Chapter 1

Mathematical introduction

1.1 Topology preliminaries

1.1.1 Open sets

Open sets *are defined as being the sets belonging to a family* τ, *if* τ *is a topology on* X. τ, *in its turn, is a* ***topology*** *on* X *if the following list of requirements is satisfied:*

- $X \in \tau$ *and* $\emptyset \in \tau$. *Both the empy set and* X *are in* τ;
- $\{O_i\}_{i \in I} \subseteq \tau \implies \bigcup_{i \in I} O_i \in \tau$. *If the family of all* O_i *(with* i *in a arbitrary index set) is a subset of the family* τ, *then every union of the subsets* O_i *is also a subset in the family* τ;
- $\{O_i\}_{i=1}^{n} \subseteq \tau \implies \bigcap_{i=1}^{n} O_i \in \tau$. *If the family of all* O_i *(with* i *in a finite set) is in the family* τ, *then every (consequently) finite union of the subsets* O_i *is also a subset in the family* τ.

1.1.2 Image of a function and inverse image

1.1.3 Continuous functions

A function $f : X \to Y$ *is said to be* ***continuous*** *if for every open set* $W \subseteq Y$, *the inverse image of* f

$$f^{-1}(W) = \{x \in X \mid f(x) \in W\} \tag{1.1}$$

is an open subset of X.

1.1.4 Homeomorphism

A ***homeomorphism*** *is a continuous bijective function between topological spaces that has a continuous inverse function.* Homeomorphism are the isomorphism in the category of topological spaces. They are the mappings that preserve *all* topological properties of a given space.

A function $f : X \to Y$ between topological spaces X and Y is a homeomorphism if

- f is continuous;
- f is a bijection (f maps every element of X into only one element of Y, and no element of Y is "unmapped");
- f^{-1} is continuous.

That is why homeomorphism are sometimes called **bicontinuous functions**. *If there exists a function such that these three properties hold, we say X and Y are **homeomorphic**.*

*Alternatively, a **topological property**, or **topological invariant** may be defined as a property that is unchanged by homeomorphisms.*

1.1.5 Cartesian product

Let A and B be two sets, for which the elements of A are denoted by a and the elements of B denoted by b. *The **cartesian product** (abbreviated by the symbol $\times$) of A and B is a new set, say, C, which corresponds to the set formed by all ordered pairs (a, b).* In other words,

$$C = A \times B = \{(a, b) \mid a \in A, b \in B\}. \tag{1.2}$$

a and b may as well be n- and m-tuples, where the corresponding $c \in C$ will be represented by a pair of tuples.

Since the cartesian product of two sets is itself a new set, one can evidently perform the cartesian product of this new set with another abirtrary set, which enables the generalization of the Cartesian product to a product of n sets, *the **n-ary Cartesian product**, defined as*

$$\begin{aligned}\prod_{i=1}^{n} X_i &\coloneqq X_1 \times \ldots \times X_i \times \ldots \times X_n = \\ &= \{(x_1, \ldots, x_i, \ldots, x_n) \mid x_i \in X_i, \forall i \in \{1, 2, \ldots, n\}\}.\end{aligned} \tag{1.3}$$

Cartesian products need not be finite, and the index of summation doesn't need to belong to a countable set. *One can define an **infinite Cartesian product** as*

$$\prod_{i \in I} X_i = \left\{ f : I \to \bigcup_{i \in I} X_i \mid \forall i \in I, f(i) \in X_i \right\}, \tag{1.4}$$

the set of all functions f defined on I such that its image $f(i)$ is itself an element of X_i.

1.1.6 Product space

A special case of the previously defined Cartesian product are product spaces. *A **product space** X is the space defined by the infinite Cartesian product*

$$X \coloneqq \prod_i X_i, i \in I, \tag{1.5}$$

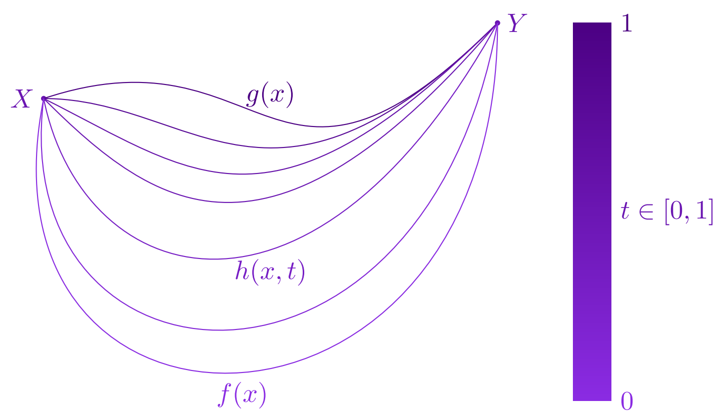
where I is any index set, X_i are the canonical projections $p_i : X \to X_i$, and the family of X_i is equipped with a product topology.

In its turn, *a **product topology** on X is the topology with the fewest open sets for which p_i are all continuous.*

1.1.7 Homotopy

Let X and Y be topological spaces, and f and g continuous functions, both mapping the space X into the space Y. *A **homotopy** between f and g is defined to be a continuous function $h : X \times [0, 1] \to Y$ such that if $x \in X$, then $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$.*

In other words, a homotopy h can be parameterized by a real number $t \in [0, 1]$, such that $h(x, t)$ will be a continuous function mapping the space X in the space Y for every value in its domain, where $h(x, 0) = f(x)$ and $h(x, t) = g(x)$. To simplify its visualization, t can be regarded as the "time", and the mapping $f(x)$ will be smoothly deformed until it reaches its final value $g(x)$.



Alternatively, one can also view t as an "extra dimension", where $h(x, t)$ will start from a "basis", the mapping $f(x)$, and be smoothly deformed along the "extra dimension" t, until it reaches its "top", namely, $g(x)$.

*Two maps are said to be **homotopic** if and only if there exists a homotopy connecting them.*

1.1.8 Pointed spaces

Base points *are points in a space that one names and keeps track after successive operations, remaining unchanged throughout the whole discussion.*

The space containing a specific base point, say, x_0*, is called* ***pointed space****.*

If a map f *between the topologies of* X *(with base point* x_0*) and* Y *(with base point* y_0*) is continuous with respect to their topology and* $f(x_0) = y_0$*,* f *is usually called* ***based map****,*

$$f : (X, x_0) \to (Y, y_0),\ x_0 \in X,\ y_0 \in Y. \tag{1.6}$$

1.1.9 Homotopy group

Let S^n be the n-sphere, where we choose a as its base point. Let also X be another topological space, where its base point is chosen to be b. *The* n*-th* ***homotopy group*** *of* X *with respect to* a *is defined to be the set of homotopy classes of maps*

$$\pi_n(X) \coloneqq \{f : S^n \to X\} \tag{1.7}$$

that map base point a *into base point* b*.*

For two topological spaces to be homeomorphic, they must share the same homotopy group. However, two spaces sharing the same homotopy group are not guaranteed to be homeomorphic.

Examples

Let T be the topological space of the torus. Its homotopy group, $\pi_1(T)$ is

1.1.10 CW complexes

1.1.11 Euler characteristics

Chapter 2

Ordered media

For almost all of our purposes here *an **ordered medium** can be regarded as a region of space described by a function $f(r)$ that assigns to every point of the region an order parameter.* The possible values of the order parameter constitute a space known as the ordered- parameter space (or manifold of internal states).

2.1 Order parameter

2.2 Topology of defects

2.2.1 Spins confined on the plane

The reason why we are going to consider this example is because it simplifies a lot the actual case for two reasons: it obviously only allow spins to be located at a plane, cutting down one position coordinate, and makes mapping to order parameter space fairly easier.

Spins are intrinsic angular momenta, carried by elementary particles, such as electrons, composite particles, such as protons, and consequently atomic nuclei and atoms. Spins will usually behave as magnetic dipole moments, having intrinsic magnetic momenta associated with its intrinsic properties (the spin number of the particles and its charge).

In macroscopic media, dipoles will be separated one another by distances much smaller than macroscopic distances of usual interest. Consequently, in what follows, we may consider each point of our medium in a region of real space $\mathbb{R}^2$ to have a certain value of spin, which will be itself another quantity represented by a vector in $\mathbb{R}^2$, just like any magnetic moment (or any arbitrary 2D vector). However, if we consider our material to be *homogeneous*, all particles will have the same magnetic moment module (previously known/measured), leaving us with only one quantity to completely describe the magnetic moment value of each point in space (note that the *material* is said to be *homogeneous*, not the spin field/configuration).

A schematic representation is depicted below.

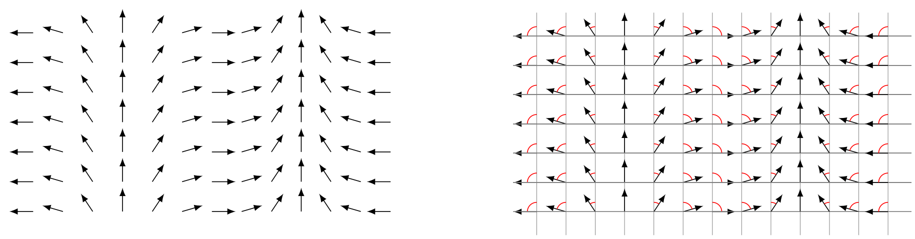


Figure 2.1: Example of 2D spin configuration. Each position (x, y) is associated with a direction, forming then a 2D vector field, $\vec{v} : \mathbb{R}^2 \to \mathbb{R}^2$. Spins, however, have all the same magnitude, being completely defined by their angle with respect to an fixed axis. One is then enabled to map the order parameter as $s : \mathbb{R}^2 \to [0, 1]$.

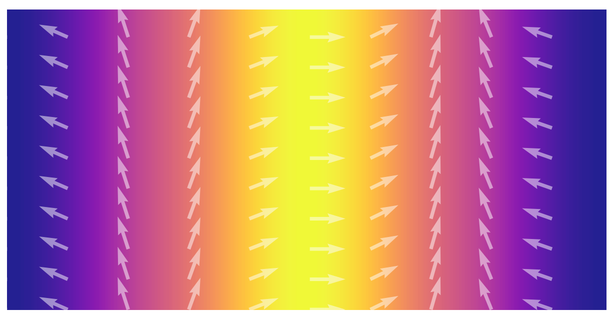


Figure 2.2: Sequential linear colormap of order parameter (background), $s(x, y) = \text{color}(\theta)$, as defined above. The order parameter in this case, as already discussed, is a mapping $s : \mathbb{R}^2 \to [0, 1]$. Superposed to the colormap, some vectors are drawn to represent the direction of the magnetic moment on (x, y), the vector origin.

As said above, magnetic moments will then be a vector with fixed length in $\mathbb{R}^2$. This makes it possible to completely determine them, for each point in space, with only one parameter - its angle according to a certain direction. One can then define a mapping that takes a point in $\mathbb{R}^2$ into a real value in the range $[0, 2\pi[$ (or any infinite bounded open set of real numbers), covering all configuration possibilities unambiguously. These possible configurations will sweep the surface of a circle in $\mathbb{R}^2$, meaning that the *spin-space* will be *homeomorphic* to a sphere in $\mathbb{R}^2$, called S^1. This is

$$s(x, y) = \left\{ f : \mathbb{R}^2 \to S^1 \mid \forall (x, y) \in \mathbb{R}^2, s(x, y) \in S^1 \right\}, \tag{2.1}$$

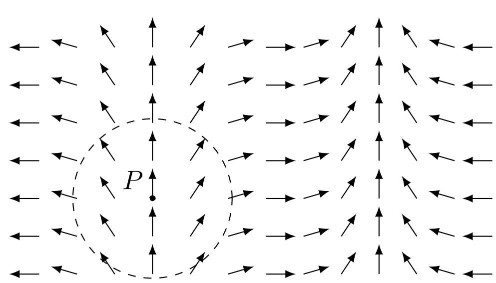
*where $s(x, y)$ is here called the **order parameter** of our space.*

2.2.2 Topological quantum number

Let P be a point in our medium, $\mathbb{R}^2$, corresponding to, perhaps, a singularity of the magnetic moment direction at that point (or equivalently the order parameter itself). In a open region D^2 around point P, no other singularity may be present. *The **topological quantum number** of a point P will be defined as the number of times the order parameter "walks" through its image as one follows the path described by the boundary ∂D^2 of the open region centered in P in a predetermined orientation.* Usually, one defines the positive orientation as being the counterclockwise direction. If the order parameter increases as one "walks" through its image, a positive topological quantum number is assured. Otherwise, a negative topological quantum number is obtained. If the order parameter increases for some part of the "walk" and then decreases equally, its value will be null.

Since the order parameter won't be singular at any other place except, perhaps, at point P, it *must* at the end of the contour return continuously to the same value as one had when started the "walk". For this exact reason, *only integer values will be accepted to characterize the topological quantum number*, hence the "quantum" in its name. If one could have non-integer values of the topological quantum number, the order parameter at the end of the contour s_f should be different from the s_0 one started (for one to have "incomplete walks through the image's path"), providing two values of the order parameter on the same point, contradicting our hypothesis of non-singularity.

Much better than any extensive discussion, a schematic representation on how to calculate the topological quantum number is depicted below.



Let us get a plot of the vector field on the boundary ∂D^2 of the disk around P, in order to visualize the orientation of the spins in each point of the contour.

Now, let us "walk" throughout the contour and keep track of the order parameter's image. For each value of theta (which is continuous) one will have a corresponding value of $s(\theta)$, making it possible to plot a graph representing the evolution of s as one "walks" throughout the contour. There are several methods to analyze the results, some of the most popular being represented below

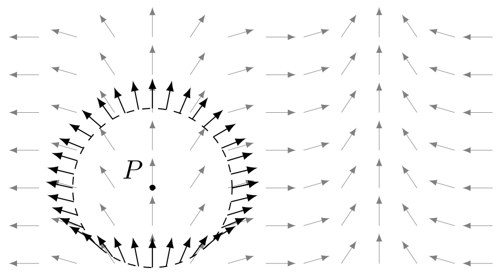


Figure 2.3: Vector field explicited over the contour ∂D^2. Lightly exposed on the background, the vector field throughout space, for comparison and guide eyes.

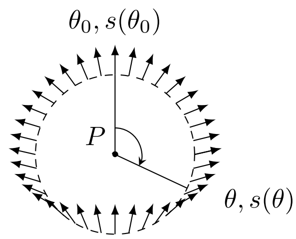


Figure 2.4:

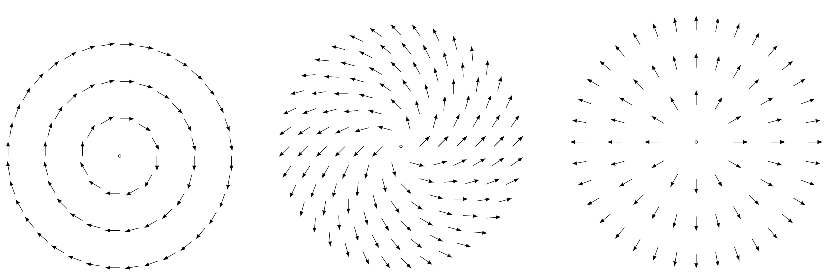


Figure 2.5: Topological defects with the same winding number $n = 1$. However, with different phases β, corresponding to (a), (b) and (c)

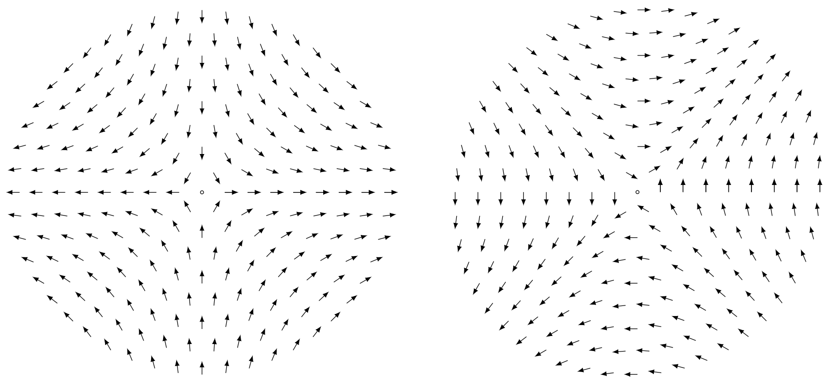


Figure 2.6: Once again, topological defects with the same winding number. This time, $n = -1$. Where the phases β correspond to (a), (b) and (c)

2.2.3 Kosterlitz-Thouless transition

2.2.4 Uniaxial nematics on the plane

2.2.5 Three-dimensional spins confined on a plane

2.2.6 Light propagation (Gaussian mode)

Hermite-Gaussian modes

Laguerre-Gaussian modes

Gravitational lenses

2.2.7 extra

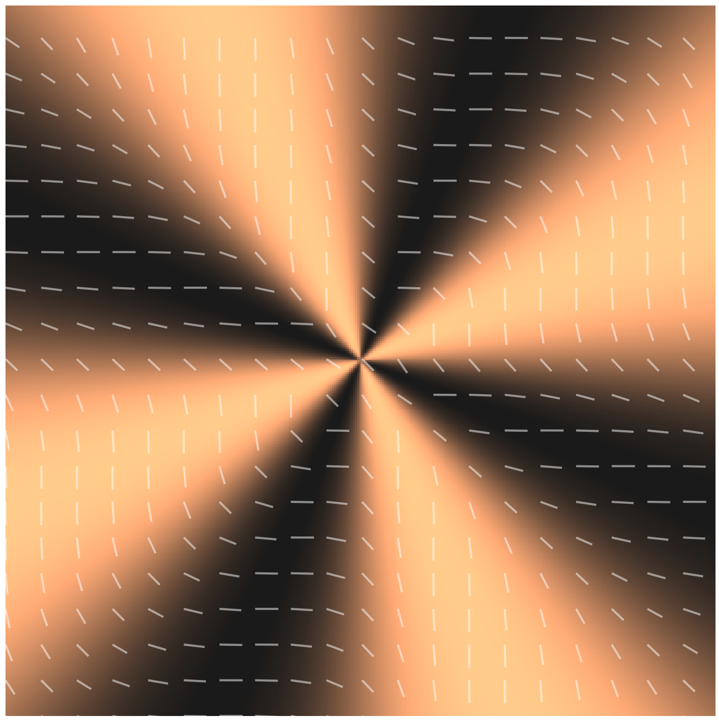


Figure 2.7: Nematic (headless) director profile, corresponding to a $n = 1$ winding number topological defect, with $\beta = 0$ phase. Even though this colormap configuration may seem like what one sees in an experimental POM image of a nematic sample containing a topological defect of such charge, these angles (and, consequently, colors) correspond to the director orientation with respect to the y axis (director contained in the plane of the text). The similarity occurs because $\theta = 0$ here is represented by black (minimum value) while $\theta = \pi/2$ by the brightest color, which is exactly what occurs in reality, except for the angle (order parameter) being mapped with respect to the z axis (α). Of course, for a axisymmetric sample with respect to the z axis, θ variations won't matter. Further is discussed in the text.