THE THOM ISOMORPHISM

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ABSTRACT. Script for a talk of the Wednesday Seminar of the GK1821 at Freiburg during the Summer Semester 2021. The main reference is [Ati67, §2].

CONTENTS

—parts in gray will be omitted during the talk—

1. Recollection from previous talks

We start off with some recollections from [Ati67, §2]. We refer to [Ati67] and the appendix for previous conventions and other preliminaries.

1.1. **The functor $K(-)$ [Ati67, p. 44].** Let **CHaus** be the category of compact Hausdorff topological spaces and let **Ring** be the category of commutative unital rings. We have defined a functor

$$K\colon \mathbf{CHaus} \to \mathbf{Ring}$$

which can be explicitly described as follows. If $E \in \mathrm{Vect}(X)$ is a vector bundle on a compact Hausdorff space X, then we denote by $[E]$ its stable equivalence class, i.e.,

$$[E] := \{F \in \mathrm{Vect}(X) \mid \exists n \in \mathbb{N} \text{ s.t. } E \oplus n \cong F \oplus n\},$$

where we denote also by n the trivial vector bundle of rank n on X. Then the underlying set of the ring $K(X)$ consists of formal differences $[E] - [F]$ for vector bundles $E, F \in \mathrm{Vect}(X)$. The sum is given by

$$([E] - [F]) + ([E'] - [F']) = [E \oplus E'] - [F \oplus F'],$$

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and the product is given by

$$([E]-[F])([E']-[F'])=[(E\otimes E')\oplus(F\otimes F')]-[(E\otimes F')\oplus(E'\otimes F)].$$

The element zero can be represented by $[0]-[0]$ and the element one by $[1]-[0]$, where again by 0 we mean $X\times\{0\}$ and by 1 we mean $X\times\mathbb{C}$. Moreover, since X is compact and Hausdorff, we can represent every element of $K(X)$ as $[E]-[n]$ for some $E\in\mathrm{Vect}(X)$ and some $n\in\mathbb{N}$.

If $f\colon X\to Y$ is a continuous map between compact Hausdorff topological spaces, then $K(f)=: f^*$ is given by

$$\begin{aligned} f^*\colon K(Y)&\to K(X)\\ [E]-[F]&\mapsto[f^*(E)]-[f^*(F)], \end{aligned}$$

and this ring homomorphism only depends on the homotopy class of f.

1.2. **The functor $\tilde{K}(-)$ [Ati67, p. 66].** Let **CHaus**$^+$ be the category of pointed compact Hausdorff topological spaces and let **Rng** be the category of commutative non-unital rings. We have defined a functor

$$\tilde{K}\colon \mathbf{CHaus}^+\to\mathbf{Rng}$$

as follows. Let (X,x_0) be pointed compact Hausdorff topological space and let $i\colon\{x_0\}\to X$ denote the inclusion of the base point.

Appendix A. Conventions and preliminaries

A.1. **Construction of $K(X)$.** The Grothendieck group is defined via universal property, but let us agree on a specific construction in order to have a precise description of the elements in the ring $K(X)$. We follow both [Ati67] and [Hat03] and consider the construction 1 described by Jin in the first talk of the seminar, which is the second construction discussed by Atiyah in [Ati67, p. 42].

Lemma 1. *Let M be a commutative monoid. Then Jin's construction 1 agrees with Atiyah's second construction of $K(M)$.*

Proof. In both cases $K(M)$ is the quotient of $M\times M$ by an equivalence relation, so it suffices to show that the equivalence relations agree. In Atiyah's construction we have

$$(x,y)\sim_A(x',y') :\Leftrightarrow \exists z,z'\in M,,(x+z,y+z)=(x'+z',y'+z').$$

In Jin's construction we have

$$(x,y)\sim_J(x',y') :\Leftrightarrow \exists z\in M,,x+y'+z=x'+y+z.$$

If $(x,y)\sim_A(x',y')$, then we have $x+z=x'+z'$ and $y+z=y'+z'$ for some $z,z'\in M$. Associativity and commutativity of M imply that

$$x+y'+z+z'=x'+y'+z'+z'=x'+y+z+z',$$

hence $(x,y) \sim_J (x',y')$. Conversely, if $(x,y) \sim_J (x',y')$, then we have $x+y'+z = x'+y+z$ for some $z \in M$. In particular we have

$$\begin{aligned}(x+(x+y'+z), y+(x+y'+z)) &= (x+(x'+y+z), y'+(x+y+z)) \\ &= (x'+(x+y+z), y'+(x+y+z)),\end{aligned}$$

so $(x,y) \sim_A (x',y')$ as well. □

Given (an isomorphism class of) a vector bundle $E \in \mathrm{Vect}(X)$, we denote by $[E]$ its image in $K(X)$, that is, $[E] = [(E,0)]$. Since $-[E] = [(0,E)]$, we can write every element $[(E,F)] \in K(X)$ as $[E]-[F]$. We can find some vector bundle G such that $F \oplus G$ is trivial [Ati67, Corollary 1.4.14]. With the notation introduced earlier we can write $[F \oplus G] = [\underline{n}]$ for some $n \in \mathbb{N}$. Then we would have

$$[E]-[F] = [E]+[G]-([F]+[G]) = [E \oplus G] - [\underline{n}],$$

showing that every element of $K(X)$ can be written as $[H]-[\underline{n}]$ for some vector bundle H on X and some natural number $n \in \mathbb{N}$ [Ati67, p. 44].

Suppose now that E and F are such that $[E]=[F]$, that is, $[(E,0)] = [(F,0)]$. By definition of the equivalence relation that we are using, there exists some vector bundle G such that $E \oplus G \cong F \oplus G$. Applying [Ati67, Corollary 1.4.14] again we deduce that $E \oplus \underline{n} \cong F \oplus \underline{n}$ for some $n \in \mathbb{N}$. In this case we say that E and F are *stably equivalent*. This brings us to Hatcher's description of $K(X)$ [Hat03, p. 39], namely, as formal differences $E - E'$ in which we identify $E_1 - E_1'$ with $E_2 - E_2'$ if and only if $E_1 \oplus E_2'$ and $E_2 \oplus E_1'$ are stably equivalent, that is, if and only if $[E_1 \oplus E_2'] = [E_2 \oplus E_1']$. Since $[E_1 \oplus E_2'] = [E_1] + [E_2']$ and $[E_2 \oplus E_1'] = [E_2]+[E_1']$, we do have $E_1 - E_1' = E_2 - E_2'$ in Hatcher's sense if and only if $[(E_1,E_1')] = [(E_2,E_2')]$ in Atiyah's sense. We will try to follow Atiyah's notation most of the time.

References

[Ati67] M. F. Atiyah. *K-theory*. Lecture notes by D. W. Anderson. W. A. Benjamin, Inc., New York-Amsterdam, 1967.

[Hat03] A. Hatcher. *Vector Bundles and K-Theory*. 2003. http://www.math.cornell.edu/~hatcher.

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