

Torelli's Theorem

Remarks on Sections III.12–13 of Milne's *Abelian Varieties*

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- For all $1 \leq r \leq g$ we get an induced map

$$f: C^{(r)} \rightarrow J, \quad P_1 \cdot \dots \cdot P_r \mapsto [P_1 + \dots + P_r - rP]$$

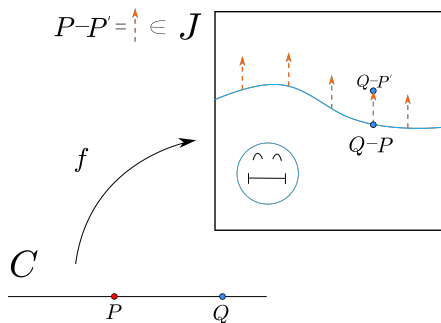
birational onto its image $W^r \subseteq J$, which is a closed subvariety.

Canonical polarization

- For $r = g - 1$ we get a divisor $\Theta = W^{g-1} \subseteq J$, the image of $f: C^{(g-1)} \rightarrow J$, $P_1 \cdot \dots \cdot P_{g-1} \mapsto [P_1 + \dots + P_{g-1} - (g-1)P]$.

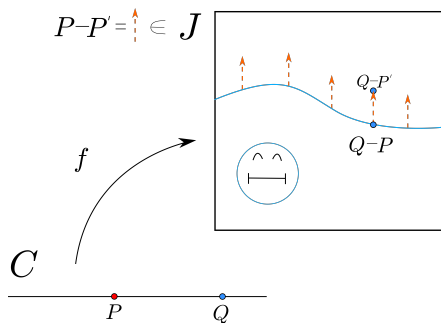
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- The induced $\lambda: J \rightarrow J^\vee$ is an isomorphism, so Θ gives us a principal polarization of J called the *canonical polarization*.

Statement — Existence

Let C and C' be curves as before and let $\beta: (J, \lambda) \xrightarrow{\sim} (J', \lambda')$ be an isomorphism such that $\lambda' \circ \beta = \beta^\vee \circ \lambda$.

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Then there exists an isomorphism $\alpha: C \xrightarrow{\sim} C'$ such that

$$\begin{array}{ccc} C & \xrightarrow{f} & J \\ \alpha \downarrow & & \downarrow \beta \\ C' & \xrightarrow{f'} & J' \end{array}$$

commutes up to a sign and a translation by some $c \in J'(k)$.

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We can now state uniqueness distinguishing two cases:

- If C is not hyperelliptic, then the sign, α and c are uniquely determined by β , P and P' .
- If C is hyperelliptic, then the sign can be chosen arbitrarily, and α and c are uniquely determined by β , P , P' and the chosen sign.

Proof — Uniqueness (modulo case distinctions for signs)

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- Hence $(\alpha_1 - \alpha_2)(Q_1) \sim (\alpha_1 - \alpha_2)(Q_2)$ and

$$(*) \quad \alpha_1(Q_1) + \alpha_2(Q_2) \sim \alpha_2(Q_1) + \alpha_1(Q_2) \quad (\forall Q_1, Q_2).$$

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- So the degree 2 linear system given by $(*)$ contains at least two divisors, which implies that it is of dimension at least 1.
- Varying Q_1 we obtain more such linear systems, and curves of general type can have at most one such linear system.
- This contradiction shows that $\alpha_1 = \alpha_2$, thus $c_1 = c_2$.

Corollary — Torelli over a perfect field $F \subseteq k$

If C , C' and β are defined over F , then α is defined over F as well. In particular, $C \cong C'$ over F .

Sketch of proof (choose a sign if C hyperelliptic):

- (1) α is characterized by: $\exists c \in J(k)$ s.t. $f^{P'} \circ \alpha = \pm \beta \circ f^P + c$.
- (2) Replacing P by Q we get $f^Q = f^P + d$, resp. for $(-)'$. Hence

$$f^{Q'} \circ \alpha = f^{P'} \circ \alpha + d' = \pm \beta \circ f^P + c + d' = \pm \beta \circ f^Q \mp \beta(d) + c + d'.$$

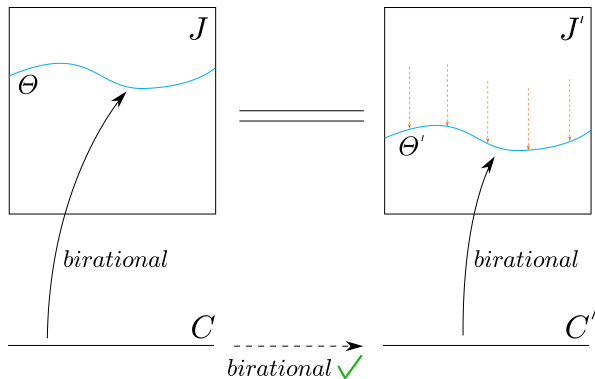
- (3) Hence α is independent of the chosen k -points P and P' .
- (4) For $\sigma \in \text{Gal}(k/F)$ we have $\sigma f^P = f^{\sigma P}$, resp. for $(-)'$. Hence

$$f^{\sigma P'} \circ \sigma \alpha = \sigma f^{P'} \circ \sigma \alpha = \pm \sigma \beta \circ \sigma f^P + \sigma c = \pm \beta \circ f^{\sigma P} + \sigma c.$$

Point (3) implies then $\sigma \alpha = \alpha$.

Idea of the existence proof

Suppose $g = 2$ as in the previous picture. If we identify J with J' via β , the fact that β is a polarized isomorphism guarantees the following situation:



Thanks for your attention!