TALK ON HILBERT SCHEMES OF POINTS ON SURFACES

PEDRO NÚÑEZ

Abstract. Script for the 7th talk of the seminar on Heisenberg algebras and Hilbert schemes of points on surfaces organized by Mara Ungureanu during the Summer Term 2021 at the University of Freiburg.

Contents

—parts in gray will be omitted during the talk—

0. Conventions and notation

We always work over $\mathbb{C}$. By a variety we mean an integral separated scheme of finite type over $\mathbb{C}$.

Appendix A. Quotients of quasi-projective varieties by finite groups

We will mostly follow the notes in http://www.math.lsa.umich.edu/~mmustata/appendix.pdf in this appendix.

Lemma 1. *Let G be a finite group. Let A be a finite type $\mathbb{C}$-algebra and assume that the group G acts on A from the left by $\mathbb{C}$-algebra automorphisms. Then the set of invariant elements A^G is a $\mathbb{C}$-subalgebra of A which is of finite type over $\mathbb{C}$.*

Proof. Let $\rho\colon G \to \mathrm{Aut}_{\mathbb{C}}(A)$ be the given left action. Let us first quickly ensure that

$$A^G := \bigcap_{g\in G} \{a \in A \mid \rho(g)(a) = a\}$$

is a $\mathbb{C}$-subalgebra of A.

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- $A^G \subseteq A$ is a subgroup. Indeed, since $\rho(g)$ is a ring morphism for every $g \in G$, we have $0 \in A^G$. And if $a_1, a_2 \in A^G$ and $g \in G$, then it follows again from $\rho(g)$ being a ring morphism that
$$\rho(g)(a_1 + a_2) = \rho(g)(a_1) + \rho(g)(a_2) = a_1 + a_2.$$
- $A^G \subseteq A$ is a subring. We have seen already that it is a subgroup. Since $\rho(g)$ is a ring morphism for every $g \in G$, we also have $1 \in A^G$, so it remains only to show that A^G is closed under products. If $a_1, a_2 \in A^G$ and $g \in G$, then using once again that $\rho(g)$ is a ring morphism we see that
$$\rho(g)(a_1 a_2) = \rho(g)(a_1)\rho(g)(a_2) = a_1 a_2.$$
- $A^G \subseteq A$ is a $\mathbb{C}$-vector subspace. We have seen already that it is a subgroup, so it remains only to show that A^G is closed under scalar product. If $a \in A^G$, $\lambda \in \mathbb{C}$ and $g \in G$, then we use the assumption that $\rho(g)$ is $\mathbb{C}$-linear to deduce that
$$\rho(g)(\lambda a) = \lambda \rho(g)(a) = \lambda a.$$

The other assertion in the lemma is that A^G is a finite type $\mathbb{C}$-algebra. The idea is to write A^G as a finite B-module for some suitable finite type $\mathbb{C}$-algebra B. Then it would follow that A^G is a finite type $\mathbb{C}$-algebra as well. Indeed, let $\beta_1, \ldots, \beta_m \in B$ be generators of B as an algebra over $\mathbb{C}$, and let $e_1, \ldots, e_l \in A^G$ be generators of A^G as a B-module. Then we can write any $a \in A^G$ as a B-linear combination
$$a = \sum_{i=1}^{l} b_i e_i,$$
and in turn each b_i as an algebraic combination
$$b_i = f_i(\beta_1, \ldots, \beta_m)$$
for some $f_i \in \mathbb{C}[\beta_1, \ldots, \beta_m]$. It follows that we can write a as an algebraic combination in the variables $\beta_1, \ldots, \beta_m, e_1, \ldots, e_l$, so these elements would form a system of generators of A^G as a $\mathbb{C}$-algebra.

In order to construct such B, we first note that the inclusion $A^G \subseteq A$ is an integral ring extension. Indeed, every $a \in A$ is a root of the monic polynomial
$$P_a(t) := \prod_{g \in G} (t - \rho(g)(a)),$$
whose coefficients are in A^G by Vieta's formulas. Let $\alpha_1, \ldots, \alpha_m \in A$ be generators of A as an algebra over $\mathbb{C}$. Let $\{c_{i,j}\}_{j=0}^{d_i}$ be the coefficients of P_{α_i} for each $i \in \{1, \ldots, m\}$. Then define B to be the $\mathbb{C}$-subalgebra of A generated by all these coefficients $\{c_{1,0}, \ldots, c_{1,d_1}, c_{2,0}, \ldots, c_{m,d_m}\}$. $\square$

References

Pedro Núñez
Albert-Ludwigs-Universität Freiburg, Mathematisches Institut
Ernst-Zermelo-Strasse 1, 79104 Freiburg im Breisgau (Germany)
Email address: pedro.nunez@math.uni-freiburg.de
Homepage: https://home.mathematik.uni-freiburg.de/nunez