Teoria dos Grafos e Computabilidade

— Tractability and Intractability —

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Laboratory of Image and Multimedia Data Science – IMScience
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Feb 2025

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Algorithm Design

- Patterns
  - Greed. $O(n \log n)$ interval scheduling.
  - Divide-and-conquer. $O(n \log n)$ closest pair of points.
  - Dynamic programming. $O(n^2)$ edit distance.
  - Duality. $O(n^3)$ maximum flow and minimum cuts.

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  - Reductions.
  - Local search.
  - Randomization.

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  - Reductions.
  - Local search.
  - Randomization.
- Anti-patterns
  - NP-completeness. $O(n^k)$ algorithm unlikely.
  - PSPACE-completeness. $O(n^k)$ certification algorithm unlikely.
  - Undecidability. No algorithm possible.

Computational Tractability

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Polynomial time	Probably not
Shortest path	Longest path
Matching	3-D matching
Minimum cut	Maximum cut
2-SAT	3-SAT
Planar four-colour	Planar three-colour
Bipartite vertex cover	Vertex cover
Primality testing	Factoring

Problem Classification

- Classify problems based on whether they admit efficient solutions or not.
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- Some extremely hard problems cannot be solved efficiently (e.g., chess on an n-by-n board).
- However, classification is unclear for a very large number of discrete computational problems.
- We can prove that these problems are fundamentally equivalent and are manifestations of the same problem!

Teoria dos Grafos e Computabilidade

— Reductions —

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Polynomial-Time Reduction

- The goal is to express statements of the type

  Problem X is at least as hard as problem Y.

- Use the notion of reductions.

Y is polynomial-time reducible to X ($Y \leq_P X$)

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Y is polynomial-time reducible to X ($Y \leq_P X$)

if an arbitrary instance of Y can be solved using a polynomial number of standard operations, plus a polynomial number of calls to a black box that solves problem X.

- $Y \leq_P X$ implies that X is at least as hard as Y.
- Such reductions are Cook reductions. Karp reductions allow only one call to the black box that solves X.

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Claim: *If $Y \leq_P X$ and X can be solved in polynomial time, then Y can be solved in polynomial time.*

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- If $Y \leq_P X$ and Y cannot be solved in polynomial-time, then X cannot be solved in polynomial time. Establish intractability
- If $X \leq_P Y$ and $Y \leq_P X$, we use notation $X \equiv_P Y$ in order to express the equivalance. Establish equivalence

Polynomial Transformation

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Polynomial transformation is polynomial reduction with just one call to oracle for Y, exactly at the end of the algorithm for X. Almost all previous reductions were of this form.

Reductions

Reduction Design a fast algorithm for one computational problem, using a supposedly fast algorithm for another problem as a subroutine.

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- We can also learn that they have a **similar structure**.

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Cook vs Karp Reductions

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The problems have a similar underling structure and it is used to design new Algorithms

Reduction Strategies

- Simple equivalence.
- Special case to general case.
- Encoding with gadgets.

Optimization versus Decision Problems

- So far, we have developed algorithms that solve optimization problems.
  - Compute the largest flow.
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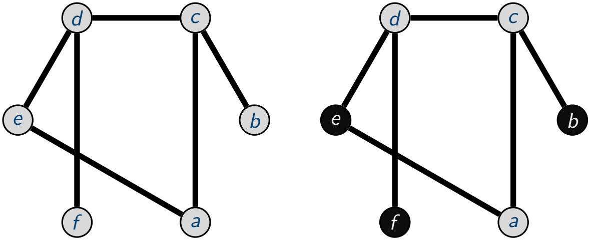
- So far, we have developed algorithms that solve optimization problems.
  - Compute the largest flow.
  - Find the closest pair of points.
  - Find the schedule with the least completion time.
- Now, we will focus on decision versions of problems, e.g..,

  Is there a flow with value at least k, for a given value of k?

Independent sets

Let $G = (V, E)$ be an undirected connected graph.

- A subset $S \subseteq V$ is an **independent set** if $\forall u, v \in S$ there exist an edge $(u, v) \in E$.
- Given G and an integer k, is there a subset of vertices $S \subseteq V$ such that $|S| \geq k$, and for each edge at most one of its endpoints is in S?



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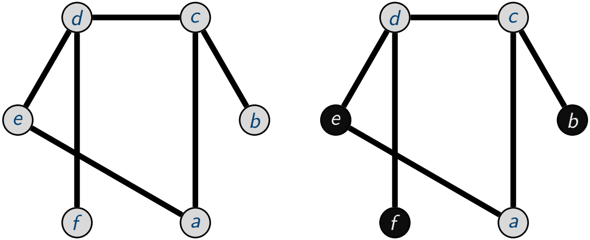

Is there an independent set of size ≥ 3? Yes.

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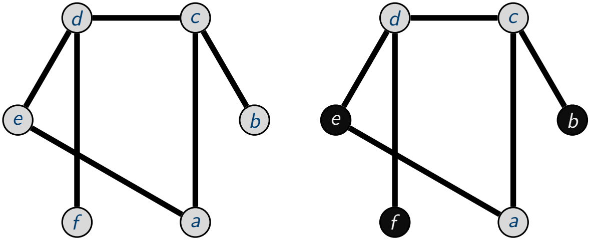
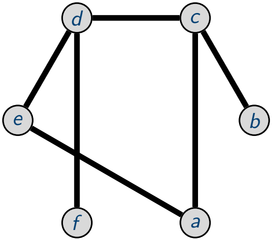


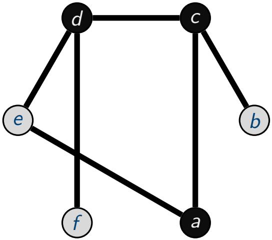
Is there an independent set of size ≥ 4? No.

Vertex cover

Let $G = (V, E)$ be an undirected connected graph.

- A subset $S \subseteq V$ is an vertex cover if $\forall (u, v) \in E$, either $u \in S$ or $v \in S$.
- Given a graph G and an integer k, is there a subset of vertices $S \subseteq V$ such that $|S| \leq k$, and for each edge, at least one of its endpoints is in S?





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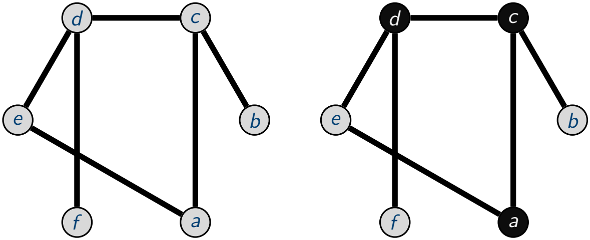

Is there a vertex cover of size ≤ 3? Yes.

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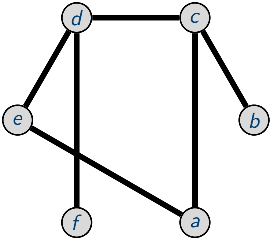
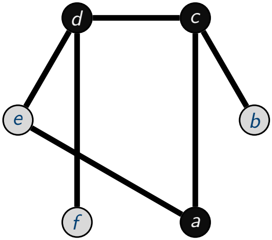


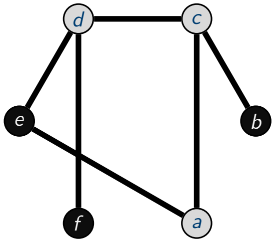


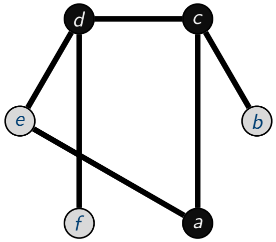
Is there a vertex cover of size ≤ 2? No.

Vertex cover

Let $G = (V, E)$ be an undirected connected graph, and S a vertex cover of G

As S is a vertex cover of G, then V-S is an independent set.





Independent Set and Vertex Cover

- Given an undirected graph $G(V, E)$, a subset $S \subseteq V$ is an independent set if no two vertices in S are connected by an edge.
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QUESTION	Does G contain an independent set of size

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VERTEX COVER

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QUESTION Does G contain a vertex cover of size at most k?

- Demonstrate simple equivalence between these two problems.

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INDEPENDENT SET

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QUESTION	Does G contain an independent set of size at least k?

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- Demonstrate simple equivalence between these two problems.
- S is an independent set in G iff $V - S$ is a vertex cover in G.

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- Independent Set $\leq_P$ Vertex Cover and Vertex Cover $\leq_P$ Independent Set.

Independent Set and Vertex Cover

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We show S is an independent set iff V - S is a vertex cover

- Let S be any independent set.
- Consider an arbitrary edge (u, v).
- S independent $\Rightarrow u \notin S$ or $v \notin S \Rightarrow u \in V - S$ or $v \in V - S$.
- Thus, V - S covers (u, v).

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- Let V - S be any vertex cover.
- Consider two nodes $u \in S$ and $v \in S$.
- Observe that $(u, v) \notin E$ since V - S is a vertex cover.
- Thus, no two nodes in S are joined by an edge $\Rightarrow$ S independent set

Set Cover

Given a set U of elements, a collection $S = \{S_1, S_2, \cdots, S_m\}$ of subsets of U.

- A subset $C \subseteq S$ is a **set cover** if the union of elements of C is equal to U.
- Given U, S, and an integer k, does there exist a collection of $\leq$ k of these sets whose union is equal to U?

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Sample application:

- m available pieces of software
- Set U of n capabilities that we would like our system to have
- The i^{th} piece of software provides the set $S_i \subseteq U$ of capabilities.
- The goal is to achieve all n capabilities using fewest pieces of software.

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- $U = \{1, 2, 3, 4, 5, 6, 7\}$ and $k = 2$

$$\begin{array}{ll} S_1 = \{3, 7\} & S_4 = \{2, 4\} \\ S_2 = \{3, 4, 5, 6\} & S_5 = \{5\} \\ S_3 = \{1\} & S_6 = \{1, 2, 6, 7\} \end{array}$$

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Vertex Cover and Set Cover

- Set cover is a packing problem: pack as many vertices as possible, subject to constraints (the edges).
- Vertex Cover is a covering problem: cover all edges in the graph with as few vertices as possible.
- There are more general covering problems.

Vertex Cover and Set Cover

- Set cover is a packing problem: pack as many vertices as possible, subject to constraints (the edges).
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INSTANCE A set U of n elements, a collection $S_1, S_2, \ldots, S_m$ of subsets of U, and an integer k.

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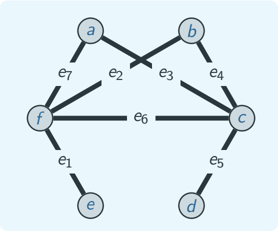

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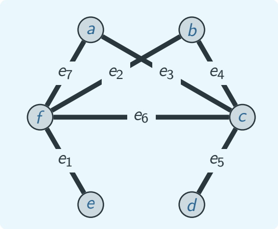

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Boolean Satisfiability

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- We are given a set $X = \{x_1, x_2, \ldots, x_n\}$ of n Boolean variables.
- Each variable can take the value 0 or 1.
- A **term** is a variable x_i or its negation $\overline{x_i}$.
- A **clause** of **length** l is a disjunction of l distinct terms $t_1 \vee t_2 \vee \cdots t_l$.
- A **truth assignment** for X is a function $\nu : X \rightarrow \{0, 1\}$.
- An assignment **satisfies** a clause C if it causes C to evaluate to 1 under the rules of Boolean logic.
- An assignment **satisfies** a collection of clauses $C_1, C_2, \ldots C_k$ if it causes $C_1 \wedge C_2 \wedge \cdots C_k$ to evaluate to 1.
  - ν is a **satisfying assignment** with respect to $C_1, C_2, \ldots C_k$.
  - set of clauses $C_1, C_2, \ldots C_k$ is **satisfiable**.

Satisfiability Problem (SAT)

INSTANCE A set of clauses $C_1, C_2, \ldots C_k$ over a set $X = \{x_1, x_2, \ldots x_n\}$ of n variables.

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- SAT and 3-SAT are fundamental combinatorial search problems.
- We have to make n independent decisions (the assignments for each variable) while satisfying a set of constraints.
- Satisfying each constraint in isolation is easy, but we have to make our decisions so that all constraints are satisfied simultaneously.

3-SAT and Independent Set

- We want to prove $\boxed{\text{3-SAT} \leq_P \textsc{Independent Set}}$.

3-SAT and Independent Set

- We want to prove 3-SAT $\leq_P$ INDEPENDENT SET.
- Two ways to think about 3-SAT:
  1. Make an independent 0/1 decision on each variable and succeed if we achieve one of three ways in which to satisfy each clause.
  2. Choose (at least) one term from each clause. Find a truth assignment that causes each chosen term to evaluate to 1. Ensure that no two terms selected **conflict**, i.e., select x_i and $\overline{x_i}$.

Proving 3-SAT $\leq_P$ Independent Set

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Given an instance Φ of 3-SAT, we construct an instance (G, k) of independent set that has an independent set of size k iff Φ is satisfiable.

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- Connect 3 literals in a clause in a triangle.
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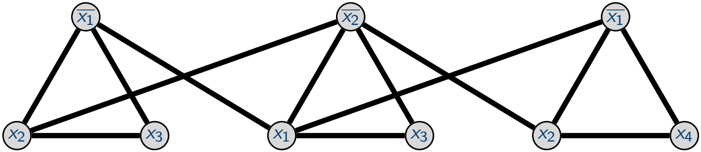

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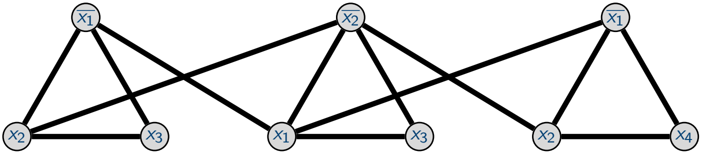

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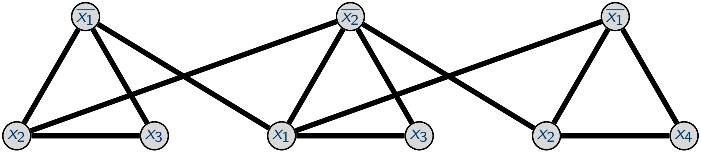
G contains independent set of size $k = |\Phi|$ iff Φ is satisfiable.

$\Rightarrow$ Let S be independent set of size k.

- S must contain exactly one vertex in each triangle.
- Set these literals to true.
- Truth assignment is consistent and all clauses are satisfied.

$\Leftarrow$ Given satisfying assignment, select one true literal from each triangle. This is an independent set of size k.



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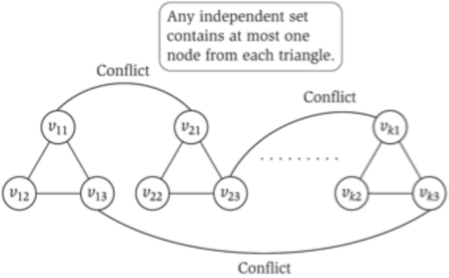

- We are given an instance of 3-SAT with k clauses of length three over n variables.
- Construct a graph $G = (V, E)$ with $3k$ nodes.
  - For each clause $C_i, 1 \leq i \leq k$, add a triangle of three nodes v_{i1}, v_{i2}, v_{i3} and three edges to G.
  - Label each node $v_{ij}, 1 \leq j \leq 3$ with the j-th term in C_i.
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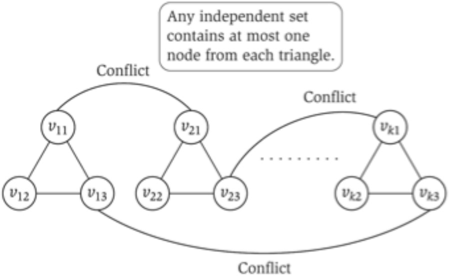

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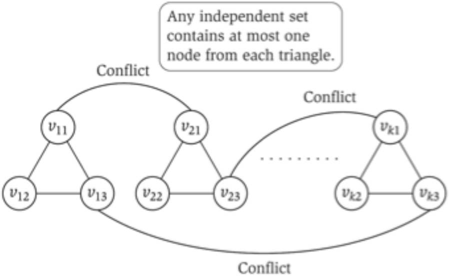

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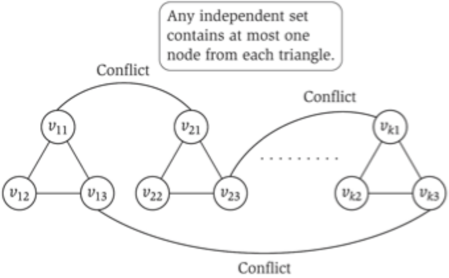

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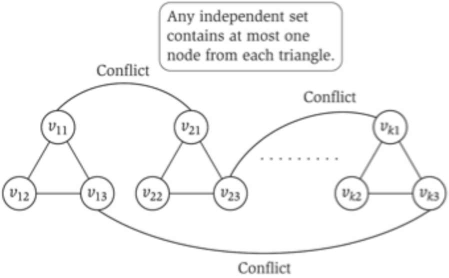

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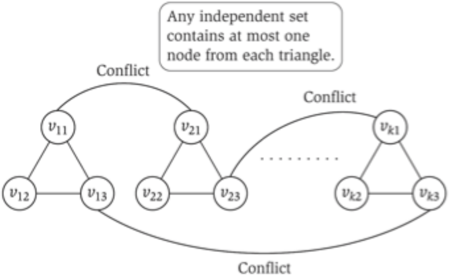

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- Independent set of size $\geq k \rightarrow$ satisfiable assignment the size of this set is k. How do we construct a satisfying truth assignment from the nodes in the independent set?

Transitivity of Reductions

Basic reduction strategies.

- Simple equivalence: Independent set $\equiv_P$ Vertex cover.
- Special case to general case: Vertex cover $\leq_P$ Set cover.
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Teoria dos Grafos e Computabilidade

— $\mathcal{NP}$ —

Silvio Jamil F. Guimarães

Graduate Program in Informatics – PPGINF
Laboratory of Image and Multimedia Data Science – IMScience
Pontifical Catholic University of Minas Gerais – PUC Minas

Finding vs. Certifying

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- We draw a contrast between finding a solution and checking a solution (in polynomial time).

We have not been able to develop efficient algorithms to solve many decision problems, let us turn our attention to whether we can check if a proposed solution is correct.

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- $\mathcal{P}$: set of problems X for which there is a polynomial time algorithm.

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- Certifier's job is to take a candidate short proof (t) that $s \in X$ and check in polynomial time whether t is a correct proof.

Certifier does not care about how to find these proofs.

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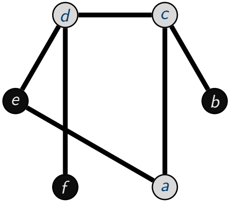

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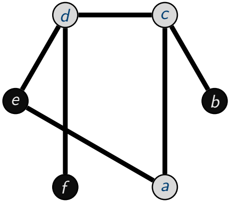

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One of the major unsolved problems in computer science.

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- $\mathcal{P} \subseteq \mathcal{NP}$ If $X \in P$, then there is a polynomial time algorithm A that solves X. B ignores t and returns $A(s)$. Why is B an efficient certifier?
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- $\mathcal{NP} \subseteq EXP$. Consider any problem X in $\mathcal{NP}$.
  - By definition, there exists a poly-time certifier C(s, t) for X.
  - To solve input s, run C(s, t) on all strings t with $|t| \leq p(|s|)$.
  - Return yes, if C(s, t) returns yes for any of these.

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Class $\mathcal{NP}$ – Class of decision problems, for which there exists a Non- Deterministic Turing Machine that can solve any yes instance in polynomial time. The machine guesses a yes solution and then verifies that it is a yes solution

Never tell to an expert in *Computational Complexity – tractability –* that you think that $\mathcal{NP}$ stands for Non Polynomial

$\mathcal{NP}$ STANDS for Non-deterministic Polynomial

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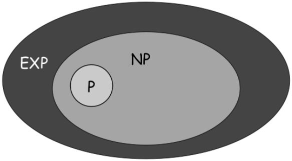
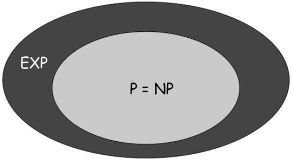


If P ≠ NP



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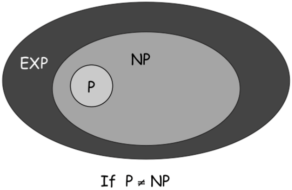

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Consensus opinion on P = NP? Probably no.

$\mathcal{P}$ vs. $\mathcal{NP}$

The Simpson's: P = NP?

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$\mathcal{P}$ vs. $\mathcal{NP}$

Futurama: P = NP?

Teoria dos Grafos e Computabilidade

— $\mathcal{NP}$-Complete —

Silvio Jamil F. Guimarães

Graduate Program in Informatics – PPGINF
Laboratory of Image and Multimedia Data Science – IMScience
Pontifical Catholic University of Minas Gerais – PUC Minas

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- Are there any $\mathcal{NP}$-Complete problems?
  1. Perhaps there are two problems X_1 and X_2 in $\mathcal{NP}$ such that there is no problem $X \in \mathcal{NP}$ where $X_1 \leq_P X$ and $X_2 \leq_P X$.
  2. Perhaps there is a sequence of problems $X_1, X_2, X_3, \ldots$ in $\mathcal{NP}$, each strictly harder than the previous one.

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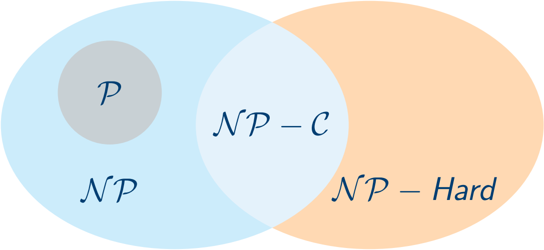

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- A **circuit** K is a labelled, directed acyclic graph such that
  1. the **sources** in K are labelled with constants (0 or 1) or the name of a distinct variable (the **inputs** to the circuit).
  2. every other node is labelled with one Boolean operator $\wedge$, $\vee$, or $\neg$.
  3. a single node with no outgoing edges represents the **output** of K.

Circuit Satisfiability

INSTANCE	A circuit K.
QUESTION	Is there a truth assignment to the inputs that causes the output to have value 1?

Proving CIRCUIT SATISFIABILITY is $\mathcal{NP}$-Complete

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- Convert B to a polynomial-sized circuit K with $n + p(n)$ sources.
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- $s \in X$ iff there is an assignment of the input bits of K that makes K satisfiable.

- Does a graph G on n nodes have a two-node independent set?

Example of Transformation to Circuit Satisfiability

- Does a graph G on n nodes have a two-node independent set?
- s encodes the graph G with $\binom{n}{2}$ bits.
- t encodes the independent set with n bits.
- Certifier needs to check if
  1. at least two bits in t are set to 1 and
  2. no two bits in t are set to 1 if they form the ends of an edge (the corresponding bit in s is set to 1).

Proving Other Problems $\mathcal{NP}$-Complete

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  3. Prove that $Y \leq_P X$.
- If we use Karp reductions, we can refine the strategy:
  1. Prove that $X \in \mathcal{NP}$.
  2. Select a problem Y known to be $\mathcal{NP}$-Complete.
  3. Consider an arbitrary instance s_Y of problem Y. Show how to construct, in polynomial time, an instance s_X of problem X such that
     (a) If $s_Y \in Y$, then $s_X \in X$ and
     (b) If $s_X \in X$, then $s_Y \in Y$.

$\mathcal{NP}$-Completeness