THE THOM ISOMORPHISM

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Abstract. Script for a talk of the Wednesday Seminar of the GK1821 at Freiburg during the Summer Semester 2021. The main reference is [Ati67, §2].

Contents

—parts in gray will be omitted during the talk—

1. Recollection from previous talks

We start off with some recollections from [Ati67, §2]. We refer to [Ati67] and the appendix for previous conventions and other preliminaries.

1.1. **Notation and topological preliminaries.** For $n \in \mathbb{N}$ we denote by S^n the n-dimensional sphere. We think of it as the pointed compact Hausdorff space $I^n/\partial I^n$, where $I^n \subseteq \mathbb{R}^n$ is the unit interval and the basepoint is given by the equivalence class of any point in the boundary ∂I^n. The unit interval $I^1 = [0,1]$ is denoted simply by I.

Given pointed compact Hausdorff spaces (X, x_0) and (Y, y_0), we define their *smash product* as

$$X \wedge Y := X \times Y / X \vee Y,$$

where $X \vee Y := X \times \{y_0\} \cup \{x_0\} \times Y$ is their *wedge sum*. The *reduced suspension* of (X, x_0) is obtained from the usual suspension $(I \times X)/(\{0\} \times X \cup \{1\} \times X)$ by further collapsing the line $I \times \{x_0\}$ joining the two vertices of the suspension along the basepoint. Using our explicit description of S^1, we may rewrite the reduced suspension of

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X as $S^1 \wedge X$, and we denote it by SX. The n-th iterated suspension is naturally homeomorphic to $S^n \wedge X$, and we denote it by $S^n X$. Finally, the *cone* of the pointed compact Hausdorff space (X, x_0) is defined as $CX := (I \times X)/(\{0\} \times X)$, and we take as basepoint the equivalence class of any point in the subspace that we are collapsing [Ati67, p. 68].

1.2. **The functor $K(-)$.** We follow [Ati67, p. 44]. Let **CHaus** be the category of compact Hausdorff topological spaces and let **Ring** be the category of commutative unital rings. We have defined a functor $K\colon \mathbf{CHaus} \to \mathbf{Ring}$ which can be explicitly described as follows. If $E \in \mathrm{Vect}(X)$ is a vector bundle on a compact Hausdorff space X, then we denote by $[E]$ its stable equivalence class, i.e.,

$$[E] := \{F \in \mathrm{Vect}(X) \mid \exists n \in \mathbb{N} \text{ s.t. } E \oplus n \cong F \oplus n\},$$

where we denote also by n the trivial vector bundle of rank n on X. Then the underlying set of the ring $K(X)$ consists of formal differences $[E] - [F]$ for vector bundles $E, F \in \mathrm{Vect}(X)$. The sum is given by

$$([E] - [F]) + ([E'] - [F']) = [E \oplus E'] - [F \oplus F'],$$

and the product is given by

$$([E] - [F])([E'] - [F']) = [(E \otimes E') \oplus (F \otimes F')] - [(E \otimes F') \oplus (E' \otimes F)].$$

The element zero can be represented by $[0] - [0]$ and the element one by $[1] - [0]$, where again by 0 we mean $X \times \{0\}$ and by 1 we mean $X \times \mathbb{C}$. Moreover, since X is compact and Hausdorff, we can represent every element of $K(X)$ as $[E] - [n]$ for some $E \in \mathrm{Vect}(X)$ and some $n \in \mathbb{N}$.

If $f\colon X \to Y$ is a continuous map between compact Hausdorff topological spaces, then $K(f) =: f^*$ is given by

$$\begin{aligned} f^*\colon K(Y) &\to K(X) \\ [E] - [F] &\mapsto [f^*(E)] - [f^*(F)], \end{aligned}$$

and this ring homomorphism only depends on the homotopy class of f.

1.3. **The functor $\tilde{K}(-)$.** We follow [Ati67, p. 66]. Let $\mathbf{CHaus}^+$ be the category of pointed compact Hausdorff topological spaces and let **Rng** be the category of commutative non-unital rings. We have defined a functor $\tilde{K}\colon \mathbf{CHaus}^+ \to \mathbf{Rng}$ as follows. Let (X, x_0) be pointed compact Hausdorff topological space and let $i\colon \{x_0\} \to X$ denote the inclusion of the base point. Then

$$\tilde{K}(X) := \ker(i^*) \subseteq K(X).$$

Remark 1. $\tilde{K}(X)$ is a non-unital subring: it is a subgroup closed under multiplication but it does not contain $1 \in K(X)$.

If $c\colon X \to \{x_0\}$ is the constant morphism to the basepoint, then c^* induces a splitting $K(X) \cong \tilde{K}(X) \oplus K(x_0)$ which is natural with respect to morphisms in $\mathbf{CHaus}^+$, hence $\tilde{K}(-)$ is a functor as claimed above. If $f\colon (X, x_0) \to (Y, y_0)$ is a morphism in $\mathbf{CHaus}^+$ given by a continuous function $f\colon X \to Y$ such that $f(x_0) = y_0$, then $\tilde{K}(f) =: f^*$ is induced by the restriction of $f^*\colon K(Y) \to K(X)$.

Remark 2. We can recover the functor $K(-)$ from the functor $\tilde{K}(-)$ by adding disjoint basepoints:

$$K(X) \cong \tilde{K}(X^+),$$

where X^+ is the pointed compact Hausdorff space obtained from adding a disjoint basepoint to the compact Hausdorff space X. This is a priori an isomorphism of non-unital rings, but we recover the unit $1 \in K(X)$ as the element corresponding to finish this!

1.4. **The functor $K(-,-)$.** We follow [Ati67, p. 66]. Let $\mathbf{CHaus}^2$ be the category of pairs (X, Y) consisting of a compact Hausdorff space X and a compact Hausdorff subspace $Y \subseteq X$, or equivalently a compact Hausdorff space X and a closed subspace $Y \subseteq X$. We have defined a functor $K\colon \mathbf{CHaus}^2 \to \mathbf{Rng}$ as follows. For an object (X, Y) in $\mathbf{CHaus}^2$, we set

$$K(X, Y) := \tilde{K}(X/Y),$$

where the basepoint of X/Y is the equivalence class of any point $y \in Y$, which we denote by Y/Y. If $f\colon (X, Y) \to (Z, W)$ is a morphism in $\mathbf{CHaus}^2$ given by a continuous function $f\colon X \to Z$ such that $f(Y) \subseteq W$, then it induced a morphism $\bar{f}\colon (X/Y, Y/Y) \to (Z/W, W/W)$ in $\mathbf{CHaus}^+$. Then $K(f) =: f^*$ is given by $\tilde{K}(\bar{f})$.

1.5. **The six-term exact sequence.**

Definition 3 ([Ati67, Definition 2.4.1])**.** For $n \in \mathbb{N}_{>0}$ we define a functor $\tilde{K}^{-n}\colon \mathbf{CHaus}^+ \to \mathbf{Rng}$ by setting

$$\tilde{K}^{-n}(X) := \tilde{K}(S^n X).$$

Then we define a functor $K^{-n}\colon \mathbf{CHaus}^2 \to \mathbf{Rng}$ via

$$K^{-n}(X, Y) := \tilde{K}^{-n}(X/Y),$$

and finally a functor $K^{-n}\colon \mathbf{CHaus} \to \mathbf{Rng}$ with the formula

$$K^{-n}(X) := K^{-n}(X, \varnothing).$$

Remark 4. The last functor takes values in $\mathbf{Ring}$, the unit of $K^{-n}(X)$ being given by... finish!

Recall from Vera's talk:

Proposition 5 ([Ati67, Proposition 2.4.4])**.** *For each pair (X, Y) in* $\mathbf{CHaus}^2$ *there is a natural exact sequence*

$$\cdots \to K^{-1}(Y) \xrightarrow{\delta} K^0(X, Y) \xrightarrow{j^*} K^0(X) \xrightarrow{i^*} K^0(Y),$$

where $i\colon Y \to X$ and $j\colon (X, \varnothing) \to (X, Y)$ are the inclusions.

Remark 6. The morphism $\delta\colon K^{-1}(Y) \to K^0(X, Y)$ can be constructed as the composition $m^* \circ \theta^{-1}$, where θ is the isomorphism $K(X \cup CY)$... finish! But the important part is that δ is induced by a morphism of spaces, cf. [Ati67, p. 77].

Using the periodicity isomorphisms $\beta\colon K^{-n}(X) \to K^{-n-2}(X)$ from [Ati67, Theorem 2.4.9] we extend the collection of functors $\{K^{-n}(-, -)\}_{n\in\mathbb{N}}$ to a collection of funcotrs $\{K^{n}(-, -)\}_{n\in\mathbb{Z}}$ inductively, i.e. $K^1 = K^{-1}$, $K^2 = K^0$, $K^3 = K^1 = K^{-1}$, and so on. Then we use the previously seen relations among the different K functors to define collections of functors $\{K^n(-)\}_{n\in\mathbb{Z}}$ and $\{\tilde{K}^n(-)\}_{n\in\mathbb{Z}}$ as well. As we saw at the end of Vera's talk, this allows us to extend the previous exact sequence to the right, and we rearrange this data into the following *six-term exact sequence*

$$\begin{tikzcd}
K^0(X,Y) \arrow[r] & K^0(X) \arrow[r] & K^0(Y) \arrow[d] \\
K^1(Y) \arrow[u] & K^1(X) \arrow[l] & K^1(X,Y) \arrow[l]
\end{tikzcd}$$

Definition 7. Let X be a compact Hausdorff space. Then we define

$$K^*(X) := K^0(X) \oplus K^1(X).$$

Similarly, for a pair (X, Y) in $\mathbf{CHaus}^2$ we define

$$K^*(X, Y) := K^0(X, Y) \oplus K^1(X, Y)$$

and finally

$$\tilde{K}^*(X) := K^*(X, \varnothing).$$

Remark 8. Ring structure on K^* finish! In fact, we get the triangle of [Ati67, p. 87] in which all morphisms are $K^*(X)$-module morphisms.

2. Thom spaces and statement of the theorem

Introduction and goal of the section: introduce Thom spaces and state the Thom isomorphism theorem.

Brief spoiler defining Thom spaces here already.

2.1. **Even dimensional spheres.** Recall the structure of the group $\tilde{K}^0(\mathbb{S}^{2n})$ determined in Vera's talk [Hat03, Corollary 2.12]. We already have a generator from this result in Hatcher's book. We can describe this canonical generator—up to a sign—in terms of the exterior algebra on an n-dimensional $\mathbb{C}$-vector space V [Ati67, p. 99].

2.2. **Thom spaces.** Generalize discussion on even dimensional spheres to Thom spaces as in [Ati67, p. 100]. Define the canonical $\lambda_E \in \tilde{K}(X^E)$ and explain its properties. Explain this also in terms of projectivizations of vector bundles plus a trivial line bundle, and recall Prof. Huber's remark that this projectivization is a way to compactify the vector bundle.

2.3. **Statement of the theorem.** State the Thom isomorphism theorem [Ati67, Corollary 2.7.12].

3. Proof for sums of line bundles

In this section we work with a direct sum of line bundles

$$E = L_1 \oplus \ldots \oplus L_m.$$

3.1. **Recall $K^0(\mathbb{P}(E))$.** This is treated in [Ati67, Proposition 2.5.3]; leave proof in gray, omitted during the talk.

3.2. **Computation of $K^*(\mathbb{P}(E))$.** This is [Ati67, Proposition 2.7.1].

3.3. **Thom isomorphism theorem in this case.** This is [Ati67, Proposition 2.7.2].

4. Proof of the general case

This corresponds to [Ati67, Proposition 2.7.8], [Ati67, Proposition 2.7.9] and [Ati67, Proposition 2.7.12].

Appendix A. Conventions and preliminaries

A.1. **Construction of $K(X)$.** The Grothendieck group is defined via universal property, but let us agree on a specific construction in order to have a precise description of the elements in the ring $K(X)$. We follow both [Ati67] and [Hat03] and consider the construction 1 described by Jin in the first talk of the seminar, which is the second construction discussed by Atiyah in [Ati67, p. 42].

Lemma 9. *Let M be a commutative monoid. Then Jin's construction 1 agrees with Atiyah's second construction of $K(M)$.*

Proof. In both cases $K(M)$ is the quotient of $M \times M$ by an equivalence relation, so it suffices to show that the equivalence relations agree. In Atiyah's construction we have

$$(x, y) \sim_A (x', y') :\Leftrightarrow \exists z, z' \in M, , (x + z, y + z) = (x' + z', y' + z').$$

In Jin's construction we have

$$(x, y) \sim_J (x', y') :\Leftrightarrow \exists z \in M, , x + y' + z = x' + y + z.$$

If $(x, y) \sim_A (x', y')$, then we have $x + z = x' + z'$ and $y + z = y' + z'$ for some $z, z' \in M$. Associativity and commutativity of M imply that

$$x + y' + z + z' = x' + y' + z' + z' = x' + y + z + z',$$

hence $(x, y) \sim_J (x', y')$. Conversely, if $(x, y) \sim_J (x', y')$, then we have $x + y' + z = x' + y + z$ for some $z \in M$. In particular we have

$$\begin{aligned}(x + (x + y' + z), y + (x + y' + z)) &= (x + (x' + y + z), y' + (x + y + z)) \\ &= (x' + (x + y + z), y' + (x + y + z)),\end{aligned}$$

so $(x, y) \sim_A (x', y')$ as well. □

Given (an isomorphism class of) a vector bundle $E \in \mathrm{Vect}(X)$, we denote by $[E]$ its image in $K(X)$, that is, $[E] = [(E, 0)]$. Since $-[E] = [(0, E)]$, we can write every element $[(E, F)] \in K(X)$ as $[E] - [F]$. We can find some vector bundle G such that $F \oplus G$ is trivial [Ati67, Corollary 1.4.14]. With the notation introduced earlier we can write $[F \oplus G] = [\underline{n}]$ for some $n \in \mathbb{N}$. Then we would have

$$[E] - [F] = [E] + [G] - ([F] + [G]) = [E \oplus G] - [\underline{n}],$$

showing that every element of $K(X)$ can be written as $[H] - [\underline{n}]$ for some vector bundle H on X and some natural number $n \in \mathbb{N}$ [Ati67, p. 44].

Suppose now that E and F are such that $[E] = [F]$, that is, $[(E, 0)] = [(F, 0)]$. By definition of the equivalence relation that we are using, there exists some vector bundle G such that $E \oplus G \cong F \oplus G$. Applying [Ati67, Corollary 1.4.14] again we deduce that $E \oplus \underline{n} \cong F \oplus \underline{n}$ for some $n \in \mathbb{N}$. In this case we say that E and F are *stably equivalent*. This brings us to Hatcher's description of $K(X)$ [Hat03, p. 39], namely, as formal differences $E - E'$ in which we identify $E_1 - E_1'$ with $E_2 - E_2'$ if and only if $E_1 \oplus E_2'$ and $E_2 \oplus E_1'$ are stably equivalent, that is, if and only if $[E_1 \oplus E_2'] = [E_2 \oplus E_1']$. Since $[E_1 \oplus E_2'] = [E_1] + [E_2']$ and $[E_2 \oplus E_1'] = [E_2] + [E_1']$, we do have $E_1 - E_1' = E_2 - E_2'$ in Hatcher's sense if and only if $[(E_1, E_1')] = [(E_2, E_2')]$ in Atiyah's sense. We will try to follow Atiyah's notation most of the time.

References

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