Torelli's Theorem

Remarks on Sections III.12–13 of Milne's *Abelian Varieties*

University of Freiburg

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- For all $1 \leqslant r \leqslant g$ we get an induced map

$$f\colon C^{(r)} \to J, \quad P_1 \cdot \ldots \cdot P_r \mapsto [P_1 + \cdots + P_r - rP]$$

birational onto its image $W^r \subseteq J$, which is a closed subvariety.

Canonical polarization

- For $r = g - 1$ we get a divisor $\Theta = W^{g-1} \subseteq J$, the image of

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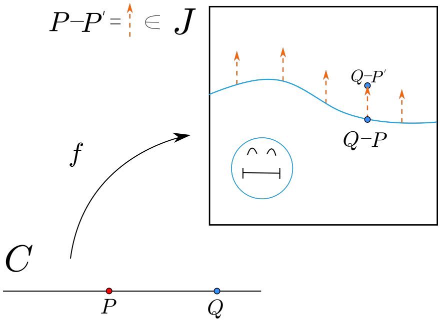

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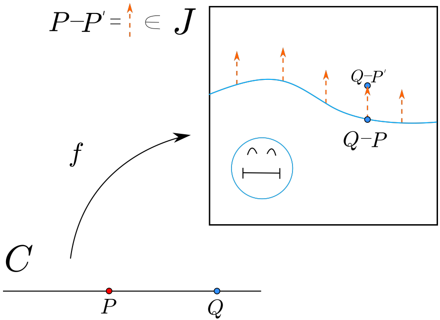

- The induced $\lambda\colon J \to J^\vee$ is an isomorphism, so Θ gives us a principal polarization of J called the *canonical polarization*.

Statement — Existence

Let C and C' be curves as before and let $\beta\colon (J,\lambda) \xrightarrow{\sim} (J',\lambda')$ be an isomorphism such that $\lambda' \circ \beta = \beta^\vee \circ \lambda$.

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Then there exists an isomorphism $\alpha\colon C \xrightarrow{\sim} C'$ such that

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\alpha"'] & J \arrow[d, "\beta"] \\ C' \arrow[r, "f'"'] & J' \end{tikzcd}$$

commutes up to a sign and a translation by some $c \in J'(k)$.

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We can now state uniqueness distinguishing two cases:

- If C is not hyperelliptic, then the sign, α and c are uniquely determined by β, P and P'.
- If C is hyperelliptic, then the sign can be chosen arbitrarily, and α and c are uniquely determined by β, P, P' and the chosen sign.

Proof — Uniqueness (modulo case distinctions for signs)

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- Hence $(\alpha_1 - \alpha_2)(Q_1) \sim (\alpha_1 - \alpha_2)(Q_2)$ and

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- Varying Q_1 we obtain more such linear systems, and curves of general type can have at most one such linear system.
- This contradiction shows that $\alpha_1 = \alpha_2$, thus $c_1 = c_2$.

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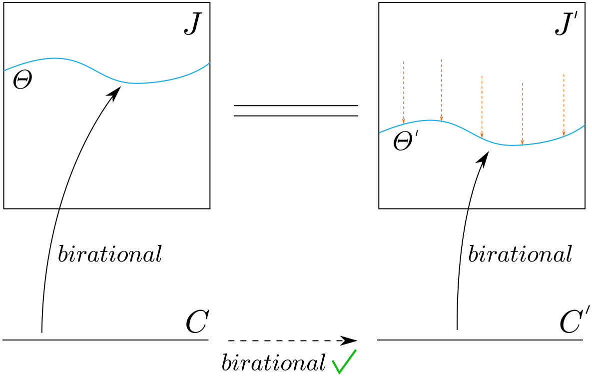
(4) For $\sigma \in \mathrm{Gal}(k/F)$ we have $\sigma f^P = f^{\sigma P}$, resp. for $(-)'$. Hence

$$f^{\sigma P'} \circ \sigma\alpha = \sigma f^{P'} \circ \sigma\alpha = \pm\sigma\beta \circ \sigma f^P + \sigma c = \pm\beta \circ f^{\sigma P} + \sigma c.$$

Point (3) implies then $\sigma\alpha = \alpha$.

Idea of the existence proof for $g = 2$

Suppose $g = 2$ as in the previous picture. If we identify J with J' via β, the fact that β is a polarized isomorphism guarantees the following situation:



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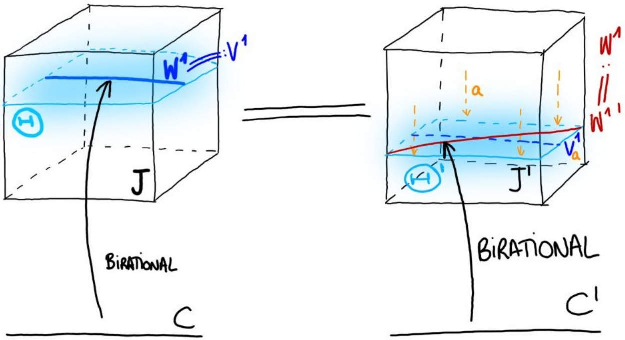

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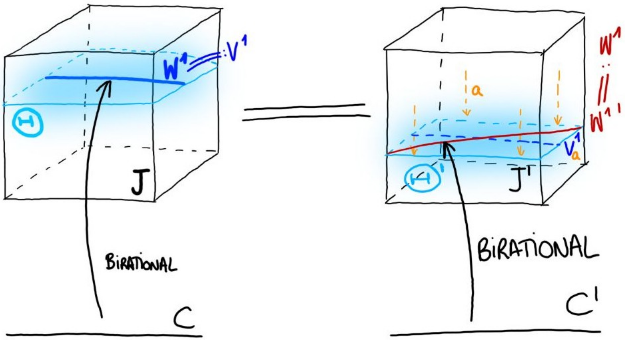

Is $V^1_a = W^1_b$ for some $b \in J(k)$?
[Yes! Modulo replacing W^1 by its image under $x \mapsto -x$.]

Thanks for listening!