CHAPTER 4

# Probability

Probability is the very guide of life.

Cicero, De Natura

## CONTENTS



This chapter starts with consideration of an experiment whose outcome cannot be predicted with certainty. We define the events of this experiment. We then introduce the concept of the probability of an event, which is the probability that the outcome of the experiment is contained in the event. An interpretation of the

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probability of an event as being a long-term relative frequency is given. Properties of probabilities are discussed. The conditional probability of one event, given the occurrence of a second event, is introduced. We see what it means for events to be independent.

## 4.1 INTRODUCTION

To gain information about the current leader in the next gubernatorial election, a representative sample of 100 voters has been polled. If 62 of those polled are in favor of the Republican candidate, can we conclude that a majority of the state's voters favor this candidate? Or, is it possible that *by chance* the sample contained a much greater proportion of this candidate's supporters than is contained in the general population and that the Democratic candidate is actually the current choice of a majority of the electorate?

To answer these questions, it is necessary to know something about the chance that as many as 62 people in a representative sample of size 100 would favor a candidate who, in fact, is not favored by a majority of the entire population. Indeed, as a general rule, to be able to draw valid inferences about a population from a sample, one needs to know how likely it is that certain events will occur under various circumstances. The determination of the likelihood, or chance, that an event will occur is the subject matter of *probability*.

## 4.2 SAMPLE SPACE AND EVENTS OF AN EXPERIMENT

The word *probability* is a commonly used term that relates to the chance that a particular event will occur when some experiment is performed, where we use the word *experiment* in a very broad sense. Indeed, an *experiment* for us is *any* process that produces an observation, or *outcome*.

We are often concerned with an experiment whose outcome is not predictable, with certainty, in advance. Even though the outcome of the experiment will not be known in advance, we will suppose that the set of all possible outcomes is known. This set of all possible outcomes of the experiment is called the *sample space* and is denoted by $S$.

**Definition** *An* experiment *is any process that produces an observation or* outcome. *The set of all possible outcomes of an experiment is called the* sample space.

### ■ Example 4.1

Some examples of experiments and their sample spaces are as follows.

(a) If the outcome of the experiment is the gender of a child, then

$$S = \{g, b\}$$

where outcome $g$ means that the child is a girl and $b$ that it is a boy.

**(b)** If the experiment consists of flipping two coins and noting whether they land heads or tails, then

$$S = \{(\mathrm{H}, \mathrm{H}), (\mathrm{H}, \mathrm{T}), (\mathrm{T}, \mathrm{H}), (\mathrm{T}, \mathrm{T})\}$$

The outcome is (H, H) if both coins land heads, (H, T) if the first coin lands heads and the second tails, (T, H) if the first is tails and the second is heads, and (T, T) if both coins land tails.

**(c)** If the outcome of the experiment is the order of finish in a race among 7 horses having positions 1, 2, 3, 4, 5, 6, 7, then

$$S = \{\text{all orderings of } 1, 2, 3, 4, 5, 6, 7\}$$

The outcome (4, 1, 6, 7, 5, 3, 2) means, for instance, that the number 4 horse comes in first, the number 1 horse comes in second, and so on.

**(d)** Consider an experiment that consists of rolling two six-sided dice and noting the sides facing up. Calling one of the dice die 1 and the other die 2, we can represent the outcome of this experiment by the pair of upturned values on these dice. If we let $(i, j)$ denote the outcome in which die 1 has value $i$ and die 2 has value $j$, then the sample space of this experiment is

$$\begin{aligned} S = \{&(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 1), (2, 2), (2, 3), (2, 4),\\ &(2, 5), (2, 6), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (4, 1), (4, 2),\\ &(4, 3), (4, 4), (4, 5), (4, 6), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6),\\ &(6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\} \end{aligned}$$

■

Any set of outcomes of the experiment is called an *event*. That is, an event is a subset of the sample space. Events will be denoted by the capital letters $A, B, C$, and so on.

## ■ Example 4.2

In Example 4.1(a), if $A = \{g\}$, then $A$ is the event that the child is a girl. Similarly, if $B = \{b\}$, then $B$ is the event that the child is a boy.

In Example 4.1(b), if $A = \{(\mathrm{H}, \mathrm{H}), (\mathrm{H}, \mathrm{T})\}$, then $A$ is the event that the first coin lands on heads.

In Example 4.1(c), if

$$A = \{\text{all outcomes in } S \text{ starting with } 2\}$$

then $A$ is the event that horse number 2 wins the race.

In Example 4.1(d), if

$$A = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$$

then $A$ is the event that the sum of the dice is 7. ■

**Definition** *Any set of outcomes of the experiment is called an* event. *We designate events by the letters* A, B, C, *and so on. We say that the event* A occurs *whenever the outcome is contained in* A.

For any two events $A$ and $B$, we define the new event $A \cup B$, called the *union* of events $A$ and $B$, to consist of all outcomes that are in $A$ or in $B$ or in both $A$ and $B$. That is, the event $A \cup B$ will occur if *either* $A$ or $B$ occurs.

In Example 4.1(a), if $A = \{g\}$ is the event that the child is a girl and $B = \{b\}$ is the event that it is a boy, then $A \cup B = \{g, b\}$. That is, $A \cup B$ is the whole sample space $S$.

In Example 4.1(c), let

$$A = \{\text{all outcomes starting with } 4\}$$

be the event that the number 4 horse wins; and let

$$B = \{\text{all outcomes whose second element is } 2\}$$

be the event that the number 2 horse comes in second. Then $A \cup B$ is the event that either the number 4 horse wins or the number 2 horse comes in second or both.

A graphical representation of events that is very useful is the *Venn diagram*. The sample space $S$ is represented as consisting of all the points in a large rectangle, and events are represented as consisting of all the points in circles within the rectangle. Events of interest are indicated by shading appropriate regions of the diagram. The colored region of Fig. 4.1 represents the union of events $A$ and $B$.

For any two events $A$ and $B$, we define the *intersection* of $A$ and $B$ to consist of all outcomes that are both in $A$ and in $B$. That is, the intersection will occur if *both* $A$ and $B$ occur. We denote the intersection of $A$ and $B$ by $A \cap B$. The colored region of Fig. 4.2 represents the intersection of events $A$ and $B$.

In Example 4.1(b), if $A = \{(\mathrm{H}, \mathrm{H}), (\mathrm{H}, \mathrm{T})\}$ is the event that the first coin lands heads and $B = \{(\mathrm{H}, \mathrm{T}), (\mathrm{T}, \mathrm{T})\}$ is the event that the second coin lands tails, then $A \cap B = \{(\mathrm{H}, \mathrm{T})\}$ is the event that the first coin lands heads and the second lands tails.

In Example 4.1(c), if $A$ is the event that the number 2 horse wins and $B$ is the event that the number 3 horse wins, then the event $A \cap B$ does not contain any outcomes and so cannot occur. We call the event without any outcomes the *null* event, and

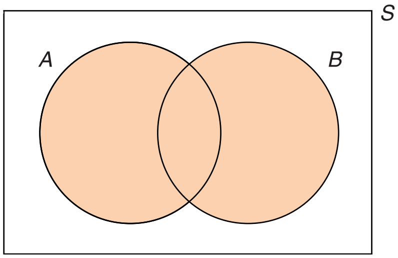



**FIGURE 4.1**
*A Venn diagram: shaded region is $A \cup B$.*

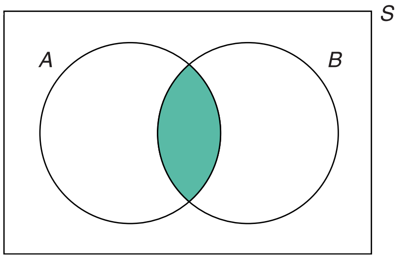



**FIGURE 4.2**
*Shaded region is $A \cap B$.*

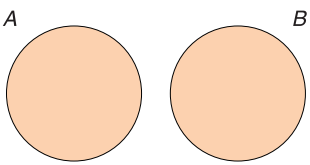



**FIGURE 4.3**
*$A$ and $B$ are disjoint events.*

designate it as Ø. If the intersection of $A$ and $B$ is the null event, then since $A$ and $B$ cannot simultaneously occur, we say that $A$ and $B$ are *disjoint*, or *mutually exclusive*. Two disjoint events are pictured in the Venn diagram of Fig. 4.3.

For any event $A$ we define the event $A^c$, called the *complement* of $A$, to consist of all outcomes in the sample space that are not in $A$. That is, $A^c$ will occur when $A$ does not, and vice versa. For instance, in Example 4.1(a), if $A = \{g\}$ is the event that the child is a girl, then $A^c = \{b\}$ is the event that it is a boy. Also note that the complement of the sample space is the null set, that is, $S^c = \emptyset$. Figure 4.4 indicates $A^c$, the complement of event $A$.

We can also define unions and intersections of more than two events. For instance, the union of events $A, B$, and $C$, written $A \cup B \cup C$, consists of all the outcomes

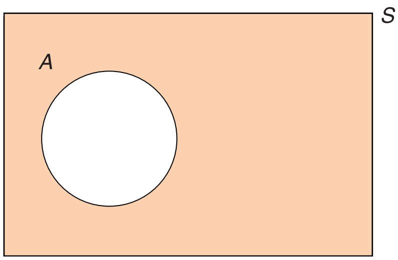



**FIGURE 4.4**
*Shaded region is $A^c$.*

of the experiment that are in $A$ or in $B$ or in $C$. Thus, $A \cup B \cup C$ will occur if at least one of these events occurs. Similarly, the intersection $A \cap B \cap C$ consists of the outcomes that are in all the events $A$, $B$, and $C$. Thus, the intersection will occur only if all the events occur.

We say that events $A$, $B$, and $C$ are *disjoint* if no two of them can simultaneously occur.

## PROBLEMS

1. A box contains three balls—one red, one blue, and one yellow. Consider an experiment that consists of withdrawing a ball from the box, replacing it, and withdrawing a second ball.
   (a) What is the sample space of this experiment?
   (b) What is the event that the first ball drawn is yellow?
   (c) What is the event that the same ball is drawn twice?
2. Repeat Prob. 1 when the second ball is drawn without replacement of the first ball.
3. Audrey and her boyfriend Charles must both choose which colleges they will attend in the coming fall. Audrey was accepted at the University of Michigan (MI), Reed College (OR), San Jose State College (CA), Yale University (CT), and Oregon State University (OR). Charles was accepted at Oregon State University and San Jose State College. Let the outcome of the experiment consist of the colleges that Audrey and Charles choose to attend.
   (a) List all the outcomes in sample space $S$.
   (b) List all the outcomes in the event that Audrey and Charles attend the same school.
   (c) List all the outcomes in the event that Audrey and Charles attend different schools.

(d) List all the outcomes in the event that Audrey and Charles attend schools in the same state.

4. An experiment consists of flipping a coin three times and each time noting whether it lands heads or tails.
   (a) What is the sample space of this experiment?
   (b) What is the event that tails occur more often than heads?
5. Family members have decided that their next vacation will be either in France or in Canada. If they go to France, they can either fly or take a boat. If they go to Canada, they can drive, take a train, or fly. Letting the outcome of the experiment be the location of their vacation and their mode of travel, list all the points in sample space $S$. Also list all the outcomes in $A$, where $A$ is the event that the family flies to the destination.
6. The New York Yankees and the Chicago White Sox are playing three games this weekend. Assuming that all games are played to a conclusion and that we are interested only in which team wins each game, list all the outcomes in sample space $S$. Also list all the outcomes in $A$, where $A$ is the event that the Yankees win more games than the White Sox.
7. Let $S = \{1, 2, 3, 4, 5, 6\}$, $A = \{1, 3, 5\}$, $B = \{4, 6\}$, and $C = \{1, 4\}$. Find
   (a) $A \cap B$
   (b) $B \cup C$
   (c) $A \cup (B \cap C)$
   (d) $(A \cup B)^c$

   *Note*: The operations within parentheses are performed first. For instance, in (c) first determine the intersection of $B$ and $C$, and then take the union of $A$ and that set.
8. A cafeteria offers a three-course meal. One chooses a main course, a starch, and a dessert. The possible choices are as follows:

| **Meal** | **Choices** |
|---|---|
| Main course | Chicken or roast beef |
| Starch course | Pasta or rice or potatoes |
| Dessert | Ice cream or gelatin or apple pie |

   An individual is to choose one course from each category.
   (a) List all the outcomes in the sample space.
   (b) Let $A$ be the event that ice cream is chosen. List all the outcomes in $A$.
   (c) Let $B$ be the event that chicken is chosen. List all the outcomes in $B$.

(d) List all the outcomes in the event $A \cap B$.
(e) Let $C$ be the event that rice is chosen. List all the outcomes in $C$.
(f) List all the outcomes in the event $A \cap B \cap C$.

9. A hospital administrator codes patients according to whether they have insurance and according to their condition, which is rated as good, fair, serious, or critical. The administrator records a 0 if a patient has no insurance and a 1 if he or she does, and then records one of the letters $g, f, s$, or $c$, depending on the patient's condition. Thus, for instance, the coding 1, $g$ is used for a patient with insurance who is in good condition. Consider an experiment that consists of the coding of a new patient.
   (a) List the sample space of this experiment.
   (b) Specify the event corresponding to the patient's being in serious or critical condition and having no medical insurance.
   (c) Specify the event corresponding to the patient's being in either good or fair condition.
   (d) Specify the event corresponding to the patient's having insurance.

10. The following pairs of events $E$ and $F$ relate to the same experiment. Tell in each case whether $E$ and $F$ are disjoint events.
   (a) A die is rolled. Event $E$ is that it lands on an even number, and $F$ is the event that it lands on an odd number.
   (b) A die is rolled. Event $E$ is that it lands on 3, and $F$ is the event that it lands on an even number.
   (c) A person is chosen. Event $E$ is that this person was born in the United States, and $F$ is the event that this person is a U.S. citizen.
   (d) A man is chosen. Event $E$ is that he is over 30 years of age, and $F$ is the event that he has been married for over 30 years.
   (e) A woman waiting in line to register her car at the department of motor vehicles is chosen. Event $E$ is that the car is made in the United States, and $F$ is the event that it is made in a foreign country.

11. Let $A$ be the event that a rolled die lands on an even number.
   (a) Describe in words the event $A^c$.
   (b) Describe in words the event $(A^c)^c$.
   (c) In general, let $A$ be an event. What is the complement of its complement? That is, what is $(A^c)^c$?

12. Two dice are rolled. Let $A$ be the event that the sum of the dice is even, let $B$ be the event that the first die lands on 1, and let $C$ be the event that the sum of the dice is 6. Describe the following events.
   (a) $A \cap B$
   (b) $A \cup B$
   (c) $B \cap C$
   (d) $B^c$

(e) $A^c \cap C$
(f) $A \cap B \cap C$

13. Let $A$, $B$, and $C$ be events. Use Venn diagrams to represent the event that of $A$, $B$, and $C$
   (a) Only $A$ occurs.
   (b) Both $A$ and $B$ occur, but $C$ does not.
   (c) At least one event occurs.
   (d) At least two of the events occur.
   (e) All three events occur.

## 4.3 PROPERTIES OF PROBABILITY

It is an empirical fact that if an experiment is continually repeated under the same conditions, then, for any event $A$, the proportion of times that the outcome is contained in $A$ approaches some value as the number of repetitions increases. For example, if a coin is continually flipped, then the proportion of flips landing on tails will approach some value as the number of flips increases. It is this long-run proportion, or *relative frequency*, that we often have in mind when we speak of the probability of an event.

Consider an experiment whose sample space is $S$. We suppose that for each event $A$ there is a number, denoted $P(A)$ and called the *probability* of event $A$, that is in accord with the following three properties.

**PROPERTY 1:** For any event $A$, the probability of $A$ is a number between 0 and 1. That is,

$$0 \leq P(A) \leq 1$$

**PROPERTY 2:** The probability of sample space $S$ is 1. Symbolically,

$$P(S) = 1$$

**PROPERTY 3:** The probability of the union of disjoint events is equal to the sum of the probabilities of these events. For instance, if $A$ and $B$ are disjoint, then

$$P(A \cup B) = P(A) + P(B)$$

The quantity $P(A)$ represents the probability that the outcome of the experiment is contained in event $A$. Property 1 states that the probability that the outcome of the experiment is contained in $A$ is some value between 0 and 1. Property 2 states that, with probability 1, the outcome of the experiment will be an element of sample space $S$. Property 3 states that if events $A$ and $B$ cannot simultaneously occur, then the probability that the outcome of the experiment is contained in either $A$ or $B$ is equal to the sum of the probability that it is in $A$ and the probability that it is in $B$.

If we interpret $P(A)$ as the long-run relative frequency of event $A$, then the stated conditions are satisfied. The proportion of experiments in which the outcome is contained in $A$ would certainly be a number between 0 and 1. The proportion of experiments in which the outcome is contained in $S$ is 1 since all outcomes are contained in sample space $S$. Finally, if $A$ and $B$ have no outcomes in common, then the proportion of experiments whose outcome is in either $A$ or $B$ is equal to the proportion whose outcome is in $A$ plus the proportion whose outcome is in $B$. For instance, if the proportion of time that a pair of rolled dice sums to 7 is 1/6 and the proportion of time that they sum to 11 is 1/18, then the proportion of time that they sum to either 7 or 11 is $1/6 + 1/18 = 2/9$.

Properties 1, 2, and 3 can be used to establish some general results concerning probabilities. For instance, since $A$ and $A^c$ are disjoint events whose union is the entire sample space, we can write

$$S = A \cup A^c$$

Using properties 2 and 3 now yields the following.

$$\begin{aligned} 1 &= P(S) && \text{by property 2} \\ &= P(A \cup A^c) \\ &= P(A) + P(A^c) && \text{by property 3} \end{aligned}$$

Therefore, we see that

$$P(A^c) = 1 - P(A)$$

In words, the probability that the outcome of the experiment is not contained in $A$ is 1 minus the probability that it is. For instance, if the probability of obtaining heads on the toss of a coin is 0.4, then the probability of obtaining tails is 0.6.

The following formula relates the probability of the union of events $A$ and $B$, which are not necessarily disjoint, to $P(A)$, $P(B)$, and the probability of the intersection of $A$ and $B$. It is often called the *addition rule of probability*.

### Addition Rule

For any events $A$ and $B$,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

To see why the addition rule holds, note that $P(A \cup B)$ is the probability of all outcomes that are either in $A$ or in $B$. On the other hand, $P(A) + P(B)$ is the probability of all the outcomes that are in $A$ plus the probability of all the outcomes that are in $B$. Since any outcome that is in both $A$ and $B$ is counted twice in $P(A) + P(B)$

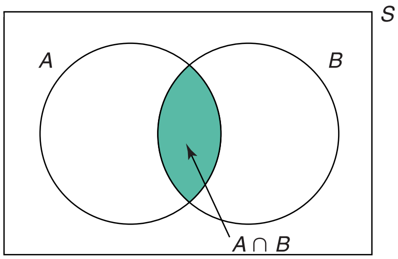



**FIGURE 4.5**
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

and only once in $P(A \cup B)$ (see Fig. 4.5), it follows that

$$P(A) + P(B) = P(A \cup B) + P(A \cap B)$$

Subtracting $P(A \cap B)$ from both sides of the preceding equation gives the addition rule.

Example 4.3 illustrates the use of the addition rule.

## ■ Example 4.3

A certain retail establishment accepts either the American Express or the VISA credit card. A total of 22 percent of its customers carry an American Express card, 58 percent carry a VISA credit card, and 14 percent carry both. What is the probability that a customer will have at least one of these cards?

### Solution

Let $A$ denote the event that the customer has an American Express card, and let $B$ be the event that she or he has a VISA card. The given information yields

$$P(A) = 0.22 \quad P(B) = 0.58 \quad P(A \cap B) = 0.14$$

By the additive rule, the desired probability $P(A \cup B)$ is

$$P(A \cup B) = 0.22 + 0.58 - 0.14 = 0.66$$

That is, 66 percent of the establishment's customers carry at least one of the cards that it will accept. ■

As an illustration of the interpretation of probability as a long-run relative frequency, we have simulated 10,000 flips of a perfectly symmetric coin. The total numbers of heads and tails that occurred in the first 10, 50, 100, 500, 2000, 6000, 8000, and 10,000 flips, along with the proportion of them that was heads, are presented in Table 4.1. Note how the proportion of the flips that lands heads becomes very close to 0.5 as the number of flips increases.

**Table 4.1** 10,000 Flips of a Symmetric Coin

| $n$ | Number of heads in first $n$ flips | Number of tails in first $n$ flips | Proportion of first $n$ flips that lands on heads |
|---|---|---|---|
| 10 | 3 | 7 | 0.3 |
| 50 | 21 | 29 | 0.42 |
| 100 | 46 | 54 | 0.46 |
| 500 | 248 | 252 | 0.496 |
| 2,000 | 1,004 | 996 | 0.502 |
| 6,000 | 3,011 | 2,989 | 0.5018 |
| 8,000 | 3,974 | 4,026 | 0.4968 |
| 10,000 | 5,011 | 4,989 | 0.5011 |

**Table 4.2** 10,000 Rolls of a Symmetric Die

| | $i$ | | | | | |
|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 |
| Frequency of outcome | 1724 | 1664 | 1628 | 1648 | 1672 | 1664 |
| Relative frequency | 0.1724 | 0.1664 | 0.1628 | 0.1648 | 0.1672 | 0.1664 |

*Note*: 1/6 = 0.166667.

The results of 10,000 simulated rolls of a perfectly symmetric die are presented in Table 4.2.

## PROBLEMS

1. Suppose the sample space of an experiment is

$$S = \{1, 2, 3, 4, 5, 6\}$$

Let $A_i$ denote the event consisting of the single outcome $i$, and suppose that

$$P(A_1) = 0.1 \qquad P(A_4) = 0.15$$
$$P(A_2) = 0.2 \qquad P(A_5) = 0.1$$
$$P(A_3) = 0.15 \qquad P(A_6) = 0.3$$

That is, the outcome of the experiment is 1 with probability 0.1, it is 2 with probability 0.2, it is 3 with probability 0.15, and so on. Let events $E$, $F$, and $G$ be as follows:

$$E = \{1, 3, 5\} \quad F = \{2, 4, 6\} \quad G = \{1, 4, 6\}$$

## Historical Perspective

(Bettmann)

Pierre Fermat

(Bettmann)

Blaise Pascal

The notion that chance, or probability, can be treated numerically is relatively recent. Indeed, for most of recorded history it was felt that what occurred in life was determined by forces that were beyond one's ability to understand. It was only during the first half of the 17th century, near the end of the Renaissance, that people became curious about the world and the laws governing its operation. Among the curious were the gamblers. A group of Italian gamblers, unable to answer certain questions concerning dice, approached the famous scientist Galileo. Galileo, though busy with other work, found their problems to be of interest and not only provided solutions but also wrote a short treatise on games of chance.

A few years later a similar story took place in France, where a gambler known as Chevalier de Mere resided. De Mere, a strong amateur mathematician as well as a gambler, had an acquaintance with the brilliant mathematician Blaise Pascal. It was to Pascal that de Mere turned for help in his more difficult gaming questions. One particular problem, known as the *problem of the points*, concerned the equitable division of stakes when two players are interrupted in the midst of a game of chance. Pascal found this problem particularly intriguing and, in 1654, wrote to the mathematician Pierre Fermat about it. Their resulting exchange of letters not only led to a solution of this problem but also laid the framework for the solution of many other problems connected with games of chance. Their celebrated correspondence, cited by some as the birth date of probability, stimulated interest in probability among some of the foremost European mathematicians of the time. For instance, the young Dutch genius Ludwig Huyghens came to Paris to discuss the new subject, and activity in this new field grew rapidly.

Find

**(a)** $P(E), P(F), P(G)$ **(b)** $P(E \cup F)$
**(c)** $P(E \cup G)$ **(d)** $P(F \cup G)$
**(e)** $P(E \cup F \cup G)$ **(f)** $P(E \cap F)$
**(g)** $P(F \cap G)$ **(h)** $P(E \cap G)$
**(i)** $P(E \cap F \cap G)$

**2.** If $A$ and $B$ are disjoint events for which $P(A) = 0.2$ and $P(B) = 0.5$, find
**(a)** $P(A^c)$
**(b)** $P(A \cup B)$
**(c)** $P(A \cap B)$
**(d)** $P(A^c \cap B)$

**3.** Phenylketonuria is a genetic disorder that produces mental retardation. About one child in every 10,000 live births in the United States has phenylketonuria. What is the probability that the next child born in a Houston hospital has phenylketonuria?

4. A certain person encounters three traffic lights when driving to work. Suppose that the following represent the probabilities of the total number of red lights that she has to stop for:

$$P(0 \text{ red lights}) = 0.14$$
$$P(1 \text{ red light}) = 0.36$$
$$P(2 \text{ red lights}) = 0.34$$
$$P(3 \text{ red lights}) = 0.16$$

(a) What is the probability that she stops for at least one red light when driving to work?
(b) What is the probability that she stops for more than two red lights?

5. If $A$ and $B$ are disjoint events, is the following possible?

$$P(A) + P(B) = 1.2$$

What if $A$ and $B$ are not disjoint?

6. If the probability of drawing a king from a deck of pinochle cards is 1/6 and the probability of drawing an ace is 1/6, what is the probability of drawing either an ace or a king?

7. Suppose that the demand for Christmas trees from a certain dealer will be

| | |
|---|---|
| 1100 | with probability 0.2 |
| 1400 | with probability 0.3 |
| 1600 | with probability 0.4 |
| 2000 | with probability 0.1 |

Find the probability that the dealer will be able to sell his entire stock if he purchases
(a) 1100 trees
(b) 1400 trees
(c) 1600 trees
(d) 2000 trees

8. The Japanese automobile company Lexus has established a reputation for quality control. Recent statistics indicate that a newly purchased Lexus ES 350 will have

| | |
|---|---|
| 0 defects | with probability 0.12 |
| 1 defect | with probability 0.18 |
| 2 defects | with probability 0.25 |
| 3 defects | with probability 0.20 |
| 4 defects | with probability 0.15 |
| 5 or more defects | with probability 0.10 |

If you purchase a new Lexus ES 350, find the probability that it will have

(a) 2 or fewer defects
(b) 4 or more defects
(c) Between (inclusive) 1 and 3 defects

Let $p$ denote the probability it will have an even number of defects. Whereas the information given above does not enable us to specify the value of $p$, find the

(d) Largest
(e) Smallest

value of $p$ which is consistent with the preceding.

9. When typing a five-page manuscript, a certain typist makes

| | |
|---|---|
| 0 errors | with probability 0.20 |
| 1 error | with probability 0.35 |
| 2 errors | with probability 0.25 |
| 3 errors | with probability 0.15 |
| 4 or more errors | with probability 0.05 |

If you give such a manuscript to this typist, find the probability that it will contain

(a) 3 or fewer errors
(b) 2 or fewer errors
(c) 0 errors

10. The following table is a modern version of a *life table*, which was first developed by John Graunt in 1662. It gives the probabilities that a newly born member of a certain specified group will die in his or her $i$th decade of life, for $i$ ranging from 1 to 10. The first decade starts with birth and ends with an individual's 10th birthday. The second decade starts at age 10 and ends at the 20th birthday, and so on.

Life Table

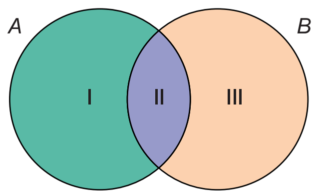
| Decade | Probability of death | Decade | Probability of death |
|---|---|---|---|
| 1 | 0.062 | 6 | 0.124 |
| 2 | 0.012 | 7 | 0.215 |
| 3 | 0.024 | 8 | 0.271 |
| 4 | 0.033 | 9 | 0.168 |
| 5 | 0.063 | 10 | 0.028 |

For example, the probability that a newborn child dies in her or his fifties is 0.124. Find the probability that a newborn will

(a) Die between the ages of 30 and 60
(b) Not survive to age 40
(c) Survive to age 80

11. The family picnic scheduled for tomorrow will be postponed if it is either cloudy or rainy. The weather report states that there is a 40 percent chance of rain tomorrow, a 50 percent chance of cloudiness, and a 20 percent chance that it will be both cloudy and rainy. What is the probability that the picnic will be postponed?
12. In Example 4.3, what proportion of customers has neither an American Express nor a VISA card?
13. It is estimated that 30 percent of all adults in the United States are obese and that 3 percent suffer from diabetes. If 2 percent of the population both is obese and suffers from diabetes, what percentage of the population either is obese or suffers from diabetes?
14. Welds of tubular joints can have two types of defects, which we call $A$ and $B$. Each weld produced has defect $A$ with probability 0.064, defect $B$ with probability 0.043, and both defects with probability 0.025. Find the proportion of welds that has
    (a) Either defect $A$ or defect $B$
    (b) Neither defect
15. A customer that goes to the suit department of a certain store will purchase a suit with probability 0.3. The customer will purchase a tie with probability 0.2 and will purchase both a suit and a tie with probability 0.1. What proportion of customers purchases neither a suit nor a tie?
16. Anita has a 40 percent chance of receiving an $A$ grade in statistics, a 60 percent chance of receiving an $A$ in physics, and an 86 percent chance of receiving an $A$ in either statistics or physics. Find the probability that she
    (a) Does not receive an $A$ in either statistics or physics
    (b) Receives $A$'s in both statistics and physics
17. This problem uses a Venn diagram to present a formal proof of the addition rule. Events $A$ and $B$ are represented by circles in the Venn diagram.

    In terms of $A$ and $B$, describe the region labeled
    (a) I
    (b) II
    (c) III




A Venn diagram: dividing up $A \cup B$

Express, in terms of $P(\mathrm{I})$, $P(\mathrm{II})$, and $P(\mathrm{III})$,

**(d)** $P(A \cup B)$

**(e)** $P(A)$

**(f)** $P(B)$

**(g)** $P(A \cap B)$

**(h)** Conclude that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

## 4.4 EXPERIMENTS HAVING EQUALLY LIKELY OUTCOMES

For certain experiments it is natural to assume that each outcome in the sample space $S$ is equally likely to occur. That is, if sample space $S$ consists of $N$ outcomes, say, $S = \{1, 2, \ldots, N\}$, then it is often reasonable to suppose that

$$P(\{1\}) = P(\{2\}) = \cdots = P(\{N\})$$

In this expression, $P(\{i\})$ is the probability of the event consisting of the single outcome $i$; that is, it is the probability that the outcome of the experiment is $i$.

Using the properties of probability, we can show that the foregoing implies that the probability of any event $A$ is equal to the proportion of the outcomes in the sample space that is in $A$. That is,

$$P(A) = \frac{\text{number of outcomes in } S \text{ that are in } A}{N}$$

### ■ Example 4.4

In a survey of 420 members of a retirement center, it was found that 144 are smokers and 276 are not. If a member is selected in such a way that each of the members is equally likely to be the one selected, what is the probability that person is a smoker?

**Solution**

There are 420 outcomes in the sample space of the experiment of selecting a member of the center. Namely, the outcome is the person selected. Since there are 144 outcomes in the event that the selected person is a smoker, it follows that the probability of this event is

$$P\{\text{smoker}\} = \frac{144}{420} = \frac{12}{35}$$

■

## ■ Example 4.5

Suppose that when two dice are rolled, each of the 36 possible outcomes given in Example 4.1(d) is equally likely. Find the probability that the sum of the dice is 6 and that it is 7.

**Solution**

If we let $A$ denote the event that the sum of the dice is 6 and $B$ that it is 7, then

$$A = \{(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)\}$$

and

$$B = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$$

Therefore, since $A$ contains 5 outcomes and $B$ contains 6, we see that

$$P(A) = P\{\text{sum is } 6\} = 5/36$$
$$P(B) = P\{\text{sum is } 7\} = 6/36 = 1/6$$ ■

## ■ Example 4.6

One man and one woman are to be selected from a group that consists of 10 married couples. If all possible selections are equally likely, what is the probability that the woman and man selected are married to each other?

**Solution**

Once the man is selected, there are 10 possible choices of the woman. Since one of these 10 choices is the wife of the man chosen, we see that the desired probability is 1/10. ■

When each outcome of the sample space is equally likely to be the outcome of the experiment, we say that an element of the sample space is *randomly selected*.

## ■ Example 4.7

An elementary school is offering two optional language classes, one in French and the other in Spanish. These classes are open to any of the 120 upper-grade students in the school. Suppose there are 32 students in the French class, 36 in the Spanish class, and a total of 8 who are in both classes. If an upper-grade student is randomly chosen, what is the probability that this student is enrolled in at least one of these classes?

**Solution**

Let $A$ and $B$ denote, respectively, the events that the randomly chosen student is enrolled in the French class and is enrolled in the Spanish class. We will determine $P(A \cup B)$, the probability that the student is enrolled in either French or Spanish, by using the addition rule

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Since 32 of the 120 students are enrolled in the French class, 36 of the 120 are in the Spanish class, and 8 of the 120 are in both classes, we have

$$P(A) = \frac{32}{120}, \quad P(B) = \frac{36}{120}, \quad \text{and} \quad P(A \cap B) = \frac{8}{120}$$

Therefore,

$$P(A \cup B) = \frac{32}{120} + \frac{36}{120} - \frac{8}{120} = \frac{1}{2}$$

That is, the probability that a randomly chosen student is taking at least one of the language classes is 1/2. ■

## ■ Example 4.8

Table 4.3 lists the earnings frequencies of all full-time workers who are at least 15 years old, classified according to their annual salary and gender.

**Table 4.3** Earnings of Workers by Sex, 1989

| Earnings group (in $1000) | Number | | Distribution (percent) | |
|---|---|---|---|---|
| | Women | Men | Women | Men |
| <5 | 427,000 | 548,000 | 1.4 | 1.1 |
| 5–10 | 440,000 | 358,000 | 1.4 | .7 |
| 10–15 | 1,274,000 | 889,000 | 4.1 | 1.8 |
| 15–20 | 1,982,000 | 1,454,000 | 6.3 | 2.9 |
| 20–30 | 6,291,000 | 5,081,000 | 20.1 | 10.2 |
| 30–40 | 6,555,000 | 6,386,000 | 20.9 | 12.9 |
| 40–50 | 5,169,000 | 6,648,000 | 16.5 | 13.4 |
| 50–100 | 8,255,000 | 20,984,000 | 26.3 | 42.1 |
| >100 | 947,000 | 7,377,000 | 3.0 | 14.9 |
| Total | 31,340,000 | 49,678,000 | 100.0 | 100.0 |

*Source*: Department of Commerce, Bureau of the Census.

Suppose one of these workers is randomly chosen. Find the probability that this person is

(a) A woman (b) A man
(c) A man earning under \$30,000 (d) A woman earning over \$50,000

**Solution**

(a) Since 31,340,000 of the $31{,}340{,}000 + 49{,}678{,}000 = 81{,}018{,}000$ workers are women, it follows that the probability that a randomly chosen worker is a woman is

$$\frac{31{,}340{,}000}{81{,}018{,}000} \approx .3868$$

That is, there is approximately a 38.7 percent chance that the randomly selected worker is a woman.

(b) Since the event that the randomly selected worker is a man is the complement of the event that the worker is a woman, we see from (a) that the probability is approximately $1 - 0.3868 = 0.6132$.

(c) Since (in thousands) the number of men earning under \$30,000 is

$$548 + 358 + 889 + 1454 + 5081 = 8330$$

we see that the desired probability is $8330/81{,}018 \approx .1028$. That is, there is approximately a 10.3 percent chance that the person selected is a man with an income under \$30,000.

(d) The probability that the person selected is a woman with an income above \$50,000 is

$$\frac{8255 + 947}{81{,}018} \approx .1136$$

That is, there is approximately an 11.4 percent chance that the person selected is a woman with an income above \$50,000. ■

## PROBLEMS

1. In an experiment involving smoke detectors, an alarm was set off at a college dormitory at 3 a.m. Out of 216 residents of the dormitory, 128 slept through the alarm. If one of the residents is randomly chosen, what is the probability that this person did not sleep through the alarm?
2. Among 32 dieters following a similar routine, 18 lost weight, 5 gained weight, and 9 remained the same weight. If one of these dieters is randomly chosen, find the probability that he or she

(a) Gained weight
(b) Lost weight
(c) Neither lost nor gained weight

3. One card is to be selected at random from an ordinary deck of 52 cards. Find the probability that the selected card is
(a) An ace (b) Not an ace
(c) A spade (d) The ace of spades

4. The following table lists the 10 countries with the highest production of meat.

| Country | Meat production (thousands of metric tons) |
|---|---|
| China | 20,136 |
| United States | 17,564 |
| Russia | 12,698 |
| Germany | 6,395 |
| France | 3,853 |
| Brazil | 3,003 |
| Argentina | 2,951 |
| Britain | 2,440 |
| Italy | 2,413 |
| Australia | 2,373 |

Suppose a World Health Organization committee is formed to discuss the long-term ramifications of producing such quantities of meat. Suppose further that it consists of one representative from each of these countries. If the chair of this committee is then randomly chosen, find the probability that this person will be from a country whose production of meat (in thousands of metric tons)
(a) Exceeds 10,000
(b) Is under 3500
(c) Is between 4000 and 6000
(d) Is less than 2000

5. Suppose that distinct integer values are written on each of 3 cards. These cards are then randomly given the designations $A$, $B$, and $C$. The values on cards $A$ and $B$ are then compared. If the smaller of these values is then compared with the value on card $C$, what is the probability that it is also smaller than the value on card $C$?

6. A bag containing pennies and dimes has 4 times as many dimes as pennies. One coin is drawn. Assuming that the drawn coin is equally likely to be any of the coins, what is the probability that it is a dime?

7. A total of 44 out of 100 patients at a rehabilitation center are signed up for a special exercise program that consists of a swimming class and a

calisthenics class. Each of these 44 patients takes at least one of these classes. Suppose that there are 26 patients in the swimming class and 28 in the calisthenics class. Find the probability that a randomly chosen patient at the center is

(a) Not in the exercise program
(b) Enrolled in both classes

8. Of the families in a certain community, 20 percent have a cat, 32 percent have a dog, and 12 percent have both a cat and a dog.
   (a) If a family is chosen at random, what is the probability it has neither a dog nor a cat?
   (b) If the community consists of 1000 families, how many of them have either a cat or a dog?
9. Of the students at a girls' school, 60 percent wear neither a ring nor a necklace, 20 percent wear a ring, and 30 percent wear a necklace. If one of them is randomly chosen, find the probability that she is wearing
   (a) A ring or a necklace
   (b) A ring and a necklace
10. A sports club has 120 members, of whom 44 play tennis, 30 play squash, and 18 play both tennis and squash. If a member is chosen at random, find the probability that this person
    (a) Does not play tennis
    (b) Does not play squash
    (c) Plays neither tennis nor squash
11. In Prob. 10, how many members play either tennis or squash?
12. If two dice are rolled, find the probability that the sum of the dice is
    (a) Either 7 or 11
    (b) One of the values 2, 3, or 12
    (c) An even number
13. Suppose 2 people are randomly chosen from a set of 20 people that consists of 10 married couples. What is the probability that the 2 people are married to each other? (*Hint*: After the initial person is chosen, the next one is equally likely to be any of the remaining people.)
14. Find the probability that a randomly chosen worker in Example 4.8
    (a) Earns under \$15,000
    (b) Is a woman who earns between \$20,000 and \$40,000
    (c) Earns under \$50,000
15. A real estate agent has a set of 10 keys, one of which will open the front door of a house he is trying to show to a client. If the keys are tried in a completely random order, find the probability that
    (a) The first key opens the door
    (b) All 10 keys are tried

16. A group of 5 girls and 4 boys is randomly lined up.
    (a) What is the probability that the person in the second position is a boy?
    (b) What is the probability that Charles (one of the boys) is in the second position?
17. The following data are from the U.S. National Oceanic and Atmospheric Administration. They give the average number of days in each month with precipitation of 0.01 inch or more for Washington, D.C.

| Jan. | Feb. | Mar. | Apr. | May | June | July | Aug. | Sept. | Oct. | Nov. | Dec. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 10 | 9 | 11 | 10 | 11 | 10 | 10 | 9 | 8 | 7 | 8 | 9 |

Find the probability you will encounter rain if you are planning to visit Washington, D.C., next
    (a) January 5
    (b) August 12
    (c) April 15
    (d) May 15
    (e) October 12

## 4.5 CONDITIONAL PROBABILITY AND INDEPENDENCE

We are often interested in determining probabilities when some partial information concerning the outcome of the experiment is available. In such situations, the probabilities are called *conditional probabilities*.

As an example of a conditional probability, suppose two dice are to be rolled. Then, as noted in Example 4.1(d), the sample space of this experiment is the set of 36 outcomes $(i, j)$, where both $i$ and $j$ range from 1 through 6. The outcome $(i, j)$ results when the first die lands on $i$ and the second on $j$.

Suppose that each of the 36 possible outcomes is equally likely to occur and thus has probability $1/36$. (When this is the case, we say that the dice are *fair*.) Suppose further that the first die lands on 4. Given this information, what is the resulting probability that the sum of the dice is 10? To determine this probability, we reason as follows. Given that the first die lands on 4, there are 6 possible outcomes of the experiment, namely,

$$(4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6)$$

In addition, since these outcomes initially had the same probabilities of occurrence, they should still have equal probabilities. That is, given that the first die lands on 4, the *conditional* probability of each of the outcomes should be 1/6. Since in only one of the outcomes is the sum of the dice equal to 10, namely, the outcome (4, 6), it follows that the conditional probability that the sum is 10, given that the first die lands on 4, is 1/6.

If we let $B$ denote the event that the sum of the dice is 10 and let $A$ denote the event that the first die lands on 4, then the probability obtained is called the *conditional probability of B given that A has occurred*. It is denoted by

$$P(B|A)$$

A general formula for $P(B|A)$ can be derived by an argument similar to the one used earlier. Suppose that the outcome of the experiment is contained in $A$. Now, in order for the outcome also to be in $B$, it must be in both $A$ and $B$; that is, it must be in $A \cap B$. However, since we know that the outcome is in $A$, it follows that $A$ becomes our new (or reduced) sample space, and the probability that event $A \cap B$ occurs is the probability of $A \cap B$ relative to the probability of $A$. That is (see Fig. 4.6),

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

This definition of conditional probability is consistent with the interpretation of probability as being a long-run relative frequency. To show this, suppose that a large number, call it $n$, of repetitions of the experiment are performed. We will now argue that if we consider only those experiments in which $A$ occurs, then $P(B|A)$ will equal the long-run proportion of them in which $B$ also occurs. To see this, note that since $P(A)$ is the long-run proportion of experiments in which $A$ occurs, it follows that in $n$ repetitions of the experiment, $A$ will occur approximately $nP(A)$ times. Similarly, in approximately $nP(A \cap B)$ of these experiments, both $A$ and $B$ will occur. Hence, out of the approximately $nP(A)$ experiments for which the outcome is contained in $A$, approximately $nP(A \cap B)$ of them will also have their outcomes in $B$. Therefore, of those experiments whose outcomes are in $A$, the proportion whose outcome is also in $B$ is approximately equal to

$$\frac{nP(A \cap B)}{nP(A)} = \frac{P(A \cap B)}{P(A)}$$

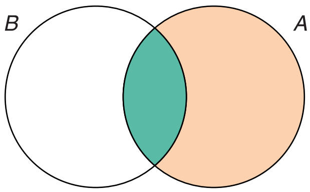



**FIGURE 4.6**
$P(B|A) = \frac{P(A \cap B)}{P(A)}$.

Since this approximation becomes exact as $n$ becomes larger and larger, we see that we have given the appropriate definition of the conditional probability of $B$ given that $A$ has occurred.

## ■ Example 4.9

As a further check of the preceding formula for the conditional probability, use it to compute the conditional probability that the sum of a pair of rolled dice is 10, given that the first die lands on 4.

### Solution

Letting $B$ denote the event that the sum of the dice is 10 and $A$ the event that the first die lands on 4, we have

$$\begin{aligned} P(B|A) &= \frac{P(A \cap B)}{P(A)} \\ &= \frac{P(\{(4,6)\})}{P(\{(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)\})} \\ &= \frac{1/36}{6/36} = \frac{1}{6} \end{aligned}$$

Therefore, we obtain the same result as before. ■

## ■ Example 4.10

The organization that employs Jacobi is organizing a parent-daughter dinner for those employees having at least one daughter. Each of these employees is asked to attend along with one of his or her daughters. If Jacobi is known to have two children, what is the conditional probability that they are both girls given that Jacobi is invited to the dinner? Assume the sample space $S$ is given by

$$S = \{(g,g),(g,b),(b,g),(b,b)\}$$

and that all these outcomes are equally likely, where the outcome $(g,b)$ means, for instance, that Jacobi's oldest child is a girl and youngest is a boy.

### Solution

Since Jacobi is invited to the dinner, we know that at least one of Jacobi's children is a girl. Letting $B$ denote the event that both of them are girls and $A$ the event that at least one is a girl, we see that the desired probability is $P(B|A)$. This is determined as follows:

$$
\begin{aligned}
P(B|A) &= \frac{P(A \cap B)}{P(A)} \\
&= \frac{P(\{g,g\})}{P(\{(g,g),(g,b),(b,g)\})} \\
&= \frac{1/4}{3/4} = \frac{1}{3}
\end{aligned}
$$

That is, the conditional probability that both of Jacobi's children are girls given that at least one is a girl is 1/3. Many students incorrectly suppose that this conditional probability is 1/2, reasoning that the Jacobi child not attending the dinner is equally likely to be a boy or a girl. Their mistake lies in assuming that these two possibilities are equally likely, for initially there were 4 equally likely outcomes. The information that at least one of the children is a girl is equivalent to knowing that the outcome is not $(b,b)$. Thus we are left with the 3 equally likely outcomes, $(g,g)$, $(g,b)$, $(b,g)$, showing that there is only a 1/3 chance that Jacobi has two girls. ■

## ■ Example 4.11

Table 4.4 lists the number (in thousands) of students enrolled in a California State College, categorized by sex and age.

**(a)** Suppose a student is randomly chosen. What is the probability this student is a woman?

Find the conditional probability that a randomly chosen student is

**(b)** Over 35, given that this student is a man
**(c)** Over 35, given that this student is a woman
**(d)** A woman, given that this student is over 35
**(e)** A man, given that this student is between 20 and 21

### Solution

**(a)** Since there are 6663 women out of a total of 12,544 students, it follows that the probability that a randomly chosen student is a woman is

$$\frac{6663}{12{,}544} = 0.5312$$

**(b)** Since there are a total of 5881 males, of whom 684 are over age 35, the desired conditional probability is

$$P(\text{over } 35|\text{man}) = \frac{684}{5881} = 0.1163$$

**Table 4.4** Enrollment

| Sex and age | |
|---|---|
| Total | 12,544 |
| Male | 5,881 |
| 14 to 17 years old | 91 |
| 18 and 19 years old | 1,309 |
| 20 and 21 years old | 1,089 |
| 22 to 24 years old | 1,080 |
| 25 to 29 years old | 1,016 |
| 30 to 34 years old | 613 |
| 35 years old and over | 684 |
| Female | 6,663 |
| 14 to 17 years old | 119 |
| 18 and 19 years old | 1,455 |
| 20 and 21 years old | 1,135 |
| 22 to 24 years old | 968 |
| 25 to 29 years old | 931 |
| 30 to 34 years old | 716 |
| 35 years old and over | 1,339 |

**(c)** By similar reasoning to that used in (b), we see that

$$P(\text{over } 35|\text{woman}) = \frac{1339}{6663} = 0.2010$$

**(d)** Since there are a total of $684 + 1339 = 2023$ students who are over age 35, of whom 1339 are women, it follows that

$$P(\text{woman}|\text{over } 35) = \frac{1339}{2023} = 0.6619$$

**(e)** Since there are a total of $1089 + 1135 = 2224$ students who are between 20 and 21, of whom 1089 are men, it follows that

$$P(\text{man}|\text{between 20 and 21}) = \frac{1089}{2224} = 0.4897$$

■

Since

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

we obtain, upon multiplying both sides by $P(A)$, the following result, known as the *multiplication rule*.

### Multiplication Rule

$$P(A \cap B) = P(A)P(B|A)$$

This rule states that the probability that both $A$ and $B$ occur is equal to the probability that $A$ occurs multiplied by the conditional probability of $B$ given that $A$ occurs. It is often quite useful for computing the probability of an intersection.

## ■ Example 4.12

Suppose that two people are randomly chosen from a group of 4 women and 6 men.

**(a)** What is the probability that both are women?
**(b)** What is the probability that one is a woman and the other a man?

### Solution

**(a)** Let $A$ and $B$ denote, respectively, the events that the first person selected is a woman and that the second person selected is a woman. To compute the desired probability $P(A \cap B)$, we start with the identity

$$P(A \cap B) = P(A)P(B|A)$$

Now since the first person chosen is equally likely to be any of the 10 people, of whom 4 are women, it follows that

$$P(A) = \frac{4}{10}$$

Now given that the first person selected is a woman, it follows that the next selection is equally likely to be any of the remaining 9 people, of whom 3 are women. Therefore,

$$P(B|A) = \frac{3}{9}$$

and so

$$P(A \cap B) = \frac{4}{10} \cdot \frac{3}{9} = \frac{2}{15}$$

**(b)** To determine the probability that the chosen pair consists of 1 woman and 1 man, note first that this can occur in two disjoint ways. Either the first person chosen is a man and the second chosen is a woman, or vice versa. Let us determine the probabilities for each of these cases. Letting $A$

denote the event that the first person chosen is a man and $B$ the event that the second person chosen is a woman, we have

$$P(A \cap B) = P(A)P(B|A)$$

Now, since the first person is equally likely to be any of the 10 people, of whom 6 are men,

$$P(A) = \frac{6}{10}$$

Also, given that the first person is a man, the next selection is equally likely to be any of the remaining 9 people, of whom 4 are women, and so

$$P(B|A) = \frac{4}{9}$$

Therefore,

$$P(\text{man then woman}) = P(A \cap B) = \frac{6}{10} \cdot \frac{4}{9} = \frac{4}{15}$$

By similar reasoning, the probability that the first person chosen is a woman and the second chosen is a man is

$$P(\text{woman then man}) = \frac{4}{10} \cdot \frac{6}{9} = \frac{4}{15}$$

Since the event that the chosen pair consists of a woman and a man is the union of the above two disjoint events, we see that

$$P(1 \text{ woman and } 1 \text{ man}) = \frac{4}{15} + \frac{4}{15} = \frac{8}{15}$$ ■

The conditional probability that $B$ occurs given that $A$ has occurred is not generally equal to the (unconditional) probability of $B$. That is, knowing that $A$ has occurred generally changes the chances of $B$'s occurrence. In the cases where $P(B|A)$ is equal to $P(B)$, we say that $B$ is *independent* of $A$.

Since

$$P(A \cap B) = P(A)P(B|A)$$

we see that $B$ is independent of $A$ if

$$P(A \cap B) = P(A)P(B)$$

Since this equation is symmetric in $A$ and $B$, it follows that if $B$ is independent of $A$, then $A$ is also independent of $B$.

It can also be shown that if $A$ and $B$ are independent, then the probability of $B$ given that $A$ does not occur is also equal to the (unconditional) probability of $B$. That is, if $A$ and $B$ are independent, then

$$P(B|A^c) = P(B)$$

Thus, when $A$ and $B$ are independent, any information about the occurrence or nonoccurrence of one of these events does not affect the probability of the other.

---

Events $A$ and $B$ are *independent* if

$$P(A \cap B) = P(A)P(B)$$

If $A$ and $B$ are independent, then the probability that a given one of them occurs is unchanged by information as to whether the other one has occurred.

---

## ■ Example 4.13

Suppose that we roll a pair of fair dice, so each of the 36 possible outcomes is equally likely. Let $A$ denote the event that the first die lands on 3, let $B$ be the event that the sum of the dice is 8, and let $C$ be the event that the sum of the dice is 7.

**(a)** Are $A$ and $B$ independent?
**(b)** Are $A$ and $C$ independent?

### Solution

**(a)** Since $A \cap B$ is the event that the first die lands on 3 and the second on 5, we see that

$$P(A \cap B) = P(\{(3,5)\}) = \frac{1}{36}$$

On the other hand,

$$P(A) = P(\{(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)\}) = \frac{6}{36}$$

and

$$P(B) = P(\{(2,6),(3,5),(4,4),(5,3),(6,2)\}) = \frac{5}{36}$$

Therefore, since $1/36 \neq (6/36) \cdot (5/36)$, we see that

$$P(A \cap B) \neq P(A)P(B)$$

and so events $A$ and $B$ are not independent.

Intuitively, the reason why the events are not independent is that the chance that the sum of the dice is 8 is affected by the outcome of the first die. In particular, the chance that the sum is 8 is enhanced when the first die is 3, since then we still have a chance of obtaining the total of 8 (which we would not have if the first die were 1).

**(b)** Events $A$ and $C$ are independent. This is seen by noting that

$$P(A \cap C) = P(\{3, 4\}) = \frac{1}{36}$$

while

$$P(A) = \frac{1}{6}$$

and

$$P(C) = P(\{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}) = \frac{6}{36}$$

Therefore,

$$P(A \cap C) = P(A)(C)$$

and so events $A$ and $C$ are independent.

It is rather intuitive that the event that the sum of the dice is 7 should be independent of the event that the first die lands on 3. For no matter what the outcome of the first die, there will always be exactly one outcome of the second die that results in the sum being equal to 7. As a result, the conditional probability that the sum is 7 given the value of the first die will always equal 1/6. ■

## ■ Example 4.14

Consider Table 4.4, presented in Example 4.11. Suppose that a female student is randomly chosen, as is, independently, a male student. Find the probability that both students are between 22 and 24 years old.

### Solution

Since 1080 of the 5881 male students are between 22 and 24 years old, it follows that

$$P(\{\text{male is between 22 and 24}\}) = \frac{1080}{5881} \approx 0.1836$$

Similarly, since 968 of the 6663 female students are between 22 and 24 years old, we see that

$$P(\{\text{female is between 22 and 24}\}) = \frac{968}{6663} \approx 0.1453$$

Since the choices of the male and female students are independent, we obtain

$$P(\{\text{both are between ages 22 and 24}\}) = \frac{1080}{5881} \cdot \frac{968}{6663} \approx 0.0267$$

That is, there is approximately a 2.7 percent chance that both students are between 22 and 24 years of age. ■

While so far we have discussed independence only for pairs of events, this concept can be extended to any number of events. The probability of the intersection of any number of independent events will be equal to the product of their probabilities.

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If $A_1, \ldots, A_n$ are independent, then

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1)P(A_2)\cdots P(A_n)$$

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## ■ Example 4.15

A couple is planning on having three children. Assuming that each child is equally likely to be of either sex and that the sexes of the children are independent, find the probability that

**(a)** All three children will be girls.
**(b)** At least one child will be a girl.

### Solution

**(a)** If we let $A_i$ be the event that their $i$th child is a girl, then

$$\begin{aligned} P(\text{all girls}) &= P(A_1 \cap A_2 \cap A_3) \\ &= P(A_1)P(A_2)P(A_3) \quad \text{by independence} \\ &= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8} \end{aligned}$$

**(b)** The easiest way to compute the probability of at least one girl is by first computing the probability of the complementary event—that all the children are boys. Since, by the same reasoning as used in part (a),

$$P(\text{all boys}) = \frac{1}{8}$$

we see that

$$P(\text{at least one girl}) = 1 - P(\text{all boys}) = \frac{7}{8}$$ ■

## PROBLEMS

1. It is estimated that 30 percent of all adults in the United States are obese, 3 percent of all adults suffer from diabetes, and 2 percent of all adults both are obese and suffer from diabetes. Determine the conditional probability that a randomly chosen individual
   (a) Suffers from diabetes given that he or she is obese
   (b) Is obese given that she or he suffers from diabetes
2. Suppose a coin is flipped twice. Assume that all four possibilities are equally likely to occur. Find the conditional probability that both coins land heads given that the first one does.
3. Consider Table 4.3 as presented in Example 4.8. Suppose that one of the workers is randomly chosen. Find the conditional probability that this worker
   (a) Is a woman given that he or she earns over $25,000
   (b) Earns over $25,000 given that this worker is a woman
4. Fifty-two percent of the students at a certain college are females. Five percent of the students in this college are majoring in computer science. Two percent of the students are women majoring in computer science. If a student is selected at random, find the conditional probability that
   (a) This student is female, given that the student is majoring in computer science
   (b) This student is majoring in computer science, given that the student is female

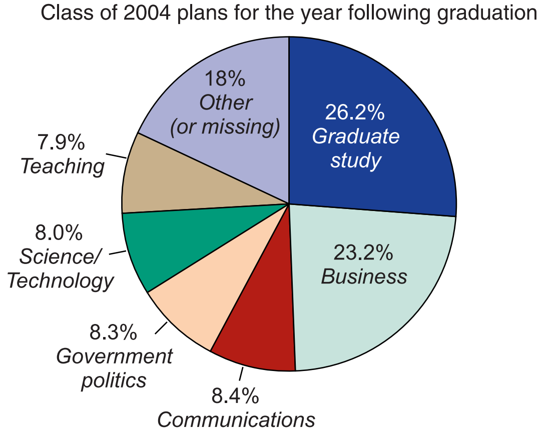
Problems 5 and 6 refer to the data in the following table, which describes the age distribution of residents in a northern California county.

| Age | Number |
|---|---|
| 0–9 | 4200 |
| 10–19 | 5100 |
| 20–29 | 6200 |
| 30–39 | 4400 |
| 40–49 | 3600 |
| 50–59 | 2500 |
| 60–69 | 1800 |
| Over 70 | 1100 |

5. If a resident is randomly selected from this county, determine the probability that the resident is
   (a) Less than 10 years old
   (b) Between 10 and 20 years old

(c) Between 20 and 30 years old
(d) Between 30 and 40 years old

6. Find the conditional probability that a randomly chosen resident is
   (a) Between 10 and 20 years old, given that the resident is less than 30 years old
   (b) Between 30 and 40 years old, given that the resident is older than 30
7. A games club has 120 members, of whom 40 play chess, 56 play bridge, and 26 play both chess and bridge. If a member of the club is randomly chosen, find the conditional probability that she or he
   (a) Plays chess given that he or she plays bridge
   (b) Plays bridge given that she or he plays chess
8. Refer to Table 4.4, which is presented in Example 4.11. Determine the conditional probability that a randomly chosen student is
   (a) Less than 25 years old, given that the student is a man
   (b) A man, given that this student is less than 25 years old
   (c) Less than 25 years old, given that the student is a woman
   (d) A woman, given that this student is less than 25 years old
9. Following is a pie chart detailing the after-graduation plans of the 2004 graduating class of Harvard University.




Suppose a student from this class is randomly chosen. Given that this student is not planning to go into either business or teaching, what is the probability that this student
   (a) Is planning to go into graduate study?
   (b) Is planning to go into either teaching or graduate study?
   (c) Is planning to go into either communications or graduate study?
   (d) Is not planning to go into science/technology?

(e) Is not planning to go into either communications or business?
(f) Is not planning to go into either science/technology or government/politics?

10. Many psychologists believe that birth order and personality are related. To study this hypothesis, 400 elementary school children were randomly selected and then given a test to measure confidence. On the results of this test each of the students was classified as being either confident or not confident. The numbers falling into each of the possible categories are:

| | Firstborn | Not firstborn |
|---|---|---|
| Confident | 62 | 60 |
| Not confident | 105 | 173 |

That is, for instance, out of 167 students who were firstborn children, a total of 62 were rated as being confident. Suppose that a student is randomly chosen from this group.

(a) What is the probability that the student is a firstborn?
(b) What is the probability that the student is rated confident?
(c) What is the conditional probability that the student is rated confident given that the student is a firstborn?
(d) What is the conditional probability that the student is rated confident given that the student is not a firstborn?
(e) What is the conditional probability that the student is a firstborn given that the student is confident?

11. Two cards are randomly selected from a deck of 52 playing cards. What is the conditional probability they are both aces given that they are of different suits?

12. In the U.S. Presidential election of 1984, 68.3 percent of those citizens eligible to vote registered; and of those registering to vote, 59.9 percent actually voted. Suppose a citizen eligible to vote is randomly chosen.

(a) What is the probability that this person voted?
(b) What is the conditional probability that this person registered given that he or she did not vote?

*Note*: In order to vote, first you must register.

13. There are 30 psychiatrists and 24 psychologists attending a certain conference. Two of these 54 people are randomly chosen to take part in a panel discussion. What is the probability that at least one psychologist is chosen? (*Hint*: You may want to first determine the probability of the complementary event that no psychologists are chosen.)

14. A child has 12 socks in a drawer; 5 are red, 4 are blue, and 3 are green. If 2 socks are chosen at random, find the probability that they are
   (a) Both red
   (b) Both blue
   (c) Both green
   (d) The same color
15. Two cards are chosen at random from a deck of 52 playing cards. Find the probability that
   (a) Neither one is a spade
   (b) At least one is a spade
   (c) Both are spades
16. There are $n$ socks in a drawer, of which 3 are red. Suppose that if 2 socks are randomly chosen, then the probability that they are both red is 1/2. Find $n$.
*17. Suppose the occurrence of $A$ makes it more likely that $B$ will occur. In that case, show that the occurrence of $B$ makes it more likely that $A$ will occur.

   That is, show that if

$$P(B|A) > P(B)$$

   then it is also true that

$$P(A|B) > P(A)$$

18. Two fair dice are rolled.
   (a) What is the probability that at least one of the dice lands on 6?
   (b) What is the conditional probability that at least one of the dice lands on 6 given that their sum is 9?
   (c) What is the conditional probability that at least one of the dice lands on 6 given that their sum is 10?
19. There is a 40 percent chance that a particular company will set up a new branch office in Chicago. If it does, there is a 60 percent chance that Norris will be named the manager. What is the probability that Norris will be named the manager of a new Chicago office?
20. According to a geologist, the probability that a certain plot of land contains oil is 0.7. Moreover, if oil is present, then the probability of hitting it with the first well is 0.5. What is the probability that the first well hits oil?
21. At a certain hospital, the probability that a patient dies on the operating table during open heart surgery is 0.20. A patient who survives the operating table has a 15 percent chance of dying in the hospital from the aftereffects of the operation. What fraction of open-heart surgery patients survive both the operation and its aftereffects?

**22.** An urn initially contains 4 white and 6 black balls. Each time a ball is drawn, its color is noted and then it is replaced in the urn along with another ball of the same color. What is the probability that the first 2 balls drawn are black?

**23.** Reconsider Prob. 7.

**(a)** If a member is randomly chosen, what is the probability that the chosen person plays either chess or bridge?

**(b)** How many members play neither chess nor bridge?

If two members are randomly chosen, find the probability that

**(c)** They both play chess.

**(d)** Neither one plays chess or bridge.

**(e)** Both play either chess or bridge.

**24.** Consider Table 4.4 as given in Example 4.11. Suppose that a female student and a male student are independently and randomly chosen.

**(a)** Find the probability that exactly one of them is over 30 years old.

**(b)** Given that exactly one of them is over 30 years old, find the conditional probability that the male is older.

**25.** José and Jim go duck hunting together. Suppose that José hits the target with probability 0.3 and Jim, independently, with probability 0.1. They both fire one shot at a duck.

**(a)** Given that exactly one shot hits the duck, what is the conditional probability that it is José's shot? That it is Jim's?

**(b)** Given that the duck is hit, what is the conditional probability that José hit it? That Jim hit it?

**26.** A couple has two children. Let $A$ denote the event that their older child is a girl, and let $B$ denote the event that their younger child is a boy. Assuming that all 4 possible outcomes are equally likely, show that $A$ and $B$ are independent.

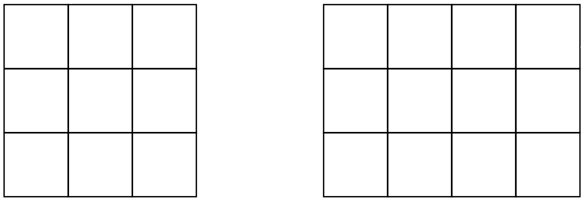
**27.** A simplified model for the movement of the price of a stock supposes that on each day the stock's price either moves up 1 unit with probability $p$ or moves down 1 unit with probability $1 - p$. The changes on different days are assumed to be independent. Suppose that for a certain stock $p$ is equal to 1/2. (Therefore, for instance, if the stock's price at the end of today is 100 units, then its price at the end of tomorrow will equally likely be either 101 or 99.)

**(a)** What is the probability that after 2 days the stock will be at its original price?

**(b)** What is the probability that after 3 days the stock's price will have increased by 1 unit?

**(c)** If after 3 days the stock's price has increased by 1 unit, what is the conditional probability that it went up on the first day?

28. A male New York resident is randomly selected. Which of the following pairs of events $A$ and $B$ can reasonably be assumed to be independent?
   (a) $A$: He is a journalist.
   $B$: He has brown eyes.
   (b) $A$: He had a headache yesterday.
   $B$: He was in an accident yesterday.
   (c) $A$: He is wearing a white shirt.
   $B$: He is late to work.
29. A coin that is equally likely to land on heads or on tails is successively flipped until tails appear. Assuming that the successive flips are independent, what is the probability that the coin will have to be tossed at least 5 times? (*Hint*: Fill in the missing word in the following sentence. The coin will have to be tossed at least 5 times if the first ______ flips all land on heads.)
30. A die is thrown until a 5 appears. Assuming that the die is equally likely to land on any of its six sides and that the successive throws are independent, what is the probability that it takes more than six throws?
31. Suppose that the probability of getting a busy signal when you call a friend is 0.1. Would it be reasonable to suppose that the probability of getting successive busy signals when you call two friends, one right after the other, is 0.01? If not, can you think of a condition under which this would be a reasonable supposition?
32. Two fields contain 9 and 12 plots of land, as shown here.



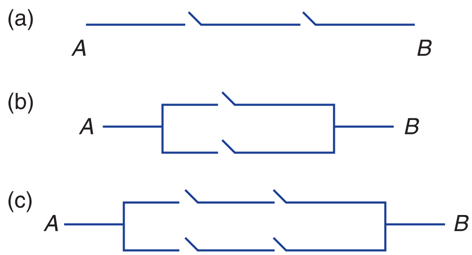
   For an agricultural experiment, one plot from each field will be selected at random, independently of each other.
   (a) What is the probability that both selected plots are corner plots?
   (b) What is the probability that neither plot is a corner plot?
   (c) What is the probability at least one of the selected plots is a corner plot?
33. A card is to be randomly selected from a deck of 52 playing cards. Let $A$ be the event that the card selected is an ace, and let $B$ be the event that the card is a spade. Show that $A$ and $B$ are independent.

**34.** A pair of fair dice is rolled. Let $A$ be the event that the sum of the dice is equal to 7. Is $A$ independent of the event that the first die lands on 1? on 2? on 4? on 5? on 6?

**35.** What is the probability that two strangers have the same birthday?

**36.** A U.S. publication reported that 4.78 percent of all deaths in 1988 were caused by accidents. What is the probability that three randomly chosen deaths were all due to accidents?

**37.** Each relay in the following circuits will close with probability 0.8. If all relays function independently, what is the probability that a current flows between $A$ and $B$ for the respective circuits? (The circuit in part (a) of the figure, which needs both of its relays to close, is called a *series circuit*. The circuit in part (b), which needs at least one of its relays to close, is called a *parallel circuit*.)




*Hint*: For parts (b) and (c) use the addition rule.

**38.** An urn contains 5 white and 5 black balls. Two balls are randomly selected from this urn. Let $A$ be the event that the first ball is white and $B$ be the event that the second ball is black. Are $A$ and $B$ independent events? Explain your reasoning.

**39.** Suppose in Prob. 38 that the first ball is returned to the urn before the second is selected. Will $A$ and $B$ be independent in this case? Again, explain your answer.

**40.** Suppose that each person who is asked whether she or he is in favor of a certain proposition will answer yes with probability 0.7 and no with probability 0.3. Assume that the answers given by different people are independent. Of the next four people asked, find the probability that

(a) All give the same answer.
(b) The first two answer no and the final two yes.
(c) At least one answers no.
(d) Exactly three answer yes.
(e) At least one answers yes.

**41.** The following data, obtained from the U.S. National Oceanic and Atmospheric Administration, give the average number of days with precipitation of 0.01 inch or more in different months for the cities of Mobile, Phoenix, and Los Angeles.

Average Number of Days with Precipitation of 0.01 Inch or More

| City | January | April | July |
|---|---|---|---|
| Mobile | 11 | 7 | 16 |
| Phoenix | 4 | 2 | 4 |
| Los Angeles | 6 | 3 | 1 |

Suppose that in the coming year you are planning to visit Phoenix on January 4, Los Angeles on April 10, and Mobile on July 15.

(a) What is the probability that it will rain on all three trips?
(b) What is the probability it will be dry on all three trips?
(c) What is the probability that you encounter rain in Phoenix and Mobile but not in Los Angeles?
(d) What is the probability that you encounter rain in Mobile and Los Angeles but not in Phoenix?
(e) What is the probability that you encounter rain in Phoenix and Los Angeles but not in Mobile?
(f) What is the probability that it rains in exactly two of your three trips?

42. Each computer chip produced by machine $A$ is defective with probability 0.10, whereas each chip produced by machine $B$ is defective with probability 0.05. If one chip is taken from machine $A$ and one from machine $B$, find the probability (assuming independence) that

(a) Both chips are defective.
(b) Both are not defective.
(c) Exactly one of them is defective.

If it happens that exactly one of the two chips is defective, find the probability that it was the one from

(d) Machine $A$
(e) Machine $B$

43. Genetic testing has enabled parents to determine if their children are at risk for cystic fibrosis (CF), a degenerative neural disease. A child who receives a CF gene from both parents will develop the disease by his or her teenage years and will not live to adulthood. A child who receives either zero or one CF gene will not develop the disease; however, if she or he does receive one CF gene, it may be passed on to subsequent offspring. If an individual has a CF gene, then each of his or her children will receive that gene with probability 1/2.

(a) If both parents possess the CF gene, what is the probability that their child will develop cystic fibrosis?
(b) What is the probability that a 25-year-old person who does not have CF but whose sibling does, carries the gene?

# *4.6 BAYES' THEOREM

For any two events $A$ and $B$, we have the following representation for $A$:

$$A = (A \cap B) \cup (A \cap B^c)$$

That this is valid is easily seen by noting that for an outcome to be in $A$, either it must be in both $A$ and $B$ or it must be in $A$ but not in $B$ (see Fig. 4.7). Since $A \cap B$ and $A \cap B^c$ are mutually exclusive (why?), we have by Property 3 (see Sec. 4.3)

$$P(A) = P(A \cap B) + P(A \cap B^c)$$

Since

$$P(A \cap B) = P(A|B)P(B) \quad \text{and} \quad P(A \cap B^c) = P(A|B^c)P(B^c)$$

we have thus shown the following equality:

$$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c) \tag{4.1}$$

This equality states that the probability of event $A$ is a weighted average of the conditional probability of $A$ given that $B$ occurs and the conditional probability of $A$ given that $B$ does not occur; each conditional probability is weighted by the probability of the event on which it is conditioned. It is a very useful formula for it often enables us to compute the probability of an event $A$ by first "conditioning" on whether a second event $B$ occurs.

Before illustrating the use of Eq. (4.1), we first consider the problem of how to reevaluate an initial probability in light of additional evidence. Suppose there is a certain hypothesis under consideration; let $H$ denote the event that the hypothesis is true and $P(H)$ the probability that the hypothesis is true. Now, suppose that additional evidence, call it $E$, concerning this hypothesis becomes available. We thus want to determine $P(H|E)$, the conditional probability that the

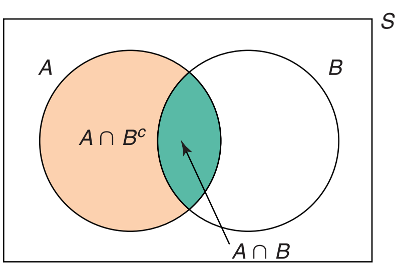



**FIGURE 4.7**
$A = (A \cap B) \cup (A \cup B^c)$

hypothesis is true given the new evidence $E$. Now, by the definition of conditional probability,

$$P(H|E) = \frac{P(H \cap E)}{P(E)} = \frac{P(E|H)P(H)}{P(E)}$$

By making use of Eq. (4.1), we can compute $P(E)$ by conditioning on whether the hypothesis is true. This yields the following identity, known as *Bayes' theorem*.

### Bayes' Theorem

$$P(H|E) = \frac{P(E|H)P(H)}{P(E|H)P(H) + P(E|H^c)P(H^c)}$$

## ■ Example 4.16

An insurance company believes that people can be divided into two classes—those who are prone to have accidents and those who are not. The data indicate that an accident-prone person will have an accident in a 1-year period with probability 0.1; the probability for all others is 0.05. Suppose that the probability is 0.2 that a new policyholder is accident-prone.

**(a)** What is the probability that a new policyholder will have an accident in the first year?

**(b)** If a new policyholder has an accident in the first year, what is the probability that he or she is accident-prone?

### Solution

Let $H$ be the event that the new policyholder is accident-prone, and let $A$ denote the event that she or he has an accident in the first year. We can compute $P(A)$ by conditioning on whether the person is accident-prone:

$$\begin{aligned} P(A) &= P(A|H)P(H) + P(A|H^c)P(H^c) \\ &= (0.1)(0.2) + (0.05)(0.8) = 0.06 \end{aligned}$$

Therefore, there is a 6 percent chance that a new policyholder will have an accident in the first year.

We compute $P(H|A)$ as follows:

$$\begin{aligned} P(H|A) &= \frac{P(H \cap A)}{P(A)} \\ &= \frac{P(A|H)P(H)}{P(A)} \\ &= \frac{(0.1)(0.2)}{0.06} = \frac{1}{3} \end{aligned}$$

Therefore, given that a new policyholder has an accident in the first year, the conditional probability that the policyholder is prone to accidents is 1/3. ■

## ■ Example 4.17

A blood test is 99 percent effective in detecting a certain disease when the disease is present. However, the test also yields a *false-positive* result for 2 percent of the healthy patients tested. (That is, if a healthy person is tested, then with probability 0.02 the test will say that this person has the disease.) Suppose 0.5 percent of the population has the disease. Find the conditional probability that a randomly tested individual actually has the disease given that his or her test result is positive.

### Solution

Let $D$ denote the event that the person has the disease, and let $E$ be the event that the test is positive. We want to determine $P(D|E)$, which can be accomplished by using Bayes' theorem as follows:

$$\begin{aligned} P(D|E) &= \frac{P(E|D)P(D)}{P(E|D)P(D) + P(E|D^c)P(D^c)} \\ &= \frac{(0.99)(0.005)}{(0.99)(0.005) + (0.02)(0.995)} = 0.199 \end{aligned}$$

Thus, there is approximately a 20 percent chance that a randomly chosen person from the population who tests positive actually has the disease. (The reason why it is so low is that the chance that a randomly chosen person is free of the disease yet tests positive is greater than the chance that the person has the disease and tests positive.) ■

## PROBLEMS

1. There are two coins on a table. When both are flipped, one coin lands on heads with probability 0.5 while the other lands on heads with probability 0.6. A coin is randomly selected from the table and flipped.
   (a) What is the probability it lands on heads?
   (b) Given that it lands on tails, what is the conditional probability that it was the fair coin (that is, the one equally likely to land heads or tails)?
2. Suppose that when answering a question on a multiple-choice test, a student either knows the answer or guesses at it. If he guesses at the

answer, then he will be correct with probability 1/5. If the probability that a student knows the answer is 0.6, what is the conditional probability that the student knew the answer given that he answered it correctly?

3. The inspector in charge of a criminal investigation is 60 percent certain of the guilt of a certain suspect. A new piece of evidence proving that the criminal was left-handed has just been discovered. Whereas the inspector knows that 18 percent of the population is left-handed, she is waiting to find out whether the suspect is left-handed.
   (a) What is the probability that the suspect is left-handed?
   (b) If the suspect turns out to be left-handed, what is the probability that the suspect is guilty?
4. Urn 1 contains 4 red and 3 blue balls, and urn 2 contains 2 red and 2 blue balls. A ball is randomly selected from urn 1 and placed in urn 2. A ball is then drawn from urn 2.
   (a) What is the probability that the ball drawn from urn 2 is red?
   (b) What is the conditional probability that the ball drawn from urn 1 is red given that a blue ball is drawn from urn 2?
5. Consider a diagnostic test that is 97 percent accurate on both those who have and those who do not have the disease. (That is, if a person has the disease, then with probability 0.97 the diagnosis will be positive; and if the person does not have the disease, then with probability 0.97 the diagnosis will be negative.) Suppose 2 percent of the population has the disease. What is the conditional probability that a randomly selected member of the population has the disease if that person's diagnosis was positive?
6. There are three cards in a hat. One is colored red on both sides, one is black on both sides, and one is red on one side and black on the other. The cards are thoroughly mixed in the hat, and one card is drawn and placed on a table. If the side facing up is red, what is the conditional probability that the other side is black?
7. A total of 52 percent of voting-age residents of a certain city are Republicans, and the other 48 percent are Democrats. Of these residents, 64 percent of the Republicans and 42 percent of the Democrats are in favor of discontinuing affirmative action city hiring policies. A voting-age resident is randomly chosen.
   (a) What is the probability that the chosen person is in favor of discontinuing affirmative action city hiring policies?
   (b) If the person chosen is against discontinuing affirmative action hiring policies, what is the probability she or he is a Republican?
8. A person's eye color is determined by a single pair of genes. If both genes are blue-eyed genes, then the person will have blue eyes; if they are both brown-eyed genes, then the person will have brown eyes;

and if one gene is blue-eyed and the other is brown-eyed, then the person will have brown eyes. (Because of this latter fact we say that the brown-eyed gene is *dominant* over the blue-eyed one.) A newborn child independently receives one eye gene from each parent, and the gene that the child receives from a parent is equally likely to be either of the two eye genes of that parent. Suppose that Susan has blue eyes and both her parents have brown eyes.

(a) What is the eye gene pair of Susan's mother? of her father?

(b) Susan's brown-eyed sister is pregnant. If her sister's husband has blue eyes, what is the probability the baby will have blue eyes? *Hint*: What is the probability that Susan's sister has a blue-eyed gene?

9. Twelve percent of all U.S. households are in California. A total of 1.3 percent of all U.S. households earn over 250,000 dollars per year, while a total of 3.3 percent of all California households earn over 250,000 dollars per year. A U.S. household is randomly chosen.

(a) What percentage of non-California households earn over 250,000 dollars per year?

(b) Given that the chosen household earns over 250,000 dollars per year, what is the probability it is a California household?

## *4.7 COUNTING PRINCIPLES

As seen in Sec. 4.4, we often determine probabilities by counting the number of different outcomes in a specified event. The key to doing this effectively is to make use of the following rule, known as the *basic principle of counting*.

### Basic Principle of Counting

Suppose an experiment consists of two parts. If part 1 can result in any of $n$ possible outcomes and if for each outcome of part 1 there are $m$ possible outcomes of part 2, then there is a total of $nm$ possible outcomes of the experiment.

That the basic principle is valid can easily be seen by enumerating all possible outcomes of the experiment:

$$\begin{array}{llll}
(1,1), & (1,2), & \ldots, & (1,m) \\
(2,1), & (2,2), & \ldots, & (2,m) \\
\vdots & & & \\
(n,1), & (n,2), & \ldots, & (n,m)
\end{array}$$

where we say that the outcome of the experiment is $(i, j)$ if part 1 of the experiment resulted in its $i$th possible outcome and part 2 then resulted in its $j$th possible outcome. Since the preceding display contains $n$ rows, each of which consists of $m$ outcomes, it follows that there is total of $m + m \cdots + m = nm$ outcomes.

## ■ Example 4.18

One man and one woman are to be selected from a group consisting of 12 women and 8 men. How many different choices are possible?

**Solution**

By regarding the choice of the woman as the first part of the experiment and the choice of the man as the second, we see from the basic principle that there are $12 \cdot 8 = 96$ possible outcomes. ■

## ■ Example 4.19

Two people are to be selected from a group that consists of 10 married couples. How many different choices are possible? If each choice is equally likely, what is the probability that the two people selected are married to each other?

**Solution**

Since the first person selected is any of the 20 people and the next one is then any of the remaining 19, it follows from the basic principle that there are $20 \cdot 19 = 380$ possible outcomes. Now, for each married couple there are 2 outcomes that result in that couple's selection. Namely, the husband could be the first person selected and the wife the second, or the reverse. Thus, there are $2 \cdot 10 = 20$ different outcomes that result in a married couple's selection. Hence, assuming that that all possible outcomes are equally likely, it follows that the probability that the people selected are married to each other is $20/380 = 1/19$. ■

When the experiment consists of more than two parts, the basic principle can be generalized as follows.

### Generalized Basic Principle of Counting

Suppose an experiment consists of $r$ parts. Suppose there are $n_1$ possible outcomes of part 1 and then $n_2$ possible outcomes of part 2 and then $n_3$ possible outcomes of part 3, and so on. Then there is a total of $n_1 \cdot n_2 \cdots n_r$ possible outcomes of the experiment.

As an application of the generalized principle, suppose that we want to determine the number of different ways that the three letters $a$, $b$, $c$ can be arranged in a linear order. By direct enumeration we can see that there are 6 possible arrangements:

$$abc, \ acb, \ bac, \ bca, \ cab, \ cba$$

This result could also have been obtained by using the generalized basic principle of counting. That is, there are 3 choices for the first element in the ordering, there are then 2 choices for the second, and then 1 choice for the third position. Hence, there are $3 \cdot 2 \cdot 1 = 6$ possible outcomes.

Suppose now that we want to determine the number of different arrangements of $n$ objects. By the same reasoning, we see that there is a total of

$$n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$$

different arrangements. Each of these arrangements is called a *permutation*. It is convenient to introduce for the foregoing expression the notation $n!$, which is read "$n$ factorial." That is,

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$$

Thus, for instance,

$$\begin{aligned}
1! &= 1 \\
2! &= 2 \cdot 1 = 2 \\
3! &= 3 \cdot 2 \cdot 1 = 6 \\
4! &= 4 \cdot 3 \cdot 2 \cdot 1 = 24
\end{aligned}$$

and so on. In addition, it is convenient to define $0!$ to be equal to 1.

## ■ Example 4.20

If four people are in a room, what is the probability that no two of them celebrate their birthday on the same day of the year?

### Solution

Since each person can celebrate his or her birthday on any of the 365 days of the year, it follows from the generalized basic principle that there is a total of

$$365 \cdot 365 \cdot 365 \cdot 365 = (365)^4$$

possible outcomes. (We are ignoring the possibility that someone was born on February 29.) Let us now determine the number of outcomes in which no two individuals have the same birthday. This will occur if the birthday of the first person is any of the 365 days, the birthday of the second person is then any one of the remaining 364 days, the birthday of the third person is then any one of the remaining 363 days, and the birthday of the final person is then any of the remaining 362 days. Thus, by the generalized basic principle of counting, we see that there is a total of

$$365 \cdot 364 \cdot 363 \cdot 362$$

different outcomes in which all the 4 birthdays are different. Hence, if we assume that all of the possible outcomes are equally likely, we obtain the probability that no two people have the same birthday:

$$\frac{365 \cdot 364 \cdot 363 \cdot 362}{365 \cdot 365 \cdot 365 \cdot 365} = 0.983644$$

The same approach can be used to find the probability that a group of $n$ people will all have different birthdays, for any integer $n$. It is an interesting fact that when $n = 23$, this probability is less than 1/2. That is, if there are 23 people in a room, then it is more likely than not that at least two of them will celebrate the same birthday. ■

Suppose now that we are interested in choosing 3 of the 5 items $a, b, c, d, e$. How many different choices are possible? To answer this we can reason as follows. Since there are 5 possible choices for the first item and then 4 possible choices for the next one and finally 3 possible choices for the final item, it follows that there are $5 \cdot 4 \cdot 3$ possible choices when the order in which the items are chosen is considered relevant. However, in this set of ordered choices, every group of three items will appear 3! times. For instance, consider the group consisting of the items $a$, $b$, and $c$. Each of the permutations

$$abc,\ acb,\ bac,\ bca,\ cab,\ cba$$

of these three elements will be included in the set of possible choices when the order of selection is considered relevant. Therefore it follows that the number of different groups of size 3 that can be formed from 5 items, when the order of selection is not considered relevant, is

$$\frac{5 \cdot 4 \cdot 3}{3 \cdot 2 \cdot 1} = 10$$

Suppose now that we are interested in determining the number of different groups of size $r$ that can be chosen from a set of $n$ elements. By the same reasoning as before, it follows that there are

$$\frac{n \cdot (n-1) \cdots (n-r+1)}{r!}$$

different groups. Since, $n(n-1) \cdots (n-r+1)$ can be written as $n!/(n-r)!$, we can express this number as $n!/[(n-r)!\, r!\,]$.

### Notation and Terminology

Define $\binom{n}{r}$, for $r \leq n$, by

$$\binom{n}{r} = \frac{n!}{(n-r)!\,r!} = \frac{n(n-1)\cdots(n-r+1)}{r!}$$

Call $\binom{n}{r}$ the number of combinations of $n$ things taken $r$ at a time; it represents the number of different groups of size $r$ that can be selected from a set of size $n$ when the order of selection is not of importance.

## Example 4.21

**(a)** How many different groups of size 2 can be selected from the items $a$, $b$, $c$?
**(b)** How many different groups of size 2 can be chosen from a set of 6 people?
**(c)** How many different groups of size 3 can be chosen from a set of 6 people?

### Solution

**(a)** There are $\binom{3}{2} = \frac{3\cdot 2}{2\cdot 1} = 3$ different groups of 2 items that can be selected from the items $a, b, c$: $a$ and $b$, $a$ and $c$, and $b$ and $c$.

**(b)** and **(c)** From a set of 6 people there are

$$\binom{6}{2} = \frac{6\cdot 5}{2\cdot 1} = 15$$

different groups of size 2 that can be chosen, and

$$\binom{6}{3} = \frac{6\cdot 5\cdot 4}{3\cdot 2\cdot 1} = 20$$

different groups of size 3. ■

## Example 4.22

A random sample of size 3 is to be selected from a set of 10 items. What is the probability that a prespecified item will be selected?

### Solution

There are $\binom{10}{3}$ different groups that can be chosen. The number of different groups that contain the specified item is equal to the number of choices of the additional 2 items from the remaining 9 items after the specified item is chosen. Thus, there are $\binom{9}{2}$ different groups that contain the given item. So, assuming that a random sample is one in which each group is equally likely to be selected, we see that the desired probability that a given item is selected is

$$\frac{\binom{9}{2}}{\binom{10}{3}} = \frac{9 \cdot 8}{2 \cdot 1} \div \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = \frac{9 \cdot 8 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 10 \cdot 9 \cdot 8} = \frac{3}{10}$$

That is, there is a 3-in-10 chance that a given item will be selected. ■

## ■ Example 4.23

A committee of 4 people is to be selected from a group of 5 men and 7 women. If the selection is made randomly, what is the probability the committee will consist of 2 men and 2 women?

### Solution

We will assume that "the selection is made randomly" means that each of the $\binom{12}{4}$ possible combinations is equally likely to be chosen. Because there are $\binom{5}{2}$ possible choices of 2 men and $\binom{7}{2}$ possible choices of 2 women, it follows from the basic principle of counting that there are $\binom{5}{2}\binom{7}{2}$ possible outcomes that contain 2 men and 2 women. Therefore, the desired probability is

$$\frac{\binom{5}{2}\binom{7}{2}}{\binom{12}{4}} = \frac{5 \cdot 4 \cdot 7 \cdot 6 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 2 \cdot 1 \cdot 12 \cdot 11 \cdot 10 \cdot 9} = \frac{14}{33}$$ ■

It follows from the formula

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

that

$$\binom{n}{r} = \binom{n}{n-r}$$

## ■ Example 4.24

Compare $\binom{8}{5}$ and $\binom{12}{10}$.

### Solution

$$\binom{8}{5} = \binom{8}{3} = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} = 56$$

$$\binom{12}{10} = \binom{12}{2} = \frac{12 \cdot 11}{2 \cdot 1} = 66$$ ■

The identity $\binom{n}{r} = \binom{n}{n-r}$ can be seen by a "counting argument." Suppose we want to select $r$ items from a set of $n$ items. Since this can be done either by directly specifying the $r$ items to be selected or equivalently by specifying the $n-r$ items that are not to be selected, it follows that the number of choices of $r$ items is equal to the number of choices of $n-r$ items. For instance, any choice of 8 of the first 10 integers corresponds to a choice of the 2 integers that are not chosen.

## ■ Example 4.25

Suppose that $n+m$ digits, $n$ of which are equal to 1 and $m$ of which are equal to 0, are to be arranged in a linear order. How many different arrangements are possible? For instance, if $n=2$ and $m=1$, then there are 3 possible arrangements:

$$1, 1, 0 \quad 1, 0, 1 \quad 0, 1, 1$$

**Solution**

Each arrangement will have a digit in position 1, another digit in position 2, another in position 3, . . . , and finally a digit in position $n+m$. Each arrangement can therefore be described by specifying the $n$ positions that contain the digit 1. That is, each different choice of $n$ of the $n+m$ positions will result in a different arrangement. Therefore, there are $\binom{n+m}{n}$ different arrangements.

Of course, we can also describe an arrangement by specifying the $m$ positions that contain the digit 0. This results in the solution $\binom{n+m}{m}$, which is equal to $\binom{n+m}{n}$. ■

## PROBLEMS

1. How many different 7-place license plates are possible when the first 3 places are for letters and the last 4 are for digits?
2. How many different batting orders are possible for a baseball team consisting of 9 players?
3. $9! = 362{,}880$. What is the value of $10!$?
4. There is a certain type of combination lock that has a dial that can be stopped at any of the numbers 1 through 36. To open the lock you have to twirl the dial clockwise until a certain number is reached, then twirl it counterclockwise until a second number is reached, and then twirl it clockwise until a third number is reached. If you have forgotten the three numbers (which need not be different from each other), how many different possibilities might you have to try before the lock opens?
5. Telephone area codes in the United States and Canada consist of a sequence of three digits. The first digit is an integer between 2 and 9; the second digit is either 0 or 1; the third digit is any integer between

1 and 9. How many area codes are possible? How many area codes starting with a 4 are possible?

6. A well-known nursery tale starts as follows:

   As I was going to St. Ives
   I met a man with 7 wives.
   Each wife had 7 sacks,
   Each sack had 7 cats,
   Each cat had 7 kittens.

   How many kittens did our traveler meet?

7. (a) If four workers are to be assigned to four jobs how many different assignments are possible?
   (b) How many assignments are possible if workers 1 and 2 are both qualified only for jobs 1 and 2 and workers 3 and 4 are both qualified only for jobs 3 and 4?

8. Use the formula

$$\binom{n}{r} = \frac{n!}{(n-r)!\,r!}$$

   to find $\binom{n}{0}$, where $n$ is a positive integer. Recall that 0! is defined to equal 1. Since $\binom{n}{r}$ is supposed to equal the number of groups of size $r$ that can be formed from a set of $n$ objects, do you think the answer makes sense?

9. Calculate the following:

$$\binom{8}{4}, \binom{9}{2}, \binom{7}{6}, \binom{10}{3}$$

10. Consider a group of 20 people. If everyone shakes hands with everyone else, how many handshakes take place?

11. A student must choose four courses from among French, Spanish, History, Physics, and English Literature.
    (a) How many different choices are possible?
    (b) If the student chooses randomly, what is the probability that both French and Spanish are chosen?

12. A delivery company has 10 trucks, of which 3 have faulty brakes. If an inspector randomly chooses 2 of the trucks for a brake check, what is the probability that none of the trucks with faulty brakes are chosen?

13. A company regularly receives large shipments of computer chips. The company's policy is to randomly select and test 10 of the chips. If 2 or more of these are found to be defective, then the shipment is returned; otherwise the shipment is accepted. Suppose that a shipment of 100 chips contains 14 that are defective.

(a) What is the probability that the sample inspected has no defective chips?
(b) What is the probability that the sample inspected has 1 defective chip?
(c) What is the probability this shipment will be rejected?

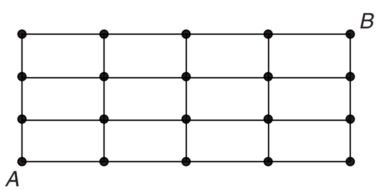
14. In a state lottery, a player must choose 8 of the numbers from 1 to 40. The Lottery Commission then performs an experiment that selects 8 of these 40 numbers. Assuming that the choice of the Lottery Commission is equally likely to be any of the $\binom{40}{8}$ combinations, what is the probability that a player has
(a) All 8 of the selected numbers?
(b) Seven of the selected numbers?
(c) At least 6 of the selected numbers?

15. An approved jury list contains 22 men and 18 women. What is the probability that a random selection of 12 of these people will result in a jury with
(a) Six women and 6 men?
(b) Eight women and 4 men?
(c) At least 10 men?

16. The second Earl of Yarborough is reported to have bet at odds of 1000 to 1 that a bridge hand of 13 cards would contain at least one card that is 10 or higher. (By 10 or higher we mean that it is either ten, jack, queen, king, or ace.) Nowadays, we call a hand that has no cards higher than 9 a *Yarborough*. What is the probability that a randomly selected bridge hand is a Yarborough?

17. An instructor gives her class a set of 10 problems and tells the class that the final exam (in 1 week) will consist of a random selection of 5 of the problems. If a student has figured out how to do 7 of the problems by the time of the exam, what is the probability he or she will correctly answer
(a) All 5 problems?
(b) At least 4 of the problems?

18. Consider the grid of points shown here.

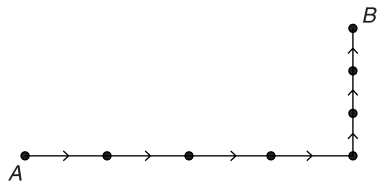



Suppose that starting at the point labeled $A$ you can at each move either go one step up or one step to the right. You keep doing this until the point labeled $B$ is reached. How many different paths from

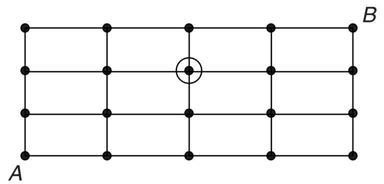
$A$ to $B$ are possible? *Hint*: To go from $A$ to $B$ you have to go 4 steps to the right and 3 steps up. Indeed, any path can be specified by an arrangement of 4 $r$'s and 3 $u$'s. For instance, the arrangement

$$r,\ r,\ r,\ r,\ u,\ u,\ u$$

specifies the following path:




19. Suppose, in Prob. 18, that a path from $A$ to $B$ is randomly chosen. What is the probability it goes through the point circled in the following grid? (*Hint*: How many paths are there from $A$ to the circled point? How many from the circled point to $B$?)




## KEY TERMS

**Experiment**: Any process that produces an observation.

**Outcome**: The observation produced by an experiment.

**Sample space**: The set of all possible outcomes of an experiment.

**Event**: Any set of outcomes of the experiment. An event is a subset of sample space $S$. The event is said to occur if the outcome of the experiment is contained in it.

**Union of events**: The union of events $A$ and $B$, denoted by $A \cup B$, consists of all outcomes that are in $A$ or in $B$ or in both $A$ and $B$.

**Intersection of events**: The intersection of events $A$ and $B$, denoted by $A \cap B$, consists of all outcomes that are in both $A$ and $B$.

**Complement of an event**: The complement of event $A$, denoted by $A^c$, consists of all outcomes that are not in $A$.

**Mutually exclusive or disjoint**: Events are mutually exclusive or disjoint if they cannot occur simultaneously.

**Null event**: The event containing no outcomes. It is the complement of sample space $S$.

**Venn diagram**: A graphical representation of events.

**Probability of an event**: The probability of event $A$, denoted by $P(A)$, is the probability that the outcome of the experiment is contained in $A$.

**Addition rule of probability**: The formula

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

**Conditional probability**: The probability of one event given the information that a second event has occurred. We denote the conditional probability of $B$ given that $A$ has occurred by $P(B|A)$.

**Multiplication rule**: The formula

$$P(A \cap B) = P(A)P(B|A)$$

**Independent**: Two events are said to be independent if knowing whether a specific one has occurred does not change the probability that the other occurs.

## SUMMARY

Let $S$ denote all possible outcomes of an experiment whose outcome is not predictable in advance. $S$ is called the *sample space* of the experiment.

Any set of outcomes, or equivalently any subset of $S$, is called an *event*. If $A$ and $B$ are events, then $A \cup B$, called the *union* of $A$ and $B$, is the event consisting of all outcomes that are in $A$ or in $B$ or in both $A$ and $B$. The event $A \cap B$ is called the *intersection* of $A$ and $B$. It consists of all outcomes that are in both $A$ and $B$.

For any event $A$, we define the event $A^c$, called the *complement* of $A$, to consist of all outcomes in $S$ that are not in $A$. The event $S^c$, which contains no outcomes, is designated by $\varnothing$. If $A \cap B = \varnothing$, meaning that $A$ and $B$ have no outcomes in common, then we say that $A$ and $B$ are *disjoint* (also called *mutually exclusive*).

We suppose that for every event $A$ there is a number $P(A)$, called the *probability of* $A$. These probabilities satisfy the following three properties.

PROPERTY 1: $0 \leq P(A) \leq 1$

PROPERTY 2: $P(S) = 1$

PROPERTY 3: $P(A \cup B) = P(A) + P(B) \quad \text{when } A \cap B = \varnothing$

The quantity $P(A)$ represents the probability that the outcome of the experiment is in $A$. If so, we say that $A$ occurs.

The identity

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

is called the *addition rule of probability*.

We sometimes assume that all the outcomes of an experiment are equally likely. Under this assumption, it can be shown that

$$P(A) = \frac{\text{Numbers of outcomes in } A}{\text{Numbers of outcomes in } S}$$

The conditional probability of $B$ given that $A$ has occurred is denoted by $P(B|A)$ and is given by the following equation:

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

Multiplying both sides of this equation by $P(A)$ gives the following identity, known as the *multiplication rule*:

$$P(A \cap B) = P(A)P(B|A)$$

If

$$P(A \cap B) = P(A)P(B)$$

then we say that events $A$ and $B$ are *independent*. If $A$ and $B$ are independent, then the probability that one of them will occur is unchanged by information as to whether the other has occurred.

## REVIEW PROBLEMS

1. Of 12 bottles in a case of wine, 3 are bad. Suppose 2 bottles are randomly chosen from the case. Find the probability that
   (a) The first bottle chosen is good.
   (b) The second bottle chosen is good.
   (c) Both bottles are good.
   (d) Both bottles are bad.
   (e) One is good, and one is bad.
2. A basketball player makes each of her foul shots with probability 0.8. Suppose she is fouled and is awarded two foul shots. Assuming that the results of different foul shots are independent, find the probability that she
   (a) Makes both shots
   (b) Misses both shots
   (c) Makes the second shot given that she missed the first
3. Suppose that a basketball player makes her first foul shot with probability 0.8. However, suppose that the probability that she makes her second shot depends on whether the first shot is successful. Specifically, suppose that if she is successful on her first shot, then her second will be successful with probability 0.85, whereas if she misses her first shot, then the second will be successful with probability 0.7. Find the probability that she
   (a) Makes both shots
   (b) Misses both shots
   (c) Makes the first but misses the second shot
4. Of the registered voters in a certain community 54 percent are women and 46 percent are men. Sixty-eight percent of the registered women voters and 62 percent of the registered men voters voted in the last local election. If a registered voter from this community is randomly chosen, find the probability that this person is
   (a) A woman who voted in the last election
   (b) A man who did not vote in the last election
   (c) What is the conditional probability that this person is a man given that this person voted in the last election?

5. A kindergarten class consists of 24 students—13 girls and 11 boys. Each day one of these children is chosen as "student of the day." The selection is made as follows. At the beginning of the school year, the names of the children are written on slips of paper, which are then put in a large urn. On the first day of school, the urn is shaken and a name is chosen to be student of the day. The next day this process is repeated with the remaining 23 slips of paper, and so on. When each student has been selected once (which occurs on day 24), the process is repeated.
   (a) What is the probability that the first selection is a boy?
   (b) If the first selection is a boy, what is the probability that the second is a girl?
6. Two cards are chosen from an ordinary deck of 52 playing cards. Find the probability that
   (a) Both are aces.
   (b) Both are spades.
   (c) They are of different suits.
   (d) They are of different denominations.
7. What is the probability of the following outcomes when a fair coin is independently tossed 6 times?
   (a) H H H H H H
   (b) H T H T H T
   (c) T T H H T H
8. Find the probability of getting a perfect score just by guessing on a true/false test with
   (a) 2 questions
   (b) 3 questions
   (c) 10 questions
9. A cafeteria offers a three-course meal. One chooses a main course, a starch, and a dessert. The possible choices are given in this table.

| **Meal** | **Choices** |
|---|---|
| Main course | Chicken or roast beef |
| Starch course | Rice or potatoes |
| Dessert | Melon or ice cream or gelatin |

   Let the outcome of an experiment be the dinner selection of a person who makes one selection from each of the courses.
   (a) List all the outcomes in sample space *S*.
   (b) Suppose the person is allergic to rice and melon. List all the outcomes in the event corresponding to a choice that is acceptable to this person.

(c) If the person randomly chooses a dessert, what is the probability it is ice cream?
(d) If the person makes a random choice in each of the courses, what is the probability that chicken, rice, and melon are chosen?

10. The following is a breakdown by age and sex of the population of the United States. The numbers in each class are in units of 1 million.

| | Sex | |
|---|---|---|
| **Age** | **Females** | **Males** |
| Under 25 years | 48.8 | 50.4 |
| Over 25 years | 74.5 | 66.6 |

Suppose a person is chosen at random. Let $A$ be the event that the person is male and $B$ be the event that the person is under age 25. Find
(a) $P(A)$ and $P(A^c)$ (b) $P(B)$ and $P(B^c)$
(c) $P(A \cap B)$ (d) $P(A \cap B^c)$
(e) $P(A|B)$ (f) $P(B|A)$

11. A person has three keys of which only one fits a certain lock. If she tries the keys in a random order, find the probability that
(a) The successful key is the first one tried.
(b) The successful key is the second one tried.
(c) The successful key is the third one tried.
(d) The second key works given that the first one did not.

12. Two cards from an ordinary playing deck constitute a blackjack if one card is an ace and the other is a face card, where a face card is 10, jack, queen, or king. What is the probability that a random selection of two cards yields a blackjack? (*Hint*: You might try to compute the probability that the first card is an ace and the second a face card, and the probability that the first is a face card and the second an ace.)

13. A delivery company has 12 trucks, of which 4 have faulty brakes. If an inspector randomly chooses 2 of the trucks for a brake check, what is the probability that neither one has faulty brakes?

14. Suppose that $A$ and $B$ are independent events, and

$$P(A) = 0.8 \quad P(B^c) = 0.4$$

Find
(a) $P(A \cap B)$
(b) $P(A \cup B)$
(c) $P(B)$
(d) $P(A^c \cap B)$

15. A deck of 52 cards is shuffled, and the cards are turned face up, one at a time.
   (a) What is the probability that the first card turned up is the ace of spades?
   (b) Let $A$ denote the event that the first card turned up is not the ace of spades, and let $B$ denote the event that the second card turned up is the ace of spades. Therefore, $A \cap B$ is the event that the second card turned up is the ace of spades. Compute the probability of this event by using

$$P(A \cap B) = P(A)P(B|A)$$

   (c) Fill in the missing word in the following intuitive argument for the solution obtained in part (b): Since all orderings are equally likely, the second card turned up is ________ likely to be any of the 52 cards.
   (d) What is the probability that the 17th card turned up is the ace of spades?
16. Floppy disks go through a two-stage inspection procedure. Each disk is checked first manually and then electronically. If the disk is defective, then a manual inspection will spot the defect with probability 0.70. A defective disk that passes the manual inspection will be detected electronically with probability 0.80. What percentage of defective disks is not detected?
17. Assume that business conditions in any year can be classified as either good or bad. Suppose that if business is good this year, then with probability 0.7 it will also be good next year. Also suppose that if business is bad this year, then with probability 0.4 it will be good next year. The probability that business will be good this year is 0.6. Find the probabilities that the following statements are true.
   (a) Business conditions both this year and next will be good.
   (b) Business conditions will be good this year and bad next year.
   (c) Business conditions will be bad both years.
   (d) Business conditions will be good next year.
   (e) Given that business conditions are good next year, what is the conditional probability that they were good this year?
18. Both John and Maureen have one gene for blue eyes and one for brown eyes. A child of theirs will receive one gene for eye color from Maureen and one from John. The gene received from each parent is equally likely to be either of the parent's two genes. Also, the gene received from John is independent of the one received from Maureen. If a child receives a blue gene from both John and Maureen, then that child will have blue eyes; otherwise, it will have brown eyes. Maureen and John have two children.

(a) What is the probability that their older child has blue eyes?
(b) What is the probability that the older child has blue and the younger has brown eyes?
(c) What is the probability that the older has brown and the younger has blue eyes?
(d) What is the probability that one child has blue eyes and the other has brown eyes?
(e) What is the probability they both have blue eyes?
(f) What is the probability they both have brown eyes?

19. It is estimated that for the U.S. adult population as a whole, 55 percent are above ideal weight, 20 percent have high blood pressure, and 60 percent either are above ideal weight or have high blood pressure. Let $A$ be the event that a randomly chosen member of the population is above his or her ideal weight, and let $B$ be the event that this person has high blood pressure. Are $A$ and $B$ independent events?

20. A card is randomly selected from a deck of playing cards. Let $A$ be the event that the card is an ace, and let $B$ be the event that it is a spade. State whether $A$ and $B$ are independent, if the deck is
(a) A standard deck of 52 cards
(b) A standard deck, with all 13 hearts removed
(c) A standard deck, with the hearts from 2 through 9 removed

21. A total of 500 married working couples were polled about whether their annual salaries exceeded \$75,000. The following information was obtained:

| | Husband | |
|---|---|---|
| **Wife** | **Less than \$75,000** | **More than \$75,000** |
| **Less than \$75,000** | 212 | 198 |
| **More than \$75,000** | 36 | 54 |

Thus, for instance, in 36 couples, the wife earned over \$75,000 and the husband earned less than \$75,000. One of the couples is randomly chosen.
(a) What is the probability that the husband earns less than \$75,000?
(b) What is the conditional probability that the wife earns more than \$75,000 given that the husband earns more than this amount?
(c) What is the conditional probability that the wife earns more than \$75,000 given that the husband earns less than this amount?
(d) Are the salaries of the wife and husband independent?

22. The probability that a new car battery functions for over 10,000 miles is 0.8, the probability it functions for over 20,000 miles is 0.4, and the

probability it functions for over 30,000 miles is 0.1. If a new car battery is still working after 10,000 miles, find the conditional probability that

(a) Its total life will exceed 20,000 miles.
(b) Its additional life will exceed 20,000 miles.

23. Of the drivers who stop at a certain gas station, 90 percent purchase either gasoline or oil. A total of 86 percent purchase gasoline, and 8 percent purchase oil.
(a) What percentage of drivers purchase gasoline and oil?
(b) Find the conditional probability that a driver
(i) Purchases oil given that she or he purchases gasoline
(ii) Purchases gasoline given that he or she purchases oil
(iii) Suppose a driver stops at a gas station. Are the events that the driver purchases oil and that the driver purchases gasoline independent?

The following table gives participation rates at various artistic and leisure activities for individuals in different age categories. The data are for the year 2000, and the numbers represent the proportion of the population being considered who satisfied the stated criteria.

| | Attended at least once | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| **Characteristic** | **Jazz performance** | **Classical music performance** | **Opera performance** | **Musical plays** | **Plays** | **Ballet performance** | **Visited at least once—art museum or gallery** | **Read—novel, short stories, poetry, or plays** |
| Average | 10 | 13 | 3 | 17 | 12 | 4 | 22 | 56 |
| 18–24 years old | 14 | 11 | 2 | 15 | 11 | 4 | 22 | 57 |
| 25–34 years old | 15 | 12 | 2 | 16 | 12 | 5 | 26 | 59 |
| 35–44 years old | 10 | 16 | 4 | 21 | 14 | 6 | 27 | 62 |
| 45–54 years old | 8 | 15 | 4 | 20 | 13 | 3 | 22 | 57 |
| 55–64 years old | 5 | 11 | 3 | 18 | 10 | 4 | 19 | 50 |
| 65–74 years old | 3 | 13 | 3 | 13 | 10 | 4 | 16 | 50 |
| 75 years old and over | 1 | 10 | 1 | 8 | 7 | 2 | 10 | 48 |
| Male | 10 | 11 | 2 | 15 | 11 | 3 | 21 | 48 |
| Female | 9 | 14 | 3 | 19 | 12 | 5 | 23 | 63 |

*Source*: U.S. National Endowment for the Arts.

Problems 24 to 26 refer to the preceding table.

24. Suppose an 18- to 24-year-old is randomly chosen, as is a 35- to 44-year-old. Find the probability that
(a) Both attended a jazz performance.
(b) Exactly one attended a jazz performance.

(c) Given that exactly one of them attended a jazz performance, what is the conditional probability that it was the younger person who attended?

25. Suppose that a man and a woman are randomly chosen. Find the probability that
    (a) Exactly one attended a ballet performance.
    (b) At least one attended an opera.
    (c) Both attended a musical play.
26. Suppose an individual is randomly chosen. Is the given table informative enough for us to determine the probability that this individual attended both a jazz and a classical music performance? If not, under what assumption would we be able to determine this probability? Compute the probability under this assumption, and then tell whether you think it is a reasonable assumption in this situation.
27. There is a 60 percent chance that event $A$ will occur. If $A$ does not occur, then there is a 10 percent chance that $B$ will occur.
    (a) What is the probability that at least one of the events $A$ or $B$ occur?
    (b) If $A$ is the event that the Democratic candidate wins the presidential election in 2012 and $B$ is the event that there is a 6.2 or higher earthquake in Los Angeles sometime in 2013, what would you take as the probability that both $A$ and $B$ occur? What assumption are you making?
28. Suppose that distinct integer values are written on each of three cards. Suppose you are to be offered these cards in a random order. When you are offered a card you must immediately either accept it or reject it. If you accept a card, the process ends. If you reject a card then the next card (if a card still remains) is offered. If you reject the first two cards offered, then you must accept the final card.
    (a) If you plan to accept the first card offered, what is the probability that you will accept the highest valued card?
    (b) If you plan to reject the first card offered, and to then accept the second card if and only if its value is greater than the value of the first card, what is the probability that you will accept the highest valued card?
29. Let $A, B, C$ be events such that $P(A) = .2, P(B) = .3, P(C) = .4$. Find the probability that at least one of the events $A$ and $B$ occur if
    (a) $A$ and $B$ are mutually exclusive.
    (b) $A$ and $B$ are independent.

    Find the probability that all of the events $A, B, C$ occur if
    (c) $A, B, C$ are independent.
    (d) $A, B, C$ are mutually exclusive.
30. Two percent of women of age 45 who participate in routine screening have breast cancer. Ninety percent of those with breast cancer have

positive mammographies. Ten percent of the women who do not have breast cancer will also have positive mammographies. Given a woman has a positive mammography, what is the probability she has breast cancer?

31. Identical, also called monozygotic, twins form when a single fertilized egg splits into two genetically identical parts. The twins share the same DNA set and are thus always of the same sex. Fraternal, also called dizygotic, twins develop when two separate eggs are fertilized and implant in the uterus. The genetic connection is no more or less the same as siblings born at separate times. If 64 percent of all twin pairs are of the same sex, what percentage of twin pairs are identical twins?

    *Hint*: Compute the probability that a twin pair is of the same sex by conditioning on whether the pair is monozygotic.