Teoria dos Grafos e Computabilidade

— Greedy graph algorithms —

Silvio Jamil F. Guimarães

Graduate Program in Informatics – PPGINF
Laboratory of Image and Multimedia Data Science – IMScience
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Graphs

- Model pairwise relationships (edges) between objects (nodes or vertices).
- **Undirected graph** $G = (V, E)$: set V of nodes and set E of edges, where $E \subseteq V \times V$. Elements of E are unordered pairs.
- **Directed graph** $G = (V, E)$: set V of nodes and set E of edges, where $E \subseteq V \times V$. Elements of E are ordered pairs.

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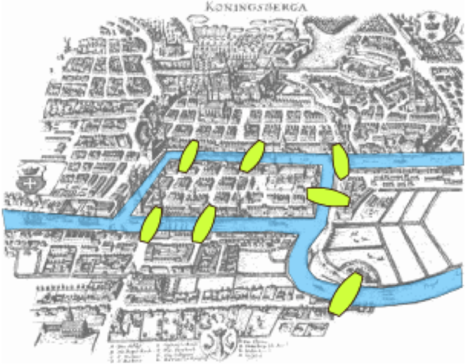

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— Shortest Paths —

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Shortest Path Problem

- $G = (V, E)$ is a connected directed graph. Each edge e has a length $l_e \geq 0$.
- V has n nodes and E has m edges.
- Length of a path P is the sum of lengths of the edges in P.
- Goal is to determine the shortest path from some start node s to each node in V.
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Shortest Paths

INSTANCE	A directed graph $G(V, E)$, a function $l : E \rightarrow \mathbb{R}^+$, and a node $s \in V$
SOLUTION	A set $\{P_u, u \in V\}$, where P_u is the shortest path in G from s to u.

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- Maintain a set S of explored nodes: for each node $u \in S$, we have determined the length $d(u)$ of the shortest path from s to u.

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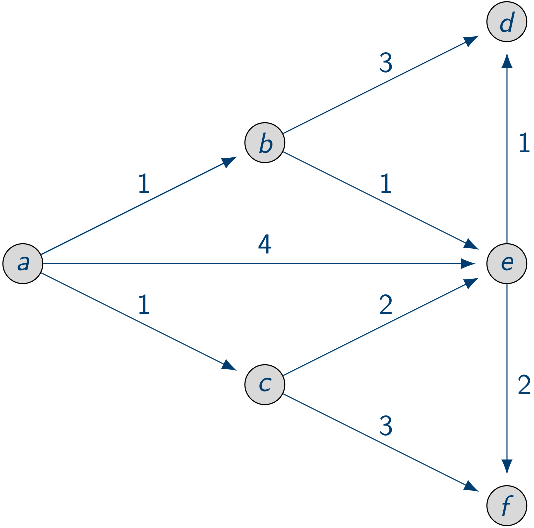
- Can modify algorithm to compute the shortest paths themselves: **record the predecessor** u that minimizes $d'(v)$.

Example of Dijkstra's Algorithm



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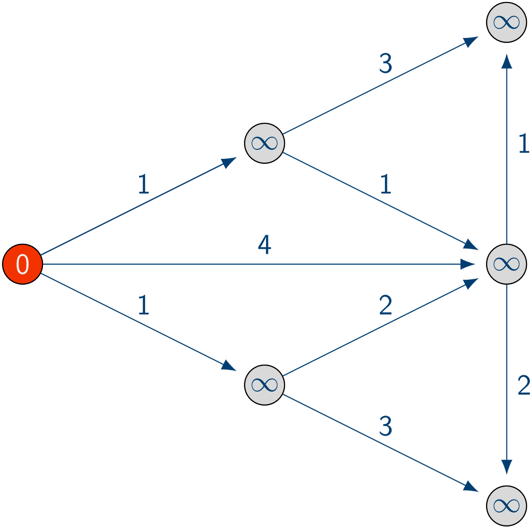

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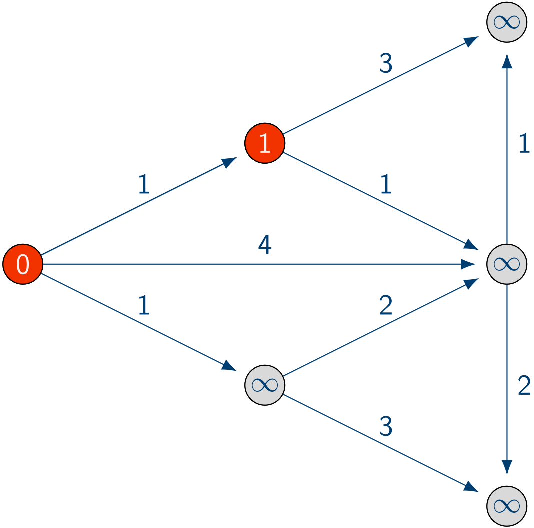

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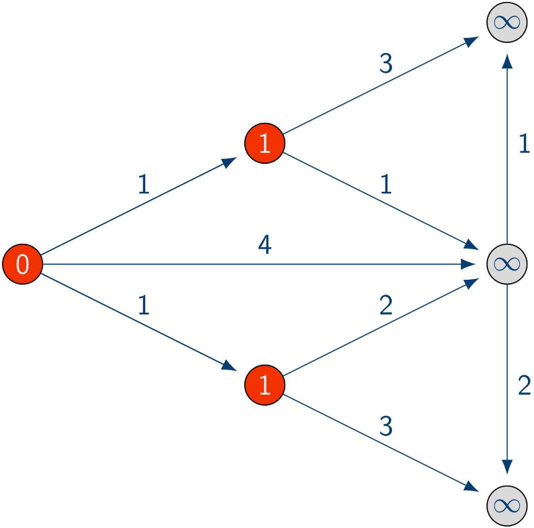

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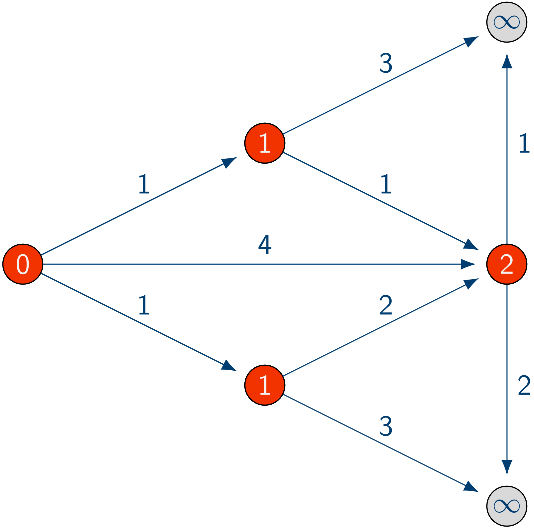

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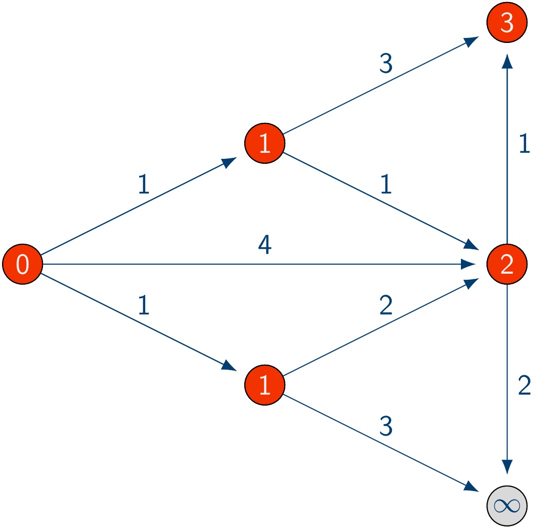

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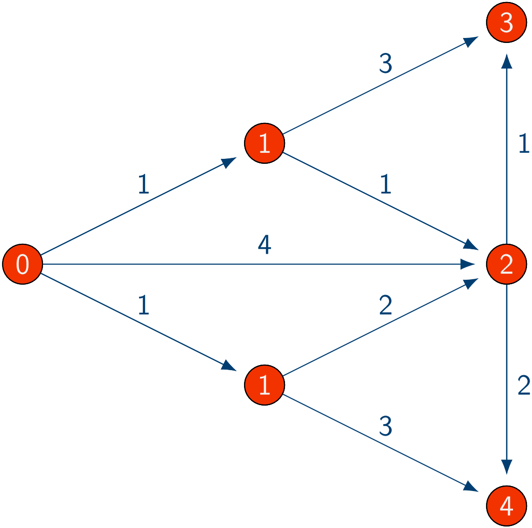

Proof of Correctness

- Let P_u be the shortest path computed for a node u.
- Claim: P_u is the shortest path from s to u.
- Prove by induction on the size of S.
  - Base case: $|S| = 1$. The only node in S is s.
  - Inductive step: we add the node v to S. Let u be the v's predecessor on the path P_v. Could there be a shorter path P from s to v?

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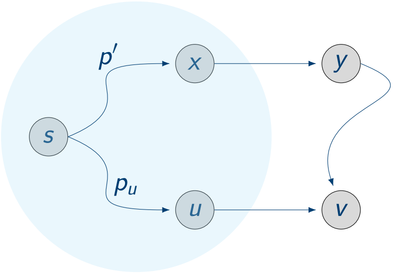

The alternate $s - v$ path P through x and y already too long by the time it had left the set S

Comments about Dijkstra's Algorithm

- Algorithm **cannot handle negative** edge lengths.
- Union of shortest paths output form a tree. Why?

Implementing Dijkstra's Algorithm

Algorithm: Shortest path algorithm – Dijkstra

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- In each iteration, for each node $v \notin S$, compute
$$\min_{e=(u,v),u \in S} d(u) + l_e.$$
- Running time **per iteration** is $O(m)$, yielding an overall running time of $O(nm)$.

A Faster implementation of Dijkstra's Algorithm

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- Store the minima $d'(v)$ for each node $v \in V - S$ in a **priority queue**.
- Determine the next node v to add to S using ExtractMin.
- After adding v, for each neighbour w of v, compute $d(v) + l_{(v,w)}$.
- If $d(v) + l_{(v,w)} < d'(w)$, update w's key using ChangeKey.

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- How many times are ExtractMin and ChangeKey invoked? $n - 1$ and m times, respectively. Total running time is $O(m \log n)$.

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— Minimum Spanning Trees —

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Network Design

- Connect a set of nodes using a set of edges with certain properties.
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- Input is usually a graph and the desired network (the output) should use subset of edges in the graph.
- Example: connect all nodes using a cycle of shortest total length. This problem is the NP-complete traveling salesman problem.

Minimum Spanning Tree (MST)

- Given an undirected graph $G = (V, E)$ with a cost $c_e > 0$ associated with each edge $e \in E$.
- Find a subset T of edges such that the graph (V, T) is connected and the cost $\sum_{e \in T} c_e$ is as small as possible.

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SOLUTION A set $T \subseteq E$ of edges such that (V, T) is connected and the $\sum_{e \in T} c_e$ is as small as possible.

- Claim: If T is a minimum-cost solution to this network design problem then (V, T) is a tree.
- A subset T of E is a spanning tree of G if (V, T) is a tree.

Greedy Algorithm for the MST Problem

- Template: process edges in some order. Add an edge to T if tree property is not violated.

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  Increasing cost order *Process edges in increasing order of cost. Discard an edge if it creates a cycle.*

  Dijkstra-like *Start from a node s and grow T outward from s: add the node that can be attached most cheaply to current tree.*

  Decreasing cost order *Delete edges in order of decreasing cost as long as graph remains connected.*

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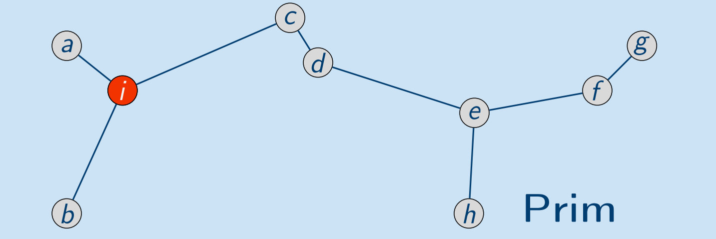
  Increasing cost order *Process edges in increasing order of cost. Discard an edge if it creates a cycle.* *Kruskal's algorithm*

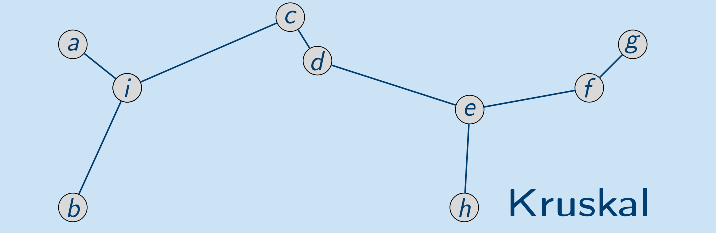
  Dijkstra-like *Start from a node s and grow T outward from s: add the node that can be attached most cheaply to current tree.* *Prim's algorithm*

  Decreasing cost order *Delete edges in order of decreasing cost as long as graph remains connected.* *Reverse-Delete algorithm*

- Which of these algorithms works? All of them!

Example of Prim's and Kruskal's Algorithms





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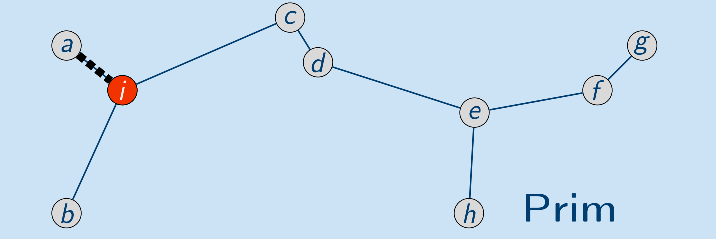

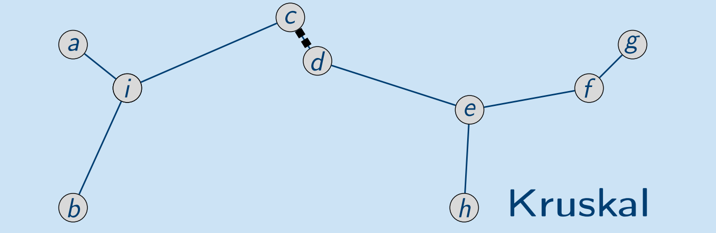

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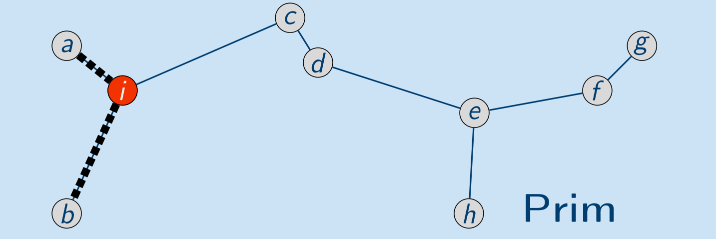

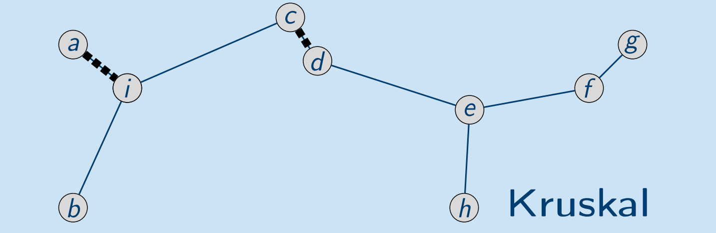

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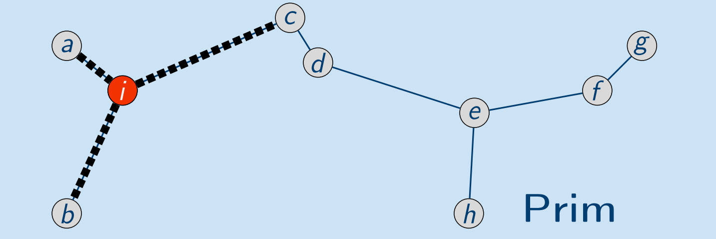

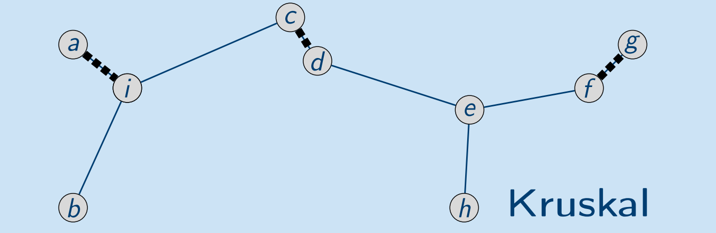

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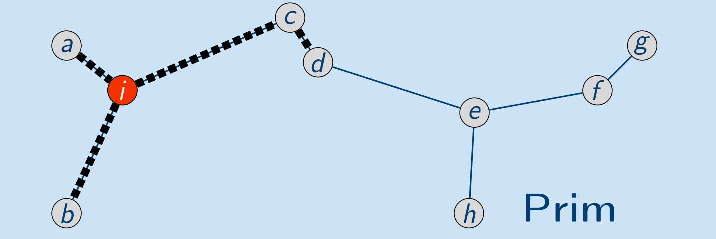

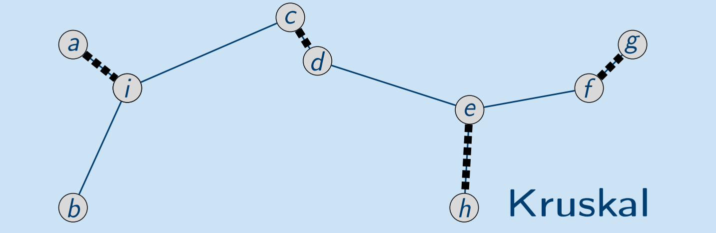

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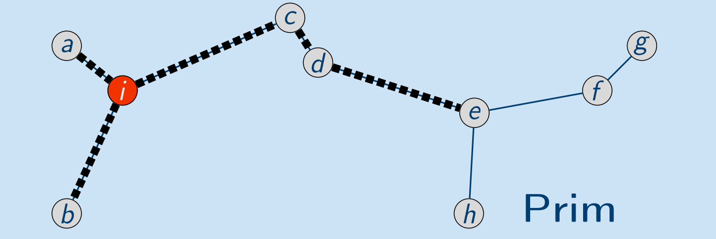
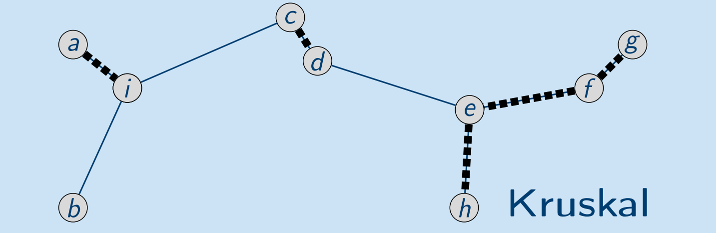
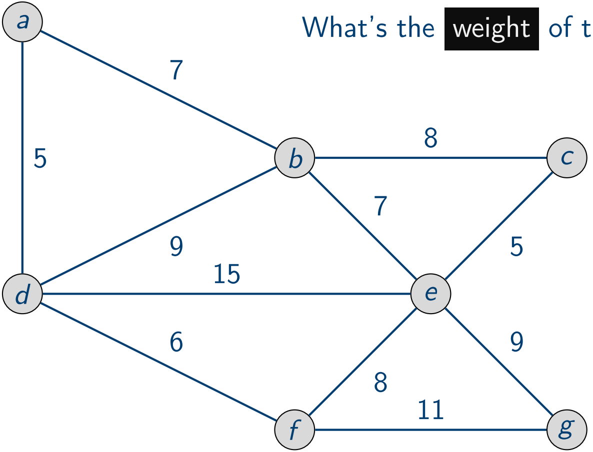




Example of Prim's Algorithm

What's the weight of the MST?



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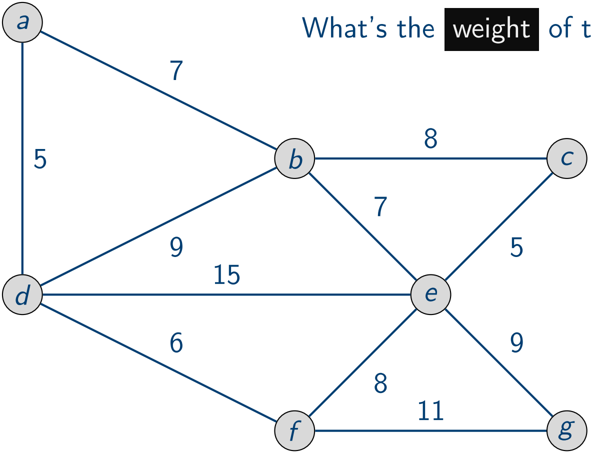

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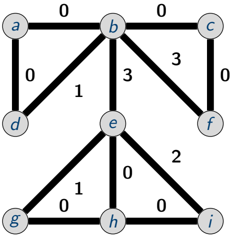

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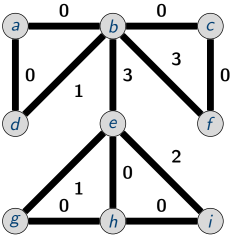

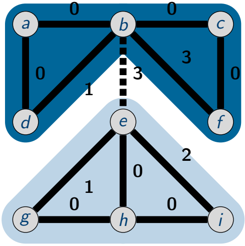

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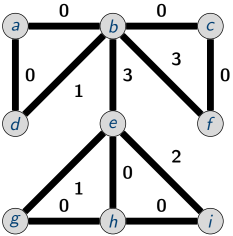

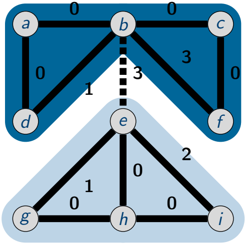

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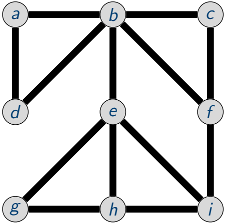

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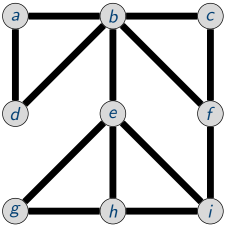

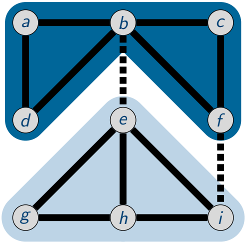

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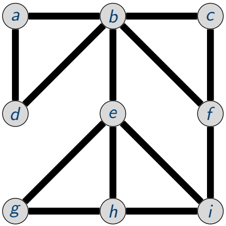

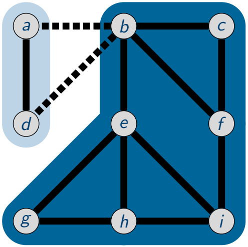

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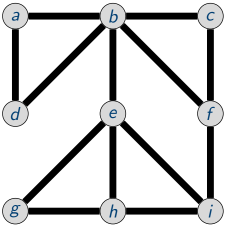
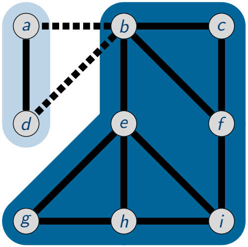
- cut(S) is a cut because deleting the edges in cut(S) disconnects S from $V - S$.





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- Proof: exchange argument. If a supposed MST T does not contain e, show that there is a tree with smaller cost than T that contains e.

Using the Cut Property

- Let F be the set of all edges that satisfy the cut property.
- Is the graph induced by F connected?
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- How many edges can F contain? $n - 1$
- F is the unique MST.
- Kruskal's and Prim's algorithms compute F **efficiently**.

Optimality of Kruskal's Algorithm

- Kruskal's algorithm:
  - Start with an empty set T of edges.
  - Process edges in E in non decreasing order of cost.
  - Add the next edge e to T only if adding e does not create a cycle. Discard e if it creates a cycle.
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  1. For every edge e added, demonstrate the existence of S and $V - S$ such that e and S satisfy the cut property.
  2. Prove that the algorithm computes a spanning tree.

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- Prim's algorithm: Maintain a tree (S, U)
  - Start with an **arbitrary** node $s \in S$ and $U = \emptyset$.
  - Add the node v to S and the edge e to U that **minimize**

$$\min_{e=(u,v), u \in S, v \notin S} c_e \equiv \min_{e \in \text{cut}(S)} c_e.$$

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  1. Prove that every edge inserted satisfies the cut property.
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Optimality of the Reverse-Delete Algorithm

- Reverse-Delete algorithm: Maintain a set E' of edges.
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- Claim: the Reverse-Delete algorithm outputs an MST.
  1. Show that every edge deleted belongs to no MST.
  2. Prove that the graph remaining at the end is a spanning tree.

Comments on MST Algorithms

- To handle **multiple edges** with the **same weight**, perturb each length by a random infinitesimal amount.
- **Any** algorithm that constructs a spanning tree by including edges that satisfy the cut property and deleting edges that satisfy the cycle property will yield an **MST**!

Teoria dos Grafos e Computabilidade

— Implementation —

Silvio Jamil F. Guimarães

Graduate Program in Informatics – PPGINF
Laboratory of Image and Multimedia Data Science – IMScience
Pontifical Catholic University of Minas Gerais – PUC Minas

Feb 2025

Implementing Prim's Algorithm

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  - Stop when $S = V$.
- Sorting edges takes $O(m \log n)$ time.
- Implementation is very similar to Dijkstra's algorithm.
- Maintain S and store attachment costs $a(v) = \min_{e \in \mathrm{cut}(S)} c_e$ for every node $v \in V - S$ in a priority queue.
- At each step, extract minimum v from priority queue and update the attachment costs of the neighbours of v.
- Total of $n - 1$ ExtractMin and m ChangeKey operations, yielding a running time of $O(m \log n)$.

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- Sorting edges takes $O(m \log n)$ time.
- Key question: "Does adding $e = (u, v)$ to T create a cycle?"
  - Maintain set of connected components of T.
  - FIND(u): return the name of the connected component of T that u belongs to.
  - UNION(A, B): merge connected components A and B.
- Answering the question: Adding e creates a cycle if and only if FIND(u) = FIND(v). If not, execute UNION(FIND(u), FIND(v)).

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- We will show two implementations of UNION-FIND:
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- Total running time of Kruskal's algorithm is $O(m \log n)$.

Teoria dos Grafos e Computabilidade

— Huffman code —

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How do we know when the next symbol begins?

Use a separation symbol (like the pause in Morse), or make sure that there is no ambiguity by ensuring that no code is a prefix of another one

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What is the meaning of 1001000001 ?

Suppose frequencies are known in a text of 1G characters: $f_a = 0.4$, $f_e = 0.2$, $f_k = 0.2$, $f_l = 0.1$, $f_u = 0.1$. What is the size of the encoded text?

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2*fa + 2*fe + 3*fk + 2*fl + 4*fu = 2.4 G bits

How to **greedly** compute a prefix tree to encode an alphabet?

Data compression: how to create a prefix tree

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Suppose frequencies are known:
$f_a = 0.32$, $f_e = 0.25$, $f_k = 0.20$, $f_l = 0.18$, $f_u = 0.05$.

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Algorithm: Huffman code

input : A set S of elements with their frequencies.
output: A prefix tree

```
if S = 2 then
    return a tree with root and 2 leaves;
else
    let y and z be lowest-frequency letters in S;
    S' = S;
    remove y and z from S';
    insert new letter w in S' with f_w = f_y + f_z;
    T'= Huffman(S');
    T = add two children y and z to leaf w from T';
    return T;
end
```