Let $A$ be our $\mathbb{R}$-algebra and $\sigma\colon \mathbb{C} \to \mathbb{C}$ complex conjugation. Then we have the pushout diagram:

$$\begin{tikzcd}
 & & A_{\mathbb{C}} \\
A \arrow[r, "j"] \arrow[urr, bend left, "j"] & A_{\mathbb{C}} \arrow[ur, dashed, "\tilde{\sigma}"'] & \\
\mathbb{R} \arrow[u, "s"] \arrow[r, "i"] & \mathbb{C} \arrow[u, "t"] \arrow[uur, bend right, "t\circ\sigma"'] &
\end{tikzcd}$$

We know $A_{\mathbb{C}}^{\times} = t(\mathbb{C}^{\times})$, $s(\mathbb{R}^{\times}) \subseteq A^{\times}$ and $j(A^{\times}) \subseteq t(\mathbb{C}^{\times})$. We need $A^{\times} \subseteq s(\mathbb{R}^{\times})$ as well.

If $u \in A^{\times}$, then $j(u) = t(\lambda)$ for some $\lambda \in \mathbb{C}^{\times}$. Then

$$t(\lambda) = j(u) = \tilde{\sigma}(j(u)) = \tilde{\sigma}(t(\lambda)) = t(\sigma(\lambda)),$$

so $\lambda = \sigma(\lambda)$ because $t$ is injective. Hence $\lambda = i(\mu)$ for some $\mu \in \mathbb{R}^{\times}$ and $j(u) = j(s(\mu))$. And by injectivity again this implies $u = s(\mu)$.