Chapter 1

Mathematical introduction

1.1 Symbols

- I means the **unit interval**, defined to be the closed interval $[0, 1]$. Or, in other words, the set of real numbers which are greather than or equal to zero and less than or equal to one;
- $X = \{\ldots\}$ we describe the elements of the set X by what is inside the curly brackets (extensional definition);

1.2 Topology preliminaries

1.2.1 Open sets

1.2.2 Continuous functions

*A function $f : X \to Y$ is said to be **continuous** if for every open set $W \subseteq Y$, the inverse image of f*

$$f^{-1}(W)\{x \in X \mid f(x) \in W\} \tag{1.1}$$

is an open subset of X.

1.2.3 Homeomorphism

*A **homeomorphism** is a continuous bijective function between topological spaces that has a continuous inverse function.* Homeomorphism are the isomorphism in the category of topological spaces. They are the mappings that preserve *all* topological properties of a given space.

A function $f : X \to Y$ between topological spaces X and Y is a homeomorphism if

- f is continuous;

- f is a bijection (f maps every element of X into only one element of Y, and no element of Y is "unmapped");
- f^{-1} is continuous.

That is why homeomorphism are sometimes called **bicontinuous functions**. *If there exists a function such that these three properties hold, we say X and Y are* ***homeomorphic****.*

Alternatively, a ***topological property****, or* ***topological invariant*** *may be defined as a property that is unchanged by homeomorphisms.*

1.2.4 Cartesian product

Let A and B be two sets, for which the elements of A are denoted by a and the elements of B denoted by b. *The* ***cartesian product*** *(abbreviated by the symbol $\times$) of A and B is a new set, say, C, which corresponds to the set formed by all ordered pairs (a, b).* In other words,

$$C = A \times B = \{(a, b) \mid a \in A, b \in B\}. \tag{1.2}$$

a and b may as well be n- and m-tuples, where the corresponding $c \in C$ will be represented by a pair of tuples.

1.2.5 Product space

A ***product space*** *X is the space defined by*

$$X \coloneqq \prod_i X_i, i \in [0, 1], \tag{1.3}$$

where X_i are the canonical projections $p_i : X \to X_i$, and the family of X_i is equipped with a product topology.

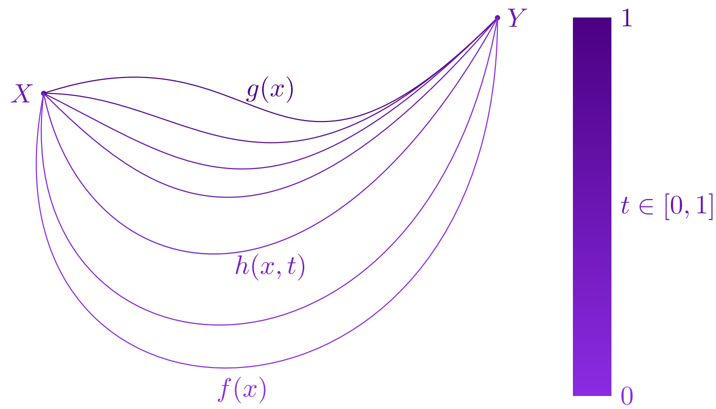
In its turn, *a* ***product topology*** *on X is the topology with the fewest open sets for which p_i are all continuous.*

1.2.6 Homotopy

Let X and Y be topological spaces, and f and g continuous functions, both mapping the space X into the space Y. *A* ***homotopy*** *between f and g is defined to be a continuous function $h : X \times [0, 1] \to Y$ such that if $x \in X$, then $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$.*

In other words, a homotopy h can be parameterized by a real number $t \in [0, 1]$, such that $h(x, t)$ will be a continuous function mapping the space X in the space Y for every value in its domain, where $h(x, 0) = f(x)$ and $h(x, t) = g(x)$. To simplify its visualization, t can be regarded as the "time", and the mapping $f(x)$ will be smoothly deformed until it reaches its final value $g(x)$.

Alternatively, one can also view t as an "extra dimension", where $h(x, t)$ will start from a "basis", the mapping $f(x)$, and be smoothly deformed along the "extra dimension" t, until it reaches its "top", namely, $g(x)$.



Two maps are said to be ***homotopic*** *if and only if there exists a homotopy connecting them.*

Chapter 2

Ordered media

For almost all of our purposes here *an **ordered medium** can be regarded as a region of space described by a function $f(r)$ that assigns to every point of the region an order parameter*. The possible values of the order parameter constitute a space known as the ordered- parameter space (or manifold of internal states).

2.1 Order parameter

2.2 Topology of defects

2.2.1 Spins on the plane