

# ZARISKI CANCELLATION

PEDRO NÚÑEZ

ABSTRACT. Following [Hoc72] we provide an example of rings<sup>1</sup>  $B, C$  such that  $B \not\cong C$  but  $B[t] \cong C[t]$ , where  $t$  is an indeterminate. As a preparation for this counterexample we also study the notion of projective module and the hairy ball theorem.

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—parts in gray will be omitted during the talk—

## 1. INTRODUCTION

Let  $R$  be a ring. We can form the polynomial ring  $R[t]$  in one variable  $t$  with coefficients in  $R$ . This construction is functorial, and hence

$$R \cong S \Rightarrow R[t] \cong S[t].$$

The goal of this talk is to show with an explicit counterexample due to Hochster [Hoc72] that the converse is not true.

In the process of constructing this counterexample we will come across a projective module which, as a consequence of the hairy ball theorem, is not a free module. Therefore we will discuss projective modules and the hairy ball theorem before jumping into the counterexample.

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## 2. PROJECTIVE MODULES

**Definition 2.1.** Let  $\mathcal{C}$  be a category. An object  $P \in \mathcal{C}$  is called *projective* if the following lifting problem can always be solved:

$$\begin{array}{ccc} & & M \\ & \nearrow & \downarrow \text{epi} \\ P & \longrightarrow & N \end{array}$$

**Lemma 2.2.** Let  $\mathcal{A}$  be an abelian category and let  $P \in \mathcal{A}$  be an object. The following are equivalent:

- (1)  $P$  is projective.
- (2)  $\text{Hom}_{\mathcal{A}}(P, -)$  is exact.
- (3) Every short exact sequence of the form

$$0 \rightarrow N \rightarrow M \rightarrow P \rightarrow 0$$

splits.

*Proof.* We start with (1)  $\Rightarrow$  (2). Assume  $P$  is projective and consider a short exact sequence

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0.$$

Since  $\text{Hom}_{\mathcal{A}}(P, -)$  is always left exact, we only need to show that the induced map  $\text{Hom}_{\mathcal{A}}(P, B) \rightarrow \text{Hom}_{\mathcal{A}}(P, C)$  is surjective. But  $B \rightarrow C$  is an epimorphism, so this is precisely what  $P$  being projective means by definition.

Next we show (2)  $\Rightarrow$  (3). Assume  $\text{Hom}_{\mathcal{A}}(P, -)$  is exact and consider a short exact sequence

$$0 \rightarrow N \rightarrow M \rightarrow P \rightarrow 0.$$

Applying  $\text{Hom}_{\mathcal{A}}(P, -)$  we get a surjection  $\text{Hom}_{\mathcal{A}}(P, M) \rightarrow \text{Hom}_{\mathcal{A}}(P, P)$ , and the identity on  $P$  comes then from the desired section  $\sigma: P \rightarrow M$ .

Let us check finally that (3)  $\Rightarrow$  (1). We are given the following situation:

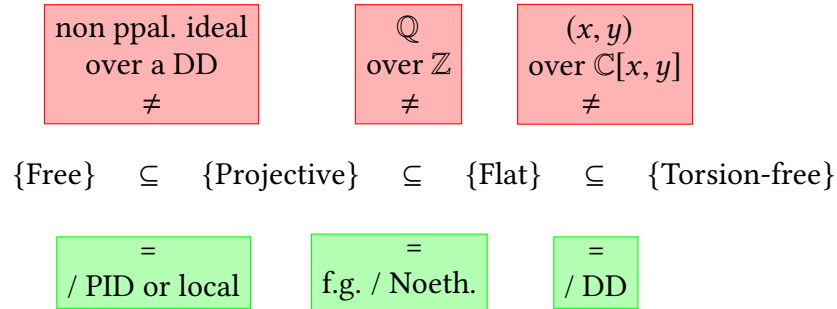
$$\begin{array}{ccc} & & M \\ & & \downarrow \text{epi} \\ P & \longrightarrow & N \end{array}$$

All finite limits exist in  $\mathcal{A}$ , so we may consider the cartesian square

$$\begin{array}{ccc} P \times_N M & \xrightarrow{g} & M \\ f \downarrow & & \downarrow \text{epi} \\ P & \longrightarrow & N \end{array}$$

Epimorphisms are stable under pullback in abelian categories, so  $f$  is also an epimorphism. By assumption, we can find a section  $\sigma: P \rightarrow P \times_N M$  splitting the corresponding short exact sequence. The composition  $g \circ \sigma: P \rightarrow M$  is then the desired lift.  $\square$

Let us look now at the abelian category of modules over a ring  $R$ . What does it mean for an  $R$ -module to be projective?



In the above diagram, ppal. stands for principal, DD stands for Dedekind domain, PID stands for principal ideal domain, f.g. stands for finitely generated and Noeth. stands for Noetherian.

#### REFERENCES

- [Hoc72] M. Hochster. Nonuniqueness of coefficient rings in a polynomial ring. *Proc. Amer. Math. Soc.*, 34:81–82, 1972. [↑ 1](#)

PEDRO NÚÑEZ

ALBERT-LUDWIGS-UNIVERSITÄT FREIBURG, MATHEMATISCHES INSTITUT  
ERNST-ZERMELO-STRASSE 1, 79104 FREIBURG IM BREISGAU (GERMANY)

Email address: [pedro.nunez@math.uni-freiburg.de](mailto:pedro.nunez@math.uni-freiburg.de)

Homepage: <https://home.mathematik.uni-freiburg.de/nunez>