ZARISKI CANCELLATION

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ABSTRACT. Following [Hoc72] we provide an example of rings[1] B, C such that $B \not\cong C$ but $B[t] \cong C[t]$, where t is an indeterminate. As a preparation for this counterexample we also study the notion of projective module and the hairy ball theorem.

CONTENTS

—parts in gray will be omitted during the talk—

1. INTRODUCTION

Let R be a ring. We can form the polynomial ring $R[t]$ in one variable t with coefficients in R. This construction is functorial, and hence

$$R \cong S \Rightarrow R[t] \cong S[t].$$

The goal of this talk is to show with an explicit counterexample due to Hochster [Hoc72] that the converse is not true.

In the process of constructing this counterexample we will come across a projective module which, as a consequence of the hairy ball theorem, is not a free module. Therefore we will discuss projective modules and the hairy ball theorem before jumping into the counterexample.

2. Projective modules

Definition 2.1. Let C be a category. An object $P \in C$ is called *projective* if the following lifting problem can always be solved:

$$
\begin{array}{ccc}
 & & M \\
 & \nearrow & \downarrow \text{epi} \\
P & \longrightarrow & N
\end{array}
$$

Lemma 2.2. *Let $\mathcal{A}$ be an abelian category and let $P \in \mathcal{A}$ be an object. The following are equivalent:*

(1) P is projective.
(2) $\mathrm{Hom}(P, -)$ is exact.
(3) Every short exact sequence of the form

$$0 \to N \to M \to P \to 0$$

splits.

Proof. We start with $(1) \Rightarrow (2)$. Assume P is projective and consider a short exact sequence

$$0 \to A \to B \to C \to 0.$$

Since $\mathrm{Hom}(P, -)$ is always left exact, we only need to show that the induced map $\mathrm{Hom}(P, B) \to \mathrm{Hom}(P, C)$ is surjective. But $B \to C$ is an epimorphism, so this is precisely what P being projective means by definition.

Next we show $(2) \Rightarrow (3)$. Assume $\mathrm{Hom}(P, -)$ is exact and consider a short exact sequence

$$0 \to N \to M \to P \to 0.$$

Applying $\mathrm{Hom}(P, -)$ we get a surjection $\mathrm{Hom}(P, M) \to \mathrm{Hom}(P, P)$, and the identity on P comes then from the desired section $\sigma \colon P \to M$.

The implication $(3) \Rightarrow (1)$ is left as an exercise during the talk. The hint is that epimorphisms are stable under pullback in abelian categories, and the solution follows in gray. We are given the following situation:
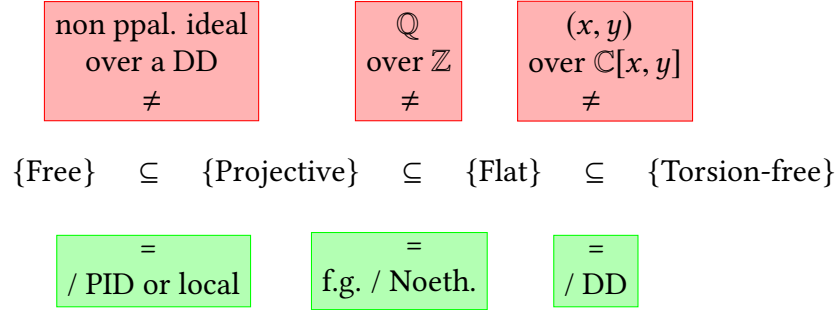
$$
\begin{array}{ccc}
 & & M \\
 & & \downarrow \text{epi} \\
P & \longrightarrow & N
\end{array}
$$

All finite limits exist in $\mathcal{A}$, so we may consider the cartesian square

$$
\begin{array}{ccc}
P \times_N M & \xrightarrow{\ g\ } & M \\
f \downarrow & & \downarrow \text{epi} \\
P & \longrightarrow & N
\end{array}
$$

Epimorphisms are stable under pullback in abelian categories, so f is also an epimorphism. By assumption, we can find a section $\sigma \colon P \to P \times_N M$ splitting the corresponding short exact sequence. The composition $g \circ \sigma \colon P \to M$ is then the desired lift. $\qquad \square$

Let us look now at the abelian category of modules over a ring R. What does it mean for an R-module to be projective?

non ppal. ideal over a DD $\neq$		$\mathbb{Q}$ over $\mathbb{Z}$ $\neq$	(x, y) over $\mathbb{C}[x, y]$ $\neq$		
{Free}	$\subseteq$	{Projective} $\subseteq$ {Flat}	$\subseteq$	{Torsion-free}	
$=$ / PID or local		$=$ f.g. / Noeth.	$=$ / DD		

In the above diagram, ppal. stands for principal, DD stands for Dedekind domain, PID stands for principal ideal domain, f.g. stands for finitely generated and Noeth. stands for Noetherian.

Proof. To do! $\qquad \square$

Remark 2.3. The example of $\mathbb{Q}$ as a $\mathbb{Z}$-module also shows that being projective depends on the base ring, since $\mathbb{Q}$ is projective over itself.

Remark 2.4. A ring R is called *hereditary* if all submodules of projective R-modules are again projective. Dedekind domains can be characterised as hereditary integral domains.

Remark 2.5. The example of (x, y) as a $\mathbb{C}[x, y]$-module also shows that $\mathbb{C}[x, y]$ is not an hereditary ring, since $\mathbb{C}[x, y]$ is projective over itself but its submodule (x, y) is not flat, thus not projective.

Remark 2.6. The left-most equality in the diagram above is also true for finitely generated modules over polynomial rings with coefficients on a principal ideal domain. This is am important result, first asked by Serre in the case of fields, and later proven independently by Quillen and Suslin, see the Wikipedia article on the Quillen–Suslin theorem. It corresponds geometrically to the statement that vector bundles on affine space are all trivial.

Remark 2.7 ([Fra18]). Baer's criterion [Fra18, Prop. 1.1.1] allows us to characterise the dual notion of injective modules only in terms of Ext^1_R and quotients of the ring R by its ideals. Indeed, an R-module N is injective if

and only if $\text{Ext}^1_R(R/I, N) = 0$ for every ideal $I \subseteq R$ [Fra18, Prop. 1.1.4]. In general, no similar criterion for projectivity is available. In fact, the *Whitehead problem* —which states that if an abelian group A has $\text{Ext}^1_{\mathbb{Z}}(A, \mathbb{Z}) = 0$, then it is free— is undecidable in ZFC due to a result of Shelah.

On the other hand, the notion of flatness, which agrees with projectivity for finitely generated modules over Noetherian rings, does have a general criterion similar to the previous one for injectivity. Namely, an R-module M is flat if and only if $\text{Tor}^R_1(M, R/I) = 0$ for every ideal $I \subseteq R$ [Fra18, Prop. 1.2.3].

Nevertheless, projectivity admits a different kind of criterion that will be useful for us later on:

Proposition 2.8 ([Fra18, Lemma 1.1.2])**.** *Let R be a ring. An R-module P is projective if and only if it is a direct summand of some free module F. Moreover, if P is finitely generated, then we can also choose F to be finitely generated.*

Proof. If P is (finitely generated) projective, consider a surjection $F \twoheadrightarrow P$ from a (finitely generated) free module F. The resulting short exact sequence

$$0 \to K \to F \to P \to 0$$

splits by Lemma 2.2, hence P is a direct summand of F.

Suppose conversely that $F = P \oplus K$ and consider a surjection $\varphi \colon M \twoheadrightarrow N$ and a morphism $f \colon P \to N$. Then $\varphi \oplus \text{id}_K \colon M \oplus K \twoheadrightarrow N \oplus K$ is again a surjection. Since F is free, the morphism $f \oplus \text{id}_K \colon F \to N \oplus K$ can be lifted to a morphism $\bar{f} \oplus \text{id}_K \colon F \to M \oplus K$, so that $\bar{f} \colon P \to M$ is the desired lifting of the original surjection $\varphi \colon M \twoheadrightarrow N$.   □

As a consequence of this criterion we also obtain a characterisation of projectivity for finitely presented modules in terms of Ext^1_R and finitely generated modules:

Corollary 2.9 ([Fra18, Cor. 1.1.28])**.** *Let R be a ring. A finitely presented R-module P is projective if and only if $\text{Ext}^1_R(P, T) = 0$ for every finitely generated R-module T.*

Proof. Pick a presentation

$$0 \to K \to F \to P \to 0$$

with K finitely generated and F finitely generated and free. Then take the long exact sequence of $\text{Ext}^\bullet_R(P, -)$ and use the assumption to find a splitting of the short exact sequence, exhibiting therefore P as a direct summand of a free module.   □

3. Hairy ball theorem

In this section we will prove the hairy ball theorem following [EG79].

Theorem 3.1 (Hairy ball theorem). *Every continuous vector field on the 2-sphere $\mathbb{S}^2$ has at least one zero.*

*Proof.* We will use the following definitions and identifications:

- $\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$.
- $\mathbb{R}^2 = \{(x, y, z) \in \mathbb{R}^3 \mid z = 0\}$.
- $\mathbb{D}^2 = \{(x, y, 0) \in \mathbb{R}^2 \subseteq \mathbb{R}^3 \mid x^2 + y^2 \leqslant 1\}$.
- $\mathbb{S}^1 = \{(x, y, 0) \in \mathbb{R}^2 \subseteq \mathbb{R}^3 \mid x^2 + y^2 = 1\}$.
- $\mathbb{S}^2_+ = \{(x, y, z) \in \mathbb{S}^2 \subseteq \mathbb{R}^3 \mid z \geqslant 0\}$.
- $\mathbb{S}^2_- = \{(x, y, z) \in \mathbb{S}^2 \subseteq \mathbb{R}^3 \mid z \leqslant 0\}$.
- For $\vec{v} \in \mathbb{R}^2 \setminus \{0\}$ we denote by $R_{\vec{v}} \colon \mathbb{R}^2 \to \mathbb{R}^2$ the reflection with respect to the direction defined by $\vec{v}$.

We prove the theorem by contradiction, so suppose that we are given a non-vanishing continuous vector field X on $\mathbb{S}^2$.

Let $h \colon \mathbb{D}^2 \to \mathbb{S}^2_-$ be the inverse of the homeomorphism $\mathbb{S}^2_- \cong \mathbb{D}^2$ induced by the stereographic projection from the north pole $(0, 0, 1) \in \mathbb{S}^2$. Let $p \in \mathbb{D}^2$ be a point. Let L_p be the line joining $(0, 0, 1)$ and p, which then intersects $\mathbb{S}^2_-$ precisely at $h(p)$. Let $\vec{v}_p$ denote the corresponding tangent vector $X(h(p))$. We consider the line L'_p which is parallel to L_p and passes through the point $h(p) + \vec{v}_p \in \mathbb{R}^3$. With this notation, we define a new function

$$W \colon \mathbb{D}^2 \longrightarrow \mathbb{R}^2$$

$$p \longmapsto (\mathbb{R}^2 \cap L'_p) - p.$$
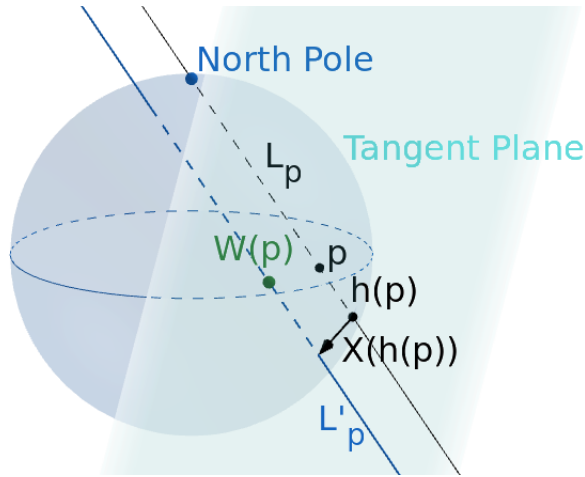
Since X is a non-vanishing continuous vector field, $W\colon \mathbb{D}^2 \to \mathbb{R}^2$ is a non-vanishing continuous function. Therefore it induces a continuous function

$$F\colon \mathbb{D}^2 \longrightarrow \mathbb{S}^1$$

$$p \longmapsto \frac{F(p)}{|F(p)|}.$$

We repeat the same process with the south pole and $\mathbb{S}^2_+$. This gives us a non-vanishing continuous function $W^*\colon \mathbb{D}^2 \to \mathbb{R}^2$ that we can again normalise into a continuous function $F^*\colon \mathbb{D}^2 \to \mathbb{S}^1$.

We denote by f and f^* the restrictions to $\mathbb{S}^1 \subseteq \mathbb{D}^2$ of F and F^* respectively. By definition f and f^* factor through the contractible space $\mathbb{D}^2$:

$$
\begin{array}{ccc}
\mathbb{S}^1 & \xrightarrow{\ f,f^*\ } & \mathbb{S}^1 \\
 & \searrow \quad \nearrow_{F,F^*} & \\
 & \mathbb{D}^2 &
\end{array}
$$

Therefore, they are both nullhomotopic.

The functions $f, f^*\colon \mathbb{S}^1 \to \mathbb{S}^1$ are also related to each other. Namely, ...

Let $H\colon \mathbb{S}^1 \times [0,1] \to \mathbb{S}^1$ be a homotopy from f to the constant map $c\colon \mathbb{S}^1 \to \mathbb{S}^1$ with constant value the poitn $(-1, 0, 0) \in \mathbb{S}^1 \subseteq \mathbb{R}^3$.   □

References

[EG79] Murray Eisenberg and Robert Guy. A proof of the hairy ball theorem. *Amer. Math. Monthly*, 86(7):572–574, 1979. ↑ 5

[Fra18] Jens Franke. Homological Methods in Commutative Algebra, 2018. Lecture notes by Ferdinand Wagner, available at github.com/Nicholas42/AlgebraFranke. ↑ 3, 4

[Hoc72] M. Hochster. Nonuniqueness of coefficient rings in a polynomial ring. *Proc. Amer. Math. Soc.*, 34:81–82, 1972. ↑ 1

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