Torelli's Theorem

Remarks on Sections III.12–13 of Milne's *Abelian Varieties*

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28th July 2020

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- For all $1 \leqslant r \leqslant g$ we get an induced map

$$f \colon C^{(r)} \to J, \quad P_1 \cdot \ldots \cdot P_r \mapsto [P_1 + \cdots + P_r - rP]$$

birational onto its image $W^r \subseteq J$, which is a closed subvariety.

Canonical polarization

- For $r = g - 1$ we get a divisor $\Theta = W^{g-1} \subseteq J$, the image of

  $f\colon C^{(g-1)} \to J, \quad P_1 \cdot \ldots \cdot P_{g-1} \mapsto [P_1 + \cdots + P_{g-1} - (g-1)P].$

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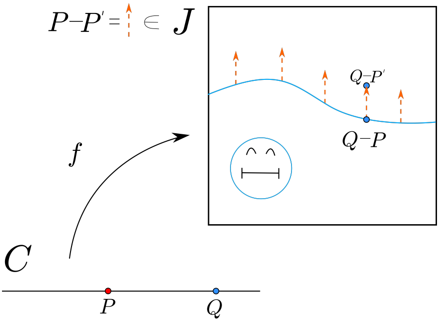

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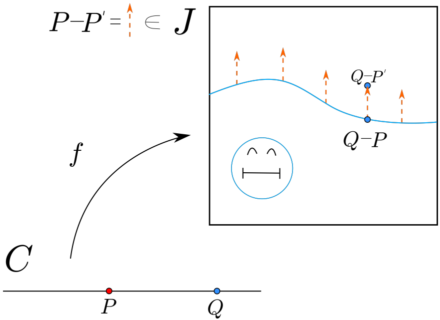

- The induced $\lambda\colon J \to J^\vee$ is an isomorphism, so Θ gives us a principal polarization of J called the *canonical polarization*.

Statement — Existence

Let C and C' be curves as before and let $\beta\colon (J,\lambda) \xrightarrow{\sim} (J',\lambda')$ be an isomorphism such that $\lambda' \circ \beta = \beta^\vee \circ \lambda$.

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Then there exists an isomorphism $\alpha\colon C \xrightarrow{\sim} C'$ such that

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\alpha"'] & J \arrow[d, "\beta"] \\ C' \arrow[r, "f'"'] & J' \end{tikzcd}$$

commutes up to a sign and translation by some $c \in J'(k)$.

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A curve C as before is called *hyperelliptic* if there is a $2:1$ branched covering $\gamma\colon C \to \mathbb{P}^1$.

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- There are hyperelliptic curves of any genus $g \geqslant 2$.
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We can now state uniqueness distinguishing two cases:

- If C is not hyperelliptic, then α, the sign and the element $c \in J'(k)$ by which we have to translate are uniquely determined by β, P and P'.
- If C is hyperelliptic, then the sign can be chosen arbitrarily, and α and c are uniquely determined by β, P, P' and the chosen sign.

Thanks for your attention! Here are some references: