

# Derived Categories and Birational Geometry

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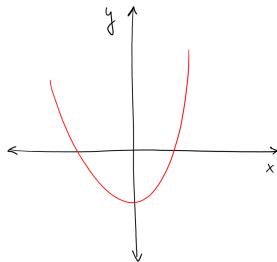
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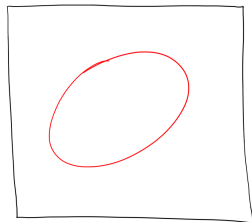
Mathematics Seminar at CUNEF University



# Projective algebraic varieties



$$\{(x, y) \in \mathbb{A}^2 \mid y = x^2 - 1\}$$



$$\{[x : y : z] \in \mathbb{P}^2 \mid yz = x^2 - z^2\}$$

- Work over  $\mathbb{C}$ , because we want FTA, Bézout's theorem, etc.
- Zariski topology: closed subsets are zero loci of polynomials.
- Work with projective varieties, because we want compactness.
- We assume varieties to be irreducible.

# Morphisms of varieties

- A morphism of algebraic varieties is a continuous map which locally looks polynomial. So  $x \mapsto x^2$  is, but  $x \mapsto e^x$  isn't.
- An isomorphism of algebraic varieties is a morphism which is invertible (the inverse should be algebraic as well).
- **Goal:** Classify varieties up to isomorphism.

This turns out to be too hard!

↪ First try to classify up to a coarser equivalence relation.

# Birational equivalence

- Two varieties are *birationally equivalent* if they contain isomorphic dense open subsets, i.e., they are the same except possibly over some (lower-dimensional) proper closed subset.
- Isomorphic varieties are birationally equivalent, but not vice-versa.

**Example:** Let  $X = \mathbb{A}^2$  and  $\tilde{X} = \{(x, y, t) \in \mathbb{A}^3 \mid y = xt\}$ . Then

$$\begin{aligned}\pi: \tilde{X} &\rightarrow X \\ (x, y, t) &\mapsto (x, xt)\end{aligned}$$

induces  $\tilde{X} \setminus E \cong X \setminus \{x = 0\}$ , where  $E = \{(0, 0, t) \mid t \in \mathbb{C}\}$ .

So  $\tilde{X} \sim_{\text{bir}} X$ , but they are not isomorphic.

## Canonical line bundle/divisor

- If  $X$  is an  $n$ -dimensional smooth variety, its *canonical line bundle* is

$$\omega_X := \bigwedge^n \Omega_X, \text{ where } \Omega_X \text{ is its (holomorphic) cotangent bundle.}$$

- A divisor is a  $\mathbb{Z}$ -linear combination of codim. 1 subvarieties.
- The *canonical divisor*  $K_X$  is the divisor of zeros and poles of any non-zero rational section of  $\omega_X$ .

**Example:** On  $\mathbb{P}^1 = \{[x_0 : x_1] \mid (x_0, x_1) \neq (0, 0)\}$  we have coordinates  $x := x_1/x_0$  when  $x_0 \neq 0$  and  $y := x_0/x_1$  when  $x_1 \neq 0$ . The rational differential form  $dx = d(y^{-1}) = -y^{-2}dy$  has a pole of order 2 at the point  $H := \{x_0 = 0\} = \{y = 0\}$ , hence  $K_{\mathbb{P}^1} = -2H$ .

# Minimal Model Program (MMP)

- To classify up to birational equivalence, we want to pick a single representative  $X'$  in the birational equivalence class  $[X]$ .
- If  $\pi: \tilde{X} \rightarrow X$  is a blow-up,  $\tilde{X} \sim_{\text{bir}} X$ . Among them,  $X$  is simpler.
- We have the relation  $K_{\tilde{S}} = \pi^* K_S + E$ , and  $K_{\tilde{S}} \cdot E = -1$ .
- **MMP's idea:** Look for curves  $C$  such that  $K_X \cdot C < 0$ . If you find one, you can contract it (Castelnuovo/Mori). Then repeat.
- Conjecturally, this process terminates. The variety we are left with at the end is our chosen representative.

**Definition:** A projective variety  $X$  is called *minimal* when  $K_X$  has non-negative intersection with every irreducible curve in  $X$ .

# From singular cohomology to sheaf cohomology

- We can use (topological) singular cohomology to compute invariants such as *Betti numbers*

$$b_i := \dim H^i(X, \mathbb{C}).$$

- We can let the cohomology coefficients  $\mathbb{C}$  vary along the topological space  $X$   
 $\rightsquigarrow$  sheaves and sheaf cohomology.
- With sheaf cohomology we can get more refined invariants such as *Hodge numbers*

$$h^{p,q} := \dim H^q(X, \Omega_X^p), \text{ which satisfy } \sum_{p+q=i} h^{p,q} = b_i.$$



# From sheaf cohomology to derived categories

Sheaf cohomology is the right derived functor of global sections  $\Gamma(X, \mathcal{F})$ , so in order to compute  $H^i(X, \mathcal{F})$  we follow these steps:

1. Replace  $\mathcal{F}$  by an injective resolution, that is, a cochain complex of sheaves

$$\mathcal{I}^\bullet = \dots \rightarrow \mathcal{I}^{n-1} \xrightarrow{d^{n-1}} \mathcal{I}^n \xrightarrow{d^n} \mathcal{I}^{n+1} \rightarrow \dots$$

which are well-behaved with respect to taking global sections.

2. Take  $i$ -th cohomology of this cochain complex

$$H^i(X, \mathcal{F}) := \ker(d^i) / \operatorname{im}(d^{i-1}).$$

Step 2. loses too much information. The *derived category*  $D^b(X)$  fixes that: work with cochain complexes directly!

## Semiorthogonal decompositions (SOD)

- The category  $D^b(X)$  has a natural *triangulated structure*.
- An *orthogonal decomposition* of  $D^b(X)$  would consist of
  - triangulated subcategories  $\mathcal{A}, \mathcal{B} \subseteq D^b(X)$ ;
  - such that  $\text{Hom}(\mathcal{A}, \mathcal{B}) = \text{Hom}(\mathcal{B}, \mathcal{A}) = 0$ , i.e.,  
 $\text{Hom}(a, b) = \text{Hom}(b, a) = 0$  for all  $a \in \mathcal{A}$  and all  $b \in \mathcal{B}$ ;
  - and such that the smallest triangulated subcategory of  $D^b(X)$  containing both of them is  $D^b(X)$  itself.
- **Fact:** If  $X$  is connected, then  $D^b(X)$  does not admit any orthogonal decomposition. (Bridgeland '99.)
- A *semiorthogonal decomposition* is the same thing, but without requiring  $\text{Hom}(\mathcal{A}, \mathcal{B}) = 0$ , only requiring  $\text{Hom}(\mathcal{B}, \mathcal{A}) = 0$ . It is denoted

$$D^b(X) = \langle \mathcal{A}, \mathcal{B} \rangle.$$

## Kawamata's DK hypothesis

# Birational Geometry

classify varieties



## MMP operations



inequalities of  
canonical divisors

## Derived Categories

compute invariants

↑

## SODs of derived categories



embeddings of  
derived categories

## DK-hypothesis

 $\longleftrightarrow$ 

## Example: blow-up

$$\begin{array}{ccc}
 E & \xhookrightarrow{i} & \tilde{S} \\
 \pi \downarrow & & \downarrow q \\
 \{p\} & \xhookrightarrow{j} & S
 \end{array}$$

$$q: \tilde{S} = \mathrm{Bl}_p(S) \rightarrow S$$

$$\Leftrightarrow$$

$$\mathrm{D}^b(\tilde{S}) = \langle \mathrm{D}^b(p), \mathrm{D}^b(S) \rangle$$

$$\updownarrow$$

$$\updownarrow$$

$$\begin{aligned}
 K_{\tilde{S}} &= q^* K_S + E \\
 (K_{\tilde{S}} &\geq K_S)
 \end{aligned}$$

$$\Leftrightarrow$$

$$\begin{aligned}
 &q^*(\mathrm{D}^b(S)) \\
 &i_*(\mathcal{O}_E(-E) \otimes \pi^* \mathrm{D}^b(p))
 \end{aligned}$$

# Indecomposability conjecture

The previous discussion suggests that

minimal varieties  $\Leftrightarrow$  indecomposable derived categories.

This is not strictly true, but the following is a folklore conjecture:

## Conjecture

Let  $X$  be a minimal smooth projective variety with  $p_g > 0$ .  
Then  $D^b(X)$  is indecomposable.

## Main known results on indecomposability

- Bridgeland '99: Calabi–Yau varieties have indecomposable derived categories.
- Kawatani–Okawa '18: the base locus of the canonical linear system controls indecomposability.
- Pirozhkov '23: stronger notion of indecomposability (NSSI); examples are finite covers of abelian varieties and varieties fibered in NSSI varieties over NSSI bases.

Theorem (Kawatani–Okawa '18, Okawa '23, Pirozhkov '25, ...)

A minimal smooth projective surface has indecomposable derived category if and only if  $(p_g, q) \neq (0, 0)$ .

# Hyperelliptic varieties: definition

- A *hyperelliptic variety*  $X = A/G$  is the quotient of an abelian variety  $A$  by a finite group of automorphisms  $G \subseteq \operatorname{Aut}(A)$  acting freely and without translations on  $A$ .
- It follows that they are smooth projective minimal varieties with torsion canonical divisor, i.e.,  $mK_X \sim 0$  for some  $m \in \mathbb{Z}_{>0}$ .
- Equivalently, they are smooth projective varieties which are not abelian but admit an abelian variety as a finite étale cover.
- 1-dimensional hyperelliptic varieties do not exist, and 2-dimensional hyperelliptic varieties are bielliptic surfaces.

# Hyperelliptic varieties: conjecture and main result

## Conjecture

Let  $X$  be a hyperelliptic variety. Then  $D^b(X)$  is indecomposable.

The *irregularity* of  $X$  is  $q_X := h^1(X, \mathcal{O}_X)$  ( $< \dim X$  if  $X$  hyperelliptic).

## Theorem

The conjecture holds in the following cases:

1.  $X$  is cyclic, i.e.,  $X = A/G$  with  $G$  cyclic.
2.  $X$  has irregularity  $q_X = \dim X - 2$  or  $\dim X - 1$ .
3. The fiber(s) of the Albanese morphism of  $X$  have trivial canonical bundle.

In particular, the conjecture holds if  $\dim X \leq 3$ .



# Main approach: Albanese morphism + induction

- The Albanese morphism is a universal morphism into an abelian variety  $\mathrm{alb}_X: X \rightarrow \mathrm{Alb}(X)$ .
- By [Kawamata '85], if  $X$  is hyperelliptic, then the Albanese morphism is an étale fiber bundle with smooth connected fibers.
- In our paper we show that the fibers are either abelian varieties or hyperelliptic varieties again.
- Combining this with [Pirozhkov '23] and induction on the dimension, we can deduce indecomposability in the first two cases of the theorem.

## Threefolds on the Noether Line: definition

Two key birational invariants of a projective variety  $X$  are its geometric genus  $p_g(X) := h^0(X, \omega_X)$  and its canonical volume

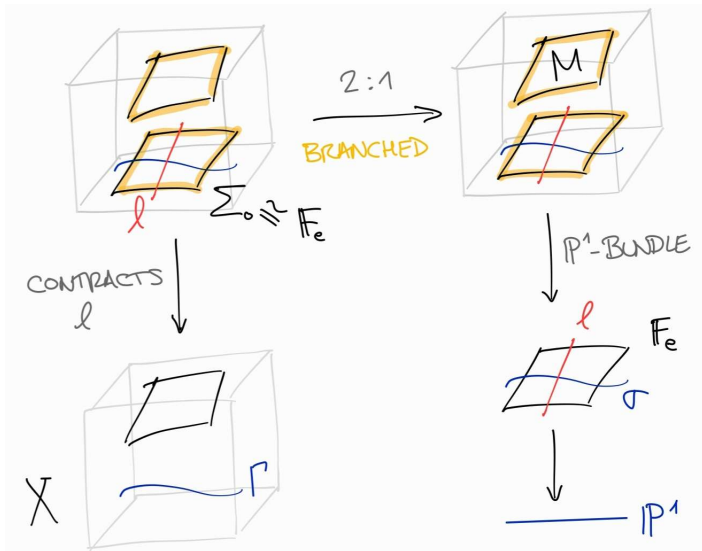
$$\mathrm{vol}(X) := \lim_{m \rightarrow \infty} \frac{h^0(X, \omega_X^{\otimes m})}{m^n / n!}, \quad \text{where } n := \dim(X).$$

By work of Jungkai Chen, Meng Chen and Chen Jiang ('20), and others, we know that projective threefolds of general type satisfy

$$\mathrm{vol}(X) \geq \frac{4}{3} p_g(X) - \frac{10}{3} \quad (\text{Noether Inequality}).$$

**Definition:** A projective threefold of general type is said to be *on the Noether line* if equality holds above.

## Kobayashi's construction ('92)



# Threefolds on the Noether Line: current result

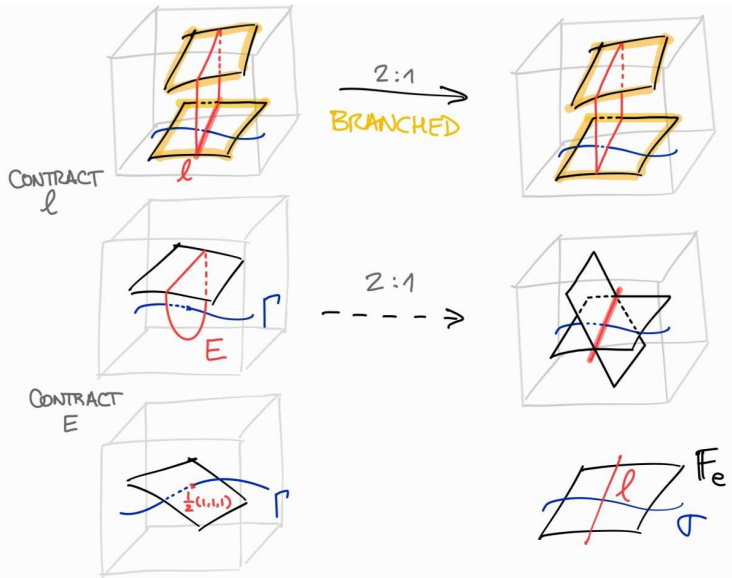
## Theorem

Let  $X$  be a *general*<sup>\*</sup> minimal smooth projective threefold on the first<sup>\*\*</sup> Noether Line. Then  $D^b(X)$  is indecomposable.

- \* The moduli space of such threefolds has several irreducible components, and this statement applies to one of the top-dimensional irreducible components.
- \*\* There are three Noether Lines, and threefolds on the second and third Noether Lines are necessarily singular.

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# Current work in progress: singular cases



Thanks for your attention!