SEMINAR ON TOPOS THEORY AND LOGIC

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1. Presheaves and sheaves on topological spaces

First half ([MLMoe, Ch. I, § 1; Ch. II, § 1], [SGA4, Exp. I, § 1]):

- Introduce the category of open subsets of a topological space X, and define the category of *presheaves (of sets)* on X.
- Give examples including the empty set, the singleton, and the Sierpiski space.
- Recall how (co)limits of presheaves are computed, and determine the initial and terminal objects of these categories.
- Construct the *direct and inverse image functors* of presheaves under a continuous map of topological spaces. Discuss their exactness properties, in particular the left exactness of inverse images.
- Specializing to the case when the continuous map is the inclusion of a point, define the notion of *stalk* of a presheaf at a point.

Second half ([MLMoe, Ch. II, §§ 1-3]):

- Introduce the notion of *separated presheaf* and *sheaf* on a topological space X, and define the associated full subcategories of the category of presheaves. Explain how separated presheaves interact with connected components.
- Give examples including the empty set, the singleton, and the Sierpinsky space.
- Explain briefly why the presheaf represented by a fixed topological space Y, as well as the presheaf of local sections of a continuous map $f : P \to X$, are both examples of sheaves; in particular, all representable presheaves on X are in fact sheaves.
- Construct the *sheafification* of a presheaf and prove its universal property. Deduce that the sheafification functor is left adjoint to the inclusion.
- Discuss the exactness properties of this adjunction, in particular the left exactness of the sheafification functor; stress the analogy with the properties of the adjunction between inverse and direct image of presheaves.

2. Functoriality of sheaves

First half ([SGA4, Exp. III, § 3-4]):

- Recall the two kinds of adjunctions introduced in the previous talk: (inverse and direct image of presheaves under a continuous map; sheafification and inclusion) and their exactness properties (in particular, the left exactness of left adjoints).

- Construct the *direct and inverse image* of sheaves under a continuous map. Discuss the exactness properties of this adjunction, in particular the left exactness of inverse images.
- Explain how adjunction interacts with composition of functors in general. Apply this to the case of topological space to define a category where objects are the categories of sheaves $Sh(X)$ associated to topological spaces X and morphisms $Sh(X) \to Sh(Y)$ are adjoint pairs
$$f^* : Sh(Y) \leftrightarrows Sh(X) : f_*$$
with f^* (left) exact. Construct a natural functor from topological spaces to this category.

Second half ([MLMoe, Ch. IX, §§ 1-5], [SGA4, Exp. III, § 3-4]):

- Introduce the notion of *sober* topological space and discuss its relations with the usual separatedness axioms. Define the *soberization* of a topological space and prove its universal property.
- Consider the functor sending a sober topological space to its associated topos (or locale) and construct its left adjoint. Deduce that the associated topos functor is fully faithful. Deduce that the category of topological spaces can be embedded into the category of topoi up to soberization. In particular, explain the relation between homeomorphisms of topological spaces and equivalences of the underlying topoi.

3. Presites and pretopologies

First half ([KaSc, § 17.1], [SGA4, Exp. II § 1], [Vis, § 2.3.1]):

- Recall the main properties of the category of presheaves of sets on a given category.
- Define the notion of *pretopology* on a category (admitting finite limits) and of *presite*.
- Give examples including topological spaces, the *chaotic* pretopology, and the *discrete* pretopology.
- Check that the fibered product of two covering families of a given object is again a covering family of the same object.

Second half ([SGA4, Exp. II §§ 2-3], [Vis, §§ 2.3.3, 2.3.5]):

- Introduce the notion of *separated presheaf* and *sheaf* on a presite, and define the corresponding full subcategories of the category of presheaves.
- Construct the *sheafification* of a presheaf and prove its universal property. Deduce that the sheafification functor is left adjoint the the inclusion.
- Discuss the exactness properties of this adjunction, in particular the left exactness of the sheafification functor.
- Give examples that different pretopologies on the same category may give rise to the same category of sheaves.

4. Sites and topologies

First half ([MLMoe, § III.2], [SGA4, Exp. I, § 4; Exp. II, §§ 1-3]):

- Recall the notion of presite and pretopology from the previous talk, and the example that different pretopologies on the same categories may have the same associated category of sheaves.
- Using the Yoneda Lemma, explain how to every covering family of an object in a presite can be attached a sub-functor of the corresponding representable functor, and introduce the notion of *sieve* on an object in a category.
- Define the notion of *Grothendieck topology* on a category and of *site*.
- Give examples including the *chaotic* topology, the *discrete* topology, and the topology *associated* to a given pretopology.

Second half ([SGA4, Exp. II, § 2]):

- Define the topology *generated* by a family of sieves. Dually, define the *finest* topology for which all presheaves in a given family are separated/sheaves.
- Show that this defines an order-reversing correspondence between Grothendieck topologies and categories of sheaves. Deduce that a pretopology and the associated topology define the same category of sheaves.
- Introduce the notion of *canonical* and *sub-canonical* topology. Explain how to characterize them via the Yoneda embedding.

5. Topoi

First half ([MLMoe, § IV.1], [SGA4, Exp. IV, §§ 1-2]):

- Recall the notion of presite and site from the previous talks.
- Define the notion of *topos* as a category equivalent to the category of sheaves on a site.
- Give some examples, including the category of sets, the trivial category, the category of G-sets, and categories of sheaves on topological spaces.
- Sketch the proof that topoi are the same as left-exact localizations of a presheaf category.

Second half ([MLMoe], [SGA4, Exp. IV, § 1]):

- Introduce *Giraud's axioms* and exaplin why they are satisfied in any topos.
- Sketch the proof of Giraud's theorem characterising topoi in terms of Giraud's axioms.

6. Morphisms of topoi

First half ([MLMoe, Ch. VI, §§ 1, 6-9], [SGA4, Exp. IV, § 1]):

- Recall the definition of topos from the previous talk.
- Introduce the notion of a *(geometric) morphism* of topoi.
- Give examples including those arising from sheafification and those arising from continuous maps of topological spaces.
- Explain how to describe (geometric) morphisms in terms of *continuous functors*.
- Define *open morphisms* and explain how to characterize them.

Second half ([MLMoe, Ch. VI, § 5; Ch. IX, § 11], [SGA4, Exp. III, §§ 6-7; Exp. VI, § 9]):

- Introduce the notion of *point* of a topos.
- Explain what it means for a topos to have *enough points*, and give some examples including the empty topos, presheaf topoi, and topoi associated to topological spaces.
- Mention some examples of topoi without (enough) points.
- Define the notion of *coherent* topos, and give examples including the topos associated to a site of finite type. State Deligne's Theorem on points of coherent topoi.

7. Logic of topoi

First half ([MLMoe, Ch. I, § 4; Ch. III, § 7; Ch. IV, § 1]):

- Introduce the notion of *subobject classifier* in a category.
- Discuss some examples, including the category of sets, and the arrow category of sets.
- Explain how topoi are characterized in terms of subobject classifiers.

Second half ([MLMoe, Ch. X, §§ 1-3]):

- Recall the notion of *first-order theory* and the category of *set-theoretic interpretations and models* of such a theory.
- Introduce the category of *topos-theoretic interpretations and models* of a first-order theory.
- Define the *pull-back* of interpretations along (geometric) morphisms of topoi. Point out that this functor does not preserve models in general. Describe explicitly the pull-back of a formula under an open morphism and deduce that in this case the pull-back functor preserves models.
- Introduce the notion of *geometric formula* in a first-order theory and of *geometric theory*. Describe explicitly the pull-back of a geometric formula under any morphism of topoi and deduce that the pull-back functors preserve models of geometric theories.

8. The syntactic site

First half ([MLMoe, Ch. X, § 4]):

- Introduce the category of *definable objects* for a topos-theoretic model of a geometric theory, describing composition in detail.
- Construct a Grothendieck topology on the category of definable objects and show that it is sub-canonical.

Second half ([MLMoe, Ch. X, § 5]):

- Construct the *syntactic site* of a geometric first-order theory and prove that its topology is sub-canonical.
- For every topos-theoretic model of the theory, construct a morphism from the syntactic category to the category of definable objects and explain its properties.

9. The classifying topos

First half ([MLMoe, Ch. X, § 6, 7]):

- Recall the construction of the syntactic site of a first-order geometric theory.
- Define the associated *classifying topos* and prove its universal property.
- Define the *universal model* of the theory and explain how to recover any other model from it.
- Prove that implications between geometric formulas on topos-theoretic models of the theory can be tested on the universal model.

Second half ([MLMoe, Ch. IX, § 11; Ch. X, § 7], [SGA4, Exp. VI, § 9]):

- Recall the notion of coherent topos. Point out that the classifying topos is coherent as it arises from a site of finite type.
- Recall Deligne's Theorem on points of coherent topoi. Deduce that the classifying topos has enough points.
- Deduce that implications between geometric formulas on topos-theoretic models of the theory can be tested on the set-theoretic models of the theory.
- In particular, recover Gödel's completeness theorem for geometric first order theories.

References

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[MLMoe] S. Mac Lane, I. Moerdijk, *Sheaves in Geometry and Logic: A First Introduction to Topos Theory.* Springer Verlag, 1992.

[SGA4] M. Artin, A. Grothendieck, J.-L. Verdier, *Théorie des topos et cohomologie étale des schémas (SGA 4).* Available online.

[Vis] A. Vistoli, *Notes on Grothendieck topologies, fibered categories, and descent theory.* Available online.