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A SURVEY ON ZARISKI CANCELLATION PROBLEM

Neena Gupta

Statistics and Mathematics Unit, Indian Statistical Institute,
203 *B. T. Road, Kolkata* 700 108, *India*
e-mails: neenag@isical.ac.in, rnanina@gmail.com

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In this survey article we describe known results and open questions on the Zariski cancellation problem, highlighting recent developments on the problem. We also discuss its close relationship with some of the other central problems on polynomial rings.

Key words : Polynomial ring, cancellation problem; affine fibration; linearization problem; epimorphism problem.

1. Introduction

Let k be a field. The polynomial ring $k[X_1, \ldots, X_n]$ is one of the oldest rings that mathematicians have attempted to investigate. The study of polynomial rings and their quotients, equivalently the study of affine spaces and their closed subspaces, is the driving motivation of the area called "Affine Algebraic Geometry". There are fascinating problems on polynomial rings which are still open. Though these problems are very easy to state, it is normally difficult to approach them. The most celebrated problems on polynomial rings include the Jacobian problem (first asked by Ott-Heinrich Keller in 1939), the Zariski cancellation problem, the epimorphism or embedding problem of Abhyankar-Sathaye, the affine fibration problem of Dolgačev-Veĭsfeĭler, the linearization problem of Kambayashi, the problem of characterisation of polynomial rings by a few chosen properties, and the study of the Automorphism groups of polynomial rings.

One of the major difficulties in the study of the polynomial rings in more than two variables is the paucity of our knowledge about their automorphism groups. After Jung [25] and Kulk [47] showed that all automorphisms of the polynomial ring in two variables over a field are "tame", there was very little progress on the automorphism problem for a long time. A major development in affine algebraic geometry in the present century was the Shestakov-Umirbaev theory [44] to detect “wild” automorphisms on a polynomial ring in three variables over a field; the application of the theory to prove that an automorphism constructed by Nagata in 1972 was a “wild” automorphism, as conjectured by Nagata, and the simplification and development of the theory by Kuroda [28].

The articles of Kraft [27], Freudenburg and Russell [16] and Miyanishi [32] give a survey on problems in affine algebraic geometry; the monograph of Essen [15] gives an account of polynomial automorphisms and the Jacobian conjecture; the Epimorphism problem and its offshoots are discussed in the articles by Russell and Sathaye [39], and by Dutta and Gupta [13]; the problems on affine fibrations are discussed by Bhatwadekar and Dutta in [7] and [12].

In this article, we shall discuss some recent developments on the cancellation problem and its connections with other problems. We shall first begin with the history of the problem. Before that we introduce some notation.

Throughout the article, our rings will be assumed to be commutative with unity. We shall use the notation $R^{[n]}$ for a polynomial ring in n variables over a commutative ring R. Thus, $E = R^{[n]}$ will mean that $E = R[t_1, \dots, t_n]$ for some elements $t_1, \dots, t_n$ in E which are algebraically independent over R. Unless otherwise stated, capital letters like $X_1, X_2, \dots, X_n, Y_1, \dots, Y_m, X, Y, Z, T$ will be used as variables of polynomial rings.

Let R be a ring and A and B be two R-algebras. The notation $A \cong_R B$ would mean that A is isomorphic to B as an R-algebra.

An R-algebra B is said to be *cancellative* over R if, whenever $A[X] \cong_R B[X]$ for some R-algebra A, then $A \cong_R B$.

2. The Zariski Cancellation Problem

The following cancellation problem was first raised by Zariski in 1949 at the Paris Colloquium on Algebra and Theory of Numbers [41]. He asked [34]:

Question 1 : Let L and L' be two finitely generated fields over a field k and let $L(x)$, $L'(x)$ be simple transcendental extensions of L and L' respectively. Suppose that $L(x) \cong_k L'(x)$. Does it follow that $L \cong_k L'$?

In particular, when $L' = k(x_1, \dots, x_n)$, the rational function field in n variables, Q.1 takes the following form:

Question $1'$: Let k be a field. Suppose that $L(y) \cong_k k(x_1, \ldots, x_{n+1})$ for some simple transcendental extension $L(y)$ of a field L. Does it follow that $L \cong_k k(x_1, \ldots, x_n)$?

Q. 1 does have an affirmative answer under certain cases but not in general. In fact, Beauville, Colliot-Thélène, Sansuc and Swinnerton-Dyer constructed a family of examples in [6] which show that even the special case Q.$1'$ does not have an affirmative answer in general. They constructed a field F containing some subfield k such that $F(y_1, y_2, y_3) \cong_k k(x_1, x_2, \ldots, x_5)$ but $F \ncong_k k(x_1, x_2)$. In [43], Shepherd-Barron showed that actually $F(y_1, y_2) \cong_k k(x_1, x_2, \ldots, x_4)$. It is not known whether $F(y_1) \cong_k k(x_1, x_2, x_3)$. If yes, then $L = F$ provides a counter-example to Q.$1'$ and if not, then $L = F(y_1)$ provides a counter-example to Q.$1'$.

From around the early 1970's, analogous cancellation problems over rings were taken up by mathematicians like Abhyankar, Coleman, Eakin, Enochs, Heinzer, Hochster *et al*. (see [9], [1], [14] and [24]). The cancellation problem over rings may be formulated as follows:

Question 2 : Let k be a field and B be a k-algebra. Suppose that A is a k-algebra and that the polynomial rings $A[X]$ and $B[X]$ are isomorphic as k-algebras. Does it follow that $A \cong_k B$? In other words, is the k-algebra B cancellative?

Abhyankar *et al*., showed in 1972 [1] that Q. 2 has an affirmative answer for any domain B of transcendence degree one over a field k. A counter-example to Q. 2 was constructed by Hochster in 1972 [24] for $k = \mathbb{R}$ (the field of real numbers) and $\dim B = 4$. His example was based on the fact that the projective module defined by the tangent bundle over the real sphere with coordinate ring $S = \mathbb{R}[X, Y, Z]/(X^2 + Y^2 + Z^2 - 1)$ is stably free but not a free S-module. Around 1989, Danielewski constructed explicit examples [11] to show that if B is a domain of transcendence degree two over the field of complex numbers $\mathbb{C}$, then B need not be cancellative over $\mathbb{C}$.

We now discuss a very important case of Question 2: the case when B is a polynomial ring over the field k.

Question $2'$: Let k be a field and A be an affine k-algebra. Suppose that $A[X] \cong_k k[X_1, \ldots, X_{n+1}]$. Does it follow that $A \cong_k k[X_1, \ldots, X_n]$? In other words, is the polynomial ring $k[X_1, \ldots, X_n]$ cancellative?

Recall that the famous Serre conjecture asserts that all stably free modules over $B = k[X_1, \ldots, X_n]$ are free. By 1974, this conjecture had been proved for $n \leq 3$ (by Seshadri for $n = 2$ [42] and by Murthy and Towber [33] for $k = \mathbb{C}$ and $n = 3$) and, in 1976, Quillen [35] and Suslin [45] independently settled the conjecture completely. Therefore, counter-examples like Hochster's are certainly impossible for Question $2'$.

For an algebraically closed field k, Question $2'$ is equivalent to the following geometric version:

Question $2''$: Let k be an algebraically closed field and $\mathbb{V}$ be an affine k-variety such that $\mathbb{V} \times \mathbb{A}^1_k \cong_k \mathbb{A}^{n+1}_k$. Does it follow that $\mathbb{V} \cong_k \mathbb{A}^n_k$? In other words, is the affine n-space $\mathbb{A}^n_k$ cancellative?

In view of the importance of this elegant version of the original Q. 1, Question $2'$ (or Q. $2''$) has become popular as the Zariski cancellation problem (cf. [18], [31], [27] and [32]). Question $2'$ has inspired many fruitful explorations over the past 45 years. Some of the major research accomplishments during the 1970's, like the characterisation of the affine plane, originated from the efforts to investigate the problem. The problem has interesting connections with the embedding problem, affine fibration problem, linearization problem, stable coordinate problem, and so on. Question $2'$ is still open in characteristic zero and is of great interest to people in the area of affine algebraic geometry.

The polynomial ring $k[X]$ was shown to be cancellative over a field k of any characteristic by Abhyankar *et al*. [1]. In fact, as mentioned earlier, they proved the more general result that any domain B of transcendence degree one over a field k is cancellative over k. They also showed that, for any UFD R, the polynomial ring $R[X]$ is cancellative over R.

The proof of the cancellative property of $k[X]$, when k is a field, is not very difficult. But for the polynomial ring $k[X, Y]$ the problem is more intricate. In an attempt to solve the cancellation problem for $\mathbb{C}[X, Y]$, Ramanujam established in 1971 his celebrated topological characterisation of the affine plane [36]. In [30], Miyanishi (1975) gave an algebraic characterisation of the polynomial ring $k[X, Y]$. This algebraic characterisation was used by Fujita *et al*. [18], [31] to prove the cancellation property of $k[X, Y]$ over fields of characteristic zero and by Russell [37] over perfect fields of arbitrary characteristic. Later, using methods of Mumford and Ramanujam, Gurjar [23] gave a topological proof of the cancellation property of $\mathbb{C}[X, Y]$. More recently, a simplified proof of the cancellation property of $k[X, Y]$ for an algebraically closed field k was given by Crachiola and Makar-Limanov in [10]. The arguments in this paper were used by Bhatwadekar and the author [8] to establish the cancellation property of $k[X, Y]$ over any arbitrary field k. However, after Russell's work in 1981, the Cancellation Problem for higher dimensional affine varieties had remained unanswered for more than three decades.

In 1987, Asanuma [3] constructed a three-dimensional affine ring over a field of positive characteristic as a counterexample on $\mathbb{A}^2$-fibration problem in positive characteristic. This was envisaged in [4] as a possible candidate for a counter-example to either the Zariski cancellation problem or the linearization problem for the affine threespace in positive characteristic. In [20], the author showed that

Asanuma's ring is indeed a counter-example to the cancellation problem. Thus, when ch. $k > 0$, the affine 3-space $\mathbb{A}_k^3$ is not cancellative. Subsequently in [22], the author showed that when ch. $k > 0$, the affine n-space $\mathbb{A}_k^n$ is not cancellative for any $n \geq 3$. Thus, over a field of positive characteristic, the Zariski cancellation problem has been completely answered in all dimensions. However, over a field of characteristic zero, the problem still remains open for $n \geq 3$.

We shall now discuss the known counter-examples to the Zariski cancellation problem (i.e., Q.2′). We begin with Asanuma's example.

3. Asanuma's Dilemma: The First Counter-Example

A finitely generated flat S-algebra A is said to be an $\mathbb{A}^n$*-fibration* (over S) if $A \otimes_S k(P) = k(P)^{[n]}$ for each prime ideal P of S. The affine fibration problem of Dolgačev-Veĭsfeĭler asks if every $\mathbb{A}^n$-fibration over a regular local ring S is necessarily a polynomial ring over S. A nontrivial theorem of Sathaye [40] showed that *any* $\mathbb{A}^2$*-fibration over a PID* S *containing* $\mathbb{Q}$ *must be isomorphic to* $S^{[2]}$.

In [3], Asanuma made the next major breakthrough in the affine fibration problem. He proved a stable structure theorem ([3, Corollary 3.5]) for any affine fibration over a regular local ring. In the same paper, Asanuma also constructed the first counter-example to the $\mathbb{A}^2$-fibration problem over a PID in positive characteristic. We present below a version of Asanuma's example.

Let k be a field of characteristic $p\ (> 0)$ and let $R = k[X, Y, Z, T]/(X^m Y + Z^{p^e} + T + T^{sp})$, where m, e, s are positive integers such that $p^e \nmid sp$ and $sp \nmid p^e$. Let x denote the image of X in R. Then $k[x] \subset R$ and the following properties are satisfied by R (cf. [3, Theorem 5.1]):

(1) $R \otimes_{k[x]} k(P) = k(P)^{[2]}$ for all $P \in \operatorname{Spec} k[x]$.

(2) $R^{[1]} = k[x]^{[3]} = k^{[4]}$.

(3) $R \neq k[x]^{[2]}$.

(1) shows that R is an $\mathbb{A}^2$-fibration over $k[x]$; (2) shows that R is a stably polynomial ring over $k[x]$, in particular, a stably polynomial ring over k; and (3) shows that R is not a polynomial ring over $k[x]$. Thus R is a non-trivial $\mathbb{A}^2$-fibration over $k[x]$.

The ring R was soon to acquire a wider significance. In a subsequent paper [4, Theorem 2.2], using the ring R, Asanuma constructed non-linearizable algebraic torus actions on $\mathbb{A}_k^n$ over any infinite field k of positive characteristic when $n \geq 4$. He then asked whether R is a polynomial ring and explained the significance of his question as follows [4, Remark 2.3]:

"If R is a polynomial ring then it will give an example of a non-linearizable torus action on k^3 in positive characteristic. On the other hand if R is not a polynomial ring then it will clearly give a counter-example to the Cancellation Problem."

Thus either way one would answer a major problem in affine algebraic geometry. This dichotomy has been popularized by Russell as "Asanuma's Dilemma" (cf. [38, Problem 2, p 9]).

In [20], using techniques of Crachiola and Makar-Limanov, the author has shown that R is not a polynomial ring at least when $m \geq 2$. Consequently, it follows that the Zariski cancellation problem does not have an affirmative answer in positive characteristic. In a subsequent paper, the author made further investigations on the Asanuma ring.

4. Subsequent Developments and Examples

We now discuss the concepts behind Asanuma's example and see how it is related to other interesting problems in the area of affine algebraic geometry.

We shall use the following terminology. A polynomial $g \in k[Z,T]$ will be called a *line* if $k[Z,T]/(g) = k^{[1]}$ and a line g will be called a *nontrivial line* if $k[Z,T] \neq k[g]^{[1]}$.

We briefly recall the Epimorphism problem for the affine plane. Let $f \in k[Z,T]$ be a polynomial such that $k[Z,T] = k[f]^{[1]}$. Then clearly $k[Z,T]/(f) = k^{[1]}$. The Epimorphism problem asks the converse: if $f \in k[Z,T]$ is such that $k[Z,T]/(f) = k^{[1]}$, is then $k[Z,T] = k[f]^{[1]}$? In other words, does there exist a non-trivial line?

The famous Epimorphism theorem of Abhyankar-Moh [2] (also proved independently by Suzuki [46]) asserts that there does not exist any nontrivial line over any field of characteristic zero. However, as early as in 1957, Segre [41] had exhibited an example of a nontrivial line over any field of positive characteristic. Later Nagata gave a family of examples [34]. For more details on the Epimorphism Problem see [13].

Asanuma's 3-dimensional ring R can be considered as a special case of the general class of threefolds in $\mathbb{A}^4_k$ defined by the zero locus of a polynomial of the form $X^mY - f(Z,T)$, where $f(Z,T)$ is a Segre-Nagata nontrivial line. This led us to a problem which was asked independently by Russell.

Q. Let f be any nontrivial line and A be a ring defined by the relation $x^ry = f$. Is the ring A necessarily not a polynomial ring over the field k?

As classification of nontrivial lines is still an open problem, Russell's question has to be approached abstractly. In [21], the author has answered Russell's question affirmatively. The generalisation is more transparent and conceptual, and has simplified the earlier proofs in [20]. The proofs are independent of the characteristic.

The problems investigated by the author in [21] may be formulated as follows.

Q. Let k be a field of any characteristic and $F \in k[X, Z, T]$. Let

$$A = k[X, Y, Z, T]/(X^r Y - F(X, Z, T)), \text{ where } r > 1.$$

Find conditions on F for which

(I) A is a stably polynomial ring.

(II) A is itself a polynomial ring.

(III) The polynomial $X^r Y - F$ is a coordinate in $k[X, Y, Z, T]$.

A solution to problem I is as follows [21, Theorem 4.2].

***Theorem* 4.1** — *Let k be a field of any characteristic and $F \in k[X, Z, T]$ a polynomial in three variables. Let*

$$A = k[X, Y, Z, T]/(X^r Y - F(X, Z, T)), \textit{ where } r \geq 1.$$

Then A is a stably polynomial ring if $F(0, Z, T)$ is a line in $k[Z, T]$. That is, $A^{[1]} = k[x]^{[3]} = k^{[4]}$ if $k[Z, T]/(F(0, Z, T)) = k^{[1]}$.

Note that the condition in Theorem 4.1 depends only on $F(0, Z, T)$ and not on the general form of $F(X, Z, T)$.

The problems II and III have been shown to be equivalent and their precise solution is described below ([21, Theorem 3.11]):

***Theorem* 4.2** — *Let k be a field of any characteristic and $F \in k[X, Z, T]$, a polynomial in three variables. Let $f(Z, T) = F(0, Z, T)$, $G = X^r Y - F(X, Z, T) \in k[X, Y, Z, T]$, where $r > 1$ and $A = k[X, Y, Z, T]/(G)$. Then the following statements are equivalent:*

(i) *$f(Z, T)$ is a variable in $k[Z, T]$.*

(ii) *$A = k[x]^{[2]}$, where x denotes the image of X in A.*

(iii) *$A = k^{[3]}$.*

(iv) *G is a variable in $k[X, Y, Z, T]$.*

(v) *G is a variable in $k[X, Y, Z, T]$ along with X.*

Theorem 4.2 answers in one statement several very different looking questions that had been of long interest in the field.

Theorem 4.1 shows that the ring A is a stably polynomial ring if $F(0, Z, T)$ is a line and now by the equivalence of (i) and (iii) in Theorem 4.2, it follows that A is not a polynomial ring if $F(0, Z, T)$ is not a coordinate of $k[Z, T]$. Thus, if $F(0, Z, T)$ is a nontrivial line in $k[Z, T]$, then A is a stably polynomial ring but not a polynomial ring. This gives a recipe for constructing counter-examples to the Zariski cancellation problem (ZCP). The proofs of Theorems 4.1 and 4.2 are independent of the characteristic of the field. But we know by the Abhyankar-Moh-Suzuki theorem that a non-trivial line never exists in characteristic zero. Thus, for obtaining counter-examples to the cancellation problems for affine 3-space, an application of Theorems 4.1 and 4.2 can be made only in positive characteristic.

The equivalence of (i) and (iii) answers in one stroke Russell's question affirmatively. It also explains the nontriviality of another important threefold $\mathbb{V}$, namely, the Russell-Koras threefold defined over the complex number field by the zero locus of the polynomial $x^2y + x + z^2 + t^3 = 0$. This threefold has played a pivotal role in the linearization problem of $\mathbb{A}^3_{\mathbb{C}}$. We briefly discuss this problem.

The Linearization Conjecture asserts that any algebraic action of k^* on $k^{[n]}$ is linearizable. But it was shown that if $\mathbb{V}$ were isomorphic to the affine threespace $\mathbb{A}^3_{\mathbb{C}}$, then $\mathbb{V}$ would admit a $\mathbb{C}^*$-action which is not linearizable. It was therefore important to know whether $\mathbb{V} \cong \mathbb{A}^3_{\mathbb{C}}$. Koras and Russell had constructed a family of such examples envisaged as possible counter-examples to the linearization problem for $\mathbb{A}^3_{\mathbb{C}}$, the hypersurface $\mathbb{V}$ being one of the simplest. For quite some time, no known invariant could distinguish $\mathbb{V}$ from the affine threespace. $\mathbb{V}$ is smooth, contractible and diffeomorphic to $\mathbb{R}^6$. The coordinate ring of $\mathbb{V}$ is a UFD which can be embedded in the polynomial ring $\mathbb{C}^{[3]}$. Analogous properties characterize the affine plane.

Subsequently, in a remarkable paper [29], Makar-Limanov discovered an invariant: the ring obtained as the intersection of the rings of invariants of all $\mathbb{G}_a$-actions. He called it the ring of absolute constants; now it is commonly known as the Makar-Limanov invariant. This invariant distinguished the hypersurface $\mathbb{V}$ from the affine 3-space. In a subsequent paper [26], Kaliman and Makar-Limanov proved that none of the Russell-Koras threefolds is an affine 3-space. This led Russell and Koras to complete their proof of the conjecture that every algebraic $\mathbb{C}^*$-action of $\mathbb{C}^3$ is linearizable.

A major part of the proof of (i) $\iff$ (iii) of Theorem 4.2 is based on techniques adapted from Makar-Limanov's arguments. The implication (iii) $\implies$ (i) in Theorem 4.2 now gives us the precise reason for the non-trivially of $\mathbb{V}$: namely that the polynomial $Z^2 + T^3$ is not a coordinate in $k[Z, T]$.

The equivalence of (iii) and (iv) is a special case of Abhyankar-Sathaye conjecture for the affine four space $\mathbb{A}^4_k$. The conjecture asserts that if $G \in k[X, Y, Z, T]$ is a polynomial in four variables such that $k[X, Y, Z, T]/(G) = k^{[3]}$ then $k[X, Y, Z, T] = k[G]^{[3]}$. The result also extends, partially, a theorem of Sathaye-Russell on the embedding of linear surfaces in $\mathbb{A}^3_k$ to linear hypersurfaces in $\mathbb{A}^4_k$.

In [21, Theorem 3.11], the author has also shown that each of the equivalent statements of Theorem 4.2 is equivalent to 5 other statements each involving the triviality of a ring of invariants called the "Derksen invariant", associated to "exponential maps". We now briefly say a few words about these concepts.

An important tool for studying polynomial rings has been the ring of invariants of a polynomial ring under a suitable algebraic group action. For instance, Hilbert's 14th problem was related to the finite generation of the ring of invariants of an algebraic group action. Some of the recent low-dimensional counterexamples to Hilbert's 14th problem in characteristic zero were realised as the rings of invariants of $\mathbb{G}_a$-actions (see [17] for details).

The concept of an exponential map is a ring-theoretic formulation of a $\mathbb{G}_a$-action (see [20, p. 280]); in the case when the ground field is of characteristic zero, this concept is equivalent to that of a derivation which is "locally nilpotent" (see [17]). Two invariants of a ring defined by its locally nilpotent derivations or exponential maps, the Makar-Limanov invariant mentioned earlier and the Derksen invariant, have turned out to be powerful tools for settling certain central questions in affine algebraic geometry. The Derksen invariant of a ring R, denoted by $\mathrm{DK}(R)$, is the subring of R generated by the rings of invariants of all the nontrivial exponential maps of R. If $A = k^{[n]}$, then it is easy to see that $\mathrm{DK}(A) = A$ for $n \geq 2$ (cf. [20, Lemma 2.4]).

A crucial step in the proof of the theorem mentioned above [21, Theorem 3.11] is the following theorem on the Derksen invariant of the ring A [21, Proposition 3.7]:

***Theorem* 4.3** — *Let k be a field and A be an integral domain defined by*

$$A = k[X, Y, Z, T]/(X^r Y - F(X, Z, T)), \text{ where } r > 1.$$

Set $f(Z, T) := F(0, Z, T)$. Let x, y, z and t denote, respectively, the images of X, Y, Z and T in A. Suppose that $\mathrm{DK}(A) \neq k[x, z, t]$.

Then the following statements hold.

(i) *There exist $Z_1, T_1 \in k[Z, T]$ and $a_0, a_1 \in k^{[1]}$ such that $k[Z, T] = k[Z_1, T_1]$ and $f(Z, T) = a_0(Z_1) + a_1(Z_1)T_1$.*

(ii) *If* $k[Z,T]/(f) = k^{[1]}$*, then* $k[Z,T] = k[f]^{[1]}$.

As a consequence of Theorem 4.3, it follows that if f is a non-trivial line, i.e., $k[Z,T]/(f) = k^{[1]}$ but $k[Z,T] \neq k[f]^{[1]}$, then $\mathrm{DK}(A) = k[x,z,t] \subsetneqq A$. Hence A cannot be $k^{[3]}$.

Another result proved in [21, Theorem 4.11] describes the isomorphism classes of the rings of Asanuma type.

***Theorem* 4.4** — *Let* k *be a field of positive characteristic. For any integer* $r \geq 2$ *and any non-trivial line* f *in* $k[Z,T]$*, set*

$$A(r,f) := k[X,Y,Z,T]/(X^rY - f(Z,T)).$$

Then $A(r,f)$ *is isomorphic to* $A(s,g)$ *if and only if* $r = s$ *and there exists a* k*-algebra automorphism* θ *of* $k[Z,T]$ *such that* $\theta(g) = \delta f$ *for some* $\delta \in k^*$.

Note that by Theorem 4.1, $A(r,f)^{[1]} = k^{[4]}$. Therefore, by Theorem 4.4, if we vary the integer r, we will get an infinite family of rings which are not isomorphic to each other but are all stably isomorphic to the polynomial ring $k^{[3]}$. Again the proof is independent of the characteristic of the field. But as non-trivial lines exist only in positive characteristic, we get an infinite family of counter-examples to the Zariski cancellation problem but only over fields of positive characteristic.

5. Higher Dimensional Counter-Examples

In [22], the author generalised the construction of the affine threefold $x^ry = F(x,z,t)$ to construct counter-examples to the Zariski cancellation problem in higher dimension in positive characteristic, i.e., to show that the polynomial ring $k[X_1,\ldots,X_n]$ is not cancellative for any $n \geq 3$. This settles the Zariski cancellation problem completely in positive characteristic. She proved [22, Theorem 3.7]:

***Theorem* 5.1** — *Let* k *be a field of any characteristic and* A *an integral domain defined by*

$$A = k[X_1,\ldots,X_m,Y,Z,T]/({X_1}^{r_1}\cdots{X_m}^{r_m}Y - f(Z,T)),$$

where $r_i \geq 2$ *for each* i*. Suppose that* $k[Z,T]/(f(Z,T)) = k^{[1]}$*. Then*

$$A^{[1]} = k[X_1,\ldots,X_m]^{[3]} = k^{[m+3]}.$$

Moreover, if ch. $k > 0$ *and* $f(Z,T)$ *is a non-trivial line in* $k[Z,T]$ *then*

$$A \neq k^{[m+2]}.$$

The proof involves a suitable generalisation of Theorem 4.3.

6. A Candidate Counter-Example in Characteristic Zero

Shastri had proved that any knot admits a polynomial embedding in the affine threespace $\mathbb{A}^3_{\mathbb{R}}$. He also gave explicit construction for the embedding of the trefoil knot in $A^3_{\mathbb{R}}$. Let $\phi : \mathbb{R}[X,Y,Z] \to \mathbb{R}[T]$ be the polynomial embedding of a non-trivial knot and $\ker(\phi) = (f,g)$ for some $f,g \in k[X,Y,Z]$. Using Shastri's embedding ϕ, Asanuma constructed the ring $B = \mathbb{R}[t][X,Y,Z,U,V]/(t^dU - f, t^dV - g)$ and proved that $B^{[1]} = \mathbb{R}^{[5]}$ (cf. [5, Corollary 4.2]). He further asked [5, Remark 7.8]:

Question : Is $B = \mathbb{R}^{[4]}$?

Again this question of Asanuma has twin consequences. Either way, one answers a major problem in Affine algebraic geometry:

If $B = \mathbb{R}^{[4]}$, then there exist non-linearizable $\mathbb{R}^*$-actions on the affine four space $\mathbb{A}^{[4]}_{\mathbb{R}}$. If $B \neq \mathbb{R}^{[4]}$, then clearly B is a counter-example to the ZCP!!

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