Teoria dos Grafos e Computabilidade

— Dynamic programming —

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— Dynamic programming: fundamentals —

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Dynamic programming.

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   - Pro: simple to develop algorithm skeleton.
   - Pro: usually reduces time for a problem known to be solvable in polynomial time.
   - Con: conquer step can be very hard to implement efficiently.
4. Dynamic programming
   - More powerful than greedy and divide-and-conquer strategies.
   - Implicitly explore space of all possible solutions.
   - Solve multiple sub-problems and build up correct solutions to larger and larger sub-problems.
   - Careful analysis needed to ensure number of sub-problems solved is polynomial in the size of the input.

History of Dynamic Programming

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- Bellman pioneered the systematic study of dynamic programming in the 1950s.
- Dynamic programming = "planning over time."
- The Secretary of Defense at that time was hostile to mathematical research.
- Bellman sought an impressive name to avoid confrontation.
  - "it's impossible to use dynamic in a pejorative sense"
  - "something not even a Congressman could object to" Reference:
  - Bellman, R. E., *Eye of the Hurricane, An Autobiography*.

Applications of Dynamic Programming

- Computational biology: Smith-Waterman algorithm for sequence alignment.
- Operations research: Bellman-Ford algorithm for shortest path routing in networks.
- Control theory: Viterbi algorithm for hidden Markov models.
- Computer science (theory, graphics, AI, . . .): Unix `diff` command for comparing two files.

Teoria dos Grafos e Computabilidade

— Weighted interval scheduling —

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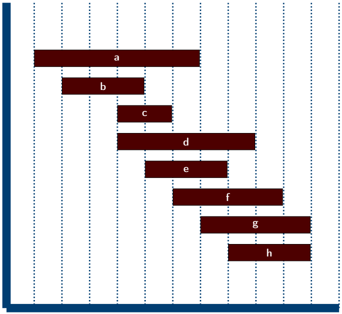
INTERVAL SCHEDULING

INSTANCE Nonempty set $\{(s(i), f(i)), 1 \leq i \leq n\}$ of start and finish times of n jobs.

SOLUTION The largest subset of mutually compatible jobs.



Review: Interval Scheduling

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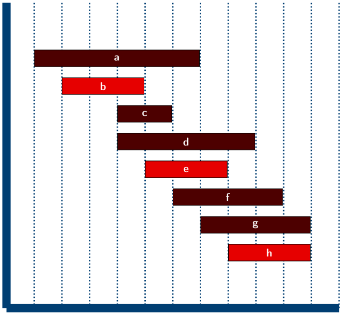
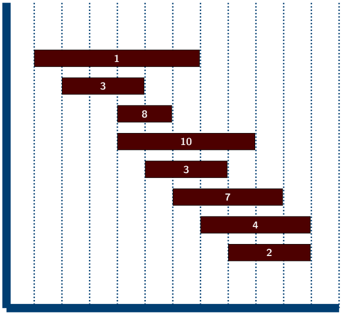


- Two jobs are compatible if they do not overlap.
- This problem models the situation where you have a resource, a set of fixed jobs, and you want to schedule as many jobs as possible.
- Greedy algorithm sort jobs in non decreasing order of finish times. Add next job to current subset only if it is compatible with previously-selected jobs.

Weighted Interval Scheduling

INSTANCE Nonempty set $\{(s_i, f_i), 1 \leq i \leq n\}$ of start and finish times of n jobs and a weight $v_i \geq 0$ associated with each job.

SOLUTION A set S of mutually compatible jobs such that $\sum_{i \in S} v_i$ is maximised.



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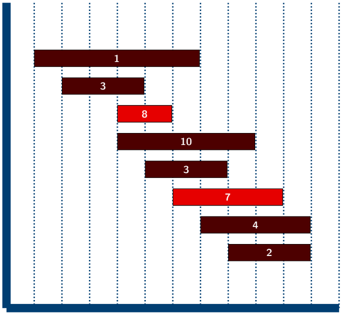
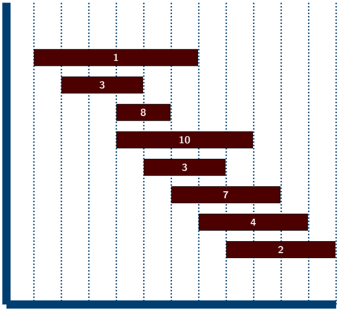


- Two jobs are compatible if they do not overlap.
- This problem models the situation where you have a resource, a set of fixed jobs, and you want to schedule as many weighted jobs as possible.
- Greedy algorithm can produce arbitrarily bad results for this problem.

Approach

- Sort jobs in increasing order of finish time and relabel: $f_1 \leq f_2 \leq \ldots \leq f_n$.
- Request i comes before request j if $i < j$.



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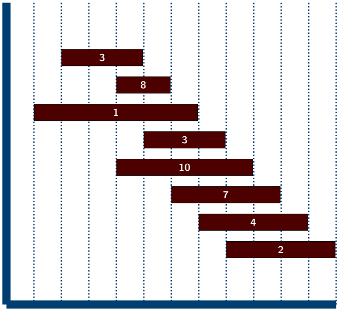

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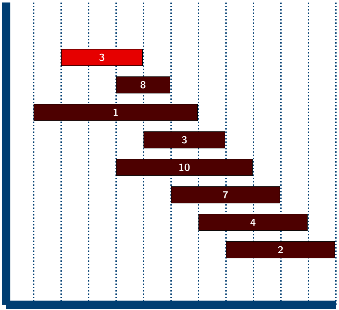

- p(j) is the largest index $i < j$ such that job i is compatible with job j. $p(j) = 0$ if there is no such job i.
- We will develop optimal algorithm from very obvious statements about the problem.

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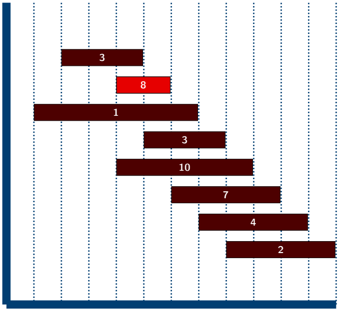

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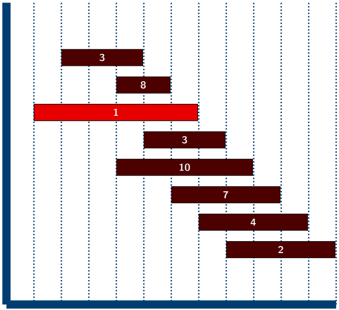

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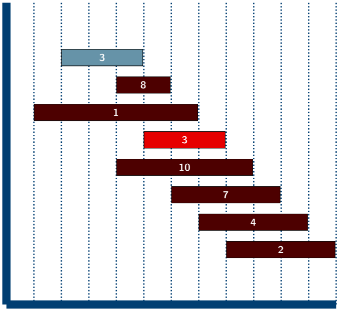

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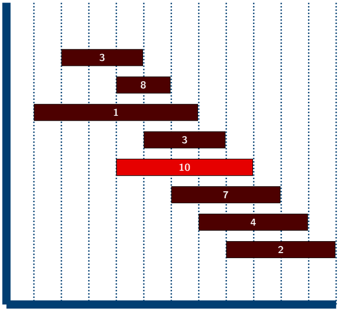

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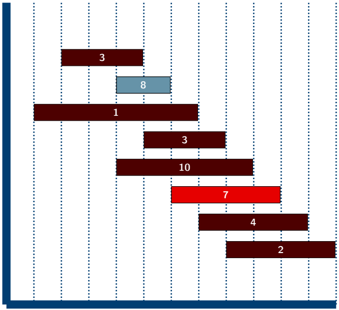

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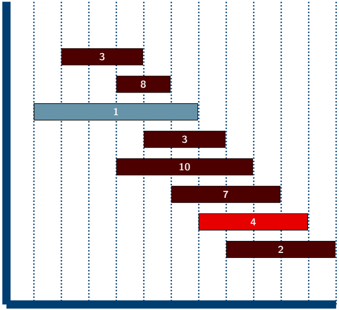

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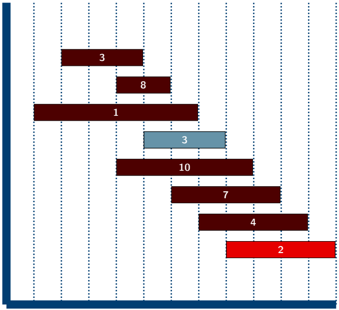

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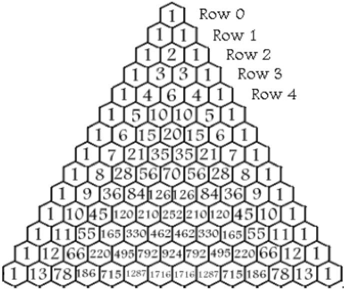

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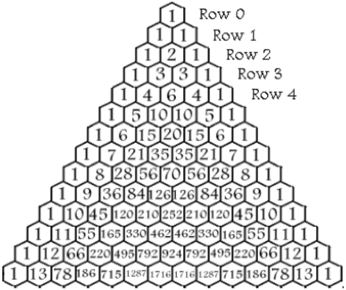

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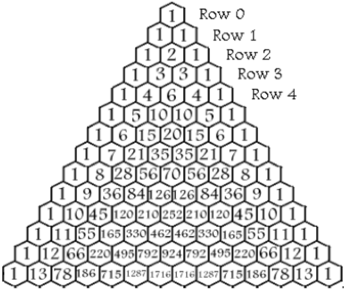

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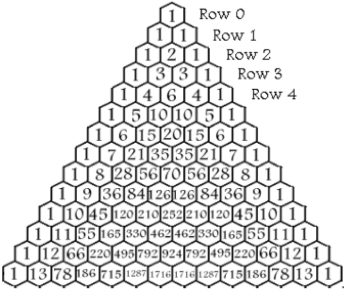

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- Proof: either we select the nth element or not ...

Sub-problems

- Let $\mathcal{O}$ be the optimal solution. Two cases to consider.

  Case 1 *job n is not in $\mathcal{O}$.*

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- $\mathcal{O}$ must be the best of these two choices!
- Suggests finding optimal solution for sub-problems consisting of jobs $\{1, 2, \ldots, j-1, j\}$, for all values of j.

- Let $\mathcal{O}_j$ be the optimal solution for jobs $\{1, 2, \ldots, j\}$ and $\mathsf{OPT}(j)$ be the value of this solution ($\mathsf{OPT}(0) = 0$).

Recursion

- Let $\mathcal{O}_j$ be the optimal solution for jobs $\{1, 2, \ldots, j\}$ and $\mathrm{OPT}(j)$ be the value of this solution ($\mathrm{OPT}(0) = 0$).
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- When does request j belong to $\mathcal{O}_j$? If and only if $v_j + \mathrm{OPT}(p(j)) \geq \mathrm{OPT}(j-1)$.

Recursive Algorithm

Algorithm: Compute-opt

input : A set of weighted jobs R, index j and largest compatible indices.

output: A set of compatible jobs A

```
if j = 0 then
    return 0
else
    return max
      (v_j+Compute−opt(p(j)),Compute−opt(j-1))
end
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- Correctness of algorithm follows by induction.
- What is the running time of the algorithm?

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- Correctness of algorithm follows by induction.
- What is the running time of the algorithm? Can be exponential in n.
- When $p(j) = j - 2$, for all $j \geq 2$: recursive calls are for $j - 1$ and $j - 2$.

Example of Recursive Algorithm

$$\text{OPT}(6) =$$
$$\text{OPT}(5) =$$
$$\text{OPT}(4) =$$
$$\text{OPT}(3) =$$
$$\text{OPT}(2) =$$
$$\text{OPT}(1) =$$
$$\text{OPT}(0) = 0$$

Example of Recursive Algorithm

$$\begin{aligned}
OPT(6) &= \max(v_6 + OPT(p(6)), OPT(5)) = \max(1 + OPT(3), OPT(5)) \\
OPT(5) &= \\
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\mathrm{OPT}(5) &= \max(v_5 + \mathrm{OPT}(p(5)), \mathrm{OPT}(4)) = \max(2 + \mathrm{OPT}(3), \mathrm{OPT}(4))\\
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\mathrm{OPT}(1) &= v_1 = 2\\
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\end{aligned}
$$

Example of Recursive Algorithm

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OPT(3) &= \max(v_3 + OPT(p(3)), OPT(2)) = \max(4 + OPT(1), OPT(2)) = 6 \\
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- Optimal solution is

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- Optimal solution is job 5, job 3, and job 1.

Memoisation

- Store OPT(j) values in a cache and reuse them rather than recompute them.

Algorithm: M-Compute-opt

input : A set of weighted jobs R, index j and largest compatible indices.

output: A set of compatible jobs A

```
if j = 0 then
    return 0;
else if M[j] is not empty then
    return M[j];
else
    M[j]= max
        (v_j+M-Compute−opt(p(j)),M-Compute−opt(j-1));
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end
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Running Time of Memoisation

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- Time spent in a single call to `M-Compute-Opt` is $O(1)$ apart from time spent in recursive calls.

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- Claim: running time of this algorithm is $O(n)$ (after sorting).
- Time spent in a single call to `M-Compute-Opt` is $O(1)$ apart from time spent in recursive calls.
- Total time spent is the order of the number of recursive calls to `M-Compute-Opt`.

Running Time of Memoisation

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- How many such recursive calls are there in total?

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- How many such recursive calls are there in total?
- Use number of filled entries in M as a measure of progress.
- Each time `M-Compute-Opt` issues two recursive calls, it fills in a new entry in M.
- Therefore, total number of recursive calls is $\boxed{O(n)}$.

From Recursion to Iteration

- Unwind the recursion and convert it into iteration.
- Can compute values in M iteratively in $O(n)$ time.
- `Find-Solution` works as before.

Algorithm: Iterative weighted interval scheduling

input : A set of weighted jobs R, index j and largest compatible indices.

output: A set of compatible jobs A

```
M[0] = 0;
foreach j ∈ [1, n] do
    M[j] = max(v_j+M[p(j)],M[j-1]);
end
```

Basic Outline of Dynamic Programming

- To solve a problem, we need a collection of sub-problems that satisfy a few properties:
  1. There are a polynomial number of sub-problems.
  2. The solution to the problem can be computed easily from the solutions to the sub-problems.
  3. There is a natural ordering of the sub-problems from "smallest" to "largest".
  4. There is an easy-to-compute recurrence that allows us to compute the solution to a sub-problem from the solutions to some smaller sub-problems.

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  4. There is an easy-to-compute recurrence that allows us to compute the solution to a sub-problem from the solutions to some smaller sub-problems.
- Difficulties in designing dynamic programming algorithms:
  1. Which sub-problems to define?
  2. How can we tie up sub-problems using a recurrence?
  3. How do we order the sub-problems (to allow iterative computation of optimal solutions to sub-problems)?

Teoria dos Grafos e Computabilidade

— Some exercises —

Silvio Jamil F. Guimarães

Graduate Program in Informatics – PPGINF
Laboratory of Image and Multimedia Data Science – IMScience
Pontifical Catholic University of Minas Gerais – PUC Minas

Feb 2025

Maximum subarray problem

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Some properties of this problem are:

- If the array contains all non-negative numbers, then the problem is trivial
- If the array contains all non-positive numbers, then a solution is any subarray of size 1;
- Several different sub-arrays may have the same maximum sum.

Maximum subarray problem

Maximum subarray problem

A>0	x	B>0

- If $x > 0$ then the answer is $\boxed{\mathrm{A} + \mathrm{x} + \mathrm{B}}$
- If $x < 0$ then the answer may be
  1. $\boxed{\max\{A, B\}}$ if $A + x < 0$
  2. $\boxed{\max\{A, B, A + x + B\}}$ if $A + x > 0$

Longest increasing subsequence

The longest increasing subsequence problem is to find a subsequence of a given sequence in which the subsequence's elements are in sorted order, lowest to highest, and in which the subsequence is as long as possible.

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L:	1	2	3	3	4	4	3	0	1

$$max\{L(1), L(2), ..., L(n)\}$$

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Longest increasing subsequence

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- There are n possible places where you can place an advertisement given by $x_1, x_2, \cdots, x_n$ in $[0, M]$.
- Placing an advertisement at x_i gives value r_i.
- You cannot put two advertisements at distance < 5kms from each other.

06	07	12	14	18	19

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Placing billboards

Teoria dos Grafos e Computabilidade

— Segmented Least Squares —

Silvio Jamil F. Guimarães

Graduate Program in Informatics – PPGINF
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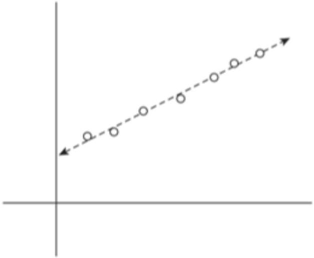
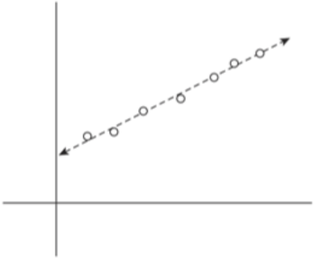
Least Squares Problem



- Given scientific or statistical data plotted on two axes.
- Find the "best" line that "passes" through these points.

Least Squares Problem



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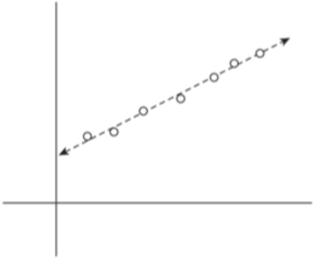

- Given scientific or statistical data plotted on two axes.
- Find the "best" line that "passes" through these points.
- How do we formalise the problem?

LEAST SQUARES

INSTANCE Set $P = \{(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\}$ of n points.

SOLUTION Line $L : y = ax + b$ that minimises

$$\text{Error}(L, P) = \sum_{i=1}^{n} (y_i - ax_i - b)^2.$$

Least Squares Problem

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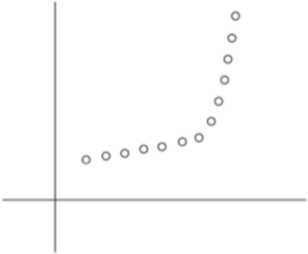
$$\text{Error}(L, P) = \sum_{i=1}^{n} (y_i - ax_i - b)^2.$$

- Solution is achieved by

$$a = \frac{n \sum_i x_i y_i - \left(\sum_i x_i\right)\left(\sum_i y_i\right)}{n \sum_i x_i^2 - \left(\sum_i x_i\right)^2} \text{ and } b = \frac{\sum_i y_i - a \sum_i x_i}{n}$$

Segmented Least Squares



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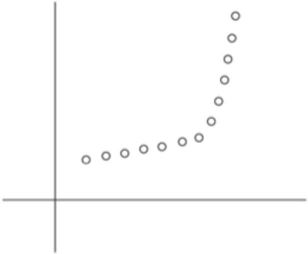
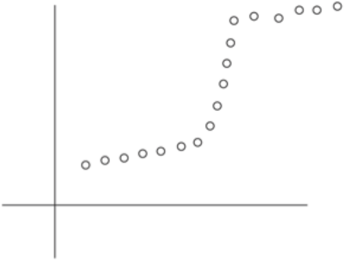
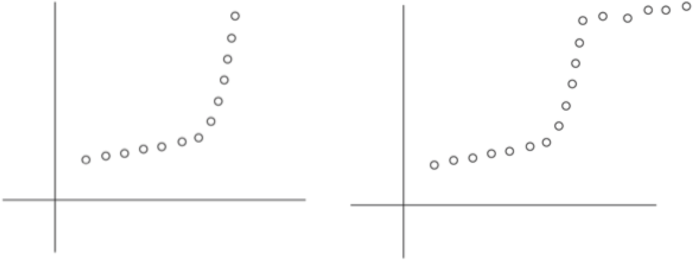
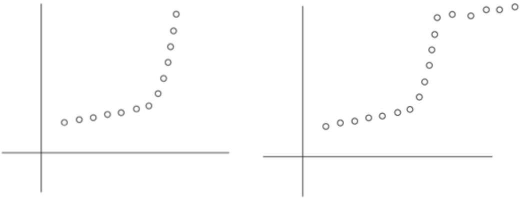






- Want to fit multiple lines through P.
- Each line must fit contiguous set of x-coordinates.
- Lines must minimise total error.

Segmented Least Squares



SEGMENTED LEAST SQUARES

INSTANCE Set $P = \{p_i = (x_i, y_i), 1 \leq i \leq n\}$ of n points, $x_1 < x_2 < \cdots < x_n$.

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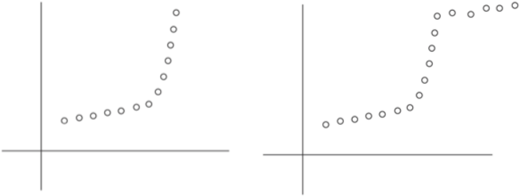
$$\sum_{j=1}^{k} \text{Error}(L_j, P_j)$$

Segmented Least Squares



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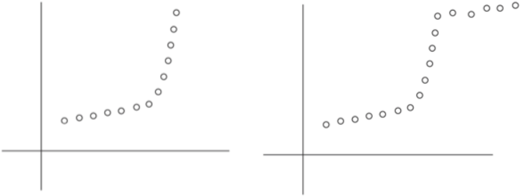

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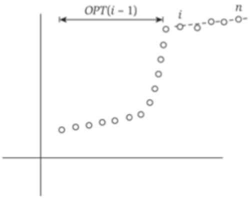
A subset P' of P is a **segment** if $1 \leq i < j \leq n$ exist such that $P' = \{(x_i, y_i), (x_{i+1}, y_{i+1}), \ldots, (x_{j-1}, y_{j-1}), (x_j, y_j)\}$.

Formulating the Recursion: I

- Observation: p_n is part of some segment in the optimal solution. This segment starts at some point p_i.
- Let $\mathrm{OPT}(i)$ be the optimal value for the points $\{p_1, p_2, \ldots, p_i\}$.
- Let $e_{i,j}$ denote the minimum error of any line that fits $\{p_i, p_2, \ldots, p_j\}$.
- We want to compute $\mathrm{OPT}(n)$.



- If the last segment in the optimal partition is $\{p_i, p_{i+1}, \ldots, p_n\}$, then

$$\mathrm{OPT}(n) = e_{i,n} + C + \mathrm{OPT}(i-1)$$

Formulating the Recursion: II

- Consider the sub-problem on the points $\{p_1, p_2, \ldots p_j\}$
- To obtain $\mathrm{OPT}(j)$, if the last segment in the optimal partition is $\{p_i, p_{i+1}, \ldots, p_j\}$, then

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- Since i can take only j distinct values,

$$\mathrm{OPT}(j) = \min_{1 \leq i \leq j} (e_{i,j} + C + \mathrm{OPT}(i-1))$$

- Segment $\{p_i, p_{i+1}, \dots p_j\}$ is part of the optimal solution for this sub-problem if and only if the minimum value of $\mathrm{OPT}(j)$ is obtained using index i. solution

Dynamic Programming Algorithm

$$OPT(j) = \begin{cases} 0, & \text{if } j = 0 \\ \min_{1 \leq i \leq j}(e_{ij} + c + OPT[i-1]), & \text{otherwise} \end{cases}$$

Algorithm: Segmented least squares: an iterative algorithm

input : A set of n points p_i
output: A set of compatible jobs A

```
M[0] = 0;
for j=1 to n do
    for i=1 to j do
        compute the e_ij for the segment p_i, ..., p_j;
    end
end
for j=1 to n do
    M[j] = min_{1<=i<=j}(e_ij + c + M[i-1]);
end
```

Dynamic Programming Algorithm

$$OPT(j) = \begin{cases} 0, & \text{if } j = 0 \\ \min_{1 \le i \le j}(e_{ij} + c + OPT[i-1]), & \text{otherwise} \end{cases}$$

Algorithm: Segmented least squares: an iterative algorithm

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    for i=1 to j do
        compute the e_ij for the segment p_i, ..., p_j;
    end
end
for j=1 to n do
    M[j] = min_{1≤i≤j}(e_ij + c + M[i − 1]);
end
```

- Running time is $O(n^3)$, can be improved to $O(n^2)$.
- We can find the segments in the optimal solution by backtracking.

Teoria dos Grafos e Computabilidade

— Sequence alignment —

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Sequence Similarity

- Given two strings, measure how similar they are.
- Given a database of strings and a query string, compute the string most similar to query in the database.
- Applications:
  - Online searches (Web, dictionary).
  - Spell-checkers.
  - Computational biology
  - Speech recognition.
  - Basis for Unix `diff`.

Defining Sequence Similarity

o	c	u	r	r	a	n	c	e	-
o	c	c	u	r	r	e	n	c	e

6 mismatches, 1 gap

Defining Sequence Similarity

o	c	u	r	r	a	n	c	e	-
o	c	c	u	r	r	e	n	c	e

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o	c	-	u	r	r	a	n	c	e
o	c	c	u	r	r	e	n	c	e

1 mismatch, 1 gap

Defining Sequence Similarity

o	c	u	r	r	a	n	c	e	-
o	c	c	u	r	r	e	n	c	e

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o	c	-	u	r	r	a	n	c	e
o	c	c	u	r	r	e	n	c	e

1 mismatch, 1 gap

o	c	-	u	r	r	-	a	n	c	e
o	c	c	u	r	r	e	-	n	c	e

0 mismatches, 3 gaps

Defining Sequence Similarity

o	c	u	r	r	a	n	c	e	-
o	c	c	u	r	r	e	n	c	e

6 mismatches, 1 gap

o	c	-	u	r	r	a	n	c	e
o	c	c	u	r	r	e	n	c	e

1 mismatch, 1 gap

o	c	-	u	r	r	-	a	n	c	e
o	c	c	u	r	r	e	-	n	c	e

0 mismatches, 3 gaps

- Edit distance model: how many changes must you to make to one string to transform it into another?
- Changes allowed are deleting a letter, adding a letter, changing a letter.

Edit distance

INSTANCE Let two string $x = x_1x_2x_3 \ldots x_m$ and $y = y_1y_2 \ldots y_n$

SOLUTION An alignment of minimum cost.

Edit distance

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o	c	-	u	r	r	a	n	c	e
o	c	c	u	r	r	e	n	c	e

- A matching of these sets is a set M of ordered pairs such that
  1. in each pair (i, j), $1 \leq i \leq m$ and $1 \leq j \leq m$ and
  2. no index from x (respectively, from y) appears as the first (respectively, second) element in more than one ordered pair.
- A matching M is an alignment if there are no "crossing pairs" in M: if $(i, j) \in M$ and $(i', j') \in M$ and $i < i'$ then $j < j'$.

EDIT DISTANCE

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o	c	c	u	r	r	e	n	c	e

- A matching M is an alignment if there are no "crossing pairs" in M: if $(i, j) \in M$ and $(i', j') \in M$ and $i < i'$ then $j < j'$.
- The pair x_i-y_j and $x_{i'}$-yj' cross if $i < i'$', but $j - j'$.

Edit distance

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- A matching M is an alignment if there are no "crossing pairs" in M: if $(i,j) \in M$ and $(i',j') \in M$ and $i < i'$ then $j < j'$.

$$cost(M) = \underbrace{\sum_{(x_i,y_j)\in M} \alpha_{x_iy_j}}_{\text{mismatch}} + \underbrace{\sum_{i:x_i \ \text{unmatched}} \delta + \sum_{j:x_j \ \text{unmatched}} \delta}_{\text{gaps}}$$

Dynamic Programming Approach

- Consider index $m \in x$ and index $n \in y$. Is $(m, n) \in M$?

Dynamic Programming Approach

- Consider index $m \in x$ and index $n \in y$. Is $(m, n) \in M$?
- Claim: $(m, n) \notin M \Rightarrow m \in x$ not matched or $n \in y$ not matched.

Dynamic Programming Approach

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- Claim: $(m, n) \notin M \Rightarrow m \in x$ not matched or $n \in y$ not matched.
- OPT(i, j) = min cost of aligning $x = x_1 \dots x_i$ and $y = y_1 \dots y_j$.
  - Case 1: OPT matches x_i-y_j so $(i, j) \in M$:

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$$\mathrm{OPT}(i, j) = \alpha_{x_i y_j} + \mathrm{OPT}(i-1, j-1)$$

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Dynamic Programming Approach

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  - Case 1: OPT matches x_i-y_j so $(i, j) \in M$:

$$\mathrm{OPT}(i, j) = \alpha_{x_i y_j} + \mathrm{OPT}(i-1, j-1)$$

    .
  - Case 2a: OPT leaves x_i unmatched, so i not matched:

$$\mathrm{OPT}(i, j) = \delta + \mathrm{OPT}(i-1, j)$$

    .

Dynamic Programming Approach

- Consider index $m \in x$ and index $n \in y$. Is $(m, n) \in M$?
- Claim: $(m, n) \notin M \Rightarrow m \in x$ not matched or $n \in y$ not matched.
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    .
  - Case 2a: OPT leaves x_i unmatched, so i not matched:

    $$\mathrm{OPT}(i, j) = \delta + \mathrm{OPT}(i-1, j)$$

    .
  - Case 2b: OPT leaves y_j unmatched, so j not matched:

    $$\mathrm{OPT}(i, j) = \delta + \mathrm{OPT}(i, j-1)$$

    .

Dynamic Programming Approach

- Consider index $m \in x$ and index $n \in y$. Is $(m, n) \in M$?
- Claim: $(m, n) \notin M \Rightarrow m \in x$ not matched or $n \in y$ not matched.
- $\text{OPT}(i, j)$ = min cost of aligning $x = x_1 \dots x_i$ and $y = y_1 \dots y_j$.
  - Case 1: OPT matches x_i-y_j so $(i, j) \in M$:
    $$\text{OPT}(i, j) = \alpha_{x_i y_j} + \text{OPT}(i-1, j-1)$$
    .
  - Case 2a: OPT leaves x_i unmatched, so i not matched:
    $$\text{OPT}(i, j) = \delta + \text{OPT}(i-1, j)$$
    .
  - Case 2b: OPT leaves y_j unmatched, so j not matched:
    $$\text{OPT}(i, j) = \delta + \text{OPT}(i, j-1)$$
    .
  - $(i, j) \in M$ if and only if minimum is achieved by the first term.
- What are the base cases?

Dynamic Programming Approach

- Consider index $m \in x$ and index $n \in y$. Is $(m, n) \in M$?
- Claim: $(m, n) \notin M \Rightarrow m \in x$ not matched or $n \in y$ not matched.
- $\mathrm{OPT}(i, j) =$ min cost of aligning $x = x_1 \dots x_i$ and $y = y_1 \dots y_j$.
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    $$\mathrm{OPT}(i, j) = \alpha_{x_i y_j} + \mathrm{OPT}(i-1, j-1)$$

    .
  - Case 2a: OPT leaves x_i unmatched, so i not matched:

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    .
  - Case 2b: OPT leaves y_j unmatched, so j not matched:

    $$\mathrm{OPT}(i, j) = \delta + \mathrm{OPT}(i, j-1)$$

    .
  - $(i, j) \in M$ if and only if minimum is achieved by the first term.
- What are the base cases? $\mathrm{OPT}(i, 0) = \mathrm{OPT}(0, i) = i\delta$.

Dynamic Programming Algorithm

$$\mathrm{OPT}(i,j) = \begin{cases} j\delta, & \text{if } i = 0 \\ \min \begin{cases} \alpha_{x_i y_j} + \mathrm{OPT}(i-1, j-1), \\ \delta + \mathrm{OPT}(i-1, j), \\ \delta + \mathrm{OPT}(i, j-1) \end{cases} & \text{otherwise} \\ i\delta, & \text{if } j = 0 \end{cases}$$

- Running time is $O(mn)$. Space used in $O(mn)$.

Dynamic Programming Algorithm

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- Running time is $O(mn)$. Space used in $O(mn)$.
- Can compute $\mathrm{OPT}(m, n)$ in $O(mn)$ time and $O(m+n)$ space (*Hirschberg 1975*, Chapter 6.7).

Dynamic Programming Algorithm

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$$

- Running time is $O(mn)$. Space used in $O(mn)$.
- Can compute $\mathrm{OPT}(m, n)$ in $O(mn)$ time and $O(m+n)$ space (*Hirschberg 1975*, Chapter 6.7).
- Can compute *alignment* in the same bounds by combining dynamic programming with divide and conquer.

Dynamic Programming Algorithm

Longest commom subsequence

The longest commom subsequence problem is the task of finding the longest subsequence which is in two sequences x and y.

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Formally, $w_0 w_1 \ldots w_{i-1}$ is a subsequence of $x_0 x_1 \ldots x_{m-1}$ if there exists a strictly increasing sequence of integers $(k_0, k_1, \ldots, k_{i-1})$ such that for $0 \leq k \leq i-1$. A word w is a longest common subsequence of x and y if w is a subsequence of x, a subsequence of y and its length is maximal.

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C	T	A	C	C	G	A	
T	A	C	A	T	T	G	T

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C	T	A	C	C	G	A	
T	A	C	A	T	T	G	T

Some properties of this problem are:

- the lenght of the longest subsequence must be maximal;
- may have several longest subsequences with the same size;
- it is possible to identify the subsequence by backtracking

Dynamic Programming Algorithm

$$\text{OPT}(i,j) = \begin{cases} 0, & \text{if } i = 0 \\ 1 + \text{OPT}(i-1, j-1), & \text{if } x_i = y_j \\ \max \begin{cases} \text{OPT}(i-1, j), \\ \text{OPT}(i, j-1) \end{cases} & \text{otherwise} \\ 0, & \text{if } j = 0 \end{cases}$$

	C	T	A	C	C
T					
A					
C					
A					
C					
G					

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		C	T	A	C	C
	0	0	0	0	0	0
T						
A						
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		C	T	A	C	C
	0	0	0	0	0	0
T	0	0	1	1	1	1
A						
C						
A						
C						
G						

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		C	T	A	C	C
	0	0	0	0	0	0
T	0	0	1	1	1	1
A	0	0	1	2	2	2
C						
A						
C						
G						

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A						
C						
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	0	0	0	0	0	0
T	0	0	1	1	1	1
A	0	0	1	2	2	2
C	0	1	1	2	3	3
A	0	1	1	2	3	3
C	0	1	1	2	3	4
G						

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	0	0	0	0	0	0
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A	0	0	1	2	2	2
C	0	1	1	2	3	3
A	0	1	1	2	3	3
C	0	1	1	2	3	4
G	0	1	1	2	3	4

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A	0	1	1	2	3	3
C	0	1	1	2	3	4
G	0	1	1	2	3	4

C	T	A	C	C	

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The longest palindrome problem is the task of finding the longest subsequence which is a palindrome.

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How to find the size of the longest palindrome?

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How to find the size of the longest palindrome?

Longest increasing subsequence

The **longest increasing subsequence** –LIS– problem is to find a subsequence of a given sequence in which the subsequence's elements are in **sorted order**, lowest to highest, and in which the subsequence is as long as possible.

Longest increasing subsequence

The longest increasing subsequence –LIS– problem is to find a subsequence of a given sequence in which the subsequence's elements are in sorted order, lowest to highest, and in which the subsequence is as long as possible.

2	3	9	5	9	8	4	0	1

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L:	1	2	3	3	4	4	3	0	1

$$max\{L(1), L(2), ..., L(n)\}$$

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2	3	9	5	9	8	4	0	1
1	2	3	3	4	4	3	0	1

How to find the size of the LIS by using another strategy?

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1 2 3 3 4 4 3 0 1

How to find the size of the LIS by using another strategy?

Teoria dos Grafos e Computabilidade

— Shortest Path Problem —

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Shortest Path Problem

- $G = (V, E)$ is a connected directed graph. Each edge e has a length $l_e \geq 0$.
- V has n nodes and E has m edges.
- Length of a path P is the sum of lengths of the edges in P.
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Shortest Paths

INSTANCE A directed graph $G(V, E)$, a function $l : E \rightarrow \mathbb{R}^+$, and a node $s \in V$

SOLUTION A set $\{P_u, u \in V\}$, where P_u is the shortest path in G from s to u.

Dijkstra's Algorithm

- Maintain a set S of explored nodes: for each node $u \in S$, we have determined the length $d(u)$ of the shortest path from s to u.

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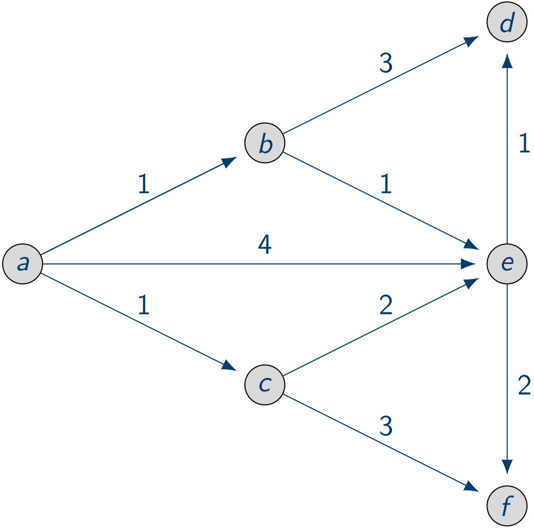
- Can modify algorithm to compute the shortest paths themselves: record the predecessor u that minimises $d'(v)$.

Example of Dijkstra's Algorithm



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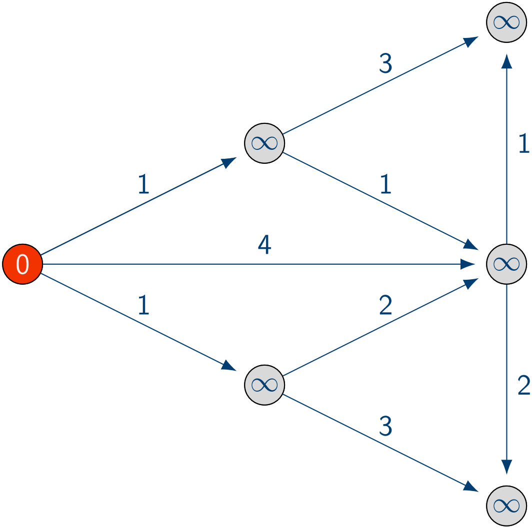

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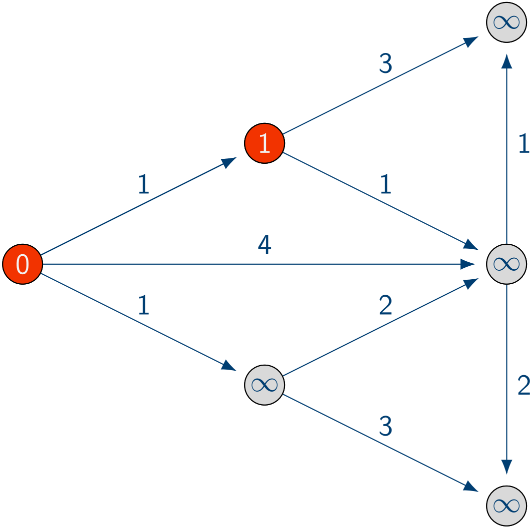

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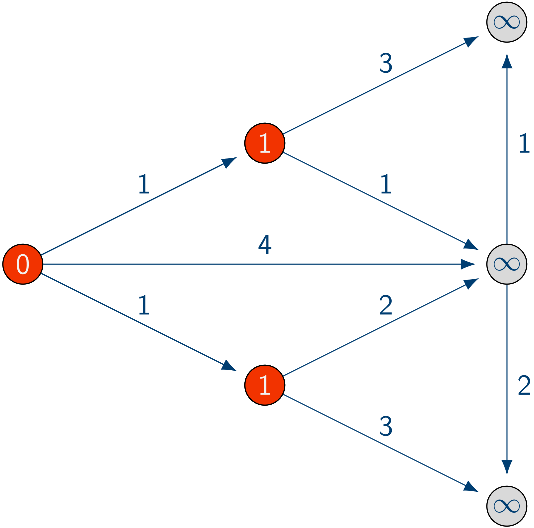

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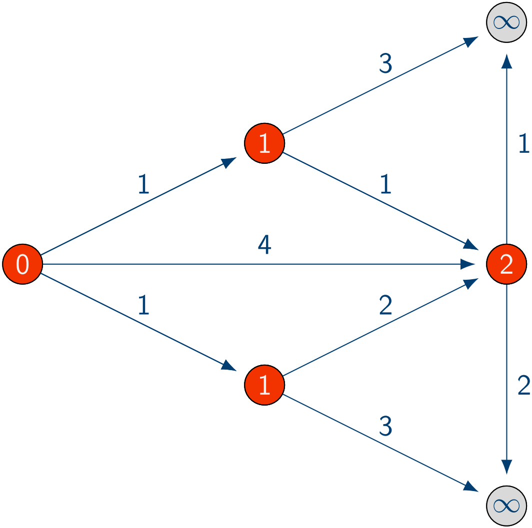

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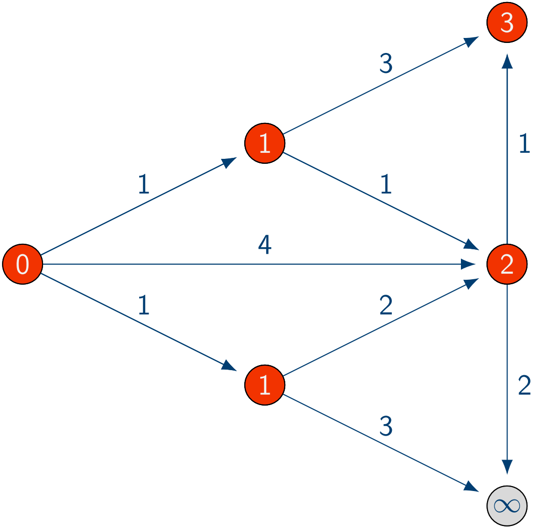

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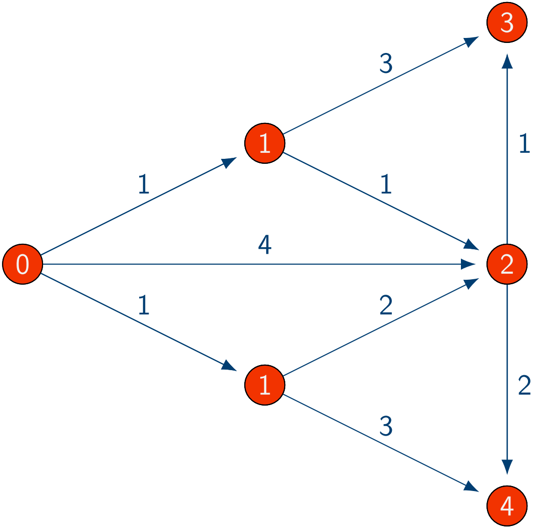

Proof of Correctness

- Let P_u be the shortest path computed for a node u.
- Claim: P_u is the shortest path from s to u.
- Prove by induction on the size of S.
  - Base case: $|S| = 1$. The only node in S is s.
  - Inductive step: we add the node v to S. Let u be the v's predecessor on the path P_v. Could there be a shorter path P from s to v?

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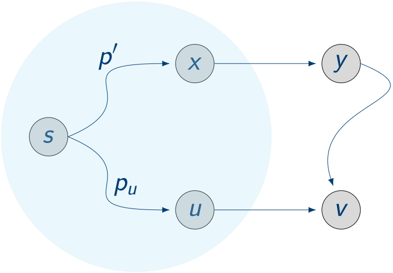

The alternate $s - v$ path P through x and y already too long by the time it had left the set S

Comments about Dijkstra's Algorithm

- Algorithm cannot handle negative edge lengths.
- Union of shortest paths output form a tree. Why?

Implementing Dijkstra's Algorithm

Algorithm: Shortes path algorithm – Dijkstra)

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- How many iterations are there of the while loop? $n - 1$.
- In each iteration, for each node $v \notin S$, compute

$$\min_{e=(u,v),u\in S} d(u) + l_e.$$

A Faster implementation of Dijkstra's Algorithm

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- Store the minima $d'(v)$ for each node $v \in V - S$ in a priority queue.
- Determine the next node v to add to S using ExtractMin.
- After adding v, for each neighbour w of v, compute $d(v) + l_{(v,w)}$.
- If $d(v) + l_{(v,w)} < d'(w)$, update w's key using ChangeKey.

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- How many times are ExtractMin and ChangeKey invoked? $n - 1$ and m times, respectively.

Single Source Shortest Path Problem

- $G = (V, E)$ is a connected directed graph. Each edge e has a length l_e. Note that the weights may be negative.
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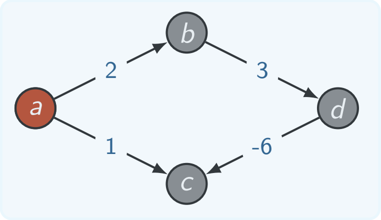
Bellman-Ford Algorithm

Dijkstra – Can fail if negative edge costs.



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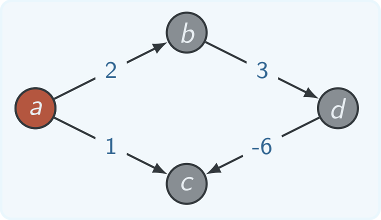

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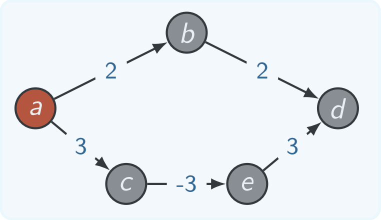

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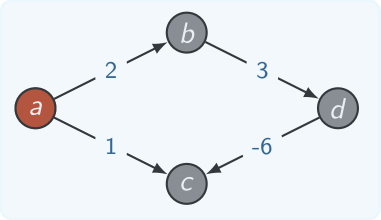

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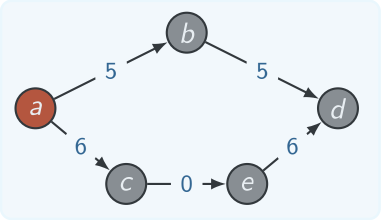
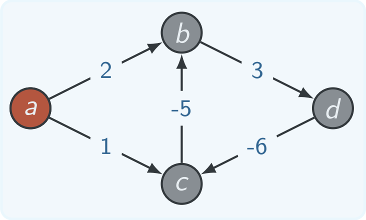


Bellman-Ford Algorithm

If some path from s to t contains a negative cost cycle, there does not exist a shortest s-t path; otherwise, there exists one that is simple.



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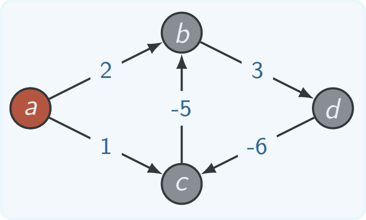

The Bellman-Ford algorithm is a way to find single source shortest paths in a graph with negative edge weights (but no negative cycles).

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$$OPT(i, v) = \begin{cases} 0, & \text{if } i = 0 \\ \min\begin{cases} OPT(i-1, v) \\ \min\{OPT(i-1, w) + c_{vw}\} \end{cases}, & \text{otherwise} \end{cases}$$

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A Faster implementation of Dijkstra's Algorithm

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- Computational cost: O(mn)

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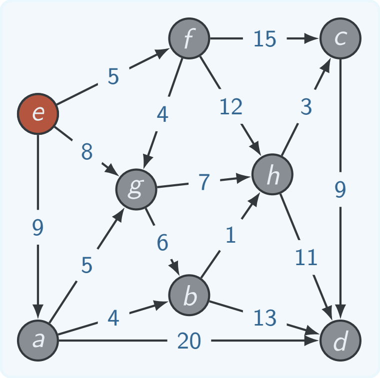
How to detect negative cycles?

Shortest path – an example



Compute the shortest path from e to all other nodes!