

ZARISKI CANCELLATION

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ABSTRACT. Following [Hoc72] we provide an example of rings¹ B, C such that $B \not\cong C$ but $B[t] \cong C[t]$, where t is an indeterminate. As a preparation for this counterexample we also study the notion of projective module and the hairy ball theorem.

CONTENTS

1. Introduction	1
2. Projective modules	2
References	3

—parts in gray will be omitted during the talk—

1. INTRODUCTION

Let R be a ring. We can form the polynomial ring $R[t]$ in one variable t with coefficients in R . This construction is functorial, and hence

$$R \cong S \Rightarrow R[t] \cong S[t].$$

The goal of this talk is to show with an explicit counterexample due to Hochster [Hoc72] that the converse is not true.

In the process of constructing this counterexample we will come across a projective module which, as a consequence of the hairy ball theorem, is not a free module. Therefore we will discuss projective modules and the hairy ball theorem before jumping into the counterexample.

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2. PROJECTIVE MODULES

Definition 2.1. Let \mathcal{C} be a category. An object $P \in \mathcal{C}$ is called *projective* if the following lifting problem can always be solved:

$$\begin{array}{ccc} & M & \\ & \downarrow \text{epi} & \\ P & \longrightarrow & N \end{array}$$

Lemma 2.2. Let \mathcal{A} be an abelian category and let $P \in \mathcal{A}$ be an object. The following are equivalent:

- (1) P is projective.
- (2) $\text{Hom}_{\mathcal{A}}(P, -)$ is exact.
- (3) Every short exact sequence of the form

$$0 \rightarrow N \rightarrow M \rightarrow P \rightarrow 0$$

splits.

Proof. We start with (1) \Rightarrow (2). Assume P is projective and consider a short exact sequence

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0.$$

Since $\text{Hom}_{\mathcal{A}}(P, -)$ is always left exact, we only need to show that the induced map $\text{Hom}_{\mathcal{A}}(P, B) \rightarrow \text{Hom}_{\mathcal{A}}(P, C)$ is surjective. But $B \rightarrow C$ is an epimorphism, so this is precisely what P being projective means by definition.

Next we show (2) \Rightarrow (3). Assume $\text{Hom}_{\mathcal{A}}(P, -)$ is exact and consider a short exact sequence

$$0 \rightarrow N \rightarrow M \rightarrow P \rightarrow 0.$$

Applying $\text{Hom}_{\mathcal{A}}(P, -)$ we get a surjection $\text{Hom}_{\mathcal{A}}(P, M) \rightarrow \text{Hom}_{\mathcal{A}}(P, P)$, and the identity on P comes then from the desired section $\sigma: P \rightarrow M$.

Let us check finally that (3) \Rightarrow (1). We are given the following situation:

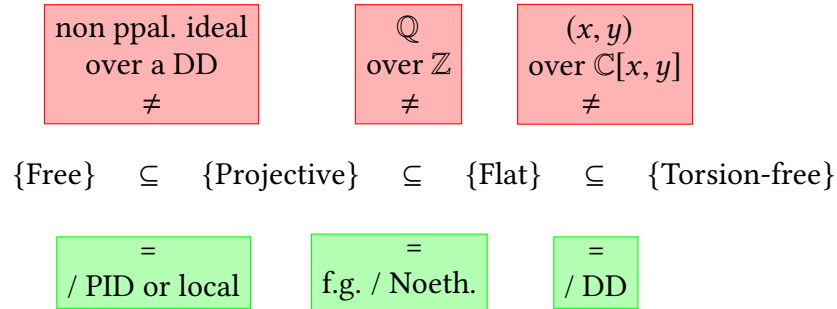
$$\begin{array}{ccc} & M & \\ & \downarrow \text{epi} & \\ P & \longrightarrow & N \end{array}$$

All finite limits exist in \mathcal{A} , so we may consider the cartesian square

$$\begin{array}{ccc} P \times_N M & \xrightarrow{g} & M \\ f \downarrow & & \downarrow \text{epi} \\ P & \longrightarrow & N \end{array}$$

Epimorphisms are stable under pullback in abelian categories, so f is also an epimorphism. By assumption, we can find a section $\sigma: P \rightarrow P \times_N M$ splitting the corresponding short exact sequence. The composition $g \circ \sigma: P \rightarrow M$ is then the desired lift. \square

Let us look now at the abelian category of modules over a ring R . What does it mean for an R -module to be projective?



REFERENCES

- [Hoc72] M. Hochster. Nonuniqueness of coefficient rings in a polynomial ring. *Proc. Amer. Math. Soc.*, 34:81–82, 1972. [↑ 1](#)

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