THE REVOLUTION FUNCTION: A CONSTRUCTIVE DEFINITION OF HYPERSOLIDS AND HYPERSURFACES OF REVOLUTION IN $\mathbb{R}^n$ AND SOME PROPERTIES OF SETS OF REVOLUTION

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Abstract

At first, we defined the revolution operation through a function $\mathcal{R}$ whose entries are two non-empty subsets $A, C \subseteq \mathbb{R}^n$ and named $\mathcal{R}(A, C)$ the set of revolution under (A, C). The objective was for $\mathcal{R}(A, C)$ to represent all the possible rotations of A around C, similarly to the case of the surfaces of revolution, where A is taken to be a curve and C is a line. After, we proved some properties of the sets $\mathcal{R}(A, C)$, showing their consistency with the theory of solids and surfaces of revolution. With this, we used $\mathcal{R}(A, C)$ to generalize the solids and surfaces of revolution to higher dimensional analogs embedded in $\mathbb{R}^n$, which were called hypersolids and hypersurfaces of revolution. Afterward, we studied the measure of hypersolids of revolution and lastly, we gave parametrizations for hypersurfaces of revolution embedded in $\mathbb{R}^n$.

Keywords. Rotations; Isometry; Revolution; Hypersolids of Revolution; Rotational Hypersolids; Hypersurfaces of Revolution; Rotational Hypersurfaces; Parametrization; Lebesgue Measure; Special Orthogonal Group.

Contents

Introduction

In 1854, at the request of Gauss, Riemann produced the article "Ueber die Hypothesen, welche der Geometrie zu Grunde liegen" [1, 2] [1]. In this paper, the axiomatic foundations of Riemannian geometry emerged, and the investigation of geometry in higher-dimensional spaces was proposed for the first time. From there, a significant part of modern mathematics has focused on studying spaces and objects of dimension n with $n \geq 4$. Following this trend, many authors published results about n-dimensional analogs of surfaces of revolution, commonly named hypersurfaces of revolution or rotational hypersurfaces. Some of these results include connections of the rotational hypersurfaces to solutions of minimal surfaces and hypersurfaces of constant mean curvature. For example, Simon Brendle posted in 2013 a solution to Lawson's conjecture [3], which states that the Clifford torus, a rotational surface, is the only minimally embedded torus in the 3-sphere.

Despite the recent interest of many mathematicians in the topic, the precise definition of a rotational hypersurface varies across different papers. Most of the articles in the theme focused on studying the geometrical properties of some family of hypersurfaces; when this family is in some way a generalization of surfaces of revolution, usually the authors refer to them as either hypersurfaces of revolution or rotational hypersurfaces. Consequently, as one might expect from the heterogeneous nature of the motivations for the definitions, though these generalizations share numerous similarities, they are not exactly equivalent. For examples of different definitions, see [4], [5] and [6].

Nevertheless, in this paper, our purpose was not to study the geometrical properties of these objects, but rather to formally construct an analog class to the surfaces and solids of revolution in $\mathbb{R}^n$. Since the term rotational hypersurfaces is often found in literature naming objects sometimes slightly different from what we defined here, we preferred the name hypersurfaces of revolution, which is less used and usually more directly linked to the 2-dimensional surfaces of revolution. Here, it must be made clear that our aim with this construction is not, by any means, to deconstruct alternative reasonings for the definition of these objects.

[1] Paper translated by William K. Clifford: On the Hypotheses which Lie at the Bases of Geometry.

With that clarified, let us describe what was done here. To construct the hypersolids and hypersurfaces of revolution, we initially defined a function $\mathcal{R}$ over $\mathbb{R}^n$ closely related to the idea of a rotation, which we called the revolution function. Afterward, we defined a set $S \subseteq \mathbb{R}^n$ as a set of revolution when S is the image of $\mathcal{R}$ under some entry. Following, we demonstrated some results about the function $\mathcal{R}$ and the set S; in particular, a lower-dimensional case of the main theorem shows that when S has dimension 2 or 3, it corresponds to what we consider as a surface or a solid of revolution, respectively. Lastly, before addressing the hypersolids and hypersurfaces of revolution, we briefly commented on a natural extension of the function $\mathcal{R}$ for general metric spaces and on some relations between $\mathcal{R}$, rotations and the special orthogonal group $SO(n)$.

Remark 0.1. *Evidently, our initial objective was to define hypersolids and hypersurfaces of revolution and the function $\mathcal{R}$ was originally intended just as a means to this end. However, we shall mention here that, at some point, it became clear that our major contribution in this paper was the use of $\mathcal{R}$ to characterize general revolution sets. The methods involving $\mathcal{R}$ were considerably more efficient in proving some properties of these sets than the standard approaches with rotations and $SO(n)$. The definition of the hypersolids and hypersurfaces of revolution and subsequent results emerged naturally from the characterizations of general sets of revolution.*

In section 2, we defined a k-solid of revolution as a set $S\colon \dim(S) = k$ with S being a hypersolid and the image of $\mathcal{R}$ under some entry. After this, we studied the Lebesgue measure of a subclass of the sets S that are images of $\mathcal{R}$ and proved an n-dimensional analog to the well-known result $\pi \int_D f^2$ that holds for solids of revolution.

Later, in section 3, we similarly defined the concept of a k-surface of revolution as a hypersurface S of dimension k with S being the image of $\mathcal{R}$ for some entry. Subsequently, we shifted our attention to studying the revolution of the image set of parametric functions $\alpha\colon D \to \mathbb{R}^n$, a class of sets that includes the hypersurfaces of revolution. In particular, we gave a parametrization of the image of α under the revolution function, which is also an n-dimensional analog to the well-known parametrization $(f(t), g(t)\cos\theta, g(t)\sin\theta)$ that applies for surfaces of revolution.

Notations

$\mathcal{P}(X)$ – The power set of X defined by $\mathcal{P}(X) := \{Y : Y \subseteq X\}$.

$d\,(A, B)$ – Distance function of the points A and B. Unless otherwise stated, we will consider for all purposes d as a metric over $\mathbb{R}^n$ consistent with the Euclidean norm ℓ^2.

$\overline{AB}$ – Line segment that goes from A to B.

$|\overline{AB}|$ – Length of the line segment AB; $|\overline{AB}| := d(A, B)$.

$\overline{AB} \perp \overline{CD}$ – $\overline{AB}$ is perpendicular to $\overline{CD}$.

graph f – The graph of $f\colon A \to B$, defined by graph $f := \{(\mathbf{x}, f(\mathbf{x}))\colon \mathbf{x} \in A\} = A \times f(A)$.

$S^n_O(r)$ – Sphere n-dimensional of radius r and center O. Defining $O \in \mathbb{R}^{n+1}$ to be the point of coordinates $O := (o_1, o_2, \ldots, o_{n+1})$, $S^n_O(r)$ is the set that satisfies: $S^n_O(r) = \left\{\mathbf{x} \in \mathbb{R}^{n+1} : \sum_{k=1}^{n+1} (x_k - o_k)^2 = r^2\right\}$.

$B^n_O(r)$ – Closed n-dimensional ball of radius r and center O. This is a region delimited by an $(n-1)$-sphere. Resuming the definitions given about the n-sphere, the n-closed ball can be expressed by: $B^n_O(r) = \left\{\mathbf{x} \in \mathbb{R}^n : \sum_{k=1}^{n} (x_k - o_k)^2 \leq r^2\right\}$. Throughout the article, we will use the term n-ball as an analog to the idea of a closed n-ball.

Definition 0.1. *Let m, n be positive integers with $m \leq n$. We say the map $E_m\colon \mathbb{R}^m \to \mathbb{R}^n$ defined by $E_m(\mathbf{x}) := (\mathbf{x}, 0, \ldots, 0)$ is the canonical embedding map of $\mathbb{R}^m$ in $\mathbb{R}^n$. Additionally, we define $\mathbb{R}^m_* := E_m(\mathbb{R}^m)$.*

1 The revolution function and rotations in $\mathbb{R}^n$

Rotations have been for a long time, an essential study object in mathematics. It can be said that rotations are a quite ubiquitous tool in physics and mathematics since the applications and studies related to them have been dissociated towards numerous problems and analysis purposes.

Conceptually, the naive definitions of rotations can vary across different fields, according to the goals and objects of study. The term "rotation" is widely used to refer to ideas that are similar, but not exactly compatible to the most standard definition of rotation as a rigid motion of the space. This definition of rotation is standard in linear algebra and can be more precisely characterized up to translation as a linear transformation represented by an element of the special orthogonal group, typically denoted by $SO(n) := \{M \in \mathbb{R}^{n \cdot n} | MM^T = M^T M = I_n,\ \det(M) = 1\}$ in dimension n [7].

However, this definition is not quite appropriate for the constructions we need here. For this reason, we will define another function that is closely related to the idea of rotation, but better suited for the purposes of this paper. This function will be denoted by $\mathcal{R}$ and called the revolution function. Intuitively, $\mathcal{R}$ should represent the image of all the possible rotations of a region $A \subseteq \mathbb{R}^n$ around a region $S \subseteq \mathbb{R}^n$, i.e., with any angle combination.

1.1 The revolution function and the orthogonality principle of the revolutions

Definition 1.1 (The revolution function)**.** *Let $\mathcal{P}^*(X) := \{Y : Y \subseteq X\} \backslash \{\emptyset\}$ be the power set of X minus the empty set, and consider $A, C \subseteq \mathbb{R}^n$ regions of $\mathbb{R}^n$ and $a \in A$ a point. We define the function $\mathcal{R}\colon \mathcal{P}^*(\mathbb{R}^n) \times \mathcal{P}^*(\mathbb{R}^n) \to \mathcal{P}^*(\mathbb{R}^n)$ by:*

$$\mathcal{R}(\{a\}, C) := \{s \in \mathbb{R}^n \colon d(a,c) = d(s,c)\ \forall c \in C\} = \bigcap_{c \in C} \mathcal{R}(\{a\}, \{c\})$$

$$\mathcal{R}(A, C) := \bigcup_{a \in A} \mathcal{R}(\{a\}, C)$$

In the above equality, $\mathcal{R}$ will be called the revolution function and we define the revolution of A about C to be the set $\mathcal{R}(A, C)$. Additionally, we will name $\mathcal{R}(A, C)$ as the revolution set under the pair (A, C), and C will be called the isometry center of the revolution or simply revolution center.

Remark 1.1. *Here, it is worth mentioning that the function $\mathcal{R}$ could also be constructed using the image set of $SO(n)$ composed with two translations (the first to move C to the origin and then the to second move it back to its original place after performing the rotation). Nevertheless, this would be rather inefficient and hard to define for a general $C \subseteq \mathbb{R}^n$. Besides, the matrix methods are absolutely inconvenient to study the properties of the function $\mathcal{R}$.*

For initial analysis of the function $\mathcal{R}$, consider two points $P, O \in \mathbb{R}^n$. The revolution of P about O is then:

$$\mathcal{R}(\{P\}, \{O\}) = \{s \in \mathbb{R}^n : d\ (P, O) = d\ (s, O)\} = S_O^{(n-1)}(|\overline{OP}|)$$

This is, an $(n-1)$-sphere of radius $|\overline{OP}|$ and center O.

In order to get a better understanding of how $\mathcal{R}$ behaves over general sets $A, C \subseteq \mathbb{R}$, we shall first prove the Proposition 1.1. When the lowest-dimensional affine space containing C is not $\mathbb{R}^k_*$ (Definition 0.1), this result serves as an indispensable tool for computing $\mathcal{R}(A, C)$. This proposition will later be used in the proof of Theorem 1.1, which will give a characterization of the set $\mathcal{R}(A, C)$.

Proposition 1.1. *Take $A, C \subseteq \mathbb{R}^n$ and let $M\colon \mathbb{R}^n \to \mathbb{R}^n$ be an isometry (a bijective distance-*

preserving function). Then:

$$M(\mathcal{R}(A, C)) = \mathcal{R}(M(A), M(C))$$

In particular:

$$\mathcal{R}(A, C) = M^{-1}(\mathcal{R}(M(A), M(C)))$$

Proof. Since M is an isometry, the inverse of M defined by M^{-1} is also an isometry.

First, define $A' := M(A)$ and $C' := M(C)$. Additionally, take two points $a \in A$ and $c \in C$ and consider $a' := M(a), c' := M(c)$ and $d(a, c) = d(a', c') := r$. From now on, when $a, c \in \mathbb{R}^n$ are points of $\mathbb{R}^n$, we will abbreviate $\mathcal{R}(\{a\}, \{c\})$ as simply $\mathcal{R}(a, c)$. From this construction, it holds:

$$M^{-1}(\mathcal{R}(a', c')) = M^{-1}(S_{c'}^{(n-1)}(r))$$

Here, $x' \in S_{c'}^{(n-1)}(r) \Leftrightarrow d(x', c') = r$ and $M^{-1}(x') \in S_c^{(n-1)}(r) \Leftrightarrow d(M^{-1}(x'), c) = r$. Letting $x := M^{-1}(x)$, we have $d(x, c) = r \Leftrightarrow d(x', c') = r$, and thus $x \in S_c^{(n-1)}(r) \Leftrightarrow x' \in S_{c'}^{(n-1)}(r)$. Therefore:

$$\mathcal{R}(a, c) = S_c^{(n-1)}(r) = M^{-1}(S_{c'}^{(n-1)}(r)) = M^{-1}(\mathcal{R}(a', c'))$$

$$\Leftrightarrow \mathcal{R}(a, c) = \mathcal{R}(M^{-1}(a'), M^{-1}(c')) = M^{-1}(\mathcal{R}(a', c')) \tag{1}$$

Generalizing $M(a) := a'$ and $M(c) := c'$ for each pair $(a, c) \in A \times C$, by (1):

$$\mathcal{R}(A, C) = \bigcup_{a \in A} \bigcap_{c \in C} \mathcal{R}(a, c) = \bigcup_{a' \in A'} \bigcap_{c' \in C'} M^{-1}(\mathcal{R}(a', c'))$$

$$= M^{-1}\left(\bigcup_{a' \in A'} \bigcap_{c' \in C'} \mathcal{R}(a', c')\right) = M^{-1}(\mathcal{R}(A', C')) = M^{-1}(\mathcal{R}(M(A), M(C)))$$

From this:

$$\mathcal{R}(A, C) = M^{-1}(\mathcal{R}(M(A), M(C)))$$

$$\Leftrightarrow M(\mathcal{R}(A, C)) = \mathcal{R}(M(A), M(C))$$

□

Remark 1.2. *As stated previously, in general, when the lowest-dimensional affine space containing C is not $\mathbb{R}^k_*$ for any k, we can use an isometric transformation M to "move" C to a more convenient position to compute the set $\mathcal{R}(A, C)$. With a well-chosen transformation M, the set $\mathcal{R}(M(A), M(C))$ should be way easier to compute than $\mathcal{R}(A, C)$.*

From this point forward, it will be interesting to look at some specific embedding of A and C in $\mathbb{R}^n$ to analyze the function $\mathcal{R}(A, C)$. In particular, we will be frequently using the image of C under an isometric map for which $M(C) \subseteq \mathbb{R}^m_*$ for some $m \leq n$ (Definition 0.1). Similarly, it will always be considered that $\mathcal{R}$ has domain $\mathcal{P}^*(\mathbb{R}^n) \times \mathcal{P}^*(\mathbb{R}^n)$ and range $\mathcal{P}(\mathbb{R}^n)$. In addition, for some of the next passages, it will be convenient to use a particular notation we will introduce now.

Definition 1.2. *Given two regions $S_1, S_2 \subseteq \mathbb{R}^n$ and $A, C \subseteq S_1$ (or $A, C \in S_1$), we define the revolution of A about C from S_1 to S_2 to be:*

$$\frac{S_1 \to S_2}{A \circ C} := \mathcal{R}(A, C) \cap S_2$$

The term S_1 in the notation is used just to carry the information that $A, C \subseteq S_1$ (or $A, C \in S_1$).

These two definitions and Proposition 1.1 should suffice to prove the rest of the statements present in this chapter. In particular, we will now prove a weaker version of Theorem 1.1, which will follow right after. The Lemma to be proved now, though, is not required to prove Theorem 1.1 and can be skipped without major issues; the goal here is just to provide a better intuition about the meaning of Theorem 1.1 before presenting its proof. Lemma 1.1 is basically the one-dimensional case of Theorem 1.1.

Lemma 1.1 (Optional)**.** *Let $P, O, Q \in \mathbb{R}^n$ be points such that $O \neq Q$. Let $S \supset \{O, Q\}$ be the line containing O and Q. Then:*

- $\mathcal{R}(P, (\{O, Q\})) = \mathcal{R}(P, S)$.
- *There is a point $T = proj_S(P)$ and an affine space A: $\dim A = n - 1$ such that $A \supset \overline{PT}$ and $A \perp S$. For the point T and the affine space A, it holds that:*

$$\mathcal{R}(P, S) = \frac{\mathbb{R}^n \to A}{P \circ T} = S^{n-2}_T(|\overline{PT}|) \subset A$$

Proof. Given that translation and rotation are both isometric mappings, by Proposition 1.1, if

M is a finite composition of translations and rotations:

$$\mathcal{R}(S_1, S_2) = M^{-1}(\mathcal{R}(M(S_1), M(S_2))$$

That is, we can move the sets S_1 and S_2 to make the revolution and then apply the inverse transformation to move the revolution set back to its original locus.

From this, for the three points $P, O, Q \in \mathbb{R}^n$, there exists a transformation M generated by a finite composition of translations and rotations such that $M(P) = P' = (0, b_1, 0, \ldots, 0)$, $M(O) = O' = (a_1, 0, \ldots, 0)$ and $M(Q) = Q' = (a_2, 0, \ldots, 0)$, with $a_2 \neq a_1$. The revolution of P' about O' and Q' is then:

$$\mathcal{R}(P', (\{O', Q'\})) = \mathcal{R}(P', O') \cap \mathcal{R}(P', Q')$$

Analyzing each set, we have:

$$\mathcal{R}(P', O') = \left\{ \mathbf{x} \in \mathbb{R}^n | (x_1 - a_1)^2 + \sum_{i=2}^{n} (x_i)^2 = (a_1)^2 + (b_1)^2 \right\} \tag{2}$$

$$\mathcal{R}(P', Q') = \left\{ \mathbf{x} \in \mathbb{R}^n | (x_1 - a_2)^2 + \sum_{i=2}^{n} (x_i)^2 = (a_2)^2 + (b_1)^2 \right\} \tag{3}$$

By (2) and (3), developing the equations:

$$(x_1 - a_1)^2 - (a_1)^2 = (b_1)^2 - \sum_{i=2}^{n} (x_i)^2 \tag{4}$$

$$(x_1 - a_2)^2 - (a_2)^2 = (b_1)^2 - \sum_{i=2}^{n} (x_i)^2 \tag{5}$$

From (4) and (5):

$$(x_1 - a_1)^2 - (a_1)^2 = (x_1 - a_2)^2 - (a_2)^2$$

$$\Leftrightarrow -2a_1x_1 = -2a_2x_1$$

$$\Leftrightarrow x_1 = 0 \tag{6}$$

Replacing (6) into (2) and (3), we get:

$$\mathcal{R}(P',(\{O',Q'\})) = \left\{\mathbf{x} \in \mathbb{R}^n | x_1 = 0, \sum_{i=2}^{n}(x_i)^2 = (b_1)^2\right\} \tag{7}$$

At the above equality, it can be said that the specific values of a_1 and a_2 didn't play any role in determining the set $\mathcal{R}(P',(\{O',Q'\}))$. That is, we could have chosen any values $s_1, s_2 \in \mathbb{R}$ with $s_1 \neq s_2$ to substitute a_1 and a_2 and the set would still be the same. In particular, all the possible variations of $s_1, s_2 \in \mathbb{R}$ with $s_1 \neq s_2$ cover the line that contains O' and Q'. From this, taking S' to be the line that contains O' and Q', we have that:

$$\mathcal{R}(P',(\{O',Q'\})) = \bigcap_{s' \in S'} \mathcal{R}(P',s') = \mathcal{R}(P',S')$$

$$\Leftrightarrow \mathcal{R}(P',S') = \left\{\mathbf{x} \in \mathbb{R}^n | x_1 = 0, \sum_{i=2}^{n}(x_i)^2 = (b_1)^2\right\}$$

Admitting $T' := (0,\ldots,0) = \text{proj}_{S'}(P')$, we have $\overline{P'T'} \perp S'$. Let $A' \subset \mathbb{R}^n\colon \dim(A') = n-1$ be the affine space such that $A' \supset \overline{P'T'}$ and $A' \perp S'$. From this construction, we can rewrite the above equality as:

$$\Leftrightarrow \mathcal{R}(P',S') = \left\{\mathbf{x} \in \mathbb{R}^n | x_1 = 0, \sum_{i=2}^{n}(x_i)^2 = (b_1)^2\right\} = S^{n-2}_{T'}(|b_1|) \subset A' \tag{8}$$

This is, the set $\mathcal{R}(P',S')$ is the $(n-2)$-sphere of radius $|b_1|$ and center T' contained in A'. Defining $S := M^{-1}(S')$, from Proposition 1.1, it holds:

$$\mathcal{R}(P,(\{O,Q\})) = M^{-1}(\mathcal{R}(P',(\{O',Q'\}))) = M^{-1}(\mathcal{R}(P',S'))$$

$$= M^{-1}(\mathcal{R}(M(P),M(S))) = \mathcal{R}(P,S)$$

$$\Leftrightarrow \mathcal{R}(P,(\{O,Q\})) = \mathcal{R}(P,S) \tag{9}$$

This proves the first part of the Lemma.

For the second part, let us define $A := M^{-1}(A')$ and $T := M^{-1}(T')$. Since rotation and translation preserve dimension, orientation and the angles relations of $\mathbb{R}^n$, it holds that $T =$

$\text{proj}_S(P)$, $A \perp S$ and $\dim(A) = n-1$. Thus, from (8) and (9):

$$\mathcal{R}(P,S) = M^{-1}(S_{T'}^{n-2}(|b_1|) \subset A')$$

$$\Leftrightarrow \mathcal{R}(P,S) = S_T^{n-2}(|b_1|) \subset A$$

$$\Leftrightarrow \mathcal{R}(P,S) = \frac{\mathbb{R}^n \to A}{P \circ T} = S_T^{n-2}(|b_1|) \subset A$$

Substituting $|b_1| = |\overline{P'T'}| = |\overline{PT}|$:

$$\mathcal{R}(P,S) = \frac{\mathbb{R}^n \to A}{P \circ T} = S_T^{n-2}(|\overline{PT}|) \subset A$$

This is, the revolution of P about S is the $(n-2)$-sphere of center T and radius $|\overline{PT}|$ located on the affine space A that is perpendicular to S and contains $\overline{PT}$. This special point $T = \text{proj}_S(P)$ will be called the point of partial revolution of P with respect to S. This finishes the proof. $\square$

Example 1.1 (Informal). *For merely illustrative purposes, a revolution of P from $\mathbb{R}^2_*$ to $\mathbb{R}^3$ with isometry center in two or more distinct points of a line is visually represented as:*

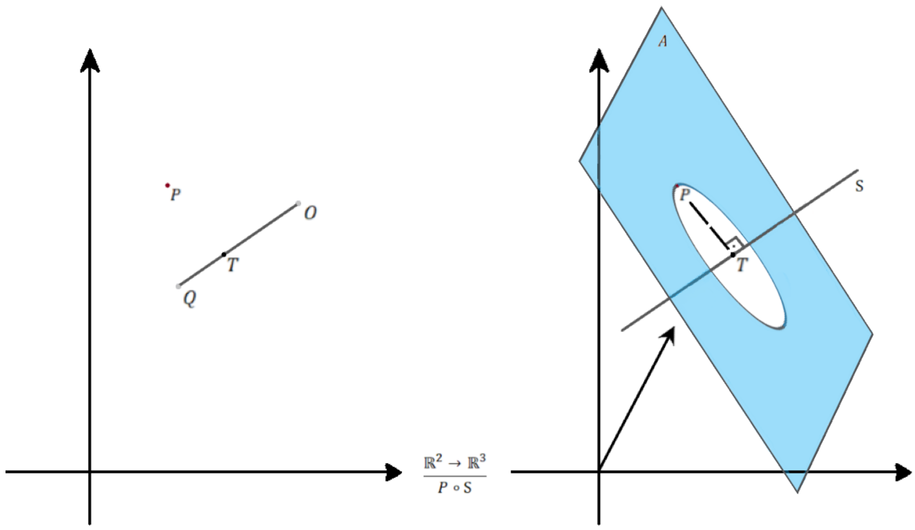

Figure 1: Revolution of P about $\{O, Q\}$ from $\mathbb{R}^2_*$ to $\mathbb{R}^3$

With the problem solved for $\dim(S) = 1$, we will now generalize Lemma 1.1 for $\dim(S) = l$:

Theorem 1.1. *Let $P \in \mathbb{R}^n$ be a point and $C \subseteq \mathbb{R}^n$ be a region of $\mathbb{R}^n$. Additionally, let $S \supseteq C$: $\dim(S) = l$ be the lowest-dimensional affine space of $\mathbb{R}^n$ containing C. Then:*

- $\mathcal{R}(P, C) = \mathcal{R}(P, S)$
- *(Revolutions' orthogonality principle) There is a point* $T = proj_S(P)$ *and an affine space* $A\colon \dim(A) = n - l$ *such that* $A \supseteq \overline{PT}$ *and* $A \perp S$*. For the point* T *and the affine space* A*, it holds that:*

$$\mathcal{R}(P, S) = \frac{\mathbb{R}^n \to A}{P \circ T}$$

$$= S_T^{(n-l-1)}(|\overline{PT}|) \subset A \quad \textit{when } l < n$$

Proof. The strategy here will be to make use of our hypothesis that we only need S to be the lowest-dimensional affine space containing C to "choose" a minimum subset B of C (with the lowest possible cardinality), such that S remains the lowest-dimensional affine space containing B. After this, we will use again that rotation and translation preserve distance, dimension, orientation, and angles to apply Proposition 1.1 and "move" B to a more convenient position for performing the calculations. With the image of B under this transformation, we will be able to prove the desired properties about the revolution. Finally, in the end, these properties will be generalized for the initial sets by using the inverse of the transformation applied in B. This way, let us start.

Given that $S \supseteq C | \dim(S) = l$ is the lowest-dimensional affine space containing C, there exists a subset of points $B \subseteq C\colon |B| = l + 1$ with the property that S is still the lowest-dimensional affine space such that $S \supseteq B$. Admit $B := \{B_k | k \in [1, l+1] \cap \mathbb{N}\}$ to be this set.

Let $(e_i)_{i=1}^n$ be the canonical basis of $\mathbb{R}^n$. From the construction of B, there exists a transformation M equivalent to a finite composition of rotations and translations such that for each $B_k \in B$, it holds that $M(B_k) = B'_k = \sum_{i=1}^{k-1} b_{i,k} \cdot e_i = (b_{1,k}, b_{2,k}, \ldots, b_{k-1,k}, 0, \ldots, 0)$, with $b_{k-1,k} \neq 0$. We define then $B' := M(B)$.

Remark 1.3. *One can prove the existence of the transformation* M *by inducting on* l*, i.e., showing that for* $l = 0$ *it exists (which is obvious since one can just translate the point to the origin) and then proving that if for* $l = j$ *it exists, then for* $l = j + 1$ *it exists as well. This last part can be done by just considering one additional point in* $\mathbb{R}^n$ *that does not belong to* $\mathbb{R}_*^j$ *and then using a particular rotation matrix:*

$$A = \begin{pmatrix} I_j & D \\ 0 & E \end{pmatrix}$$

$A \in SO(n)$ to move this last point to $\mathbb{R}_*^{j+1}$. In the above notation 0 is the null $(n-j) \times j$ matrix and $D = D_{j\times(n-j)}$ and $E = E_{(n-j)\times(n-j)}$ are matrices. The translation here is used only once, to move the first point to the origin; after this, the composition of M takes only rotation matrices (elements of $SO(n)$). However, we shall refrain from explicitly writing the full proof of the existence of M because it does not add much to the rest of the text and constructing A is indeed a little painful.

Remark 1.4 (Informal)**.** *The intuition behind the transformation M is that it initially occurs a translation of B_1 to the origin and then, we make successive rotations of the space, always using the fact that $(j+2)$ points are not in an affine space of dimension j to place B_k over $\mathbb{R}_*^{k-1}$ and ensure that $B_k \not\subset \mathbb{R}_*^{k-2}$.*

Continuing, let $M(P) := P' = (p_1, p_2, \ldots, p_n)$. With this defined, the revolution of P' about B' is:

$$\begin{aligned}\mathcal{R}(P', B') &= \bigcap_{k=1}^{l+1} \mathcal{R}(P', B'_k) \\ &= \bigcap_{k=1}^{l+1} \left\{ \mathbf{x} \in \mathbb{R}^n | \sum_{i=1}^{k-1} (x_i - b_{i,k})^2 + \sum_{i=k}^{n} (x_i)^2 = \sum_{i=1}^{k-1} (p_i - b_{i,k})^2 + \sum_{i=k}^{n} (p_i)^2 \right\} \end{aligned} \tag{10}$$

Developing the equation in (10):

$$\begin{aligned}&\sum_{i=1}^{k-1} (x_i - b_{i,k})^2 - (p_i - b_{i,k})^2 + \sum_{i=k}^{n} (x_i)^2 - (p_i)^2 = 0 \\ &\Leftrightarrow \sum_{i=1}^{k-1} (x_i)^2 - 2b_{i,k}(x_i - p_i) - (p_i)^2 + \sum_{i=k}^{n} (x_i)^2 - (p_i)^2 = 0 \\ &\Leftrightarrow \sum_{i=1}^{k-1} 2b_{i,k}(x_i - p_i) = \sum_{i=1}^{n} (x_i)^2 - (p_i)^2 \\ &\Leftrightarrow \sum_{i=1}^{k-1} b_{i,k}(x_i - p_i) = \sum_{i=1}^{n} \frac{(x_i)^2 - (p_i)^2}{2}\end{aligned}$$

Substituting and equaling for all values of k (notice in particular that $B'_1 = \mathbf{0}$):

$$0 = b_{1,2}(x_1 - p_1) = b_{1,3}(x_1 - p_1) + b_{2,3}(x_2 - p_2) = \ldots = \sum_{i=1}^{l} b_{i,k}(x_i - p_i) = \sum_{i=1}^{n} \frac{(x_i)^2 - (p_i)^2}{2}$$

By knowing $b_{k-1,k} \neq 0$ for all k, we have:

$$x_1 = p_1; x_2 = p_2; \ldots; x_l = p_l; \sum_{i=l+1}^{n} (x_i)^2 = \sum_{i=l+1}^{n} (p_i)^2$$

From this result, we can rewrite $\mathcal{R}(P', B')$ as:

$$\mathcal{R}(P', B') = \left\{ \mathbf{x} \in \mathbb{R}^n | x_i = p_i \text{ for } i \in [1, l] \cap \mathbb{N}, \sum_{i=l+1}^{n} (x_i)^2 = \sum_{i=l+1}^{n} (p_i)^2 \right\} \tag{11}$$

At this point, we can state again that the specific values of $b_{i,k}$ did not influence the revolution set $\mathcal{R}(P', B')$. In fact, the single important property of the values $b_{i,k}$ is that $b_{k-1,k} \neq 0$ for all k. Keeping this restriction and varying the rest of the coordinates prior to $b_{k-1,k}$, the image set of all possible locus of the points B_k is exactly the space $\mathbb{R}^l_*$. Thus:

$$\mathcal{R}(P', B') = \bigcap_{B'_k \in B'} \mathcal{R}(P', B'_k) = \bigcap_{L' \in \mathbb{R}^l_*} \mathcal{R}(P', L') = \mathcal{R}(P', \mathbb{R}^l_*)$$

Defining $C' = M(C)$, it holds that $B' \subseteq C' \subseteq \mathbb{R}^l_*$. Therefore, from the equality above:

$$\bigcap_{B'_k \in B'} \mathcal{R}(P', B'_k) \subseteq \bigcap_{c' \in C'} \mathcal{R}(P', c') \subseteq \bigcap_{L' \in \mathbb{R}^l_*} \mathcal{R}(P', L')$$

$$\Leftrightarrow \mathcal{R}(P', B') = \mathcal{R}(P', C') = \mathcal{R}(P', \mathbb{R}^l_*) \tag{12}$$

Let $T' := \text{proj}_{\mathbb{R}^l_*}(P') = \sum_{i=1}^{l} p_i \cdot e_i = (p_1, p_2, \ldots, p_l, 0, \ldots, 0)$ and A' be the affine space such that $A' \supseteq \overline{P'T'}$, $A' \perp \mathbb{R}^l_*$ and $\dim(A') = n - l$. From the coordinates of T', $|\overline{P'T'}| = \sqrt{\sum_{i=l+1}^{n} (p_i)^2}$. Thereof, given that the coordinates x_i of $\mathbf{x}$ for $1 \leq i \leq l$ are fixed and equal to the coordinates of T', by (11) and (12), it holds that:

$$\mathcal{R}(P', \mathbb{R}^l_*) = \left\{ \mathbf{x} \in \mathbb{R}^n | x_i = p_i \text{ for } i \in [1, l] \cap \mathbb{N}, \sum_{l+1}^{n} (x_i)^2 = \sum_{l+1}^{n} (p_i)^2 \right\}$$

$$= S^{(n-l-1)}_{T'}(|\overline{P'T'}|) \subset A' \quad \text{when } l < n \tag{13}$$

From this, define $S := M^{-1}(\mathbb{R}^l_*)$, $A := M^{-1}(A')$ and $T := M^{-1}(T')$. From the fact that M preserves orientation and the angles relations, we have that $A \perp S$ and $T = \text{proj}_S(P)$. From this construction, by (13) and Proposition 1.1:

$$\mathcal{R}(P,S) = M^{-1}(\mathcal{R}(P',\mathbb{R}^l_*)) = M^{-1}(\mathcal{R}(P',C')) = \mathcal{R}(P,C)$$

This proves the first part of the Theorem. Also, for $l < n$:

$$\mathcal{R}(P,S) = M^{-1}(\mathcal{R}(P',\mathbb{R}^l_*)) = M^{-1}(S^{(n-l-1)}_{T'}(|\overline{P'T'}|) \subset A')$$

$$= S^{(n-l-1)}_{T}(|\overline{PT}|) \subset A$$

For $l = n$, $P \in S$ and so $\mathcal{R}(P,S) = P$ holds trivially. From this:

$$\mathcal{R}(P,S) = \frac{\mathbb{R}^n \to A}{P \circ T}$$

$$= S^{(n-l-1)}_{T}(|\overline{PT}|) \subset A \quad \text{when } l < n$$

Therefore, when $l < n$, the revolution of P about S is the $(n-l-1)$-sphere of center T and radius $|\overline{PT}|$ located on the affine space A that is perpendicular to S and contains $\overline{PT}$. This special point $T = \text{proj}_S(P)$ will be called the point of partial revolution of P with respect to S. This finishes the proof. □

Remark 1.5 (Informal)**.** *Although rotations specifically are not our object of study here, it is worth pointing out that there is an analog of the properties we proved for rotations in* $\mathbb{R}^n$*. We will briefly use the definitions stated in the Theorem to characterize the analogous properties for rotations. The first part of the Theorem asserts that the set of fixed points under a rotation is always an affine space (it is specifically* S*, where the point* P *is being rotated about), indicating that it is redundant studying rotations in* $\mathbb{R}^n$ *around non-affine spaces. Additionally, the number of Euclidean degrees of freedom associated with the rotation about* S *is* $\dim(A) = n - \dim(S)$*, and if one considers the angles as the degrees of freedom, then it would be* $\dim(A) - 1$ *(for angles, this only makes sense when* $\dim(S) \leq n-2$*). All the degrees of freedom are centered at the point* T *and their image set lies in the affine space* A *(it is like rotating* P *about* T *inside the affine space* A*). Furthermore, all the angles relations between* A*,* S*, and* $\overline{PT}$ *remain the same for rotations and one can still make use of isometric mappings to auxiliate the calculus for rotations and then move the image back with the inverse transformation.*

One should note that demonstrating an equivalence between the image set of $SO(n)$ *and the* *image set of the revolution function (obviously, excluding the case in which* $\dim(C) = n - 1$ *to avoid reflections) would connect all these results for* $\mathcal{R}$ *and* $SO(n)$*, and potentially other ones we did not study here. Now, speculating a little, the methods involving the* $\mathcal{R}$ *function may (or may not, this is fundamentally only speculation) be useful to study* $SO(n)$ *for certain purposes. For instance, all these results proved above were demonstrated in a much friendlier way than it would be using the standard approaches for* $SO(n)$*; notably, we did not rely on results from spectral theory, quarternions or heavy algebraic matrix manipulations to deduce any theorem.*

Remark 1.6 (Informal)**.** *Although we restricted our attention to* $\mathbb{R}^n$*, the function* $\mathcal{R}$ *can be naturally extended to general metric spaces. For instance, by substituting* $(\mathbb{R}^n, \ell^2)$ *by a metric space* (M, d)*, we can define* $\mathcal{R}_M \colon \mathcal{P}^*(M) \times \mathcal{P}^*(M) \to \mathcal{P}^*(M)$ *analogously to how we did for* $\mathbb{R}^n$*. However, Theorem 1.1 will not hold for general metric spaces and, probably, without the addition of further structure, it will be difficult to prove any meaningful property about* $\mathcal{R}_M$ *besides an analog of Proposition 1.1. Speculating again, under well-chosen conditions for* (M, d)*,* $\mathcal{R}_M$ *might have some interesting properties to be studied like in* $(\mathbb{R}^n, \ell^2)$*.*

From Theorem 1.1, we can assume without loss of generality that any revolution has an affine space as center of isometry. So, consider $A \subseteq \mathbb{R}^n$ a region of $\mathbb{R}^n$ and $S\colon \dim(S) = m$ an affine space. Moreover, for each $K_a \in A$, let $P_a := \text{proj}_S(K_a)$ and B_a be the affine space such that $\dim(B_a) = n - m$, $B_a \supseteq \overline{K_a P_a}$ and $B_a \perp S$. Then, by Theorem 1.1:

$$\mathcal{R}(A, S) = \bigcup_{K_a \in A} \mathcal{R}(K_A, S) = \bigcup_{K_a \in A} \frac{\mathbb{R}^n \to B_a}{K_a \circ P_a}$$

Thus, when $m \neq n$, $\mathcal{R}(A, S)$ is a the union of the sets $S_{P_a}^{(n-m-1)}(|\overline{K_a P_a}|) \subset B_a$ for each K_a in A.

In the next section, this last result will be used to perform the revolution of functions' graphs belonging to the space $\mathbb{R}^m_*$, $m \leq n$ around $\mathbb{R}^{m-1}_*$. In particular, we will study the measure of revolution sets limited by these graphs. Then after this, in the last section, we will also utilize the previous expression to deduce the revolution of the image set of parametric functions. In the case of the parametric functions, the results will be proved for $C := span(e_{p(i)})_{i=1}^k$ with $k < n$ and $p\colon \{1, \dots k\} \to \{1, \dots, n\}$ injective (notice that when p is the identity function, $C = \mathbb{R}^k_*$). In particular, we obtain there a parametrization of the set resulting from the revolution of the original function.

2 Measure of sets of revolution and hypersolids of revolution

Over the development of the revolution function, one should have noticed that when $\mathcal{R}(S, C) \subseteq \mathbb{R}^3$ is a solid in $\mathbb{R}^3$ and C is a line or a point, this solid is indeed a solid of revolution. This way, naturally, when $\mathcal{R}(S, C)$ is a hypersolid in $\mathbb{R}^n$ with dimension k, we will name $\mathcal{R}(S, C)$ a k-solid of revolution or simply hypersolid of revolution. This motivates the definition:

Definition 2.1 (k-solid of revolution)**.** *Let $U \subseteq \mathbb{R}^n$ and $C \subseteq \mathbb{R}^n$ be an affine space. We say U is a k-solid of revolution in $\mathbb{R}^n$ if and only if U is a hypersolid with* $\dim(U) = k$ *and $U = \mathcal{R}(S, C)$ for some $S, C \subseteq \mathbb{R}^n$ with* $\dim(C) \neq n - 1$ *and $S \nsubseteq C$.*

Remark 2.1. *In the above definition, we discarded the cases* $\dim(C) = n - 1$ *and $S \subseteq C$ to avoid $\mathcal{R}(S, C)$ being a reflection of S or the identity function $I(S) = S$. Moreover, we are being cautious and saying "if U is a hypersolid with* $\dim(U) = k$*" instead of just stating if* $\dim(U) = k$ *because declaring that U is a hypersolid of revolution essentially requires U to be first a hypersolid. The issue though is that the definition of a hypersolid can vary from one context to another; for this reason, we (on purpose) conditioned the definition of a k-solid of revolution to the definition one is utilizing for a hypersolid. But in the end, this will not matter for what we will study here, any standard definition of a hypersolid will be included in the hypothesis we will require.*

Our purpose in this section is to study the Lebesgue measure of a subclass of the sets of type $\mathcal{R}(S, C) \subseteq \mathbb{R}^n$, which includes the hypersolids of revolution. Specifically, we will consider $C := \mathbb{R}^{m-1}_*$ and $S \subset \mathbb{R}^m_*$ a region delimited by C and by the graph of a measurable function $f\colon D \subseteq \mathbb{R}^{m-1} \to \mathbb{R}$ (considering the canonical embedding) within the domain D (which will also be taken to be measurable). With this, we will prove that sets $\mathcal{R}(S, C) \subseteq \mathbb{R}^3$ have Lebesgue measure consistent with the well-known formula $\pi \int_D f^2 \, d\mu$ for the solids of revolution. In particular, we will demonstrate an analogous version of this formula for any given integer value of m and n, with $m \leq n$.

2.1 Measure of sets of revolutions $\mathbb{R}^m_* \to \mathbb{R}^n$

Given two positive integers $k \leq n$, let $(\mathbb{R}^k, \Sigma^k, \mu^k)$ be a measure space with μ^k being the Lebesgue measure over $\mathbb{R}^k$. Define m, p to be positive integers such that $m + p = n$ and let $S := S_0 \times S_1$ for $S_0 \in \Sigma^m, S_1 \in \Sigma^p$. From the construction of the Lebesgue measure, we have that $S \in \Sigma^n$ and $\mu^n(S) = \mu^m(S_0) \cdot \mu^p(S_1)$ (this directly follows from the definition of the Lebesgue outer measure).

Let D be a measurable subset of $\mathbb{R}^{m-1}$ with respect to the measure μ^{m-1} and consider $f\colon D \subseteq$

$\mathbb{R}^{m-1} \to \mathbb{R}$ a measurable function. For each $\mathbf{x} \in D$, take $L_{\mathbf{x}} := [\min\{0, f(\mathbf{x})\}, \max\{0, f(\mathbf{x})\}]$ and define $S := \bigcup_{\mathbf{x}\in D} \mathbf{x} \times L_{\mathbf{x}}$. Additionally, let $E_m : \mathbb{R}^m \to \mathbb{R}^n$ be the canonical embedding map (Definition 0.1) defined by $E_m(\mathbf{x}) := (\mathbf{x}, 0, \dots, 0)$ for $\mathbf{x} \in \mathbb{R}^m$ and let $S^* := E_m(S)$. From this, consider the revolution of S^* about $\mathbb{R}^{m-1}_*$ given by the set $U := \mathcal{R}(S^*, \mathbb{R}^{m-1}_*)$.

For each $\mathbf{x} \in D$, let $P_{\mathbf{x}} := E_{m-1}(\mathbf{x})$ and $F_{\mathbf{x}} := (\mathbf{x}, f(\mathbf{x}), 0, \dots, 0)$. With this, it holds that $|\overline{P_{\mathbf{x}}F_{\mathbf{x}}}| = |f(\mathbf{x})|$, $S^* = \bigcup_{\mathbf{x}\in D} \overline{P_{\mathbf{x}}F_{\mathbf{x}}}$ and also that $P_{\mathbf{x}}$ is point of partial revolution of all the points in the line $\overline{P_{\mathbf{x}}F_{\mathbf{x}}}$. Defining $A_{\mathbf{x}}$ to be the $(n-m+1)$-dimensional affine space such that $A_{\mathbf{x}} \supset \overline{P_{\mathbf{x}}F_{\mathbf{x}}}$ and $A_{\mathbf{x}} \perp \mathbb{R}^{m-1}_*$, by Theorem 1.1, we have:

$$\mathcal{R}(S^*, \mathbb{R}^{m-1}_*) = \bigcup_{\mathbf{x}\in D} \frac{\mathbb{R}^m_* \to A_{\mathbf{x}}}{\overline{P_{\mathbf{x}}F_{\mathbf{x}}} \circ P_{\mathbf{x}}} = \bigcup_{\mathbf{x}\in D} (B^{(n-m+1)}_{P_{\mathbf{x}}}(|f(\mathbf{x})|) \subset A_{\mathbf{x}})$$

$$= \bigcup_{\mathbf{x}\in D} \mathbf{x} \times B^{(n-m+1)}_0(|f(\mathbf{x})|)$$

In the above equation, $B^{(n-m+1)}_0(|f(\mathbf{x})|)$ is centered at the origin and is subset of $\mathbb{R}^{n-m+1}$. Taking the indicator function $1_U \colon \mathbb{R}^n \to \mathbb{R}$ of U, we can use Tonelli's theorem [8] to develop $\mu^n(U)$:

$$\mu^n(U) = \int_{\mathbb{R}^n} 1_U \; d\mu^n = \int_{\mathbb{R}^{m-1}} \int_{\mathbb{R}^{n-m+1}} 1_U \; d\mu^{n-m+1} \; d\mu^{m-1}$$

$$= \int_{\mathbb{R}^{m-1}\setminus D} \int_{\mathbb{R}^{n-m+1}} 1_U \; d\mu^{n-m+1} \; d\mu^{m-1} + \int_D \int_{\mathbb{R}^{n-m+1}} 1_U \; d\mu^{n-m+1} \; d\mu^{m-1}$$

$$= \int_D \mu^{n-m+1}(B^{(n-m+1)}_0(|f|)) \; d\mu^{m-1} \tag{14}$$

To develop the integral in (14), we can use that the measure μ^d of a d-ball of radius r satisfies the equation below [9]:

$$\mu^d(\mathbb{B}^d(r)) = \frac{\pi^{\frac{d}{2}} \cdot r^d}{\Gamma\left(\frac{d}{2}+1\right)}$$

Here Γ represents the Gamma function. Relying on this fact, we can substitute the result in (14):

$$\int_D \mu^{n-m+1}(B^{(n-m+1)}_0(|f|)) \; d\mu^{m-1}$$

$$= \frac{\pi^{\frac{n-m+1}{2}}}{\Gamma\left(\frac{n-m+3}{2}\right)} \int_D |f|^{n-m+1} d\mu^{m-1}$$

$$\Rightarrow \mu^n(U) = \frac{\pi^{\frac{n-m+1}{2}}}{\Gamma\left(\frac{n-m+3}{2}\right)} \int_D |f|^{n-m+1} d\mu^{m-1}$$

Notice this expression is consistent with the results for the solids of revolutions. For instance, if we substitute $n = 3$ and $m = 2$ we would have the traditional formula $\pi \int_D f^2 d\mu$ of the solids of revolution.

3 Revolution of the image of parametric functions and hypersurfaces of revolution

Now, in this section, we will shift our focus to describing the revolution of the image of parametric functions whose image is contained in $\mathbb{R}^n$. Given the revolution of the image set of a function $\alpha\colon D \subseteq \mathbb{R}^N \to \mathbb{R}^n$, our objective is to find another parametrized function F such that the image set of F is equal to the set of revolution $\mathcal{R}(\alpha(D), C)$. Although it will be used $C := \mathbb{R}^k_*$ (Definition 0.1) over the passages, this time, the process applies analogously for any $C = \operatorname{span}(e_{p(j)})_{j=1}^k$ where $p\colon \{1, \dots, k\} \to \{1, \dots, n\}$ is an injective function. We will briefly comment on this at the end but basically, the decision to use $C := \mathbb{R}^k_*$ was made to maintain notations clearer throughout the text.

Moreover, likewise we mentioned in section 2 for the solids of revolution (Definition 2.1), one should have realized that when $\mathcal{R}(S, C) \subseteq \mathbb{R}^3$ is a surface with C being a line or a point, $\mathcal{R}(S, C)$ is a surface of revolution. At this point, we are again being cautious with saying "when $\mathcal{R}(S, C) \subseteq \mathbb{R}^3$ is a surface" to ensure $\mathcal{R}(S, C)$ is first of all, a surface in the context one is considering. So, excluding again the cases in which $\dim(C) = n - 1$ or $S \subseteq C$ to avoid reflections or the identity function, we have the definition:

Definition 3.1 (k-surface of revolution). *Let $U \subseteq \mathbb{R}^n$ and $C \subseteq \mathbb{R}^n$ be an affine space. We say U is a k-surface of revolution in $\mathbb{R}^n$ if and only if U is a hypersurface with $\dim(U) = k$ and $U = \mathcal{R}(S, C)$ for some $S, C \subseteq \mathbb{R}^n$ with $\dim(C) \neq n - 1$ and $S \nsubseteq C$.*

Since every surface or hypersurface is the image of some parametric function, naturally the image set of F emerges as a generalization of the surfaces of revolution when the image of F is indeed a hypersurface. However, the parametrization given by F, as one might expect, does not depend on any kind of regularity conditions of the parametric function α, holding regardless of whether $\mathcal{R}(\alpha(D), C)$ is a hypersurface or not.

3.1 Revolution of the image of parametric functions $\alpha\colon D \subseteq \mathbb{R}^N \to \mathbb{R}^n$

For each positive integer $i \leq n$, let $f_i\colon D \subseteq \mathbb{R}^N \to \mathbb{R}$ be a function. Additionally, define $\alpha\colon D \to \mathbb{R}^n$ by $\alpha(\mathbf{x}) := (f_1(\mathbf{x}), \ldots, f_n(\mathbf{x}))$ for $\mathbf{x} \in D$ and consider $C := \mathbb{R}^k_*$, with $k < n$. From this construction, take the revolution of $\alpha(D)$ around C given by the set $\mathcal{R}(\alpha(D), C)$.

For this case, Theorem 1.1 provides a nice generalization of the well-known parametric equation $\beta(x, y) = (f(x), g(x)\cos y, g(x)\sin y)$ for surfaces of revolution. As one might anticipate, we can use the fact that $\mathcal{R}(\alpha(D), C)$ is a union of $(n-k-1)$-spheres along C to utilize the parametrization of an m-sphere in the last coordinates and obtain a parametrization of the hypersurface resulting from the revolution. For instance, the parametric equation of the unit circumference $U = (\cos y, \sin y)$ is used in $\beta(x, y)$ to complete the revolution of $\gamma(x) = (f(x), g(x), 0)$.

But first, we shall note that:

$$\mathcal{R}(\alpha(D), C) = \bigcup_{\mathbf{x} \in D} \mathcal{R}(\alpha(\mathbf{x}), C)$$

In this way, from Theorem 1.1, we have that $P_{\mathbf{x}} := (f_1(\mathbf{x}), \ldots, f_k(\mathbf{x}), 0, \ldots, 0)$ is the point of partial revolution of $\alpha(\mathbf{x})$ with respect to C. From this, let $A_{\mathbf{x}} = \operatorname{span}(e_j)_{j=k+1}^n + P_{\mathbf{x}}$, i.e., $A_{\mathbf{x}}$ is the affine space orthogonal to C of dimension $(n-k)$ that contains $\overline{\alpha(\mathbf{x})P_{\mathbf{x}}}$. In order to optimize notation, define the function $R\colon D \to \mathbb{R}$ by $R(\mathbf{x}) := |\overline{\alpha(\mathbf{x})P_{\mathbf{x}}}| = \sqrt{\sum_{j=k+1}^n [f_j(\mathbf{x})]^2}$. Developing $\mathcal{R}(\alpha(\mathbf{x}), C)$, by Theorem 1.1:

$$\mathcal{R}(\alpha(\mathbf{x}), C) = \frac{\mathbb{R}^n \to A_{\mathbf{x}}}{\alpha(\mathbf{x}) \circ P_{\mathbf{x}}} = S^{n-k-1}_{P_{\mathbf{x}}}(R(\mathbf{x})) \subset A_{\mathbf{x}} \tag{15}$$

Define $\boldsymbol{\theta} := \sum_{l=1}^m \theta_l \cdot e_l = (\theta_1, \ldots, \theta_m)$, with $\boldsymbol{\theta} \in \mathbb{R}^m$. To further develop (15), we will need the following expression [10], which is a parametrization of the m-sphere of radius r embedded in $\mathbb{R}^{m+1}$ and centered at the origin:

$$S^m_0(\boldsymbol{\theta}) = r(\cos\theta_1, \sin\theta_1\cos\theta_2, \sin\theta_1\sin\theta_2\cos\theta_3, \ldots, \sin\theta_1 \ldots \sin\theta_{m-1}\cos\theta_m, \sin\theta_1 \ldots \sin\theta_m)$$

$$= r(g_1(\boldsymbol{\theta}), \ldots, g_{m+1}(\boldsymbol{\theta})) \tag{16}$$

This is, the j-th coordinate is given by the parametric function $g_j\colon \mathbb{R}^m \to \mathbb{R}$:

$$j = 1 \Rightarrow g_j(\boldsymbol{\theta}) = \cos\theta_1 \tag{17}$$

$$2 \le j \le m \Rightarrow g_j(\boldsymbol{\theta}) = \cos\theta_j \cdot \prod_{l=1}^{j-1} \sin\theta_l \tag{18}$$

$$j = m+1 \Rightarrow g_j(\boldsymbol{\theta}) = \prod_{l=1}^{m} \sin\theta_l \tag{19}$$

Applying the parametrization of the m-sphere in (15) by substituting $m = n-k-1$ and placing $S^{n-k-1}_{P_{\mathbf{x}}}(R(\mathbf{x}))$ over $A_{\mathbf{x}}$, it holds:

$$\mathcal{R}(\alpha(\mathbf{x}), C) = S^m_{P_{\mathbf{x}}}(R(\mathbf{x})) \subset A_{\mathbf{x}}$$

$$= \bigcup_{\boldsymbol{\theta}\in\mathbb{R}^m} (f_1(\mathbf{x}), \dots, f_k(\mathbf{x}), R(\mathbf{x}) \cdot g_1(\boldsymbol{\theta}), \dots, R(\mathbf{x}) \cdot g_{m+1}(\boldsymbol{\theta}))$$

where:

$m = n - k - 1$

$R(\mathbf{x}) = \sqrt{\sum_{j=k+1}^{n} [f_j(\mathbf{x})]^2}$;

$g_j(\boldsymbol{\theta})$ is defined as in (17), (18), (19) for $1 \le j \le m+1$

Remark 3.1. *Although we used the order $g_1, g_2, \dots, g_{m+1}$ in the remaining coordinates of the parametrization, as the functions g are the parametrization of the coordinates of an m-sphere, it does not matter which order one places the functions in the parametrization. This is true basically because any sphere of any dimension is invariant under rotation around its own center. So, switching coordinates (which is a rotation) will not change the image of the parametric function $S^m_0(\boldsymbol{\theta})$. Actually, any parametrization of an m-sphere would work in that situation.*

From this, we can define the set $\mathcal{R}(\alpha(\mathbf{x}), C)$ in terms of its parameters by the function $F\colon D \times \mathbb{R}^m \to \mathbb{R}^n$ given by:

$$F(\mathbf{x}, \boldsymbol{\theta}) := (f_1(\mathbf{x}), \dots, f_k(\mathbf{x}), R(\mathbf{x}) \cdot g_1(\boldsymbol{\theta}), \dots, R(\mathbf{x}) \cdot g_{m+1}(\boldsymbol{\theta})) \tag{20}$$

$$\Rightarrow \mathcal{R}(\alpha(\mathbf{x}), C) = F(\mathbf{x} \times \mathbb{R}^m)$$

$$\Rightarrow \mathcal{R}(\alpha(D), C) = F(D \times \mathbb{R}^m)$$

This is, F is a parametrization of $\mathcal{R}(\alpha(D), C)$ and the image set of F is the set $\mathcal{R}(\alpha(D), C)$. One should realize this is consistent with the parametrizations of the surfaces of revolution. For instance, if we set $n = 3$, $k = 1$ and $\alpha(x) := (f(x), g(x), 0)$, we have then $F(x, y) = (f(x), |g(x)| \cos y, |g(x)| \sin y) \simeq (f(x), g(x) \cos y, g(x) \sin y)$, where $\simeq$ indicates that these two parametric equations share the same image set.

Remark 3.2 (Informal)**.** *It is worth noting here that when* $\dim C < k$*, the function* F *has more than* $n-1$ *parameters, which is typically the maximum number utilized for functions whose image lies in* $\mathbb{R}^n$*. For this reason, though we shall not formally elaborate on this idea or conditions for it to hold, the set* $\mathcal{R}(\alpha(D), C)$ *can easily have positive or even infinite Lebesgue measure in* $\mathbb{R}^n$*, depending on whether* $\alpha(D)$ *is bounded or not.*

Remark 3.3. *Although we used* $C := \mathbb{R}^k_*$*, the process for finding the function* F *is analogous for any space* $C = span(e_{p(j)})_{j=1}^k$ *where* $p\colon \{1, \dots, k\} \to \{1, \dots, n\}$ *is an injective function. In the case* p *is not the identity function, one just has to find* $P_{\mathbf{x}} = proj_C \alpha(\mathbf{x})$ *and applies the same process we did where the coordinates of the new* $A_{\mathbf{x}}$ *lie; i.e., substituting the original parametric functions of* α *by the parametrization with the* m*-sphere of radius* $|\overline{\alpha(\mathbf{x}) P_{\mathbf{x}}}|$*. Since it is an* m*-sphere, the order one chooses for the functions* g_j *in the coordinates does not change the image set of* F *as we already mentioned. As an example of this process, when* $k = 1$ *and* $p(1) = 2$*, this would give* $(|f(x)| \cos y, g(x), |f(x)| \sin y) \simeq (f(x) \cos y, g(x), f(x) \sin y)$ *for the revolution of* $(f(x), g(x), 0)$*. We did not write the proof with* $C = span(e_{p(j)})_{j=1}^k$ *because this would unnecessarily complicate notation.*

Remark 3.4. *About the parametrization of the* j*-sphere, by considering the parametric equation of the circumference* $S = (\cos\theta, \sin\theta)$*, we could derive the general form of the parametric equation of a* j*-sphere by inducting on* j *with the process of revolution we gave. In this case, by performing the revolution of a* j*-sphere in* $\mathbb{R}^{j+2}$ *around an affine space* $C\colon \dim(C) = j$ *that cuts the* j*-sphere in half, one can show that this will generate a* $(j+1)$*-sphere with the parametric equation given before.*

Example 3.1 (Clifford Torus)**.** *Let* $S^d(r)$ *be a* d*-sphere of radius* r *and consider* $\mathbb{T} := S^m\left(\sqrt{\frac{m}{n}}\right) \times S^{n-m}\left(\sqrt{\frac{n-m}{n}}\right)$ *with* $1 \leq m \leq n-1$*. Define* $\boldsymbol{\theta_0} := \sum_{k=1}^m \theta_k \cdot e_k = (\theta_1, \dots, \theta_m)$ *with* $\boldsymbol{\theta_0} \in \mathbb{R}^m$*, and* $\boldsymbol{\theta_1} := \sum_{k=1}^{n-m} \theta_k \cdot e_k = (\theta_1, \dots, \theta_{n-m})$ *with* $\boldsymbol{\theta_1} \in \mathbb{R}^{n-m}$*. A natural embedding of* $S^d(r)$ *is given by equation (16). From this, an embedding of* $\mathbb{T}$ *is given by:*

$$T^n := \left\{ \sqrt{\frac{m}{n}} \sum_{j=1}^{m+1} g_j(\boldsymbol{\theta_0}) \cdot e_j + \sqrt{\frac{n-m}{n}} \sum_{j=1}^{n-m+1} h_j(\boldsymbol{\theta_1}) \cdot e_{j+m+1} | \boldsymbol{\theta_0} \in \mathbb{R}^m, \boldsymbol{\theta_1} \in \mathbb{R}^{n-m} \right\}$$

$$= \left(\sqrt{\frac{m}{n}} g_1(\boldsymbol{\theta_0}), \ldots, \sqrt{\frac{m}{n}} g_{m+1}(\boldsymbol{\theta_0}), \sqrt{\frac{n-m}{n}} h_1(\boldsymbol{\theta_1}), \ldots, \sqrt{\frac{n-m}{n}} h_{n-m+1}(\boldsymbol{\theta_1}) \right)$$

where g_j and h_j are defined analogously to (17), (18), (19).

Define a function $\alpha\colon \mathbb{R}^m \to \mathbb{R}^{n+2}$ by $\alpha(\boldsymbol{\theta_0}) := \left(\sqrt{\frac{m}{n}} g_1(\boldsymbol{\theta_0}), \ldots, \sqrt{\frac{m}{n}} g_{m+1}(\boldsymbol{\theta_0}), 0, \ldots, 0, \sqrt{\frac{n-m}{n}} \right)$ and let $C \subseteq \mathbb{R}^{n+2}$ such that $C := span(e_i)_{i=1}^{m+1}$. By directly applying the result (20)*, we have that $T^n = \mathcal{R}(\alpha(\mathbb{R}^m), C)$ and thus, T^n is an n-surface of revolution. This surface T^n is a subset of the unit $(n+1)$-sphere, and is a higher dimensional generalization of the Clifford torus [11].*

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