Torelli's Theorem

Remarks on Sections III.12–13 of Milne's *Abelian Varieties*

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- For all $1 \leqslant r \leqslant g$ we get an induced map

$$f\colon C^{(r)} \to J, \quad P_1 \cdot \ldots \cdot P_r \mapsto [P_1 + \cdots + P_r - rP]$$

birational onto its image $W^r \subseteq J$, which is a closed subvariety.

Canonical polarization

- For $r = g-1$ we get a divisor $\Theta = W^{g-1} \subseteq J$, the image of

  $f\colon C^{(g-1)} \to J, \quad P_1 \cdot \ldots \cdot P_{g-1} \mapsto [P_1 + \cdots + P_{g-1} - (g-1)P].$

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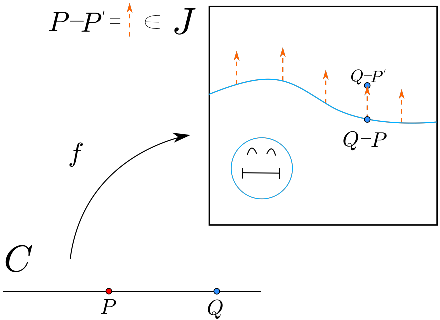

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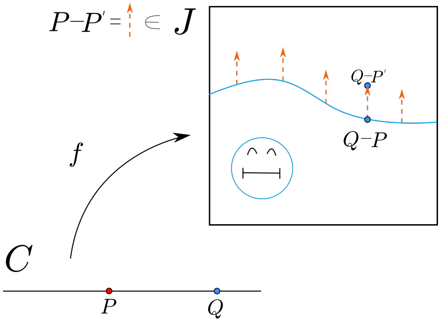

- The induced $\lambda\colon J \to J^\vee$ is an isomorphism, so Θ gives us a principal polarization of J called the *canonical polarization*.

Statement — Existence

Let C and C' be curves as before and let $\beta\colon (J, \lambda) \xrightarrow{\sim} (J', \lambda')$ be an isomorphism such that $\lambda' \circ \beta = \beta^\vee \circ \lambda$.

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Then there exists an isomorphism $\alpha\colon C \xrightarrow{\sim} C'$ such that

$$\begin{array}{ccc} C & \xrightarrow{f} & J \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ C' & \xrightarrow[f']{} & J' \end{array}$$

commutes up to a sign and translation by some $c \in J'(k)$.

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A curve C as before is called *hyperelliptic* if there is a (unique) $2:1$ branched covering $\pi\colon C \to \mathbb{P}^1$.

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We can now state uniqueness distinguishing two cases:

- If C is not hyperelliptic, then α, the sign and the element $c \in J'(k)$ by which we have to translate are uniquely determined by β, P and P'.
- If C is hyperelliptic, then the sign can be chosen arbitrarily, and α and c are uniquely determined by β, P, P' and the chosen sign.

Proof — Uniqueness (modulo case distinctions for signs)

- Suppose α_1, α_2, c_1 and c_2 were such that $f' \circ \alpha_i = \beta \circ f + c_i$.
- Then $f' \circ (\alpha_1 - \alpha_2) \colon C \to J'$ is constant, so $\alpha_1 - \alpha_2$ sends every pair of points in C to the same fibre of $f' \colon C' \to J'$.
- Hence $(\alpha_1 - \alpha_2)(Q_1) \sim (\alpha_1 - \alpha_2)(Q_2)$ and

$$(*) \quad \alpha_1(Q_1) + \alpha_2(Q_2) \sim \alpha_2(Q_1) + \alpha_1(Q_2).$$

- Suppose $\alpha_1 \neq \alpha_2$. Then $\exists Q_1 \in C(k)$ s.t. $\alpha_1(Q_1) \neq \alpha_2(Q_1)$. Since α_1 is an isomorphism, there are also plenty $Q_2 \in C(k)$ s.t. $\alpha_1(Q_1) \neq \alpha_1(Q_2)$, hence $\alpha_1(Q_1) \notin \{\alpha_2(Q_1), \alpha_1(Q_2)\}$.
- So the degree 2 linear system given by $(*)$ contains at least two divisors, hence is at least of dimension 1.
- Varying Q_1 we obtain more such linear systems, and curves of general type can have at most one such linear system.
- This contradiction shows that $\alpha_1 = \alpha_2$, thus $c_1 = c_2$.

Thanks for your attention! Here are some references: