THE THOM ISOMORPHISM

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Abstract. Script for a talk of the Wednesday Seminar of the GK1821 at Freiburg during the Summer Semester 2021. The main reference is [Ati67, §2].

Contents

—parts in gray will be omitted during the talk—

1. Setting and conventions

- We work with complex vector spaces and complex vector bundles only [Ati67, p. 1].
- We use the usual word *rank* instead of *dimension*, which is the one used in [Ati67, p. 3].
- All base spaces are implicitly assumed to be compact and Hausdorff, although reminders will appear now and then. The usual notation for a base space will be X.
- We use $\mathrm{Vect}(X)$ to denote the set of isomorphism classes of vector bundles X, and $\mathrm{Vect}_n(X)$ to denote the subset of $\mathrm{Vect}(X)$ given by bundles of rank n [Ati67, p. 17]. Note that $(\mathrm{Vect}(X), \oplus)$ is a commutative monoid [Ati67, p. 17], and $(\mathrm{Vect}(X), \oplus, \otimes)$ is a semiring.
- Given a commutative monoid A, we denote by $K(A)$ its Grothendieck group [Ati67, p. 42]. If A is also a semiring, we regard $K(A)$ as a ring with the induced ring structure [Ati67, p. 43].
- We denote $K(X) := K(\mathrm{Vect}(X))$, which is then a commutative ring with one [Ati67, p. 43]. We think of elements of $K(X)$ as

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formal differences $[E] - [F]$ of vector bundles E and F on X [Ati67, p. 44].
- We write $\underline{n}$ for the trivial bundle of rank n.
- We denote by $\mathcal{C}$ the category of compact topological spaces, by $\mathcal{C}^+$ the category of compact spaces with distinguished basepoints and by $\mathcal{C}^2$ the category of compact pairs.

2. Recollection from previous talks

We have a functor $\mathcal{C}^2 \to \mathcal{C}^+$ that sends a pair (X, Y) to X/Y, with basepoint Y/Y. If $Y = \varnothing$, then we interpret the resulting object as X with a disjoint basepoint. We also have a functor $\mathcal{C} \to \mathcal{C}^2$ sending $X \mapsto (X, \varnothing)$. Hence, the composition of the two functors gives $X \mapsto X^+$, where X^+ is the disjoint union of X with a basepoint.

For X in $\mathcal{C}^+$ we define $\tilde{K}(X)$ to be the kernel of the map $i^*\colon K(X) \to K(x_0)$, where $i\colon x_0 \to X$ is the inclusion of the basepoint. If $c\colon X \to x_0$ is the collapsing map, then c^* induces a splitting

$$K(X) \cong \tilde{K}(X) \oplus K(x_0).$$

Indeed, we need...

Appendix A. More on conventions and preliminaries

A.1. **Construction of $K(X)$.** The Grothendieck group is defined via universal property, but let us agree on a specific construction in order to have a precise description of the elements in the ring $K(X)$. We follow both [Ati67] and [Hat03] and consider the construction 1 described by Jin in the first talk of the seminar, which is the second construction discussed by Atiyah in [Ati67, p. 42].

Lemma 1. *Let M be a commutative monoid. Then Jin's construction 1 agrees with Atiyah's second construction of $K(M)$.*

Proof. In both cases $K(M)$ is the quotient of $M \times M$ by an equivalence relation, so it suffices to show that the equivalence relations agree. In Atiyah's construction we have

$$(x, y) \sim_A (x', y') :\Leftrightarrow \exists z, z' \in M, , (x + z, y + z) = (x' + z', y' + z').$$

In Jin's construction we have

$$(x, y) \sim_J (x', y') :\Leftrightarrow \exists z \in M, , x + y' + z = x' + y + z.$$

If $(x, y) \sim_A (x', y')$, then we have $x + z = x' + z'$ and $y + z = y' + z'$ for some $z, z' \in M$. Associativity and commutativity of M imply that

$$x + y' + z + z' = x' + y' + z + z' = x' + y + z + z',$$

hence $(x, y) \sim_J (x', y')$. Conversely, if $(x, y) \sim_J (x', y')$, then we have $x + y' + z = x' + y + z$ for some $z \in M$. In particular we have

$$\begin{aligned}(x + (x + y' + z), y + (x + y' + z)) &= (x + (x' + y + z), y' + (x + y + z)) \\ &= (x' + (x + y + z), y' + (x + y + z)),\end{aligned}$$

so $(x, y) \sim_A (x', y')$ as well. □

Given (an isomorphism class of) a vector bundle $E \in \operatorname{Vect}(X)$, we denote by $[E]$ its image in $K(X)$, that is, $[E] = [(E, 0)]$. Since $-[E] = [(0, E)]$, we can write every element $[(E, F)] \in K(X)$ as $[E] - [F]$. We can find some vector bundle G such that $F \oplus G$ is trivial [Ati67, Corollary 1.4.14]. With the notation introduced earlier we can write $[F \oplus G] = [\underline{n}]$ for some $n \in \mathbb{N}$. Then we would have

$$[E] - [F] = [E] + [G] - ([F] + [G]) = [E \oplus G] - [\underline{n}],$$

showing that every element of $K(X)$ can be written as $[H] - [\underline{n}]$ for some vector bundle H on X and some natural number $n \in \mathbb{N}$ [Ati67, p. 44].

Suppose now that E and F are such that $[E] = [F]$, that is, $[(E, 0)] = [(F, 0)]$. By definition of the equivalence relation that we are using, there exists some vector bundle G such that $E \oplus G \cong F \oplus G$. Applying [Ati67, Corollary 1.4.14] again we deduce that $E \oplus \underline{n} \cong F \oplus \underline{n}$ for some $n \in \mathbb{N}$. In this case we say that E and F are *stably equivalent*. This brings us to Hatcher's description of $K(X)$ [Hat03, p. 39], namely, as formal differences $E - E'$ in which we identify $E_1 - E_1'$ with $E_2 - E_2'$ if and only if $E_1 \oplus E_2'$ and $E_2 \oplus E_1'$ are stably equivalent, that is, if and only if $[E_1 \oplus E_2'] = [E_2 \oplus E_1']$. Since $[E_1 \oplus E_2'] = [E_1] + [E_2']$ and $[E_2 \oplus E_1'] = [E_2] + [E_1']$, we do have $E_1 - E_1' = E_2 - E_2'$ in Hatcher's sense if and only if $[(E_1, E_1')] = [(E_2, E_2')]$ in Atiyah's sense. We will try to follow Atiyah's notation most of the time.

References

[Ati67] M. F. Atiyah. *K-theory*. Lecture notes by D. W. Anderson. W. A. Benjamin, Inc., New York-Amsterdam, 1967.

[Hat03] A. Hatcher. *Vector Bundles and K-Theory*. 2003. http://www.math.cornell.edu/~hatcher.

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