Chapter 1

Mathematical introduction

1.1 Topology preliminaries

1.1.1 Open sets

Open sets *are defined as being the sets belonging to a family* τ*, if* τ *is a topology on* X*.* τ*, in its turn, is a* ***topology*** *on* X *if the following list of requirements is satisfied:*

- $X \in \tau$ *and* $\emptyset \in \tau$*. Both the empy set and* X *are in* τ*;*

- $\{O_i\}_{i \in I} \subseteq \tau \implies \cup_{i \in I} O_i \in \tau$*. If the family of all* O_i *(with* i *in a arbitrary index set) is a subset of the family* τ*, then every union of the subsets* O_i *is also a subset in the family* τ*;*

- $\{O_i\}_{i=1}^{n} \subseteq \tau \implies \cap_{i=1}^{n} O_i \in \tau$*. If the family of all* O_i *(with* i *in a finite set) is in the family* τ*, then every (consequently) finite union of the subsets* O_i *is also a subset in the family* τ*.*

1.1.2 Image of a function and inverse image

1.1.3 Continuous functions

A function $f : X \to Y$ *is said to be* ***continuous*** *if for every open set* $W \subseteq Y$*, the inverse image of* f

$$f^{-1}(W) = \{x \in X \mid f(x) \in W\} \tag{1.1}$$

is an open subset of X*.*

1.1.4 Homeomorphism

A ***homeomorphism*** *is a continuous bijective function between topological spaces that has a continuous inverse function.* Homeomorphism are the isomorphism in the category of topological spaces. They are the mappings that preserve *all* topological properties of a given space.

A function $f : X \to Y$ between topological spaces X and Y is a homeomorphism if

- f is continuous;
- f is a bijection (f maps every element of X into only one element of Y, and no element of Y is "unmapped");
- f^{-1} is continuous.

That is why homeomorphism are sometimes called **bicontinuous functions**. *If there exists a function such that these three properties hold, we say X and Y are* ***homeomorphic****.*

Alternatively, a ***topological property****, or* ***topological invariant*** *may be defined as a property that is unchanged by homeomorphisms.*

1.1.5 Cartesian product

Let A and B be two sets, for which the elements of A are denoted by a and the elements of B denoted by b. *The* ***cartesian product*** *(abbreviated by the symbol $\times$) of A and B is a new set, say, C, which corresponds to the set formed by all ordered pairs (a, b).* In other words,

$$C = A \times B = \{(a, b) \mid a \in A, b \in B\}. \tag{1.2}$$

a and b may as well be n- and m-tuples, where the corresponding $c \in C$ will be represented by a pair of tuples.

Since the cartesian product of two sets is itself a new set, one can evidently perform the cartesian product of this new set with another abirtrary set, which enables the generalization of the Cartesian product to a product of n sets, *the* ***n-ary Cartesian product****, defined as*

$$\begin{aligned}\prod_{i=1}^{n} X_i &\coloneqq X_1 \times \ldots \times X_i \times \ldots \times X_n = \\ &= \{(x_1, \ldots, x_i, \ldots, x_n) \mid x_i \in X_i, \forall i \in \{1, 2, \ldots, n\}\}.\end{aligned} \tag{1.3}$$

Cartesian products need not be finite, and the index of summation doesn't need to belong to a countable set. *One can define an* ***infinite Cartesian product*** *as*

$$\prod_{i \in I} X_i = \{f : I \to \cup_{i \in I} X_i \mid \forall i \in I, f(i) \in X_i\}, \tag{1.4}$$

the set of all functions f defined on I such that its image $f(i)$ is itself an element of X_i.

1.1.6 Product space

A special case of the previously defined Cartesian product are product spaces. *A **product space** X is the space defined by the infinite Cartesian product*

$$X \coloneqq \prod_i X_i, i \in I, \tag{1.5}$$

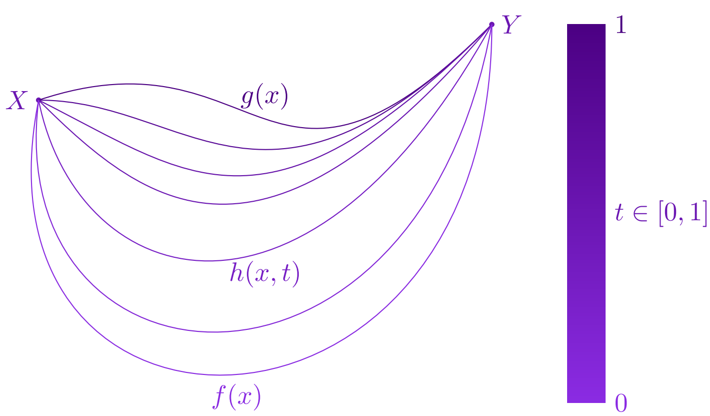
where I is any index set, X_i are the canonical projections $p_i : X \to X_i$, and the family of X_i is equipped with a product topology.

In its turn, *a **product topology** on X is the topology with the fewest open sets for which p_i are all continuous.*

1.1.7 Homotopy

Let X and Y be topological spaces, and f and g continuous functions, both mapping the space X into the space Y. *A **homotopy** between f and g is defined to be a continuous function $h : X \times [0,1] \to Y$ such that if $x \in X$, then $h(x,0) = f(x)$ and $h(x,1) = g(x)$.*

In other words, a homotopy h can be parameterized by a real number $t \in [0,1]$, such that $h(x,t)$ will be a continuous function mapping the space X in the space Y for every value in its domain, where $h(x,0) = f(x)$ and $h(x,t) = g(x)$. To simplify its visualization, t can be regarded as the "time", and the mapping $f(x)$ will be smoothly deformed until it reaches its final value $g(x)$.



Alternatively, one can also view t as an "extra dimension", where $h(x,t)$ will start from a "basis", the mapping $f(x)$, and be smoothly deformed along the "extra dimension" t, until it reaches its "top", namely, $g(x)$.

*Two maps are said to be **homotopic** if and only if there exists a homotopy connecting them.*

Chapter 2

Ordered media

For almost all of our purposes here *an **ordered medium** can be regarded as a region of space described by a function $f(r)$ that assigns to every point of the region an order parameter.* The possible values of the order parameter constitute a space known as the ordered- parameter space (or manifold of internal states).

2.1 Order parameter

2.2 Topology of defects

2.2.1 Spins confined on the plane

2.2.2 Uniaxial nematics on the plane

2.2.3 Three-dimensional spins confined on a plane

2.2.4 Light propagation (Gaussian mode)

Hermite-Gaussian modes

Laguerre-Gaussian modes

2.2.5 extra