathbb{P}^n)^\vee \\ \downarrow & & \swarrow \\ L \cong \mathbb{P}^1 & \subseteq & (\mathbb{P}^n)^\vee \end{array}$$



The fibres of the morphism  $\tilde{X} \rightarrow L$  have only ordinary double points, i.e. non-degenerate critical points.

On the differential geometry side, Morse theory gives us information about the topology of  $\tilde{X}$ .

Finally, on the algebraic topology side we gave a brief introduction to singular homology, which will be the invariant appearing in the statement of our goal result.

**Example 44.1** For  $n > 0$  we have

$$H_i(S^n, R) = \begin{cases} R & \text{if } i = 0, n \\ 0 & \text{if } i \neq 0, n \end{cases}$$

The next aim is to formulate Poincaré duality. For this we will need singular cohomology.

#### 44.1. Singular cohomology.

**Definition 44.2** *Singular cohomology* with coefficients in  $R$  is defined as the cohomology of the singular cochain complex, which is the dual of the singular chain complex.

*Remark 44.3.* If  $R$  is a field, then dualizing is exact. This is not the case anymore for the integers.

For  $R = \mathbb{Z}$  we have the universal coefficient theorem

$$0 \rightarrow \text{Ext}^1(H_{n-1}(X, \mathbb{Z}), \mathbb{Z}) \rightarrow H^n(X, \mathbb{Z}) \rightarrow \text{Hom}(H_n(X, \mathbb{Z}), \mathbb{Z}) \rightarrow 0.$$

**Example 44.4** Since the homology groups of spheres are free finitely generated, cohomology with  $\mathbb{Z}$  coefficients is dual of homology.

Define also relative singular cohomology. The dual of  $S_*(X, A, R) := S_*(X, R)/S_*(A, R)$  is the kernel of the restriction map  $S^*(X, R) \rightarrow S^*(A, R)$ . Relative cohomology is then defined as the cohomology of this complex. In particular, we get a long exact sequence associated to the pair  $(X, A)$ .

Cohomology has a ring structure given already on the level of cochains by the *cup product* (see Hatcher).

**Definition 44.5** Let  $S_c^*(X, R) \subseteq S^*(X, R)$  be the subcomplex of cochains  $\sigma$  such that there is a compact  $K \subseteq X$  such that  $\sigma$  vanishes on all chains on  $X \setminus K$ . The cohomology of this complex is called *cohomology with compact supports*. Equivalently,

$$H_c^i(X, R) = \varinjlim_K H^i(X, X \setminus K, R).$$

By excision, the right hand side only depends on a neighbourhood of  $K$  in  $X$ .

*Remark 44.6.* If  $X$  is compact, then cohomology with compact support agrees with usual singular cohomology.

*Remark 44.7.* Cohomology with compact supports is not homotopy invariant, e.g.  $\mathbb{R}^n$  and a point.

There is a pairing  $\cap: S_k(X, R) \times S_c^l(X, R) \rightarrow S_{k-l}(X, R)$  inducing the *cap product* (see Hatcher).

**44.2. Orientations.** Consider the  $\mathbb{R}$ -vector space  $V$  of dimension  $d$ . An *orientation* on  $V$  is a choice of an equivalence class of bases, where two bases are equivalent if the transition matrix has positive determinant. Alternatively, take  $\det V = \Lambda^d V$ , which is a 1-dimensional  $\mathbb{R}$ -vector space. An orientation of  $V$  is a choice of an  $\mathbb{R}_{>0}$  equivalence classes of bases of  $\det V$ .

**Example 44.8** If  $V$  is a complex vector space of dimension 1 and  $v \in V$  is non-zero, then  $v$  gives rise to the  $\mathbb{R}$ -basis  $(v, iv)$ . Consider the basis change given by multiplication with  $\alpha = a + ib \in \mathbb{C}^\times$ . Over  $\mathbb{R}$  we get the base change matrix

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

The determinant is always positive, hence  $v$  induces a canonical orientation of  $V$  as an  $\mathbb{R}$ -vector space.

*Exercise 44.9.* If  $V$  is a  $\mathbb{C}$ -vector space of higher dimension, then any complex basis gives rise to a canonical orientation of  $V$  as an  $\mathbb{R}$ -vector space.

**Definition 44.10** Let  $X$  be a smooth manifold. An *orientation* of  $X$  is the choice of an orientation of the tangent bundle  $TU$  for all charts  $U \subseteq X$  such that the transition maps are orientation preserving.  $X$  is called *orientable* if an orientation exists.

**Example 44.11**  $\mathbb{R}P^2$  is not orientable.

**Lemma 44.12** Let  $X$  be a complex manifold. Then  $X$  is orientable.

*Proof.* Consider the transition matrix  $(\varphi_{a,b})_{a,b}$  from one holomorphic chart to the other. The induced base change maps on  $TU$  are  $\mathbb{C}$ -linear, and by the exercise above they preserve the orientation.  $\square$

**Definition 44.13** Let  $X$  be a smooth manifold of dimension  $d$ . We consider *local homology*  $H_d(X, X \setminus \{x\}, R)$  of  $X$  at  $x$ . By excision, we may replace  $X$  by a ball in a coordinate chart, hence by homotopy equivalence we have

$$H_d(X, X \setminus \{x\}, R) \cong H_d(B^d, S^{d-1}, R) \cong R.$$

An  $R$ -orientation of  $X$  at  $x$  is a choice of a generator of  $H_d(X, X \setminus \{x\}, R)$ . An *orientation* of  $X$  is a function  $\mu: x \mapsto \mu_x$  which assigns to every point  $x$  an  $R$ -orientation of  $X$  at  $x$  satisfying the local consistency condition: every  $x \in X$  has a neighbourhood  $U \subseteq X$  homeomorphic to a ball in  $\mathbb{R}^n$  containing a smaller ball  $B$  of finite radius around  $x$  such that the local orientations  $\mu_y$  for all  $y \in B$  are the image of one generator of  $H_d(X, X \setminus B, R)$  under the maps  $H_d(X, X \setminus B, R) \rightarrow H_d(X, X \setminus \{y\}, R)$ .

**Lemma 44.14** If  $X$  is a smooth orientable manifold, then the choice of an orientation on  $X$  gives us an  $R$ -orientation for every ring  $R$ .

*Proof.* It suffices to do  $R = \mathbb{Z}$ . Everything else follows by basechange. The choice of an orientation defines compatible isomorphisms with  $H_d(B^d, S^{d-1}, \mathbb{Z})$ .  $\square$

**Theorem 44.15** Let  $X$  be a connected compact orientable smooth manifold of dimension  $d$ . Then we have isomorphisms

$$H_d(X, R) \xrightarrow{\sim} H_d(X, X \setminus \{x\}, R)$$

for all  $x \in X$ . Moreover,  $H_n(X, R) = 0$  for all  $n > d$ .

*Proof.* See Hatcher, theorem 3.26.  $\square$

**Definition 44.16** A *fundamental class* for  $X$  is a class  $[X] \in H_d(X, R)$  such that  $[X]$  induces an  $R$ -orientation at  $x$  for all  $x \in X$ .

For the non-compact case we have

**Theorem 44.17** Let  $X$  be an oriented connected smooth manifold of dimension  $d$  and  $K \subseteq X$  compact. Then there is a unique class in  $H_d(X, X \setminus K, R)$  whose image in  $H_d(X, X \setminus \{x\}, R)$  agrees with the orientation for all  $x \in K$ . Moreover,  $H_n(X, X \setminus K, R) = 0$  for  $n > d$ .

*Proof.* See Hatcher, lemma 3.27.  $\square$

**44.3. Poincare duality.**  $X$  smooth compact orientable manifold of dimension  $d$ . Caping with the fundamental class  $[X]$  gives us a map

$$[X] \cap (-): H^l(X, R) \rightarrow H_{d-l}(X, R)$$

for all  $l \leq d$ .

**Theorem 44.18 (Poincare duality)** This is an isomorphism.

*Proof.* See Hatcher, theorem 3.30.  $\square$

If  $R = F$  is a field, then Poincare duality takes the form

$$H^l(X, F) \cong H^{d-l}(X, F)^\vee.$$

45. [WS] DERIVED CATS 5 (NICOLA) - THE RECONSTRUCTION THEOREM OF BONDAL AND ORLOV - 08.01.2020

### References:

- (1) D. Huybrechts, *Fourier-Mukai transforms in algebraic geometry*.
- (2) Notes by Johan.
- (3) nCatLab.

**Notation 45.1.** (1)  $X$  and  $Y$  will be smooth projective varieties, let's say over  $\mathbb{C}$ , although we will usually write  $k$ . The proof works over any field, with some minor simplifications in the algebraically closed case.

(2)  $\mathbf{D}^b(X)$  is the bounded derived category of coherent sheaves on  $X$ .

(3)  $n = \dim(X)$ .

(4)  $\omega_X$  is the canonical line bundle.

(5) We will usually talk about closed points omitting the word closed.

### Recall from last talk:

(1)  $H^i(X, \mathcal{F}) \cong \mathrm{Hom}_{\mathbf{D}^b(X)}(\mathcal{O}_X, \mathcal{F}[i])$  and the cup product can be expressed as composition.

(2)  $\mathrm{Hom}(\mathcal{F}, \mathcal{G}[\leq 0]) = 0$ .

(3)  $\mathbf{D}^b(X)$  has a Serre functor and Serre functors commute with  $k$ -linear equivalences.

**Goal:** Let  $F: \mathbf{D}^b(X) \xrightarrow{\sim} \mathbf{D}^b(Y)$  be an exact  $k$ -linear equivalence (from now on denoted  $\mathbf{D}^b(X) \cong \mathbf{D}^b(Y)$ ) and assume either  $\omega_X$  or  $\omega_X^{-1}$  is ample. Then  $X \cong Y$ .

We will focus here on the  $\omega_X$  ample case.

#### 45.1. Warm up.

**Proposition 45.2** *If  $\mathbf{D}^b(X) \xrightarrow{F} \mathbf{D}^b(Y)$ , then  $\dim X = \dim Y$  and  $\text{ord}(\omega_Y) = \text{ord}(\omega_X)$ .*

*Proof.* For a closed point  $x \in X$ , let  $\kappa(x)$  be the skyscraper sheaf with value the residue field  $\kappa(x)$  at  $x$ . Note that  $\kappa(x) \cong \kappa(x) \otimes \omega_X \cong \mathbb{S}_X(\kappa(x))[-n]$ . Since  $F$  is a  $k$ -linear equivalence, it commutes with Serre functors, so

$$F(\kappa(x)) \cong F(\mathbb{S}_X(\kappa(x))[-n]) \cong \mathbb{S}_Y(F(\kappa(x)))[-n] = F(\kappa(x)) \otimes \omega_Y[\dim(Y) - n].$$

The LHS is non-zero and bounded, so there is a maximal (minimal)  $i$  such that  $0 \neq H^i(F(\kappa(x))) = H^{i+\dim(Y)-n}(F(\kappa(x))) \otimes \omega_Y$ . This implies that  $\dim(Y) = n$  as well.

If  $\omega_X^k = \mathcal{O}_X$ , then  $\mathbb{S}_X^k[-kn] \cong \text{id}$ , so  $\mathbb{S}_Y^k[-kn] \cong F \circ \text{id} \circ F^{-1} \cong \text{id}$ . Hence  $\omega_Y^k = \mathcal{O}_Y$ .  $\square$

**45.2. Idea of the proof.**  $\omega_X$  is ample if and only if  $X \cong \text{Proj}(\oplus_{d \geq 0} H^0(X, \omega_X^d))$  (see tag 01Q3 in the Stacks Project). Let  $B_X$  denote this canonical ring from now on.

- (1) Prove  $B_X \cong B_Y$ .
- (2)  $\omega_Y$  is ample.
- (3) Then  $X \cong \text{Proj}(B_X) \cong \text{Proj}(B_Y) \cong Y$ .

**Problem:** we don't have a clear access to some key geometric objects related to  $Y$ , such as

- (1) points,
- (2) line bundles,
- (3) topology of  $Y$ .

The plan is then to try to recover these objects from  $\mathbf{D}^b(Y)$ .

**Definition 45.3** Let  $\mathbf{D}$  be a  $k$ -linear triangulated category with a Serre functor  $\mathbb{S}_D$ . An object  $P \in \mathbf{D}$  is called *point-like* of codimension  $d$  if:

- (1)  $\mathbb{S}(P) = P[d]$ .
- (2)  $\text{Hom}(P, P[i]) = 0$  for all  $i < 0$ .
- (3)  $\kappa(P) := \text{Hom}(P, P)$  is a field.

**Proposition 45.4**  $\{\kappa(x)[m] \mid x \in X, m \in \mathbb{Z}\} \hookrightarrow \{\text{point-like objects}\}$  and equality holds if  $\omega_X$  is ample.

**Definition 45.5**  $\mathbf{D}$  as before. An object  $L \in \mathbf{D}$  is called *invertible* if for all  $P \in \mathbf{D}$  point-like objects there is an integer  $n(L, P) \in \mathbb{Z}$  such that

$$\text{Hom}(L, P[i]) \cong \begin{cases} \kappa(P) & \text{if } i = n(L, P) \\ 0 & \text{otherwise.} \end{cases}$$

**Proposition 45.6**  $\{\text{invertible objects}\} \hookrightarrow \{\mathcal{L}[m] \mid \mathcal{L} \in \text{Pic}(X), m \in \mathbb{Z}\}$ , and equality holds if  $\omega_X$  is ample.

*Remark 45.7.* These definitions are intrinsic, so such objects need to be preserved by  $F$ .

**45.3. Proof of the theorem.** There are 6 steps:

- (1) We may assume that  $F(\mathcal{O}_X) = \mathcal{O}_Y$ . Indeed, since  $\mathcal{O}_X$  is invertible, so is  $F(\mathcal{O}_X)$ , which is therefore of the form  $\mathcal{M}[m]$  for  $\mathcal{M} \in \text{Pic}(X)$ . By shifting and tensoring with its dual we may assume  $F(\mathcal{O}_X) = \mathcal{O}_Y$ .
- (2)  $F(\omega_X^k) = \omega_Y^k$ . Indeed, since  $\dim Y = n$  as well, we have

$$F(\omega_X^k) = F(\mathbb{S}_X^k(\mathcal{O}_X)[-kn]) \cong \mathbb{S}_Y^k(F(\mathcal{O}_X))[-kn] = \omega_Y^k.$$

- (3)  $B_X \cong B_Y$ . On one hand we have

$$H^0(X, \omega_X^k) = \text{Hom}(\mathcal{O}_X, \omega_X^k) \cong \text{Hom}(F(\mathcal{O}_X), F(\omega_X^k)) = H^0(Y, \omega_Y^k).$$

So both canonical rings agree as groups. Now we need to show that the multiplications agree. For this we express multiplication of global sections as composition:

$$s_1 s_2 = \mathbb{S}_X^k(s_2)[-kn] \circ s_1.$$

The functor  $F$  commutes with shifts, Serre functors and preserves composition, hence product of global sections is preserved as well.

- (4) It remains to show that  $\omega_Y$  is ample to conclude as sketched before. In this step we show that

$$\{\text{point-like objects}\} \cong \{\kappa(y)[m] \mid y \in Y, m \in \mathbb{Z}\}$$

form a spanning class, see below.

- (5) We have a bijection  $\phi: X \rightarrow Y$ . Skyscraper sheaves are flasque, hence have no higher cohomology, so we have

$$0 \neq H^0(X, \kappa(x)) = \text{Hom}(\mathcal{O}_X, \kappa(x)) = \text{Hom}(F(\mathcal{O}_X), F(\kappa(x))) = H^m(X, \kappa(y))$$

and the RHS is non-zero iff  $m = 0$ . Thus the bijection.

- (6) To show that  $\omega_Y$  is ample, we use that  $\mathcal{L}$  is ample iff  $U_\alpha = X \setminus Z(\alpha)$  is a basis for the topology as  $\alpha$  runs through  $\alpha \in H^0(\mathcal{O}_X, \mathcal{L}^k)$ . Note that

$$U_\alpha = \{x \in X \mid \mathcal{O}_X \xrightarrow{\alpha} \mathcal{L}^k \rightarrow \kappa(x) \text{ is non-zero for all } k \in \mathbb{N}\}.$$

We have  $F(\kappa(x)) = \kappa(\phi(x))$  and if  $\alpha$  yields a non-zero morphism  $\mathcal{O}_X \xrightarrow{\alpha} \mathcal{L} \rightarrow \kappa(x)$ , then applying  $F$  to this morphism we should get a non-zero morphism

$$\mathcal{O}_Y \xrightarrow{F(\alpha)} F(\mathcal{L}) \rightarrow \kappa(\phi(x)),$$

which implies that  $F(\mathcal{L}) = \mathcal{M}[m]$  with  $m = 0$ , i.e.  $F(\mathcal{L})$  is an honest line bundle. Hence  $\phi(U_\alpha) = V_{F(\alpha)}$ . This shows that  $\phi$  is a homeomorphism, taking thus bases to bases, hence  $\omega_Y$  is ample by the previous criterion.

#### 45.4. Step 4 in the proof.

**Definition 45.8** With  $\mathbf{D}$  as above, a set  $\Omega \subseteq \mathbf{D}$  is called a *spanning class* if for all  $B \in \mathbf{D}$  we have

- (1)  $(\text{Hom}(A, B[i]) = 0 \text{ for all } A \in \Omega \text{ and all } i \in \mathbb{Z}) \Rightarrow B = 0$ .
- (2) Similarly in the other entry of the hom functor.

*Remark 45.9.* Since we have Serre functors, checking one of the two conditions is enough.

**Proposition 45.10**  *$Y$  smooth projective variety implies that the shifts of skyscraper sheaves form a spanning class of  $\mathbf{D}^b(Y)$ .*

*Proof.* Assume  $\mathcal{F}^\bullet \in \mathbf{D}^b(Y)$  is not 0 and let  $m \in \mathbb{Z}$  be a maximal integer such that  $H^m(\mathcal{F}^\bullet) =: \mathcal{H}^m \neq 0$ . Take  $x \in X$  in the support of  $\mathcal{H}^m$ . Then there exists a surjection  $\mathcal{H}^m \twoheadrightarrow \kappa(x)$ . Replacing  $\mathcal{F}^\bullet$  by a nice truncation and composing with a shift of this morphism we get the non-zero morphism that we wanted.  $\square$

For all  $y \in Y$  there exists some  $x_y \in X$  and some  $m_y \in \mathbb{Z}$  such that  $F(\kappa(x_y)[m_y])$ . Assume  $\exists P \in \mathbf{D}^b(Y)$  point-like but  $\not\cong \kappa(y)[m]$  for any  $y$  or  $m$ . Then  $\exists x_P$  and  $m_P$  such that  $F(\kappa(x_P)[m_P]) = P$  but  $x_P \neq x_y$ . Now conclude using that  $\text{Hom}(\kappa(x_1), \kappa(x_2)[m]) = 0$  for all  $x_1 \neq x_2$  and all  $m \in \mathbb{Z}$ .

## 46. [LT] LECTURE 20 - 09.01.20

$X$  connected oriented compact manifold of dimension  $d$ . Then Poincare duality tells us that capping with the fundamental class yields isomorphisms

$$H^l(X, R) \xrightarrow{\sim} H_{d-l}(X, R).$$

This allows us to pull functoriality from cohomology to homology and viceversa.

**Definition 46.1** Let  $i: Y \subseteq X$  be an inclusion of connected compact complex manifolds of complex codimension  $r$ . We call the composition

$$H^l(Y, \mathbb{Z}) \cong H_{2 \dim Y - l}(Y, \mathbb{Z}) \xrightarrow{i_*} H_{2 \dim Y - l}(X, \mathbb{Z}) \cong H^{2 \dim X - 2 \dim Y + l}(X, \mathbb{Z})$$

the *Gysin map*. The image of the standard generator of  $H^0(Y, \mathbb{Z}) \cong \mathbb{Z}$  in  $H^{2r}(X, \mathbb{Z})$  is called the *cycle class* of  $Y$ .

*Remark 46.2.* With these definitions one can already state the *Hodge conjecture*.

For the non-compact case we use the pairing

$$H_n(X, X \setminus K; R) \times H^l(X, X \setminus K; R) \rightarrow H_{n-l}(X; R).$$

Recall that an orientation on  $X$  induces an orientation on all  $H_d(X, X \setminus K; R)$  for all  $K \subseteq X$  compact.

**Theorem 46.3** Let  $X$  be an oriented connected smooth manifold of dimension  $d$ . Then Poincare duality tells us that capping with the fundamental classes relative to complements of compact subsets is an isomorphism

$$H_c^l(X, R) \rightarrow H_{d-l}(X, R).$$

**Example 46.4**

$$H_c^l(\mathbb{R}, R) \cong H_{d-l}(\mathbb{R}^n, R) \cong \begin{cases} R & \text{if } d = l \\ 0 & \text{if } d \neq l. \end{cases}$$

Finally we also have the version in Theorem 3.44 of Hatcher's book.

**46.1. Homological interpretation of the local picture.** Recall:

**Theorem 46.5 (Theorem A)** Consider  $f: X \rightarrow V \subseteq \mathbb{C}$  holomorphic and proper with  $X$  a complex manifold. Let  $x_0 \in f^{-1}(0)$  be an ordinary double point. Assume that  $f$  is submersive away from  $x_0$ . Take  $\Delta \subseteq V$  disk centered at 0. Let  $t \in \Delta^*$  be a point away from zero.

Then  $X_\Delta$  deformation retracts onto  $X_t \cup_{S_t^{n-1}} B_t^n$ , where  $S_t^{n-1} \subseteq X_t$  is a vanishing  $(n-1)$ -sphere and  $B_t^n$  is a (real)  $n$ -ball.

Now in terms of homology we can say the following:

**Corollary 46.6 (Corollary A)** With the hypotheses above, let  $i: X_t \rightarrow X_\Delta$  be the inclusion.

- (1)  $i_*: H_k(X_t, \mathbb{Z}) \rightarrow H_k(X_\Delta, \mathbb{Z})$  is an isomorphism for  $k < n-1$ .
- (2)  $i_*$  is surjective for  $k = n-1$ .
- (3)  $\ker(i_*)$  is generated by the class of  $S_t^{n-1} \subseteq X_t$ , for  $k = n-1$ .

*Proof.* We have

$$\begin{array}{ccc} X_t & \xhookrightarrow{i} & X_\Delta \\ & \searrow & \uparrow \downarrow \text{def. ret.} \\ & & X_t \cup_{S_t^{n-1}} B_t^n \end{array}$$

Homotopy invariance of homology gives us

$$\begin{array}{ccc} H_k(X_t) & \xrightarrow{i_*} & H_k(X_\Delta) \\ & \searrow & \uparrow \sim \\ & & H_k(X_t \cup_{S_t^{n-1}} B_t^n) \end{array}$$

We may therefore replace  $X_\Delta$  by  $X_t \cup_{S_t^{n-1}} B_t^n$ . (1) and (2) are equivalent to the assertion that  $H_k(X_t \cup_{S_t^{n-1}} B_t^n, X_t) = 0$  for  $k < n$  by the LES of relative homology. By excision this relative homology group is isomorphic to

$$H_k(B_t^n, S_t^{n-1})$$

which is indeed 0 for  $k < n$ .

For assertion (3) we argue again by excision:

$$\begin{array}{ccccc} H_n(B_t^n, S_t^{n-1}) & \xrightarrow{\sim} & H_{n-1}(S_t^{n-1}) & \longrightarrow & 0 \\ \downarrow \sim & & \downarrow & & \downarrow \\ H_n(X_t \cup_{S_t^{n-1}} B_t^n, X_t) & \longrightarrow & H_{n-1}(X_t) & \longrightarrow & H_{n-1}(X_\Delta) \end{array}$$

□

#### Example 46.7 (Picture)

**46.2. Global picture.** Recall the notation we had.  $X \subseteq \mathbb{P}^N$  a nonsingular projective variety of dimension  $n$ .  $X^\vee = \{\xi \in (\mathbb{P}^N)^\vee \mid \xi \text{ tangent to } X \text{ at some } x \in X\} \subseteq (\mathbb{P}^N)^\vee$  its dual variety.  $L \cong \mathbb{P}^1 \hookrightarrow (\mathbb{P}^N)^\vee$  a Lefschetz pencil on  $X$ .  $A := \cap_{t \in L} H_t = H_0 \cap H_\infty \subseteq \mathbb{P}^N$  axis of  $L$ .

**Fact:**  $X \cap A$  nonsingular of codimension 2 in  $X$ .

**Fact:** Let  $\tilde{X} = \{(x, \xi) \in \mathbb{P}^N \times (\mathbb{P}^N)^\vee \mid x \in X, \xi \in L, x \in X \cap H_\xi\}$ . Then we have projections  $p: \tilde{X} \rightarrow X$  and  $f: \tilde{X} \rightarrow L \cong \mathbb{P}^1$ . Then  $\tilde{X}$  is nonsingular and  $p: p^{-1}(X \setminus A) \xrightarrow{\sim} X \setminus A$  is an isomorphism.

For all  $t \in L$  we have isomorphisms  $p: \tilde{X}_t = f^{-1}(t) \xrightarrow{\sim} X \cap H_t = X_t$ .

For all  $t \in L$  the fibre  $X_t$  has at worst one singular point and that point is an ordinary double point.

Let  $s_1, \dots, s_n \in L$  be critical values of  $f$ . For all  $r$  let  $\Delta_r \subseteq \mathbb{P}^1 = L$  be a small disk centered at  $s_r$ . Let  $0, \infty \in \mathbb{P}^1 \setminus \cup_r \Delta_r$  with  $0 \neq \infty$ . [Picture] For all  $r$  let  $t_r \in \Delta_r^*$  and  $\gamma_r$  a simple path from 0 to  $t_r$ . For all  $r_0 \neq r_1$ ,  $\gamma_{r_0} \cap \gamma_{r_1} = \{0\}$  and  $s_{r_1} \notin \gamma_{r_0}$  and  $s_{r_1} \notin \gamma_{r_0}$ .

**Remark 46.8.**  $\mathbb{C} = \mathbb{P}^1 \setminus \{\infty\}$  deformation retracts onto  $\cup_r (\Delta_r \cup \gamma_r)$ .

**Theorem 46.9 (Theorem B)**  $\tilde{X} \setminus X_\infty$  is homotopy equivalent to the union of  $X_0$  and a finite set of  $n$ -balls glued along  $(n-1)$ -spheres.

*Proof.* We proceed in several steps.

- (1)  $\tilde{X} \setminus X_\infty$  deformation retracts onto  $\cup_r (\tilde{X}_{\gamma_r} \cup \tilde{X}_{\Delta_r})$ . Indeed,  $f$  is proper, submersive away from  $s_1, \dots, s_n$ . By Ehresmann,  $f$  is a locally trivial fibration away from  $\{s_1, \dots, s_n\}$ . Apply the homotopy lifting property to lift the deformation retract of the previous remark along  $f$ .
- (2)  $\tilde{X}_{\gamma_r}$  deformation retracts onto  $X_{t_r}$ . Indeed: Ehresmann

$$\begin{array}{ccc}
 \tilde{X}_{\gamma_r} & \xrightarrow{\text{diffeo.}} & \tilde{X}_{t_r} \times \gamma_r = X_{t_r} \times \gamma_r \\
 & \searrow & \swarrow \\
 & \gamma_r &
 \end{array}$$

Lift the deformation retract of  $\gamma_r$  to  $\{t_r\}$  along  $f$ .

- (3)  $X_{t_r}$  is diffeomorphic to  $X_0$ . This follows from (2) and the contractibility of  $\gamma_r$ .
- (4) Combine the previous 3 points with Theorem A to conclude.

□

**Corollary 46.10 (Corollary B)** For all  $0 \in \mathbb{P}^1 \setminus \{\infty\}$  such that  $X_0$  is nonsingular, consider the inclusion  $i: X_0 \hookrightarrow \tilde{X} \setminus X_\infty$ .

- (1)  $i_*: H_k(X_0) \rightarrow H_k(\tilde{X} \setminus X_\infty)$  is an isomorphism for  $k < n - 1$ .
- (2) It is surjective for  $k = n - 1$ .
- (3) For  $k = n - 1$ , the kernel is generated by classes of vanishing spheres.

*Proof.* Consider the diagram

$$\begin{array}{ccc}
 X_0 & \xhookrightarrow{i} & \tilde{X} \setminus X_\infty \\
 & \searrow & \uparrow \downarrow \text{def. ret. (Thm. B)} \\
 & & X_0 \cup_{\cup_r S_r^{n-1}} (\cup_r B_r^n)
 \end{array}$$

Hence we may replace  $\tilde{X} \setminus X_\infty$  by the guy below. Now use homotopy invariance, excision and induction on the number of critical values combined with corollary A. □

**46.3. Lefschetz hyperplane theorem.** With the notation as above, if  $0 \in \mathbb{P}^1 \setminus \{\infty\}$  is such that  $X_0$  is nonsingular, then the restriction map  $i^*: H^k(X) \rightarrow H^k(X_0)$  is:

- (1) An isomorphism for  $k < n - 1$ .
- (2) Injective for  $k = n - 1$ .

47. [FS] CHRISTIAN LEHN - A NEW PROOF OF THE GLOBAL TORELLI THEOREM FOR IRREDUCIBLE SYMPLECTIC MANIFOLDS - 10.01.20

jt. work in progress with B. Bakker.

**Torelli problem:** can a variety be reconstructed from its Hodge structure?

- Example 47.1**
- (1) Complex tori  $\leftrightarrow H^1(T, \mathbb{Z})$ .
  - (2) (Smooth projective) Curves  $\leftrightarrow H^1(C, \mathbb{Z}) + \text{polarization (see Torelli)}$ .
  - (3) K3 surfaces  $\leftrightarrow H^2(S, \mathbb{Z}) + \text{intersection pairing (see Torelli)}$ .
  - (4) Cubic fourfolds  $\leftrightarrow H^4(X, \mathbb{Z}) + \text{" (see Voisin)}$ .
  - (5) Irreducible  $2n$  dimensional symplectic manifold  $\leftrightarrow (H^2(X, \mathbb{Z}), q) + \dots$  (due to Verbitsky).

**Definition 47.2** A compact Kaehler manifold  $X$  is called an *irreducible symplectic manifold* if  $X$  is simply connected and  $H^0(\Omega_X^2) = \mathbb{C}\sigma$  is generated by a symplectic form  $\sigma$ .

There is a quadratic form  $q: H^2(X, \mathbb{Z}) \rightarrow \mathbb{Z}$  which is non-degenerate with signature  $(3, n)$ . The pair  $(H^2(X, \mathbb{Z}), q)$  is a semi-polarized Hodge structure (positivity condition in the usual definition is not quite true) such that  $h^{2,0} = 1$ ,  $q|_{H^{2,0}} = 0$  and  $(H^{2,0} \oplus H^{0,2})^\perp = H^{1,1}$ .



Fix  $\Lambda$  a lattice of signature  $(3, n)$ . The *period domain* is

$$D_\Lambda = \{x \in \mathbb{P}(\Lambda \otimes \mathbb{C}) \mid (x, x) = 0, (x, \bar{x}) > 0\}.$$

$D_\Lambda \ni \omega$ , so we have a Hodge structure  $H_\omega$  s.t.  $H_\omega^{2,0} = \omega$ .

Let us see now how to phrase the Torelli problem. For  $X$  ISM we fix an isometry  $\mu: H^2(X, \mathbb{Z}) \rightarrow \Lambda$ . Such an isometry is called a *marking*. We consider the marked moduli space

$$\mathcal{M}_\Lambda = \{(X, \mu) \mid X \text{ ISM}, \mu \text{ marking}, X \sim_{\text{defo}} X_0\} / \sim$$

endowed with a complex structure given by deformation theory. We obtain a *period map*

$$\begin{aligned} \mathcal{P}: \mathcal{M}_\Lambda &\rightarrow D_\Lambda \\ (X, \mu) &\mapsto \mu_{\mathbb{C}}(H^{2,0}(X)) \end{aligned}$$

**Theorem 47.3** Let  $X_0$  be an ISM and  $\Lambda$  a lattice isometric to  $H^2(X, \mathbb{Z})$ . Take a non-empty connected component  $\emptyset \neq \mathbb{N} \subseteq \mathcal{M}_\Lambda$ . Then:

- (1)  $\mathcal{P}|_{\mathbb{N}}: \mathbb{N} \rightarrow D_\Lambda$  is generically injective, where generically here means on the complement of a countable union of proper subvarieties [Ver].
- (2)  $\mathcal{P}: \mathbb{N} \rightarrow D_\Lambda$  is surjective [Huy].
- (3)  $\mathcal{P}^{-1}(\omega) \ni (X, \mu), (X', \mu') \Rightarrow$  bimeromorphic map between  $X$  to  $X'$  [Ver]. The converse is due to [Huy].
- (4)  $\text{Mon}(\mathbb{N}) \subseteq O(\Lambda)$  (by parallel transport, from closed loops on the base we get automorphisms on the  $H^2$  of the fibre; the quick way would be just to define it as the stabilizer) [Ver].

The proof by Huy. is 20 years old, the one by Ver. is 10 years old, but both of them use *twisted families*. This is something coming from differential geometry, so it does not work as well with singular varieties. For this reason, one of the key points of this work is to get rid of the use of these twisted families at each point of the proof.

**Today:** proof of the theorem under the hypothesis  $b_2(X_0) \geq 5$  [BL].

**Ingredients:**

- (1) Deformation theory [Namibara, BL, KS].
- (2) Ergodic complex structures [Verbitsky].
- (3) Theory of deformations [KLSV].
- (4) Theory of MBM classes [Amerik-Ver].

We will start by explaining the ergodicity ingredient, which is somehow the main substitute of the twisted families and it is also where the  $b_2$  hypothesis comes into play.

**47.1. Proof.** *Zero step:* We have  $\mathcal{M}_\Lambda \rightarrow D_\Lambda$ , both spaces acting on  $\Gamma := O(\Lambda)$ .  $g \cdot (X, \mu) = (X, g\omega\mu)$ .

**Definition 47.4** Let  $\omega \in D_\Lambda$ . We define the *rational rank* as

$$\text{rrk}(\omega) := \dim_{\mathbb{Q}}((H_\omega^{2,0} \oplus H_\omega^{0,2}) \cap \Lambda_{\mathbb{Q}}) \in \{0, 1, 2\}.$$

**Theorem 47.5 (Verbitsky)** Assuming  $b_2 \geq 5$  we have:

- (1)  $\text{rrk}(\omega) = 2$  iff  $\Gamma \cdot \omega$  is closed in  $D_\Lambda$  (iff  $H^{1,1}(X) = \text{NS}(X)_{\mathbb{C}}$ , i.e. maximal Neron-Severi).
- (2)  $\text{rrk}(\omega) = 1$  iff  $\overline{\Gamma \cdot \omega} = \bigcup_{\lambda \in \Lambda} G_\lambda$ , for  $G_\lambda \subseteq D_\Lambda$  totally real.
- (3)  $\text{rrk}(\omega) = 0$  iff the orbit  $\Gamma \cdot \omega$  is dense ("ergodicity").

**Corollary 47.6 (Almost surjectivity)**  $\mathcal{P}(\mathcal{M}_\Lambda) \subseteq D_\Lambda$  is dense (by  $\text{rrk} = 0$  case) and open in the euclidean sense (by deformation theory),  $\Gamma$ -invariant.

*First step:* Look at the subset of periods  $D_i = \{\omega \in D_\Lambda \mid \text{rk}(\omega) = i\}$ . Consider  $D_0 \supseteq D_\eta^d := \{\omega \in D_0 \mid \text{NS}(H_\omega) = \mathbb{Z}v, v^2 = d > 0, \omega \text{ very general}\}$ . Here  $\eta$  is a reference to the generic point, standing for the last assumption on  $\omega$  being very general.

We show that  $\forall \omega \in D_\eta^d$ , the number of points in the fibre  $\mathcal{P}^{-1}(\omega) \subseteq \mathcal{M}_\Lambda$  is finite. Some ingredients here:

- (1) Projectivity criterion:  $(X, \mu) \in \mathcal{P}^{-1}(\omega) \Rightarrow X$  projective.
- (2) Matsusaka:

$$\#\{X \mid X \text{ proj. ISM}, X \sim_{\text{def}} X_0, \text{Pic}(X) = \mathbb{Z}L, H^2(X) \cong H_\omega\} < \infty.$$

- (3)  $\omega$  very general  $\Rightarrow \text{Aut}(H_\omega) = \pm \text{id}$ . Hence  $\#\mathcal{P}^{-1}(\omega) < \infty$ .

*Second step:*  $\#\mathcal{P}^{-1}(\omega) = \text{const.}$  for  $\omega \in D_0$ . By step 1,  $\exists \omega_0 \in D_0$  such that  $\#\mathcal{P}^{-1}(\omega)$  is  $n < \infty$ . Take  $\omega \in D_0$  and suppose  $\#\mathcal{P}^{-1}(\omega) > n$ . Use the ergodicity condition [picture]. This yields a contradiction with Hausdorffness (which can be assumed). Hence  $\mathcal{P}: \mathcal{M}_0 \rightarrow D_0$  is a finite topological cover. This implies that  $\mathcal{P}: \mathcal{N}_0 = \mathcal{N} \cap \mathcal{M}_0 \xrightarrow{\sim} D_0$ .

#### 48. [LT] LECTURE 21 - 14.01.20

*Notation 48.1.*  $X \subseteq \mathbb{P}^N$  projective variety,  $n = \dim X$ .  $L = \mathbb{P}^1 \subseteq (\mathbb{P}^N)^\vee$  Lefschetz pencil,  $A$  its axis,

$$\begin{array}{ccc} & \tilde{X} & \\ p \swarrow & & \searrow f \\ X & & \mathbb{P}^1, \end{array}$$

$0, \infty \in \mathbb{P}^1$  regular values of  $f$ ,  $0 \neq \infty$ , and  $\forall t \in \mathbb{P}^1, p: \tilde{X}_t = f^{-1}(t) \xrightarrow{\sim} X_t = X \cap H_t$ .

$$\begin{array}{ccccc} p^{-1}(\tau) & \hookrightarrow & \tilde{X} & \xleftarrow{\circ} & \tilde{X} \setminus p^{-1}(\tau) \\ \downarrow & & \downarrow p & & \downarrow \sim \\ Z_0 := X \cap A & \hookrightarrow & X & \xleftarrow{\circ} & X \setminus Z \\ & & & & \uparrow \circ \\ & & & & X \setminus Z \\ & & & & \uparrow \circ \\ & & & & X \setminus X_0 \end{array}$$

$$\begin{array}{ccccc} Z & \xrightarrow{h_0} & X_0 & \xleftarrow{i_0} & X \\ & & & & \uparrow \circ \\ & & & & X \setminus Z \\ & & & & \uparrow \circ \\ & & & & X \setminus X_0 \end{array}$$

**48.1. Lefschetz hyperplane theorem (LHT).** The restriction map  $i_0^*: H^k(X) \rightarrow H^k(X_0)$  along the zero hyperplane section of our Lefschetz pencil is

- an isomorphism for  $k < n - 1$ , and
- injective for  $k = n - 1$ .

Ingredients of LHT:

- (i) Induction hypothesis: LHT is satisfied for each non-singular projective variety  $Y \subseteq \mathbb{P}^N$  of dimension  $\leq n - 1$ .
- (ii) Thom isomorphism:  $H^k(\tilde{X}, \tilde{X} \setminus X_\infty) \cong H^k(X_\infty)$ .
- (iii) Gysin LES:  $\cdots \rightarrow H^k(\tilde{X}) \rightarrow H^k(\tilde{X} \setminus X_\infty) \rightarrow H^{k-1}(X_\infty) \rightarrow H^{k+1}(\tilde{X}) \rightarrow \cdots$ .
- (iv) Blow-up formula:  $H^{k+1}(\tilde{X}) \cong H^{k+1}(X) \oplus H^{k-1}(Z)$ .

- (v)  $\tilde{i}_0^*: H^k(\tilde{X} \setminus \tilde{X}_\infty) \rightarrow H^k(\tilde{X}_0)$  is an isomorphism for  $k < n - 1$  and injective for  $k = n - 1$ .
- (vi)  $\tilde{i}_{\infty,*}: H^{k-1}(X_\infty) \rightarrow H^{k+1}(\tilde{X} \setminus \tilde{X})$  is injective for  $k \leq m - 1$ .
- (vii)  $\tilde{j}_\infty^*: H^k(\tilde{X}) \rightarrow H(\tilde{X} \setminus \tilde{X}_\infty)$  is surjective for  $k \leq n - 1$ .
- (viii)  $h_{0,*} \circ h_0^* = [X_0] \cup (-) = i_0^* \circ i_{0,*}$ .

**Example 48.2**  $H^*(\mathbb{P}^n)$ . For  $n = 0$  we have just a point, so

$$H^k(\mathbb{P}^0) = \begin{cases} \mathbb{Z} & \text{if } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

For higher  $n$  we proceed by induction. We have  $\mathbb{P}^n \subseteq \mathbb{P}^{n+1}$  sitting as a linear subspace, and  $\mathbb{P}^{n+1} \setminus \mathbb{P}^n \cong \mathbb{C}^{n+1}$  is contractible. Using (iii) we get a LES

$$\dots \rightarrow H^k(\mathbb{C}^{n+1}) \rightarrow H^{k-1}(\mathbb{P}^n) \rightarrow H^{k+1}(\mathbb{P}^{n+1}) \rightarrow H^{k+1}(\mathbb{C}^{n+1}) \rightarrow \dots$$

from which we deduce that

$$H^k(\mathbb{P}^n) = \begin{cases} \mathbb{Z} & \text{if } 0 \leq k \leq 2n \text{ and } 2 \mid k, \\ 0 & \text{otherwise.} \end{cases}$$

By Poincaré duality, the same is true for the homology groups.

Ingredient (i) is okay, and ingredients (ii), (iii), (iv) and (viii) will be treated the next time. Ingredient (v) is the dual of the statement in homology from the last time.

*Proof of (vi).* By (iv) we have a diagram

$$\begin{array}{ccc} H^{k-1}(X_\infty) & \xrightarrow{i_\infty^*} & H^{k+1}(\tilde{X}) \cong H^{k+1}(X) \oplus H^{k-1}(Z) \\ & \searrow h_\infty^* & \downarrow \\ & & H^{k-1}(Z) \end{array}$$

Here  $Z \subseteq X_\infty$  is a nonsingular hyperplane section, and  $\dim(X_\infty) = n - 1$ . By (i),  $h_\infty^*$  is injective in degrees  $k - 1 \leq n - 2$ .  $\square$

*Proof of (vii).* Consider the LES

$$\dots \rightarrow H^k(\tilde{X}) \xrightarrow{a} H^k(X \setminus X_\infty) \xrightarrow{b} H^{k-1}(X_\infty) \xrightarrow{c} H^{k+1}(\tilde{X}) \rightarrow \dots$$

By (v),  $c$  is injective for  $k \leq n - 1$ . By exactness of the sequence,  $\text{im}(b) = \ker(c) = 0$ . Hence  $\text{im}(a) = \ker(b) = H^k(\tilde{X} \setminus X_\infty)$ . Therefore  $a$  is surjective.  $\square$

*Proof of LHT.* We split the proof into several steps:

- (1)  $\tilde{i}_0^*: H^k(\tilde{X}) \rightarrow H^k(\tilde{X}_0)$  is surjective for  $k < n - 1$ . For this consider the commutative diagram

$$\begin{array}{ccc} & H^k(\tilde{X} \setminus \tilde{X}_\infty) & \\ a \nearrow & & \searrow b \\ H^k(\tilde{X}) & \xrightarrow{i_0^*} & H^k(\tilde{X}_0) \end{array}$$

By (vii),  $a$  is surjective in the desired range. By (v),  $b$  is surjective in the desired range. Hence  $i_0^*$  is surjective in the desired range.

- (2)  $i_0^*: H^k(X) \rightarrow H^k(X_0)$  is surjective for  $k < n - 1$ . For this consider the diagram

$$\begin{array}{ccc}
H^k(\tilde{X}) & \xrightarrow{\sim} & H^k(X) \oplus H^{k-2}(Z) \\
\downarrow (1) & & \downarrow \\
H^k(\tilde{X}_0) = H^k(X_0) & \xleftarrow{h_{0,*}} & H^{k-2}(Z)
\end{array}$$

which will commute by the proof of (iv). Hence it suffices to show that  $H^k(X_0) \stackrel{(1)}{=} \text{im}(h_{0,*}) \subseteq \text{im}(i_0^*)$ . By (i) we have that  $h_0^*: H^m(X_0) \rightarrow H^m(Z)$  is surjective for  $m < n-2$ . Hence  $\text{im}(h_0^*) = H^{k-2}(Z)$  for  $k < n-1$ . Hence  $\text{im}(h_{0,*}: H^{k-2}(Z) \rightarrow H^k(X_0))$  is the same as  $\text{im}(h_{0,*} \circ h_0^*: H^k(X_0) \rightarrow H^k(X_0))$  which is the same as  $\text{im}(i_0^* \circ i_0^*)$ .

- (3)  $i_0^*: H^k(X) \rightarrow H^k(X_0)$  is injective for  $k < n-1$ . Let  $k < n-1$  and let  $\alpha \in H^k(X)$  be such that  $i_0^*(\alpha) = 0 \in H^k(X_0)$ . We will show that  $\alpha = 0$ . Consider the commutative diagram

$$\begin{array}{ccccc}
H^k(X) & \xrightarrow{i_0^*} & H^k(X_0) & & \\
\downarrow p^* & & \searrow & & \\
H^k(\tilde{X}) & \xrightarrow{\tilde{j}_\infty^*} & H^k(\tilde{X} \setminus X_\infty) & \xrightarrow{\tilde{i}_0^*} & H^k(\tilde{X}_0) \\
\uparrow \tilde{i}_{\infty,*} & & & & \\
H^{k-2}(X_\infty) & & & & 
\end{array}$$

The map  $\tilde{i}_0^*$  is injective by (v). Hence  $\tilde{j}_\infty^* p^* \alpha = 0$ . By (iii), there exists some  $\beta \in H^{k-2}(X_\infty)$  such that  $p^* \alpha = \tilde{i}_{\infty,*} \beta$ . (viii) will then imply that  $\alpha \stackrel{\text{next time}}{=} p_* p^* \alpha = p_* \tilde{i}_{\infty,*} \beta \in H^k(X)$ , hence  $p^* i_{\infty,*} \beta = p^* \alpha = i_{\infty,*} \beta$  by choice of  $\beta$ . By (iv) we have  $i_\infty^* \beta \in H^k(\tilde{X}) \cong H^k(X) \oplus H^{k-2}(Z) = \text{im}(p^*) \oplus H^{k-2}(Z)$ , which maps to  $h_\infty^* \beta$  in  $H^{k-2}(Z)$ , where  $h_\infty^*: H^{k-2}(X_\infty) \rightarrow H^{k-2}(Z)$ . Since  $p^* i_{\infty,*} \beta = i_{\infty,*} \beta$ , we have  $h_\infty^* \beta = 0$ . But then by (i)  $\beta = 0$ , therefore  $\alpha = 0$  as well.

□

*Remark 48.3.* Last time we said (although not in the exact same words) that  $H_n(\tilde{X} \setminus X_\infty, X_0)$  was generated by the classes of cones over vanishing cycles.

**Proposition 48.4**  $H_n(X, X_0)$  is generated by the images in  $X$  under  $p: \tilde{X} \rightarrow X$  of cones over vanishing cycles.

*Proof.* Exercise! Use ingredients from LHT to show that  $p_*: H_n(\tilde{X} \setminus X_\infty, X_0) \rightarrow H_n(X, X_0)$  is surjective. □

**48.2. Cohomology of non-singular hypersurface in projective space.** Let  $X \subseteq \mathbb{P}^n$  be a non-singular hypersurface cut out by  $f$  an irreducible homogeneous polynomial of degree  $d$ . Consider the Veronese embedding. For  $V$  a  $\mathbb{C}$ -vector space, it is given by

$$\begin{aligned}
v_d: \mathbb{P}(V) &\rightarrow \mathbb{P}(\text{Sym}^d(V)) \\
[v] &\mapsto [v^d]
\end{aligned}$$

**Fact:**  $v_d$  is a homeomorphism onto its image (a closed immersion in fact).

**Example 48.5**  $v_2: \mathbb{P}^2 \rightarrow \mathbb{P}^5$  is given in coordinates by

$$[x : y : z] \mapsto [x^2 : y^2 : z^2 : xy : xz : yz].$$

**Fact:**  $v_d(X) = v_d(\mathbb{P}^n) \cap H$  for some hyperplane  $H$  in  $\mathbb{P}^{\binom{n+1}{d}-1}$ . So computing  $H^*(\mathbb{P}^n, X)$  is equivalent to computing  $H^*(v_d(\mathbb{P}^n), v_d(\mathbb{P}^n) \cap H)$ . By the LHT,

$$H^k(X) = \begin{cases} \mathbb{Z} & \text{if } k \leq n-2 \text{ and } 2 \mid k, \\ \mathbb{Z} & \text{if } k \geq n \text{ and } 2 \mid n-k, \\ ? & \text{if } k = n-1, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

#### 49. [WS] RATIONALITY 1 (FABIAN) - THE RATIONALITY PROBLEM - 15.01.20

- (1) Motivation.
- (2) Basics and conventions.
- (3) Rational varieties.
- (4) Related problems.
- (5) Criterion against rationality.

49.1. **Motivation.** Let  $k$  be some field and let  $f_1, \dots, f_n \in k[x_1, \dots, x_n]$ . We want to understand solutions of

$$(1) \begin{cases} f_1(x_1, \dots, x_n) = 0, \\ \vdots \\ f_n(x_1, \dots, x_n) = 0. \end{cases}$$

The easiest case is when the  $f_i$  are linear forms. Then we can rely on linear algebra.

Assume  $k = \bar{k}$ .

**Definition 49.1**  $X \subseteq \mathbb{P}^n(k)$  is *algebraic* if  $X = V(s)$  is the vanishing locus of some set  $S \subseteq k[x_0, \dots, x_n]$  of homogeneous polynomials. We say that  $X$  is a *projective variety* if it is algebraic and irreducible. A *quasi-projective variety* is an open subset of a projective variety.

Parametrization by  $\mathbb{P}^n(k)$  yields an explicit solution to the equations.

#### 49.2. Basics and conventions.

**Definition 49.2**  $X \subseteq \mathbb{P}^n(k)$  a projective variety. Then its *homogeneous coordinate ring* is defined as  $k[X] = k[x_0, \dots, x_n]/I(X)$ , where  $I(X)$  is the ideal generated by homogeneous polynomials vanishing on points of  $X$ . The *function field*  $k(X)$  of  $X$  is the degree zero part of the field of fractions of  $k[X]$ .

**Definition 49.3** Let  $X$  and  $Y$  be quasi-projective varieties. A *morphism*  $X \rightarrow Y$  is a morphism of locally ringed spaces  $(X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ . A *rational map*  $\varphi: X \dashrightarrow Y$  is an equivalence class of pairs  $(U, \varphi)$  with  $U \subseteq X$  open and  $\varphi: U \rightarrow Y$  a morphism, where two such pairs are related if there is a non-empty open set inside the intersection of the corresponding open sets on which both functions agree. The domain of  $\varphi$  is the union of all such  $U$ .

**Remark 49.4.** (1) There are no non-constant regular functions on projective varieties, so morphisms are too strict for classification purposes. Rational maps are better.  
 (2) Composition of rational maps need not be a rational map again.  
 (3) Rational maps with dominant image are in bijection with field extensions of the function fields.

- (4) The codimension of  $X \setminus \text{dom}(\varphi)$  in  $X$  is at least 2. Indeed,  $X \subseteq \mathbb{P}^m$ ,  $Y \subseteq \mathbb{P}^n$ ,  $U \subseteq X$  open with  $\varphi = [f_0, f_1, \dots, f_n]$  on  $U$ . WLOG the  $f_i$  are polynomials of the same degree without common divisors. Then  $\text{dom}(\varphi) = X \setminus \{f_i = 0\}$ ,  $n \geq 2$ .

**Example 49.5** Let  $Q := V(x_0x_3 - x_1x_2)$  be the quadric in  $\mathbb{P}^3$ . Let  $p = [0, 0, 0, 1]$ . Then we have a rational map  $\pi_p: Q \dashrightarrow \mathbb{P}^2$  given by  $[a_0, \dots, a_3] \mapsto [a_0, a_1, a_2]$ . This is the stereographic projection at  $p$ , and it is not a morphism. It restricts to an isomorphism from  $\{a_0 \neq 0\} \subseteq Q$  to  $U_0 = \{a_0 \neq 0\} \subseteq \mathbb{P}^2$ . Hence  $k(Q) \cong k(x, y)$ .

**Definition 49.6**  $X, Y$  quasi-projective varieties,  $\varphi: X \dashrightarrow Y$  is called *birational* if there is some  $\psi: Y \dashrightarrow X$  such that both compositions are defined and equal to the respective identities as rational maps. If such a map exists, we say that  $X$  and  $Y$  are birational.

*Remark 49.7.* Birational maps are in bijection with automorphisms of the function fields.

**Example 49.8** Consider  $\varphi: \mathbb{A}^2 \xrightarrow{\sim} U_0 \subseteq \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$  sending  $(x, y) \mapsto [1 : x : y] \mapsto [x : y]$ . The blowup of  $\mathbb{A}^2$  at the origin is the closure of the graph of  $\varphi$ , denoted  $\tilde{\mathbb{A}}^2$  or  $\text{Bl}_O(\mathbb{A}^2)$ . It comes with a map  $\pi: \tilde{\mathbb{A}}^2 \rightarrow \mathbb{A}^2$ . We call  $\{(0, 0)\} \times \mathbb{P}^1 \subseteq \tilde{\mathbb{A}}^2$  the *exceptional divisor*. Outside of it,  $\pi$  restricts to an isomorphism.

Hironaka showed that we can always resolve singularities with a finite sequence of blowups along certain centers.

#### 49.3. Rational varieties.

**Definition 49.9** A variety  $X$  is rational if for some  $n \in \mathbb{N}$  there is a birational map  $\mathbb{P}^n \dashrightarrow X$ .

The *rationality problem* consists of determining whether a variety  $X$  is rational. From now on, assume  $k = \mathbb{C}$ .

**Example 49.10** The following are rational:

- (1)  $X = \mathbb{P}^n$  and  $X = \mathbb{A}^n$ .
- (2)  $X$  same,  $X \supseteq (\mathbb{C}^\times)^n \subseteq \mathbb{P}^n$  open dense.
- (3) Grassmannians.
- (4) Smooth quadrics.
- (5) Smooth cubic surface.

#### 49.4. Related problems.

**Definition 49.11**  $X$  and  $Y$  varieties. A morphism  $\varphi: X \rightarrow Y$  is called *dominant* if its image is dense in  $Y$ . A rational map  $\varphi: X \dashrightarrow Y$  is called dominant if a defining morphism is dominant.

*Remark 49.12.* Dominant rational maps are in bijection with inclusions of function fields.

**Definition 49.13** An algebraic variety  $X$  is called *unirational* if there is a dominant rational map generically finite  $\mathbb{P}^n \dashrightarrow X$  for some  $n \in \mathbb{N}$ .

*Question 49.14.* Given  $X$ , is it unirational?

**Lüroth's problem:** Does unirationality imply rationality?

The goal of this seminar is to provide a counterexample in dimension 3.

**Theorem 49.15** If  $X$  is unirational, then  $\Gamma(X, \Omega_X^{\otimes n}) = 0$  and  $\Gamma(X, \Lambda^m \Omega_X) = 0$  for all  $m \geq 1$ .

**Example 49.16** Combining the previous theorem with the adjunction formula, if  $X \subseteq \mathbb{P}^N$  is a hypersurface of degree  $d > N$ , then  $X$  is not rational.

## 50. [LT] LECTURE 22 - 16.01.20

## 50.1. Eilenberg-Steenrod theorem and axioms.

**Definition 50.1** A *generalized cohomology theory* on pairs of finite cell complexes is a sequence of functors  $(h^n)_{n \geq 0}$  satisfying

- (a) Homotopy invariance.
- (b) Exactness, meaning that for all pairs  $(X, A)$  we get a long exact sequence

$$\cdots \rightarrow h^n(X) \rightarrow h^n(A) \rightarrow h^{n+1}(X, A) \rightarrow \cdots$$

- (c) Excision, meaning that for all pairs  $(X, A)$  and all subsets  $B \subseteq A$  such that  $\bar{B} \subseteq A^\circ$  we have an isomorphism

$$h^n(X, A) \cong h^n(X \setminus B, A \setminus B).$$

- (d) Additivity, meaning that if  $X = \coprod_\alpha X_\alpha$ , then

$$h^n(X) \cong \bigoplus_\alpha h^n(X_\alpha).$$

**Theorem 50.2 (Eilenberg-Steenrod)** If  $(\varphi_n: h^n \rightarrow k^n)_{n \geq 0}$  is a natural transformation of generalized cohomology theories such that for all pairs  $(X, A)$  the diagram

$$\begin{array}{ccc} h^n(A) & \longrightarrow & h^{n+1}(X, A) \\ \downarrow \varphi_n & & \downarrow \varphi_{n+1} \\ k^n(A) & \longrightarrow & k^{n+1}(X, A) \end{array}$$

commutes and for all  $n \in \mathbb{N}$  there are isomorphisms  $\varphi_n: h^n(*, \emptyset) \xrightarrow{\sim} k^n(*, \emptyset)$ , then  $\varphi_n$  is a natural isomorphism for all  $n \geq 0$ .

*Proof.* See [?, Theorem 3.19]. □

## 50.2. Kuenneth formula.

**Definition 50.3** Let  $X$  and  $Y$  be topological spaces. The *external cup product* is given by

$$\begin{aligned} H^*(X, R) \times H^*(Y, R) &\longrightarrow H^*(X \times Y, R) \\ (\alpha, \beta) &\longmapsto \alpha \times \beta := p_X^*(\alpha) \cup p_Y^*(\beta) \end{aligned}$$

*Exercise 50.4.* The external cup product is bilinear and therefore factors through the tensor product over  $R$ .

*Remark 50.5.* Let  $\Delta: X \rightarrow X \times X$  be the diagonal embedding. Then we have a commutative diagram

$$\begin{array}{ccc} H^*(X, R) \otimes_R H^*(X, R) & \xrightarrow{\times} & H^*(X \times X, R) \\ & \searrow \cup & \downarrow \Delta^* \\ & & H^*(X, R) \end{array}$$

**Theorem 50.6 (Kuenneth)** Let  $X$  and  $Y$  be finite cell complexes. If  $H^*(Y, R)$  is free of finite rank over  $R$ , then the external cup product induces an isomorphism

$$H^*(X, R) \otimes_R H^*(Y, R) = \bigoplus_{p+q=n} H^p(X, R) \otimes_R H^q(Y, R) \xrightarrow{\sim} H^n(X \times Y, R).$$

*Proof.* Fix  $Y$  such that  $H^*(Y, R)$  is free. We have two families of functors of pairs  $h^n(X, A) := \bigoplus_{p+q=n} H^p(X, A; R) \otimes_R H^q(Y, R)$  and  $k^n(X, A) := H^n(X \times Y, A \times Y; R)$ . Now

- (1) The external cup product defines natural transformations

$$\varphi_n: h^n \rightarrow k^n$$

for all  $n \in \mathbb{N}$ .

- (2) One can check that  $\varphi$  is compatible with the cohomology maps as required by Eilenberg-Steenrod's theorem.  
 (3) Both functors agree on points  $(*, \emptyset)$ .  
 (4) They are both cohomology theories.

For the details, see [?, Theorem 3.16].  $\square$

**Example 50.7** In our Lefschetz set up, we have

$$\begin{array}{ccc} E = p^{-1}(Z) & \longrightarrow & \tilde{X} \\ \downarrow & & \downarrow p \\ X \cap A = Z & \longrightarrow & X \end{array}$$

with isomorphisms  $p^{-1}(Z) \cong Z \times L \cong Z \times \mathbb{P}^1$  over  $Z$ . Hence  $H^*(E) \cong H^*(Z \times \mathbb{P}^1) \cong H^*(Z) \otimes H^*(\mathbb{P}^1)$ . Since  $H^*(\mathbb{P}^1)$  is free of finite rank, this is in turn isomorphic to  $H^*(Z) \otimes (H^0(\mathbb{P}^1) \oplus H^2(\mathbb{P}^1)) \cong H^*(Z) \oplus H^{*-2}(Z)$ .

### 50.3. Leray-Hirsch theorem.

**Theorem 50.8 (Leray-Hirsch)** *Let  $p: E \rightarrow B$  be a locally trivial fibration trivialized by a finite open cover of  $B$ . Assume that  $B$  is connected and that  $f^{-1}(b) =: F$  and  $B$  are finite cell complexes. Let  $\alpha_1, \dots, \alpha_r \in H^*(E, R)$  be such that for all  $b \in B$  we obtain a basis  $\alpha_1|_F, \dots, \alpha_r|_F$  of  $H^*(F, R)$ . Then there is an isomorphism*

$$\begin{aligned} H^*(B, R) \otimes_R H^*(F, R) &\xrightarrow{\sim} H^*(E, R) \\ \sum_k (\beta \otimes \alpha_k|_F) &\mapsto \sum_k p^*(\beta) \cup \alpha_k \end{aligned}$$

*Idea of proof.* Locally we apply the Kuenneth formula. To globalize, use an induction argument with Mayer-Vietoris, see [?, Section 4.D].  $\square$

**Corollary 50.9 (Projective bundle formula)** *Let  $p: E \rightarrow X$  be a vector bundle of rank  $r + 1$  and let  $\pi: \mathbb{P}(E) \rightarrow X$  be the associated projective bundle of rank  $r$ . Then  $H^*(X)[t]/t^{r+1} \xrightarrow{\sim} H^*(\mathbb{P}(E))$  for  $|t| = 2$ .*

### 50.4. Thom isomorphism for oriented vector bundles.

**Definition 50.10** A real vector bundle is *oriented* if it is equipped with an orientation on each fibre  $E_x = p^{-1}(x)$  such that each trivialization preserves orientations fibrewise, where  $\{x\} \times \mathbb{R}^r \xrightarrow{\sim} \mathbb{R}^r$  carries the standard orientation.

**Theorem 50.11 (Thom isomorphism)** *Let  $p: E \rightarrow X$  be an oriented real vector bundle of rank  $r > 0$  on a finite cell complex  $X$ . Then*

- (1) *There is a class  $\eta_E \in H^r(E, E \setminus X; \mathbb{Z})$  which restricts to the orientation class in  $H^r(E_x, E_x \setminus \{0\}; \mathbb{Z})$  for each  $x \in X$ .*  
 (2) *For all  $0 \leq k < r$  we have  $H^k(E, E \setminus X; \mathbb{Z}) = 0$ .*  
 (3) *For all  $k \geq r$  we have an isomorphism*

$$\begin{aligned} H^{k-r}(X, \mathbb{Z}) &\xrightarrow{\sim} H^k(E, E \setminus X; \mathbb{Z}) \\ \alpha &\mapsto p^*(\alpha) \cup \eta_E \end{aligned}$$



*Idea of the proof.* As in the proof of the Leray-Hirsch theorem, we reduce to the case in which  $p: E \rightarrow X$  is trivial by induction on the number of opens in a trivialization and Mayer-Vietoris. Assume hence that  $E \cong_X X \times \mathbb{R}^r$  is trivial. Since  $\mathbb{R}^r$  is contractible,  $E$  deformation retracts onto  $X \times B^r$  and  $E \setminus (X \times \{0\})$  deformation retracts onto  $X \times S^{r-1}$ . The long exact sequence

$$\cdots \rightarrow H^k(E) \rightarrow H^k(E \setminus X) \rightarrow H^{k+1}(E, E \setminus X) \rightarrow \cdots$$

is isomorphic to

$$\begin{array}{ccccc} H^k(X \times B^r) & \longrightarrow & H^k(X \times S^{r-1}) & \longrightarrow & H^{k+1}(X \times B^r, X \times S^{r-1}) \\ \downarrow \sim & & \downarrow \sim & & \downarrow \sim \\ H^k(X) & \longrightarrow & \oplus_{p+q=k} H^p(X) \otimes H^q(S^{r-1}) & \longrightarrow & \oplus_{p+q=k+1} H^p(X) \otimes H^q(D^r, S^{r-1}) \\ & & \downarrow \sim & & \downarrow \sim \\ & & H^k(X) \oplus H^{k-r+1}(X) & & H^{k-r+1}(X) \end{array}$$

□

**Definition 50.12** We call  $\eta_E$  in (1) the *Thom class* of  $p: E \rightarrow X$ , and the isomorphism in (3) the *Thom isomorphism* of  $p: E \rightarrow X$ .

### 50.5. Tubular neighbourhood theorem.

**Theorem 50.13 (Tubular neighbourhood)** Let  $i: Z \hookrightarrow X$  be a closed embedding of smooth manifolds. There exists an open neighbourhood  $U$  of  $Z$  in  $X$ , a retract  $U \rightarrow Z$  and a homeomorphism  $h: U \rightarrow N_{Z/X}$  onto the normal bundle of  $Z$  in  $X$  such that the following diagram commutes:

$$\begin{array}{ccccc} & & \text{---} & & \\ & & \text{---} & & \\ Z & \hookrightarrow & U & \xrightarrow{r} & Z \\ & & \downarrow h & & \\ Z & \xrightarrow{\sigma_0} & N_{Z/X} & \xrightarrow{\pi} & Z \\ & & \text{---} & & \\ & & \text{---} & & \end{array}$$

Here  $\sigma_0$  denotes the zero section and  $\pi$  the projection.

**Remark 50.14.** The normal bundle  $N_{Z/X}$  is intuitively the subbundle of  $T_X$  spanned by the vectors orthogonal to  $Z$ .

**50.6. Application to Thom isomorphism for non-singular varieties.** Let  $Z \subseteq X$  be a closed smooth subvariety of the smooth variety  $X$ , and let  $r > 0$  be its codimension. Then  $N_{Z/X}$  is locally free of rank  $r$  over  $Z$ . We have

$$\begin{aligned} H^{k-2r}(Z) &\cong H^k(N_{Z/X}, N_{Z/X} \setminus Z) && \text{Thom iso. for bundles} \\ &\cong H^k(U, U \setminus Z) && \text{Tubular nbhd } Z \subseteq U \subseteq X \\ &\cong H^k(X, X \setminus Z) && \text{Excision} \end{aligned}$$

We call the composite  $H^{k-2r}(Z) \rightarrow H^k(X, X \setminus Z)$  the *Thom isomorphism* for  $(X, Z)$ .

50.7. **Gysin sequence.** Another application of the Thom isomorphism theorem is the Gysin long exact sequence. One can check that

$$\begin{array}{ccc} & H^{k-2r}(X, X \setminus Z) & \\ \nearrow & & \searrow \\ H^k(Z) & \xrightarrow{i_*} & H^{k+2r}(X) \end{array}$$

commutes.

Hence we have a long exact sequence

$$\dots \rightarrow H^{k-2r}(Z) \rightarrow H^k(X) \rightarrow H^k(X \setminus Z) \rightarrow H^{k+1-2r}(Z) \rightarrow \dots$$

called the *Gysin sequence*.

## 51. [CM] INTRODUCTION TO INFINITY CATEGORIES (BRAD) - 20.01.20

51.1. **Motivation.** Last time we introduced discrete Huber pairs  $(A, A^+)$ . From this we defined adic spaces  $(X, \mathcal{O}_X, \mathcal{O}_X^+)$ , which is a ringed space equipped with a family of valuations locally isomorphic to  $\mathrm{Spa}(A, A^+)$ .

**Goal:** give  $\mathrm{D}((A, A^+)_{\blacksquare})$  to  $\mathrm{D}((\mathcal{O}_X, \mathcal{O}_X^+)_{\blacksquare})$  for general discrete adic space  $X$ .

**Problem:** Cannot glue naively, so we need  $\infty$ -categories.

51.2. **Enrichment. Idea:**  $\infty$ -categories should have sets of objects and for all  $x, y \in \mathrm{Ob}(\mathcal{C})$  a "space"  $\mathrm{Map}_{\mathcal{C}}(x, y)$  of morphisms.

**Definition 51.1** (Topological categories) For all  $x, y \in \mathrm{Ob}(\mathcal{C})$ ,  $\mathrm{Map}_{\mathcal{C}}(x, y) \in \mathbf{Top}$ . For all  $x \in \mathrm{Ob}(\mathcal{C})$  we specify  $\mathrm{id}_x \in \mathrm{Map}_{\mathcal{C}}(x, x)$ . For all  $x, y, z \in \mathrm{Ob}(\mathcal{C})$ ,  $\mathrm{Map}_{\mathcal{C}}(y, z) \times \mathrm{Map}_{\mathcal{C}}(x, y) \rightarrow \mathrm{Map}_{\mathcal{C}}(x, z)$  continuous, associative and unital composition.

**Example 51.2**  $\mathbf{Top}$  is a  $\mathbf{Top}$ -categories.

Now "invert" equivalences. Let  $F: \mathcal{C} \rightarrow \mathcal{D}$  be a topological functor. We say that  $F$  is a Dwyer-Kan equivalence if for all  $x, y \in \mathcal{C}$ ,  $F_{x,y}: \mathrm{Map}_{\mathcal{C}}(x, y) \rightarrow \mathrm{Map}_{\mathcal{D}}(Fx, Fy)$  is a homotopy equivalence ("faithfulness") and for all  $z \in \mathcal{D}$  there is some  $x \in \mathcal{C}$  such that  $Fx \simeq z$ , where this last symbol means isomorphic in the homotopy category, in which the morphisms are the  $\pi_0$  of the previous map spaces.

$$\begin{array}{ccc} \mathrm{ho}(\mathbf{Top}) = \mathbf{Top}/\simeq & \begin{array}{c} \parallel K \parallel \\ \text{geom. real.} \uparrow \end{array} & X \\ \downarrow \sim & & \downarrow \text{sing. chains} \\ \mathrm{ho}(\mathbf{sSet}) = \mathbf{sSet}/\simeq & K & \mathrm{Sing}(X) \end{array}$$

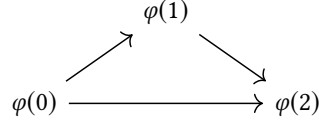
We can replace  $\mathbf{Top}$ -category with  $\mathbf{sSet}$ -category.

51.3. **Nerves.** Let  $\mathcal{C}$  be an ordinary category. Its *nerve*  $\mathbf{N}(\mathcal{C})$  is the simplicial set defined by

$$\forall n \geq 0, \quad \mathbf{N}(\mathcal{C})_n := \mathrm{Fun}([n], \mathcal{C})$$

where  $[n] = \{0 \rightarrow \dots \rightarrow n\}$  is the standard  $n$ -simplex category. Face and degeneracy maps are given by inserting identity maps and composing adjacent maps. This gives a fully faithful functor  $\mathbf{N}: \mathbf{Cat} \hookrightarrow \mathbf{sSet}$ .

**Example 51.3**  $\varphi \in \mathbf{N}(\mathcal{C})_2$  corresponding to  $\varphi: [2] \rightarrow \mathcal{C}$ .



If  $\mathbf{C}$  is a 1-category, then there is a unique way to "fill-in" the previous triangle, i.e. there is a unique extension

$$\begin{array}{ccc}
 \Lambda_1^2 = \Delta^2 \setminus \{0 \rightarrow 2\} & \xrightarrow{\quad} & \mathbf{C} \\
 \downarrow & \nearrow \text{dashed} & \\
 \Delta^2 & & 
 \end{array}$$

In an  $\infty$ -category there will be many ways to do it. They will all be homotopic, but still different.

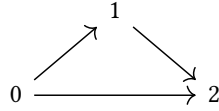
To generalize to  $\infty$ -categories, we replace  $[n]$  by simplicial categories, which we will denote  $\mathfrak{C}(\Delta^n)$ .  $\text{Ob}(\mathfrak{C}(\Delta^n)) = \text{Ob}(\Delta^n) = \{0, \dots, n\}$ . For all  $0 \leq i, j \leq n$ , consider the category

$$P_{i,j} = \begin{cases} \text{Objects are subsets of } [i, j] \text{ containing } \{i, j\} \\ \text{Morphisms are inclusions of subsets.} \end{cases}$$

Then  $\text{Map}_{\mathfrak{C}(\Delta^n)}(i, j) = \mathbf{N}(P_{i,j})$ .

**Example 51.4** When  $n = 1$ ,  $\text{Ob}(\mathfrak{C}(\Delta^1)) = \{0, 1\}$ .  $P_{0,0} = *$ ,  $P_{0,1} = \{[0, 1]\} = *$ , and  $\text{Map}_{\mathfrak{C}(\Delta^1)}(0, 1) = *$ . Hence  $\mathfrak{C}(\Delta^1) = \Delta^1$ .

**Example 51.5** When  $n = 2$ , for the endomorphism spaces we get again singletons. Same for adjacent vertices, so the new data is  $P_{0,2} = \{\{0, 2\} \subseteq \{0, 1, 2\}\}$ .  $\mathbf{N}(P_{0,2})$  has a single nondegenerate 1-simplex  $\alpha$ . Hence  $\mathfrak{C}(\Delta^2)$  specifies



with  $\alpha$  filling in the previous triangle.

**Example 51.6** For  $n = 3$  we get 4 zero-simplices, 6 one-simplices, 4 two-simplices and 1 three-simplex.

While  $\Delta^n$  encodes strict composition, these  $\mathfrak{C}(\Delta^n)$  are encoding composition up to coherent homotopy.

Let  $\mathbf{C}$  be a **sSet**-category. Then  $F \in \text{Fun}_{\text{sSet}}(\mathfrak{C}(\Delta^n), \mathbf{C})$  encodes a "homotopy coherent diagram"

$$F: \Delta^n \rightarrow \mathbf{C}$$

For all  $0 \leq i \leq n$ , we have  $F(i) \in \mathbf{C}$  an object, and for all  $0 \leq i \leq j \leq n$  we have  $F(i) \rightarrow F(j)$ , and for all  $0 \leq i \leq j \leq k \leq n$  we get  $F(i) \rightarrow F(k)$ ,  $F(i) \rightarrow F(j) \rightarrow F(k)$  and a homotopy between those.

**Theorem 51.7 (Cordier-Porter, 1986)** If  $\mathbf{C}$  is a simplicial category such that for all  $x, y \in \mathbf{C}$  the simplicial set  $\text{Map}_{\mathbf{C}}(x, y)$  is a Kan complex, then the simplicial set  $\mathbf{N}_{\Delta}(\mathbf{C})$  given by  $\mathbf{N}_{\Delta}(\mathbf{C})_n = \text{Fun}_{\text{sSet}}(\mathfrak{C}(\Delta^n), \mathbf{C})$  satisfies the following inner-horn filler condition

$$\begin{array}{ccc}
 \Lambda_k^n & \xrightarrow{\quad} & \mathbf{N}_{\Delta}(\mathbf{C}) \\
 \downarrow & \nearrow \text{dashed} & \\
 \Delta^n & & 
 \end{array}$$

for all  $0 < k < n$ . For all  $0 \leq k \leq n$ ,  $\Delta_k^n$  is the simplicial set obtained from  $\Delta^n$  by deleting the interior and the face opposite to the  $k$ -th vertex.

A Kan complex is a simplicial set satisfying this condition for  $0 \leq k \leq n$ .

**Definition 51.8 (Quasi-category)** A quasi-category (or simply  $\infty$ -category if we follow Lurie's convention) is a simplicial set  $C$  satisfying the inner horn condition.

Composition in quasi-categories is encoded weakly:

$$\begin{array}{ccc} \Delta_1^2 & \longrightarrow & C \\ \downarrow & \nearrow & \\ \Delta^2 & & \end{array}$$

If we have a quasi-category  $C$  and two maps  $\varphi, \psi: \Delta^2 \rightarrow C$  restricting to a same map  $\iota$  on  $\Delta_1^2$ , then they differ by a homotopy, and two such homotopies differ by a higher homotopy, and so on. So  $g \circ f$  is not unique, but it is a point on a contractible space of possibilities.

$\text{Map}(\Delta^n, C) \xrightarrow{\iota^*} \text{Map}(\Delta_k^n, C)$ . The inclusion of the horn is a cofibration. If  $C$  is a quasi-category, then  $\iota^*$  is a weak homotopy equivalence. Different choices of composition correspond to points in the homotopy fiber of this map, which is weakly contractible, hence contractible in our setting.

#### 51.4. Working with quasi-categories.

**Definition 51.9 (Mapping spaces)**  $C$  a quasi-category,  $x, y \in C_0 \stackrel{\text{def}}{=} \text{Ob}(C)$ .  $\text{Map}_C(x, y)$  is given as the following pullback  $\mathbf{sSet}$ :

$$\begin{array}{ccccc} \text{Map}_C(x, y) & \longrightarrow & C_1 & \xlongequal{\quad} & \text{hom}(\Delta^1, C) \\ \downarrow & & \downarrow (s, t) & & \downarrow \text{restrict along } \partial\Delta^1 \hookrightarrow \Delta^1 \\ * & \xrightarrow{(x, y)} & C_0 \times C_0 & \xlongequal{\quad} & \text{hom}(\partial\Delta^1, C) \end{array}$$

*Fact 51.10.*  $\text{Map}_C(x, y)$  is a Kan complex.

**Definition 51.11 (Functor categories)**  $K \in \mathbf{sSet}$ ,  $C$  a q-cat.  $\text{Fun}(K, C) = \text{Map}_{\mathbf{sSet}}(K, C) \in \mathbf{sSet}$ , where

$$\text{Map}_{\mathbf{sSet}}(K, C)_n = \text{hom}_{\mathbf{sSet}}(K \times \Delta^n, C)$$

*Remark 51.12.*  $\text{Fun}(K, C)_1$  are just natural transformations if  $K, C$  are 1-categories. In fact, if  $K, C$  are 1-categories, then  $N(\text{Fun}(K, C)) \cong \text{Fun}(N(K), N(C))$ .

**Definition 51.13 (Localization)**  $C$  a q-cat.,  $S \subseteq \text{Mor}(C) = C_1 = \text{hom}_{\mathbf{sSet}}(\Delta^1, C)$ . A localization of  $C$  with respect to  $S$  is a functor  $\lambda: C \rightarrow C[S^{-1}]$  (i.e. a zero-simplex in  $\text{Fun}(C, C[S^{-1}])$ ) such that  $C[S^{-1}]$  is a q-cat, and for all q-cat  $D$  we have

$$\text{Fun}(C[S^{-1}], D) \xrightarrow{\lambda^*} \text{Fun}(C, D)$$

is fully faithful and its essential image is  $F: C \rightarrow D \mid F(S) \subseteq \text{eq}(D)\}$ .

**Theorem 51.14 (Lurie)** Localizations exist and are unique up to contractible choice (modulo set-theoretical non-sense).

**Example 51.15**

$$\begin{array}{ccc} N(\text{Cat}) & \xrightarrow{\quad} & N(\text{Cat})[eq^{-1}] \\ & \searrow N & \downarrow \text{fully faithful} \\ & & \mathbf{QCat} = \infty\text{-cat of q-cats} \end{array}$$

### 51.5. Limits.

**Definition 51.16** (Final objects)  $\mathcal{C}$  a quasi-category. We say that  $*$   $\in \text{Ob}(\mathcal{C})$  is *final* if  $\forall x \in \text{Ob}(\mathcal{C}), \text{Map}_{\mathcal{C}}(x, *)$  is contractible.

For all 1-category  $K$  (indexing category) we can form  $K^{\flat} = K \cup \{\infty\} = K$  with a new freely adjoint final object. If  $F: K \rightarrow \mathcal{C}$  is a functor from  $K$  to the q-cat  $\mathcal{C}$ , then  $\lim_{k \in K} F(k)$ , if it exists, is  $\tilde{F}(\infty)$ , where  $\tilde{F}$  is the final object in  $X = \text{Fun}_F(K^{\flat}, \mathcal{C}) = \{F': K^{\flat} \rightarrow \mathcal{C} \mid F'|_K \simeq F\} \subseteq \text{Fun}(K^{\flat}, \mathcal{C}) \rightarrow \text{Fun}(K, \mathcal{C})$ .

**51.6. Adjunctions.** Use triangle equations to define adjunctions. Instead of equality, we ask for coherent homotopy between the identities and the corresponding compositions. Such an adjunction induces natural equivalences of bifunctors as usual.

### 52. [LT] LECTURE 23 - 21.01.20

**52.1. Blow-up formula.** If  $\tilde{X}$  is the blow-up of  $X$  along  $Z \subseteq X$ , then  $H^k(\tilde{X}) \cong H^k(X) \oplus H^{k-2}(Z)$ .

The *degree* of a map  $f: M \rightarrow N$  between oriented, connected, compact manifolds of the same dimension  $n \in \mathbb{N}$  is defined as the integer corresponding to  $\mathbb{Z} = H_n(M) \xrightarrow{f_*} H_n(N) = \mathbb{Z}$ .

**Lemma 52.1**  $\deg(p: \tilde{X} \rightarrow X) = 1$ .

*Proof.*  $p$  is an isomorphism over  $U = X \setminus Z$ . For all  $x \in U$  we have a pullback square

$$\begin{array}{ccc} p^{-1}\{x\} & \xrightarrow{\tilde{i}_x} & \tilde{X} \\ \downarrow q & & \downarrow p \\ \{x\} & \xrightarrow{i_x} & X \end{array}$$

By base change we have  $i_x^* p_* = q_* \tilde{i}_x^*$ . Hence

$$\begin{aligned} i^* p_*([\tilde{X}]^*) &= q_* \tilde{i}_x^*([\tilde{X}]^*) \\ &= q_*([p^{-1}\{x\}]^*) \\ &= [\{x\}]^* q \text{ iso, preserves the orientation} \end{aligned}$$

Therefore  $i_x^*(\deg(p)[X]^*) = \deg(p)i_x^*([X]^*) = \deg(p)[\{x\}]^* = [\{x\}]^*$ , and so  $\deg(p) = 1$ .  $\square$

*Proof of (3).*  $p^*: H^k(X) \rightarrow H^k(\tilde{X})$  is injective. Indeed,  $p_* \circ p^* = \deg(p)$ . So if  $p^*(\alpha) = 0$ , then  $\alpha = p_* p^*(\alpha) = 0$ . By the projective bundle formula proven last time,  $p^*: H^k(X, X \setminus Z) \rightarrow H^k(\tilde{X}, \tilde{X} \setminus E)$  is injective. Indeed,

$$\begin{array}{ccc} H^k(X, X \setminus Z) & \longrightarrow & H^k(\tilde{X}, \tilde{X} \setminus E) \\ \downarrow \text{Thom.iso.} & & \downarrow \sim \\ H^{k-4}(Z) & \hookrightarrow & H^{k-2}(E) \cong H^{k-2}(Z) \\ & \searrow & \downarrow \sim \\ & & H^{k-2}(Z) \oplus H^{k-4}(Z) \end{array}$$

Consider now the diagram

$$\begin{array}{ccccccc}
H^k(X, X \setminus Z) & \longrightarrow & H^k(X) & \longrightarrow & H^k(U) & \longrightarrow & H^{k+1}(X, X \setminus Z) \\
\downarrow & & \downarrow & & \downarrow \sim & & \downarrow \\
H^k(\tilde{X}, \tilde{X} \setminus E) & \longrightarrow & H^k(\tilde{X}) & \longrightarrow & H^k(\tilde{U}) & \longrightarrow & H^{k+1}(\tilde{X}, \tilde{X} \setminus E)
\end{array}$$

$H^k(X) \oplus H^{k-2}(E) \cong H^k(X) \oplus H^k(\tilde{X}, \tilde{X} \setminus E) \rightarrow H^k(\tilde{X})$  is surjective with kernel

$$\text{im}(H^{k-4}(Z) \xrightarrow{(i_*, -q^*)} H^k(X) \oplus H^{k-2}(E)),$$

where  $q: E \rightarrow Z$  is the projection [diagram chase].  $\square$

**52.2. Extraordinary cup-product.** Let  $i: Z \rightarrow X$  be a closed subvariety of codimension  $r$ , both  $Z$  and  $X$  non-singular. We will define an operation

$$H^k(X, X \setminus Z) \times H^k(Z) \xrightarrow{\cup} H^{k+l}(X, X \setminus Z)$$

satisfies

- (1) For all  $\alpha \in H^k(X)$  and for all  $\beta \in H^l(X, X \setminus Z)$  and for all  $\gamma \in H^n(Z)$  we have

$$\alpha \cup (\beta \cup \gamma) = (\alpha \cup \beta) \cup \gamma.$$

- (2) If  $r: H^k(X, X \setminus Z) \rightarrow H^k(X)$ , then for all  $\alpha \in H^k(X, X \setminus Z)$  and for all  $\beta \in H^l(X, X \setminus Z)$  we have

$$r(\alpha) \cup \beta = \alpha \cup i^*(r(\beta)).$$

- (3) For all  $\alpha \in H^k(X)$  and for all  $\beta \in H^l(X, X \setminus Z)$  we have

$$\alpha \cup \beta = (-1)^{kl} \beta \cup i^*(\alpha)$$

For more details, see Iversen's Cohomology of sheaves, page 128.

Recall the Thom isomorphism  $H^{k-2r}(Z) \xrightarrow{\sim} H^k(N_{Z/X}, N_{Z/X} \setminus Z) \cong H^k(X, X \setminus Z)$ , already used previously today in one of the commutative diagrams. We can express this composition as  $\alpha \mapsto \tau_Z \cup \alpha$ , and we call  $\tau_Z \in H^{2k}(X, X \setminus Z)$  the *Thom class* of  $Z \subseteq X$ . We have

$$\begin{array}{ccc}
& H^k(X, X \setminus Z) & \\
\tau_Z \cup \nearrow & & \searrow r \\
H^{k-2r}(Z) & \xrightarrow{i_*} & H^k(X)
\end{array}$$

i.e. for all  $\alpha$  we have  $i_*(\alpha) = r(\tau_Z \cup \alpha)$ .

### 52.3. Projection formula.

**Proposition 52.2** For all  $\alpha \in H^k(Z)$  and all  $\beta \in H^l(X)$  we have

$$i_*(\alpha \cup i^*(\beta)) = i_*(\alpha) \cup \beta \in H^{k+2+l}(X).$$

*Proof.* Using property (3) above we have

$$i_*(\alpha \cup i^*(\beta)) = r(\tau_Z(\alpha \cup i^*(\beta))) = (-1)^{(k+2r)l} r(\beta \cup \tau_Z \cup \alpha),$$

which is equal to

$$(-1)^{(k+2r)l} \beta \cup r(\tau_Z \cup \alpha) = r(\tau_Z \cup \alpha) \cup \beta = i_*(\alpha) \cup \beta.$$

$\square$

**Corollary 52.3** For all  $\alpha \in H^k(X)$  we have

$$i_* i^*(\alpha) = r(\tau_Z) \cup \alpha.$$

*Proof.*

$$\begin{aligned} i_* i^*(\alpha) &= i_* i^*(1_X \cup \alpha) \\ &= i_*(i^*(1_X) \cup i^*(\alpha)) \\ &= i_*(1_Z \cup i^*(\alpha)) \\ &= i_*(1_Z) \cup \alpha \\ &= r(\tau_Z) \cup \alpha \end{aligned}$$

□

**Corollary 52.4** If  $i: Z \hookrightarrow X$  is the inclusion of a nonsingular hyperplane section, then for all  $\alpha \in H^k(X)$  we have  $i_* i^*(\alpha) = [Z]^* \cup \alpha$ .

The idea of the proof is to show that  $r(\tau_Z) = [Z]^*$  in this special case and apply the previous corollary.

**Definition 52.5** The Euler class is defined as

$$e_Z := i^* r(\tau_Z) \in H^{2r}(Z).$$

**Proposition 52.6** For all  $\alpha \in H^k(Z)$  we have  $i^* i_*(\alpha) = e_Z \cup \alpha$ .

*Proof.*  $\tau_Z \cup$  is an isomorphism. On the one hand we have

$$\begin{aligned} \tau_Z \cup i^* i_*(\alpha) &= \tau_Z \cup i^*(r(\tau_Z) \cup \alpha) \\ &\stackrel{(2)}{=} r(\tau_Z) \cup (\tau_Z \cup \alpha) \end{aligned}$$

On the other hand,

$$\begin{aligned} \tau_Z \cup (e_Z \cup \alpha) &= \tau_Z \cup (i^* r(\tau_Z) \cup \alpha) \\ &\stackrel{(1)}{=} (\tau_Z \cup i^* r(\tau_Z)) \cup \alpha \\ &\stackrel{(2)}{=} (r(\tau_Z) \cup \tau_Z) \cup \alpha \end{aligned}$$

Finally apply (1) again. □

**Corollary 52.7** If  $i: Z \hookrightarrow X$  is the inclusion of a nonsingular hyperplane section, then for all  $\alpha \in H^k(Z)$  we have  $i^* i_*(\alpha) = i^*([Z]^*) \cup \alpha$ .

The idea of the proof is as before: show that  $e_Z = i^* r(\tau_Z) = i^*([Z]^*)$  in this special case.

Combining the last two corollaries we get (7) from the proof of the LHT.

#### 52.4. Review of the proof of LHT.

**Theorem 52.8** Choose a Lefschetz pencil and restrict along the zero hyperplane section  $i_0^*: H^k(X) \rightarrow H^k(X_0)$ . This restriction is

- an isomorphism for  $k < n - 1$ , and
- injective for  $k = n - 1$ .

*Proof.* (1)  $\tilde{i}_0^*: H^k(\tilde{X}) \rightarrow H^k(\tilde{X}_0)$  is surjective for  $k < n - 1$ . Consider the diagram

$$\begin{array}{ccc}
& H^k(\tilde{X} \setminus X_\infty) & \\
a \nearrow & & \searrow b \\
H^k(\tilde{X}) & \xrightarrow{\tilde{i}_0^*} & H^k(\tilde{X}_0)
\end{array}$$

in which both  $a$  and  $b$  are surjective in the desired range. Hence the bottom arrow is surjective in the desired range as well.

- (2)  $i_0^*: H^k(X) \rightarrow H^k(X_0)$  is surjective for  $k < n - 1$ . Consider the diagram

$$\begin{array}{ccc}
H^k(\tilde{X}) & \xrightarrow{\sim} & H^k(X) \oplus H^{k-2}(Z) \\
\downarrow & & \downarrow \\
H^k(X_0) & \xleftarrow{h_{0,*}} & H^{k-2}(Z)
\end{array}$$

in which the vertical arrows are surjective in the desired range. Hence the bottom arrow is surjective in the desired range as well. The claim is that  $\text{im}(h_{0,*}) \subseteq \text{im}(i_{0,*})$ .

By induction hypothesis,  $h_0^*: H^m(X_0) \rightarrow H^m(Z)$  is surjective for  $m < n - 2$ . Hence  $\text{im}(h_0^*) = H^{k-2}(Z)$  for  $k < n - 1$ . Hence  $\text{im}(h_{0,*}) = \text{im}(h_{0,*} \circ h_0^*)$  which by (7) is equal to  $\text{im}(i_0^* \circ i_{0,*}) \subseteq \text{im}(i_0^*)$ .

- (3)  $i_0^*$  is injective for  $k \leq n - 1$ . Let  $\alpha \in H^k(X)$  such that  $i_0^*(\alpha) = 0$ .

$$\begin{array}{ccccc}
H^k(X) & \xrightarrow{i_0^*} & H^k(X_0) & & \\
\downarrow & & \downarrow & & \\
H^{k-2}(X_\infty) & \xrightarrow{\tilde{i}_{\infty,*}} & H^k(\tilde{X}) & \xrightarrow{\tilde{j}_\infty^*} & H^k(\tilde{X} \setminus X_\infty) \xrightarrow{(4)} H^k(\tilde{X}_0)
\end{array}$$

Hence  $\tilde{j}_\infty^* p^*(\alpha) = 0$ .

By (2), there exists some  $\beta \in H^{k-2}$  such that  $p^*(\alpha) = \tilde{i}_{\infty,*}\beta$ . By (7),  $\alpha = p_* p^*(\alpha) = p_* \tilde{i}_{\infty,*}\beta = i_{\infty,*}\beta$ . Hence  $p^* i_{\infty,*}\beta = p^* \alpha = \tilde{i}_{\infty,*}\beta$ . By (3), we get a diagram

$$\begin{array}{ccc}
H^k(\tilde{X}) & \xrightarrow{\sim} & H^k(X) \oplus H^{k-2}(Z) \\
& \searrow & \parallel \\
& & \text{im}(p^*) \oplus H^{k-2}(Z) \\
& \searrow & \downarrow \\
& & H^{k-2}(Z)
\end{array}$$

□

### 53. [WS] RATIONALITY 2 (TANUJ) - LUEROOTH'S PROBLEM - 22.01.20

Let  $k = \bar{k}$ .

A map  $(\dashrightarrow)$  is a rational map, as opposed to an honest morphism  $(\rightarrow)$ .

**Proposition 53.1** *If  $f: X \dashrightarrow Y$  with  $X$  and  $Y$  being smooth and projective, then  $\text{codim}(X \setminus V) \geq 2$ , where  $V$  is a maximal open set on which the map is defined.*

**Remark 53.2.** For a smooth curve,  $f: C \dashrightarrow X$  can always be extended to a morphism.

Let  $f: X \dashrightarrow Y$ . We say that  $f$  is *dominant* if  $\text{im}(f)$  is dense in  $Y$ . We say that  $X$  is *rational* if it is birational to  $\mathbb{P}^N$  for some  $N$ . We say that  $X$  is *unirational* if  $\exists \mathbb{P}^n \dashrightarrow X$



generically finite dominant map for some  $N$ , or equivalent, there exists a dominant map  $\exists \mathbb{P}^{N'} \dashrightarrow X$ .

**Luroth's problem:** if  $X$  is unirational, is  $X$  rational as well?

For curves, the answer is yes in all characteristics. For surfaces, the answer is yes in characteristic zero but no in positive characteristics.

*Question 53.3 (Philosophical).* Is being rational all that great?

The goal of the seminar is to provide a counterexample for Luroth's problem in dimension 3.

**Setting:**  $X$  and  $Y$  smooth projective varieties over  $k = \bar{k}$  of characteristic zero.  $S$  a surface.

*Fabian's HW:* if  $Y$  is unirational, then  $H^0(Y, \Omega^p) = 0$  and  $H^0(Y, (\Omega^p)^{\otimes n}) = 0$ .

**Proposition 53.4**  $f: X \dashrightarrow Y$  dominant and generically finite, then  $H^0(Y, \Omega_Y^p) \subseteq H^0(X, \Omega_X^p)$ .

*Proof.* Let  $V$  be a maximal open such that  $f: V \rightarrow Y$  is a morphism. We can pullback forms:

$$f^*: H^0(Y, \Omega_Y^p) \rightarrow H^0(V, \Omega_V^p).$$

This  $f^*$  is injective, and since the complement of  $V$  is of codimension at least 2, we get  $H^0(V, \Omega_V^p) \cong H^0(X, \Omega_X^p)$  by Hartog's extension theorem.  $\square$

**Corollary 53.5 (Fabian's Homework)**  $X = \mathbb{P}^N$ , and  $H^0(\mathbb{P}^N, \Omega_{\mathbb{P}^N}^p) = 0$ . Hence  $H^0(Y, \Omega_Y^p) = 0$ .

*Proof.* To see that on  $\mathbb{P}^N$  there are no global sections, use the Euler sequence.  $\square$

**Corollary 53.6** Suppose  $f$  is birational instead. Then  $H^0(X, \Omega_X^p) \cong H^0(Y, \Omega_Y^p)$ . Hence  $h^{p,0}$  are birational invariants.

**Corollary 53.7 (Luroth's problem for curves)**  $f: \mathbb{P}^1 \dashrightarrow C$  is a morphism because we are in the case of curves. Thus we can pullback forms and  $H^0(C, \omega_C) = 0$ . Hence  $C = \mathbb{P}^1$ .

**53.1. Surfaces.** [Picture: Hodge diamond for a surface, Hodge and Serre symmetries] The outer part of the Hodge diamond is a birational invariant. Example: Hodge diamond of  $\mathbb{P}^2$ . For surfaces,  $h^{1,0} =: q$  is an important birational invariant, called the *irregularity* of the surface. The guy in the middle is not a birational invariant, since it goes up with blow-ups.

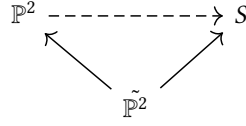
We want a criterium for rationality of surfaces. For sure we need  $q(S) = 0$ . We can hope that if  $q(S) = 0$  and  $h^{2,0}(S) = 0$ , then  $S$  is rational. Enriques gave a counterexample (Enriques' surfaces). But Castelnuovo realized that if  $\dim(H^0(S, \omega_S^{\otimes 2})) = 0$  and  $q(S) = 0$ , then  $S$  is rational.

**Theorem 53.8 (Castelnuovo's rationality criterion)**  $S$  surface,  $q(S) = 0$ ,  $P_2(S) := H^0(S, \omega_S^{\otimes 2}) = 0$ , then  $S$  is rational.

For a proof, see Vakil's notes on algebraic surfaces.

**Fact 53.9 (Resolution of indeterminacy).** Whenever we have a rational map  $S \dashrightarrow X$  between a surface and a variety  $X$ , we can resolve the indeterminacies by blowing up  $S$  at the points on which the map is not defined. Hence we obtain a morphism  $\tilde{S} \rightarrow X$ .

In our setting unirational



**Proposition 53.10** (Luroth's problem for surfaces) *Stated above.*

*Proof.* We are given  $\mathbb{P}^2 \dashrightarrow S$  and we want to show that  $S$  is rational. By resolving the indeterminacy we get a morphism  $\tilde{\mathbb{P}}^2 \rightarrow S$ . The claim is that  $f$  is surjective. Indeed, it is dominant and proper, hence surjective.

Suppose  $S$  is not rational. Then  $q(S) > 0$  or  $P_2(S) > 0$  by Castelnuovo's rationality criterion. By pulling back differentials we would get  $q(\tilde{\mathbb{P}}^2) > 0$  or  $P_2(\tilde{\mathbb{P}}^2) > 0$ , but this is a contradiction, because these are birational invariants and  $\tilde{\mathbb{P}}^2$  is rational.  $\square$

In positive characteristic we have the following counterexample:

**Example 53.11** (Shioda) Let  $k$  be a field of characteristic  $p > 2$ . Let  $X$  be the hypersurface  $S = \{x_1^{p+1} - x_2^{p+1} = x_3^{p+1} - x_4^{p+1}\} \subseteq \mathbb{P}^3(k)$ . By adjunction,  $\omega_S = \mathcal{O}_{\mathbb{P}^3}(p+1-4)|_S$  has degree non-negative, so it has sections. But we get a unirational map by some field theory.

**53.2. [Philosophical] Is rationality all that good?** We will talk about *families*, by which we will mean surjective smooth maps  $X \rightarrow \Delta$  where  $\Delta$  is the unit disk in  $\mathbb{C}$ . Let  $\mathbf{P}$  be a property.

**Definition 53.12** We will call  $\mathbf{P}$  an *open* (resp. *closed*) property if the set  $\{g \in \Delta \mid X_g\}$  is open (resp. closed) in  $\Delta$  for every such family.

We may also replace  $\Delta$  by any other base.

**Question 53.13.** Is rationality an open or closed condition in families?

No, neither of the two! This was answered by a recent work of Hassett, Piratka and Tschinkel in 2018.

Instead, if we look at *rationally connected* varieties, this turns out to be an open and closed condition.

54. [LT] LECTURE 24 - ARTIN VANISHING - 23.01.20

See [?] and [?].

55. [FS] MIRKO MAURI - DUAL COMPLEX OF LOG CALABI-YAU PAIRS ON MORI FIBRE SPACES - 24.01.20

The idea is to encode the intersection of divisors on a CW complex.

**Definition 55.1** Let  $(X^{n+1}, \Delta = \sum_{i=1}^{m+1})$  snc (dlt, qdlt). The *dual complex* of  $\Delta$ , denoted  $D(\Delta)$ , is a regular cell complex defined as follows:

- (1) 0-cells are in bijection with the irreducible components  $\Delta_i$ .
- (2)  $k$ -cells are in bijection with irreducible components of intersections  $\Delta_{i_0} \cap \cdots \cap \Delta_{i_k}$ .

### 55.1. Algebraic version of the Poincare conjecture.

**Conjecture 55.2 (Kollar-Xu)** *Let  $(X, \Delta)$  dlt log Calabi-Yau, i.e.  $K_X + \Delta \sim_{\mathbb{Q}} 0$ . Then the dual complex  $D(\Delta)$  is a sphere or a finite quotient of a sphere.*

**Example 55.3 (Toric varieties)** Let  $X$  be a toric variety, i.e. a variety with an action of an algebraic group and a dense orbit, together with some fan  $\Sigma$  from which it is obtained. Let  $\Delta$  be a toric divisor, i.e. the complement of the big torus. For example,  $\mathbb{P}^2 \supseteq \mathbb{C}^\times \times \mathbb{C}^\times$ , whose complement are the three axes, hence the dual complex is an  $S^1$ . This is a classical example of log CY pairs.  $\text{Cone}(D(\Sigma)) \cong |\Sigma| \cong \mathbb{R}^n$ .

In this case we are in fact building the variety from the dual complex.

**Example 55.4** Let  $(X, \Delta)$  and  $(X_2, \Delta_2)$  be a log Calabi-Yau. For curves, the only example is  $\mathbb{P}^1$  with two points. Then  $D(X_1 \times X_2, \pi_1^* \Delta_1 + \pi_2^* \Delta_2) = D(\Delta_1) * D(\Delta_2)$ , where  $*$  denotes the join of topological spaces. This is an important reality check, because if we had e.g. a product there, then we would have a counterexample to the conjecture above.

**Theorem 55.5 (Kollar-Xu)** *The conjecture holds if  $\dim X \leq 4$  or ( $\dim X = 5$  and  $(X, \Delta)$  snc).*

**Conjecture 55.6** *If  $(X, \Delta)$  is snc, then*

$$\text{Tors}(H^*(D(\Delta), \mathbb{Z})) = 0.$$

**55.2. Motivation.** Let  $(Y, E) \rightarrow X$  be a (minimal dlt modification) log resolution of some (log canonical) isolated singularity. Then [Stepomov...] the (homeo) homotopy type of  $D(E)$  is an invariant of the singularity.  $K_Y + E \sim_{\mathbb{Q},f} 0$ .

**55.3. Mirror symmetry.** Let  $\pi: \mathcal{X} \rightarrow B$  be a family of CY varieties over a small disk. Assume  $\pi$  is a minimal degeneration, so  $K_{\mathcal{X}} \sim_{\mathbb{Q}} 0$ . The general member will be smooth and CY, but over zero we may have some singular object, let's say  $F := \pi^{-1}(0) = \sum F_i$  dlt (modulo reducing the scheme structure). Then [Nicaise-Xu] the Borekavich analytification of  $\mathcal{X}_\eta^{an}$  retracts onto essential skeleton  $\simeq D(F)$  and it is the basis of a non-archimedean SYL fibration.

**55.4. What is the relation with the conjecture.** Let  $v_0 \in D(\Delta)$ .  $\text{Link}(v_0, D(\Delta)) = \overline{\text{Star}(v_0)} \setminus \text{Star}(v_0)$ , where  $\text{Star}$  is the union of the cell adjacent to  $v_0$ , which is equal to  $D(\Delta_0, \sum_{j \neq 0} \Delta_j|_{\Delta_0})$ .

If  $D(\sum_{j \neq i} E_j|_{E_i})$  is a sphere, then  $D(E)$  is a manifold.

The tricky part is to determine whether it is a manifold or not. A way to tackle this problem is:

### 55.5. Connectivity theorems.

**Theorem 55.7 (Kollar)** *Let  $(X, \Delta)$  be a dlt log CY. If  $D(\Delta)$  is not connected, then  $D(\Delta) \simeq S^0$ .*

*Proof.* Let  $(X, \Delta) \dashrightarrow (Y, \Delta_Y) \xrightarrow{\pi} Z$ . Here the dashed arrow is the result of  $K_X$ -MMP,  $\pi$  is a Mori fibre space and  $(Y, \Delta_Y)$  is log CY. Then use the assumption:

- (1) Support of  $\Delta_Y$  is not connected, so  $\exists \Delta_0 \subseteq \Delta_Y$  dominant.
- (2)  $\Delta_0|_{F_{gen}}, \Delta_1|_{F_{gen}}$  ample and they need to intersect, unless we are on a curve. So we are on  $\mathbb{P}^1$  with two points (if it was an elliptic curve, the assumptions would not hold).

□

**Theorem 55.8 (-)** Let  $(X, \Delta) \rightarrow Z$  be dlt log CY Mori fibre space with  $\rho(Y) \leq 2$  or  $\dim(Z) \leq 2$ . Then  $D(\Delta) \simeq$  sphere, standard simplex or  $\mathbb{R}P^2 * S^*$ .

See [?] for the details.

56. [CM] GLOBALIZATION II (BRAD) - 27.01.20

Discrete Huber pairs = affine scheme analogue.

**56.1. Recollections.** Discrete Huber pairs  $(A, A^+)$  consist of a discrete ring  $A$  and an integrally closed subring  $A^+$ . We defined a functor

$$\{ \text{discrete Huber pairs} \} \rightarrow \{ \text{analytic rings} \}$$

sending  $(A, A^+)$  to the colimit

$$\text{colim}_{(B, B^+) \rightarrow (A, A^+), B, B^+ f.g. / \mathbb{Z}} (B, B^+) \blacksquare$$

$X = \text{Spa}(A, A^+) = \{v: A \rightarrow \Gamma \cup \{0\} \mid v \text{ valuation}, v(A^+) \leq 1\} / \sim$ . Rational opens  $U = U(\frac{g_1, \dots, g_n}{f}) := \{x \in X \mid \forall k |g_k(x)| \leq |f(x)| \neq 0\}$  form the base for the topology on  $\text{Spa}(A, A^+)$ .  $\mathcal{O}_{\text{Spa}(A, A^+)}(U) = A[\frac{1}{f}]$  for  $U$  as above.

**56.2. Goal.**

**Theorem 56.1** (Theorem 9.8)  $\text{Spa}(A, A^+) \mapsto D((A, A^+) \blacksquare)$  is a sheaf of  $\infty$ -categories.

$$\begin{array}{ccc} & U \cap V & \longrightarrow V \\ \text{Example 56.2} & \downarrow & \downarrow \\ & U & \longrightarrow X \end{array}$$

with  $U, V \subseteq X = \text{Spa}(A, A^+)$  rational, then

$$\begin{array}{ccc} D((A, A^+) \blacksquare) & \longrightarrow & D((\mathcal{O}_X(U), \mathcal{O}_X^+(U)) \blacksquare) \\ \downarrow & & \downarrow \\ D((\mathcal{O}_X(V), \mathcal{O}_X^+(V)) \blacksquare) & \longrightarrow & D((\mathcal{O}_X(U \cap V), \mathcal{O}_X^+(U \cap V)) \blacksquare) \end{array}$$

is cartesian.

**Corollary 56.3** There is an essentially unique way to extend the functor of Theorem 9.8 to discrete adic spaces

$$X \mapsto D((\mathcal{O}_X, \mathcal{O}_X^+) \blacksquare) = \lim_{\text{Spa}(A, A^+) \rightarrow X} D((A, A^+) \blacksquare)$$

**Remark 56.4.** If we are willing to accept infinity-categorical sheaf-theoretic nonsense, then it suffices to check the sheaf condition on open covers of  $\text{Spa}(A, A^+)$  by rational opens.

**56.3. Ingredients in the proof.** The theorem follows from the following ingredients:

- (1)  $\forall$  rational open  $j: U = \text{Spec}(B, B^+) \hookrightarrow \text{Spa}(A, A^+) = X$  we have an adjunction

$$j^* = (-) \otimes_{(A, A^+) \blacksquare} (B, B^+) \blacksquare: D((A, A^+) \blacksquare) \leftrightarrow D((B, B^+) \blacksquare): j_*$$

where  $j_*$  is fully faithful.

- (2) If  $X = \cup_{1 \leq k \leq n} U_k$  rational open cover,  $M \in D((A, A^+) \blacksquare)$  such that  $M|_{U_k} = 0$  for each  $k$ , then  $M = 0$ .

**Notation 56.5.**  $(A, A^+)$  discrete Huber pair.  $X = \text{Spa}(A, A^+)$ .

**Proposition 56.6** (Proposition 10.1) *Let  $U \subseteq X$  be a rational open.*

- (1) *The functor  $j_*: D((\mathcal{O}_X(U), \mathcal{O}_X^+(U))_{\blacksquare}) \rightarrow D((A, A^+)_{\blacksquare})$  is fully faithful.*
- (2)  *$\exists j^* = (-) \otimes_{(A, A^+)_{\blacksquare}} (\mathcal{O}_X(U), \mathcal{O}_X^+(U))_{\blacksquare} \dashv j_*$  adjunction.*

*Proof.* Let  $U$  as above.

- (1) Consider the diagram

$$\begin{array}{ccc}
 D((\mathcal{O}_X(U), \mathcal{O}_X^+(U))_{\blacksquare}) & \xrightarrow{7.5} & D(\text{Cond}(\mathcal{O}_X(U))) \\
 \downarrow j_* & & \parallel \\
 & & D(\text{Cond}(A[\frac{1}{f}])) \\
 & & \downarrow 5.9 \\
 D((A, A^+)_{\blacksquare}) & \xrightarrow{7.5} & D(\text{Cond}(A))
 \end{array}$$

- (2) Follows from 7.7.

□

**Definition 56.7** A *left-adjointable square* is an essentially commutative square of  $\infty$ -categories

$$\begin{array}{ccc}
 C & \xrightarrow{f_*} & D \\
 \downarrow p_* & \swarrow \alpha \text{ htpy} & \downarrow q_* \\
 C' & \xrightarrow{g_*} & D'
 \end{array}$$

such that  $\exists$  adjunctions  $f^* \dashv f_*$ ,  $g^* \dashv g_*$  and the composite

$$g^* g_* \xrightarrow{\eta} g^* q_* f_* f^* \xrightarrow{\alpha, \sim} g^* g_* p^* f^* \xrightarrow{\varepsilon} p_* f^*$$

is a natural isomorphism.

**Proposition 56.8** (Proposition 10.2) *Consider the cartesian square*

$$\begin{array}{ccc}
 V & \hookrightarrow & Y = \text{Spa}(B, B^+) \\
 \downarrow & & \downarrow \\
 U & \xrightarrow{\text{ratl open}} & X
 \end{array}$$

*Then we have a left-adjointable and cartesian square*

$$\begin{array}{ccc}
 D((\mathcal{O}_X(V), \mathcal{O}_X^+(V))_{\blacksquare}) & \hookrightarrow & D((B, B^+)_{\blacksquare}) \\
 \downarrow & & \downarrow \\
 D((\mathcal{O}_X(U), \mathcal{O}_X^+(U))_{\blacksquare}) & \hookrightarrow & D((A, A^+)_{\blacksquare})
 \end{array}$$

*Proof.* Let  $U \subseteq X$  be a rational open as above.  $U = \{x \in X \mid |f(x)| \neq 0\} \cap \bigcap_k \{x \in X \mid |\frac{g_k}{f}(x)| \leq 1\}$ . This is in turn equal to  $U(\frac{f}{f}) \cap \bigcap_k U(\frac{g_k/f}{1})$ . By induction on  $n$ , we may assume that  $U$  is either  $U(\frac{f}{f})$  or  $U(\frac{g}{f})$ .

In the first case  $U = U(\frac{f}{f})$ , note that  $\mathcal{O}_X^+(U)$  is the integral closure of  $A^+$  in  $A[\frac{1}{f}]$ . Since integral closures and localizations commute, this is just  $A^+[\frac{1}{f}]$ . Hence  $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))_{\blacksquare} = (A[\frac{1}{f}], A^+[\frac{1}{f}])_{\blacksquare}$  is just  $(A, A^+)_{\blacksquare} \otimes_A A[\frac{1}{f}]$ .

$D((\mathcal{O}_X(U), \mathcal{O}_X^+(U))_{\blacksquare}) \subseteq D((A, A^+)_{\blacksquare})$  is the full subcategory spanned by  $M$  such that  $M \xrightarrow{f} M$  quasi-iso and  $M \otimes_{(A, A^+)_{\blacksquare}}^L (\mathcal{O}_X(U), \mathcal{O}_X^+(U))_{\blacksquare} \cong M[\frac{1}{f}]$ . This is compatible with base change along  $(A, A^+) \rightarrow (B, B^+)$ .

For  $U = U(\frac{g}{f}) = \{x \in X \mid |g(x)| \leq 1\}$ , see Scholze's notes.  $\square$

To complete the proof of 9.8, we need to be able to refine rational open covers to a standard form:

**Lemma 56.9 (Lemma 10.4)** *For any rational open cover  $X = \cup_k U_k$  we can find  $s_1, \dots, s_N \in A$  such that*

- (1)  $(s_1, \dots, s_N) = A$  is the unit ideal.
- (2)  $\forall l \exists k$  s.t.  $U(\frac{s_1, \dots, s_N}{s_l}) \subseteq U_k$ .

*Proof.* Claim:  $\forall x \in X$  closed and  $\forall x \in U \subseteq X$  open nbhd,  $\exists f, g_1, \dots, g_n \in A$  generating the unit ideal such that  $U(\frac{g_1, \dots, g_n}{f}) \subseteq U$ . [The fact that  $X$  is a spectral space would imply this for all other points.]

Proof of the claim: rational opens are a base. WLOG,  $U = U(\frac{g_1, \dots, g_n}{f})$  for some  $f, g_1, \dots, g_n \in A$ . Let  $\Gamma = \Gamma_x =$  value gp at  $x$  and let  $\Gamma'$  be the subgroup of  $\Gamma$  generated by  $|f(x)|$  such that  $f \in A$  and  $|f(x)| > 1$ . This gives us a valuation  $x': A \rightarrow \Gamma' \cup \{0\}$  sending  $f \mapsto |f(x)|$  if  $|f(x)| \in \Gamma'$  or 0 otherwise.  $x'$  is a specialization of  $x$ , i.e.  $x' \in \overline{\{x\}}$ . But  $x$  was a closed point, so  $x' = x$  and  $\Gamma' = \Gamma$ . If  $x \in U(\frac{g_1, \dots, g_n}{f})$ , then  $|f(x)| \neq 0$ . This implies that  $\exists h \in A$  such that  $|h(x)| > 1$  and  $|fh(x)| \geq 1$ . We find that  $x \in U(\frac{g_1 h, \dots, g_n h}{fh}) \subseteq U(\frac{g_1, \dots, g_n}{f})$ , and  $(g_1 h, \dots, g_n h, 1) = A$  vacuously. This proves the claim.

Now for all  $k$  let  $U_k = U(\frac{g_{1k}, \dots, g_{mk}, f_k}{f_k})$  with  $(g_{1k}, \dots, g_{mk}, f_k) = A$ . [Apply previous claim and refine  $\cup_k U_k$ .]

Let  $T = \{h_1 \cdots h_n \mid \forall k, h_k \in \{g_{1k}, \dots, g_{mk}, f_k\}\}$  and  $S \subseteq T$  be the subset of all  $h_1 \cdots h_n \in T$  such that  $h_k = f_k$  for some  $k$ , and write  $S = \{s_1, \dots, s_N\}$ . Claim: for all  $k$  there exists  $l$  such that  $U(\frac{s_1, \dots, s_N}{s_k}) \stackrel{(*)}{=} U(\frac{T}{s_k}) \stackrel{(**)}{\subseteq} U(\frac{g_{1l}, \dots, g_{nl}}{f_l})$ .

For  $(*)$ ,  $S \subseteq T \Rightarrow U(\frac{T}{s_k}) \subseteq U(\frac{S}{s_k})$ . Conversely, for all  $x \in X$  there is some  $k$  such that  $x \in U_k$ , hence  $|f_k(x)| \neq 0$ . For all  $t = h_1 \cdots h_n \in T$ ,  $s := \frac{t f_k}{h_k}$ , which implies that  $|t(x)| \leq |s(x)|$ .

For  $(**)$ ,  $s_k = h_1 \cdots h_l \cdots h_n$  for some  $h_\alpha, l$ . Hence for all  $x \in X$  we have

$$|s_k(x)| \neq 0 \Rightarrow |f_l(x)| \neq 0.$$

$\square$

**Notation 56.10.** For all  $U = \text{Spa}(B, B^+) \hookrightarrow X$  rational open, we have an adjunction

$$L_U: C := D((A, A^+)_{\blacksquare}) \leftrightarrow D((B, B^+)_{\blacksquare}) =: C_U: i_U$$

and  $i_U$  is fully faithful.

**Lemma 56.11 (Lemma 10.3)** *For all  $U, V \subseteq X$  rational opens, we have*

- (1)  $C_{U \cap V} = C_U \cap C_V \subseteq C$ .
- (2)  $L_{U \cap V} \cong L_U \circ L_V \cong L_V \circ L_U$ .
- (3) If  $X = \cup_k U_k$  rational open cover, and  $M \in C$  such that  $\forall k$  we have  $M|_{U_k} = 0$ , then  $M = 0$ .

*Proof.* (1) Let  $W = U \cap V$  for  $U$  a rational open. We have an essentially commutative square

$$\begin{array}{ccc}
C_W & \longrightarrow & C_U \\
\downarrow & & \downarrow \\
C_V & \longrightarrow & C
\end{array}$$

which is cartesian by 10.2. So we get a cartesian square

$$\begin{array}{ccc}
C_U \cap C_V & \hookrightarrow & C_U \\
\downarrow & & \downarrow \\
C_V & \hookrightarrow & C
\end{array}$$

- (2) (2) follows from 10.2.  
(3) For (3) see Scholze's notes.

□

### 57. [WS] THE GEOMETRY OF NONSINGULAR CUBIC THREEFOLDS OVER $\mathbb{C}$ (BRAD) - 29.01.20

Let  $d, n \in \mathbb{N}_{>0}$  and  $X_{d,n}$  be a nonsingular hypersurface of degree  $d$  in  $\mathbb{P}^{n+1}$ . For brevity,  $X = X_{d,n}$ ,  $P = \mathbb{P}^{n+1}$ .

**Motivation:** Recall from last week the Luertoh problem: unirational  $\Rightarrow$  rational? If  $\dim(X) \leq 2$ , the answer is affirmative (in characteristic 0). Necessary conditions for rationality:

$$h^{p,0} = \begin{cases} 1 & \text{if } p = 0 \\ 0 & \text{else} \end{cases}$$

**Example 57.1**  $X$  not unirational for  $d > n - 2$  with  $n > 3$ :  $\omega_X \cong \mathcal{O}_X(d - n - 2)$ , so  $d - n - 2 > 0$  implies that  $\Gamma(\omega_X) \neq 0$  and therefore  $h^{n,0} \neq 0$ .

#### Goals:

- (1) Computing  $H^k(X_{d,n})$ .
- (2) Looking for evidence for/against (uni)rationality.
- (3)  $X_{3,3}$  is unirational.

#### 57.1. Cohomology on degree $k \neq n$ .

*Remark 57.2.* The pair  $(P, X)$  is homeomorphic to a pair of the form  $(Q, Q \cap H)$  with  $H, Q \subseteq \mathbb{P}^N$ ,  $H$  a hyperplane. Indeed, we have a Veronese embedding of degree  $d$

$$\begin{aligned}
\mathbb{P}(V) &\hookrightarrow \mathbb{P}(\text{Sym}^d(V)) \\
[v] &\mapsto [v^d]
\end{aligned}$$

**Theorem 57.3 (Lefschetz)** Let  $Y \subseteq \mathbb{P}^N$  be a nonsingular  $n$ -dimensional projective variety and  $i: Z := Y \cap H \hookrightarrow Y$  hyperplane section with  $Z$  nonsingular, then:

- (1)  $i^*: H^k(Y) \rightarrow H^k(Z)$  is an isomorphism for  $k < n$ ,
- (2)  $i_*: H_k(Z) \rightarrow H_k(Y)$  is an isomorphism for  $k < n$ .

**Corollary 57.4** (1) We have

$$b_k = \begin{cases} 1 & \text{for } 0 \leq k \leq 2n, k \text{ even}, k \neq n, \\ ? & \text{for } k = n, \\ 0 & \text{else.} \end{cases}.$$

- (2)  $h^{p,q}(X) = h^{p,q}(\mathbb{P}^{n+1})$  for  $p + q < n$ .

*Proof.* (1) Combine the remark with the theorem and Poincare duality, with the fact that  $H^*(P) \cong \mathbb{Z}[t]/t^{n+2}$ .

(2)  $i^*$  is compatible with Hodge decomposition

$$H^k(X) \cong \oplus_{p+q=k} H^{p,q}(X)$$

□

**Corollary 57.5**  $X_{3,2}$  cubic surface is rational.

*Proof.*  $h^{1,0} = \frac{1}{2}b_1 = 0$ .  $\omega_X = \mathcal{O}_X(-1)$  has no global sections different from zero. By Castelnuovo's criterion,  $X_{3,2}$  is rational. □

**57.2. Chern classes.**  $1^{st}$  Chern classes of line bundles:

$$0 \rightarrow \mathbb{Z} \xrightarrow{2\pi i} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^\times \rightarrow 0$$

$$\rightsquigarrow c_1: \text{Pic}(X) \cong H^1(X, \mathcal{O}_X^\times) \rightarrow H^2(X, \mathbb{Z})$$

Higher Chern classes: generalize to invariants of (complex) vector bundles

$$c_k: \mathbf{Vec}(X)/\simeq \rightarrow H^{2k}(X, \mathbb{Z})$$

characterized by the following axioms:

- (1) For all  $E \in \mathbf{Vec}(X)$ ,  $c_0(E) = 1 \in H^0(X, \mathbb{Z}) \cong \mathbb{Z}$ .
- (2) For all  $f: X \rightarrow Y$  and for all  $E \in \mathbf{Vec}(Y)$  and for all  $k$  we have

$$c_k(f^*E) = f^*c_k(E)$$

- (3) (Whitney sum) for all  $0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0$  exact we have

$$c(E) = c(E') \cup c(E''),$$

where  $c(F) = \sum_k c_k(F)$  is the *total Chern class*.

- (4)  $c(\mathcal{O}_{\mathbb{P}^n}(1)) = 1 + t$  where  $t \in H^2(\mathbb{P}^n)$  is the Poincare dual of  $[\mathbb{P}^{n-1}] \in H_{2n-2}(\mathbb{P}^n)$ .

**Theorem 57.6 (Gauss-Bonnet-Chern)** Let  $X$  be a connected complex compact  $n$ -manifold.  $\chi(X) = c_n(\mathcal{T}_X) \cap [X] \in H_0(X, \mathbb{Z}) \cong \mathbb{Z}$ , where  $\chi(X) = \sum_k (-1)^k b_k(X)$ .

**Upshot:** We can recover  $b_n(X_{d,n})$  from  $c_n(\mathcal{T}_X)$ .

**57.3.  $H^n(X_{d,n})$ .** Using the Euler sequence

$$0 \rightarrow \mathcal{O}_P \rightarrow \mathcal{O}_P(1)^{\oplus n+2} \rightarrow \mathcal{T}_P \rightarrow 0$$

we get that  $c(\mathcal{O}_P(1)^{\oplus n+2}) = c(\mathcal{O}_P)c(\mathcal{T}_P)$ , hence  $c(\mathcal{O}_P(1))^{n+2} = c(\mathcal{T}_P)$ , hence

$$(1+t)^{n+2} = c(\mathcal{T}_P)$$

$X_{d,n} = V(f)$ ,  $f \in \Gamma(P, \mathcal{O}_P(d)) \rightsquigarrow f: \mathcal{O}_P(-d) \xrightarrow{\sim} \mathcal{I}_X$ . Normal bundle  $i^{-1}(\mathcal{I}_X/\mathcal{I}_X^2)^\vee = i^{-1}((\mathcal{O}_P/\mathcal{I}_X) \otimes \mathcal{I}_X)^\vee = i^*\mathcal{O}_P(-d)^\vee$ .  $\mathcal{N}_{X/P} = \mathcal{O}_X(d)$ .

$$0 \rightarrow \mathcal{T}_X \rightarrow i^*\mathcal{T}_P \rightarrow \mathcal{N}_{X/P} \rightarrow 0$$

Hence  $i^*c(\mathcal{T}_P) = c(\mathcal{T}_X)c(\mathcal{N}_{X/P}) = c(\mathcal{T}_X)c(\mathcal{O}_X(d)) = c(\mathcal{T}_X)(1 + c_1(\mathcal{O}_X(1)^{\otimes d})) = c(\mathcal{T}_X)(1 + dt_X)$ , where  $t_X = i^*t$ . Hence

$$c(\mathcal{T}_X) = \frac{(1+t_X)^{n+2}}{(1+dt_X)} \stackrel{\text{Taylor}}{=} (1+t_X)^{n+2} \sum_k (-dt_X)^k$$

Binomial formula tells us that

$$c_n(\mathcal{T}_X) = \sum_{k=0}^n (-1)^k \binom{n+2}{n-k} d^k t_X^n$$



$t_X^n \cap [X]$  is the number of points in  $H_1 \cap \cdots \cap H_n \cap X$  for hyperplanes  $H_1, \dots, H_n$  in general position, which by Bezout is  $d$ . Hence

$$\chi(X) + \sum_{k=0}^n (-1)^k \binom{n+2}{n-k} d^{k+1}$$

**Corollary 57.7** *The Hodge diamond of  $X_{3,3}$  is*

$$\begin{array}{cccc} & & & \\ & & 0 & \\ & 0 & & 0 \\ & & 1 & \\ 0 & & & 0 \\ & 5 & & 5 \\ & & 0 & \end{array}$$

#### 57.4. Unirationality of $X_{3,3}$ .

*Notation 57.8.* Henceforth,  $X = X_{3,3}$ ,  $P = \mathbb{P}^4$ .

**Definition 57.9** A line in  $\mathbb{P}^4$  is a linear subspace of dimension 1, i.e., an intersection of 3 hyperplanes in  $\mathbb{P}^4$ .

*Fact 57.10.* There exists a line  $L_0 \subseteq X$ . In fact, the set  $F(X)$  of lines in  $X$  is an algebraic variety, called the *Fano variety* of lines. It is a nonsingular surface.

The fact follows from the Cayley-Salman theorem that  $X_{3,2}$  contains 27 lines.

**Corollary 57.11**  $X_{3,3}$  is unirational.

*Proof.* The fact implies that  $\exists L_0 \subseteq X$ . We'll construct a generically 2-to-1 dominant rational map

$$\varphi: \mathbb{P}(\mathcal{T}_X|_{L_0}) \dashrightarrow X$$

This is sufficient.  $\mathcal{T}_X|_{L_0}$  is trivial over a dense open  $U \subseteq L_0 \cong \mathbb{P}^1$ , hence  $\mathbb{P}(\mathcal{T}_X|_{L_0})$  is birational to  $\mathbb{P}^{n-1} \times \mathbb{P}^1$  which is rational.

Step 1) Construct  $\varphi$ . For all  $x \in L_0$  and all  $0 \neq v \in T_x X$  there exists a unique line  $L_{x,v}$  through  $x$  such that  $T_x L_{x,v} \subseteq T_x X$  is spanned by  $v$ . Each point of  $\mathbb{P}(\mathcal{T}_X|_{L_0})$  is represented by  $L_{x,v}$  for some  $x, v$ . For generic  $x, v$  we have  $L_{x,v} \not\subseteq X$ .  $L_{x,v} \not\subseteq X$  implies by Bezout that there exists a unique  $y \in X$  such that  $X \cap L_{x,v} = 2x + y$ . Now  $\varphi: \mathbb{P}(\mathcal{T}_X|_{L_0}) \dashrightarrow X$  maps  $[v] \mapsto y$ , where  $y$  is defined as above and  $\varphi$  is defined away from the locus where  $L_{x,v} \subseteq X$ .

Step 2) For all  $x \in \text{dom}(\varphi)$ , the cardinality of  $\varphi^{-1}(\varphi(x))$  is at most 2. Let  $y = \varphi(x)$ , let  $\overline{yL_0}$  be the plane spanned by  $y$  and  $L_0$ .  $C_y := X \cap \overline{yL_0} \subseteq \mathbb{P}^2$  is a curve of degree 3 (unless  $\overline{yL_0} \subseteq X$ , which generically does not happen, so we do not need to worry about it).  $L_0 \subseteq C_y$  implies that  $C_y$  is singular, because nonsingular cubic curves do not contain lines. Hence our intersection  $X \cap \overline{yL_0} = L_0 \cup Q$  for some quadric  $Q$ .

[Picture]

The number of points in  $\varphi^{-1}(y)$  is the number of points  $x' \in L_0$  such that  $L_{x',y}$  tangent to  $X$  at  $x'$ , which is at most 2.

Step 3)  $\dim(\mathbb{P}(\mathcal{T}_X|_{L_0})) = \dim(X) = 3$ , and  $\varphi$  is generically 2 to 1. By dimension theory,  $\varphi$  is dominant.

□

### 57.5. Intermediate Jacobians.

*Remark 57.12.* We need additional invariants to show that  $X_{3,3}$  is irrational.

**Theorem 57.13 (Hodge)** *We have a decomposition  $H^k(X, \mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}(X)$  and  $H^{p,q}(X) = \overline{H^{q,p}(X)}$  in  $H^k(X, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \cong H^k(X, \mathbb{C})$ .*

**Hodge filtration:**  $F^p H^k(X) := \bigoplus_{q \geq p} H^{q,k-q}(X)$ .

$$0 = F^{k+1} H^k \subseteq F^k H^k \subseteq \dots \subseteq F^0 H^k = H^k$$

$$H^k = F^p H^k \oplus \overline{F^{k-p+1} H^k}$$

**Proposition 57.14**  $H^{2k-1}(X, \mathbb{Z})/\text{tors}$  is a lattice in  $H^{2k-1}(X, \mathbb{C})/F^k H^{2k-1}$ .

*Proof.* Consider the diagram

$$\begin{array}{ccccc} H^{2k-1}(X, \mathbb{Z}) & \hookrightarrow & H^{2k-1}(X, \mathbb{R}) & \hookrightarrow & H^{2k-1}(X, \mathbb{C}) \\ & & & \searrow \text{\tiny $\mathbb{R}$linear iso (*)} & \downarrow \\ & & & & H^{2k-1}(X, \mathbb{C})/F^k H^{2k-1} \end{array}$$

(\*):  $\alpha \otimes 1 \in H^{2k-1}(X, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$  is fixed by conjugation and  $H^{2k-1} = F^k \oplus \overline{F^k}$ . Hence  $\alpha \otimes 1 = \beta + \bar{\beta}$  for some  $\beta \in F^k$ . □

**Definition 57.15** The  $k^{\text{th}}$  intermediate Jacobian

$$J^k(X) := \frac{H^{2k-1}(X, \mathbb{C})}{F^k + H^{2k-1}(X, \mathbb{Z})}$$

is a quotient of a  $\mathbb{C}$ -vector space by a lattice, hence a complex torus.

**Example 57.16 (Jacobians of curves)** Let  $X$  be a nonsingular projective curve of genus  $g = h^{1,0}(X)$ . Then

$$\begin{aligned} J(X) = J^1(X) &= \frac{H^1(X)}{F^1 + H^1(X, \mathbb{Z})} = \frac{H^1(X)}{H^{1,0}(X) + H^1(X, \mathbb{Z})} \cong \\ &\cong \frac{H^{0,1}(X)}{H^1(X, \mathbb{Z})} \cong \mathbb{C}^g / \Lambda. \end{aligned}$$

Goal: describe the lattice  $\Lambda$  explicitly.

$$\begin{array}{c} \alpha \\ \curvearrowright \\ H_1(X, \mathbb{Z}) \longrightarrow H_1(X, \mathbb{C}) \cong H^1(X, \mathbb{C})^\vee \longrightarrow H^{1,0}(X)^\vee \\ \gamma \longmapsto (\omega \mapsto \int_\gamma \omega) \end{array}$$

Choose a basis  $\gamma_1, \dots, \gamma_{2g}$  of  $H_1(X, \mathbb{Z})$ ,  $\omega_1, \dots, \omega_g$  of  $H^{1,0}(X)$ ,  $l_1, \dots, l_g$  dual basis for  $H^{1,0}(X)^\vee$ .

$$\alpha(\gamma_k) = \sum_j^g \left( \int_{\gamma_k} \omega_j \right) l_j$$

and

$$p := \left( \int_{\gamma_k} \omega_j \right)_{jk}$$

is the *period matrix* with respect to these bases. Then

$$\Lambda = \alpha(H_1(X, \mathbb{Z})) = P\mathbb{Z}^{2g}$$

and

$$J(X) \cong \mathbb{C}^g / P\mathbb{Z}^{2g}.$$

*Remark 57.17.* (1)  $J(X)$  underlies a principally polarized abelian variety.

(2) Torelli:  $J(X)$  determines  $X$  up to isomorphism.

(3) Next time: if  $X$  is nonsingular 3-fold, then  $X$  is rational implies that  $J^2(X)$  product of Jacobians of curves.

58. [FS] AMADOR - AUTOMORPHISM GROUP OF FIELDS WITH OPERATORS - 31.01.20

*Question 58.1.* What can we say about the lattice of normal subgroups of  $\text{Aut}(\mathbb{C})$ ?

E.g. we have  $\text{Aut}(\mathbb{C}/\bar{\mathbb{Q}}) \triangleleft \text{Aut}(\mathbb{C})$ . The quotient is  $\text{Gal}(\mathbb{Q})$ , i.e. very hard to understand. We will focus on the LHS, i.e. on  $\text{Aut}(\mathbb{C}/\bar{\mathbb{Q}})$ .

**Theorem 58.2** (Lascar '93) *If the continuum hypothesis holds, then for every  $f \in \text{Aut}(\mathbb{C}/\bar{\mathbb{Q}}) \setminus \{\text{id}\}$ , every element  $g \in \text{Aut}(\mathbb{C}/\bar{\mathbb{Q}})$  is a product of 4 conjugates of  $f$  and  $f^{-1}$ . In particular,  $\text{Aut}(\mathbb{C}/\bar{\mathbb{Q}})$  is simple.*

What is so special about CH? Under CH, we can write

$$\mathbb{C} = \bigcup_{\alpha < \aleph_1} k_\alpha$$

where  $k_\alpha$  is countable and algebraically closed (e.g.  $\bar{\mathbb{Q}}$ ) and  $\text{tr. deg}(k_{\alpha+1}/k_\alpha)$  is infinite. Since  $\aleph_1$  is the continuum by hypothesis, the result of the union must be  $\mathbb{C}$ .

What is so special about countable?  $k$  countable, then the topological subgroup  $\text{Aut}(k) \subseteq \text{Bij}(k) \cong \text{Bij}(\mathbb{N})$  is closed, where  $\text{Bij}(\mathbb{N})$  is a Baire space called the *Polish group*. It is a separable topological space which admits a complete metric. Explicitly, if  $f, g \in \text{Bij}(\mathbb{N})$ , then

$$d(f, g) = \sum_{n \in \mathbb{N}} \frac{a_n}{2^n},$$

where  $a_n = 1$  if  $f(n) \neq g(n)$  and 0 otherwise.

In '97, Lascar provided a different proof with no topology and not under CH. There are two steps in this proof:

Step 1) Which automorphisms are bounded?  $f \in \text{Aut}(\mathbb{C})$  is bounded if there is an algebraically closed subfield  $k$  of small (finite, countable) transcendence degree such that for all  $a \in \mathbb{C}$ ,  $f(a)$  is algebraic over  $k(a)$ . Are there non-trivial bounded automorphisms? E.g. in characteristic  $p > 0$ , Frobenius is bounded. But in characteristic 0, the only bounded automorphism is the identity.

Step 2)  $f \in \text{Aut}(\mathbb{C}/\bar{\mathbb{Q}}) \setminus \{\text{id}\}$ . Given  $g \in \text{Aut}(\mathbb{C}/\bar{\mathbb{Q}})$ , we need to show that there are  $g_1, g_2, g_3, g_4 \in \text{Aut}(\mathbb{C}/\bar{\mathbb{Q}})$  such that

$$g = f^{g_1} \cdot (f^{-1})^{g_2} \cdot (f^{-1})^{g_3} \cdot f^{g_4}.$$