K-THEORY WEDNESDAY SEMINAR PROGRAMME

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Abstract. Detailed programme for the Wednesday Seminar on K-theory held by the GK 1821 "Cohomological Methods in Geometry" during the Summer Term 2021 at the University of Freiburg.

Introduction

The aim of this Wednesday Seminar is to provide an introduction to algebraic and topological K-theory. It is divided into five main parts:

(1) talks 1 to 3 are devoted to the basic K-theory of rings;
(2) talks 4 to 6 introduce the (complex) K-theory of topological spaces;
(3) talks 7 to 9 are devoted to the K-theory of algebraic varieties, leading to the Hirzebruch–Riemann–Roch and Grothendieck–Riemann–Roch theorem;
(4) talks 10 to 11 present the Thom isomorphism in topological K-theory and the Atiyah–Singer index theorem;
(5) finally, talks 12 to 13 are meant to explore some applications of topological K-theory.

There are many analogies between the algebraic and the topological side of the theory, arising from the analogy between projective modules and vector bundles. For example:

- the Serre–Swan theorem relates topological K^0-groups with suitable algebraic K_0-groups;
- in the case of (affine) algebraic varieties, there is an explicit correspondence between projective modules and algebraic vector bundles;
- the formulations of the Hirzebruch-Riemann-Roch theorem and of the Atiyah–Singer theorem are perfectly parallel, and their proofs follow similar patterns.

I hope all these analogies will emerge in the course of the seminar and will contribute to form a clear general picture of the theory.

Each talk should last 90 minutes (taking into account some minutes for questions and discussion at the end of each half). I have opted for this solution in order to cover as many interesting results as possible in the Seminar, but also because I think that a rich and detailed presentation is often more intelligible and satisfying than a synthetic one. I have tried to give each talk a nice and balanced structure; in the case where a talk contains a long proof, I have summarized the logical steps of the proof and indicated how much space to give to each step. In some cases, it may happen that the programme of a talk is a bit too abundant: if this happens, you can skip some intermediate results and proofs in order to save the main result of the talk. Also, if you figure out a better way to present your talk than the one I suggested, feel free to follow your own approach. Finally, many talks contain applications of the theory to computing concrete examples of K-groups: since some of the examples that we will collect along the way are relevant in more than one talk, please try to include all the indicated examples in your talk.

You can write me for any question or doubt concerning the talks, especially those concerning the programme; if you have specific technical questions, I can direct you to Prof. Goette and Prof. Huber.

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1. K_0 of rings (21.04)

References: [Ros, Chapter 1].

1.1. **Defining K_0 from projective modules.** [Ros, Section 1.1]

Recall the definition of *projective module* over a ring R (Definition 1.1.1) and its main characterisation (Theorem 1.2, with a very brief sketch of proof); put particular emphasis on the case of finitely generated modules.

Explain how projective R-modules form a monoid but not a group, then state Theorem 1.3 and sketch the proof; add the remark on functoriality.

Finally, introduce the group $K_0(R)$ (Definition 1.1.5) and explain the functoriality of this construction; comment on the choice of working only with finitely generated modules (Example 1.1.6).

1.2. **Characterising K_0 via idempotents.** [Ros, Section 1.2]

Explain how to construct finitely generated projective R-modules in terms of *idempotent matrices*; state Lemma 1.2.1 and prove it.

Then introduce the ring $M(R)$, the group $GL(R)$ and its subgroup $\mathbf{Idem}(R)$ (Definition 1.2.2); state Theorem 1.2.3 and prove it by reducing to the computation of Lemma 1.2.1.

1.3. **Basic properties of K_0.** [Ros, Section 1.2]

Using Theorem 1.2.3, deduce the *Morita invariance* property of K_0 (Theorem 1.2.4) and its behaviour under *direct products* (Exercise 1.2.8) and *colimits* (Theorem 1.2.5, with a brief sketch of proof).

1.4. **Examples: principal ideal domains.** [Ros, Sections 1.3-1.4]

Determine the structure of $K_0(R)$ when R is a principal ideal domain (Theorem 1.3.1, with a brief sketch of proof).

2. K_1 of rings (28.04)

References: [Ros, Chapter 2].

2.1. **Recollection of previous talks.** Recall the definitions of $M(R)$ and $GL(R)$ from talk 1.

2.2. **Defining K_1.** [Ros, Section 2.1]

Define the group of *elementary matrices* $E(R)$ (Definition 2.1.1); state Lemma 2.1.2, Corollary 2.1.3, and Proposition 2.1.4 and briefly sketch the proofs.

Then introduce the group $K_1(R)$ (Definition 1.2.5) and explain the functoriality of this construction. Point out that the same formal properties as in the case of K_0 hold: *Morita invariance* (Exercise 2.1.8), behaviour under *direct products* (Exercise 2.1.6) and *colimits* (Exercise 2.1.9).

2.3. **Example: fields and Euclidean rings.** [Ros, Section 2.2]

Explain the relation between $K_1(R)$ and the *group of units* $R^\times$ when R is commutative (Proposition 2.2.1, with proof); define the group $SK_1(R)$ measuring their difference as in the subsequent remark.

Determine the structure of $K_1(R)$ when R is a field (Proposition 2.2.2, with sketch of proof) or a Euclidean ring (Theorem 2.3.2, with sketch of proof) by showing the vanishing of $SK_1(R)$ in these cases.

If time permits, mention the existence of a similar result in the case of local rings (Theorem 2.2.5 and Corollary 2.2.6, without proof).

3. K-theory of regular rings (05.05)

References: [Ros, Chapter 3].

Work over a commutative Noetherian ring R for the whole talk.

3.1. **Recollection of previous talks.** Recall briefly the definition of $K_0(R)$ from talk 1 and the definition of $K_1(R)$ from talk 2.

3.2. **G_0 and G_1 of rings.** [Ros, Section 3.1]

Define the notion of a *category with exact sequences* (Definition 3.1.1). Discuss the following examples: projective R-modules (Example 3.1.2(2)), finitely generated R-modules (Example 3.1.2(3)), and R-modules with a finite projective resolution (Example 3.1.3(4)). Then define the notion of *regular ring* and explain why the second and third example above are equivalent for R regular.

Define the groups $K_0(\mathcal{P})$ and $K_1(\mathcal{P})$ associated to a category with exact sequences $\mathcal{P}$ (Definition 3.16); in particular, define the groups $G_0(R)$ and $G_1(R)$. Explain the natural identifications $K_i(R) = K_i(\mathbf{Proj}_R)$ (Theorem 3.1.7, without proof) and deduce the existence of natural group homomorphisms $K_i(R) \to G_i(R)$.

3.3. **K-groups of regular rings.** [Ros, Section 3.1]

State the Resolution Theorems for K_0 (Theorem 3.1.13) and for K_1 (Theorem 3.1.14). Use them, together with Lemma 3.1.15, to deduce that $K_i(R) = G_i(R)$ for a regular ring R (Theorem 3.16, with a sketch of proof).

3.4. **Example: polynomial rings and Laurent polynomial rings.** [Ros, Section 3.2]

Explain that the following properties of rings are preserved passing from a ring R to the polynomial ring $R[t]$ or the Laurent polynomial ring $R[t, t^{-1}]$: noetherianity (Theorem 3.2.1 and Corollary 3.2.2, without proof) and regularity (Theorem 3.2.3 and Corollary 3.2.4, without proof).

Discuss the isomorphisms $G_0(R) \simeq G_0(R[t])$ and $G_0(R) \simeq G_0(R[t, t^{-1}])$ (Theorem 3.2.12, with a sketch of proof). In the same spirit, discuss the isomorphisms $G_1(R) \simeq G_1(R[t])$ (Theorem 3.2.16) and $G_1(R[t, t^{-1}]) \simeq G_1(R) \oplus G_0(R)$ (Theorem 3.2.19).

As an application, compute the K-groups of the polynomial ring $k[t_1, \dots, t_n]$ and of the Laurent polynomial ring $k[t, t^{-1}]$ over a field k starting from the groups $K_0(k)$ and $K_1(k)$ determined respectively in talk 1 and in talk 2.

4. K^0 of topological spaces (12.05)

References: [Atiyah, Chapters 1-2].

4.1. **Recollection of previous talks.** Recall the notion of category with exact sequences and the construction of the associated K_0 group from talk 3.

4.2. **Vector bundles on topological spaces.** [Atiyah, Sections 1.1-1.2]

To fix notation, introduce the notion of *complex vector bundle* over a topological space X and define its rank; recall the notion of *section* of a vector bundle and of homomorphism of vector bundles (Section 1.1). Then show how to extend all operations of linear algebra to this setting, as well as how to pull-back bundles along continuous maps (Section 1.2, without going into the details).

Recall the notion of monomorphism and epimorphism of vector bundles, of sub-bundles and quotient bundles (Section 1.3); in particular, explain how the kernel, cokernel, image of a strict morphism acquire natural structures of vector bundles (Proposition 1.3.2, without proof). Recall the notion of exact sequence of vector bundles (p. 25).

4.3. **Vector bundles on compact Hausdorff spaces.** [Atiyah, Section 1.4]

From now on, work only over compact Hausdorff spaces.

Start by recalling the existence of partitions of unity (p. 15). Deduce that every vector bundle admits a surjection from a suitable trivial bundle (Lemma 1.4.12 and Corollary 1.4.13).

Explain how to construct a *Hermitian metrics* on a bundle using partitions of unity (Lemma 1.4.10). Deduce that every short exact sequence of vector bundles splits (Corollary 1.4.11) and that in particular every vector bundle is a direct summand of a suitable trivial bundle.

4.4. **Defining K_0 from vector bundles.** [Atiyah, Section 2.1]

Explain why the category $Vect_{\mathbb{C}}(X)$ is a category with exact sequences; introduce the group $K^0(X)$ (p. 43) and explain why it is a ring (see the discussion at pp. 42-43).

Show that every element of $K_0(X)$ can be written as $[H] - [I_n]$ for some vector bundle H, where I_n denotes the trivial bundle of rank n; similarly, show that two vector bundles E and F have the same class in $K^0(X)$ if and only if they are *stably equivalent* (p. 44).

Discuss the contravariant functoriality of K^0 with respect to continuous maps.

4.5. **Homotopy invariance and first examples.** [Atiyah, Section 1.4]

Go back to the setting of vector bundles on compact Hausdorff spaces.

Explain that sections of vector bundles defined on a closed subspace can be extended to an open neighbourhood (Lemma 1.4.1). Deduce that the pull-backs of a given vector bundle along homotopic continuous maps are isomorphic (Lemma 1.4.3); in particular, homotopy equivalences induce bijections between isomorphism classes of vector bundles (Lemma 1.4.4(i)).

As a consequence, deduce that two homotopic continuous maps induce the same morphism of K^0-groups; in particular, homotopy equivalences induce isomorphisms of K^0-groups. As a first application, compute $K^0(X)$ when X is contractible.

To get a more interesting example, discuss briefly the relation between vector bundles on the unit circle and connected components of $GL_n(\mathbb{C})$ (Lemma 1.4.9, discuss it briefly only for X a point). As an application, compute the group $K^0(\mathbb{S}^1)$.

4.6. **The Serre-Swan theorem.** [Atiyah, Section 1.4]

Finally, show that the section functor $\Gamma(X, -)$ induces an equivalence between the category of vector bundles over X and the category of finitely generated projective modules over the ring $\mathcal{C}(X, \mathbb{C})$ of continuous complex-valued functions on X (Proposition at p. 31, with a sketch of proof). Deduce the Serre-Swan theorem on the equality $K^0(X) = K_0(\mathcal{C}(X, \mathbb{C}))$ (see [Ros, Theorem 1.6.3]).

If time permits, mention also that we will define a group $\tilde{K}^{-1}(X)$ in talk 6; this turns out to be a quotient of the algebraic K-group $K_1(\mathcal{C}(X, \mathbb{C}))$ (see [Swan, Theorem 17.1]).

5. Bott periodicity (19.05)

References: [Atiyah, Chapters 1-2].

5.1. **Recollection of previous talks.** Recall the definition of $K^0(X)$ for a compact Hausdorff space X and the structure of $K^0(X)$ for X a point and for $X = \mathbb{S}^1$.

5.2. **Projective bundles, and statement of the theorem.** [Atiyah, Section 2.2]

Explain how to construct the *projective bundle* attached to a complex vector bundle $p : E \to X$ and how to find a canonical line sub-bundle $H^* \subset p^*E$ (pp. 46-47). Then state the *Bott periodicity theorem* in its general form (Theorem 2.2.1).

5.3. **Clutching functions.** [Atiyah, Section 1.4]

As a preliminary discussion, explain how one can define vector bundles on a compact space $X = X_1 \cup X_2$ by *clutching* bundles on X_1 and X_2 along an isomorphism ϕ between their restrictions to $X_1 \cap X_2$ (pp. 20-22) and that the final result only depends on the homotopy class of ϕ (Lemma 1.4.6, without proof).

Then explain how, after fixing a Hermitian metrics on a line bundle L, every vector bundle E over $\mathbb{P}(L \oplus I)$ is identified by a certain clutching function f whose homotopy class is uniquely determined by the isomorphism class of E (p. 47).

5.4. **Proof of the main theorem.** [Atiyah, Section 2.2]

Explain the following intermediate steps towards the proof of Theorem 2.2.1:

- introduce the setting of *Laurent clutching function* (pp. 48-51) and explain that every clutching function f can be replaced by a suitable Laurent clutching function f_n defined in terms of the Fourier coefficients of f (Lemma 2.2.4, without proof);
- define the homomorphism $\mathcal{L}^n(p)$ associated to a polynomial clutching function p of degree $\leq n$, and explain the main properties of this construction (Proposition 2.2.5 and Lemma 2.2.6, without proof);
- combine the previous results to deduce the key formula (Proposition 2.2.7, with a brief sketch of proof) which is part of the statement of Theorem 2.2.1;
- next, consider a linear clutching function p: define the associated projection endomorphisms Q^0 and Q^∞ and state their main properties (Proposition 2.2.8, without proof), then deduce a direct sum decomposition for the class of the vector bundle associated to p (Corollary 2.2.9, with a brief sketch of proof).

Then give the final argument of the proof of the theorem (pp. 61-63): combining the two last points above, deduce the formula for the class associated to a generic polynomial clutching function p (pp. 60-61); using the first point above define a ring homomorphism between the two sides of the theorem, and then show that it is an isomorphism by explaining how to construct an inverse (pp. 61-63).

5.5. **Example: the sphere $\mathbb{S}^2$.** [Atiyah, Section 2.2]

State and prove the main consequences of the main theorem (Corollaries 2.2.2 and 2.2.3). As an example, compute the group $\tilde{K}^0(\mathbb{S}^2)$ using the K^0-group of a point discussed in talk 4.

6. Cohomological properties of K-theory (02.06)

References: [Atiyah, Chapter 2]; [Hatcher, Chapter 2].

6.1. **Recollection from previous talks.** Recall the statement of the Bott periodicity theorem both in its general (Theorem 2.2.1) and specific form (Corollary 2.2.3).

6.2. **Relative K-groups, reduced K-groups, and negative K-groups.** [Atiyah, Section 2.4]

Introduce the *reduced group* $\tilde{K}^0(X)$ attached to a compact Hausdorff space X and the *relative group* $K(X, Y)$ associated to a pair of such spaces $X \supset Y$ (p. 66); discuss functoriality of this construction with respect to continuous maps of pairs.

Then recall the smash product construction and the notion of *reduced suspension* (p. 67); use the latter to define the *negative K-groups* (Definition 2.4.1).

Finally, state and prove Lemma 2.4.2, and deduce Corollary 2.4.3.

6.3. **The long exact sequence of K-groups.** [Atiyah, Section 2.4]

State Proposition 2.4.4, asserting that the negative K-groups associated to a pair of compact spaces $X \supset Y$ fit into to a natural long exact sequence; give a sketch of the proof (omitting the proof of Lemma 2.4.6).

Introduce the product pairing of negative K-groups (p. 77) and use it to rephrase the Bott periodicity theorem (Corollary 2.2.3) in the form of Theorem 2.4.9 (with brief sketch of proof). Summarize the picture obtained so far in the form of a *six-terms exact sequence* (p. 78).

Deduce Corollaries 2.4.7 and 2.4.8 from the main result of Proposition 2.4.4.

6.4. **Examples: spheres.** [Hatcher, Section 2.2]

Using the product pairing of reduced K-groups discussed above, deduce the natural isomorphism $\tilde{K}^0(X) \simeq \tilde{K}^0(S^2X)$ (Theorem 2.11). Use this to compute the groups $\tilde{K}^0(\mathbb{S}^n)$ for all $n \geq 0$, starting from the groups $K^0(\mathbb{S}^0)$ and $K^0(\mathbb{S}^1)$ computed in talk 4.

6.5. **Examples: projective bundles.** [Atiyah, Section 2.5]

Explain how to compute $K^0(\mathbb{P}(E))$ in terms of $K^0(X)$ when E is a decomposable vector bundle on X (Proposition 2.5.3, explain the general idea of the proof). As a concrete application, determine the structure of $K^0(\mathbb{P}(\mathbb{C}^n))$ (Corollary 2.5.4).

7. K-theory of algebraic varieties (09.06)

References: [Sh2, Chapter 6].

Work over an algebraically closed field of characteristic 0.

7.1. **Recollection from previous talks.** Recall the definition of K_0 for a category with exact sequences from talk 3, as well as its specializations in the ring-theoretic setting (talk 1) and in the topological setting (talk 4).

7.2. **Coherent sheaves on algebraic varieties.** [Sh2, Sections 6.1-6.3]

Recall preliminarily the notion of *affine variety* and of *quasi-projective variety*; explain that every variety is built up by glueing its Zariski open subsets together.

Then recall the construction of the *structure sheaf* $\mathcal{O}_X$ of an algebraic variety X and stress the fact that the restriction of $\mathcal{O}_X$ to an affine open subset $U \subset X$ is completely determined by its ring of sections $\mathcal{O}_X(U)$.

Introduce the category of sheaves of $\mathcal{O}_X$-modules (Definition on p. 57) and explain how to extend the usual operations of linear algebra to this setting.

Finally, recall how to attach a sheaf $\tilde{M}$ of $\mathcal{O}_X$-modules to any R-module M when $X = Spec(R)$, and introduce the notion of coherent sheaf on a general X; remark that every coherent sheaf on X is of the form $\tilde{M}$ when X is affine.

7.3. **Vector bundles and locally free sheaves.** [Sh2, Section 6.1]

Introduce the notion of *algebraic vector bundle* over an algebraic variety X (Definition on p. 54) and define its rank. Explain how to pull-back vector bundles along morphisms of varieties (p. 55). Then explain how to construct a vector bundle by glueing trivial bundles (p. 55).

Discuss the example of the *tangent bundle* $\mathcal{T}_X$ on a smooth variety X.

Define *sections* of vector bundles (Definition on p. 57) and explain how to attach to every vector bundle over X a locally free sheaf of $\mathcal{O}_X$-modules; show that this constructions defines a bijection between the two sides (Theorem 6.2, with sketch of proof).

7.4. **Defining K^0 from algebraic vector bundles.** [Sh2, Section 6.1]

Introduce the notions of sub-bundles (Definition at page 60) and quotient bundles (p. 61). Along the way, discuss the local form of a short exact sequence of vector bundles (Proposition on p. 60, only explain how to reduce the statement to a problem of free modules over a ring); remark that this result does not hold globally.

Explain why the category of algebraic vector bundles over X is a category with exact sequences. Introduce the corresponding group $K^0(X)$ (see also [FL, p. 102]).

7.5. **Examples: affine varieties.** [Sh2, Section 6.1]

Recall that, if $X = Spec(R)$ is an affine variety, then a finitely generated $\mathcal{O}_X$-module is locally free if and only if it is projective; combining this with the general bijection between vector bundles and locally free sheaves and with the correspondence between coherent sheaves over X and finitely generated R-modules, deduce that $K^0(X) = K_0(R)$. Point out the formal analogy with the Serre–Swan theorem discussed in talk 4.

Using the examples discussed in talks 1 and 3, compute the structure of $K^0(X)$ for $X = \mathbb{A}^n_k$ and $X = \mathbb{A}^1_k \setminus \{0\}$.

8. K-groups and Chow rings (16.06)

References: [EH, Chapters 1, 5, 14]; [FL, Chapters 1 and 5].

8.1. **Recollection from previous talks.** Recall the construction of the group $K^0(X)$ and its ring structure from talk 7.

8.2. **The λ-ring structure of $K_0(X)$.** [FL, Section 1.1]

Point out that the exterior product of vector bundles induces a sequence of ring endomorphisms $\lambda_i : K^0(X) \to K^0(X)$; explain that this makes $K^0(X)$ into a *λ-ring with positive structure* ([FL], pp. 3-4).

Recall the notion of *determinant* of a vector bundle. Show that it defines a group homomorphism $K^0(X) \to Pic(X)$ giving a left inverse to the natural homomorphism $Pic(X) \to K^0(X)$; thus the Picard group $Pic(X)$ can be identified with the group of degree-1 elements of $K^0(X)$ (see [FL], Section 1.1).

8.3. **Review of the Chow ring and Chern classes.** [EH, Sections 1.2, 5.2-5.4]

Recall the definition of the *Chow ring* $A(X)$ of a variety X (Definition 1.3) based on the notion of rational equivalence (Definition 1.2).

Explain the definition of the first Chern class of a line bundle (Section 5.1) and then give the axiomatic definition of *Chern classes* (Theorem 5.3).

Recall the Splitting Principle (Section 5.1); introduce the notation of *Chern roots* of a vector bundle and briefly explain how to work with them.

Note: all these notions and results (except the notion of Chern roots) were already discussed in the previous Wednesday Seminar on Enumerative Geometry.

8.4. **The Chern character and the Todd class.** [EH, Section 14.2]

Using the Chern roots, define the *Chern character* $K^0(X) \to A(X)$ (Subsection 14.2.1); show that the definition makes sense. State Grothendieck's theorem asserting that the induced ring homomorphism $K^0(X) \otimes \mathbb{Q} \to A(X) \otimes \mathbb{Q}$ is an isomorphism when X is smooth and projective (Theorem 14.3).

Using the Chern roots again, define the *Todd class* of a vector bundle (Subsection 14.2.2) and show that it takes sums to products.

9. The Riemann-Roch theorem (23.06)

References: [Gath, Chapters 7, 8, 10].

9.1. **Recollection of previous talks.** Recall the construction of the Chern character and of the Todd class from talk 8.

9.2. **The classical Riemann-Roch theorem for curves.** [Gath, Sections 7.5, 8.3]

Recall the classical version of the *Riemann-Roch theorem* for line bundles on smooth projective curves (Theorem 7.7.3, see the discussion preceding the statement). Using Serre duality (Remark 8.3.4) rephrase the statement of the theorem in terms of sheaf cohomology (Corollary 8.3.3).

9.3. **Statement of the Hirzebruch-Riemann-Roch theorem.** [Gath, Section 10.4]

Recall that to every coherent sheaf $\mathcal{F}$ on a variety X is attached a sequence of cohomology groups $H^i(X, \mathcal{F})$; if X is projective of dimension n then these groups have finite dimension over k and vanish for $i > n$. Define the *Euler characteristic* $\chi(X, \mathcal{F})$ of such a coherent sheaf (Definition 10.4.1) and show that it is additive under short exact sequences. By restricting to the case of locally free sheaves, show that the Euler characteristic determines a group homomorphism $K^0(X) \to \mathbb{Z}$.

State the *Hirzebruch-Riemann-Roch theorem* (Theorem 10.4.5). Explain its meaning in the case of line bundles (Example 10.4.6) and vector bundles (Example 10.4.7) on curves. Verify that both sides of the theorem are additive on short exact sequences and therefore it suffices to check the result on a family of vector bundles whose classes generate the group $K^0(X)$.

9.4. **Proof: the case of projective space.** [Gath, Section 10.4]

Consider first the case $X = \mathbb{P}^n_k$. Recall that in this case the group $K^0(X)$ is generated by the locally free sheaves $\mathcal{O}_X(d)$, $d \geq 0$. Compute explicitly both sides of the Hirzebruch-Riemann-Roch theorem for the sheaf $\mathcal{O}_X(d)$ (Example 10.4.10, with sketch of proof) thus establishing Theorem 10.4.5 in the case of projective space.

9.5. **Proof: the general case.** [Gath, Section 10.5]

Then consider a general smooth projective variety X and fix a closed embedding $i : X \hookrightarrow \mathbb{P}^n_k$.

Sketch the following preliminary reduction argument:

- recall the short exact sequence relating the tangent bundles of the two spaces and the normal bundle of the immersion (see for example [Gath] p. 115);
- recall from talk 7 that the groups $K^0(X)$ and $K^0(\mathbb{P}^n_k)$ are the same as the K^0-groups attached to the categories of coherent sheaves - or more concretely that, for every coherent sheaf $\mathcal{F}$ on X, the coherent sheaf $i_*\mathcal{F}$ on $\mathbb{P}^n_k$ admits a finite locally free resolution (Lemma 10.5.3, without proof);
- use this result to construct the homomorphism $i_! : K^0(X) \to K^0(\mathbb{P}^n_k)$, form the diagram of Remark 10.5.5 and reduce the conclusion of Theorem 10.4.5 for X to the statement of Proposition 10.5.6.

Finally, explain the idea of the proof of Proposition 10.5.6 by considering first the special case of projective bundles (Example 10.5.7, sketch only the key computations) and then the general case (proof at p. 207, sketch only the main geometric idea).

9.6. **The Grothendieck-Riemann-Roch theorem.** [Gath, Section 10.5]

As a final comment, explain how the statement of Proposition 10.5.6 can be seen as a particular case of the general *Grothendieck-Riemann-Roch theorem* (Remark 10.5.8, see also [FL, Introduction and Section 5.7]).

10. The Thom isomorphism (30.06)

References: [Atiyah, Chapter 2].

10.1. **Recollection from previous talks.** Recall the statement of Proposition 2.4.4 and the six-terms exact sequence from talk 6; introduce the notation K^* and $\tilde{K}^*$ (p. 78).

10.2. **Thom spaces.** [Atiyah, Sections 2.6-2.7]

Recall the structure of the group $\tilde{K}^0(\mathbb{S}^{2n})$ determined in talk 6, and explain how to describe its canonical generator in terms of the exterior algebra on a n-dimensional $\mathbb{C}$-vector space V.

Then introduce the *Thom space* X^E attached to a general complex vector bundle E over X; generalising the previous example, define the corresponding canonical element $\lambda_E \in \tilde{K}(X^E)$ and explain the properties of this construction (pp. 99-100).

Explain the alternative interpretation of X^E in terms of the projective bundle $\mathbb{P}(E \oplus I)$; give the explicit formula for the image of λ_E under the homomorphism $\tilde{K}^0(X^E) \to K^0(\mathbb{P}(E \oplus I))$ (pp. 100-101).

Finally, state the *Thom isomorphism theorem* (Corollary 2.7.12).

10.3. **Proof: the case of decomposable vector bundles.** [Atiyah, Section 2.7]

If E is a decomposable vector bundle over X, recall how to compute $K^0(\mathbb{P}(E))$ in terms of $K^0(X)$ from talk 6. Deduce how to write $\tilde{K}^*(\mathbb{P}(E))$ as a $K^*(X)$-algebra (Proposition 2.7.1, with a brief sketch of proof).

As a consequence, verify that the Thom isomorphism theorem holds for decomposable vector bundles (Proposition 2.7.2, with a brief sketch of proof) .

10.4. **Proof: the general case.** [Atiyah, Section 2.7]

Explain the technical result of Theorem 2.7.8 (with sketch of proof).

Combining this with the results in the decomposable case, deduce that similar results hold in general (Corollary 2.7.9 and Corollary 2.7.12, with proof).

11. The Atiyah-Singer theorem (07.07)

References: [Sha, Chapters 1-3].

11.1. **Recollection from previous talks.** Recall the statement of the Hirzebruch-Riemann-Roch theorem from talk 9.

11.2. **Chern classes, Euler classes, and the analytic index.** [Sha, Sections 1.1-1.2]

Introduce the Chern classes and Chern character attached to complex vector bundles over compact spaces (pp. 1-2); give the formula defining the Todd class and discuss the interpretation of the first Chern class of a line bundle (p. 3). Point out the strong analogy with the formalism of Chern classes in the algebraic setting, discussed in talk 8.

Introduce the *Thom class* and the *Euler class* of a vector bundle and discuss their geometric interpretation (pp. 6-7). In the case of smooth manifolds, explain how the Euler characteristic of a space is related to the Euler class of its tangent bundle (Theorem A on p. 8); discuss the reformulation in terms of de Rham cohomology (p. 9) and explain its meaning.

Then introduce the notion of *analytic index* associated to a complex of differential operators (p. 10) and state the extension of Theorem A to this general setting (Theorem A').

11.3. **Statement of the theorem.** [Sha, Section 1.3]

Introduce the general notion of *linear partial differential operator* on a smooth manifold and describe its local form (p. 12).

Define the *symbol* of an operator in terms of linear algebra (p. 13). Then introduce the notion of *elliptic operator* (p. 14) and of *elliptic sequence* (p. 16); describe geometrically the corresponding symbols (pp. 15-16).

Finally, state the *Atiyah-Singer index theorem* and discuss the remarks (p. 17).

11.4. **Applications.** [Sha, Sections 2.4-2.7]

Discuss briefly some of the applications of the index theorem to classical differential operators, choosing among: de Rham (Section 2.4), Dolbeaut (Section 2.5), Hodge (Section 2.6), Dirac (Section 2.7).

11.5. **Alternative construction of $K^0(X)$.** [Sha, Sections 3.1, 3.11]

Recall the definition of $K^0(X)$ for a locally compact space X (p. 61) and explain the alternative construction of this group in terms of triples (pp. 61-62, with a brief sketch of the correspondence between the two definitions).

Then explain how to recover the symbol of a partial differential operator in this notation (Example on p. 64). In fact, explain that every element of $K^0(TX)$ is the symbol of a suitable operator (pp. 88-90, sketch only the main ideas).

Explain that two differential operators with the same symbol also have the same index (pp. 90-92, sketch only the main ideas). Use this to define the index homomorphism $K^0(TX) \to \mathbb{Z}$.

11.6. **The topological index.** [Sha, Section 3.9]

Define the homomorphism $i_! : K^0(X) \to K^0(Y)$ associated to a closed embedding $i : X \hookrightarrow Y$ (pp. 72-73, explain briefly that the normal bundle admits a complex structure); deduce the homomorphism $i_! : K^0(TX) \to K^0(TY)$.

Introduce the *topological index* $B : K^0(TX) \to \mathbb{Z}$ (pp. 74-75), discuss its basic properties (Proposition on p. 75, without proof) and give the explicit formula in terms of the Todd class of TX (Proposition at the end of p. 75, without proof).

Using the last result, rephrase the Atiyah-Singer theorem as a relation between the analytic and the topological index (pp. 77-78).

11.7. **Outline of proof.** [Sha, sect. 3.12]

Explain how to reduce the index theorem to the statement that the index homomorphism is compatible with the homomorphism $i_!$ (p. 94, state properties (1) and (2), and prove (1)). Conclude explaining the broad idea of the proof of the latter statement (pp. 94-109, without going into the details).

12. Division algebras and parallelizable spheres (14.07)

References: [Hatcher, Chapter 2].

12.1. **Recollection from previous talks.** Recall the statement and meaning of the Bott periodicity theorem from talk 5, as well as the structure of $\tilde{K}^0(\mathbb{S}^n)$ from talk 6 (see also the remaks on p. 60).

12.2. ***H*-spaces and the Hopf invariant.** [Hatcher, sect. 2.3]

Start with the statement of the main theorem (Theorem 2.16).

Then introduce the notion of *H-space* (p. 59) and explain its relation with the theorem (Lemma 2.17, with proof). Also, recall the consequences of Bott periodicity and deduce that $\mathbb{S}^n$ is not an H-space for $n > 0$ even (p. 60, with proof).

For every $n > 0$, construct the *Hopf invariant* attached to a map $f : \mathbb{S}^{4n-1} \to \mathbb{S}^{2n}$ and show that it is well-defined (p. 61). Then discuss the Hopf invariant of a H-space multiplication (Lemma 2.18, with proof). Use this to reduce the statement of Theorem 2.16 to that of Theorem 2.19.

12.3. **Adams operations.** [Hatcher, sect. 2.3]

Recall that the exterior powers of vector bundles determine a λ-ring structure on $K^0(X)$: this was explained in talk 8 in the setting of algebraic varieties, but the same argument works for topological spaces.

Introduce the *Adams operations* acting on the ring $K^0(X)$ for every compact Hausdorff space X (Theorem 2.20): first define them on decomposable vector bundles, then use the Splitting Principle to obtain the general construction (discuss the proof of Theorem 2.20).

Explain how the Adams operations interact with pull-backs (p. 64) and then compute their action on $\tilde{K}^0(\mathbb{S}^{2n})$ (Proposition 2.21, with proof).

12.4. **Proof of the theorem.** [Hatcher, sect. 2.3]

Finally, explain how to derive the thesis of Theorem 2.19 (p. 65, including the proof of Lemma 2.22).

13. Microbundles and their K-theory (21.07)

References: [ACKKNPRR]; [KM]; [KS]; [Mil].

Microbundles are a topological version of vector bundles without linear structure. They have been introduced by Milnor to describe tangent bundles of topological manifolds. Since they are sufficiently analogous to vector bundles, it makes sense to consider their K-theory [Mil, §4]. There is a forgetful functor from KO-theory (the K-theory of $\mathbb{R}$-vector bundles) to microbundle K-theory. If M is a topological manifold, then modulo certain equivalence relations to be explained, preimages in $KO^0(M)$ of the tangent microbundle class $[\mathfrak{t}_M]$ are in one-to-one correspondence with smooth structures on M. The purpose of this talk is to give a brief overview.

The first half of the talk should introduce microbundles. It should be said first that KO-theory is formally completely analogous to topological K-theory as discussed in the seminar. Then I suggest to follow [Mil] and to give the definition, examples 2 and 3 and Theorem 2.2 as an introduction, explain the "stable smoothing Theorem" 5.12 and possibly sketch its proof, which spans the whole section 5. Another good reference is [ACKKNPRR, sect. 9].

The second half of the talk should sketch the relation between microbundle K-theory and smoothing theory. Existence and uniqueness of smooth structures [KS, IV, Thm 4.1] follows from Milnor's result and the product structure theorem [KS, I, Thm 5.1], see also [ACKKNPRR, sect 16]. To highlight the difference between microbundle K-theory and KO-theory, one could mention exotic spheres [KM], see also [ACKKNPRR, Lemma 17.1], and also microbundles without a linear structure [Mil, Thm 8.1].

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