Transport equation

The convection-diffusion equation for the transport of temperature T is

$$\frac{\partial T}{\partial t} = -U\frac{\partial T}{\partial x} + k\frac{\partial}{\partial x}(\frac{\partial T}{\partial x}) + S \tag{1}$$

where U is velocity and S a source term. For a non existing convection case, Equation (1) becomes the diffusion Equation

$$\frac{\partial T}{\partial t} = k\frac{\partial}{\partial x}(\frac{\partial T}{\partial x}) + S \tag{2}$$

that, for a steady-state, is

$$0 = k\frac{\partial}{\partial x}(\frac{\partial T}{\partial x}) + S \tag{3}$$

The diffusion process can also be ignored, resulting on a advection equation

$$\frac{\partial T}{\partial t} = -U\frac{\partial T}{\partial x} + S \tag{4}$$

These equations are solved by a Finite Volume Method (FVM) and by a Finite Difference Method (FDM).

Finite Volume Method

In the Finite Volume Method, values for the above diferential equations are calculated at discrete places on a grid of volumes, shown in Figure 1,

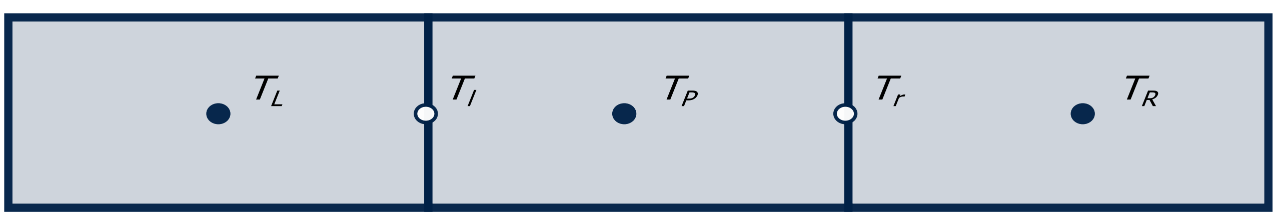

Figure 1: Temperature at center of cell, and on left and right cell borders.

where temperature at the center of a cell is T_P, T_r e T_l is temperature at the border between cells, right and left, T_R e T_L is temperature at the right and left cell centers. Integrating Equation 3 over a cell volume, and considering the rate of accumulation over the volume V equal to the flow across the surfaces of the control volume, Equation (6), where n is the unit normal vector pointing out of the control volume and A is the cross sectional area of the volume.

$$\int \left[k \frac{\partial}{\partial x} (\frac{\partial T}{\partial x}) + S \right] dV = 0 \tag{5}$$

$$\int \left[\frac{\partial}{\partial x} (k \frac{\partial T}{\partial x}) dV \right] + SV = 0 \tag{6}$$

$$\int (k \frac{\partial T}{\partial x} n) dA + SV = 0 \tag{7}$$

Equation (6) is written in terms of the flux leaving the right face r minus the flux entering the left face l in Figure (1)

$$(kA \frac{\partial T}{\partial x})_r - (kA \frac{\partial T}{\partial x})_l + SV = 0 \tag{8}$$

Equation (7) for the temperature gradient at the boundaries is discretized in terms of the temperature at the cell centers, where Δx is the distance between cell centers

$$kA \frac{T_R - T_P}{\Delta x} - kA \frac{T_P - T_L}{\Delta x} + SV = 0 \tag{9}$$

Solving Equation (8) for T_P and making an individual set of equations for each cell results in a system of equations of the form

$$-\frac{kA}{\Delta x} T_{P-1} + 2 \frac{kA}{\Delta x} T_P - \frac{kA}{\Delta x} T_{P+1} = SV \tag{10}$$

for each grid position except the first and last volumes in the grid. For the first and last volumes in the grid, the temperature T_A and T_B at the borders is also considered, as well as a half distance to the border. In matrix form, where the first and last row are the two special border cases:

$$\begin{bmatrix} \frac{kA}{\Delta x} + \frac{kA}{0.5\Delta x} & -\frac{kA}{\Delta x} & 0 & 0 & 0 \\ -\frac{kA}{\Delta x} & 2\frac{kA}{\Delta x} & -\frac{kA}{\Delta x} & 0 & 0 \\ 0 & -\frac{kA}{\Delta x} & 2\frac{kA}{\Delta x} & -\frac{kA}{\Delta x} & 0 \\ 0 & 0 & -\frac{kA}{\Delta x} & 2\frac{kA}{\Delta x} & -\frac{kA}{\Delta x} \\ 0 & 0 & 0 & -\frac{kA}{\Delta x} & \frac{kA}{\Delta x} + \frac{kA}{0.5\Delta x} \end{bmatrix} \begin{bmatrix} T_1 \\ T_{P-1} \\ T_P \\ T_{P+1} \\ T_N \end{bmatrix} = \begin{bmatrix} SV + T_A \frac{kA}{0.5\Delta x} \\ SV \\ SV \\ SV \\ SV + T_B \frac{kA}{0.5\Delta x} \end{bmatrix} \tag{11}$$

This matrix is solved for the vector of temperatures T.

Finite Difference Method

Diffusion equation

The diffusion equation (2)

$$\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2} + S \tag{12}$$

can be approximated by finite differences with a forward difference in time and a central difference in space scheme.

$$\frac{T_i^{n+1} - T_i^n}{\Delta t} = k\frac{T_{i-1}^n - 2T_i^n + T_{i+1}^n}{\Delta x^2} + S_i^n \tag{13}$$

that can be solved for the temperature at the next time step T^{n+1}

$$T_i^{n+1} = T_i^n + k\frac{\Delta t}{\Delta x^2}(T_{i-1}^n - 2T_i^n + T_{i+1}^n) + S_i^n \tag{14}$$

Advection equation

The advection equation (4), without source term is

$$\frac{\partial T}{\partial t} = -U\frac{\partial T}{\partial x} \tag{15}$$

and it can be approximated by finite differences with a forward difference in time and a upwind difference in space

$$\frac{T_i^{n+1} - T_i^n}{\Delta t} = -U\frac{T_i^n - T_{i-1}^n}{\Delta x} \tag{16}$$

References

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