Insightful Examples

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Below, all talks are pairwise independent, except for the two talks about "blowing up $\mathbb{CP}^2$ in n points". We will only have 6 talks, so there is some freedom depending on your tastes and one topic we can drop.

Acknowledgement 1 *I heartily thank Mara, Lukas and Pedro for their help when preparing this, especially with the literature!*

Contents

1 (optional) Crazy abelian groups

Perhaps because finitely generated abelian groups have such a simple well-known structure theory, it is easy to be tempted to believe that abelian groups are in some sense "understood". This is completely false. We just look at two perhaps surprising facts: Already inside the rational numbers, there is a pretty huge supply of abelian groups:

Theorem 2 (Baer) *The set of isomorphism classes of subgroups $G \subseteq \mathbb{Q}$ is uncountable.*

Probably there is not enough time to prove this, but it follows from the theory of types ([Fuchs], Theorem 1.1 in Chapter 12). Abelian groups can easily contain copies of themselves, e.g., $\mathbb{Z}$. One can also find abelian groups G such that

$$G \cong G \oplus G.$$

That's actually easy: Take $G := \bigoplus_{i\in\mathbb{N}} \mathbb{Z}$ and pick any bijection between $\mathbb{N}$ and $\mathbb{N}\times\mathbb{N}$ to construct an isomorphism of the desired type.[1] However, a lot more tricky (or unbelievable) is the construction of groups with the following weird properties.

Theorem 3 (Sasiada) *There are abelian groups G, G' such that each is a direct summand of the other, yet $G \ncong G'$.*

Theorem 4 (Jónsson) *There exist non-isomorphic abelian groups G, H such that $G \oplus G \cong H \oplus H$.*

Theorem 5 (Corner) *There exists a countable torsion-free abelian group G such that $G \cong G^3$, yet $G \ncong G^2$.*

Note that Corner's example, when choosing $H := G\oplus G$ implies the one of Jónsson.
While Corner's construction has a textbook proof (linked below), I think following Jeremy Rickard's "friendly example" approach from 2019 is *much* easier.

Literature:

- J. Rickard: Pathological abelian groups: A friendly example, https://arxiv.org/pdf/1904.09327.pdf
- L. Fuchs: Infinite abelian groups, Chapter 13, Theorem 1.6, p486

2 Zariski Cancellation

This topic combines the following themes:

2.1 Combing the 2-sphere

2.2 Non-free projective modules

2.3 Zariski cancellation

Here are two formulations of differing generality:

[1] Such isomorphisms suggest that there might be an issue with monoidal structures on a category (like the direct sum) when not carefully distinguishing between the words "*isomorphism*" and "*equal*" in a category. This links to Topic 6.

Problem 6 (for geometers) *If X, Y are complex algebraic varieties, does $X \times \mathbb{C} \cong Y \times \mathbb{C}$ imply $X \cong Y$?*

Problem 7 (for algebraists) *If R, R' are rings, does an isomorphism of polynomial rings $R[T] \cong R'[T]$ imply $R \cong R'$?*

This is the famous *Zariski Cancellation Problem*, and the answer is negative!

Discuss Hochster's paper "Nonuniqueness of coefficient rings in a polynomial ring", which provides a clever and elegant counter-example to both problems.

The construction has fun side connections to other maths:

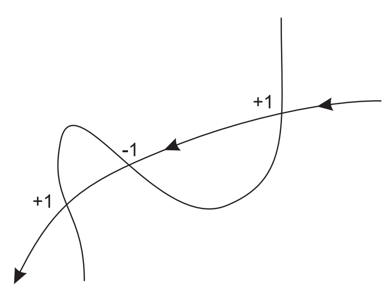
- in passing we find a projective module which is not free,
- and exploit that one cannot comb the sphere S^2.

Hochster's paper just refers to these things in passing, but perhaps see [Weibel, K-book, up and including Example 1.2.2] which both explains projective modules as well as the relevant construction used in Hochster's paper.[2]

3 Blow-ups of $\mathbb{CP}^2$ in up to 8 points

This talk and the next come as a pair and I would recommend preparing them in close communication.

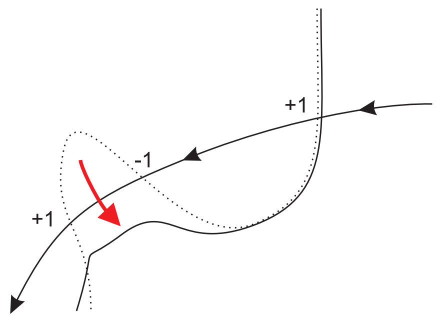
The idea of *Intersection Theory*[3] is to count how often geometric objects (within a bigger one) intersect or touch. Okay, let's be a little less vague: Imagine an oriented real surface. Then we can attach to oriented curves, at least if they meet transversally, a "count", according to whether the tangent vectors at the intersection point are oriented, i.e. span a local basis in the tangent space in the correct orientation or not. We then count the intersection with a count of $+1$ resp. -1.



[2] or see https://math.stackexchange.com/questions/3635305/proving-that-the-tangent-bundle-of-the-sphere-is-projective-but-not-free for further details.

[3] In Pedro's programme, we shall see intersection theory in terms of Chow groups. These theories are equivalent on smooth varieties when only considering intersection *numbers*. Chow groups are much richer (can see many homologically trivial algebraic cycles or torsion cycles), but only work on algebraic varities, while the intersection theory we use here also works on real manifolds, but cannot detect homologically trivial algebraic cycles (that already happens on an elliptic curve, for example).

This is useful for various purposes. One can update this theory to have very cool properties: **(1)** one can remove the need for the curves to intersect as cleanly as in the above picture. For example, it's easy to guess that the graph of $x \mapsto y := x^n$ intersects the x-axis for $n \geq 2$ albeit not tangentially, with a count of $\pm n$. This generalizes the idea that we get a nicer theory of zeros of polynomials if we count them "with multiplicity". And **(2)** one can make the "total intersection count" (i.e. when we add up all local intersection counts) invariant under deforming the curves. For example, when moving the "left" bump in the above picture below the other curve,



two intersection points disappear, but their total count contributed $0 = (+1) + (-1)$ anyway.

Even better, one can attach to curves a homology class and really the total intersection count only depends on the homology class. So far, so good. When looking at *complex* manifolds and *complex* submanifolds, an interesting phenomenon occurs: The local intersection multiplicity (in, say, the case of transversal intersection) can only be non-negative. That's essentially true because holomorphic maps always preserve the orientation.

This leads to a curious phenomenon: If you have a curve C on a surface[4], it may have a *negative self-intersection* count, say $C \cdot C = -1$. Since the local multiplicities can only be ≥ 0, this means that if C' is any curve in the same homology class of C, C' and C can impossibly meet transversally (for, if they did, the intersection count would need to be ≥ 0, and since the count only depends on the homology class $C \cdot C = C \cdot C'$).

This means that such curves are somehow "very rigidly" placed in the surrounding space. On $\mathbb{CP}^2$, the simplest surface, this phenomenon does not occur. But blow-ups create curves with negative self-intersection, so let's look at $\mathbb{CP}^2$ with one, two, three... points blown-up.

- Motivate intersection theory BRIEFLY.
- Since Intersection Theory already exists on real manifolds, you *can* bring even a negative self-intersection curve in a transversal intersection situation with another copy if you allow yourself non-holomorphic maps. The idea in David E Speyers answer to https://math.stackexchange.com/questions/51193/geometric-motivation-for-negative-self-intersection is probably quite instructive.

[4] when now...clandestinely...we have moved to complex surfaces... so, dimension 4 really...

- There is a very nice discussion of intersection theory in Griffitths & Harris, perhaps complemented nicely with the discussion in de Rham cohomology in Bott & Tu. I would recommend to *exclusively* use, perhaps, de Rham cohomology to set up intersection theory for this talk and neglect all the other viewpoints. Then the intersection pairing really descends to merely the exterior multiplication of "bump forms" (wording of Bott&Tu) and the global integration map $H^{2n}(X, \mathbb{R}) \to \mathbb{R}$. Try to avoid ever seriously defining the local multiplicities rigorously. We're here to discuss the examples, not to set up the theory in all its dryness.
  Rough idea: Sketch how a closed submanifold has an attached Poincare dual de Rham cohomology class. Define the pairing
  $$H^i(X, \mathbb{R}) \times H^{n-i}(X, \mathbb{R}) \longrightarrow H^n(X, \mathbb{R}) \cong \mathbb{R}$$
  (we can stay compact, so circumvent discussing compactly supported cohomology or stuff like that) and then, anachronistically, show how this definition of the intersection pairing reduces to the local ± 1 intersection multiplicities in the situation of transversal intersection (that's in Griffiths&Harris). Just make this plausible. At this point, we then have a well-defined intersection number, even for things which do not intersect transversally. That's the beauty of following this path for setting up the theory.

- Discuss $\mathbb{CP}^2$, blown up in up to 8 points, a little. Explicit example phenomena are found in

  1. http://math.stanford.edu/~vakil/02-245/sclass13A.pdf
  2. http://math.stanford.edu/~vakil/02-245/sclass14A.pdf

  As this talk prepares the next one, maybe already discuss the stuff in McKernan "Mori dream spaces" around where he writes "There is a standard trick in algebraic geometry", how $H^0(\text{Blowup of } \mathbb{P}^2 \text{ at points}, \mathcal{O}(dH - \sum a_i E_i))$ can be read as polynomials on $\mathbb{P}^2$ vanishing in the points.

- Discuss a little the relationship between (-1)-curves and blow-ups, e.g. sketching a little the meaning of Theorem 1.10 in McKernan ("Castelnuovo's Theorem"); keeping in mind how we want to use this theorem in the next talk. Coordinate this with the next speaker.

Remark 8 *What we omit in this talk, but maybe you want to mention it: All del Pezzo surfaces (up to $\mathbb{P}^1 \times \mathbb{P}^1$) are blow-ups of $\mathbb{CP}^2$ in ≤ 8 points.*[5]*. Also, any smooth cubic surface is the blow up of 6 points on $\mathbb{CP}^2$. This is linked to the 27 lines story of Pedro's programme*[6]*.*

[5] http://www.math.ucsd.edu/~jmckerna/Teaching/13-14/Spring/203C/l_23.pdf

[6] https://www.mathematik.uni-kl.de/~gathmann/class/alggeom-2014/alggeom-2014-c11.pdf, Remark 11.9

4 Blow-ups of $\mathbb{CP}^2$ in 9 points

This talk and the previous come as a pair and I would recommend preparing them in close communication.

This topic covers the following subtopics in one go:

4.1 Hilbert's 14th problem

4.2 Infinitude of (-1)-curves

The following goes back to Coble and Nagata.

Explain (following McKernan's "Mori dream spaces" **§1**) Hilbert's 14th problem. Then explain Nagata's counterexample.

There are **TWO** examples here packaged in one:

- a non-finitely generated ring of invariants, and
- (at the end of §1) infinitely many -1 curves [and nonpolyhedral cone of effective divisors]

Literature:
Alternative literature which can complement McKernan:

- Mukai, Counterexample to Hilbert's fourteenth problem for the 3-dimensional additive group
- Totaro, Hilbert's fourteenth problem over finite fields.... [ignore the aspect that this is over finite fields, it gives an explanation of the general story in quite broad terms]
- McKernan's text is the principal source though

5 (optional) The Figure 8 Knot, hyperbolicity & arithmeticity

Regard the "figure 8" knot in the 3-sphere S^3 (as opposed to considering knots in $\mathbb{R}^3$) and consider the complement of the knot. It turns out that this space is naturally a hyperbolic 3-manifold, i.e.

$$S^3 \setminus (\text{Figure 8 Knot}) \cong \mathbb{H}^3/\Gamma,$$

where Γ is some group of fixed-point free isometries of hyperbolic 3-space $\mathbb{H}^3$. In fact,

$$\Gamma = PSL_2\left(\mathbb{Z}\left[\frac{1+\sqrt{-3}}{2}\right]\right),$$

regarded as a subgroup of $PSL_2(\mathbb{C}) = Isom^+(\mathbb{H}^3)$, the orientation-preserving isometry group of $\mathbb{H}^3$.[7]

Based on this example, Thurston proved that for "most" knots (in a precisely understood way), they all are hyperbolic, i.e. of the form $\mathbb{H}^3/\Gamma$ for a suitable (kind of well-defined) Γ. In particular this attaches to all such knots a real number, which is a knot invariant: If two such knots have a different attached volume, the knots are not equivalent. In the computer program SnapPea you can draw a knot with your mouse and the program computes this volume (by constructing a triangulation for $S^3 \setminus K$ and finding a hyperbolic metric on it).

All this is the source of Thurston's (now proven) Geometrization Conjecture, leading to a classification of 3-manifolds.

Literature:

[Bonahon - Low-dimensional topology]

[Topological Picture Book - last chapter]

[Reid - Arithmeticity of knot complements]

There is also a YouTube-video where somebody explains this, Contact me for details.

6 (optional) Monoidal structures do not need to be specified with care, right?

A *monoidal structure* on a category $\mathcal{C}$ is a functor

$$\boxtimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$$

satisfying some axioms like associativity and having a unit. Two famous examples:

- the direct sum $\oplus$ of R-modules gives a monoidal structure
- the tensor product $\otimes$ of R-modules (for R commutative) gives a monoidal structure

It is completely common practice to never provide details on the associativity, which amounts to providing isomorphisms

$$\alpha_{X,Y,Z} : (X \boxtimes Y) \boxtimes Z \xrightarrow{\sim} X \boxtimes (Y \boxtimes Z) \tag{1}$$

for all objects X, Y, Z such that additional axioms are satisfied (like the pentagon axiom about rebracketing 4 objects).

We all like the following slogans:

Slogan 9 *Left and right object in Equation 1 are kind of the same anyway.*

Indeed, this can be made precise and true: Any monoidal category has a monoidal equivalence to one in which $(X \boxtimes Y) \boxtimes Z$ and $X \boxtimes (Y \boxtimes Z)$ are always the same object (so you can take $\alpha_{X,Y,Z}$ the identity map) and never need to worry about the order of brackets.

[7] Note that $\mathcal{O}_{\mathbb{Q}(\sqrt{-3})} = \mathbb{Z}\left[\frac{1+\sqrt{-3}}{2}\right]$ is the ring of integers of the number field $\mathbb{Q}(\sqrt{-3})$.

Slogan 10 *Isomorphic objects are kind of the same anyway.*

Indeed, this can be made precise: Every category is equivalent to "a skeleton", i.e. a category where each isomorphism class is contracted to just one object (for example, the category of finite-dimensional k-vector spaces is equivalent to the category just having $\{k^{\oplus n} \mid n \geq 0\}$ as objects).

However, if you like to tacitly make such simplifications in your proofs or thoughts, you're likely to oversimplify to something false:

Theorem 11 (Isbell) *There exist (symmetric) monoidal categories for which it is impossible to simultaneously make them, under monoidal equivalence, skeletal (i.e. $X \cong Y$ implies $X = Y$) and strictly associative (i.e. $(X \boxtimes Y) \boxtimes Z = X \boxtimes (Y \boxtimes Z)$).*

Isbell gives a concrete example. This example is essentially the justification why monoidal categories and all these axiomatics and diagrams are really needed.

As explained, the problem does *not* change if you restrict to symmetric (or braided) monoidal categories, i.e. where there is additionally an isomorphism

$$c_{X,Y} : X \boxtimes Y \xrightarrow{\sim} X \boxtimes Y$$

allowing you to swap the order.

Remark 12 *Another way to motivate being careful about associators is to regard the category of finite-dimensional $SU(2)$ representations and writing its objects as direct sums of simples, and setting up the tensor product by working out the $6j$-symbols.*

Literature:

- References: contact me (Oliver) directly.

The book MacLane, Categories for the working mathematician, has this material (in a slightly old-fashioned style) in the Chapter on "Symmetry and braiding in monoidal categories". Isbell's example is on page 164 of that book, but there are nicer presentations around

Joyal–Street's papers on *Braided Tensor Categories/Braided Monoidal Categories* also treat part of this material, in somewhat different style.

7 (optional) A category whose objects do not have an underlying set

In most categories of everyday use, the objects are just sets with some extra structure, and the morphisms are those maps preserving the structure. For example, groups, R-modules, topological spaces, rings,... are such categories.

Of course there are categories which don't have a clear concept of "underlying set", say the category of abelian sheaves on some space. However, there are often ways to get around this: By the Freyd–Mitchell Embedding Theorem every small abelian categories can be realized as a full subcategory of R-modules for some ring R, and then these modules have underlying sets. This means, for example, that you can test whether two maps are different by testing their values on all elements.

In particular, when you prove the snake lemma in small abelian categories, you are really allowed to work with elements.

However, categories are really more complicated than this. Freyd has shown that there is no way to regard the homotopy category of pointed spaces $hTop_*$ (which is in daily use in topology) as a concrete category. He proved:

Theorem 13 (Freyd) *There exists no faithful functor*

$$F : hTop_* \longrightarrow Sets.$$

The crux is not hidden in what generality of topological spaces we allow here; the proof works as soon as all finite-dimensional CW-complexes are allowed.

Theorem 14 (Freyd) *Let κ be any cardinal number. Then there exist finite-dimensional CW complexes X, Y and a non-nullhomotopic map $f : X \to Y$ such that $f\mid_{X'}$ becomes nullhomotopic for any subcomplex $X' \subseteq X$ with fewer than κ cells.*

The proof uses (necessarily) some category theory, and then a bit of group theory and (as you can imagine) a little bit of homotopy theory.

Literature:

- P. Freyd, *Homotopy is not concrete*, Repr. Theory Appl. Categ. No. 6 (2004), 1-10.

- https://amathew.wordpress.com/2012/01/26/homotopy-is-not-concrete/

ENUMERATIVE GEOMETRY WEDNESDAY SEMINAR

GK1821 "COHOMOLOGICAL METHODS IN GEOMETRY"

List of Talks (assuming 1h 30min / Talk)

Some suggestions that apply to all the talks:

- Working only over $\mathbb{C}$ sounds like a good idea, even if it is often unnecessary.
- The book *3264 and all that* written by Eisenbud and Harris [EH16] should provide a self-contained reference for the seminar. They often use schemes, but I think the book has a very geometric flavour and most arguments can be followed without knowing precisely what schemes are. There is also an introduction to schemes again with a very geometric flavour by the same authors [EH00].
- In fact I would try to avoid scheme-theoretic details altogether and instead try to draw many pictures and focus on examples with complex manifolds.
- There are connections and similarities with other areas, especially with singular homology in the first talks and with differential geometry in the last talks. I think it would be nice to hear about them every now and then.
- Every talk —except the first one— is meant to start with a brief recollection of previous talks. The idea is to refresh some important results and definitions and then apply them in some specific situation.

But these are only suggestions, feel free to do otherwise if you want/need at some point!

1. Talk 1 — Introduction and the Chow ring (16.12.20)

1.1. **Introduction and motivation.** What are enumerative problems and how can we go about solving them? See [EH16, §3.1.1]. What are —roughly speaking— intersection theory and the Chow ring, and what is their role in the process of solving an enumerative problem? See [EH16, §1.1] and [EH16, §3.1.1]. The problem that we will take as an example to introduce most of the concepts during the first four talks is the following

Question 1.1. How many lines in $\mathbb{P}^3$ intersect four given lines in general position?

Many thanks to Mara Ungureanu for her useful comments on a previous version of the programme!

Caveat 1.2. One should not jump to precise conclusions immediately after some numerical computation, see [EH16, §3.1.2]. The meaning of this caveat will become clearer with an example that we will discuss during the last two talks, namely

Question 1.3. How many lines does a smooth cubic surface $S \subseteq \mathbb{P}^3$ contain?

Using Chern classes we will see that the relevant class in the relevant Chow ring has degree 27, but this is not enough to conclude that there are precisely 27 lines.

1.2. **Algebraic varieties.** These have already appeared often in past Wednesday seminars, so hopefully we can keep this to a very brief introduction or recollection. Something that we will use in later talks is the degree of a projective variety. I think having the intuitive definition with hyperplane cuts in mind is enough, but perhaps mentioning at least that there is a definition using the Hilbert polynomial is a good idea. Other things that may be useful to mention:

- Huge open subsets (no strict Hausdorffness, no strict local $\mathbb{A}^n$-ness).
- The union of two intersecting lines is connected but not irreducible.
- Intersection of irreducible stuff is not necessarily irreducible.
- Dimension of varieties is defined by chains of irreducibles.

1.3. **Chow groups** [EH16, §1.2.1 and 1.2.2]**.** Cycles and rational equivalence. [EH16, Prop. 1.4] and [EH16, Prop. 1.10] are good to know. A nice picture to see what can happen is [EH16, Fig. 1.2].

1.4. **Ring structure** [EH16, Thm. 1.5]**.** Generic transversality and moving lemma [EH16, Thm. 1.6]. What goes wrong without the smoothness assumption? Example in [EH16, p. 20].

2. Talk 2 — Basic computational tools (13.01.21)

2.1. **Brief recollection of previous talk.** Cycles, rational equivalence and intersection product.

2.2. **Chow groups of affine spaces using what we saw in the previous talk.** For any variety X, the equivalence class $[X]$ is a free generator of $A^0(X)$. This can be argued using irreducibility and dimension. In the case of affine spaces, this free cyclic group is all there is [EH16, Prop. 1.13]. A nice picture to visualise the proof is [EH16, Fig. 1.7].

2.3. **Functoriality** [EH16, §1.3.6]**.** Proper pushforward and flat pullback without technical details. I wouldn't define properness and flatness too seriously, but it is good to know that inclusions of open subsets are flat morphisms and that any morphism between projective varieties is proper. Degree map [EH16, Prop. 1.21]. [EH16, Thm. 1.23] without details of the proof.

2.4. **Mayer–Vietoris and excision** [EH16, §1.3.4]**.**

2.5. **Affine stratifications** [EH16, §1.3.5]**.** With examples of what is or isn't a stratification, examples of quasi-affine stratifications that are not affine, etc. Totaro's theorem [EH16, Thm. 1.18] is nice to know, although it won't be used later on.

3. Talk 3 — Grassmannians and Schubert cycles (20.01.21)

3.1. **Brief recollection of the previous talk.** Proper pushforward, degree map and stratifications.

3.2. **Chow ring of projective space using what we saw in previous talks.** [EH16, Thm. 2.1] and corollaries [EH16, Cor. 2.2 and Cor. 2.3]. Bézout's theorem as a consequence of [EH16, Thm. 2.1].

3.3. **Grassmannians.** Definition and projective spaces as a particular case. Plücker embedding [EH16, §3.2.1] and affine open cover [EH16, §3.2.2]. I would suggest to focus just on the Grassmannian of lines in $\mathbb{P}^3$ to discuss these notions, since it is the one we will be using later on and I think it would make the notation more explicit and clear.

3.4. **Schubert cycles.** Again, I would stick to the case of $\mathbb{G}(1,3)$, discussed in detail in [EH16, §3.3.1]. In this case we can also draw nice pictures [EH16, Fig. 3.3]. State Kleiman's transversality [EH16, Thm. 1.7] and explain what it means for Schubert cycles with respect to different and generically positioned flags [EH16, pp. 105–106].

4. Talk 4 — How many lines intersect 4 random lines in space? (27.01.21)

4.1. **Brief recollection of previous talks.** Stratifications, $\mathbb{G}(1,3)$ and its Schubert cycles.

4.2. **Computation of the Chow ring using what we saw in previous talks.** This computation is carried out in [EH16, Thm. 3.10]. I would stress how to use Kleiman's transversality [EH16, Thm. 1.7] and the method of undetermined coefficients during the proof.

After this computation we can already answer Question 1.1. But this also seems like a good time to recall Caveat 1.2. Why is the precise answer to the question justified in this case?

4.3. **(Static) specialisation** [EH16, §3.5.1]**.** This is another useful technique to compute products of Schubert classes. As an example we can use it to compute σ_1^2 in a different way. This will require describing the tangent space of the Schubert cycle of lines intersecting a given line L at a point different from L [EH16, Exe. 3.26].

4.4. **Geometry behind the answer to Question 1.1.** Paraphrasing İzzet Coşkun: What shape do you produce when you want to cook spaguetti and you put them in a pot? What does the answer to this question have to do with Question 1.1? See [EH16, §3.4.1].

5. Talk 5 — Knutson–Tao puzzles and Chern classes (03.02.21)

The two topics in the title of this talk are not directly related to each other and they are only in the same talk because I couldn't figure out a better way to organise the contents. So this talk will probably have two very indepenedent parts:

(1) Recollections from previous talk and some more Schubert calculus.
(2) Introduction to Chern classes.

5.1. **Brief recollection of the previous talk.** Using the methods of undetermined coefficients and static specialisation to compute σ_1^2.

5.2. **Knutson–Tao puzzles.** These give a nice visual tool to compute products of Schubert classes. Besides various articles by Knutson, Tao and others, see e.g. [KTW04], there is also a nice YouTube video discussing them https://youtu.be/U8sq3BplCfI. Again, one can use the computation of σ_1^2 as an example, as is done in the video already.

The speaker may want to discuss other combinatoric methods to compute products of Schubert classes as well/instead, e.g. Young diagrams [EH16, §4.5]. Whatever they prefer.

5.3. **First Chern class of a line bundle** [EH16, §1.4]**.** A nice explicit example of computation could be $c_1(K_{\mathbb{P}^n})$. Geometric motivation to generalise and define higher Chern classes [EH16, §5.2].

5.4. **Axiomatic definition of Chern classes.** Maybe mentioning some ideas in the existence proof but without getting into details [EH16, §5.3]. It would also be nice to mention here that one can use Chern–Weil theory for a differential-geometric approach to the existence of Chern classes.

5.5. **The splitting principle** [EH16, §5.4]**.**

6. Talk 6 — How many lines does a smooth cubic surface contain? (10.02.21)

6.1. **Brief recollection from the previous talk.** Axiomatic definition of Chern classes and splitting principle.

6.2. **Derivation of some formulas using what we saw in the previous talk** [EH16, §5.5]**.**

6.3. **Tautological bundles** [EH16, §5.6]**.** Describing them just for $\mathbb{G}(1,3)$ and projective spaces should be enough. The example relevant for Question 1.3 is the dual of the tautological subbundle of rank 2 on our Grassmannian $\mathbb{G}(1,3)$. What are its Chern classes? See [EH16, §5.6.2]. Another nice example could be to compute explicitly —with coordinates— that the tautological line bundle on $\mathbb{P}^n$ is $\mathscr{O}(-1)$.

6.4. **Counting lines on a cubic surface.** Following the argument in [EH16, Thm. 5.1] and using [EH16, §5.6.2] and [EH16, §6.2.1]. Recall Caveat 1.2 and the discussion in [EH16, §3.1.2]. What does the argument in [EH16, Thm. 5.1] show and what remains to be shown in order to give a precise answer to Question 1.3? If time permits and the speaker wants, it might be nice to also briefly sketch the rest of the argument involving the Fano variety of lines [EH16, §6.1–6.2].

References

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[KTW04] Allen Knutson, Terence Tao, and Christopher Woodward. The honeycomb model of $\mathrm{GL}_n(\mathbb{C})$ tensor products. II. Puzzles determine facets of the Littlewood-Richardson cone. *J. Amer. Math. Soc.*, 17(1):19–48, 2004.

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