A Proof of the Hairy Ball Theorem

Author(s): Murray Eisenberg and Robert Guy

Source: *The American Mathematical Monthly*, Aug. - Sep., 1979, Vol. 86, No. 7 (Aug. - Sep., 1979), pp. 571-574

Published by: Taylor & Francis, Ltd. on behalf of the Mathematical Association of America

Stable URL: https://www.jstor.org/stable/2320587

9. W. Orlicz, On the continuity of linear operations in Saks spaces with an application to the theory of summability, Studia Math., 16 (1957) 69–73.

10. G. M. Petersen, Summability methods and bounded sequences, J. London Math. Soc., 31 (1956) 324–326.

11. H. L. Royden, Real Analysis, Macmillan, New York, 1963.

12. I. Schur, Uber Lineare Transformationen in der theorie der unendlichen Reihen, J. Reine Angew. Math., 151 (1920) 79–111.

13. R. Sikorski, A remark on the Mazur-Orlicz theory of summability, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astr. Phys., 3 (1955) 11–15.

14. L. L. Silverman, On the definition of the sum of a divergent series, University of Missouri Studies, Math. Ser. 1 (1913) 1–96.

15. A. K. Snyder and A. Wilansky, The bounded consistency and sequential completeness theorems (to appear).

16. O. Toeplitz, Über allgemeine lineare Mittelbildungen, Prace Mat. Fiz., 22 (1911) 113–120.

17. A. Wilansky, Functional Analysis, Blaisdell, New York, 1964.

18. K. Zeller and W. Beekmann, Theorie der Limitierungsverfahren, Springer Verlag, Berlin, Heidelberg, New York, 1970.

19. K. Zeller, Faktorfolgen bei Limitierungsverfahren, Math. Z., 56 (1952) 134–151.

DEPARTMENT OF MATHEMATICAL SCIENCES, CLEMSON UNIVERSITY, CLEMSON, SC 29631.

CORRECTION TO

"The 'Why-Don't-You-Just...?' Barrier in Discrete Algorithms"

(This MONTHLY, 86 (1979) 30-36)

HERBERT S. WILF

The average number of comparisons in the binary search tree method is $O(k \log k)$, and not $Ck^{3/2}$ as stated. The reason is that the various trees are not equally likely to occur. A proof is in [2, vol. 3, p. 427]. This does not affect the ranking of this method in the hierarchy of the methods discussed.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PENNSYLVANIA, PHILADELPHIA, PA 19104.

MATHEMATICAL NOTES

EDITED BY DEBORAH TEPPER HAIMO AND FRANKLIN TEPPER HAIMO

Material for this department should be sent to Professor Deborah Tepper Haimo, Department of Mathematical Sciences, University of Missouri, St. Louis, MO 63121.

Advice to prospective authors: The editors have recently been receiving about **ten times** as many Mathematical Notes as can be used. It will simplify our work if authors will submit only especially interesting manuscripts. Mathematical Notes should be short papers of one to four printed pages which give new insights, new and improved proofs of old theorems, brief bits of mathematical folklore that have not found a home in the literature, or (occasionally!) new results that are not too technical. The topics should be of wide current interest. Papers that have already been rejected by a research journal are only very rarely suitable as Mathematical Notes.

R.P.B.

A PROOF OF THE HAIRY BALL THEOREM

MURRAY EISENBERG AND ROBERT GUY

Introduction. On the 2-sphere

$$S^2 = \{\mathbf{p} \in \mathbf{R}^3 : \mathbf{p} = (x,y,z), x^2+y^2+z^2=1\}$$

a **continuous tangent vector field** is a continuous vector-valued map V: $S^2 \to \mathbf{R}^3$ such that at each $\mathbf{p} \in S^2$ the dot product $\mathbf{p} \cdot V(\mathbf{p}) = 0$. The vector field **vanishes** at $\mathbf{p}$ when $V(\mathbf{p}) = \mathbf{0}$ (the zero vector); V is **nonvanishing** when it vanishes at no $\mathbf{p} \in S^2$.

The theorem of the title is:

THEOREM. *Any continuous tangent vector field on S^2 must vanish somewhere.*

This theorem is due to Poincaré [**9**, Chap. 13], who deduced it from his two-dimensional precursor of the Poincaré-Hopf theorem, which equates the sum of the indices of the vector field at its singular points with the Euler characteristic (cf. [**6**, p. 35]). The well-known proof due to Brouwer [**2**], valid for all even-dimensional spheres, uses the degree of a map between spheres, defined by means of homology groups or directly by means of simplicial subdivision (e.g., see [**5**, p. 70] and [**4**, p. 343]). Milnor [**6**, p. 31] also uses degree, but as defined by methods of differential topology. The editor has informed us that Milnor has another proof, analytic in nature, which appeared in this MONTHLY [**7**]. Boothby [**1**] gives a proof involving differential forms that avoids any overt use of algebraic topology.

Our proof will be elementary in that it uses only the fundamental group of the circle. Other elementary proofs are known. Munkres [**8**, pp. 367–368] also uses that fundamental group, but in a different way. Chinn and Steenrod [**3**, pp. 123–126] give a proof quite similar to ours, but employing the index of a plane vector field with respect to a closed curve.

Further discussions of the theorem in the context of related results appear in [**10**] and [**11**].

Prerequisites. We shall use the degree, $\deg f$, of a map f: $S^1 \to S^1$ from the unit circle

$$S^1 = \{(x,y) \in \mathbf{R}^2 : x^2+y^2=1\}$$

to itself. This degree is an integer defined by means of the homomorphism that f induces between the fundamental groups of S^1 at a point and at its image (see [**8**, p. 373]). However, we do not need the explicit definition, but only three properties of degree together with two simple facts about homotopy:

(1) If continuous maps $f,g : S^1 \to S^1$ are homotopic, then $\deg f = \deg g$.

(2) Any constant map $c : S^1 \to S^1$ has degree 0.

(3) The map

$$f_2 : S^1 \to S^1 : (x,y) \mapsto (x^2-y^2, 2xy)$$

(which doubles the polar angle of each point on S^1) has degree 2.

(4) A continuous map $f : S^1 \to S^1$ is homotopic to a constant map if it has a continuous extension $F : D^2 \to S^1$ to the closed 2-disk $D^2 = \{(x,y) \in \mathbf{R}^2 : x^2+y^2 \leqslant 1\}$.

(5) Any two constant maps $S^1 \to S^1$ are homotopic.

Proof of Theorem. Suppose V is a continuous tangent vector field on S^2 that is nonvanishing. Express V in components as

$$V(\mathbf{p}) = (A(\mathbf{p}), B(\mathbf{p}), C(\mathbf{p})).$$

Identify $\mathbf{R}^2$ with the plane $\mathbf{R}^2 \times \{0\}$ in $\mathbf{R}^3$ that intersects S^2 in its equator; thereby identify D^2 with $D^2 \times \{0\}$, and the equator $S^1 \times \{0\}$ with the circle S^1.

Construct a continuous nonvanishing map

$$W : D^2 \to \mathbf{R}^2$$

as follows. The stereographic projection through the north pole $(0,0,1)$ of S^2 maps the complement of $(0,0,1)$ onto $\mathbf{R}^2$ and maps the lower hemisphere

$$S^2_- = \{(x,y,z) \in S^2 : z \leqslant 0\}$$

homeomorphically onto D^2; let $h: D^2 \to S^2_-$ be the inverse homeomorphism. Explicitly,

$$h(x,y) = \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1} \right).$$

For $\mathbf{r} = (x,y) \in D^2$, define

$$W(\mathbf{r}) = (A(h(\mathbf{r})) + C(h(\mathbf{r}))x, \quad B(h(\mathbf{r})) + C(h(\mathbf{r}))y).$$

(In geometric terms, $W(\mathbf{r})$ is the vector in $\mathbf{R}^2$ from $\mathbf{r}$ to $\mathbf{p}$, where $\mathbf{p}$ is the point at which the plane $\mathbf{R}^2$ meets the line passing through $h(\mathbf{r}) + V(h(\mathbf{r}))$ and parallel to the line joining $\mathbf{r}$ to the north pole.) Clearly W is continuous. It is nonvanishing because, by direct computation, $h(\mathbf{r}) \cdot V(h(\mathbf{r})) = 0$ and $W(\mathbf{r}) = \mathbf{0}$ imply that $A(h(\mathbf{r})) = B(h(\mathbf{r})) = C(h(\mathbf{r})) = 0$, contrary to the assumption that V is nonvanishing.

Repeat the preceding construction, but starting with stereographic projection from the south pole, to get another continuous nonvanishing map

$$W^* : D^2 \to \mathbf{R}^2$$

with

$$W^*(\mathbf{r}) = (A(h^*(\mathbf{r})) - C(h^*(\mathbf{r}))x, \quad B(h^*(\mathbf{r})) - C(h^*(\mathbf{r}))y)$$

for $\mathbf{r} = (x,y) \in D^2$. Here $h^* : D^2 \to S^2_+$ is the homeomorphism onto the upper hemisphere obtained by following $h : D^2 \to S^2_-$ with the reflection $(x,y,z) \mapsto (x,y,-z)$ about the equatorial plane.

Now consider the positively oriented unit tangent vector field $T : S^1 \to \mathbf{R}^2$ on the circle, given by

$$T(x,y) = (-y,x).$$

For $\mathbf{r} = (x,y) \in S^1$, reflect each vector $\mathbf{v} \in \mathbf{R}^2$ about $T(\mathbf{r})$ to get a vector $R_{\mathbf{r}}(\mathbf{v})$. Then

$$W^*(\mathbf{r}) = R_{\mathbf{r}}(W(\mathbf{r})) \qquad (\mathbf{r} \in S^1). \tag{6}$$

In fact, for $\mathbf{r} \in S^1$ the equality $h^*(\mathbf{r}) = h(\mathbf{r})$ together with the definitions of W and W^* yield $W(\mathbf{r}) \cdot T(\mathbf{r}) = W^*(\mathbf{r}) \cdot T(\mathbf{r})$. Then $W^*(\mathbf{r}) = R_{\mathbf{r}}(W(\mathbf{r}))$ or $W(\mathbf{r}) = W^*(\mathbf{r})$. But in the latter case $C(h(\mathbf{r})) = 0 = C(h^*(\mathbf{r}))$ and so $W(\mathbf{r}) = W^*(\mathbf{r}) = T(\mathbf{r})$.

Since neither W nor W^* vanishes, the formulas

$$F(\mathbf{r}) = W(\mathbf{r})/|W(\mathbf{r})|, \qquad F^*(\mathbf{r}) = W^*(\mathbf{r})/|W^*(\mathbf{r})|$$

define continuous maps $F, F^* : D^2 \to S^1$. Let $f, f^* : S^1 \to S^1$ be their restrictions to the circle. From (1), (2), and (4),

$$\deg f = 0 = \deg f^* \tag{7}$$

and there is a homotopy $(H_t : 0 \leqslant t \leqslant 1)$ from f to a constant map $c : S^1 \to S^1$. In view of (5) we may assume the constant value $\mathbf{v}$ of the map c is $\mathbf{v} = (-1,0)$. By (6) we have $f^*(\mathbf{r}) = R_{\mathbf{r}}(f(\mathbf{r}))$ for all $\mathbf{r} \in S^1$. Hence the formula

$$H_t^*(\mathbf{r}) = R_{\mathbf{r}}(H_t(\mathbf{r}))$$

defines a homotopy $(H_t^* : 0 \leqslant t \leqslant 1)$ from f^* to the map

$$c^* : S^1 \to S^1 : \mathbf{r} \mapsto R_{\mathbf{r}}(c(\mathbf{r})) = R_{\mathbf{r}}(\mathbf{v}).$$

From (1) and (7), $\deg c^* = 0$.

To conclude, we show, to the contrary, that $\deg c^* = 2$. We show, in fact, that $c^* = f_2$, the map defined in (3). Let $\mathbf{r} \in S^1$. Recall that $c^*(\mathbf{r}) = R_{\mathbf{r}}(\mathbf{v})$. By direct computation,

$$\mathbf{v} \cdot T(\mathbf{r}) = f_2(\mathbf{r}) \cdot T(\mathbf{r}).$$

Hence $f_2(\mathbf{r}) = \mathbf{v}$ or $f_2(\mathbf{r}) = R_{\mathbf{r}}(\mathbf{v})$. But the former case occurs only when $\mathbf{r} = (0, \pm 1)$, and then $R_{\mathbf{r}}(\mathbf{v}) = \mathbf{v} = f_2(\mathbf{r})$.

References

1. W. M. Boothby, On two classical theorems of algebraic topology, this MONTHLY, 78 (1971) 237–249.
2. L. E. J. Brouwer, Über Abbildung von Mannigfaltigkeiten, Math. Ann., 71 (1912) 97–115.
3. W. G. Chinn and N. E. Steenrod, First Concepts of Topology, Mathematical Association of America, Washington, D.C., 1966.
4. James Dugundji, Topology, Allyn and Bacon, Boston, 1966.
5. Marvin Greenberg, Lectures on Algebraic Topology, W. A. Benjamin, New York, 1967.
6. John W. Milnor, Topology from the Differentiable Viewpoint, University of Virginia Press, Charlottesville, 1965.
7. ______, Analytic proofs of the "hairy ball theorem" and the Brouwer fixed point theorem, this MONTHLY, 85 (1978) 521–524.
8. James R. Munkres, Topology: A First Course, Prentice-Hall, Englewood Cliffs, N.J., 1975.
9. H. Poincaré, Sur les courbes définies par les équations différentielles, J. Math. Pures Appl., (4) 1 (1885) 167–244.
10. A. W. Tucker, Some topological properties of disk and sphere, Proc. First Canadian Math. Congress (Montreal, 1945), University of Toronto Press, Toronto, 1946, pp. 285–309.
11. E. F. Whittlesey, Fixed points and antipodal points, this MONTHLY, 70 (1963) 807–821.

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF MASSACHUSETTS, AMHERST, MA 01003.

MINIMAL INFINITE TOPOLOGICAL SPACES

JOHN GINSBURG AND BILL SANDS

In this note we will describe a collection of five infinite topological spaces having the property that every infinite space contains one of the members of the collection as a subspace. We denote the set of natural numbers by ω. Consider the following five topologies with underlying set ω:

(i) *discrete*: all subsets of ω are open;
(ii) *indiscrete*: the only open sets are ω and $\varnothing$;
(iii) *cofinite*: the open sets are ω, $\varnothing$, and all subsets of ω whose complements are finite;
(iv) *initial segment*: the open sets are ω, $\varnothing$, and all sets of the form $[0,n] = \{k \in \omega : k \leqslant n\}$ where $n \in \omega$;
(v) *final segment*: the open sets are ω, $\varnothing$, and all sets of the form $[n,\omega] = \{k \in \omega : n \leqslant k\}$ where $n \in \omega$.

We will establish the following result.

THEOREM. *Every infinite topological space contains one of the preceding five spaces as a subspace.*

Note that no two of the five spaces are homeomorphic, and each of the five spaces is homeomorphic to all of its infinite subspaces. It follows that these five spaces form the smallest collection of infinite spaces satisfying the conclusion of the theorem.

Before proceeding with the proof, let us compare our result with the analogous situation in some other mathematical structures.

(a) Let G be a graph with infinitely many vertices. Ramsey ([3], cf. [2, page 15]) showed that