Derived Categories and Birational Geometry

Pedro Núñez

National Taiwan University

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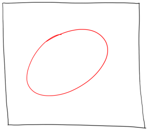
Mathematics Seminar at CUNEF University

Outline



1. Introduction
   - Birational Geometry
   - Derived Categories
2. Relation between Derived Categories and Birational Geometry
   - Relation between them: Kawamata's DK hypothesis
   - Indecomposability conjecture and known results
3. Previous and current work on indecomposability
   - Hyperelliptic varieties
     (joint work with Pieter Belmans and Andreas Demleitner)
   - Threefolds on the Noether Line
     (joint work in progress with Jungkai Chen)

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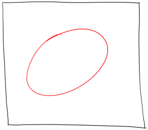

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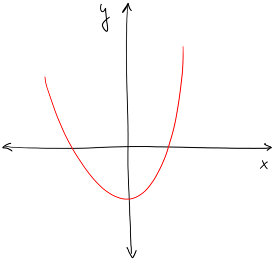

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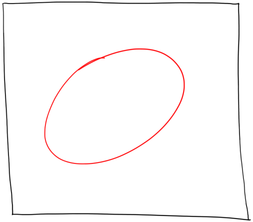

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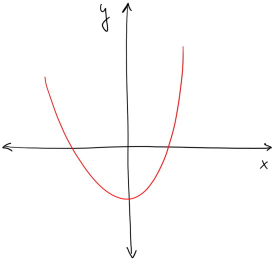

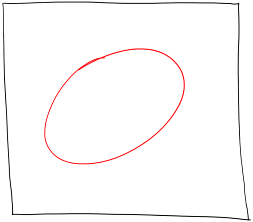

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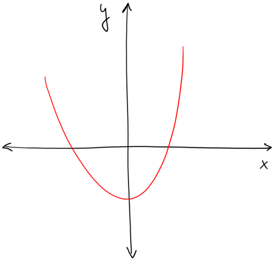

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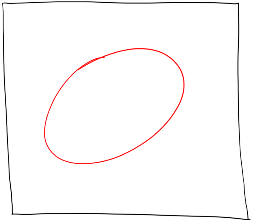

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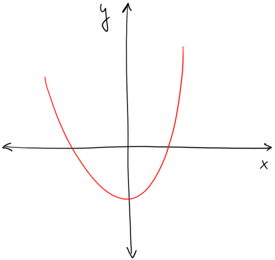

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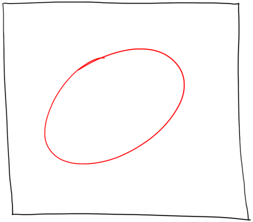

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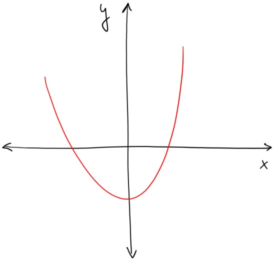

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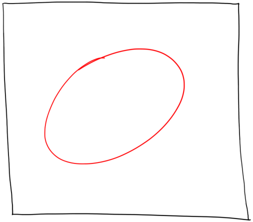

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- Work over $\mathbb{C}$, because we want FTA, Bézout's theorem, etc.
- Zariski topology: closed subsets are zero loci of polynomials.
- Work with projective varieties, because we want compactness.
- We assume varieties to be irreducible.

Birational equivalence

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- **Intermediate goal:** Classify up to birational equivalence.

Canonical line bundle/divisor

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Example: On $\mathbb{P}^1 = \{[x_0 : x_1] \mid (x_0, x_1) \neq (0, 0)\}$ we have coordinates $x := x_1/x_0$ when $x_0 \neq 0$ and $y := x_0/x_1$ when $x_1 \neq 0$. The rational differential form $dx = d(y^{-1}) = -y^{-2}dy$ has a pole of order 2 at the point $H := \{x_0 = 0\} = \{y = 0\}$, hence $K_{\mathbb{P}^1} = -2H$.

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Definition: A projective variety X is called *minimal* when K_X has non-negative intersection with every irreducible curve in X.

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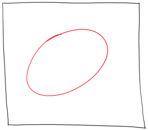

From singular cohomology to sheaf cohomology

From sheaf cohomology to derived categories

Semiorthogonal decompositions (SOD)

- The category $\mathrm{D}^b(X)$ has a natural *triangulated structure.*
- An *orthogonal decomposition* of $\mathrm{D}^b(X)$ would consist of
  - triangulated subcategories $\mathcal{A}, \mathcal{B} \subseteq \mathrm{D}^b(X)$;
  - such that $\mathrm{Hom}(\mathcal{A}, \mathcal{B}) = \mathrm{Hom}(\mathcal{B}, \mathcal{A}) = 0$, i.e., $\mathrm{Hom}(a, b) = \mathrm{Hom}(b, a) = 0$ for all $a \in \mathcal{A}$ and all $b \in \mathcal{B}$;
  - and such that the smallest triangulated subcategory of $\mathrm{D}^b(X)$ containing both of them is $\mathrm{D}^b(X)$ itself.
- **Fact:** If X is connected, then $\mathrm{D}^b(X)$ does not admit any orthogonal decomposition. (Bridgeland '99.)
- A *semiorthogonal decomposition* is the same thing, but without requiring $\mathrm{Hom}(\mathcal{A}, \mathcal{B}) = 0$, only requiring $\mathrm{Hom}(\mathcal{B}, \mathcal{A}) = 0$. It is denoted
$$\mathrm{D}^b(X) = \langle \mathcal{A}, \mathcal{B} \rangle.$$

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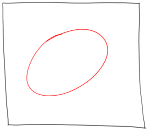

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Kawamata's DK hypothesis

$\boxed{\text{Birational Geometry}}$		$\boxed{\text{Derived Categories}}$
classify varieties	$\leftrightsquigarrow$	compute invariants
$\uparrow$		$\uparrow$
MMP operations	$\leftrightsquigarrow$	SODs of derived categories
$\updownarrow$	*DK-hypothesis*	$\updownarrow$
inequalities of canonical divisors	$\leftrightsquigarrow$	embeddings of derived categories

Example: blow-up

$$\begin{tikzcd} E \arrow[r, hook, "i"] \arrow[d, "\pi"'] & \tilde{S} \arrow[d, "q"] \\ \{p\} \arrow[r, hook, "j"'] & S \end{tikzcd}$$

$$q\colon \tilde{S} = \mathrm{Bl}_p(S) \longrightarrow S \quad \leftrightsquigarrow \quad \mathrm{D}^{\mathrm{b}}(\tilde{S}) = \langle \mathrm{D}^{\mathrm{b}}(p), \mathrm{D}^{\mathrm{b}}(S) \rangle$$

$$\updownarrow \qquad\qquad \updownarrow$$

$$K_{\tilde{S}} = q^* K_S + E \qquad q^*(\mathrm{D}^{\mathrm{b}}(S))$$

$$(\ K_{\tilde{S}} \geq K_S\ ) \quad \leftrightsquigarrow \quad i_*(\mathcal{O}_E(-E) \otimes \pi^* \mathrm{D}^{\mathrm{b}}(p))$$

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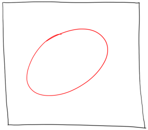

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Indecomposability conjecture

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This is not strictly true, but the following is a folklore conjecture:

Conjecture

Let X be a minimal smooth projective variety with $p_g > 0$.
Then $\mathrm{D}^b(X)$ is indecomposable.

Main known results on indecomposability

- Bridgeland '99: Calaby–Yau varieties have indecomposable derived categories.

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Theorem (Kawatani–Okawa '18, Okawa '23, Pirozhkov '25, …)

A minimal smooth projective surface has indecomposable derived category if and only if $(p_g, q) \neq (0, 0)$.

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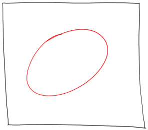

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Hyperelliptic varieties: definition

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- Equivalently, they are smooth projective varieties which are not abelian but admit an abelian variety as a finite étale cover.
- 1-dimensional hyperelliptic varieties do not exist, and 2-dimensional hyperelliptic varieties are bielliptic surfaces.

Hyperelliptic varieties: conjecture and main result

Conjecture

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The *irregularity* of X is $q_X := h^1(X, \mathcal{O}_X)$ ($< \dim X$ if X hyperelliptic).

Theorem

The conjecture holds in the following cases:

1. X is cyclic, i.e., $X = A/G$ with G cyclic.
2. X has irregularity $q_X = \dim X - 2$ or $\dim X - 1$.
3. The fiber(s) of the Albanese morphism of X have trivial canonical bundle.

In particular, the conjecture holds if $\dim X \leq 3$.

Main approach: Albanese morphism + induction

- The Albanese morphism is a universal morphism into an abelian variety $\mathrm{alb}_X\colon X \to \mathrm{Alb}(X)$.

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- In our paper we show that the fibers are either abelian varieties or hyperelliptic varieties again.
- Combining this with [Pirozhkov '23] and induction on the dimension, we can deduce indecomposability in the first two cases of the theorem.

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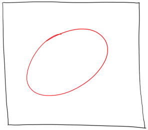

Threefolds on the Noether Line: definition

Two key birational invariants of a projective variety X are its geometric genus $p_g(X) := h^0(X, \omega_X)$ and its canonical volume

$$\mathrm{vol}(X) := \lim_{m \to \infty} \frac{h^0(X, \omega_X^{\otimes m})}{m^n/n!}, \quad \text{where } n := \dim(X).$$

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By work of Jungkai Chen, Meng Chen and Chen Jiang ('20), and others, we know that projective threefolds of general type satisfy

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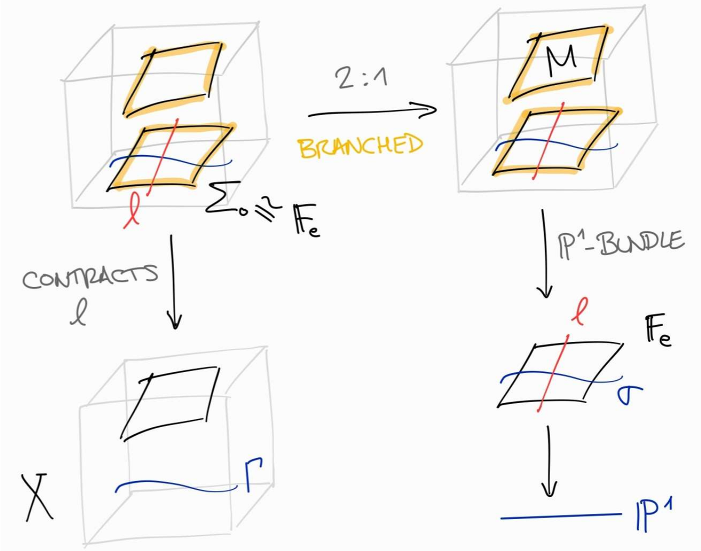
Definition: A projective threefold of general type is said to be *on the Noether line* if equality holds above.

Kobayashi's construction ('92)



Threefolds on the Noether Line: current result

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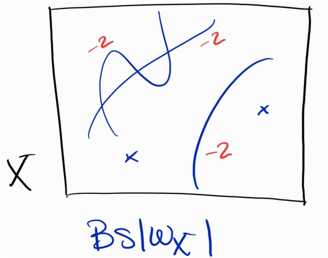
* The moduli space of such threefolds has several irreducible components, and this statement applies to one of the top-dimensional irreducible components.

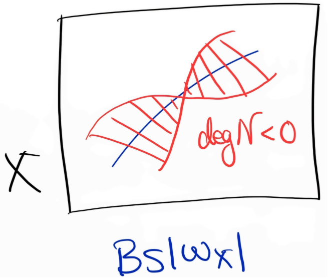
** There are three Noether Lines, and threefolds on the second and third Noether Lines are necessarily singular.

Main tool: generalization of a criterion in [KO18]

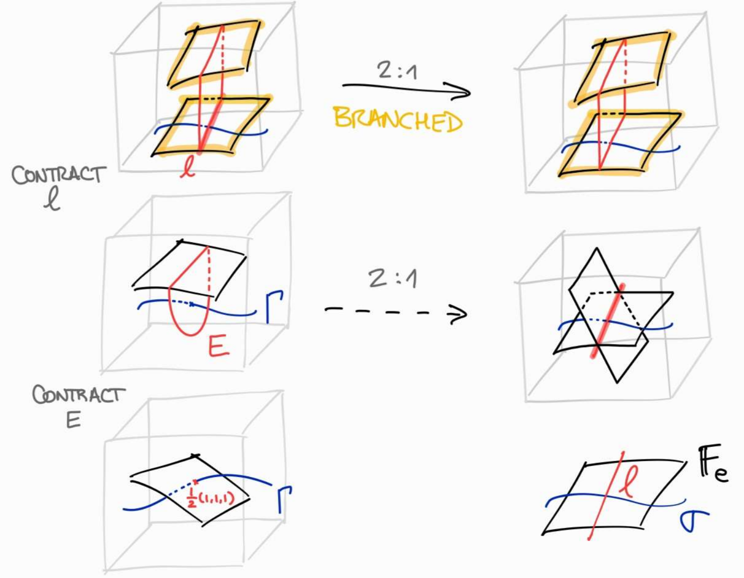
Theorem

Let X be a minimal smooth projective variety such that $\Gamma := \mathrm{Bs}\,|\omega_X|$ is a smooth (necessarily rational) curve. If its conormal bundle is big and nef, then $\mathrm{D}^{\mathrm{b}}(X)$ is indecomposable.





Current work in progress: singular cases



Thanks for your attention!