NOTES ON ENUMERATIVE GEOMETRY

Abstract. Notes based on the Wednesday Seminar held at Freiburg during the Winter Semester 2020/2021 and typesetted by Pedro Núñez. Any errors or typos were likely introduced by myself; corrections and any other suggestions are very much appreciated!

Contents

The main reference for this seminar is the book [EH16].

1. Ivan Zaccanelli: The Chow ring (16.12.2020)

1.1. **Introduction.** Some instances of enumerative problems:

- Given $f \in K[x]$ of degree $\deg(f) = d$, what is the cardinality of the set $\{x \in K \mid f(x) = 0\} =: V(f)$? Answer: it is d if the the axis $V(y)$ in the plane is not tangent to the graph of f, given by $V(y - f(x))$ and if $K = \bar{K}$. If there were some tangent intersection points we would have to take multiplicities into account.

$$\rightsquigarrow \text{We want to consider } K = \mathbb{C}.$$

- Given $f, g \in \mathbb{C}[x_1, x_2]$, with $\deg(f) = d$ and $\deg(g) = e$ such that $V(f)$ and $V(g)$ "intersect transversally", what is the cardinality of $V(f) \cap V(g)$? Answer: (Bézout) it is $d \cdot e$... if we look at this question in the projective plane $\mathbb{P}^2$.

$$\rightsquigarrow \text{We want to consider varieties in } \mathbb{P}^n.$$

Date: 20th April 2021.

- Given four lines in $\mathbb{P}^3$, how many lines intersect all of them? <u>Answer:</u> postponed to 27.01.2021.

Let us now see a "backwards proof" of Bézout's theorem:

"Backwards proof". The theorem implies that the cardinality of $V(f) \cap V(g)$ only depends on $\deg(f)$ and $\deg(g)$. So we can replace f by $\tilde{f}$ and g by $\tilde{g}$ such that $V(\tilde{f})$ is the union of d lines in the projective plane and $V(\tilde{g})$ is the union of e lines in projective plane. Moreover we can assume that these lines are in general position, thus the cardinality of the intersection is indeed $d \cdot e$. □

Some of the ideas that we want to make rigurous are:

- "Move" or identify some subvarieties.
- Show that the cardinality of the intersection does not depend on this identification, given that the intersection is transverse.

$\rightsquigarrow$ Fundamental tool: Chow ring $\mathrm{CH}^\bullet(X)$ of a quasi-projective variety X.

As a side note, let us ask the following question: is such a rigurous foundation really necessary? To answer it, let us look at some history of these kinds of problems:

- Schubert "guessed" correctly in 1879 that the number of twisted cubics tangent to 12 quadrics is 5819539783680.

  $\rightsquigarrow$ Maybe the answer is "no", at least not for him.

  Quoting Eisenbud and Harris: "He landed a jumbo jet being blindfolded."
- Hilbert includes this issue as his 15th problem in 1900.

  $\rightsquigarrow$ The answer is "yes" for him.
- Fulton wrote a bible book about this in 1984 [Ful98].

  $\rightsquigarrow$ Just cite him whenever necessary!

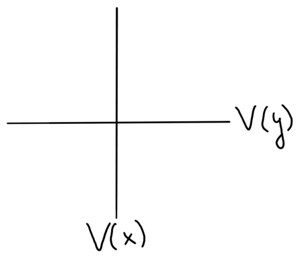
1.2. **Quick recap of algebraic varieties.** The Zariski topology on $\mathbb{A}^n$ (resp. on $\mathbb{P}^n$) has as closed subsets those of the form $V(I)$ for $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$ an ideal (resp. $V(I)$ for $I \subseteq \mathbb{C}[x_0, \ldots, x_n]$ a homogeneous ideal. The sheaf of regular functions consists of functions that can be locally written as a quotient of two polynomials (resp. homogeneous polynomials). A closed subset $X \subseteq \mathbb{A}^n$ (resp. $X \subseteq \mathbb{P}^n$) is said to be an *affine variety* (resp. a *projective variety*).

A *quasi-projective variety* is an open set inside a projective variety.

Definition 1.1 (Irreducibility)**.** A non-empty topological space is said to be *irreducible* if it cannot be written as a union of two proper closed subsets.

Example 1.2. The subset $V(xy) \subseteq \mathbb{A}^2$ is connected but not irreducible, since it can be written as

$$V(xy) = V(x) \cup V(y).$$



Proposition 1.3. *Each $X \subseteq \mathbb{P}^n$ can be written uniquely as $X = \cup_{i=1}^r X_i$, where $X_i \subseteq X$ are closed and irreducible and no X_i is superfluous.*

The X_i's in the previous proposition are called the *irreducible components* of X.

Definition 1.4 (Dimension)**.** The *dimension* of $X \subseteq \mathbb{P}^n$ is defined as the maximal length of a chain

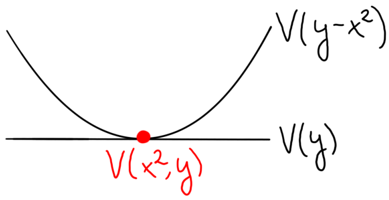
$$Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_n \subseteq X$$

with each Y_i closed and irreducible.

Definition 1.5 (Smoothness)**.** $X \subseteq \mathbb{P}^n$ is *smooth* if $\dim(X) = \dim(T_pX)$ for all $p \in X$.

Bad news: for intersection theory we must also take into account subschemes. In the setting above, $V(I) = V(I^2)$. But problems arise naturally taking intersections. In $\mathbb{A}^2$ we have

$$V(y) \cap V(y - x^2) = V((y) + (y - x^2)) = V(x^2, y) \overset{!}{\neq} V(x, y).$$



Moral: an (affine) subscheme is the datum of an (affine) variety together with the ideal defining it.

From now on, "variety" means "irreducible variety"; and "subvariety" means "irreducible closed subset in some variety".

1.3. **Chow group.** Let X be a quasi-projective variety.

Definition 1.6 (Cycle group)**.** The *group of cycles* $Z(X)$ is the free abelian group generated by the set of subvarieties in X.

So if we have $Y_1, \dots, Y_m \subseteq X$ closed and irreducible, then

$$\sum_{i=1}^{m} a_i Y_i \in Z(X).$$

Remark 1.7.

(1) $Z(X)$ is graded (as an abelian group) by the dimension or by the codimension:

$$Z(X) = \oplus_{k\in\mathbb{N}} Z_k(X) = \oplus_{l\in\mathbb{N}} Z^l(X),$$

where $Z_k(X)$ denotes the subgroup generated by subvarieties of dimension k and $Z^l(X) := Z_{\dim(X)-l}(X)$.

(2) To each subscheme $Y \subseteq X$ we can associate a cycle

$$\langle Y \rangle := \sum n_i Y_i,$$

where the Y_i are the irreducible components of the support of Y and the n_i keep track of the multiplicity of each component.

Example 1.8.

$$\langle V(x^2, y) \rangle = 2V(x, y).$$

Definition 1.9 (Chow group)**.** We define the *Chow group* of X as

$$\mathrm{CH}(X) := Z(X)/\operatorname{Rat}(X),$$

whee $\operatorname{Rat}(X)$ is the subgroup generated by cycles of the form

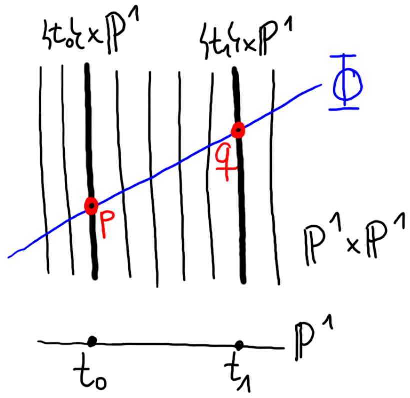
$$\langle \Phi \cap (\{t_0\} \times X) \rangle - \langle \Phi \cap (\{t_1\} \times X) \rangle,$$

where $\cap$ denotes the scheme-theoretic intersection, $t_0, t_1 \in \mathbb{P}^1$ and $\Phi \subseteq \mathbb{P}^1 \times X$ is a subvariety not contained in any fiber $\{t\} \times X$.

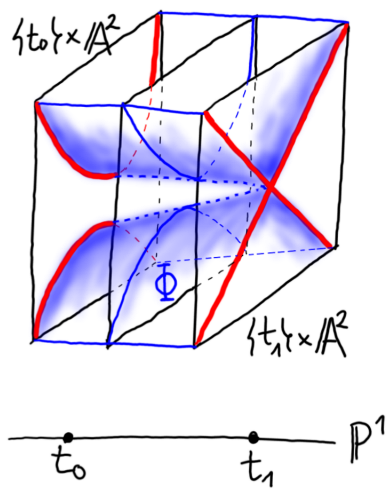
If $Z_1, Z_2 \in Z(X)$ are cycles such that $[Z_1] = [Z_2]$, then we say that they are *rationally equivalent* cycles.

Example 1.10.

(1) In $\mathbb{P}^1$, any two points are rationally equivalent:

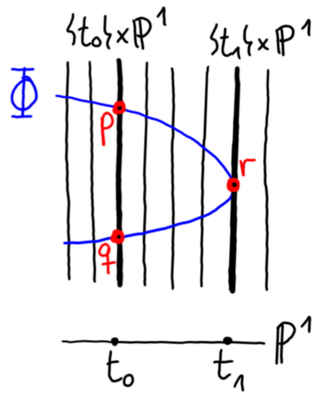


(2) In $\mathbb{A}^2$, a hyperbola is rationally equivalent to the union of two intersecting lines:



Remark 1.11.

(1) The hypothesis that $\Phi \not\subseteq \{t\} \times X$ for any $t \in \mathbb{P}^1$ is essential; otherwise, any cycle on X would be rationally equivalent to $0 = \langle \varnothing \rangle$.
(2)



$$\overset{\text{with naive intersection}}{\Rightarrow} 2[p] = [p] \Rightarrow [p] = 0.$$

We don't want this to happen! And indeed this does not happen if we consider scheme-theoretic intersections. With scheme-theoretic intersections we would instead have

$$[p] + [q] = 2[r] \Rightarrow 2[p] = 2[p],$$

which is okay.

Proposition 1.12. *The generators of* $\mathrm{Rat}(X)$ *are homogeneous with respect to the grading on* $Z(X)$.

$$\rightsquigarrow \textit{Grading on } \mathrm{CH}(X).$$

Proof. We need a

Lemma 1.13 (cf. [EH16, Theorem 0.2])**.** *If* $f\colon Z \to \mathbb{P}^1$ *is non-constant, with* Z *a variety, then for all* $p \in \mathbb{P}^1$ *the fibre* $f^{-1}(p) \subseteq Z$ *has all irreducible components of codimension* 1.

Using this, consider the map $\pi\colon \mathbb{P}^1 \times X \to \mathbb{P}^1$ given by $(t, x) \mapsto t$ and restrict it to Φ. By hypothesis, $\pi|_\Phi$ is non-constant, so each irreducible component of $\Phi \cap (\{t_0\} \times X) = \pi|_\Phi^{-1}(t_0)$ has the same dimension $\dim(\Phi) - 1$.

The same holds for $\Phi \cap (\{t_1\} \times X)$. □

1.4. Ring structure on $\mathrm{CH}^\bullet(X)$.

Definition 1.14 (Transverse intersection)**.** Two subvarieties $Z_1, Z_2 \subseteq X$ intersect *transversally* if for every $p \in Z_1 \cap Z_2$ we have that p is a smooth point and that

$$T_pZ_1 + T_pZ_2 = T_pX.$$

They intersect *generically transversally* if this happens on a dense open subset of $Z_1 \cap Z_2$.

Two cycles $A = \sum a_iA_i$ and $B = \sum b_jB_j$ intersect generically transversally if A_i and B_j intersect generically transversally for all i, j.

Theorem-Definition 1.15. *If X is a smooth quasi-projective variety, then there is a unique ring structure on* $\mathrm{CH}(X)$ *satisfying:*

($\star$) *If two subvarieties are generically transverse, then*

$$[A] \cdot [B] = [A \cap B].$$

Moreover, this makes $\mathrm{CH}^\bullet(X)$ *into a graded ring.*

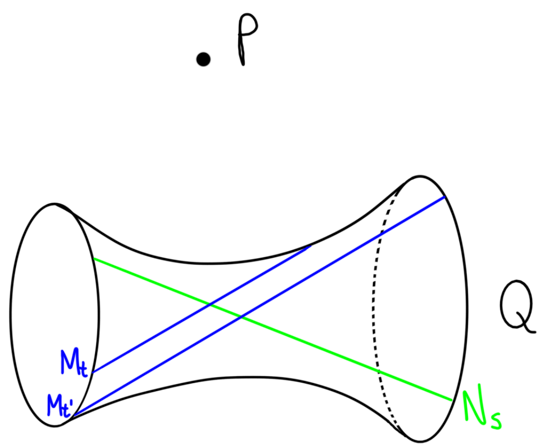
The proof of theorem follows at once from the "moving lemma":

Lemma 1.16 ([Ful98]). *Let X be a smooth quasi-projective variety.*

i) For any $\alpha, \beta \in \mathrm{CH}(X)$, there are generically transverse cycles $A, B \in Z(X)$ such that $\alpha = [A]$ and $\beta = [B]$.
ii) The class $[A \cap B]$ is independent of the choice of A and B.

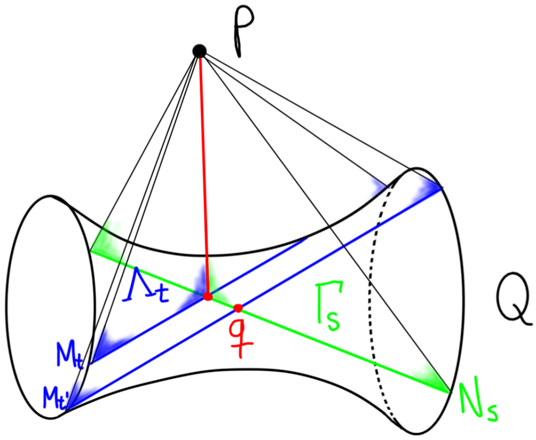
What goes wrong if X is not smooth? Both the moving lemma and the previous theorem-definition fail.

Example 1.17. Let $Q \subseteq \mathbb{P}^3 \subseteq \mathbb{P}^4$ be a smooth quadric, which is a surface ruled by two families of lines:



M_t and $M_{t'}$ are in the same family, so they are disjoint.

Let $p \in \mathbb{P}^4 \setminus \mathbb{P}^3$. Consider $X = C(p, Q) \subseteq \mathbb{P}^4$, the cone with vertex p and base Q. Then X is ruled by two families of planes.



In the second picture, $\Lambda_t = \mathrm{span}(p, M_t)$, $\Lambda_{t'} = \mathrm{span}(p, M_{t'})$ and $\Gamma_s = \mathrm{span}(p, N_s)$. We have $M_t \cap \Lambda_{t'} = \varnothing$, so they intersect transversally. And $N_s \cap \Lambda_{t'} = \{q\}$, so they intersect transversally as well. We have then $[M_t] \cdot [\Lambda_{t'}] = 0$ and $[N_s] \cdot [\Lambda_{t'}] = [q]$. But also $[M_t] = [N_s]$, because two lines in a plane are rationally equivalent, so both of them are rationally equivalent to the line $\Lambda_t \cap \Gamma_s$. Therefore we deduce that $[q] = 0$, which cannot happen on a projective variety because of the existence of the degree map [EH16, Proposition 1.21].

1.5. **General strategy for enumerative problems.** Prototype of enumerative problem: understand the set Ψ of objects of a certain type satisfying certain conditions:

Example 1.18. Lines in $\mathbb{P}^3$ intersecting 4 given lines L_1, L_2, L_3 and L_4.

•Step 1: Construct a suitable (smooth, projective) parameter space $\mathcal{H}$ for the objects we are interested in, e.g. in the case of the previous example we would look at the Grassmannian $\mathcal{H} = \mathbb{G}(1, 3)$ of lines in $\mathbb{P}^3$.

•Step 2: Show that for each condition imposed by our problem, the locus $Z_i \subseteq \mathcal{H}$ of objects satisfying that condition is closed, e.g. in our case we would have

$$Z_i = \{L \in \mathbb{G}(1, 3) \mid L \cap L_i \neq 0\},$$

so that $\Psi = \cap_{i=1}^4 Z_i \subseteq \mathcal{H}$ is closed.

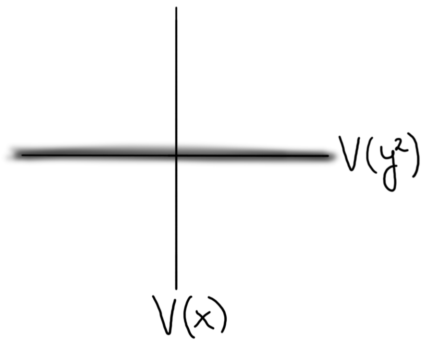
•Step 3: Describe $\mathrm{CH}^\bullet(\mathcal{H})$ and $[Z_i] \in \mathrm{CH}^\bullet(\mathcal{H})$, which in the case of our example will be done in future talks of this seminar.

•Step 4: Calculate the product $\prod_i [Z_i]$. Then $[\Psi] = [\cap_i Z_i] = \prod_i [Z_i]$, assuming that the intersections are generically transverse. So if things work out well and we are a bit careful, we can obtain the answer as the degree [EH16, Propostion 1.21] of some zero cycle $[\Psi]$ in our (smooth, projective) parameter space $\mathcal{H}$.

2. Jin Li: Basic computational tools (13.01.2021)

2.1. **Recollections.** An *algebraic set* is a closed subset in $\mathbb{A}^n$ (or $\mathbb{P}^n$) in the Zariski topology. A *scheme* is the datum of an algebraic set together with the ideal defining it.

Remark 2.1. A scheme can be non-reduced and non-irreducible, e.g. $X = V(xy^2) \subseteq \mathbb{A}^2$.



Reduced means that every irreducible component has multiplicity one.

An *algebraic variety* is a reduced an irreducible scheme in $\mathbb{A}^n$ (*affine variety*) or in $\mathbb{P}^n$ (*projective variety*).

Let X be an algebraic variety. The *group of cycles* on X, denoted $Z(X)$, is a free abelian group generated by subvarieties of X. $Z(X) = \oplus_{k\in\mathbb{N}} Z_k(X)$ is graded by dimension, and the elements in $Z_k(X)$ are called k-cycles for each $k \in \mathbb{N}$. To any closed subscheme $Y \subseteq X$ we associate a cycle

$$\langle Y \rangle := \sum_{i=1}^{s} m_i Y_i,$$

for example $\langle V(xy^2)\rangle = V(x) + 2V(y)$.

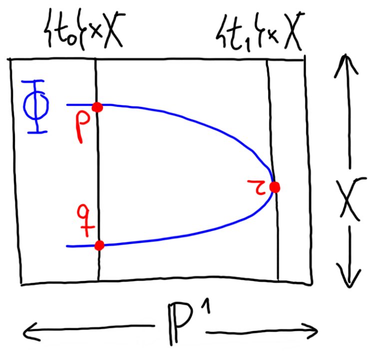
We define $\mathrm{Rat}(X) \subseteq Z(X)$ as the subgroup generated by expressions of the form

$$\langle \Phi \cap (\{t_0\} \times X)\rangle - \langle \Phi \cap (\{t_1\} \times X)\rangle, \tag{1}$$

where $t_0, t_1 \in \mathbb{P}^1$ and $\Phi \subseteq \mathbb{P}^1 \times X$ is a subvariety not contained in any fibre $\{t\} \times X$.

Two cycles $A_0, A_1 \in Z(X)$ are called *rationally equivalent* if $A_1 - A_0 \in \mathrm{Rat}(X)$.

In Equation (1) we take the scheme theoretic intersection, which means that we need to consider the multiplicities, e.g. the picture



implies that $p + q \sim 2\tau$ and not $p + q \sim \tau$.

Φ not being contained in any fibre is necessary, otherwise we would have $A \sim 0 = \langle \varnothing \rangle$ for all $A \in Z(X)$.

We define the *Chow group* of X, denoted $A(X)$ or $\mathrm{CH}(X)$, as

$$A(X) := Z(X)/\operatorname{Rat}(X).$$

For a cycle $Y \in Z(X)$, we denote by $[Y] \in A(Y)$ its equivalence class in $A(X)$. When $Y \subseteq X$ is a subscheme, we also denote simply by $[Y]$ the equivalence clas in $A(X)$ of the cycle $\langle Y \rangle$ associated to Y. For example,

$$[V(xy^2)] = [V(x)] + 2[V(y)].$$

The Chow group is graded by dimension $A(X) = \oplus_{k=0}^{\dim(X)} A_k(X)$. We have seen the following

Theorem-Definition 2.2. *If X is a smooth quasi-projective variety, then there is a unique ring structure on $A(X)$ satisfying:*

($\star$) *If two subvarieties are generically transverse, then*

$$[A] \cdot [B] = [A \cap B]. \tag{2}$$

This structure makes $A(X)$ a graded ring, graded by codimension

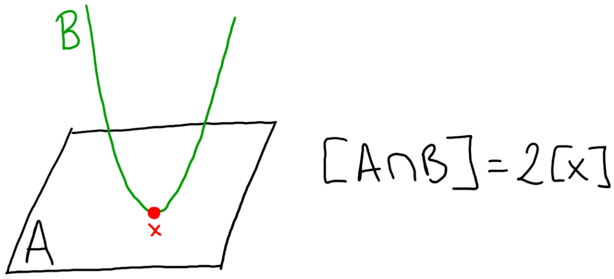
$$A(X) = \oplus_{c=0}^{\dim(X)} A^c(X), \quad c = \dim(X) - k.$$

We call it the *Chow ring* of X.

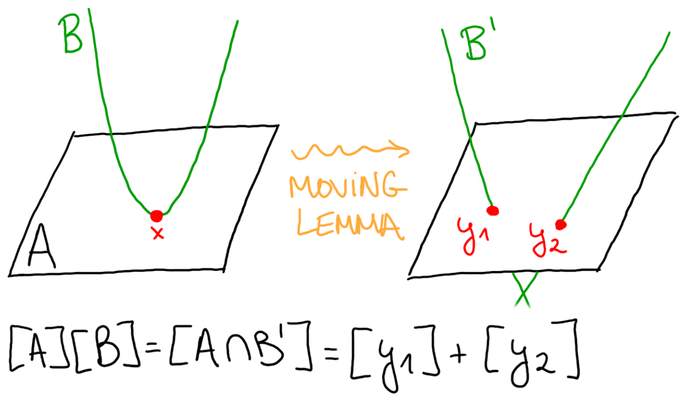
Remark 2.3. Equation (2) holds more generally as long as

$$\operatorname{codim}(A \cap B) = \operatorname{codim}(A) + \operatorname{codim}(B).$$

Example 2.4. In $\mathbb{P}^3$ we have:



But the moving lemma implies that we can move the curve within its rational equivalence class:



2.2. **The Chow ring of affine space.**

Proposition 2.5. *If X is a variety of dimension n, then its fundamental class $[X] \in A(X)$ is always non-zero.*

Proof. If $[X] = 0$, then $[X]$ could be written as a $\mathbb{Z}$-linear combination of expressions as in Equation (1). We claim that only when $\Phi = \mathbb{P}^1 \times X$ can X appear as the result of an expression of the form $\Phi \cap (\{t\} \times X)$, but in this case we would just have $X \sim X$.

Suppose then that $\Phi \subsetneq \mathbb{P}^1 \times X$. By definition of dimension and irreducibility of Φ and of $\mathbb{P}^1 \times X$, we must have

$$\dim(\Phi) \leq n = \dim(X).$$

On the other hand, since we are assuming that X appears as the result of an expression $\langle \Phi \cap (\{t\} \times X) \rangle$, we must have

$$\{t\} \times X \subseteq \Phi,$$

thus $\{t\} \times X = \Phi$, because both are irreducible, $\{t\} \times X$ has dimension n and Φ has dimension at most n. But this contradicts then the assumption that Φ is not contained in any fibre $\{t\} \times X$, hence the previous situation cannot occur. $\square$

So from this proposition we deduce that $A^0(X) = \mathbb{Z} \cdot [X]$.

Proposition 2.6. *For affine space* $\mathbb{A}^n$ *we have*

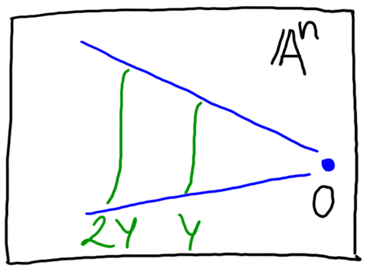
$$A(\mathbb{A}^n) = \mathbb{Z} \cdot [\mathbb{A}^n].$$

Proof. We have already seen that $A^0(\mathbb{A}^n) = \mathbb{Z} \cdot [\mathbb{A}^n]$, so it suffices to show that all other subvarieties $Y \subsetneq \mathbb{A}^n$ are rationally equivalent to $0 = \langle \varnothing \rangle$.

Given such Y, we want to construct a Φ interpolating between Y and $\varnothing$. Choose coordinates $z = (z_1, ldots, z_n)$ such that $0 \notin Y$ and let $Y = V(I)$ for some ideal I. We define

$$W^\circ := \{(t, tz) \in (\mathbb{A}^1 \setminus \{0\}) \times \mathbb{A}^n \mid z \in Y\} = V(\{f(z/t) \mid f \in I\}).$$

The fibre of W° over t is tY. Let $W := \overline{W^\circ} \subseteq \mathbb{P}^1 \times \mathbb{A}^n$ be the closure. The claim is that $\Phi = W$ is the variety interpolating between Y and $\varnothing$. The fibre of W over $t = 1$ is then just Y. So our goal is to show that there is some other fibre which is empty, and we will see that this is the fibre over $t = \infty \in \mathbb{P}^1$.



Since $0 \notin Y$, we can find some $g \in I$ which has non-zero constant term c. Define then

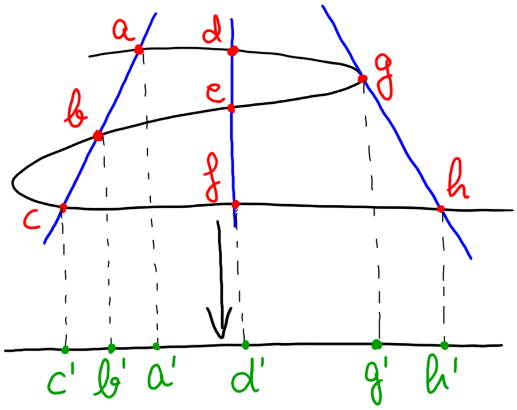
$$G(t, z) := g(z/t)$$

on $(\mathbb{A}^1 \setminus \{0\}) \times \mathbb{A}^n$. For $t = \infty$ we get $G(\infty, z) = g(0) = c$ for all $z \in \mathbb{A}^n$, so G extends to a regular function on $(\mathbb{P}^1 \setminus \{0\}) \times \mathbb{A}^n$. We have $W^\circ \subseteq V(G)$, so $W \subseteq V(G)$ as well. Since $V(G) \cap (\{\infty\} \times \mathbb{A}^n) = V(G(\infty, z)) = V(c) = \varnothing$, we also have $W \cap (\{\infty\} \times \mathbb{A}^n) = \varnothing$, therefore

$$Y \sim 0.$$

□

2.3. **Functoriality** [EH16, §1.2.6]. Let X and Y be schemes, $f \colon Y \to X$ be a proper map ("closed map with compact fibres"), e.g. any map between projective varieties. If $A \subseteq Y$ is a subvariety, then $f(A) \subseteq X$ is a subvariety. Therefore, one may be tempted to define $f_*([A]) := [f(A)]$. But this naive definition caues some problems. The key problem are the multiplicities:



With the naive definition we would have $a' + b' + c' \sim d' \sim g' + h'$, which is a contradiction because all points are rationally equivalent in $\mathbb{P}^1$ and as we will soon see they are non-zero. With the correct definition we would instead have

$$a' + b' + c' \sim 3d' \sim 2g' + h'.$$

Definition 2.7. If $\dim(f(A)) < \dim(A)$, then we set $f_*([A]) = 0$. And if $\dim(f(A)) = \dim(A)$, then we set $f_*([A]) = n[f(A)]$, where $n = [\mathbb{C}(A) : \mathbb{C}(f(A))]$ is the degree of $f|_A$. We extend f_* to all cycles by linearity, that is,

$$f_*(\sum m_i[A_i]) = \sum m_i f_*([A_i]).$$

Theorem 2.8 ([Ful98, §1.4])**.** *If $f\colon Y \to X$ is proper, then $f_*\colon Z(Y) \to Z(X)$ induces a group homomorphism $f_*\colon A_k(Y) \to A_k(X)$ for all $k \in \mathbb{N}$.*

As a particular case we obtain the existence of the degree map, which was already used in an example during the previous talk:

Proposition 2.9. *If X is proper, that is, if the structure map to a point $X \to \operatorname{Spec}\mathbb{C}$ is proper, then there exists a unique group homomorphism $\deg\colon A(X) \to \mathbb{Z}$, called the degree map, taking every closed point $p \in X$ to $1 \in \mathbb{Z}$ and vanishing on the classes of cycles of dimension greater than 0.*

If A is a k-dimensional subvariety of a smooth projective n-dimensional variety X and B is an $(n-k)$-dimensional subvariety of X such that $A \cap B$ is finite and non-empty, then the existence of the group homomorphism

$$\begin{aligned} A_k(X) &\longrightarrow \mathbb{Z} \\ [A] &\longmapsto \deg([A] \cdot [B]) > 0 \end{aligned}$$

guarantees that $[A] \neq 0$.

Let now $f\colon Y \to X$ be a flat morphism between schemes. We will not define flatness, but important examples are open inclusions and morphisms between smooth varieties with all fibres of the same dimension [GW10, Corollary 14.128]. Then

$$\begin{aligned} f^*\colon A(X) &\longrightarrow A(Y) \\ \langle A\rangle &\longmapsto \langle f^{-1}(A)\rangle \end{aligned}$$

prevserves rational equivalence and induces a group homomorphism between the Chow rings preserving the grading by codimension.

2.4. **Mayer–Vietoris and excision** [EH16, §1.3.4]**.**

Proposition 2.10. *Let X be a scheme.*

(a) (Mayer–Vietoris) If X_1, X_2 are closed subschemes of X, then there is a right exact sequence

$$A(X_1 \cap X_2) \to A(X_1) \oplus A(X_2) \to A(X_1 \cup X_2) \to 0.$$

The map on the left is given by $[W] \mapsto ([W], -[W])$, whereas the one on the right is given by $([W_1], [W_2]) \mapsto [W_1] + [W_2]$.

(b) (Excision) If $Y \subseteq X$ is a closed subscheme and $U = X \setminus Y$, then there is a right exact sequence

$$A(Y) \mapsto A(X) \mapsto A(U) \mapsto 0.$$

The map on the left is proper pushforward along the closed immersion $Y \subseteq X$, whereas the map on the right is the flat pullback along the open immersion $U \subseteq X$. If X is smooth, then $A(X) \to A(U)$ is moreover a ring homomorphism.

Corollary 2.11. *If $U \subseteq \mathbb{A}^n$ is a non-empty open subset, then*

$$A(U) = \mathbb{Z} \cdot [U].$$

Proof. If we define $Y := \mathbb{A}^n \setminus U$, then we have the excision right exact sequence

$$A(Y) \to A(\mathbb{A}^n) \to A(U) \to 0.$$

If $W \subseteq Y$ is a subvariety, then $[W] = 0$ seen as a class in $A(\mathbb{A}^n)$, as we saw earlier. Therefore the map on the left is the zero map and the result follows. □

2.5. **Stratifications** [EH16, §1.3.5]**.**

Definition 2.12. Let X be a scheme. We say that X is *stratified* by a finite collection $\{U_i\}_{i=1}^m$ of irreducible, locally closed subschemes U_i if the following hold:

i) $X = \sqcup_{i=1}^m U_i$ is a disjoint union; and
ii) each closure $\overline{U_i}$ is a union of some U_j's.

Condition $ii)$ can be rephrased by saying that

$$\overline{U_i} \cap U_j \neq \varnothing \Rightarrow U_j \subseteq \overline{U_i}.$$

The U_i are called *strata* and the $Y_i := \overline{U_i}$ are called *closed strata.* We can recover U_i from Y_i by the formula

$$U_i = Y_i \setminus \left(\bigcup_{Y_j \subsetneq Y_i} Y_j \right).$$

We say that a stratification is *affine* if for each i there exists an $l \in \mathbb{N}$ such that $U_i \cong \mathbb{A}^l$. We say that it is *quasi-affine* if for each i there exists an $l \in \mathbb{N}$ such that U_i is isomorphic to an open subset in $\mathbb{A}^l$.

Example 2.13. Projective n-space admits an affine stratification with closed strata given by

$$\mathbb{P}^0 \subseteq \mathbb{P}^1 \subseteq \ldots \subseteq \mathbb{P}^n.$$

It is affine because $\mathbb{P}^i \setminus \mathbb{P}^{i-1} \cong \mathbb{A}^i$.

Proposition 2.14 ([EH16, Proposition 1.17])**.** *If a scheme X has a quasi-affine stratification, then $A(X)$ is generated by the classes of the closed strata.*

Remark 2.15. It can happen that the clases of the strata in a quasi-affine stratification become zero in the Chow ring. For instance, consider $\mathbb{A}^n$ with $n > 0$ and the stratification

$$\mathbb{A}^n = \{0\} \cup (\mathbb{A}^n \setminus \{0\}),$$

with $U_0 = \{0\}$ and $U_1 = \mathbb{A}^n \setminus \{0\}$. Then $Y_0 = \{0\}$ has $[Y_0] = 0 \in A(\mathbb{A}^n)$, as we saw earlier in the talk.

But in the case of affine stratifications we have the following:

Theorem 2.16 (Totaro, 2014)**.** *The classes of the closed strata in an affine stratification of a scheme X form a basis of $A(X)$.*

3. Vincent Gajda: Grassmannians and Schubert varieties (20.01.21)

Goal: introduce Grassmannians (working always over $\mathbb{C}$ or over an algebraically closed field of characteristic 0).

3.1. **Recall stratifications.** We say that X is *stratified* by a finite family $\{U_i\}_{i \in I}$ of locally closed subvarieties if

$$X = \bigsqcup_{i \in I} U_i, \text{ and } \overline{U_i} = \bigsqcup_{j \in J} U_j \text{ for some } J \subseteq I.$$

A stratification is called *affine* if each U_i is isomorphic to an affine space $\mathbb{A}^{n_i}$, and *quasi-affine* if each U_i is isomorphic to a non-empty open subset inside an affine space $\mathbb{A}^{n_i}$.

Proposition 3.1. *If X is quasi-affinely stratified, then $A(X)$ is generated as a group by the classes of the closed strata, i.e. the classes of the Zariski closures of the U_i.*

Proof. We argue by induction on the number of strata. Let U_0 be a minimal stratum. By definition of stratification, it has to be a closed subset of X, and the open subset $X \setminus U_0$ admits a quasi-affine stratification with one less stratum than the original quasi-affine stratification of X. We have seen in the last talk that $A(U_0) = \mathbb{Z} \cdot [U_0]$, because U_0 is isomorphic to a non-empty open subset inside some affine space. The result follows then from the induction hypothesis and the excision exact sequence

$$\mathbb{Z} \cdot [U_0] \to A(X) \to A(X \setminus U_0) \to 0.$$

□

3.2. **Chow ring of $\mathbb{P}^n$.**

Definition 3.2. Let $X \subseteq \mathbb{P}^n$ be a projective variety of dimension k. Then we define its *degree*, denoted $\deg(X)$, as the number of points in the intersection of X with a generically transverse $(n-k)$-plane. If $X \subseteq \mathbb{P}^n$ is only a subscheme, then we add the degrees of its irreducible components with multiplicities.

Remark 3.3. One can use Hilbert polynomials to define the degree of a projective variety and make sure that it is indeed a well-defined number.

Theorem 3.4 (Kleiman)**.** *Suppose G is an algebraic group (GL_n suffices for our purposes) that acts transversally on an algebraic variety X. Let $A \subseteq X$ be a subvariety.*

(1) Let $B \subseteq X$ be another subvariety. Then there exists a dense open subset $G_0 \subseteq G$ such that for all $g \in G_0$, $g \cdot A$ intersects B generically transversally.
(2) If G is affine (again, we may just think of GL_n), then $[A] = [g \cdot A]$ for all $g \in G$.

Proof. See [EH16, Theorem 1.7]. The assumptions on the base field are important here! □

Example 3.5. GL_{n+1} acts transitively on $\mathbb{P}^n$, so all k-planes in $\mathbb{P}^n$ define the same class in the Chow ring.

Proposition 3.6. *We have*

$$A(\mathbb{P}^n) = \mathbb{Z}[\zeta]/(\zeta^{n+1}),$$

where $\zeta \in A^1(\mathbb{P}^n)$ is the class of a hyperplane. Moreover, if $X \subseteq \mathbb{P}^n$ is a subvariety of codimension k and degree d, then $[X] = d\zeta^k$.

Proof. Let $\{p\} \subseteq \mathbb{P}^1 \subseteq \ldots \subseteq \mathbb{P}^n$ be a complete flag. These are the closed strata of an affine stratification, so we know that the classes of $\mathbb{P}^i$ generates $A^{n-i}(\mathbb{P}^n)$ for each $i \in \{0, \ldots, n\}$. By Kleiman's theorem we deduce moreover that the class of any i-plane generates $A^{n-i}(\mathbb{P}^n)$. We saw in the previous talk that there exists a well-defined surjective degree map

$$\begin{aligned} \deg\colon A^n(\mathbb{P}^n) &\longrightarrow \mathbb{Z} \\ [p] &\longmapsto 1 \end{aligned}$$

We have just seen that $A^n(\mathbb{P}^n)$ is generated by a signle point, so in order for deg to be surjective we need to have $A^n(\mathbb{P}^n) = \mathbb{Z} \cdot [p]$.

Let now M be a k-plane and take a generically transverse generator of $A^k(\mathbb{P}^n)$. The intersection contains exactly 1 point, so

$$[M]\cap\colon A^k(\mathbb{P}^n) \to A^n(\mathbb{P}^n) \cong \mathbb{Z}$$

is also surjective. By the same argument as before we must have $A^k(\mathbb{P}^n) \cong \mathbb{Z}$.

Now for the ring structure, we know that an $(n-k)$-plane L is the transverse interecction of k hyperplanes, so we have

$$[L] = \zeta^k,$$

where ζ is the class of any hyperplane. This implies that the ring structure is as claimed.

Finally, if $X \subseteq \mathbb{P}^n$ is a subvariety of codimension k and degree d, then the assumption on the codimension implies that $[X] \in A^k(\mathbb{P}^n) = \mathbb{Z} \cdot \zeta^k$, so we may write $[X] = m\zeta^k$. The assumption on the degree implies then that

$$d = \deg([X] \cdot \zeta^{n-k}) = \deg(m\zeta^k\zeta^{n-k}) = m,$$

as we wanted to show. □

Corollary 3.7 (Bézout). *If $X_1, \ldots, X_k \subseteq \mathbb{P}^n$ are subvarieties of codimensions $c_1, \ldots, c_k$ respectively, with $\sum c_i \leq n$ and intersecting generically transversely, then*

$$\deg(X_1 \cap \ldots \cap X_k) = \prod \deg(X_i).$$

In particular, if $X, Y \subseteq \mathbb{P}^n$ have complementary dimensions and intersect generically transversely, then

$$|X \cap Y| = \deg(X)\deg(Y).$$

Proof. By the previous proposition we can write

$$[X_i] = \deg(X_i)\zeta^{c_i}$$

for each $i \in \{1, \ldots, k\}$. On the ohter hand,

$$[X_1 \cap \ldots \cap X_k] = \deg(X_1 \cap \ldots \cap X_k)\zeta^{\sum c_i}.$$

But

$$[X_1 \cap \ldots \cap X_k] = [X_1] \cdots [X_k] = \Big(\prod \deg(X_i)\Big)\, \zeta^{\sum c_i},$$

hence the desired equality. □

3.3. **Grassmannians.** Let V be an n-dimensional vector space over $\mathbb{C}$ and let $1 \leq k \leq n$. We want to consider the *Grassmannian* of k-planes in V:

$$G := G(k, V) := \left\{ \begin{array}{c} k\text{-dimensional} \\ \text{subspaces in } V \end{array} \right\} = \left\{ \begin{array}{c} (k-1)\text{-dimensional} \\ \text{subspaces in } \mathbb{P}(V) \end{array} \right\}$$

We would write $\mathbb{G}(k-1, V)$ instead to mean the right-hand side interpretation of $(k-1)$-dimensional subspaces in $\mathbb{P}(V)$.

We have described the Grassmannian as a set, but we want to endow it with a structure of projective variety. We consider for this the *Plücker embedding*:

$$\begin{aligned} G &\longrightarrow \mathbb{P}(\wedge^k V) \cong \mathbb{P}^{\binom{n}{k}-1} \\ \Lambda = \langle w_1, \ldots, w_n \rangle &\longmapsto [w_1 \wedge \ldots \wedge w_k] \end{aligned}$$

This map is well-defined, because if we had chosen a different basis for Λ, then the result would have only differed by multiplication with the (non-zero) determinant of the base-change matrix. And it is injective, because a vector $v \in V$ is in Λ if and only if $v \wedge (w_1 \wedge \ldots \wedge w_k) = 0$, so we can recover Λ from $[w_1 \wedge \ldots \wedge w_k]$.

If we choose a basis $\{e_1, \ldots, e_n\}$ of V, then we may identify V with $\mathbb{C}^n$ and represent Λ as the row space of a $k \times n$ matrix

$$A = \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & & \vdots \\ a_{k,1} & \cdots & a_{k,n} \end{pmatrix}$$

in which the rows are the $w_1, \ldots, w_k$. We may then write

$$w_1 \wedge \ldots \wedge w_k = \sum_{1 \leq i_1 < \ldots < i_k \leq n} p_{i_1,\ldots,i_k} e_{i_1} \wedge \ldots e_{i_k},$$

and these coefficients $p_{i_1,\ldots,i_k}$ are called the *Plücker coordinates* of Λ with respect to the basis $\{e_1, \ldots, e_n\}$. They do not depend on the chosen basis for Λ, because each $p_{i_1,\ldots,i_k}$ is given by the corresponding minor of A, and changing the basis of Λ corresponds to multiplying A on the left by an invertible $k \times k$ matrix. Therefore they are well-defined as coordinates in $\mathbb{P}^{\binom{n}{k}-1}$.

Next we want to argue that the Plücker embedding $\varphi \colon G \hookrightarrow \mathbb{P}^{\binom{n}{k}-1}$ makes G an algebraic variety. Let us call a point of the form $[w_1 \wedge \ldots \wedge w_k] \in \mathbb{P}(\wedge^k V)$ a *pure wedge product*. Then $\varphi(G)$ consists precisely of the equivalence classes of pure wedge products, so we need to find homogeneous polynomial equations describing the set of such equivalence classes. If $\eta \in \wedge^k V$ and $v \in V \setminus \{0\}$, then $v \wedge \eta = 0$ if and only if $\eta = v \wedge \eta'$ for some $\eta' \in \wedge^{k-1} V$. By induction we deduce that η is a pure wedge product if and only if

$$\dim\left(V \xrightarrow{(-)\wedge\eta} \wedge^{k+1} V\right) \geq k,$$

therefore

$$\varphi(G) = \left\{[\eta] \in \mathbb{P}(\wedge^k V) \middle| \operatorname{rank}\left(V \xrightarrow{(-)\wedge\eta} \wedge^{k+1} V\right) \leq n-k\right\}.$$

This subspace of $\mathbb{P}(\wedge^k V)$ is the zero locus of the $(n-k+1)$-minors of a matrix representing the linear map $(-) \wedge \eta$, which are homogeneous polynomials. Hence $\varphi(G)$ is an algebraic subset in $\mathbb{P}^{\binom{n}{k}-1}$. To check that G is also a variety, i.e. to check that it is irreducible, note that its ideal is the kernel of the ring homomorphism

$$\mathbb{C}[\{p_{i_1,\ldots,i_k}\}_{1 \leq i_1 < \ldots i_k \leq n}] \longrightarrow \mathbb{C}[\{x_{ij}\}_{1 \leq i \leq n, 1 \leq j \leq k}]$$

which corresponds to "writing out the Plücker coordinates" in terms of the coefficients of the matrix. The ring on the right is an integral domain, so the kernel of this morphism is a prime ideal and G is an algebraic variety. In the following we often identify G with $\varphi(G)$ as sets already.

As in the particular case of projective spaces, Grassmannians admit a *standard affine open cover* constructed as follows. Fix $\Gamma \subseteq V$ of dimension $(n-k)$, say $\Gamma = \langle e_1, \ldots, e_{n-k}\rangle$. We may extend this basis of Γ to a basis $e_1, \ldots, e_n$ of V, and we let $\eta := e_1 \wedge \ldots \wedge e_{n-k}$. Define

$$U_\Gamma := \{\Lambda \in G \mid \Lambda \cap \Gamma = 0\}.$$

The complement of U_Γ is the set of $[\omega] \in G$ such that $\omega \wedge \eta = 0$, which can be characterized as the hyperplane section where the $(e_1, \ldots, e_n)$-Plücker coordinate $p_{n-k+1,\ldots,n}$ is equal to zero. Hence U_Γ is open, and as we vary the subspace Γ we obtain an open cover of G. From this open cover we deduce that $\dim G = k(n-k)$ and that G is smooth, because we have the following

Lemma 3.8. $U_\Gamma \cong \mathbb{A}^{k(n-k)}$.

Proof. Consider a basis $(e_1, \ldots, e_n)$ of V such that $\Gamma = \langle e_{k+1}, \ldots, e_n\rangle$. Then we have

$$U_\Gamma = \{\Lambda \in G \mid p_{1,\ldots,k}(\Lambda) \neq 0\}.$$

We regard $\mathbb{A}^{k(n-k)}$ as the space of $k \times (n-k)$-matrices, and we define a map

$$f\colon \mathbb{A}^{k(n-k)} \longrightarrow U_\Gamma$$

$$C = \begin{pmatrix} c_{1,1} & \dots & c_{1,n-k} \\ \vdots & & \vdots \\ c_{k,1} & \dots & c_{k,n-k} \end{pmatrix} \longmapsto \text{RowSpace} \begin{pmatrix} 1 & 0 & \dots & 0 & c_{1,1} & \dots & c_{1,n-k} \\ 0 & 1 & \dots & 0 & c_{2,1} & \dots & c_{2,n-k} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 & c_{k,1} & \dots & c_{k,n-k} \end{pmatrix}$$

The image of f lies indeed in U_Γ, because the $(1, \dots, k)$-minor is just 1 and so $p_{1,\dots,k}(f(C)) = 1 \neq 0$.

For the surjectiviy, we represent $\Lambda \in U_\Gamma$ as the row space of a matrix $A = (K \mid A')$ in which K is an invertible $k \times k$-matrix, because $\Lambda \in U_\Gamma$ means by definition that the $(1, \dots, k)$-minor of A is non-zero. Hence we may write $K^{-1}A = (\mathrm{Id}_{k\times k} \mid A'')$ and thus the row space of A is in the image of f.

If $C \neq C'$, then $f(C) \neq f(C)$, because the matrix representation used to define f is unique for subspaces in U_Γ. Explicitly, it follows by looking at the first k rows of an expression of the form

$$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ c'_{1,1} \\ \vdots \\ c'_{1,n-k} \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ c_{1,1} \\ \vdots \\ c_{1,n-k} \end{pmatrix} + \cdots + \lambda_k \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ c_{k,1} \\ \vdots \\ c_{k,n-k} \end{pmatrix}$$

that $\lambda_1 = 1$ and the rest of λ's must be 0, hence the claimed uniqueness.

It remains to show that this bijection is algebraic. A matrix C is sent to an element in $\mathbb{P}(\wedge^k V)$ whose coordinates are $k \times k$-minors of the specified matrix, hence the image of the matrix is a polynomial expression in its entries (which are the coordinates on the left space). This shows that f is algebraic. For f^{-1}, represent $\Lambda \in U_\Gamma$ by a matrix of the specified form, so that $\Lambda = f(C)$. Then, at least up to a sign, we have

$$\pm c_{i,j} = \frac{p_{1,\dots,\hat{i},\dots,k,k+j}(\Lambda)}{p_{1,\dots,k}(\Lambda)}.$$

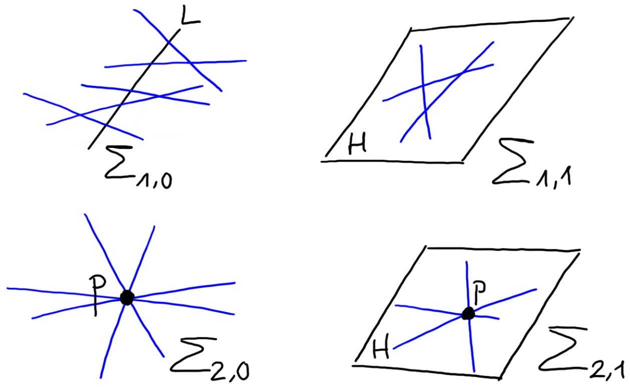
Hence f^{-1} is a regular function as well. □

3.4. **Schubert varieties in $\mathbb{G}(1,3) = G(2,4)$.** Our next step is to find a convenient stratification of $\mathbb{G}(1,3)$ in order to understand its Chow ring in the future. We fix a complete flag

$$\mathcal{V} = (\{p\} \subseteq L \subseteq H \subseteq \mathbb{P}^3).$$

We stratify $\mathbb{G}(1,3)$ by the loci of lines which have a certain dimension of intersection with respect to the subspaces in our flag $\mathcal{V}$. The closed sets of the stratification will be called *Schubert varieties* or *Schubert cycles*, whereas the open sets will be called *Schubert cells*. The Schubert varieties are given as follows:

$$
\begin{aligned}
\Sigma_{0,0} &= \mathbb{G}(1,3) \\
\Sigma_{1,0} &= \{L' \mid L' \cap L \neq \varnothing\} \\
\Sigma_{2,0} &= \{L' \mid p \in L'\} \\
\Sigma_{1,1} &= \{L' \mid L' \subseteq H\} \\
\Sigma_{2,1} &= \{L' \mid p \in L' \subseteq H\} \\
\Sigma_{2,2} &= \{L\}.
\end{aligned}
$$



We have the following inclusions:

$$
\begin{tikzcd}
 & & \Sigma_{2,0} \arrow[dr, hook] & & \\
\{L\} = \Sigma_{2,2} \arrow[r, hook] & \Sigma_{2,1} \arrow[ur, hook] \arrow[dr, hook] & & \Sigma_{1,0} \arrow[r, hook] & \Sigma_{0,0} = \mathbb{G}(1,3). \\
 & & \Sigma_{1,1} \arrow[ur, hook] & &
\end{tikzcd}
$$

We define the Schubert cells as

$$\Sigma^\circ_{a,b} = \Sigma_{a,b} \setminus \text{ smaller strata }.$$

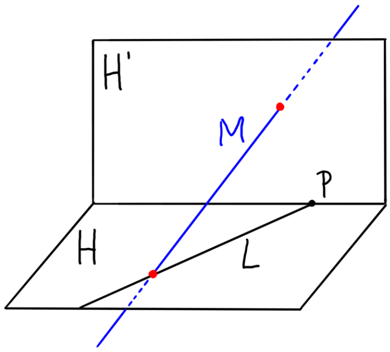
Schubert cells define an affine stratification of $\mathbb{G}(1,3)$. Let us just sketch one part of the proof as an example:

Lemma 3.9. $\Sigma^\circ_{1,0} \cong \mathbb{A}^3$.

Proof. We have

$$\Sigma^\circ_{1,0} = \Sigma_{1,0} \setminus (\Sigma_{1,1} \cup \Sigma_{2,0}).$$

Spelling out the definitions, we see that $\Sigma^\circ_{1,0}$ consists of those lines M such that $M \cap L \neq \varnothing$, $p \notin M$ and $M \not\subseteq H$. The idea to show the claimed isomorphism is to fix some $H' \ni p$ with $L \not\subseteq H'$.



Take $M \in \Sigma_{1,0}^{\circ}$; it meets $H' \setminus (H' \cap H) \cong \mathbb{A}^2$ in a unique point, and it meets $L \setminus \{p\} \cong \mathbb{A}^1$ in a unique point as well. So we define

$$\begin{aligned} \Sigma_{1,0} &\longmapsto \mathbb{A}^2 \times \mathbb{A}^1 \\ M &\longmapsto (M \cap H', M \cap L) \end{aligned}$$

□

This stratification of $\mathbb{G}(1,3)$ depends on the chosen flag, but the classes of the closed strata in the Chow ring do not depend on the chosen flag as a consequence of Kleiman's transversality [EH16, Theorem 1.7]. Indeed, any two flags $\mathcal{V}$ and $\mathcal{V}'$ are related by a GL_4 action, so the Schubert cycles $\Sigma_{a,b}(\mathcal{V})$ and $\Sigma_{a,b}(\mathcal{V}')$ are GL_4-translates. Since GL_4 acts transitively on lines in $\mathbb{P}^3$, we deduce that $[\Sigma_{a,b}(\mathcal{V})]$ does not depend on the chosen flag.

4. Fabian Kertels: How many lines intersect 4 random line in space? (27.01.2021)

Goals:

- Compute $A(G(2,4))$ ($3\times$).
- Show that the answer to the question in the title is 2 ($2\times$).

In the title, by *random* we mean in general position; and by *space* we mean in 3-dimensional projective space, say over $\mathbb{C}$ (or at least over an algebraically closed field of characteristic zero).

4.1. Recap.

Definition 4.1. For $k \leq \dim(V) = n$, the *Grsassmannian* $G(k,V)$ is defined as the set of k-dimensional vector subspaces of V together with the projective variety structure induced by the Plücker embedding:

$$\begin{aligned} G(k,n) &\longrightarrow \mathbb{P}(\wedge^k V) \\ \langle w_1, \dots, w_k \rangle &\longmapsto [w_1 \wedge \cdots \wedge w_k] \end{aligned}$$

The image of this function consists of the set

$$\{[\eta] \mid \mathrm{rk}(V \xrightarrow{(-)\wedge\eta} \wedge^{k+1} V) \leq n-k\},$$

which can be described as the vanishing locus of some minors of the corresponding matrix once we fix a basis.

To compute $A(G(k,v))$, (quasi-affine) stratifications are helpful:

Definition 4.2. A *stratification* of a variety X is a collection $\{U_i\}_{i\in I}$ of irreducible locally closed subvarieties of X with I a finite set and such that

$$X = \coprod_{i\in I} U_i \quad \text{and} \quad \overline{U_i} = \coprod_{U_j \subseteq \overline{U_i}} U_j.$$

It is an *affine stratification* if for every $i \in I$ there exists some $k_i \in \mathbb{N}$ such that $U_i \cong \mathbb{A}^{k_i}$; and it is a *quasi-affine stratification* if for every $i \in I$ there exists some $k_i \in \mathbb{N}$ such that U_i is an open subset of $\mathbb{A}^{k_i}$.

Proposition 4.3. *If X has a quasi-affine stratification $\{U_i\}_{i\in I}$, then $A(X)$ is generated by $\{[\overline{U_i}]\}_{i\in I}$.*

In the case of $\mathbb{G}(1,3) = \mathbb{G}(1,\mathbb{P}^3) = G(2,4)$ we can produce an affine stratification as follows. Take a complete flag

$$\mathcal{V} = \big(\{p\} \subseteq L \subseteq H \subseteq \mathbb{P}^3\big)$$

of $\mathbb{P}^3$, and define our closed strata (*Schubert cycles*) as

- $\Sigma_{0,0} := \mathbb{G}(1,3)$;
- $\Sigma_{1,0} := \{L' \mid L' \cap L \neq \varnothing\}$;
- $\Sigma_{2,0} := \{L' \mid p \in L'\}$;
- $\Sigma_{1,1} := \{L' \mid L' \subseteq H\}$;
- $\Sigma_{2,1} := \{L' \mid p \in L' \subseteq H\}$;
- $\Sigma_{2,2} := \{L\}$.

We had the following inclusions

$$\begin{tikzcd}
 & & \Sigma_{2,0} \arrow[dr, hook] & & \\
\{L\} = \Sigma_{2,2} \arrow[r, hook] & \Sigma_{2,1} \arrow[ur, hook] \arrow[dr, hook] & & \Sigma_{1,0} \arrow[r, hook] & \Sigma_{0,0} = \mathbb{G}(1,3), \\
 & & \Sigma_{1,1} \arrow[ur, hook] & &
\end{tikzcd}$$

and we defined the *Schubert cells* as $\Sigma^\circ_{a,b} = \Sigma_{a,b} \setminus$ smaller strata .

Corollary 4.4. *$A(\mathbb{G}(1,3))$ is generated by the Schubert classes*

$$\sigma_{a,b} := [\Sigma_{a,b}].$$

4.2. **Structure of $A(\mathbb{G}(1,3))$.**

Theorem 4.5. *We have*

$$A := A(\mathbb{G}(1,3)) = \oplus_{0\le b\le a\le 2} \mathbb{Z}\sigma_{a,b}$$

with multiplication given as follows:

- $A^1 \times A^1 \to A^2\colon \sigma_{1,0}^2 = \sigma_{1,1} + \sigma_{2,0}$;
- $A^1 \times A^2 \to A^3\colon \sigma_{1,0}\sigma_{1,1} = \sigma_{1,0}\sigma_{2,0} = \sigma_{2,1}$;
- $A^1 \times A^3 \to A^4\colon \sigma_{1,0}\sigma_{2,1} = \sigma_{2,2}$;
- $A^2 \times A^2 \to A^4\colon \sigma_{1,1}^2 = \sigma_{2,0}^2 = \sigma_{2,2}, \sigma_{1,1}\sigma_{2,0} = 0$.

Proof. Since $\mathbb{G}(1,3)$ is proper over $\mathbb{C}$, A^4 is freely generated by $\sigma_{2,2}$. It remains to prove the formulas for the multiplications, since then the group structure would follow. For example, suppose we have shown the formulas and we wanted to see that A^2 is freely generated by $\sigma_{1,1}$ and $\sigma_{2,0}$. Let $a, b \in \mathbb{Z}$ such that $a\sigma_{1,1} + b\sigma_{2,0} = 0$. We can multiply on the left by $\sigma_{1,0}$ to deduce that $a\sigma_{2,2} = 0$, hence $a = 0$. And we can multiply on the left by $\sigma_{1,0}$ to deduce that $b\sigma_{2,2} = 0$, hence $b = 0$ as well. In this way we can decude from the multiplicative formulas that

$$A^2 = \mathbb{Z}\sigma_{1,1} \oplus \mathbb{Z}\sigma_{2,0}$$

as abelian groups, and with simlar arguments we could do the same for all other degrees.

To compute the multiplicative formulas we take another flag

$$\mathcal{V}' = \big(\{p'\} \subseteq L' \subseteq H' \subseteq \mathbb{P}^3\big)$$

such that $\Sigma_{2,0} \cap \Sigma'_{2,0}$ has dimension 0, which is possible thanks to Kleiman's transversality. Then

$$\begin{aligned}
\sigma_{2,0}^2 &= |\Sigma_{2,0} \cap \Sigma'_{2,0}|\sigma_{2,2} \\
&= |\{L'' \mid p' \in L'' \text{ and } p \in L''\}|\sigma_{2,2} \\
&= |\{\overline{pp'}\}|\sigma_{2,2} = \sigma_{2,2}.
\end{aligned}$$

And we can argue similarly for the other cases in which the codimensions add up appropriately, namely, for $A^2 \times a^2$ and for $A^1 \times A^3$.

For $A^1 \times A^2$ we have

$$\Sigma_{1,0} \cap \Sigma'_{2,0} = \{L'' \mid p' \in L'' \cap L \neq \varnothing,$$

which is the $(2,1)$-Schubert cycle with respect to a flag containing the point p' and the plane $\overline{p'L}$, so that we have

$$\sigma_{1,0}\sigma_{2,0} = \sigma_{2,1}.$$

In a similar way we deduce that

$$\sigma_{1,0}\sigma_{1,1} = \sigma_{2,1}.$$

It remains to deal with $A^1 \times A^1$. From arguments as in the beginning of the proof, we already know that

$$A^2 = \mathbb{Z}\sigma_{1,1} \oplus \mathbb{Z}\sigma_{2,0}.$$

Therefore it is possible to find $a, b \in \mathbb{Z}$ such that

$$\sigma_{1,0}^2 - a\sigma_{2,0} + b\sigma_{1,1}.$$

Now

$$\begin{aligned} a\sigma_{2,2} &= (a\sigma_{2,0} + b\sigma_{1,1})\sigma_{2,0} \\ &= \sigma_{1,0}^2\sigma_{2,0} \\ &= \sigma_{1,0}(\sigma_{1,0}\sigma_{2,0}) \\ &= \sigma_{1,0}\sigma_{2,1} = \sigma_{2,2}, \end{aligned}$$

so we deduce that $a = 1$. And likewise we can deduce that $b = 1$, finishing the proof. □

Now we are ready to answer the question in the title: a line in $\mathbb{P}^3$ incident to m lines $L_1, \ldots, L_m$ corresponds to a point in $\mathbb{G}(1,3)$ contained in all the $\Sigma_{1,0}(L_i)$ for $i \in \{1, \ldots, m\}$. Here, by $\Sigma_{1,0}(L_i)$, we mean the Schubert cycle induced by some flag containing L_i. So in our case we would compute

$$\begin{aligned} |\Sigma_{1,0}(L_1) \cap \ldots \cap \Sigma_{1,0}(L_4)| &\overset{(?)}{=} \deg(\sigma_{1,0}^4) \\ &= \deg((\sigma_{1,1} + \sigma_{2,0})^2) \\ &= \deg(2\sigma_{2,2}) = 2. \end{aligned}$$

The (?) equality works in this case because Kleiman transversality allows us to apply the moving lemma without generating non-effective cycles in the process and the codimensions fit appropriately.

Alternatively, the multiplication $A^1 \times A^1 \to A^2$ can also be computed without using associativity. We would still use the *method of undetermined coefficients*: if $\sigma_{1,0}^2\sigma_{2,0} = a\sigma_{2,2}$, then Kleiman's theorem allows us to choose flags $\mathcal{V}$, $\mathcal{V}'$ and $\mathcal{V}''$ in general position, so that

$$\begin{aligned} a &= |\{M \mid M \cap L \neq \varnothing, M \cap L' \neq \varnothing, p'' \in M\}| \\ &= |\{\overline{p''L} \cap \overline{p''L'}\}| = 1. \end{aligned}$$

And likewise for the b coefficient.

4.3. **(Static) specialisation.** "Specialise" $\mathcal{V}$ and $\mathcal{V}'$ enough so that $\Sigma_{a,b}$ and $\Sigma'_{a,b}$ are still "general enough" but at the same time the intersecting can be easily read off. Using these methods one can give algorithms for computations in Chow rings of general Grassmannians, e.g. Vakil'06.

We will illustrate this method by computing $\sigma_{1,0}^2$ a third time. Pick flags $\mathcal{V}$ and $\mathcal{V}'$ so that $L \cap L' = \{p\}$ and $H = \overline{L'L}$. Then, if we knew that the corresponding Schubert cycles $\Sigma_{1,0}$ and $\Sigma'_{1,0}$ were generically transverse, we would be able to deduce

$$\begin{aligned} \Sigma_{1,0} \cap \Sigma'_{1,0} &= \{M \mid M \cap L \neq \varnothing \text{ and } M \cap L' \neq \varnothing\} \\ &= \{M \mid M \subseteq H \text{ or } p \in M\} \\ &= \Sigma_{2,0} \cup \Sigma_{1,1}. \end{aligned}$$

So we need to check that the two cycles are indeed generically transverse. First we take some $M \in \Sigma_{2,0}$, i.e. $p \in M$ and $M \notin \{L, L'\}$. We need to compute the tangent spaces of $\Sigma_{1,0}$ and $\Sigma'_{1,0}$ at M. Let V be a 4-dimensional complex vector space so that our $\mathbb{P}^3$ is $\mathbb{P}(V)$. If $T \subseteq \mathbb{P}^3$ is a linear subspace, we denote by $\tilde{T}$ the vector subspace of V such that $T = \mathbb{P}(\tilde{T})$. Then we have

$$T_M\Sigma_{1,0} = \{\varphi \in Mor_{\mathrm{Mod}_{\mathbb{C}}}(\tilde{M}, V/\tilde{M}) \mid \varphi(\tilde{p}) \subseteq \tilde{\overline{ML}}/\tilde{M}\}.$$

This is a 3-dimensional vector space, and likewise for $T_M\Sigma'_{1,0}$. Therefore both cycles are smooth at M, because we saw in the last talk that they are 3-dimensional cycles with $\Sigma^\circ_{1,0} \cong \mathbb{A}^3$. And we also have

$$\begin{aligned} T_M\Sigma_{1,0} \cap T_M\Sigma'_{1,0} &= \{\varphi \mid \varphi(\tilde{p}) \subseteq (\tilde{\overline{ML}} \cap \tilde{\overline{ML'}})/\tilde{M}\} \\ &= \{\varphi \mid \varphi(\tilde{p}) \subseteq \tilde{M}/\tilde{M}\} \\ &= \{\varphi \mid \varphi(\tilde{p}) = 0\}, \end{aligned}$$

hence the intersection has the expected codimension at M. A similar argument shows the same for $M \in \Sigma_{1,1}$, so the two cycles are indeed generically transverse.

4.4. **A geometric picture.** Brief discussion following [EH16, §3.4.1].

5. Luca Terenzi: Knutson–Tao puzzles and Chern classes (03.02.2021)

Notation: k an algebraicallly closed field of characteristic 0 (e.g. $k = \mathbb{C}$).

5.1. **Knutson–Tao puzzles.** Main character: $\mathbb{G}(1,3) = G(2,4)$, the Grassmannian of lines in $\mathbb{P}^3$. Recall from the last talk that the Chow ring $A(G(2,4))$ is freely generated as a graded abelian group by the Schubert classes

$$\sigma_{a,b} = [\Sigma_{a,b}(\mathcal{V})] \in A^{a+b}$$

for $2 \geq a \geq b \geq 0$. They satisfy the relation

$$\sigma_1^2 = \sigma_{1,0}^2 = \sigma_{1,1} + \sigma_{2,0}$$

in A^2. We established it in 2 different ways:

(1) With the method of undetermined coefficients. Write $\sigma_{1,0}^2 = a\sigma_{1,1} + b\sigma_{2,0}$ for some $a, b \in \mathbb{Z}$. Then determine a and b by multiplying this relation with suitable classes (i.e. $\sigma_{2,0}$, $\sigma_{1,1}$).
(2) By static specialization. Choose two complete flags $\mathcal{V}$ and $\mathcal{V}'$ so that
    (i) the cycles $\Sigma_{1,0}(\mathcal{V})$ and $\Sigma_{1,0}(\mathcal{V}')$ intersect generically transversely, and
    (ii) the intersection $\Sigma_{1,0}(\mathcal{V}) \cap \Sigma_{1,0}(\mathcal{V}')$ can be easily described geometrically.

    Then compute $[\Sigma_{1,0}(\mathcal{V})] \cdot [\Sigma_{1,0}(\mathcal{V}')] = [\Sigma_{1,0}(\mathcal{V}) \cap \Sigma_{1,0}(\mathcal{V}')]$.

Remark 5.1. In both cases, one cannot avoid giving an explicit geometric interpretation to some products of Schubert classes. Doing the same in general (i.e. for $G(r, n)$ with $r, n \gg 0$) seems really difficult.

Knutson–Tao puzzles allow us to compute products of general Schubert clases in a purely combinatorial way!

But first, let us change the notation for Schubert classes a little bit. Let V be a k-vector space of dimension $n \geq 0$, and let $r \leq n$ be a natural number. Consider $G(r, V) \cong G(r, n)$. Fix a complete flag $\mathcal{V}$ given by subspaces

$$0 = V_0 \subset V_1 \subset \ldots \subset V_n = V$$

with $\dim(V_i) = i$. Then, to every r-tuple

$$a = (a_1, \ldots, a_r) \in \mathbb{N}^r$$

with $n - r \geq a_1 \geq \ldots \geq a_r \geq 0$ we associate the Schubert cycle $\Sigma_a(\mathcal{V})$ given by the set

$$\{W \in G(r, n) \mid \dim(V_{n-r+i-a_i} \cap W) \geq i \text{ for all } i\}.$$

Geometric intuition: for a general $W \in G(r, n)$, we have

$$\dim(V_{n-r+j} \cap W) = \begin{cases} 0, & j < 0 \\ j, & j \geq 0. \end{cases}$$

Hence we have $W \in \Sigma_a(\mathcal{V})$ if and only if the condition $\dim(V_{n-r+j} \cap W) \geq i$ is satisfied at least a_i steps earlier than expected. Equivalently, for $W \in G(r, n)$ consider the sequence of subspaces of W given by $\mathcal{V} \cap W$, that is,

$$0 = (V_0 \cap W) \subset (V_1 \cap W) \subset \ldots \subset (V_n \cap W) = W.$$

At each step, the dimension increases by 0 or 1; since $\dim(W) = r$, it increases by 1 exactly r times. We can encode this into an n-string

$$\alpha = \alpha_W = (\alpha_1, \ldots, \alpha_n) \in \{0, 1\}^n$$

such that $\sum_i \alpha_i = r$, and the cycle $\Sigma_a(\mathcal{V})$ is the closure of the locus

$$\{W \in G(r, n) \mid \alpha_j = 1 \Leftrightarrow \exists i, , j = n - r + i - a_i\}.$$

Summary: the sequence $a = (a_1, \ldots, a_r)$ corresponds to a unique n-string $\alpha = (\alpha_1, \ldots, \alpha_n)$ as above. Hence we can write

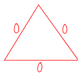
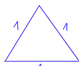
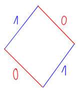
$$\Sigma_a(\mathcal{V}) = S_\alpha(\mathcal{V}), \quad \sigma_a = s_\alpha.$$

Example 5.2. In $A(G(2, 4))$ we have $\sigma_{1,0} = s_{(0,1,0,1)}$, $\sigma_{1,1} = s_{(0,1,1,0)}$ and $\sigma_{2,0} = s_{(1,0,0,1)}$. The relation $\sigma_{1,0}^2 = \sigma_{1,1} + \sigma_{2,0}$ becomes

$$s^2_{(0,1,0,1)} = s_{(0,1,1,0)} + s_{(1,0,0,1)}.$$

We will re-prove this relation soon.

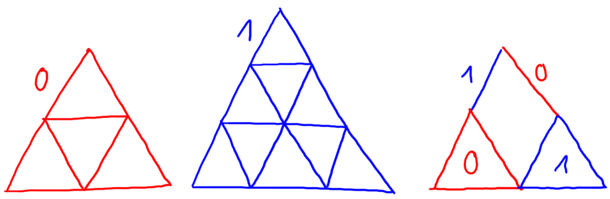
Definition 5.3. Consider the following 3 types of pieces:

-  0-triangle,
-  1-triangle, and
-  rhombus.

A *Knutson–Tao puzzle* of size $n \in \mathbb{N}$ is a decomposition of a lattice triangle of side-length n into lattice polygons such that

- all edges are labelled 0 or 1, and
- every region is a puzzle piece as above.

Example 5.4. Some valid Knutson–Tao puzzles would be

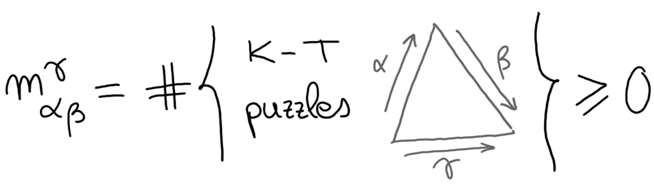


Knutson–Tao puzzles compute products of Schubert classes on Grassmannians:

Theorem 5.5. *Given two n-strings $\alpha, \beta \in \{0,1\}^n$ with $\sum_i \alpha_i = \sum_i \beta_i = r$, write in $A(G(r,n))$*

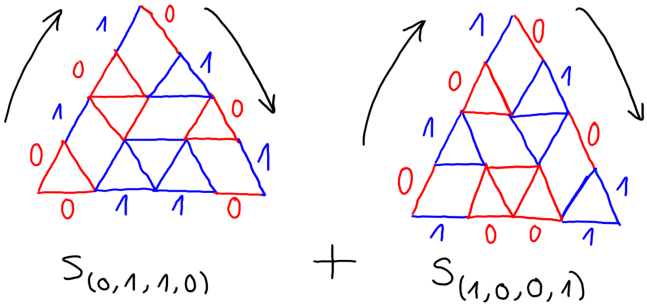
$$s_\alpha \cdot s_\beta = \sum_\gamma m^\gamma_{\alpha\beta} s_\gamma$$

for some integers $m^\gamma_{\alpha\beta} \in \mathbb{Z}$. Then



Remark 5.6. The coefficients $m^\gamma_{\alpha\beta}$ are well-defined.

Example 5.7. We compute $\sigma^2_{1,0} = s^2_{(0,1,0,1)}$ using Knutson–Tao puzzles:



The previous theorem allows us to derive some general symmetries of Schubert calculus:

Corollary 5.8. *With the same notation as in the theorem, we have the following relations:*

(i) *Rotation:* $m^{\gamma}_{\alpha^\vee\beta^\vee} = m^{\alpha}_{\beta^\vee\gamma^\vee} = m^{\beta}_{\gamma^\vee\alpha^\vee}$, *where* $\alpha^\vee_i := \alpha_{n-i}$.
(ii) *Reflection:* $m^{\gamma}_{\alpha\beta} = m^{\bar\gamma}_{\bar\beta\bar\alpha}$, *where* $\bar\alpha_i := 1-\alpha_{n-i}$.

5.2. **Introduction to Chern classes.** Main characters: X a smooth connected k-variety and $\mathscr{E}$ a rank r vector bundle on X (=locally free sheaf of $\mathscr{O}_X$-modules of rank r).

Many interesting subvarieties $Y \subseteq X$ can be written as:

- Vanishing locus of a family of sections of $\mathscr{E}$.
- Degeneracy locus of a collection of sections of $\mathscr{E}$ (i.e. locus where they are linearly dependent).

These subvarieties can be studied systematically with the theory of Chern classes, which allows us to translate questions of intersection theory into questions of linear algebra.

5.3. **The first Chern class of a line bundle.** Let $\mathscr{L}$ be a line bundle on X. A reational section $\sigma \in \Gamma(X, \mathscr{L}\otimes_{\mathscr{O}_X}\mathscr{K}_X)$ (where $\mathscr{K}_X$ is the constant sheaf of rational functions) defines a Cartier divisor D_σ as follows. Choose $U \subseteq X$ a non-empty Zariski-open subset such that $\mathscr{L}|_U \cong \mathscr{O}_U$. We can then write $\sigma_U = f_U/g_U$ for some $f_U, g_U \in \mathscr{O}_X(U)$. Define $(D_\sigma)|_U := \mathrm{div}(f_U) - \mathrm{div}(g_U)$. Then glue these objects to a divisor D_σ on X. Given another rational section τ of $\mathscr{L}$, we have $\sigma = f\tau$ for some $f \in K(X) = \Gamma(X, \mathscr{K}_X)$, and so $D_\sigma = \mathrm{div}(f) + D_\tau$, which implies that

$$D_\sigma = D_\tau$$

in A^1.

Definition 5.9. The element $c_1(\mathscr{L}) := [D_\sigma] \in A^1(X)$ is called the *first Chern class* of $\mathscr{L}$.

5.4. **Axiomatic definition of Chern classes.** More generally, for every vector bundle $\mathscr{E}$ on X it is possible to define classes $c_i(\mathscr{E}) \in A^i(X)$ for all $i \geq 0$ satisfying many useful properties:

Theorem 5.10. *There is a unique way of assigning to every smooth variety X and every vector bundle $\mathscr{E}$ on X a class*

$$c(\mathscr{E}) = 1 + c_1(\mathscr{E}) + c_2(\mathscr{E}) + \ldots \in A(X)$$

with $c_i(\mathscr{E}) \in A^i(X)$, so that:

(i) if $\mathscr{L}$ is a line bundle on X, then

$$c(\mathscr{L}) = 1 + c_1(\mathscr{L}),$$

where $c_1(\mathscr{L})$ is the class defined before;

(ii) given global sections $\tau_0, \ldots, \tau_{r-i} \in \Gamma(X, \mathscr{E})$ such that the locus $D \subseteq X$ where they are linearly dependent has codimension i, then

$$c_i(\mathscr{E}) = [D] \in A^i(X);$$

(iii) for every short exact sequence of vector bundles

$$0 \to \mathscr{E}' \to \mathscr{E} \to \mathscr{E}'' \to 0$$

we have Whitney's formula:

$$c(\mathscr{E}) = c(\mathscr{E}') \cdot c(\mathscr{E}'');$$

(iv) for every morphism of smooth varieties $\varphi \colon Y \to X$ we have

$$\varphi^* c(\mathscr{E}) = c(\varphi^* \mathscr{E}).$$

Definition 5.11. We call

- $c(\mathscr{E}) \in A(X)$ the *total Chern class* of $\mathscr{E}$, and
- $c_i(\mathscr{E}) \in A^i(X)$ the *i-th Chern class* of $\mathscr{E}$.

Corollary 5.12. *If $\mathscr{E}$ is a vector bundle on X admitting a filtration*

$$0 = \mathscr{E}_0 \subset \mathscr{E}_1 \subset \ldots \subset \mathscr{E}_n = \mathscr{E}$$

by subvector bundles $\mathscr{E}_i$ such that the corresponding quotients

$$\mathscr{L}_i := \mathscr{E}_i / \mathscr{E}_{i+1}$$

are again vector bundles, then

$$c(\mathscr{E}) = \prod_{i=1}^{n} c(\mathscr{L}_i) = c\left(\oplus_{i=1}^{n} \mathscr{L}_i\right).$$

5.5. The splitting principle.

Lemma 5.13 (Splitting construction)**.** *Let X be a smooth connected variety and $\mathscr{E}$ a vector bundle on X. Then there exist a smooth connected variety Y and a morphism $\varphi\colon Y \to X$ such that:*

(i) the map $\varphi^*\colon A(X) \to A(Y)$ is injective, and
(ii) the vector bundle $\varphi^*\mathscr{E}$ on Y admits a filtration

$$0 = \mathscr{E}_0 \subset \mathscr{E}_1 \subset \ldots \subset \mathscr{E}_r = \varphi^*\mathscr{E}$$

by subvector bundles $\mathscr{E}_i$ such that every quotient $\mathscr{L}_i := \mathscr{E}_i/\mathscr{E}_{i+1}$ is a line bundle.

Proof. (Sketch) We construct Y by induction on $r = \mathrm{rk}(\mathscr{E})$. If $r \in \{0,1\}$, then there is nothing to show. If $r \geq 2$, then define $Y_1 := \mathbb{P}(\mathscr{E}) \xrightarrow{\pi} X$. Then $\pi^*\mathscr{E}$ contains the tautological subbundle $\mathscr{S} \subseteq \pi^*\mathscr{E}$, giving a short exact sequence of vector bundles

$$0 \to \mathscr{S} \to \pi^*\mathscr{E} \to \mathscr{Q} \to 0.$$

Moreover, $\pi^*\colon A(X) \to A(Y_1)$ is injective. Now replace $(X, \mathscr{E})$ by $(Y_1, \mathscr{Q})$... □

Corollary 5.14 (Splitting principle)**.** *Every identity between combinations of Chern classes holds for all vector bundles as soon as it holds for those which are direct sums of line bundles.*

Proof. Combine Lemma 5.13 and Corollary 5.12. □

Example 5.15. We have the following identities:

(1) If $\mathscr{E}$ is a vector bundle of rank r, then
$$c_i(\mathscr{E}) = 0 \text{ for all } i > r.$$
(2) If $\mathscr{E}^\vee$ is the dual bundle of $\mathscr{E}$, then
$$c_i(\mathscr{E}^\vee) = (-1)^r c_i(\mathscr{E}).$$
(3) If $\det(\mathscr{E}) := \bigwedge^r \mathscr{E}$ is the determinant line bundle, then
$$c_1(\det(\mathscr{E})) = c_1(\mathscr{E}).$$
(4) If $\mathscr{E}_i$ has rank r_i for $i \in \{1,2\}$, then
$$c_1(\mathscr{E}_1 \otimes \mathscr{E}_2) = r_2 c_1(\mathscr{E}_1) + r_1 c_1(\mathscr{E}_2).$$

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