TALK ON HILBERT SCHEMES OF POINTS ON SURFACES

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ABSTRACT. Script for the 7th talk of the seminar on Heisenberg algebras and Hilbert schemes of points on surfaces organized by Mara Ungureanu during the Summer Term 2021 at the University of Freiburg.

CONTENTS

—parts in gray will be omitted during the talk—

0. CONVENTIONS AND NOTATION

We always work over $\mathbb{C}$. By a variety we mean an integral separated scheme of finite type over $\mathbb{C}$.

APPENDIX A. QUOTIENTS OF QUASI-PROJECTIVE VARIETIES BY FINITE GROUPS

We will mostly follow the notes in http://www.math.lsa.umich.edu/~mmustata/appendix.pdf in this appendix.

Remark A.1. Let G be a finite group and let $X = \operatorname{Spec} A$ be an affine variety. An action of G on A by $\mathbb{C}$-algebra automorphisms *from the left* is the same as an aciton of G on X by $\mathbb{C}$-scheme morphisms *from the right*. The two things are more explicitly related as follows:

$$(g \cdot f)(x) = f(x \cdot g).$$

There are various notions of quotients in algebraic geometry, cf. [MFK94, §0.1]. Fortunately, in the case of finite groups, the various notions agree.

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Definition A.2. Let G be a finite group and let X be a scheme of finite type over $\mathbb{C}$. Let $\sigma\colon X \times G \to X$ be an action of G on X on the right[1]. A *quotient of X by G* is a $\mathbb{C}$-scheme Y together with a $\mathbb{C}$-scheme morphism $\pi\colon X \to Y$ with the following properties:

i) π is G-invariant, i.e. we have $\pi \circ \sigma = \pi \circ p_1$, where $p_1\colon X \times G \to X$ is the projection.
ii) π is universal with respect to the property in i), i.e. for every $\mathbb{C}$-scheme Z and every G-invariant $\mathbb{C}$-scheme morphism $f\colon X \to Z$, there exists a unique $\mathbb{C}$-scheme morphism $h\colon Y \to Z$ such that $h \circ \pi = f$.

Remark A.3. The previous definition is that of a *categorical quotient*, cf. [MFK94, Definition 0.5].

Remark A.4. (Categorical) quotients are unique up to unique isomorphism, meaning that given any other pair (Y', π') with the same properties, there exists a unique $\mathbb{C}$-scheme isomorphism $\varphi\colon Y' \to Y$ such that $\pi' = \varphi \circ \pi$.

Lemma A.5. *Let G be a finite group. Let A be a finite type $\mathbb{C}$-algebra and assume that the group G acts on A from the left by $\mathbb{C}$-algebra automorphisms. Then the set of invariant elements A^G is a $\mathbb{C}$-subalgebra of A which is of finite type over $\mathbb{C}$.*

Proof. Let $\rho\colon G \to \mathrm{Aut}_{\mathbb{C}}(A)$ be the given left action. Let us first quickly ensure that

$$A^G := \bigcap_{g \in G} \{a \in A \mid \rho(g)(a) = a\}$$

is a $\mathbb{C}$-subalgebra of A.

- $A^G \subseteq A$ is a subgroup. Indeed, since $\rho(g)$ is a ring morphism for every $g \in G$, we have $0 \in A^G$. And if $a_1, a_2 \in A^G$ and $g \in G$, then it follows again from $\rho(g)$ being a ring morphism that
$$\rho(g)(a_1 + a_2) = \rho(g)(a_1) + \rho(g)(a_2) = a_1 + a_2.$$
- $A^G \subseteq A$ is a subring. We have seen already that it is a subgroup. Since $\rho(g)$ is a ring morphism for every $g \in G$, we also have $1 \in A^G$, so it remains only to show that A^G is closed under products. If $a_1, a_2 \in A^G$ and $g \in G$, then using once again that $\rho(g)$ is a ring morphism we see that
$$\rho(g)(a_1 a_2) = \rho(g)(a_1)\rho(g)(a_2) = a_1 a_2.$$
- $A^G \subseteq A$ is a $\mathbb{C}$-vector subspace. We have seen already that it is a subgroup, so it remains only to show that A^G is closed under scalar product. If $a \in A^G$, $\lambda \in \mathbb{C}$ and $g \in G$, then we use the assumption that $\rho(g)$ is $\mathbb{C}$-linear to deduce that
$$\rho(g)(\lambda a) = \lambda \rho(g)(a) = \lambda a.$$

[1] So that G acts on coordinate rings on the left.

The other assertion in the lemma is that A^G is a finite type $\mathbb{C}$-algebra. The idea is to write A^G as a finite B-module for some suitable finite type $\mathbb{C}$-algebra B. Then it would follow that A^G is a finite type $\mathbb{C}$-algebra as well. Indeed, let $\beta_1, \ldots, \beta_m \in B$ be generators of B as an algebra over $\mathbb{C}$, and let $e_1, \ldots, e_l \in A^G$ be generators of A^G as a B-module. Then we can write any $a \in A^G$ as a B-linear combination

$$a = \sum_{i=1}^{l} b_i e_i,$$

and in turn each b_i as an algebraic combination

$$b_i = f_i(\beta_1, \ldots, \beta_m)$$

for some $f_i \in \mathbb{C}[\beta_1, \ldots, \beta_m]$. It follows that we can write a as an algebraic combination in the variables $\beta_1, \ldots, \beta_m, e_1, \ldots, e_l$, so these elements would form a system of generators of A^G as a $\mathbb{C}$-algebra.

In order to construct such B, we first note that the inclusion $A^G \subseteq A$ is an integral ring extension. Indeed, every $a \in A$ is a root of the monic polynomial

$$P_a(t) := \prod_{g \in G} (t - \rho(g)(a)),$$

whose coefficients are in A^G by Vieta's formulas. Let $\alpha_1, \ldots, \alpha_m \in A$ be generators of A as an algebra over $\mathbb{C}$. Let $\{c_{i,j}\}_{j=0}^{d_i}$ be the coefficients of P_{α_i} for each $i \in \{1, \ldots, m\}$. Then define B to be the $\mathbb{C}$-subalgebra of A generated by all these coefficients $\{c_{1,0}, \ldots, c_{1,d_1}, c_{2,0}, \ldots, c_{m,d_m}\}$. Since each of its generators is contained in A^G, we see that B is also a $\mathbb{C}$-subalgebra of A^G. Moreover, by construction $B \subseteq A$ is an integral ring extension. The elements $\alpha_1, \ldots, \alpha_m$ still generate A as a B-algebra, so A is a finitely generated B-module [AM69, Corollary 5.2]. But B is noetherian, because it is a finitely generated $\mathbb{C}$-algebra, so every B-submodule of A must also be finitely generated as a B-module. Therefore A^G is a finitely generated B-module, which as explained earlier concludes the proof. $\square$

Lemma A.6. *In the situation of Lemma A.5, the induced $\mathbb{C}$-scheme morphism $\pi\colon \operatorname{Spec}(A) \to \operatorname{Spec}(A^G)$ is finite and surjective.*

Proof. It follows from the proof of Lemma A.5 that A is finitely generated as an A^G-module, so the induced morphism π is finite by definition [Har77, p. 84]. Surjectivity follows from [Sta21, Tag 00GQ]. $\square$

Lemma A.7. *In the situation of Lemma A.5, the fibers of π over closed points of Y are precisely the orbits of the closed points of X under the action of G. In particular, π is G-invariant.*

Proof. Let $x \in X$ be a closed point. Let us check first that the orbit $x \cdot G$ is contained in the fiber $\pi^{-1}(\pi(x))$. Let $\mathfrak{m} \subseteq A$ be the maximal ideal corresponding to x, i.e.

$$\mathfrak{m} = \{f \in A \mid f(x) = 0\}.$$

Let $g \in G$. Our goal is to show that $\pi(x) = \pi(x \cdot g)$. The maximal ideal corresponding to the point $x \cdot g$ is given by

$$\{f \in A \mid f(x \cdot g) = 0\} = \{f \in A \mid (g \cdot f)(x) = 0\} = \{g \cdot f \mid f \in \mathfrak{m}\} = g \cdot \mathfrak{m}.$$

So we need to show that

$$\mathfrak{m} \cap A^G = (g \cdot \mathfrak{m}) \cap A^G.$$

But we have

$$\begin{aligned}(g \cdot \mathfrak{m}) \cap A^G &= \{(g \cdot f) \in A^G \mid f \in \mathfrak{m}\} \\ &= \{g^{-1} \cdot (g \cdot f) \in A^G \mid f \in \mathfrak{m}\} \\ &= \{f \in A^G \mid f \in \mathfrak{m}\} \\ &= \mathfrak{m} \cap A^G.\end{aligned}$$

Hence $x \cdot G \subseteq \pi^{-1}(\pi(x))$.

Conversely, let $x_1, x_2 \in \pi^{-1}(\pi(x_1))$ be closed points with corresponding maximal ideals $\mathfrak{m}_1$ and $\mathfrak{m}_2$ respectively. The assumption that x_1 and x_2 are in the same fiber translates into the equality

$$\mathfrak{m}_1 \cap A^G = \mathfrak{m}_2 \cap A^G.$$

We use this equality to show that

$$\mathfrak{m}_1 \subseteq \bigcup_{g \in G} (g \cdot \mathfrak{m}_2).$$

Indeed, given any $f \in \mathfrak{m}_1$, we can produce a G-invariant element in the maximal ideal by looking at the (finite) product

$$\prod_{g \in G} (g \cdot f) \in \mathfrak{m}_1 \cap A^G = \mathfrak{m}_2 \cap A^G \subseteq \mathfrak{m}_2.$$

Since $\mathfrak{m}_2$ is a prime ideal, there exists some $g \in G$ such that $g \cdot f \in \mathfrak{m}_2$. Hence $\mathfrak{m}_1 \subseteq \cup_{g \in G}(g \cdot \mathfrak{m}_2)$ as claimed. Since G acts by ring morphisms, each ideal $g \cdot \mathfrak{m}_2$ is again a prime ideal. So we may apply the prime avoidance lemma to conclude that there exists some $g_1 \in G$ such that $\mathfrak{m}_1 \subseteq g_1 \cdot \mathfrak{m}_2$. By symmetry of x_1 and x_2 there exists some $g_2 \in G$ such that $\mathfrak{m}_2 \subseteq g_2 \cdot \mathfrak{m}_1$. So

$$\mathfrak{m}_1 \subseteq g_1 \cdot \mathfrak{m}_2 \subseteq g_1 g_2 \cdot \mathfrak{m}_1.$$

Since G acts by ring automorphisms, $\mathfrak{m}_1$ and $g_1 g_2 \cdot \mathfrak{m}_1$ are prime ideals of the same height. Therefore $\mathfrak{m}_1 = g_1 g_2 \cdot \mathfrak{m}_1$. From this we finally deduce that

$$g_1 \cdot \mathfrak{m}_2 \subseteq g_1 g_2 \cdot \mathfrak{m}_1 = \mathfrak{m}_1 \subseteq g_1 \cdot \mathfrak{m}_2,$$

so that $\mathfrak{m}_1 = g_1 \cdot \mathfrak{m}_2$ and $x_1 \in x_2 \cdot G$. □

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