THE THOM ISOMORPHISM

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Abstract. Script for a talk of the Wednesday Seminar of the GK1821 at Freiburg during the Summer Semester 2021. The main reference is [Ati67, §2].

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1. Recollection from previous talks

We start off with some recollections from [Ati67, §2].

1.1. **Notation and topological preliminaries.** For $n \in \mathbb{N}$ we denote by S^n the n-dimensional sphere. We think of it as the pointed compact Hausdorff space $I^n/\partial I^n$, where $I^n \subseteq \mathbb{R}^n$ is the unit cube and the basepoint is given by the equivalence class of any point in the boundary ∂I^n. The unit interval $I^1 = [0,1]$ is denoted simply by I.

Given pointed compact Hausdorff spaces (X, x_0) and (Y, y_0), we define their *smash product* as

$$X \wedge Y := X \times Y / X \vee Y,$$

where $X \vee Y := X \times \{y_0\} \cup \{x_0\} \times Y$ is their *wedge sum*. The *reduced suspension* of (X, x_0) is obtained from the usual suspension $(I \times X)/(\{0\} \times X \cup \{1\} \times X)$ by further collapsing the line $I \times \{x_0\}$ joining the two vertices of the suspension along the basepoint. Using our explicit description of S^1, we may rewrite the reduced suspension of

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X as $S^1 \wedge X$, and we denote it by SX. The n-th iterated suspension[1] is naturally homeomorphic to $S^n \wedge X$, and we denote it by $S^n X$. Finally, the *cone* of the pointed compact Hausdorff space (X, x_0) is defined as $CX := (I \times X)/(\{0\} \times X)$, and we take as basepoint the equivalence class of any point in the subspace that we are collapsing [Ati67, p. 68].

Following [Ati67, p. 66], we denote by $\mathcal{C}$ the category of compact Hausdorff spaces, by $\mathcal{C}^+$ the category of pointed compact Hausdorff spaces and by $\mathcal{C}^2$ the category consisting of pairs (X, Y) where X and Y are compact Hausdorff spaces and $Y \subseteq X$ is a subspace. Equivalently, X is a compact Hausdorff space and $Y \subseteq X$ is a closed subset.

We will consider complex vector bundles over compact Hausdorff spaces. If E is a vector bundle over X, then we will often abuse notation and denote again by E its isomorphism class. We will denote by $p\colon P(E) \to X$ the *projective bundle associated to* E, by $H^* \subseteq p^*E$ the *tautological line bundle* and by $H := (H^*)^*$ its dual, which we will call the *hyperplane bundle* [Ati67, p. 45].

1.2. **The functor** $\mathrm{Vect}(-)$**.** For a compact Hausdorff space X, we denote by $\mathrm{Vect}(X)$ the set of isomorphism classes of vector bundles on X. This set becomes a commutative monoid with respect to direct sum of vector bundles; the zero vector bundle is the neutral element of this monoid. Moreover, tensor product of vector bundles induces a multiplication which turns this commutative monoid into a commutative semiring; the trivial line bundle is the unit in this semiring.

Remark 1. We need to talk about isomorphism classes of vector bundles—and not just about vector bundles—in order to ensure that the axioms of a commutative semiring are satisfied, e.g. the addition would not be commutative otherwise.

If $f\colon X \to Y$ is a continuous function between compact Hausdorff spaces, then the pullback along f defines a morphism

$$f^*\colon\ \mathrm{Vect}(Y) \to \mathrm{Vect}(X)$$

of semirings with unit. We have $\mathrm{id}_X^* = \mathrm{id}_{\mathrm{Vect}(X)}$ and $(g \circ f)^* = f^* \circ g^*$, so we obtain a functor

$$\mathrm{Vect}\colon \mathcal{C}^{opp} \to \mathbf{SemiRing},$$

where **SemiRing** denotes the category of commutative semirings with unit and $(-)^{opp}$ denotes the opposite category.

Remark 2. Again, we need to talk about isomorphism classes of vector bundles in order to ensure that the axioms of a functor are satisfied, e.g. the composition would not be sent to the composition otherwise.

[1]Smash product is associative on compact spaces, but not in general elaborate!

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1.3. **The functor** $K(-)$**.** We follow [Ati67, p. 44] in this subsection. The Grothendieck group construction yields a functor $G\colon \mathbf{SemiRing} \to \mathbf{Ring}$, where $\mathbf{Ring}$ denotes the category of commutative rings with unit. The ring $G(A)$ corresponding to a semiring A can be described in various ways; let us fix one description for concreteness. As an abelian group, $G(A)$ is given by $A \times A/\Delta(A)$, where $\Delta\colon A \to A \times A$ is the diagonal morphism. Thus, the set $G(A)$ consists of equivalence classes of pairs $(a, b) \in A \times A$, where $(a, b) \sim (a', b')$ if and only if there exists $z, z' \in A$ such that

$$(a + z, b + z) = (a' + z', b' + z').$$

The addition in $G(A)$ is given by $[(a, b)] + [(a', b')] = [(a + a', b + b')]$. The neutral element of the group $G(A)$ is then the equivalence class of (a, a) for any $a \in A$, and the inverse of $[(a, b)]$ is $[(b, a)]$. We can think of $[(a, b)]$ as $a - b$. More precisely, if for each $a \in A$ we denote by $[a]$ its image in $G(A)$, i.e. $[a] := [(a, 0)]$, then $[(a, b)] = [a] - [b]$. With this in mind we see that the induced multiplication in $G(A)$ is given by

$$[(a, b)][(a', b')] = [(aa' + bb', ab' + a'b)],$$

and the unit of the ring $G(A)$ is $[(1_A, 0)]$. The natural monoid morphism $A \to G(A)$ given by $a \mapsto [a]$ is then also a morphism of semirings with unit. And if $\varphi\colon A \to B$ is a morphism of semirrings with unit, then the induced $G(\varphi)\colon G(A) \to G(B)$ given by $[(a, b)] \mapsto [(\varphi(a), \varphi(b))]$ is a morphism of commutative rings with unit. Thus we have a functor $G\colon \mathbf{SemiRing} \to \mathbf{Ring}$ as claimed. The functor

$$K\colon \mathcal{C}^{opp} \to \mathbf{Ring}$$

is then defined as the composition $G \circ \mathrm{Vect}$.

To make things more explicit, let X be a compact Hausdorff space. Then an element of the ring $K(X)$ is an equivalence class $[(E, F)]$ where E and F are (isomorphism classes of) vector bundles on X. As explained above, we may rewrite this element as $[E]-[F]$. The addition in $K(X)$ is then given by

$$([E] - [F]) + ([E'] - [F']) = [E \oplus E'] - [F \oplus F'],$$

and the product is given by

$$([E] - [F])([E'] - [F']) = [(E \otimes E') \oplus (F \otimes F')] - [(E \otimes F') \oplus (E' \otimes F)].$$

For $n \in \mathbb{N}$, we denote by $[n]$ the image of $X \times \mathbb{C}^n$ in $K(X)$. The element zero can be represented by $[0] - [0]$ and the element one by $[1] - [0]$, where again by 0 we mean $X \times \{0\}$ and by 1 we mean $X \times \mathbb{C}$. Moreover, since X is compact and Hausdorff, we can represent every element of $K(X)$ as $[E] - [n]$ for some $E \in \mathrm{Vect}(X)$ and some $n \in \mathbb{N}$ [Ati67, p. 44].

If $f\colon X \to Y$ is a continuous map between compact Hausdorff topological spaces, then $K(f) =: f^*$ is given by

$$\begin{aligned} f^*\colon K(Y) &\to K(X) \\ [E]-[F] &\mapsto [f^*(E)]-[f^*(F)], \end{aligned}$$

and this ring morphism only depends on the homotopy class of f.

1.4. **The functor $\tilde{K}(-)$.** We follow [Ati67, p. 66] in this subsection. Let **Rng** be the category of commutative non-unital rings. We define a functor $\tilde{K}\colon \mathcal{C}^+ \to \mathbf{Rng}$ as follows. Let (X, x_0) be a pointed compact Hausdorff topological space and let $i\colon \{x_0\} \to X$ denote the inclusion of the basepoint. Then

$$\tilde{K}(X) := \ker(i^*) \subseteq K(X).$$

Remark 3. Let $\xi = [E]-[F] \in K(X)$. Then $\xi \in \tilde{K}(X)$ if and only if $\dim(E_{x_0}) = \dim(F_{x_0})$. Indeed, it suffices to show that the difference $\dim(E_{x_0}) - \dim(F_{x_0})$ does not depend on the representative $[E]-[F]$ of the equivalence class ξ. But if $[(E,F)] = [(E',F')]$, then there exist $G, G' \in \mathrm{Vect}(X)$ such that $E \oplus G = E' \oplus G'$ and $F \oplus G = F' \oplus G'$. Hence $\dim(E_{x_0}) - \dim(F_{x_0}) = \dim(E'_{x_0}) - \dim(F'_{x_0})$.

If $c\colon X \to \{x_0\}$ is the constant morphism to the basepoint, then c^* induces a splitting $K(X) \cong \tilde{K}(X) \oplus K(x_0)$ which is natural with respect to morphisms in $\mathcal{C}^+$. Hence $\tilde{K}(-)$ is a functor as claimed above. If $f\colon (X,x_0) \to (Y,y_0)$ is a morphism in $\mathcal{C}^+$ given by a continuous function $f\colon X \to Y$ such that $f(x_0) = y_0$, then $\tilde{K}(f) =: f^*$ is induced by the restriction of $f^*\colon K(Y) \to K(X)$. One can also check directly that $f^*\colon K(Y) \to K(X)$ induces $f^*\colon \tilde{K}(Y) \to \tilde{K}(X)$ using the previous remark and the definition of pullbacks of vector bundles.

Remark 4. We can recover the functor $K(-)$ from the functor $\tilde{K}(-)$ by adding disjoint basepoints. More precisely, for any compact Hausdorff space X, there is a ring isomorphism

$$\begin{aligned} K(X) &\to \tilde{K}(X^+) \\ [E]-[F] &\mapsto [E \sqcup 0]-[F \sqcup 0] \end{aligned}$$

where by 0 we mean $\{x_0\} \times \mathbb{C}^0$, so that $E \sqcup 0$ is a vector bundle over $X^+ := X \sqcup \{x_0\}$.

Remark 5. As the previous remark already suggests, $\tilde{K}(X^+)$ is a unital ring for all non-empty X. Its unit is given by the image of the unit in $K(X)$ under the previous ring isomorphism, i.e. it is given by $[1 \sqcup 0] - [0 \sqcup 0]$. But if (X, x_0) is a connected pointed compact Hausdorff space, then $\tilde{K}(X)$ cannot have a unit $1 \neq 0$, because every element in $\tilde{K}(X)$ is nilpotent [Kar78, Theorem II.5.9].

1.5. **The functor $K(-,-)$.** We follow [Ati67, p. 66] in this subsection. We define a functor $K\colon \mathcal{C}^2 \to \mathbf{Rng}$ as follows. For an object (X,Y) in $\mathcal{C}^2$, we set

$$K(X,Y) := \tilde{K}(X/Y),$$

where the basepoint of X/Y is the equivalence class of any point $y \in Y$, which we denote by Y/Y. If $f\colon (X,Y) \to (Z,W)$ is a morphism in $\mathcal{C}^2$ given by a continuous function $f\colon X \to Z$ such that $f(Y) \subseteq W$, then it induces a morphism $\bar{f}\colon (X/Y, Y/Y) \to (Z/W, W/W)$ in $\mathcal{C}^+$. Then $K(f) =: f^*$ is given by $\tilde{K}(\bar{f})$.

Remark 6. We can recover the functor $K(-)$ from the functor $K(-,-)$ by considering the empty subspace. More precisely, for any compact Hausdorff space X, using our convention[2] that $X/\varnothing = X^+$, we have again the ring isomorphism discussed above

$$K(X) \to \tilde{K}(X^+) =: K(X, \varnothing).$$

1.6. **The six-term exact sequence.**

Definition 7 ([Ati67, Definition 2.4.1])**.** For $n \in \mathbb{N}_{>0}$ we define a functor $\tilde{K}^{-n}\colon \mathcal{C}^+ \to \mathbf{Rng}$ by setting

$$\tilde{K}^{-n}(X) := \tilde{K}(S^n X).$$

Then we define a functor $K^{-n}\colon \mathcal{C}^2 \to \mathbf{Rng}$ via

$$K^{-n}(X,Y) := \tilde{K}^{-n}(X/Y),$$

and finally a functor $K^{-n}\colon \mathcal{C} \to \mathbf{Rng}$ with the formula

$$K^{-n}(X) := K^{-n}(X, \varnothing).$$

Remark 8. For $n = 0$ we have seen the relations

$$K^0(-) = \tilde{K}^0((-)^+) = K^0(-, \varnothing).$$

Let now $n \in \mathbb{N}$ be arbitrary. Then we still have $K^{-n}(-) = K^{-n}(-, \varnothing)$ by definition. We also have $K^{-n}(-, \varnothing) = \tilde{K}^{-n}((-)^+)$ by definition, so we still have the same relations

$$K^{-n}(-) = \tilde{K}^{-n}((-)^+) = K^{-n}(-, \varnothing).$$

Remark 9. Let (X, x_0) be a pointed compact Hausdorff space. Then we have a canonical group isomorphism $K^{-1}(X) \cong \tilde{K}^{-1}(X)$. Unlike in the case of singular homology and its reduced counterpart, the existence of this isomorphism is not immediate. For example, this is already non-trivial in the case of a point. By definition we have $K^{-1}(\{x\}) = \tilde{K}(S(\{x\} \sqcup \{x_0\})) = \tilde{K}(S^1)$. The isomorphism is then true despite the

[2]For $Y \subseteq X$ we may define X/Y as the pushout $X \sqcup_Y \{*\}$. Then we really have $X/\varnothing = X \sqcup \{*\}$.

spaces being so different, because $\tilde{K}(\{x\}) = \tilde{K}(S^1) = 0$. The general case follows from the isomorphisms

$$\tilde{K}(S(X^+)) = \tilde{K}(S(X) \vee S^1) \cong \tilde{K}(SX) \oplus \tilde{K}(S^1) \cong \tilde{K}(SX),$$

see [Hat03, p. 57].

Recall from Vera's talk:

Proposition 10 ([Ati67, Proposition 2.4.4])**.** *For each pair (X, Y) in $\mathcal{C}^2$ there is a natural exact sequence*

$$\cdots \to K^{-1}(Y) \xrightarrow{\delta} K^0(X,Y) \xrightarrow{j^*} K^0(X) \xrightarrow{i^*} K^0(Y),$$

where $i\colon Y \to X$ and $j\colon (X, \varnothing) \to (X, Y)$ are the inclusions.

Remark 11 ([Ati67, p. 87]). The morphism $\delta\colon K(S(Y), \varnothing) \to K(X, Y)$ is also induced by a morphism in $\mathcal{C}^2$, namely, $\delta = \iota^*$ for the inclusion

$$\iota\colon (\{1\} \times X \cup I \times Y, \{0\} \times Y) \hookrightarrow (\{1\} \times X \cup I \times Y, \{0\} \times Y \cup \{1\} \times X),$$

modulo some topology allowing us to identify the reduced K-groups of the corresponding quotients with the reduced K-groups that we want.

Using the periodicity ringisomorphisms $\beta\colon K^{-n}(X) \to K^{-n-2}(X)$ from [Ati67, Theorem 2.4.9], or rather their reduced version $\beta\colon \tilde{K}^{-n}(X) \to \tilde{K}^{-n-2}(X)$, we extend the collection of functors $\{K^{-n}(-,-)\}_{n\in\mathbb{N}}$ to a collection of funcotrs $\{K^{n}(-,-)\}_{n\in\mathbb{Z}}$ inductively, i.e. $K^1 = K^{-1}$, $K^2 = K^0$, $K^3 = K^1 = K^{-1}$, and so on. Then we use the previously seen relations among the different K functors to define collections of functors $\{K^n(-)\}_{n\in\mathbb{Z}}$ and $\{\tilde{K}^n(-)\}_{n\in\mathbb{Z}}$ as well. This allows us to extend the previous exact sequence to the right, and identifying K^n with K^{n+2} for all $n \in \mathbb{Z}$ we may rearrange this data into the following *six-term exact sequence*

$$\begin{tikzcd}
K^0(X,Y) \arrow[r] & K^0(X) \arrow[r] & K^0(Y) \arrow[d] \\
K^1(Y) \arrow[u] & K^1(X) \arrow[l] & K^1(X,Y) \arrow[l]
\end{tikzcd}$$

2. The functor $K^*(-,-)$

Definition 12. Let X be a compact Hausdorff space. Then we define

$$K^*(X) := K^0(X) \oplus K^1(X).$$

Similarly, for a pair (X, Y) in $\mathcal{C}^2$ we define

$$K^*(X,Y) := K^0(X,Y) \oplus K^1(X,Y)$$

and finally

$$\tilde{K}^*(X) := K^*(X, \varnothing).$$

Remark 13. Ring structure on K^* finish! In fact, we get the triangle of [Ati67, p. 87] in which all morphisms are $K^*(X)$-module morphisms.

3. Thom spaces and statement of the theorem

Introduction and goal of the section: introduce Thom spaces and state the Thom isomorphism theorem.

Brief spoiler defining Thom spaces here already.

3.1. **Even dimensional spheres.** Recall the structure of the group $\tilde{K}^0(\mathbb{S}^{2n})$ determined in Vera's talk [Hat03, Corollary 2.12]. We already have a generator from this result in Hatcher's book. We can describe this canonical generator—up to a sign—in terms of the exterior algebra on an n-dimensional $\mathbb{C}$-vector space V [Ati67, p. 99].

3.2. **Thom spaces.** Generalize discussion on even dimensional spheres to Thom spaces as in [Ati67, p. 100]. Define the canonical $\lambda_E \in \tilde{K}(X^E)$ and explain its properties. Explain this also in terms of projectivizations of vector bundles plus a trivial line bundle, and recall Prof. Huber's remark that this projectivization is a way to compactify the vector bundle.

3.3. **Statement of the theorem.** State the Thom isomorphism theorem [Ati67, Corollary 2.7.12].

4. Proof for sums of line bundles

In this section we work with a direct sum of line bundles

$$E = L_1 \oplus \ldots \oplus L_m.$$

4.1. **Recall $K^0(\mathbb{P}(E))$.** This is treated in [Ati67, Proposition 2.5.3]; leave proof in gray, omitted during the talk.

4.2. **Computation of $K^*(\mathbb{P}(E))$.** This is [Ati67, Proposition 2.7.1].

4.3. **Thom isomorphism theorem in this case.** This is [Ati67, Proposition 2.7.2].

5. Proof of the general case

This corresponds to [Ati67, Proposition 2.7.8], [Ati67, Proposition 2.7.9] and [Ati67, Proposition 2.7.12].

Appendix A. Conventions and preliminaries

A.1. **Construction of $K(X)$.** The Grothendieck group is defined via universal property, but let us agree on a specific construction in order to have a precise description of the elements in the ring $K(X)$. We follow both [Ati67] and [Hat03] and consider the construction 1 described by Jin in the first talk of the seminar, which is the second construction discussed by Atiyah in [Ati67, p. 42].

Lemma 14. *Let M be a commutative monoid. Then Jin's construction 1 agrees with Atiyah's second construction of $K(M)$.*

Proof. In both cases $K(M)$ is the quotient of $M \times M$ by an equivalence relation, so it suffices to show that the equivalence relations agree. In Atiyah's construction we have

$$(x, y) \sim_A (x', y') :\Leftrightarrow \exists z, z' \in M, , (x + z, y + z) = (x' + z', y' + z').$$

In Jin's construction we have

$$(x, y) \sim_J (x', y') :\Leftrightarrow \exists z \in M, , x + y' + z = x' + y + z.$$

If $(x, y) \sim_A (x', y')$, then we have $x + z = x' + z'$ and $y + z = y' + z'$ for some $z, z' \in M$. Associativity and commutativity of M imply that

$$x + y' + z + z' = x' + y' + z' + z' = x' + y + z + z',$$

hence $(x, y) \sim_J (x', y')$. Conversely, if $(x, y) \sim_J (x', y')$, then we have $x + y' + z = x' + y + z$ for some $z \in M$. In particular we have

$$\begin{aligned}(x + (x + y' + z), y + (x + y' + z)) &= (x + (x' + y + z), y' + (x + y + z)) \\ &= (x' + (x + y + z), y' + (x + y + z)),\end{aligned}$$

so $(x, y) \sim_A (x', y')$ as well. □

Given (an isomorphism class of) a vector bundle $E \in \mathrm{Vect}(X)$, we denote by $[E]$ its image in $K(X)$, that is, $[E] = [(E, 0)]$. Since $-[E] = [(0, E)]$, we can write every element $[(E, F)] \in K(X)$ as $[E] - [F]$. We can find some vector bundle G such that $F \oplus G$ is trivial [Ati67, Corollary 1.4.14]. With the notation introduced earlier we can write $[F \oplus G] = [\underline{n}]$ for some $n \in \mathbb{N}$. Then we would have

$$[E] - [F] = [E] + [G] - ([F] + [G]) = [E \oplus G] - [\underline{n}],$$

showing that every element of $K(X)$ can be written as $[H] - [\underline{n}]$ for some vector bundle H on X and some natural number $n \in \mathbb{N}$ [Ati67, p. 44].

Suppose now that E and F are such that $[E] = [F]$, that is, $[(E, 0)] = [(F, 0)]$. By definition of the equivalence relation that we are using, there exists some vector bundle G such that $E \oplus G \cong F \oplus G$. Applying [Ati67, Corollary 1.4.14] again we deduce that $E \oplus \underline{n} \cong F \oplus \underline{n}$ for some $n \in \mathbb{N}$. In this case we say that E and F are *stably equivalent*. This brings us to Hatcher's description of $K(X)$ [Hat03, p. 39], namely, as formal differences $E - E'$ in which we identify $E_1 - E_1'$ with $E_2 - E_2'$ if and only if $E_1 \oplus E_2'$ and $E_2 \oplus E_1'$ are stably equivalent, that is, if and only if $[E_1 \oplus E_2'] = [E_2 \oplus E_1']$. Since $[E_1 \oplus E_2'] = [E_1] + [E_2']$ and $[E_2 \oplus E_1'] = [E_2] + [E_1']$, we do have $E_1 - E_1' = E_2 - E_2'$ in Hatcher's sense if and only if $[(E_1, E_1')] = [(E_2, E_2')]$ in Atiyah's sense. We will try to follow Atiyah's notation most of the time.

References

[Ati67] M. F. Atiyah. *K-theory.* Lecture notes by D. W. Anderson. W. A. Benjamin, Inc., New York-Amsterdam, 1967.

[Hat03] A. Hatcher. *Vector Bundles and K-Theory.* 2003. http://www.math.cornell.edu/~hatcher.

[Kar78] Max Karoubi. *K-theory.* Grundlehren der Mathematischen Wissenschaften, Band 226. Springer-Verlag, Berlin-New York, 1978. An introduction.

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