Chapter 1

Mathematical introduction

1.1 Topology preliminaries

1.1.1 Open sets

Open sets *are defined as being the sets belonging to a family* τ*, if* τ *is a topology on* X*.* τ*, in its turn, is a* ***topology*** *on* X *if the following list of requirements is satisfied:*

- $X \in \tau$ *and* $\emptyset \in \tau$*. Both the empy set and* X *are in* τ*;*

- $\{O_i\}_{i \in I} \subseteq \tau \implies \bigcup_{i \in I} O_i \in \tau$*. If the family of all* O_i *(with* i *in a arbitrary index set) is a subset of the family* τ*, then every union of the subsets* O_i *is also a subset in the family* τ*;*

- $\{O_i\}_{i=1}^{n} \subseteq \tau \implies \bigcap_{i=1}^{n} O_i \in \tau$*. If the family of all* O_i *(with* i *in a finite set) is in the family* τ*, then every (consequently) finite union of the subsets* O_i *is also a subset in the family* τ*.*

1.1.2 Image of a function and inverse image

Let A be a subset of X, $A \subseteq X$, and f a mapping that takes elements of A into another space Y. The image of $A \subseteq X$ under f is the set of all elements $f(A) \in Y$. Equivalently,

$$f(A) = \{y \in Y \mid y = f(x), \text{ for some } x \in A\}. \tag{1.1}$$

The inverse image of a subset $B \subseteq Y$ under the same function f is the subset of X defined by

$$f^{-1}(B) = \{x \in X \mid f(x) \in B\}. \tag{1.2}$$

1.1.3 Continuous functions

A function $f : X \to Y$ is said to be ***continuous*** *if for every open set $W \subseteq Y$, the inverse image of f*

$$f^{-1}(W) = \{x \in X \mid f(x) \in W\} \tag{1.3}$$

is an open subset of X.

1.1.4 Homeomorphism

A ***homeomorphism*** *is a continuous bijective function between topological spaces that has a continuous inverse function.* Homeomorphism are the isomorphism in the category of topological spaces. They are the mappings that preserve *all* topological properties of a given space.

A function $f : X \to Y$ between topological spaces X and Y is a homeomorphism if

- f is continuous;
- f is a bijection (f maps every element of X into only one element of Y, and no element of Y is "unmapped");
- f^{-1} is continuous.

That is why homeomorphism are sometimes called **bicontinuous functions**. *If there exists a function such that these three properties hold, we say X and Y are* ***homeomorphic****.*

Alternatively, a ***topological property****, or* ***topological invariant*** *may be defined as a property that is unchanged by homeomorphisms.*

1.1.5 Cartesian product

Let A and B be two sets, for which the elements of A are denoted by a and the elements of B denoted by b. *The* ***cartesian product*** *(abbreviated by the symbol $\times$) of A and B is a new set, say, C, which corresponds to the set formed by all ordered pairs (a, b).* In other words,

$$C = A \times B = \{(a, b) \mid a \in A, b \in B\}. \tag{1.4}$$

a and b may as well be n- and m-tuples, where the corresponding $c \in C$ will be represented by a pair of tuples.

Since the cartesian product of two sets is itself a new set, one can evidently perform the cartesian product of this new set with another abirtrary set, which enables the generalization of the Cartesian product to a product of n sets, *the* ***n-ary Cartesian product****, defined as*

$$\begin{aligned}\prod_{i=1}^{n} X_i &\coloneqq X_1 \times \ldots \times X_i \times \ldots \times X_n = \\ &= \{(x_1, \ldots, x_i, \ldots, x_n) \mid x_i \in X_i, \forall i \in \{1, 2, \ldots, n\}\}.\end{aligned} \tag{1.5}$$

Cartesian products need not be finite, and the index of summation doesn't need to belong to a countable set. *One can define an* ***infinite Cartesian product*** *as*

$$\prod_{i \in I} X_i = \left\{ f : I \to \bigcup_{i \in I} X_i \mid \forall i \in I, f(i) \in X_i \right\}, \tag{1.6}$$

the set of all functions f defined on I such that its image $f(i)$ is itself an element of X_i.

1.1.6 Product space

A special case of the previously defined Cartesian product are product spaces. *A* ***product space*** *X is the space defined by the infinite Cartesian product*

$$X \coloneqq \prod_i X_i, i \in I, \tag{1.7}$$

where I is any index set, X_i are the canonical projections $p_i : X \to X_i$, and the family of X_i is equipped with a product topology.

In its turn, *a* ***product topology*** *on X is the topology with the fewest open sets for which p_i are all continuous.*

1.1.7 Homotopy

Let X and Y be topological spaces, and f and g continuous functions, both mapping the space X into the space Y. *A* ***homotopy*** *between f and g is defined to be a continuous function $h : X \times [0, 1] \to Y$ such that if $x \in X$, then $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$.*

In other words, a homotopy h can be parameterized by a real number $t \in [0, 1]$, such that $h(x, t)$ will be a continuous function mapping the space X in the space Y for every value in its domain, where $h(x, 0) = f(x)$ and $h(x, t) = g(x)$. To simplify its visualization, t can be regarded as the "time", and the mapping $f(x)$ will be smoothly deformed until it reaches its final value $g(x)$.

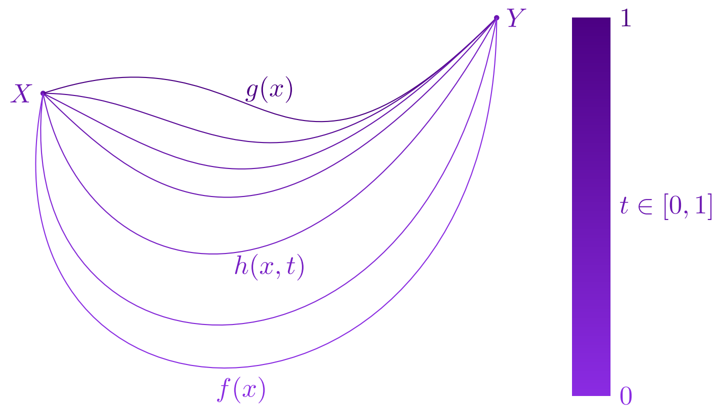
Alternatively, one can also view t as an "extra dimension", where $h(x, t)$ will start from a "basis", the mapping $f(x)$, and be smoothly deformed along the "extra dimension" t, until it reaches its "top", namely, $g(x)$.

Two maps are said to be ***homotopic*** *if and only if there exists a homotopy connecting them.*

1.1.8 Pointed spaces

Base points *are points in a space that one names and keeps track after successive operations, remaining unchanged throughout the whole discussion.*

The space containing a specific base point, say, x_0, is called ***pointed space****.*



*If a map f between the topologies of X (with base point x_0) and Y (with base point y_0) is continuous with respect to their topology and $f(x_0) = y_0$, f is usually called **based map**,*

$$f : (X, x_0) \to (Y, y_0),\ x_0 \in X,\ y_0 \in Y. \tag{1.8}$$

1.1.9 Homotopy group

Let S^n be the n-sphere, where we choose a as its base point. Let also X be another topological space, where its base point is chosen to be b. *The n-th **homotopy group** of X with respect to a is defined to be the set of homotopy classes of maps*

$$\pi_n(X) \coloneqq \{f : S^n \to X\} \tag{1.9}$$

that map base point a into base point b.

For two topological spaces to be homeomorphic, they must share the same homotopy group. However, two spaces sharing the same homotopy group are not guaranteed to be homeomorphic.

Examples

Let T be the topological space of the torus. Its homotopy group, $\pi_1(T)$ is

1.1.10 Simplicial homology

1.1.11 CW complexes

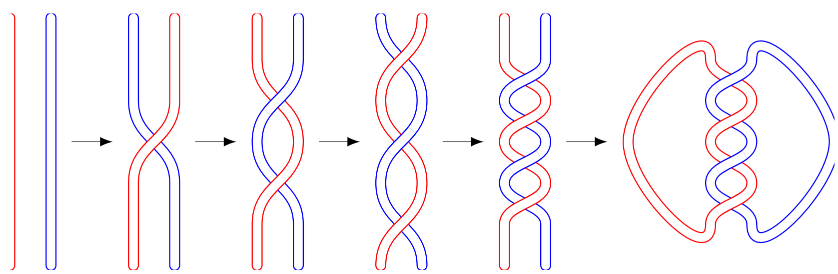
1.1.12 Euler characteristics

1.1.13 Reidemeister moves

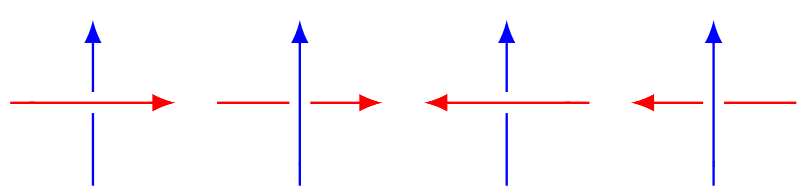
1.1.14 Linking number

Consider two curves closed in three dimensional space $\mathbb{R}^3$. Now one allows each curve to pass through themselves, but by no means by one another. The number of times each curve "winds" through each other will then be defined as the ***linking number*** *of the system.*

For simpler visualization, one can consider two straight wires, and then twist their tips, in order to create a twisted (say, DNA-like) structure. Then, one glues one tip of each curve to the other tip of the same curve, like represented below.



Any possible configuration of links can be continuously deformed (is homotopic) to one of the canonical links, so one can simply identify the corresponding canonical link to the configuration of interest, because the linking number is a topological invariant under such mappings (to be demonstrated below). Or, equivalently, it's possible to draw the link diagram, "laying" the curve over a plane and count how many times closed curve A passes above and how many times below curve B, over an given orientation for each curve. Four types of crossings will then be possible, two computing positive values to the linking number, and two of them giving a negative contribution. The resulting value will be twice the linking number. The possible crossings are shown below.



Gauss' integral definition

1.1.15 Hopf invariant

1.1.16 Medial graph

Chapter 2

Ordered media

For almost all of our purposes here *an **ordered medium** can be regarded as a region of space described by a function $f(r)$ that assigns to every point of the region an order parameter.* The possible values of the order parameter constitute a space known as the ordered- parameter space (or manifold of internal states).

2.1 Order parameter

2.2 Topology of defects

2.2.1 Spins confined on the plane

The reason why we are going to consider this example is because it simplifies a lot the actual case for two reasons: it obviously only allow spins to be located at a plane, cutting down one position coordinate, and makes mapping to order parameter space fairly easier.

Spins are intrinsic angular momenta, carried by elementary particles, such as electrons, composite particles, such as protons, and consequently atomic nuclei and atoms. Spins will usually behave as magnetic dipole moments, having intrinsic magnetic momenta associated with its intrinsic properties (the spin number of the particles and its charge).

In macroscopic media, dipoles will be separated one another by distances much smaller than macroscopic distances of usual interest. Consequently, in what follows, we may consider each point of our medium in a region of real space $\mathbb{R}^2$ to have a certain value of spin, which will be itself another quantity represented by a vector in $\mathbb{R}^2$, just like any magnetic moment (or any arbitrary 2D vector). However, if we consider our material to be *homogeneous*, all particles will have the same magnetic moment module (previously known/measured), leaving us with only one quantity to completely describe the magnetic moment value of each point in space (note that the *material* is said to be *homogeneous*, not the spin field/configuration).

A schematic representation is depicted below.

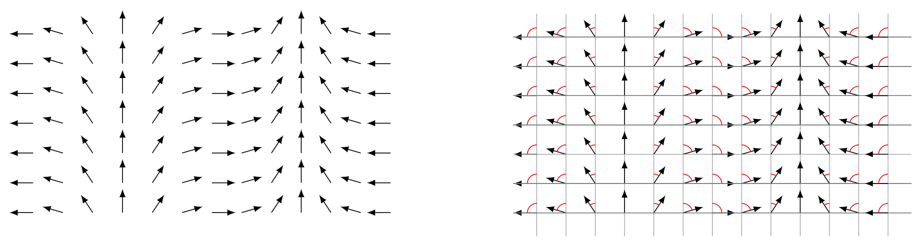


Figure 2.1: Example of 2D spin configuration. Each position (x, y) is associated with a direction, forming then a 2D vector field, $\vec{v} : \mathbb{R}^2 \to \mathbb{R}^2$. Spins, however, have all the same magnitude, being completely defined by their angle with respect to an fixed axis. One is then enabled to map the order parameter as $s : \mathbb{R}^2 \to [0, 1]$. On the right side, the definition of each angle (particle spin orientation) associated with a ordered pair (x, y) (particle position).

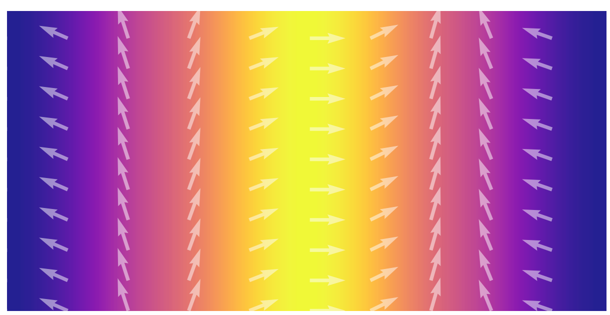


Figure 2.2: Sequential linear colormap of order parameter (background), $s(x, y) = \text{color}(\theta)$, as defined above. The order parameter in this case, as already discussed, is a mapping $s : \mathbb{R}^2 \to [0, 1]$. Superposed to the colormap, some vectors are drawn to represent the direction of the magnetic moment on (x, y), the vector origin.

As said above, magnetic moments will then be a vector with fixed length in $\mathbb{R}^2$. This makes it possible to completely determine them, for each point in space, with only one parameter - its angle according to a certain direction. One can then define a mapping that takes a point in $\mathbb{R}^2$ into a real value in the range $[0, 2\pi[$ (or any infinite bounded open set of real numbers), covering all configuration possibilities unambiguously. These possible configurations will sweep the surface of a circle in $\mathbb{R}^2$, meaning that the *spin-space* will be *homeomorphic* to a sphere in $\mathbb{R}^2$, called S^1. This is

$$s(x, y) = \left\{ f : \mathbb{R}^2 \to S^1 \mid \forall (x, y) \in \mathbb{R}^2, s(x, y) \in S^1 \right\}, \tag{2.1}$$

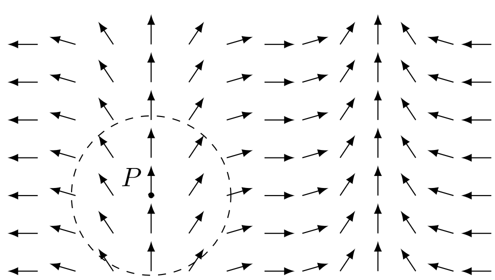
*where $s(x, y)$ is here called the **order parameter** of our space.*

2.2.2 Topological quantum number

Let P be a point in our medium, $\mathbb{R}^2$, corresponding to, perhaps, a singularity of the magnetic moment direction at that point (or equivalently the order parameter itself). In a open region D^2 around point P, no other singularity may be present. *The **topological quantum number** of a point P will be defined as the number of times the order parameter "walks" through its image as one follows the path described by the boundary ∂D^2 of the open region centered in P in a predetermined orientation.* Usually, one defines the positive orientation as being the counterclockwise direction. If the order parameter increases as one "walks" through its image, a positive topological quantum number is assured. Otherwise, a negative topological quantum number is obtained. If the order parameter increases for some part of the "walk" and then decreases equally, its value will be null.

Since the order parameter won't be singular at any other place except, perhaps, at point P, it *must* at the end of the contour return continuously to the same value as one had when started the "walk". For this exact reason, *only integer values will be accepted to characterize the topological quantum number*, hence the "quantum" in its name. If one could have non-integer values of the topological quantum number, the order parameter at the end of the contour s_f should be different from the s_0 one started (for one to have "incomplete walks through the image's path"), providing two values of the order parameter on the same point, contradicting our hypothesis of non-singularity.

Much better than any extensive discussion, a schematic representation on how to calculate the topological quantum number is depicted below.



Let us get a plot of the vector field on the boundary ∂D^2 of the disk around P, in order to visualize the orientation of the spins in each point of the contour.

Now, let us "walk" throughout the contour and keep track of the order parameter's image. For each value of the angle θ on the contour relative to a certain direction one will have a corresponding value of $s(\theta)$, making it possible to plot a graph representing the evolution of s as one "walks" throughout the contour. A popular method of analyzing the results is known as the ***hodogram***

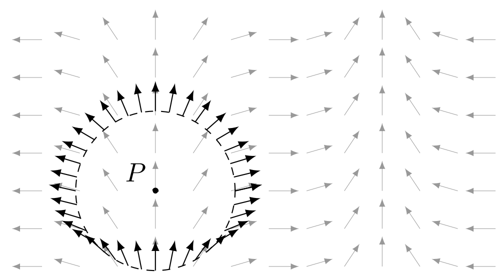


Figure 2.3: Vector field explicited over the contour ∂D^2. Lightly exposed on the background, the vector field throughout space, for comparison and guide eyes.

method*, which consists of varying θ and plot each vector tip over the order parameter space and, since the order parameter must vary continuously, as it is a continuous function of a continuous parameter, the vector heads will trace a curve over the order parameter space.* An example of the hodogram method is depicted below.

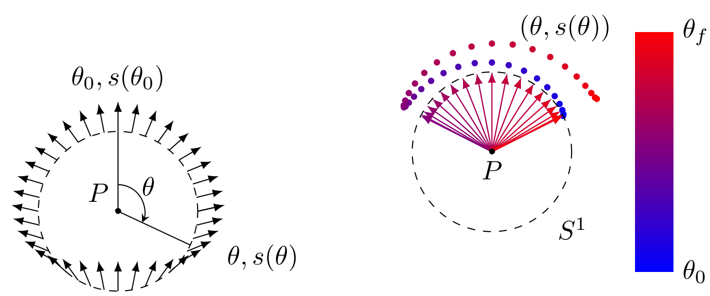


Figure 2.4: On the left, depiction of each spin value on the contour, according to the angle θ each vector makes with the direction we choose to be our start θ_0. On the right, the mapping of each spin corresponding to each θ in order parameter space. The arrow associated with θ_0 is shown to be blue, and as one increases θ, the arrow becomes red, until reaching its maximum value. Coloured dots over the S^1 sphere mean the evolution of order parameter (spin angle) on increasing θ. Note that the dots start to "wind" the S^1 sphere, until reaching a turning point, where the dots begin to "unwind" the order parameter space. This associates a $n = 0$ topological quantum number to the point P, since our "path" does not even complete a walk throughout the whole image of s.

One can imagine the parameter θ along the contour as a "time" parameter, and an arrow nailed to the center of a circle containing a pen on its tip. "As the time passes", the arrow will generally move, just like the vectors in the real system move as one "walks" throughout the contour. The pen, on its turn, will draw a curve over the circle, and we can count the times the curve makes complete turns around the circle (order parameter), and then assign this number to our topological quantum number of point P around contour ∂D^2. Note that the curve traced by the pen must end exactly where it started, for continuiry of the order parameter to be assured. This gives an intuitive view to why only integer values for the topological quantum number are permitted, as was previously demonstrated.

Since only integer values of topological quantum number and, consequently, "turns" on the order parameter space "drawn" by the values of $s(\theta)$ on our "walk" throughout the contour, *the topological quantum number is also known as the* ***winding number***.

Let us now evaluate another configuration

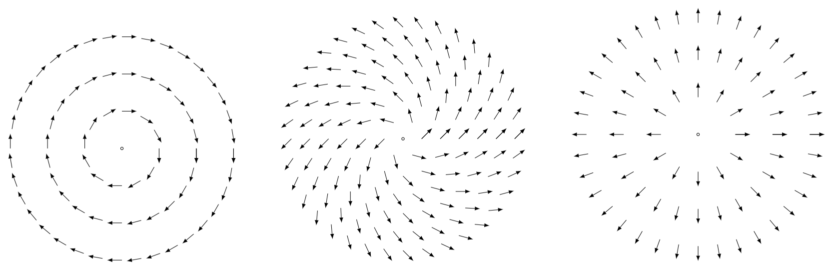


Figure 2.5: Topological defects with the same winding number $n = 1$ at each figure's center. However, with different phases β, corresponding to (a), (b) and (c)

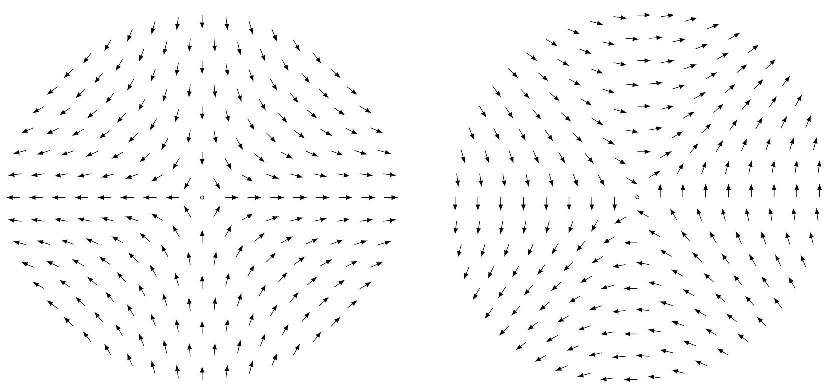


Figure 2.6: Once again, topological defects with the same winding number. This time, $n = -1$. Where the phases β correspond to (a), (b) and (c)

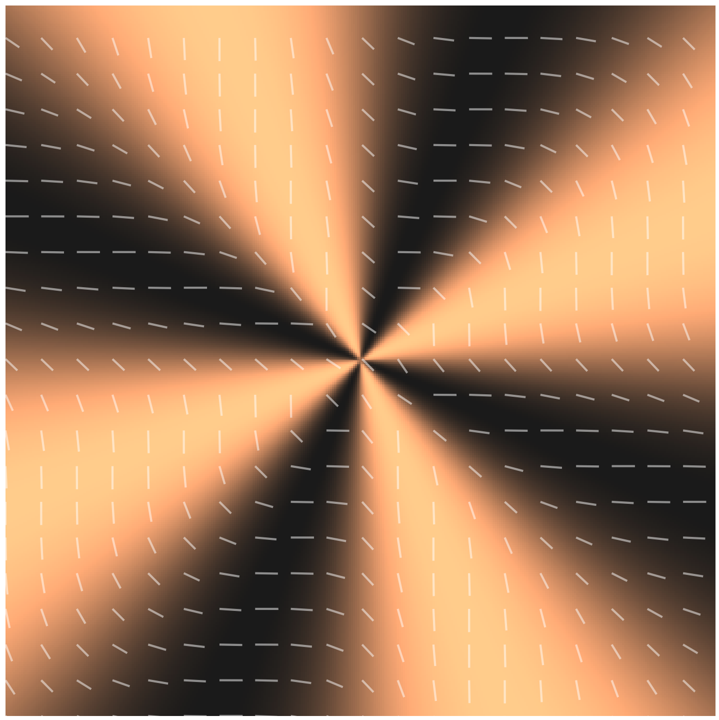


Figure 2.7: Nematic (headless) director profile, corresponding to a $n = 1$ winding number topological defect, with $\beta = 0$ phase. Even though this colormap configuration may seem like what one sees in an experimental POM image of a nematic sample containing a topological defect of such charge, these angles (and, consequently, colors) correspond to the director orientation with respect to the y axis (director contained in the plane of the text). The similarity occurs because $\theta = 0$ here is represented by black (minimum value) while $\theta = \pi/2$ by the brightest color, which is exactly what occurs in reality, except for the angle (order parameter) being mapped with respect to the z axis (α). Of course, for a axisymmetric sample with respect to the z axis, θ variations won't matter. Further is discussed in the text.

2.2.3 Kosterlitz-Thouless transition

2.2.4 Uniaxial nematics on the plane

2.2.5 Three-dimensional spins confined on a plane

2.2.6 Gaussian mode light propagation

Hermite-Gaussian modes

Laguerre-Gaussian modes

2.2.7 Gravitational lenses

2.2.8 extra

Chapter 3

Physical properties of topological defects

Chapter 4

Uniaxial liquid crystals

Chapter 5

Chirality