Derived Categories and Birational Geometry

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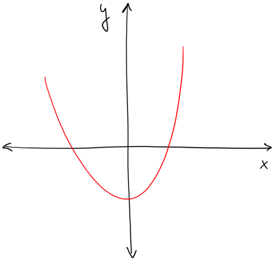
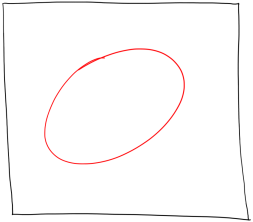
Projective algebraic varieties



$\{(x, y) \in \mathbb{A}^2 \mid y = x^2 - 1\}$



$\{[x : y : z] \in \mathbb{P}^2 \mid yz = x^2 - z^2\}$

- Work over $\mathbb{C}$, because we want FTA, Bézout's theorem, etc.
- Zariski topology: closed subsets are zero loci of polynomials.
- Work with projective varieties, because we want compactness.
- We assume varieties to be irreducible.

Morphisms of varieties

- A morphism of algebraic varieties is a continuous map which locally looks polynomial. So $x \mapsto x^2$ is, but $x \mapsto e^x$ isn't.
- An isomorphism of algebraic varieties is a morphism which is invertible (the inverse should be algebraic as well).
- **Goal:** Classify varieties up to isomorphism.

This turns out to be too hard!

$\rightsquigarrow$ First try to classify up to a coarser equivalence relation.

Birational equivalence

- Two varieties are *birationally equivalent* if they contain isomorphic dense open subsets, i.e., they are the same except possibly over some (lower-dimensional) proper closed subset.
- Isomorphic varieties are birationally equivalent, but not vice-versa.

Example: Let $\tilde{X} = \{(x, y; t) \in \mathbb{A}^2 \times \mathbb{P}^1 \mid y = xt\}$. Then

$$\begin{aligned} \pi : \tilde{X} &\longrightarrow \mathbb{A}^2 \qquad \textit{(blow-up)} \\ (x, y; t) &\longmapsto (x, xt) \end{aligned}$$

induces $\tilde{X} \setminus E \cong \mathbb{A}^2 \setminus \{(0,0)\}$, where $E = \{(0,0;t) \mid t \in \mathbb{P}^1\}$.
So $\tilde{X} \sim_{\text{bir}} \mathbb{A}^2$, but they are not isomorphic.

Canonical line bundle/divisor

- If X is an n-dimensional smooth variety, its *canonical line bundle* is

$$\omega_X := \bigwedge^n \Omega_X, \text{ where } \Omega_X \text{ is its (holomorphic) cotangent bundle.}$$

- A divisor is a $\mathbb{Z}$-linear combination of codim. 1 subvarieties.
- The *canonical divisor* K_X is the divisor of zeros and poles of any non-zero rational section of ω_X.

Example: On $\mathbb{P}^1 = \{[x_0 : x_1] \mid (x_0, x_1) \neq (0,0)\}$ we have coordinates $x := x_1/x_0$ when $x_0 \neq 0$ and $y := x_0/x_1$ when $x_1 \neq 0$. The rational differential form $dx = d(y^{-1}) = -y^{-2}dy$ has a pole of order 2 at the point $H := \{x_0 = 0\} = \{y = 0\}$, hence $K_{\mathbb{P}^1} = -2H$.

Minimal Model Program (MMP)

- To classify up to birational equivalence, we want to pick a single representative X' in the birational equivalence class $[X]$.
- If $\pi : \tilde{X} \to X$ is a blow-up, $\tilde{X} \sim_{\text{bir}} X$. Among them, X is simpler.
- We have the relation $K_{\tilde{S}} = \pi^* K_S + E$, and $K_{\tilde{S}} \cdot E = -1$.
- **MMP's idea:** Look for curves C such that $K_X \cdot C < 0$. If you find one, you can contract it (Castelnuovo/Mori). Then repeat.
- Conjecturally, this process terminates. The variety we are left with at the end is our chosen represenative.

Definition: A projective variety X is called *minimal* when K_X has non-negative intersection with every irreducible curve in X.

From singular cohomology to sheaf cohomology

- We can use (topological) singular cohomology to compute invariants such as *Betti numbers*

$$b_i := \dim H^i(X, \mathbb{C}).$$

- We can let the cohomology coefficients $\mathbb{C}$ vary along the topological space X

$\rightsquigarrow$ sheaves and sheaf cohomology.

- With sheaf cohomology we can get more refined invariants such as *Hodge numbers*

$$h^{p,q} := \dim H^q(X, \Omega_X^p), \text{ which satisfy } \sum_{p+q=i} h^{p,q} = b_i.$$

From sheaf cohomology to derived categories

Sheaf cohomolgy is the right derived functor of global sections $\Gamma(X, \mathcal{F})$, so in order to compute $H^i(X, \mathcal{F})$ we follow these steps:

1. Replace $\mathcal{F}$ by an injective resolution, that is, a cochain complex of sheaves

$$\mathcal{I}^\bullet = \cdots \longrightarrow \mathcal{I}^{n-1} \xrightarrow{d^{n-1}} \mathcal{I}^n \xrightarrow{d^n} \mathcal{I}^{n+1} \longrightarrow \cdots$$

   which are well-behaved with respect to taking global sections.

2. Take i-th cohomology of this cochain complex

$$H^i(X, \mathcal{F}) := \ker(d^i)/\mathrm{im}(d^{i-1}).$$

Step 2. loses too much information. The *derived category* $\mathrm{D}^\mathrm{b}(X)$ fixes that: work with cochain complexes directly!

Semiorthogonal decompositions (SOD)

- The category $\mathrm{D}^{\mathrm{b}}(X)$ has a natural *triangulated structure.*
- An *orthogonal decomposition* of $\mathrm{D}^{\mathrm{b}}(X)$ would consist of
  - triangulated subcategories $\mathcal{A}, \mathcal{B} \subseteq \mathrm{D}^{\mathrm{b}}(X)$;
  - such that $\mathrm{Hom}(\mathcal{A}, \mathcal{B}) = \mathrm{Hom}(\mathcal{B}, \mathcal{A}) = 0$, i.e., $\mathrm{Hom}(a, b) = \mathrm{Hom}(b, a) = 0$ for all $a \in \mathcal{A}$ and all $b \in \mathcal{B}$;
  - and such that the smallest triangulated subcategory of $\mathrm{D}^{\mathrm{b}}(X)$ containing both of them is $\mathrm{D}^{\mathrm{b}}(X)$ itself.
- **Fact:** If X is connected, then $\mathrm{D}^{\mathrm{b}}(X)$ does not admit any orthogonal decomposition. (Bridgeland '99.)
- A *semiorthogonal decomposition* is the same thing, but without requiring $\mathrm{Hom}(\mathcal{A}, \mathcal{B}) = 0$, only requiring $\mathrm{Hom}(\mathcal{B}, \mathcal{A}) = 0$. It is denoted

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathcal{A}, \mathcal{B} \rangle.$$

Kawamata's DK hypothesis

Birational Geometry		Derived Categories
classify varieties	$\leftrightsquigarrow$	compute invariants
$\uparrow$		$\uparrow$
MMP operations	$\leftrightsquigarrow$	SODs of derived categories
$\updownarrow$	*DK-hypothesis*	$\updownarrow$
inequalities of canonical divisors	$\leftrightsquigarrow$	embeddings of derived categories

Example: blow-up

$$
\begin{tikzcd}
E \arrow[r, hook, "i"] \arrow[d, "\pi"'] & \tilde{S} \arrow[d, "q"] \\
\{p\} \arrow[r, hook, "j"'] & S
\end{tikzcd}
$$

$$q\colon \tilde{S} = \mathrm{Bl}_p(S) \to S \quad \leftrightsquigarrow \quad \mathrm{D^b}(\tilde{S}) = \langle \mathrm{D^b}(p), \mathrm{D^b}(S) \rangle$$

$$\updownarrow \qquad\qquad \updownarrow$$

$$K_{\tilde{S}} = q^* K_S + E \quad \leftrightsquigarrow \quad q^*(\mathrm{D^b}(S))$$

$$(\, K_{\tilde{S}} \geq K_S \,) \qquad\qquad i_*(\mathcal{O}_E(-E) \otimes \pi^* \mathrm{D^b}(p))$$

Indecomposability conjecture

The previous discussion suggests that

minimal varieties $\leftrightsquigarrow$ indecomposable derived categories.

This is not strictly true, but the following is a folklore conjecture:

Conjecture

Let X be a minimal smooth projective variety with $p_g > 0$.
Then $\mathrm{D}^b(X)$ is indecomposable, i.e., it has no SOD.

Main known results on indecomposability

- Bridgeland '99: Calabi–Yau varieties have indecomposable derived categories.
- Kawatani–Okawa '18: the base locus of the canonical linear system controls indecomposability.
- Pirozhkov '23: stronger notion of indecomposability (NSSI); examples are finite covers of abelian varieties and varieties fibered in NSSI varieties over NSSI bases.

Theorem (Kawatani–Okawa '18, Okawa '23, Pirozhkov '25, ...)

A minimal smooth projective surface has indecomposable derived category if and only if $(p_g, q) \neq (0, 0)$.

Hyperelliptic varieties: definition

- A *hyperelliptic variety* $X = A/G$ is the quotient of an abelian variety A by a finite group of automorphisms $G \subseteq \mathrm{Aut}(A)$ acting freely and without translations on A.
- It follows that they are smooth projective minimal varieties with torsion canonical divisor, i.e., $mK_X \sim 0$ for some $m \in \mathbb{Z}_{>0}$.
- Equivalently, they are smooth projective varieties which are not abelian but admit an abelian variety as a finite étale cover.
- 1-dimensional hyperelliptic varieties do not exist, and 2-dimensional hyperelliptic varieties are bielliptic surfaces.

Hyperelliptic varieties: conjecture and main result

Conjecture

Let X be a hyperelliptic variety. Then $\mathrm{D}^{\mathrm{b}}(X)$ is indecomposable.

The *irregularity* of X is $q_X := h^1(X, \mathcal{O}_X)$ ($< \dim X$ if X hyperelliptic).

Theorem

The conjecture holds in the following cases:

1. X is cyclic, i.e., $X = A/G$ with G cyclic.
2. X has irregularity $q_X = \dim X - 2$ or $\dim X - 1$.
3. The fiber(s) of the Albanese morphism of X have trivial canonical bundle.

In particular, the conjecture holds if $\dim X \leq 3$.

Main approach: Albanese morphism + induction

- The Albanese morphism is a universal morphism into an abelian variety $\mathrm{alb}_X \colon X \to \mathrm{Alb}(X)$.
- By [Kawamata '85], if X is hyperellitpic, then the Albanese morphism is an étale fiber bundle with smooth connected fibers.
- In our paper we show that the fibers are either abelian varieties or hyperelliptic varieties again.
- Combining this with [Pirozhkov '23] and induction on the dimension, we can deduce indecomposability in the first two cases of the theorem.

Threefolds on the Noether Line: definition

Two key birational invariants of a projective variety X are its geometric genus $p_g(X) := h^0(X, \omega_X)$ and its canonical volume

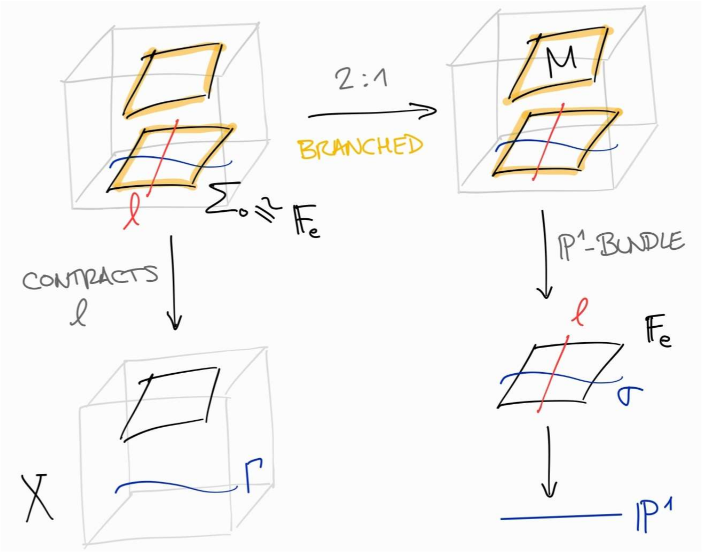
$$\mathrm{vol}(X) := \lim_{m\to\infty} \frac{h^0(X, \omega_X^{\otimes m})}{m^n/n!}, \quad \text{where } n := \dim(X).$$

By work of Jungkai Chen, Meng Chen and Chen Jiang ('20), and others, we know that projective threefolds of general type satisfy

$$\mathrm{vol}(X) \geq \frac{4}{3}p_g(X) - \frac{10}{3} \quad \text{(Noether Inequality).}$$

Definition: A projective threefold of general type is said to be *on the Noether line* if equality holds above.

Kobayashi's construction ('92)



Threefolds on the Noether Line: current result

Theorem

Let X be a *general** minimal smooth projective threefold on the first** Noether Line. Then $\mathrm{D}^b(X)$ is indecomposable.

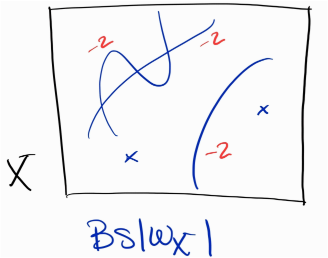
* The moduli space of such threefolds has several irreducible components, and this statement applies to one of the top-dimensional irreducible components.

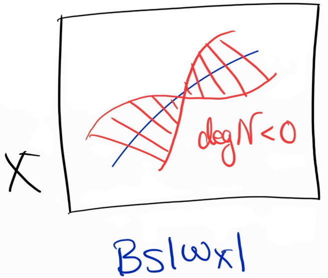
** There are three Noether Lines, and threefolds on the second and third Noether Lines are necessarily singular.

Main tool: generalization of a criterion in [KO18]

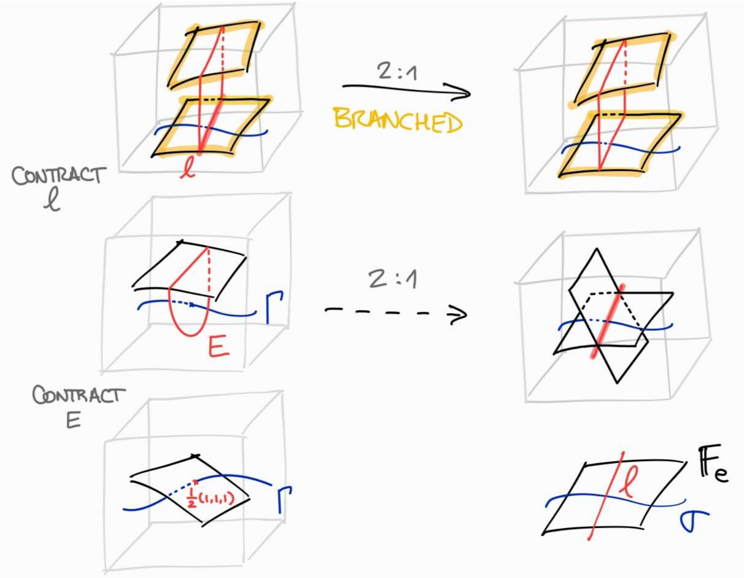
Theorem

Let X be a minimal smooth projective variety such that $\Gamma := \mathrm{Bs}\,|\omega_X|$ is a smooth (necessarily rational) curve. If its conormal bundle is big and nef, then $\mathrm{D^b}(X)$ is indecomposable.





Current work in progress: singular cases



Thanks for your attention!