Wave Coupling

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1 Hasegawa-Mima equation

$$\frac{\partial}{\partial t}\Big(\nabla^2\phi - \phi\Big) - [(\nabla\phi \times \hat{\mathbf{z}}) \cdot \nabla]\Big[\nabla^2\phi - \ln\left(\frac{n_0}{\omega_{ci}}\right)\Big] = 0 \tag{1}$$

The Hasegawa-Mima equation 1 describes the drift-wave propagation of waves in plasma and the emergence of cross-field transport, the equation is a starting point in the study of instability in plasma drift waves and how it can decay in two or more coupled waves, that emerge when the initial waves reach a limit value

The equation is based on the propagation of electrostatic waves with frequency $\omega \ll \omega_{ci}$, where $\omega_{ci} = eB/m_i$ is the ion cyclotron frequency. The wave is propagated in a magnetized and in-homogeneous plasma medium. The wave is then treated as a drift wave.

One of the solutions to this equation is using to consider the potential as an infinite sum of waves in the form of

$$\phi(\mathbf{r}, t) = \frac{1}{2}\sum_{\mathbf{k}=1}^{\infty}\Big[\phi_{\mathbf{k}}(t)e^{i\mathbf{k}\cdot\mathbf{r}} + \phi_{\mathbf{k}}^*(t)e^{-i\mathbf{k}\cdot\mathbf{r}}\Big] \tag{2}$$

the equation that describes $\phi_{\mathbf{k}}$ is

$$\frac{\mathrm{d}\phi_{\mathbf{k}}}{\mathrm{d}t} = -i\omega_{\mathbf{k}}\phi_{\mathbf{k}} + \sum_{\mathbf{k}=\mathbf{k}_2+\mathbf{k}_3}\Lambda_{\mathbf{k}_2,\mathbf{k}_3}\phi_{\mathbf{k}_2}^*\phi_{\mathbf{k}_3}^* + \gamma_{\mathbf{k}}\phi_{\mathbf{k}} \tag{3}$$

where Λ is a constant factor that depend on $\mathbf{k}, \mathbf{k}_2, \mathbf{k}_3$, and γ is a dissipative term.

2 $\mathbf{E} \times \mathbf{B}$ drift

Considering a test particle embedded in a nonuniform plasma and using the slab approximation. The movement of the particle is given by

$$\mathbf{v}_E = c\frac{\mathbf{E} \times \mathbf{B}}{B^2} \tag{4}$$

The nonuniformity is given by electrostatic potential in the plasma, $\mathbf{E} = \boldsymbol{\nabla}\phi(\mathbf{r},t)$. Considering a uniform magnetic field in the $\hat{z}$ direction and a perpendicular electric field the velocity became

$$\mathbf{v}_E = -\frac{1}{B_0}\boldsymbol{\nabla}\phi(x,y,t) \times \hat{z} \tag{5}$$

then the x and y velocities are

$$v_x = \frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{1}{B_0}\frac{\partial}{\partial y}\phi(x,y,t) \qquad \text{and} \qquad v_y = \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1}{B_0}\frac{\partial}{\partial x}\phi(x,y,t) \tag{6}$$

Comparing these equations with the Hamilton equation, it's notable that

$$H(x,y,t) = \frac{\phi(x,y,t)}{B_0} \tag{7}$$

With this set of equations, we can plug the solution of the Hasegawa-Mima equation into the movement equation to particles to study how the particles inside a tokamak are affected by drift waves described by the equation 1.

3 Three Waves

A more general system is using three wave modes produced by the equation 3. Considering the notation $\phi_{\mathbf{k}_1}$ as ϕ_1, the ODE to ϕ are

$$\frac{\mathrm{d}\phi_1}{\mathrm{d}t} = -i\omega_1\phi_1 + \Lambda^1_{2,3}\phi_2^*\phi_3^* + \gamma_1\phi_1, \tag{8}$$

$$\frac{\mathrm{d}\phi_2}{\mathrm{d}t} = -i\omega_2\phi_2 + \Lambda^2_{3,1}\phi_3^*\phi_1^* + \gamma_2\phi_2, \tag{9}$$

$$\frac{\mathrm{d}\phi_3}{\mathrm{d}t} = -i\omega_3\phi_3 + \Lambda^3_{1,2}\phi_1^*\phi_2^* + \gamma_3\phi_3 \tag{10}$$

Then the total potential is

$$\phi(\mathbf{r},t)_{HM} = \frac{1}{2}\Big[\phi_1 e^{i\mathbf{k}_1\cdot\mathbf{r}} + \phi_2 e^{i\mathbf{k}_2\cdot\mathbf{r}} + \phi_3 e^{i\mathbf{k}_3\cdot\mathbf{r}} + \phi_1^* e^{-i\mathbf{k}_1\cdot\mathbf{r}} + \phi_2^* e^{-i\mathbf{k}_2\cdot\mathbf{r}} + \phi_3^* e^{-i\mathbf{k}_3\cdot\mathbf{r}}\Big] \tag{11}$$

simplifying

$$\phi(\mathbf{r},t)_{HM} = \sum_{\mathbf{k}=1}^{3}\left[\mathrm{Re}\{\phi_{\mathbf{k}}\}\cos(\mathbf{k}\cdot\mathbf{r}) - \mathrm{Im}\{\phi_{\mathbf{k}}\}\sin(\mathbf{k}\cdot\mathbf{r})\right] \tag{12}$$

where the HM represent the Hasegawa-Mima potencial wave.

Using the equation 7 to describe the velocity of the guiding centers with the potential ϕ_{HM} we had the result

where the particles in the radial direction represented by x diverge to infinity due the

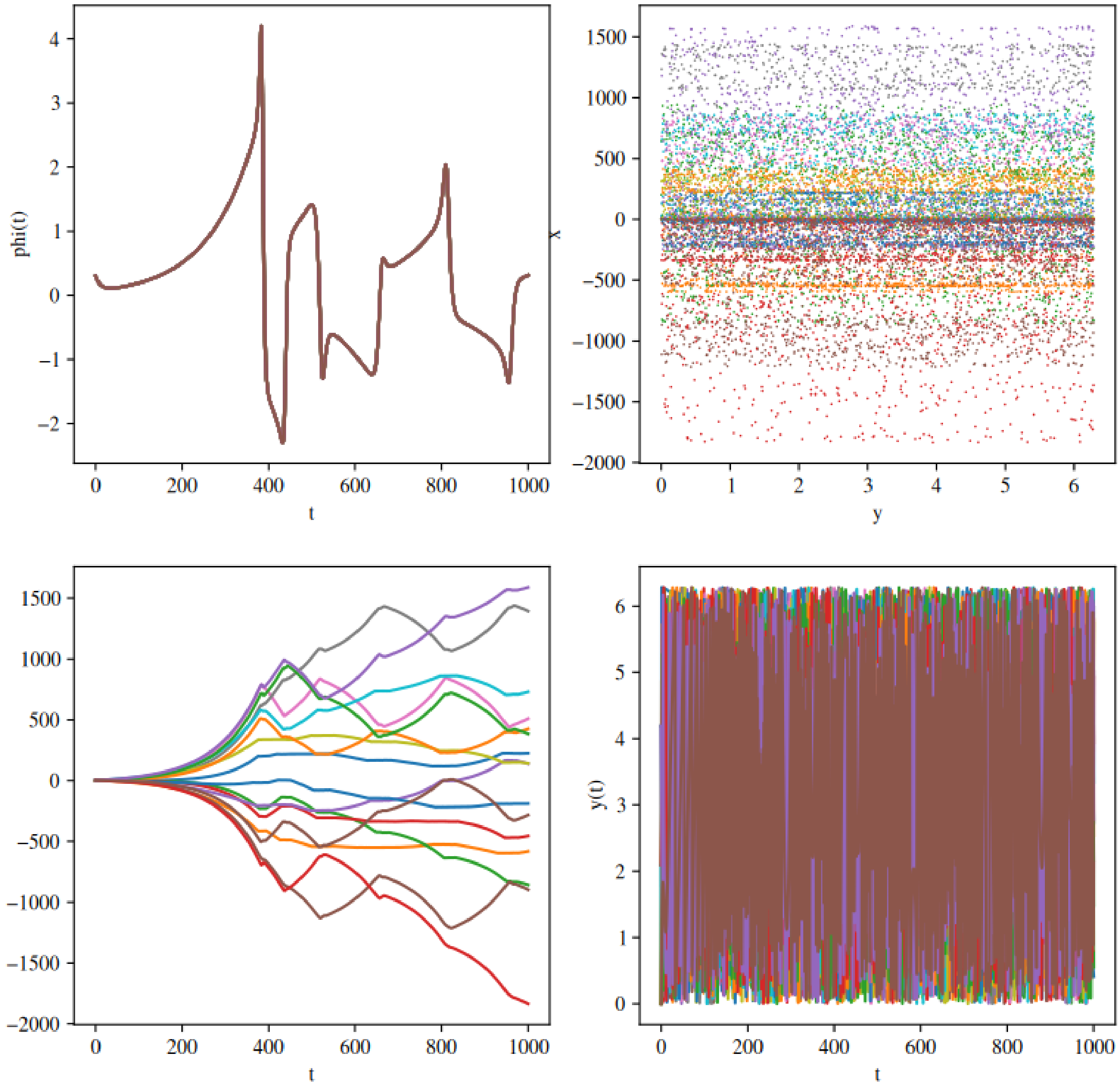

Figura 1: Eletrostatic Wave potential, Phase Space, movement in x direction and movement in y direction using just the turbulent potential

turbulence nature of the process. To represent the confinement seen in real Tokamaks we can introduce a attraction term in the, represented by a confinement potential $\phi_c = -kx$, where it will play the role of a restaurative force in classical dynamics, then our equations of motion becomes

$$v_x = \frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{1}{B_0}\frac{\partial}{\partial y}\phi_{HM}(x,y,t) \qquad \text{and} \qquad v_y = \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1}{B_0}\frac{\partial}{\partial x}\phi_{HM}(x,y,t) - k \tag{13}$$

applying this extra potential our diffusive escape is eliminated and the particles are confined, as we can see in figure 2

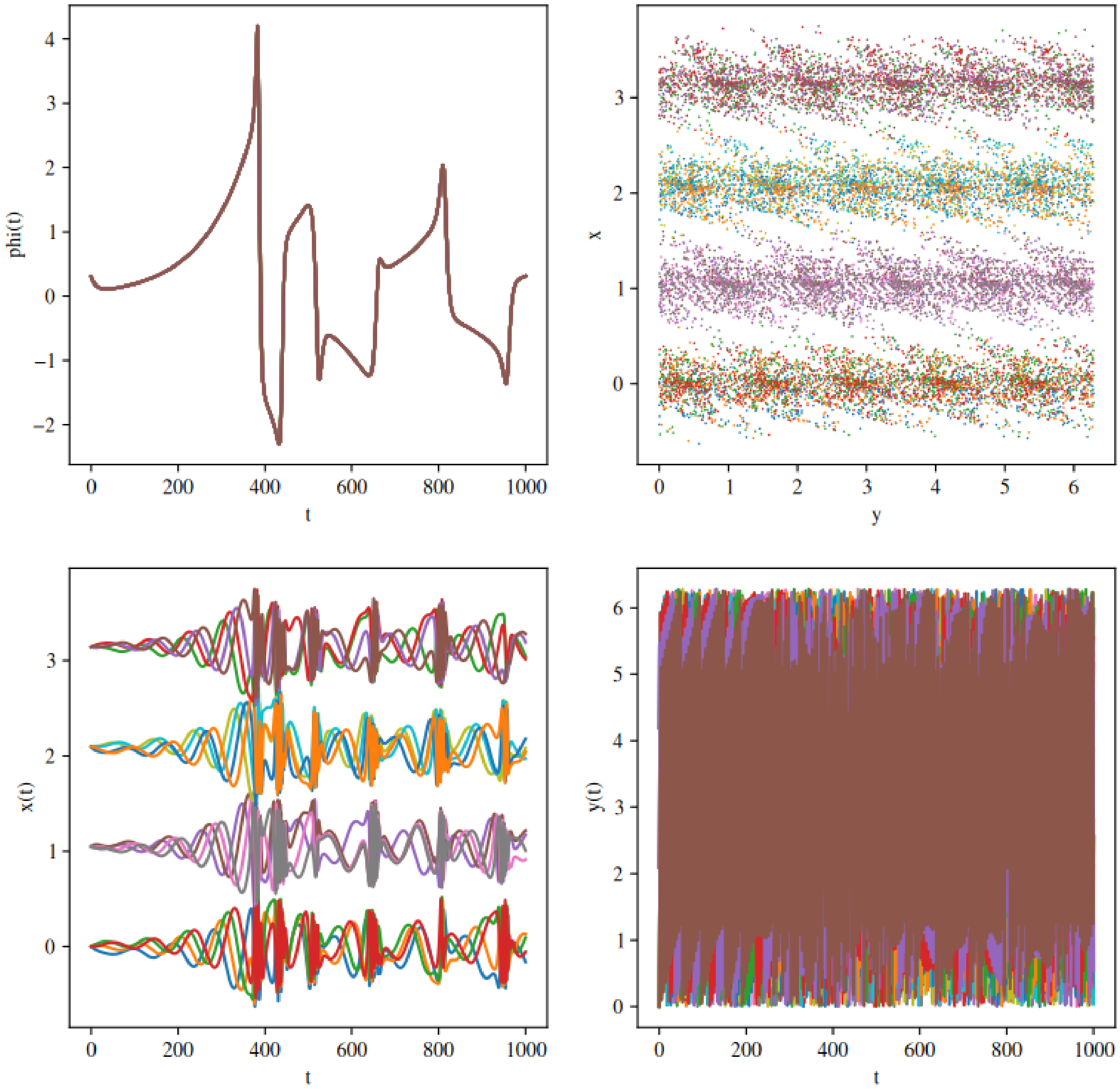

Figura 2: Eletrostatic Wave potential, Phase Space, movement in x direction and movement in y direction using the linear potential of confinement

If we use an quadratic potential, the equation of movement becomes

$$v_x = \frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{1}{B_0}\frac{\partial}{\partial y}\phi_{HM}(x,y,t) \qquad \text{and} \qquad v_y = \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1}{B_0}\frac{\partial}{\partial x}\phi_{HM}(x,y,t) - kx \tag{14}$$

and the confinement also occurs and we can see some better patterns, as in figure 3

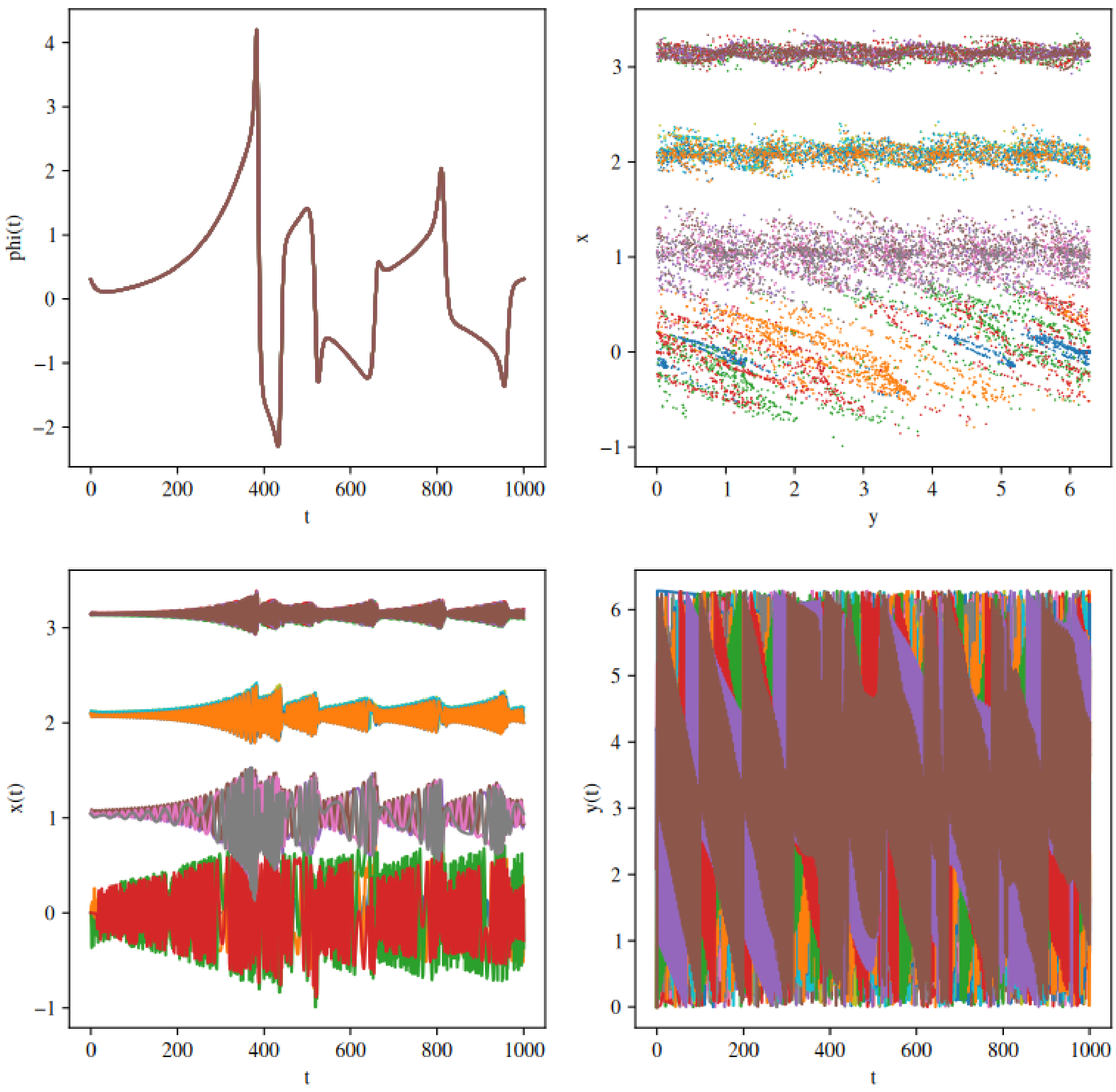

Figura 3: Eletrostatic Wave potential, Phase Space, movement in x direction and movement in y direction using the quadratic potential of confinement