ZARISKI CANCELLATION

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Abstract. Following [Hoc72] we provide an example of rings[1] B, C such that $B \not\cong C$ but $B[t] \cong C[t]$, where t is an indeterminate. As a preparation for this counterexample we also study the notion of projective module and the hairy ball theorem.

Contents

—parts in gray will be omitted during the talk—

1. Introduction

Let R be a ring. We can form the polynomial ring $R[t]$ in one variable t with coefficients in R. This construction is functorial, and hence

$$R \cong S \Longrightarrow R[t] \cong S[t].$$

The goal of this talk is to show with an explicit counterexample due to Hochster [Hoc72] that the converse is not true.

In the process of constructing this counterexample we will come across a projective module which, as a consequence of the hairy ball theorem, is not a free module. Therefore we will discuss projective modules and the hairy ball theorem before jumping into the counterexample.

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2. Projective modules

Definition 2.1. Let C be a category. An object $P \in C$ is called *projective* if the following lifting problem can always be solved:

$$\begin{tikzcd} & M \arrow[d, two heads, "\text{epi}"] \\ P \arrow[r] \arrow[ur, dashed] & N \end{tikzcd}$$

Lemma 2.2. *Let $\mathcal{A}$ be an abelian category and let $P \in \mathcal{A}$ be an object. The following are equivalent:*

(1) P is projective.
(2) $\mathrm{Hom}_{\mathcal{A}}(P, -)$ is exact.
(3) Every short exact sequence of the form

$$0 \to N \to M \to P \to 0$$

splits.

Proof. We start with (1) $\Rightarrow$ (2). Assume P is projective and consider a short exact sequence

$$0 \to A \to B \to C \to 0.$$

Since $\mathrm{Hom}_{\mathcal{A}}(P, -)$ is always left exact, we only need to show that the induced map $\mathrm{Hom}_{\mathcal{A}}(P, B) \to \mathrm{Hom}_{\mathcal{A}}(P, C)$ is surjective. But $B \to C$ is an epimorphism, so this is precisely what P being projective means by definition.

Next we show (2) $\Rightarrow$ (3). Assume $\mathrm{Hom}_{\mathcal{A}}(P, -)$ is exact and consider a short exact sequence

$$0 \to N \to M \to P \to 0.$$

Applying $\mathrm{Hom}_{\mathcal{A}}(P, -)$ we get a surjection $\mathrm{Hom}_{\mathcal{A}}(P, M) \to \mathrm{Hom}_{\mathcal{A}}(P, P)$, and the identity on P comes then from the desired section $\sigma \colon P \to M$.

Let us check finally that (3) $\Rightarrow$ (1). We are given the following situation:

$$\begin{tikzcd} & M \arrow[d, two heads, "\text{epi}"] \\ P \arrow[r] & N \end{tikzcd}$$

All finite limits exist in $\mathcal{A}$, so we may consider the cartesian square

$$\begin{tikzcd} P \times_N M \arrow[r, "g"] \arrow[d, "f"'] & M \arrow[d, two heads, "\text{epi}"] \\ P \arrow[r] & N \end{tikzcd}$$

Epimorphisms are stable under pullback in abelian categories, so f is also an epimorphism. By assumption, we can find a section $\sigma\colon P \longrightarrow P \times_N M$ splitting the corresponding short exact sequence. The composition $g \circ \sigma\colon P \longrightarrow M$ is then the desired lift. $\square$

Let us look now at the abelian category of modules over a ring R. What does it mean for an R-module to be projective?

non ppal. ideal over a DD $\neq$		$\mathbb{Q}$ over $\mathbb{Z}$ $\neq$		(x, y) over $\mathbb{C}[x, y]$ $\neq$

$$\{\text{Free}\} \quad \subseteq \quad \{\text{Projective}\} \quad \subseteq \quad \{\text{Flat}\} \quad \subseteq \quad \{\text{Torsion-free}\}$$

= / PID or local		= f.g. / Noeth.		= / DD

In the above diagram, ppal. stands for principal, DD stands for Dedekind domain, PID stands for principal ideal domain, f.g. stands for finitely generated and Noeth. stands for Noetherian.

References

[Hoc72] M. Hochster. Nonuniqueness of coefficient rings in a polynomial ring. *Proc. Amer. Math. Soc.*, 34:81–82, 1972. ↑ 1

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