ZARISKI CANCELLATION

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Abstract. Following [Hoc72] we provide an example of (commutative unital) rings R, S such that $R[t] \cong S[t]$ but $R \ncong S$, where t is an indeterminate. As a preparation for this counterexample we also discuss the notion of projective module and the hairy ball theorem.

Contents

—parts in gray will be omitted during the talk—

1. Introduction

Let R be a ring. We can form the polynomial ring $R[t]$ in one variable t with coefficients in R. This construction is functorial, and hence

$$R \cong S \Rightarrow R[t] \cong S[t].$$

The goal of this talk is to show with an explicit counterexample due to Hochster [Hoc72] that the converse is not true.

In the process of constructing this counterexample we will come across a projective module which, as a consequence of the hairy ball theorem, is not a free module. Therefore we will discuss projective modules and the hairy ball theorem before jumping into the counterexample.

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2. Projective modules

Definition 2.1. Let C be a category. An object $P \in C$ is called *projective* if the following lifting problem can always be solved:

$$\begin{tikzcd} & M \arrow[d, two heads, "\text{epi}"] \\ P \arrow[r] \arrow[ur, dashed, "\exists"] & N \end{tikzcd}$$

Lemma 2.2. *Let $\mathcal{A}$ be an abelian category and let $P \in \mathcal{A}$ be an object. The following are equivalent:*

(1) P is projective.
(2) $\mathrm{Hom}(P, -)$ is exact.
(3) Every short exact sequence of the form

$$0 \to N \to M \to P \to 0$$

splits.

Proof. We start with (1) $\Rightarrow$ (2). Assume P is projective and consider a short exact sequence

$$0 \to A \to B \to C \to 0.$$

Since $\mathrm{Hom}(P, -)$ is always left exact, we only need to show that the induced map $\mathrm{Hom}(P, B) \to \mathrm{Hom}(P, C)$ is surjective. But $B \to C$ is an epimorphism, so this is precisely what P being projective means by definition.

Next we show (2) $\Rightarrow$ (3). Assume $\mathrm{Hom}(P, -)$ is exact and consider a short exact sequence

$$0 \to N \to M \to P \to 0.$$

Applying $\mathrm{Hom}(P, -)$ we get a surjection $\mathrm{Hom}(P, M) \to \mathrm{Hom}(P, P)$, and the identity on P comes then from the desired section $\sigma \colon P \to M$.

The implication (3) $\Rightarrow$ (1) is left as an exercise during the talk. The hint is that epimorphisms are stable under pullback in abelian categories, and the solution follows in gray. We are given the following situation:

$$\begin{tikzcd} & M \arrow[d, two heads, "\text{epi}"] \\ P \arrow[r] & N \end{tikzcd}$$

All finite limits exist in $\mathcal{A}$, so we may consider the cartesian square

$$\begin{tikzcd} P \times_N M \arrow[r, "g"] \arrow[d, "f"'] & M \arrow[d, two heads, "\text{epi}"] \\ P \arrow[r] & N \end{tikzcd}$$

Epimorphisms are stable under pullback in abelian categories, so f is also an epimorphism. By assumption, we can find a section $\sigma\colon P \to P \times_N M$ splitting the corresponding short exact sequence. The composition $g \circ \sigma\colon P \to M$ is then the desired lift. □

Let us look now at the abelian category of modules over a ring R. What does it mean for an R-module to be projective?

Proposition 2.3 ([Fra18, Lemma 1.1.2]). *Let R be a ring. An R-module P is projective if and only if it is a direct summand of some free module F. Moreover, if P is finitely generated, then we can also choose F to be finitely generated.*

Proof. If P is (finitely generated) projective, consider a surjection $F \twoheadrightarrow P$ from a (finitely generated) free module F. The resulting short exact sequence

$$0 \to K \to F \to P \to 0$$

splits by Lemma 2.2, hence P is a direct summand of F.

Suppose conversely that $F = P \oplus K$ and consider a surjection $\varphi\colon M \twoheadrightarrow N$ and a morphism $f\colon P \to N$. Then $\varphi \oplus \mathrm{id}_K\colon M \oplus K \twoheadrightarrow N \oplus K$ is again a surjection. Since F is free, the morphism $f \oplus \mathrm{id}_K\colon F \to N \oplus K$ can be lifted to a morphism $\bar{f} \oplus \mathrm{id}_K\colon F \to M \oplus K$, so that $\bar{f}\colon P \to M$ is the desired lifting of the original surjection $\varphi\colon M \twoheadrightarrow N$. □

Corollary 2.4 ([Fra18, Cor. 1.1.28]). *Let R be a ring. A finitely presented R-module P is projective if and only if $\mathrm{Ext}^1_R(P, T) = 0$ for every finitely generated R-module T.*

Proof. Pick a presentation

$$0 \to K \to F \to P \to 0$$

with K finitely generated and F finitely generated and free. Then take the long exact sequence of $\mathrm{Ext}^\bullet_R(P, -)$ and use the assumption to find a splitting of the short exact sequence, exhibiting therefore P as a direct summand of a free module. □

To close this section, let us briefly discuss the relation between being projective and other well-known module properties. Let R be a ring. Recall that an R-module M is called *flat* if the functor $M \otimes_R (-)$ is exact, and it is called *torsion-free* if $0 \in M$ is the only element $m \in M$ such that there exist a non-zero divisor $r \in R$ with $rm = 0$. Then we have the following inclusions:

$$\{\text{Free}\} \quad \subseteq \quad \{\text{Projective}\} \quad \subseteq \quad \{\text{Flat}\} \quad \subseteq \quad \{\text{Torsion-free}\}$$

The axiom of choice is equivalent to the statement that every set is a projective object in the category of sets. This implies in turn that every free module is projective. Indeed, this follows then from the universal property of free modules. More generally, if a functor R preserves epimorphisms and $L \dashv R$, then L preserves projective objects [Fra18, Dual of Fact 1.1.1]. In our case R would be the forgetful functor and L would be the functor sending a set to the free module over this set.

The fact that projective modules are flat follows again from the axiom of choice, but in this case it is a strictly weaker statement. What we need now is the existence of enough projective objects in the category of R-modules, so that we can compute the left-derived functors of $M \otimes_R (-)$ with projective resolutions and argue as in [Fra18, Fact 1.2.1]. In order to ensure this, it would suffice to have enough projective objects in the category of sets.

Finally, that every flat module M is torsion-free follows from the computation in [Fra18, Example 1.2.1]. Namely, if $r \in R$ is not a zero-divisor, then

$$0 = \mathrm{Tor}_1^R(M, R/rR) \cong \ker(M \xrightarrow{r\cdot(-)} M).$$

$$\begin{array}{ccccccc} & \boxed{\begin{array}{c}\text{non ppal. ideal}\\ \text{over a DD}\\ \neq\end{array}} & & \boxed{\begin{array}{c}\mathbb{Q}\\ \text{over } \mathbb{Z}\\ \neq\end{array}} & & \boxed{\begin{array}{c}(x,y)\\ \text{over } \mathbb{C}[x,y]\\ \neq\end{array}} & \\ \{\text{Free}\} & \subseteq & \{\text{Projective}\} & \subseteq & \{\text{Flat}\} & \subseteq & \{\text{Torsion-free}\} \\ & \boxed{\begin{array}{c}=\\ \text{/ PID or local}\end{array}} & & \boxed{\begin{array}{c}=\\ \text{f.g. / Noeth.}\end{array}} & & \boxed{\begin{array}{c}=\\ \text{/ DD}\end{array}} & \end{array}$$

In the above diagram, ppal. stands for principal, DD stands for Dedekind domain, PID stands for principal ideal domain, f.g. stands for finitely generated and Noeth. stands for Noetherian.

Proof. To do! □

Remark 2.5. The example of $\mathbb{Q}$ as a $\mathbb{Z}$-module also shows that being projective depends on the base ring, since $\mathbb{Q}$ is projective over itself.

Remark 2.6. A ring R is called *hereditary* if all submodules of projective R-modules are again projective. Dedekind domains can be characterised as hereditary integral domains.

Remark 2.7. The example of (x, y) as a $\mathbb{C}[x, y]$-module also shows that $\mathbb{C}[x, y]$ is not an hereditary ring, since $\mathbb{C}[x, y]$ is projective over itself but its submodule (x, y) is not flat, thus not projective.

Remark 2.8. The left-most equality in the diagram above is also true for finitely generated modules over polynomial rings with coefficients on a principal ideal domain. This is an important result, first asked by Serre in the case of fields, and later proven independently by Quillen and Suslin, see the Wikipedia article on the Quillen–Suslin theorem. It corresponds geometrically to the statement that vector bundles on affine space are all trivial.

Remark 2.9 ([Fra18]). Baer's criterion [Fra18, Prop. 1.1.1] allows us to characterise the dual notion of injective modules only in terms of Ext^1_R and quotients of the ring R by its ideals. Indeed, an R-module N is injective if and only if $\mathrm{Ext}^1_R(R/I, N) = 0$ for every ideal $I \subseteq R$ [Fra18, Prop. 1.1.4]. In general, no similar criterion for projectivity is available. In fact, the *Whitehead problem* —which states that if an abelian group A has $\mathrm{Ext}^1_{\mathbb{Z}}(A, \mathbb{Z}) = 0$, then it is free— is undecidable in ZFC due to a result of Shelah.

On the other hand, the notion of flatness, which agrees with projectivity for finitely generated modules over Noetherian rings, does have a general criterion similar to the previous one for injectivity. Namely, an R-module M is flat if and only if $\mathrm{Tor}^R_1(M, R/I) = 0$ for every ideal $I \subseteq R$ [Fra18, Prop. 1.2.3].

3. Hairy ball theorem

In this section we will prove the hairy ball theorem following [EG79].

Theorem 3.1 (Hairy ball theorem)**.** *Every continuous vector field on the 2-sphere $\mathbb{S}^2$ has at least one zero.*

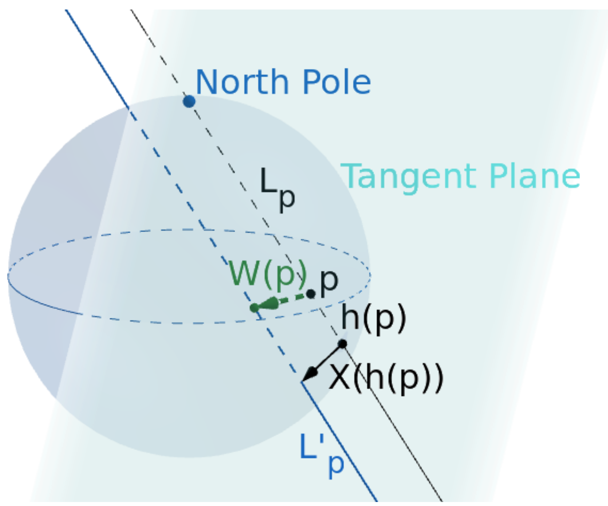
Proof. We will use the following definitions and identifications:

- $\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$.
- $\mathbb{R}^2 = \{(x, y, z) \in \mathbb{R}^3 \mid z = 0\}$.
- $\mathbb{D}^2 = \{(x, y, 0) \in \mathbb{R}^2 \subseteq \mathbb{R}^3 \mid x^2 + y^2 \leqslant 1\}$.
- $\mathbb{S}^1 = \{(x, y, 0) \in \mathbb{R}^2 \subseteq \mathbb{R}^3 \mid x^2 + y^2 = 1\}$.
- $\mathbb{S}^2_+ = \{(x, y, z) \in \mathbb{S}^2 \subseteq \mathbb{R}^3 \mid z \geqslant 0\}$.
- $\mathbb{S}^2_- = \{(x, y, z) \in \mathbb{S}^2 \subseteq \mathbb{R}^3 \mid z \leqslant 0\}$.
- For $p = (x, y) \in \mathbb{S}^1 \subseteq \mathbb{R}^2$ we denote by $R_p \colon \mathbb{R}^2 \to \mathbb{R}^2$ the linear reflection with axis the direcction defined by the vector $(-y, x)$, tangent to $\mathbb{S}^1$ at p.

We prove the theorem by contradiction, so suppose that we are given a non-vanishing continuous vector field $X\colon \mathbb{S}^2 \to T\mathbb{S}^2$ on $\mathbb{S}^2$.

Let $h\colon \mathbb{D}^2 \to \mathbb{S}^2_-$ be the inverse of the homeomorphism $\mathbb{S}^2_- \cong \mathbb{D}^2$ induced by the stereographic projection from the north pole $(0,0,1) \in \mathbb{S}^2$. Let $p \in \mathbb{D}^2$ be a point. Let L_p be the line joining $(0,0,1)$ and p, which then intersects $\mathbb{S}^2_-$ precisely at $h(p)$. We consider the line L'_p which is parallel to L_p and passes through the point $h(p) + X(h(p)) \in \mathbb{R}^3$. With this notation, we define a new function

$$\begin{aligned} W\colon \mathbb{D}^2 &\longrightarrow \mathbb{R}^2 \\ p &\longmapsto (\mathbb{R}^2 \cap L'_p) - p. \end{aligned}$$



Since X is a non-vanishing continuous vector field, $W\colon \mathbb{D}^2 \to \mathbb{R}^2$ is a non-vanishing continuous function. Therefore it induces a continuous function

$$\begin{aligned} F\colon \mathbb{D}^2 &\longrightarrow \mathbb{S}^1 \\ p &\longmapsto \frac{F(p)}{|F(p)|}. \end{aligned}$$

We repeat the same process with the south pole and $\mathbb{S}^2_+$. This gives us a non-vanishing continuous function $W^*\colon \mathbb{D}^2 \to \mathbb{R}^2$ that we can again normalise into a continuous function $F^*\colon \mathbb{D}^2 \to \mathbb{S}^1$.

We denote by f and f^* the restrictions to $\mathbb{S}^1 \subseteq \mathbb{D}^2$ of F and F^* respectively. By definition f and f^* factor through the contractible space $\mathbb{D}^2$:

$$\begin{tikzcd} \mathbb{S}^1 \arrow[rr, "{f,f^*}"] \arrow[dr, hook] & & \mathbb{S}^1 \\ & \mathbb{D}^2 \arrow[ur, "{F,F^*}"'] & \end{tikzcd}$$

Therefore, they are both nullhomotopic.

To obtain the desired contradiction, we will next find a certain relation between the functions f and f^*, which is in turn induced by a relation between $W|_{\mathbb{S}^1}$ and $W^*|_{\mathbb{S}^1}$. Namely, given a point $p = h(p) = (x_0, y_0, 0) \in \mathbb{S}^1 \subseteq \mathbb{D}^2$, we claim that

$$W^*(p) = R_p(W(p)). \tag{1}$$

Recall that this means that the vector $W^*(p)$ is the linear reflection of $W(p)$ across the axis given by the direction defined by $(-y_0, x_0, 0) \in \mathbb{R}^2$. In order to prove this, we fix an arbitrary $p = (x_0, y_0, 0) \in \mathbb{S}^1$ and we change the coordinate system, making p the origin, $(x_0, y_0, 0)$ the first basis vector in $\mathbb{R}^2$ and $(-y_0, x_0, 0)$ the second basis vector in $\mathbb{R}^2$. We express $W(p)$ and $W^*(p)$ with respect to these coordinates:

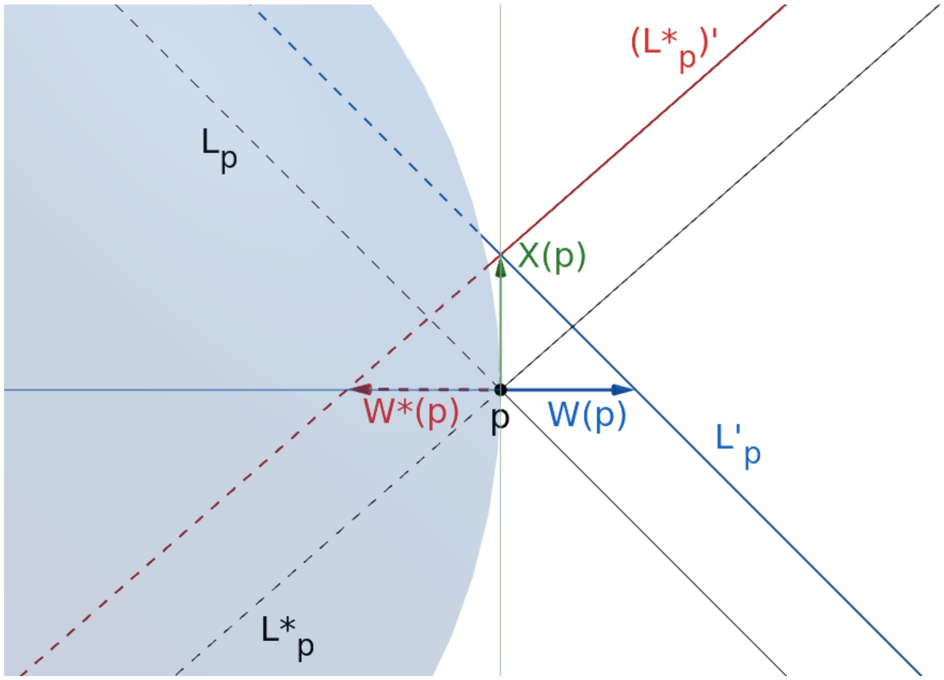
$$W(p) = (w_1, w_2, 0) \text{ and } W^*(p) = (w_1^*, w_2^*, 0).$$

The claim is then that $w_1 = -w_1^*$ and $w_2 = w_2^*$.

We check first that $w_2 = w_2^*$. Let L_p resp. L_p^* denote the line joining p to the north resp. south pole. These two lines intersect at p and define a plane Π which intersects $\mathbb{R}^2$ precisely along the x-axis of our current coordinate system. Let L_p' resp. $(L_p^*)'$ be the lines with directions equal to those of L_p resp. L_p^* containing the point $p + X(p)$. The plane Π' defined by these two lines is then parallel to Π, and therefore the intersection $\Pi' \cap \mathbb{R}^2$ is parallel to the x-axis of our current coordinate system. This implies that $w_2 = w_2^*$.

To check that $w_1 = w_1^*$ we argue projecting the whole picture into the plane spanned by the vectors $(x_0, y_0, 0)$ and $(0, 0, 1)$. From this point of view we see the following picture:

From Thales' theorem we know that L_p and L_p^* intersect at a right angle. From Proposition 29 in Book I of Euclid's *Elements* we deduce that all other angles that seem to be right angles in the picture are in fact right angles. From the fact that the inner angles of any triangle add up to two right angles we deduce in turn that all angles that look like half a right angle in the picture are in fact half a right angle. Indeed, we can start by ensuring that the inner angle between L_p and $W^*(p)$ is half a right angle, which follows from the fact that the triangle with vertices p, the usual origin of $\mathbb{R}^3$ and the north pole is isosceles with a right angle at the usual origin of $\mathbb{R}^3$. Then, knowing already that the inner angle between $W^*(p)$ and $X(p)$ is a right angle, we deduce from this that the inner angle between L_p and



$X(p)$ is also half a right angle, and so on. This implies that all four small triangles in the picture are congruent, and in particular $w_1 = w_1^*$.

We are finally ready to exhibit the desired contradiction. Let $H\colon \mathbb{S}^1 \times [0,1] \to \mathbb{S}^1$ be a homotopy from f to the constant map $c\colon \mathbb{S}^1 \to \mathbb{S}^1$ with constant value the poitn $(-1,0,0) \in \mathbb{S}^1 \subseteq \mathbb{R}^3$. The formula

$$\begin{aligned} H^*\colon \mathbb{S}^1 \times [0,1] &\longrightarrow \mathbb{S}^1 \\ (p,t) &\longmapsto R_p(H(p,t)) \end{aligned}$$

defines a homotopy between the nullhomotopic map $f^*\colon \mathbb{S}^1 \to \mathbb{S}^1$ and the map

$$\begin{aligned} c^*\colon \mathbb{S}^1 &\longrightarrow \mathbb{S}^1 \\ p &\longmapsto R_p((-1,0,0)) \end{aligned}$$

But this is a contradiction, because $c^*\colon \mathbb{S}^1 \to \mathbb{S}^1$ is the morphism going twice around $\mathbb{S}^1$ at constant speed and starting from $(1,0,0)$, which is not nullhomotopic. □

If we take the Whitney embedding theorem [Bre93, Theorem II.10.7], the tubular neighbourhood theorem [Bre93, Theorem II.11.4] and the Lefschetz–Hopf fixed point theorem [Bre93, Theorem IV.23.4] for granted, we can prove a much more general statement:

Theorem 3.2 ([Bre93, Corollary IV.23.6]). *If M is a compact smooth manifold with Euler characteristic $\chi(M) \neq 0$, then any continuous vector field on M has a zero.*

Proof. We show that if M admits a non-vanishing continuous vector field, then it has Euler characteristic $\chi(M) = 0$. So let $X\colon M \to TM$ be a non-vanishing continuous vector field on M.

The Whitney embedding theorem [Bre93, Theorem II.10.7] allows us to assume that $M \subseteq \mathbb{R}^N$ is a compact smooth submanifold. The tubular neighbourhood theorem [Bre93, Theorem II.11.4] ensures the existence of a small enough real number $\varepsilon > 0$ such that the sum in $\mathbb{R}^N$ yields a diffeomorphism

$$\theta\colon \{(x,v) \in M \times \mathbb{R}^N \mid v \perp T_xM, \|v\| < \varepsilon\} \cong \{y \in \mathbb{R}^N \mid \operatorname{dist}(M,y) < \varepsilon\}$$

from the open subset $N_\varepsilon M$ of the normal bundle consisting of normal vectors with norm less than ε to an ε-neighbourhood $B_\varepsilon M$ of M in $\mathbb{R}^N$, which we refer to as a *tubular neighbourhood* of M in $\mathbb{R}^N$. The projection from the normal bundle $\pi\colon NM \to M$ induces a smooth strong deformation retraction $r\colon B_\varepsilon M \to M \subseteq B_\varepsilon M$ given by $y \mapsto \pi(\theta^{-1}(y))$. The smooth homotopy that shows that r is a smooth strong deformation retraction [Bre93, Definition I.14.8] is

$$\begin{aligned} F\colon B_\varepsilon M \times [0,1] &\longrightarrow B_\varepsilon M \\ (\theta(x,v), t) &\longmapsto \theta(x, tv) \end{aligned}$$

Since M is compact, we may assume —up to multiplying X by a small enough scalar— that $\|X(x)\| < \varepsilon$ for all $x \in M$. We define then

$$\begin{aligned} f\colon M &\longrightarrow M \\ x &\longmapsto r(x + X(x)). \end{aligned}$$

Geometrically, we are projecting the point $x + X(x) \in \mathbb{R}^N$ onto M along the normal direction:

figure

If $f(x) = x$, then the tangent direction of $X(x)$ is zero. But $X(x)$ was by assumption a tangent vector at $x \in M$, so $X(x) = 0$, contradicting the assumption that X is non-vanishing. Thus f has no fixed points.

We have also shown as a consequence of the Whitney embedding theorem that M is an euclidean neighbourhood retract. Namely, M is a retract of the open subset $B_\varepsilon M \subseteq \mathbb{R}^N$. Therefore we may apply the Lefschetz–Hopf fixed point theorem [Bre93, Corollary IV.23.5] to deduce that □

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