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Heavy-Quark Effective Theory

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Abstract

We give an introduction to the heavy-quark effective theory and the $1/m_Q$ expansion, which provide the modern framework for a systematic, model-independent description of the properties and decays of hadrons containing a heavy quark. We discuss the applications of these concepts to spectroscopy and to the weak decays of B mesons.

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HEAVY-QUARK EFFECTIVE THEORY

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We give an introduction to the heavy-quark effective theory and the $1/m_Q$ expansion, which provide the modern framework for a systematic, model-independent description of the properties and decays of hadrons containing a heavy quark. We discuss the applications of these concepts to spectroscopy and to the weak decays of B mesons.

1 Introduction

The weak decays of hadrons containing a heavy quark are employed for tests of the Standard Model and measurements of its parameters. They offer the most direct way to determine the weak mixing angles, to test the unitarity of the Cabibbo–Kobayashi–Maskawa (CKM) matrix, and to explore the physics of CP violation. At the same time, hadronic weak decays also serve as a probe of that part of strong-interaction phenomenology which is least understood: the confinement of quarks and gluons inside hadrons.

The structure of weak interactions in the Standard Model is rather simple. Flavour-changing decays are mediated by the coupling of the charged current to the W-boson field. At low energies, the charged-current interaction gives rise to local four-fermion couplings, whose strength is governed by the Fermi constant

$$G_F = \frac{g^2}{4\sqrt{2}M_W^2} = 1.16639(2)\ \text{GeV}^{-2}\,. \tag{1}$$

According to the structure of the these interactions, the weak decays of hadrons can be divided into three classes: leptonic decays, in which the quarks of the decaying hadron annihilate each other and only leptons appear in the final state; semileptonic decays, in which both leptons and hadrons appear in the final state; and non-leptonic decays, in which the final state consists of hadrons only. Representative examples of these three types of decays are shown in Fig. 1.

The simple quark-line graphs shown in this figure are a gross oversimplification, however. In the real world, quarks are confined inside hadrons, bound by the exchange of soft gluons. The simplicity of the weak interactions is overshadowed by the complexity of the strong interactions. A complicated interplay between the weak and strong forces characterizes the phenomenology

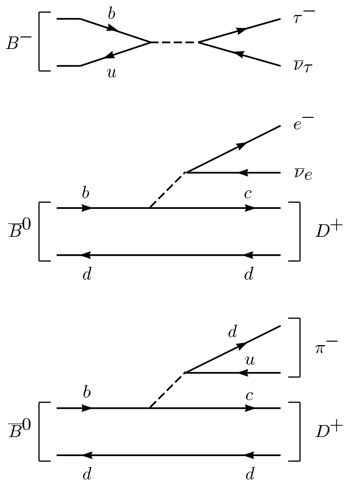

Figure 1: Examples of leptonic ($B^- \to \tau^- \bar{\nu}_\tau$), semileptonic ($\bar{B}^0 \to D^+ e^- \bar{\nu}_e$), and non-leptonic ($\bar{B}^0 \to D^+ \pi^-$) decays of B mesons.

of hadronic weak decays. As an example, a more realistic picture of a non-leptonic decay is shown in Fig. 2. Clearly, the complexity of strong-interaction effects increases with the number of quarks appearing in the final state. Bound-state effects in leptonic decays can be lumped into a single parameter (a "decay constant"), while those in semileptonic decays are described by invariant form factors, depending on the momentum transfer q^2 between the hadrons. Approximate symmetries of the strong interactions help to constrain the properties of these form factors. For non-leptonic decays, on the other hand, we are still far from having a quantitative understanding of strong-interaction effects even in the simplest decay modes.

Over the last decade, a lot of information on heavy-quark decays has been collected in experiments at e^+e^- and hadron colliders. This has led to a rather detailed knowledge of the flavour sector of the Standard Model and many of the parameters associated with it. There have been several great discoveries in this field, such as B^0–$\bar{B}^0$ mixing [1,2], charmless B decays [3−5], and rare decays induced by penguin operators [6,7]. The experimental progress in heavy-flavour physics has been accompanied by a significant progress in theory, which was related to the discovery of heavy-quark symmetry and the development of the heavy-quark effective theory (HQET). The excitement about these de-

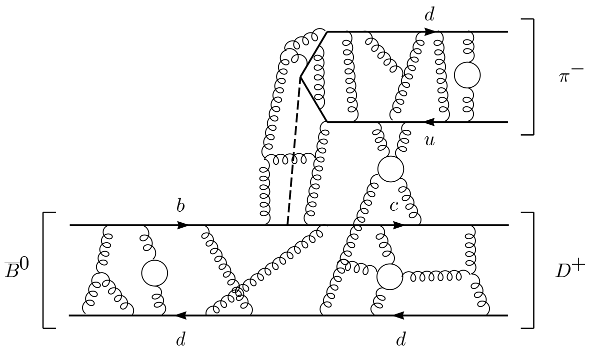

Figure 2: More realistic representation of a non-leptonic decay.

velopments is caused by the fact that they allow (some) model-independent predictions in an area in which "progress" in theory often meant nothing more than the construction of a new model, which could be used to estimate some strong-interaction hadronic matrix elements. In these notes, we explain the physical picture behind heavy-quark symmetry and discuss the construction, as well as simple applications, of the heavy-quark expansion. Because of lack of time, we will have to focus on some particularly important aspects, emphasizing the main ideas and concepts of the HQET. A more complete discussion of the applications of this formalism to heavy-flavour phenomenology can be found in some recent review articles [8,9]. The reader is also encouraged to consult the earlier review papers [10−14] on the subject.

Hadronic bound states of a heavy quark with light constituents (quarks, antiquarks and gluons) are characterized by a large separation of mass scales: the heavy-quark mass m_Q is much larger than the mass scale $\Lambda_{\rm QCD}$ associated with the light degrees of freedom. Equivalently, the Compton wave length of the heavy quark ($\lambda_Q \sim 1/m_Q$) is much smaller than the size of the hadron containing the heavy quark ($R_{\rm had} \sim 1/\Lambda_{\rm QCD}$). Our goal will be to separate the physics associated with these two scales, in such a way that all dependence on the heavy-quark mass becomes explicit. The framework in which to perform this separation is the operator product expansion (OPE) [15,16]. The HQET provides us with a convenient technical tool to construct the OPE. Before we start to explore in detail the details of this effective theory, however, we should mention two important reasons why it is desirable to separate short- and long-distance physics in the first place:

- A technical reason is that after the separation of short- and long-distance phenomena we can actually calculate a big portion of the relevant physics (i.e. all short-distance effects) using perturbation theory and renormalization-group techniques. In particular, in this way we will be able to control all logarithmic dependence on the heavy-quark mass.

- An important physical reason is that, after the short-distance physics has been separated, it may happen that the long-distance physics simplifies due to the realization of approximate symmetries, which imply non-trivial relations between observables.

The second point is particularly exciting, since it allows us to make statements beyond the range of applicability of perturbation theory. Notice that here we are not talking about symmetries of the full QCD Lagrangian, such as its local gauge symmetry, but approximate symmetries realized in a particular kinematic situation. In particular, we will find that an approximate spin–flavour symmetry is realized in systems in which a single heavy quark interacts with light degrees of freedom by the exchange of soft gluons.

At this point it is instructive to recall a more familiar example of how approximate symmetries relate the long-distance physics of several observables. The strong interactions of pions are severely constrained by the approximate chiral symmetry of QCD. In a certain kinematic regime, where the momenta of the pions are much less than 1 GeV (the scale of chiral-symmetry breaking), the long-distance physics of scattering amplitudes is encoded in a few "reduced matrix elements", such as the pion decay constant. An effective low-energy theory called chiral perturbation theory provides a systematic expansion of scattering amplitudes in powers of the pion momenta, and thus helps to derive the relations between different scattering amplitudes imposed by chiral symmetry [17]. We will find that a similar situation holds for the case of heavy quarks. Heavy-quark symmetry implies that, in the limit where $m_Q \gg \Lambda_{\rm QCD}$, the long-distance physics of several observables is encoded in few hadronic parameters, which can be defined in terms of operator matrix elements in the HQET.

2 Heavy-Quark Symmetry

2.1 The Physical Picture

There are several reasons why the strong interactions of systems containing heavy quarks are easier to understand than those of systems containing only light quarks. The first is asymptotic freedom, the fact that the effective coupling constant of QCD becomes weak in processes with large momentum transfer, corresponding to interactions at short-distance scales [18,19]. At large distances, on the other hand, the coupling becomes strong, leading to non-perturbative phenomena such as the confinement of quarks and gluons on a length scale $R_{\rm had} \sim 1/\Lambda_{\rm QCD} \sim 1$ fm, which determines the size of hadrons [20]. Roughly speaking, $\Lambda_{\rm QCD} \sim 0.2$ GeV is the energy scale that separates the regions of large and small coupling constant. When the mass of a quark Q is much larger than this scale, it is called a heavy quark. The quarks of the Standard Model fall naturally into two classes: up, down and strange are light quarks, whereas charm, bottom and top are heavy quarks.[a] For heavy quarks, the effective coupling constant $\alpha_s(m_Q)$ is small, implying that on length scales comparable to the Compton wavelength $\lambda_Q \sim 1/m_Q$ the strong interactions are perturbative and similar to the electromagnetic interactions. In fact, the quarkonium systems $(\bar{Q}Q)$, whose size is of order $\lambda_Q/\alpha_s(m_Q) \ll R_{\rm had}$, are very much hydrogen-like.

Systems composed of a heavy quark and light constituents are more complicated, however. The size of such systems is determined by $R_{\rm had}$, and the typical momenta exchanged between the heavy and light constituents are of order $\Lambda_{\rm QCD}$. The heavy quark is surrounded by a most complicated, strongly interacting cloud of light quarks, antiquarks, and gluons. In this case it is the fact that $\lambda_Q \ll R_{\rm had}$, i.e. that the Compton wavelength of the heavy quark is much smaller than the size of the hadron, which leads to simplifications. To resolve the quantum numbers of the heavy quark would require a hard probe; the soft gluons exchanged between the heavy quark and the light constituents can only resolve distances much larger than λ_Q. Therefore, the light degrees of freedom are blind to the flavour (mass) and spin orientation of the heavy quark. They experience only its colour field, which extends over large distances because of confinement. In the rest frame of the heavy quark, it is in fact only the electric colour field that is important; relativistic effects such as colour magnetism vanish as $m_Q \to \infty$. Since the heavy-quark spin participates in interactions only through such relativistic effects, it decouples. That the heavy-quark mass becomes irrelevant can be seen as follows: As $m_Q \to \infty$, the heavy quark and the hadron that contains it have the same velocity. In the rest frame of the hadron, the heavy quark is at rest, too. The wave function of the light constituents follows from a solution of the field equations of QCD subject to the boundary condition of a static triplet source of colour at the location of the heavy quark. This boundary condition is independent of m_Q, and so is the solution for the configuration of the light constituents.

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[a] Ironically, the top quark is of no relevance to our discussion here, since it is too heavy to form hadronic bound states before it decays.

It follows that, in the limit $m_Q \to \infty$, hadronic systems which differ only in the flavour or spin quantum numbers of the heavy quark have the same configuration of their light degrees of freedom [21−26]. Although this observation still does not allow us to calculate what this configuration is, it provides relations between the properties of such particles as the heavy mesons B, D, B^* and D^*, or the heavy baryons Λ_b and Λ_c (to the extent that corrections to the infinite quark-mass limit are small in these systems). These relations result from some approximate symmetries of the effective strong interactions of heavy quarks at low energies. The configuration of light degrees of freedom in a hadron containing a single heavy quark with velocity v does not change if this quark is replaced by another heavy quark with different flavour or spin, but with the same velocity. Both heavy quarks lead to the same static colour field. For N_h heavy-quark flavours, there is thus an $\mathrm{SU}(2N_h)$ spin–flavour symmetry group, under which the effective strong interactions are invariant. These symmetries are in close correspondence to familiar properties of atoms: The flavour symmetry is analogous to the fact that different isotopes have the same chemistry, since to a good approximation the wave function of the electrons is independent of the mass of the nucleus. The electrons only see the total nuclear charge. The spin symmetry is analogous to the fact that the hyperfine levels in atoms are nearly degenerate. The nuclear spin decouples in the limit $m_e/m_N \to 0$.

Heavy-quark symmetry is an approximate symmetry, and corrections arise since the quark masses are not infinite. In many respects, it is complementary to chiral symmetry, which arises in the opposite limit of small quark masses. However, whereas chiral symmetry is a symmetry of the QCD Lagrangian in the limit of vanishing quark masses, heavy-quark symmetry is not a symmetry of the Lagrangian (not even an approximate one), but rather a symmetry of an effective theory, which is a good approximation of QCD in a certain kinematic region. It is realized only in systems in which a heavy quark interacts predominantly by the exchange of soft gluons. In such systems the heavy quark is almost on shell; its momentum fluctuates around the mass shell by an amount of order Λ_{QCD}. The corresponding fluctuations in the velocity of the heavy quark vanish as $\Lambda_{\mathrm{QCD}}/m_Q \to 0$. The velocity becomes a conserved quantity and is no longer a dynamical degree of freedom [27]. Nevertheless, results derived on the basis of heavy-quark symmetry are model-independent consequences of QCD in a well-defined limit. The symmetry-breaking corrections can, at least in principle, be studied in a systematic way. A convenient framework for analyzing these corrections is provided by the heavy-quark effective theory. Before presenting a detailed discussion of the formalism, we shall first point out some of the important implications of heavy-quark symmetry for the spectroscopy and weak decays of heavy hadrons.

2.2 Spectroscopic Implications

The spin–flavour symmetry leads to many interesting relations between the properties of hadrons containing a heavy quark. The most direct consequences concern the spectroscopy of such states [28]. In the limit $m_Q \to \infty$, the spin of the heavy quark and the total angular momentum j of the light degrees of freedom inside a hadron are separately conserved by the strong interactions. Because of heavy-quark symmetry, the dynamics is independent of the spin and mass of the heavy quark. Hadronic states can thus be classified by the quantum numbers (flavour, spin, parity, etc.) of the light degrees of freedom [29]. The spin symmetry predicts that, for fixed $j \neq 0$, there is a doublet of degenerate states with total spin $J = j \pm \frac{1}{2}$. The flavour symmetry relates the properties of states with different heavy-quark flavour.

In general, the mass of a hadron H_Q containing a heavy quark Q obeys an expansion of the form

$$m_H = m_Q + \bar{\Lambda} + \frac{\Delta m^2}{2m_Q} + O(1/m_Q^2)\,. \tag{2}$$

The parameter $\bar{\Lambda}$ represents contributions arising from all terms in the Lagrangian that are independent of the heavy-quark mass [30], whereas the quantity Δm^2 originates from the terms of order $1/m_Q$ in the effective Lagrangian of the HQET. For the moment, the detailed structure of these terms is of no relevance; it will be discussed at length in the next section. For the ground-state pseudoscalar and vector mesons, one can parametrize the contributions from the $1/m_Q$ corrections in terms of two quantities, λ_1 and λ_2, in such a way that [31]

$$\Delta m^2 = -\lambda_1 + 2\Big[J(J+1) - \tfrac{3}{2}\Big]\,\lambda_2\,. \tag{3}$$

Here J is the total spin of the meson. The first term, $-\lambda_1/2m_Q$, arises from the kinetic energy of the heavy quark inside the meson; the second term describes the interaction of the heavy-quark spin with the gluon field. The hadronic parameters $\bar{\Lambda}$, λ_1 and λ_2 are independent of m_Q. They characterize the properties of the light constituents.

Consider, as a first example, the SU(3) mass splittings for heavy mesons. The heavy-quark expansion predicts that

$$\begin{aligned} m_{B_S} - m_{B_d} &= \bar{\Lambda}_s - \bar{\Lambda}_d + O(1/m_b)\,, \\ m_{D_S} - m_{D_d} &= \bar{\Lambda}_s - \bar{\Lambda}_d + O(1/m_c)\,, \end{aligned} \tag{4}$$

where we have indicated that the value of the parameter $\bar{\Lambda}$ depends on the flavour of the light quark. Thus, to the extent that the charm and bottom quarks can both be considered sufficiently heavy, the mass splittings should be similar in the two systems. This prediction is confirmed experimentally, since [32]

$$\begin{aligned} m_{B_S} - m_{B_d} &= (90 \pm 3)\ \mathrm{MeV}\,, \\ m_{D_S} - m_{D_d} &= (99 \pm 1)\ \mathrm{MeV}\,. \end{aligned} \tag{5}$$

As a second example, consider the spin splittings between the ground-state pseudoscalar ($J = 0$) and vector ($J = 1$) mesons, which are the members of the spin-doublet with $j = \frac{1}{2}$. The theory predicts that

$$\begin{aligned} m_{B^*}^2 - m_B^2 &= 4\lambda_2 + O(1/m_b)\,, \\ m_{D^*}^2 - m_D^2 &= 4\lambda_2 + O(1/m_c)\,. \end{aligned} \tag{6}$$

The data are compatible with this:

$$\begin{aligned} m_{B^*}^2 - m_B^2 &\simeq 0.49\ \mathrm{GeV}^2\,, \\ m_{D^*}^2 - m_D^2 &\simeq 0.55\ \mathrm{GeV}^2\,. \end{aligned} \tag{7}$$

Assuming that the B system is close to the heavy-quark limit, we obtain the value

$$\lambda_2 \simeq 0.12\ \mathrm{GeV}^2 \tag{8}$$

for one of the hadronic parameters in (3). This quantity plays an important role in the phenomenology of inclusive decays of heavy hadrons [8].

A third example is provided by the mass splittings between the ground-state mesons and baryons containing a heavy quark. The HQET predicts that

$$\begin{aligned} m_{\Lambda_b} - m_B &= \bar{\Lambda}_{\mathrm{baryon}} - \bar{\Lambda}_{\mathrm{meson}} + O(1/m_b)\,, \\ m_{\Lambda_c} - m_D &= \bar{\Lambda}_{\mathrm{baryon}} - \bar{\Lambda}_{\mathrm{meson}} + O(1/m_c)\,. \end{aligned} \tag{9}$$

This is again consistent with the experimental results

$$\begin{aligned} m_{\Lambda_b} - m_B &= (346 \pm 6)\ \mathrm{MeV}\,, \\ m_{\Lambda_c} - m_D &= (416 \pm 1)\ \mathrm{MeV}\,, \end{aligned} \tag{10}$$

although in this case the data indicate sizeable symmetry-breaking corrections. For the mass of the Λ_b baryon, we have used the value

$$m_{\Lambda_b} = (5625 \pm 6)\ \mathrm{MeV}\,, \tag{11}$$

which is obtained by averaging the result [32] $m_{\Lambda_b} = (5639 \pm 15)$ MeV with the value $m_{\Lambda_b} = (5623 \pm 5 \pm 4)$ MeV reported by the CDF Collaboration [33]. The dominant correction to the relations (9) comes from the contribution of the chromo-magnetic interaction to the masses of the heavy mesons,[b] which adds a term $3\lambda_2/2m_Q$ on the right-hand side. Including this term, we obtain the refined prediction that the values of the following two quantities should be close to each other:

$$\begin{aligned} m_{\Lambda_b} - m_B - \frac{3\lambda_2}{2m_B} &= (312 \pm 6)\ \text{MeV}\,, \\ m_{\Lambda_c} - m_D - \frac{3\lambda_2}{2m_D} &= (320 \pm 1)\ \text{MeV} \end{aligned} \tag{12}$$

This is clearly satisfied by the data.

The mass formula (2) can also be used to derive information on the heavy-quark (pole) masses from the observed hadron masses. Introducing the "spin-averaged" meson masses $\overline{m}_B = \frac{1}{4}\,(m_B + 3m_{B^*}) \simeq 5.31$ GeV and $\overline{m}_D = \frac{1}{4}\,(m_D + 3m_{D^*}) \simeq 1.97$ GeV, we find that

$$m_b - m_c = (\overline{m}_B - \overline{m}_D) \left\{ 1 - \frac{\lambda_1}{2\overline{m}_B \overline{m}_D} + O(1/m_Q^3) \right\}, \tag{13}$$

where $O(1/m_Q^3)$ is used as a generic notation representing terms suppressed by three powers of the b- or c-quark masses. Using theoretical estimates for the parameter λ_1, which lie in the range [34–36]

$$\lambda_1 = -(0.3 \pm 0.2)\ \text{GeV}^2\,, \tag{14}$$

this relation leads to

$$m_b - m_c = (3.39 \pm 0.03 \pm 0.03)\ \text{GeV}\,, \tag{15}$$

where the first error reflects the uncertainty in the value of λ_1, and the second one takes into account unknown higher-order corrections.

2.3 Exclusive Semileptonic Decays

Semileptonic decays of B mesons have received a lot of attention in recent years. The decay channel $\bar{B} \to D^* \ell\, \bar{\nu}$ has the largest branching fraction of all B-meson decay modes. From a theoretical point of view, semileptonic decays

[b] Because of the spin symmetry, there is no such contribution to the masses of the Λ_Q baryons.

are simple enough to allow for a reliable, quantitative description. The analysis of these decays provides much information about the strong forces that bind the quarks and gluons into hadrons. Heavy-quark symmetry implies relations between the weak decay form factors of heavy mesons, which are of particular interest. These relations have been derived by Isgur and Wise [26], generalizing ideas developed by Nussinov and Wetzel [23], and by Voloshin and Shifman [24,25].

Consider the elastic scattering of a B meson, $\bar{B}(v) \to \bar{B}(v')$, induced by a vector current coupled to the b quark. Before the action of the current, the light degrees of freedom inside the B meson orbit around the heavy quark, which acts as a static source of colour. On average, the b quark and the B meson have the same velocity v. The action of the current is to replace instantaneously (at $t = t_0$) the colour source by one moving at a velocity v', as indicated in Fig. 3. If $v = v'$, nothing happens; the light degrees of freedom do not realize that there was a current acting on the heavy quark. If the velocities are different, however, the light constituents suddenly find themselves interacting with a moving colour source. Soft gluons have to be exchanged to rearrange them so as to form a B meson moving at velocity v'. This rearrangement leads to a form-factor suppression, which reflects the fact that as the velocities become more and more different, the probability for an elastic transition decreases. The important observation is that, in the limit $m_b \to \infty$, the form factor can only depend on the Lorentz boost $\gamma = v\cdot v'$ that connects the rest frames of the initial- and final-state mesons. Thus, in this limit a dimensionless probability function $\xi(v\cdot v')$ describes the transition. It is called the Isgur–Wise function [26]. In the HQET, which provides the appropriate framework for taking the limit $m_b \to \infty$, the hadronic matrix element describing the scattering process can thus be written as

$$\frac{1}{m_B}\,\langle\bar{B}(v')|\,\bar{b}_{v'}\gamma^\mu b_v\,|\bar{B}(v)\rangle = \xi(v\cdot v')\,(v+v')^\mu\,. \qquad (16)$$

Here, b_v and $b_{v'}$ are the velocity-dependent heavy-quark fields of the HQET, whose precise definition will be discussed in Sec. 3. It is important that the function $\xi(v\cdot v')$ does not depend on m_b. The factor $1/m_B$ on the left-hand side compensates for a trivial dependence on the heavy-meson mass caused by the relativistic normalization of meson states, which is conventionally taken to be

$$\langle\bar{B}(p')|\bar{B}(p)\rangle = 2m_B v^0\,(2\pi)^3\,\delta^3(\vec{p}-\vec{p}\,')\,. \qquad (17)$$

Note that there is no term proportional to $(v-v')^\mu$ in (16). This can be seen by contracting the matrix element with $(v-v')_\mu$, which must give zero since $\not{v} b_v = b_v$ and $\bar{b}_{v'}\not{v}' = \bar{b}_{v'}$.

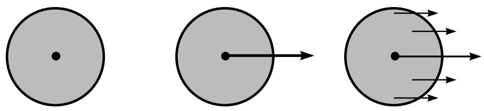

Figure 3: Elastic transition induced by an external heavy-quark current.

It is more conventional to write the above matrix element in terms of an elastic form factor $F_{\rm el}(q^2)$ depending on the momentum transfer $q^2 = (p-p')^2$:

$$\langle \bar B(v')|\,\bar b\,\gamma^\mu b\,|\bar B(v)\rangle = F_{\rm el}(q^2)\,(p+p')^\mu\,, \tag{18}$$

where $p^{(\prime)} = m_B v^{(\prime)}$. Comparing this with (16), we find that

$$F_{\rm el}(q^2) = \xi(v\cdot v')\,, \qquad q^2 = -2m_B^2(v\cdot v'-1)\,. \tag{19}$$

Because of current conservation, the elastic form factor is normalized to unity at $q^2=0$. This condition implies the normalization of the Isgur–Wise function at the kinematic point $v\cdot v'=1$, i.e. for $v=v'$:

$$\xi(1) = 1\,. \tag{20}$$

It is in accordance with the intuitive argument that the probability for an elastic transition is unity if there is no velocity change. Since for $v=v'$ the daughter meson is at rest in the rest frame of the parent meson, the point $v\cdot v'=1$ is referred to as the zero-recoil limit.

We can now use the flavour symmetry to replace the b quark in the final-state meson by a c quark, thereby turning the B meson into a D meson. Then the scattering process turns into a weak decay process. In the infinite mass limit, the replacement $b_{v'}\to c_{v'}$ is a symmetry transformation, under which the effective Lagrangian is invariant. Hence, the matrix element

$$\frac{1}{\sqrt{m_B m_D}}\,\langle D(v')|\,\bar c_{v'}\gamma^\mu b_v\,|\bar B(v)\rangle = \xi(v\cdot v')\,(v+v')^\mu \tag{21}$$

is still determined by the same function $\xi(v\cdot v')$. This is interesting, since in general the matrix element of a flavour-changing current between two pseudoscalar mesons is described by two form factors:

$$\langle D(v')|\,\bar c\,\gamma^\mu b\,|\bar B(v)\rangle = f_+(q^2)\,(p+p')^\mu - f_-(q^2)\,(p-p')^\mu\,. \tag{22}$$

Comparing the above two equations, we find that

$$\begin{aligned} f_\pm(q^2) &= \frac{m_B \pm m_D}{2\sqrt{m_B m_D}}\,\xi(v\cdot v')\,, \\ q^2 &= m_B^2 + m_D^2 - 2m_B m_D\, v\cdot v'\,. \end{aligned} \qquad (23)$$

Thus, the heavy-quark flavour symmetry relates two a priori independent form factors to one and the same function. Moreover, the normalization of the Isgur–Wise function at $v\cdot v' = 1$ now implies a non-trivial normalization of the form factors $f_\pm(q^2)$ at the point of maximum momentum transfer, $q^2_{\rm max} = (m_B - m_D)^2$:

$$f_\pm(q^2_{\rm max}) = \frac{m_B \pm m_D}{2\sqrt{m_B m_D}}\,. \qquad (24)$$

The heavy-quark spin symmetry leads to additional relations among weak decay form factors. It can be used to relate matrix elements involving vector mesons to those involving pseudoscalar mesons. A vector meson with longitudinal polarization is related to a pseudoscalar meson by a rotation of the heavy-quark spin. Hence, the spin-symmetry transformation $c^\Uparrow_{v'} \to c^\Downarrow_{v'}$ relates $\bar B\to D$ with $\bar B\to D^*$ transitions. The result of this transformation is [26]:

$$\begin{aligned} \frac{1}{\sqrt{m_B m_{D^*}}}\,\langle D^*(v',\varepsilon)|\,\bar c_{v'}\gamma^\mu b_v\,|\bar B(v)\rangle &= i\epsilon^{\mu\nu\alpha\beta}\,\varepsilon^*_\nu\, v'_\alpha v_\beta\;\xi(v\cdot v')\,, \\ \frac{1}{\sqrt{m_B m_{D^*}}}\,\langle D^*(v',\varepsilon)|\,\bar c_{v'}\gamma^\mu\gamma_5\, b_v\,|\bar B(v)\rangle &= \Big[\varepsilon^{*\mu}\,(v\cdot v'+1) - v'^\mu\,\varepsilon^*\!\cdot v\Big]\,\xi(v\cdot v')\,, \end{aligned} \qquad (25)$$

where ε denotes the polarization vector of the D^* meson. Once again, the matrix elements are completely described in terms of the Isgur–Wise function. Now this is even more remarkable, since in general four form factors, $V(q^2)$ for the vector current, and $A_i(q^2)$, $i=0,1,2$, for the axial vector current, are required to parametrize these matrix elements. In the heavy-quark limit, they obey the relations [37]

$$\begin{aligned} \frac{m_B \pm m_{D^*}}{2\sqrt{m_B m_{D^*}}}\,\xi(v\cdot v') &= V(q^2) = A_0(q^2) = A_1(q^2) \\ &= \left[1 - \frac{q^2}{(m_B+m_D)^2}\right]^{-1} A_1(q^2)\,, \\ q^2 &= m_B^2 + m_{D^*}^2 - 2m_B m_{D^*}\, v\cdot v'\,. \end{aligned} \qquad (26)$$

Equations (23) and (26) summarize the relations imposed by heavy-quark symmetry on the weak decay form factors describing the semileptonic decay processes $\bar{B} \to D\,\ell\,\bar{\nu}$ and $\bar{B} \to D^*\ell\,\bar{\nu}$. These relations are model-independent consequences of QCD in the limit where $m_b, m_c \gg \Lambda_{\rm QCD}$. They play a crucial role in the determination of the CKM matrix element $|V_{cb}|$. In terms of the recoil variable $w = v\cdot v'$, the differential semileptonic decay rates in the heavy-quark limit become [38]:

$$\begin{aligned}
\frac{\mathrm{d}\Gamma(\bar{B} \to D\,\ell\,\bar{\nu})}{\mathrm{d}w} &= \frac{G_F^2}{48\pi^3}\,|V_{cb}|^2\,(m_B + m_D)^2\,m_D^3\,(w^2-1)^{3/2}\,\xi^2(w)\,,\\
\frac{\mathrm{d}\Gamma(\bar{B} \to D^*\ell\,\bar{\nu})}{\mathrm{d}w} &= \frac{G_F^2}{48\pi^3}\,|V_{cb}|^2\,(m_B - m_{D^*})^2\,m_{D^*}^3\,\sqrt{w^2-1}\,(w+1)^2\\
&\quad\times\left[1 + \frac{4w}{w+1}\,\frac{m_B^2 - 2w\,m_B m_{D^*} + m_{D^*}^2}{(m_B - m_{D^*})^2}\right]\xi^2(w)\,. \qquad (27)
\end{aligned}$$

These expressions receive symmetry-breaking corrections, since the masses of the heavy quarks are not infinitely heavy. Perturbative corrections of order $\alpha_s^n(m_Q)$ can be calculated order by order in perturbation theory. A more difficult task is to control the non-perturbative power corrections of order $(\Lambda_{\rm QCD}/m_Q)^n$. The HQET provides a systematic framework for analysing these corrections. For the case of weak-decay form factors, the analysis of the $1/m_Q$ corrections was performed by Luke [39]. Later, Falk and the present author have also analysed the structure of $1/m_Q^2$ corrections for both meson and baryon weak decay form factors [31]. We shall not discuss these rather technical issues in detail, but only mention the most important result of Luke's analysis. It concerns the zero-recoil limit, where an analogue of the Ademollo–Gatto theorem [40] can be proved. This is Luke's theorem [39], which states that the matrix elements describing the leading $1/m_Q$ corrections to weak decay amplitudes vanish at zero recoil. This theorem is valid to all orders in perturbation theory [31,41,42]. Most importantly, it protects the $\bar{B} \to D^*\ell\,\bar{\nu}$ decay rate from receiving first-order $1/m_Q$ corrections at zero recoil [38]. (A similar statement is not true for the decay $\bar{B} \to D\,\ell\,\bar{\nu}$, however. The reason is simple but somewhat subtle. Luke's theorem protects only those form factors not multiplied by kinematic factors that vanish for $v = v'$. By angular momentum conservation, the two pseudoscalar mesons in the decay $\bar{B} \to D\,\ell\,\bar{\nu}$ must be in a relative p wave, and hence the amplitude is proportional to the velocity $|\vec{v}_D|$ of the D meson in the B-meson rest frame. This leads to a factor (w^2-1) in the decay rate. In such a situation, form factors that are kinematically suppressed can contribute [37].)

2.4 Model-Independent Determination of $|V_{cb}|$

We will now discuss the most important application of the HQET in the context of semileptonic decays of B mesons. A model-independent determination of the CKM matrix element $|V_{cb}|$ based on heavy-quark symmetry can be obtained by measuring the recoil spectrum of D^* mesons produced in $\bar{B} \to D^* \ell\,\bar{\nu}$ decays [38]. In the heavy-quark limit, the differential decay rate for this process has been given in (27). In order to allow for corrections to that limit, we write

$$\begin{aligned}\frac{\mathrm{d}\Gamma(\bar{B} \to D^* \ell\,\bar{\nu})}{\mathrm{d}w} &= \frac{G_F^2}{48\pi^3}\,(m_B - m_{D^*})^2\, m_{D^*}^3 \sqrt{w^2-1}\,(w+1)^2 \\ &\quad \times \left[1 + \frac{4w}{w+1}\,\frac{m_B^2 - 2w\,m_B m_{D^*} + m_{D^*}^2}{(m_B - m_{D^*})^2} \right] |V_{cb}|^2\, \mathcal{F}^2(w)\,,\end{aligned} \tag{28}$$

where the hadronic form factor $\mathcal{F}(w)$ coincides with the Isgur–Wise function up to symmetry-breaking corrections of order $\alpha_s(m_Q)$ and $\Lambda_{\mathrm{QCD}}/m_Q$. The idea is to measure the product $|V_{cb}|\,\mathcal{F}(w)$ as a function of w, and to extract $|V_{cb}|$ from an extrapolation of the data to the zero-recoil point $w = 1$, where the B and the D^* mesons have a common rest frame. At this kinematic point, heavy-quark symmetry helps to calculate the normalization $\mathcal{F}(1)$ with small and controlled theoretical errors. Since the range of w values accessible in this decay is rather small ($1 < w < 1.5$), the extrapolation can be done using an expansion around $w = 1$:

$$\mathcal{F}(w) = \mathcal{F}(1)\left[1 - \widehat{\varrho}^2\,(w-1) + \dots \right]. \tag{29}$$

The slope $\widehat{\varrho}^2$ is treated as a fit parameter.

Measurements of the recoil spectrum have been performed first by the ARGUS [43] and CLEO [44] Collaborations in experiments operating at the $\Upsilon(4s)$ resonance, and more recently by the ALEPH [45] and DELPHI [46] Collaborations at LEP. As an example, Fig. 4 shows the data reported by the CLEO Collaboration. The results obtained by the various experimental groups from a linear fit to their data are summarized in Table 1. The weighted average of these results is

$$\begin{aligned}|V_{cb}|\,\mathcal{F}(1) &= (34.6 \pm 1.7) \times 10^{-3}\,, \\ \widehat{\varrho}^2 &= 0.82 \pm 0.09\,.\end{aligned} \tag{30}$$

The effect of a positive curvature of the form factor has been investigated by Stone [47], who finds that the value of $|V_{cb}|\,\mathcal{F}(1)$ may change by up to $+4\%$. We

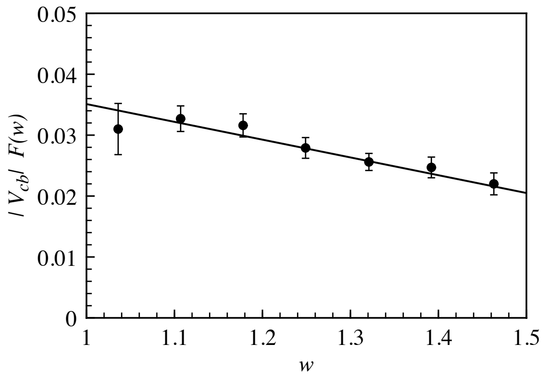

Figure 4: CLEO data for the product $|V_{cb}|\,\mathcal{F}(w)$, as extracted from the recoil spectrum in $\bar{B} \to D^*\ell\,\bar{\nu}$ decays [44]. The line shows a linear fit to the data.

thus increase the above value by $(2 \pm 2)\%$ and quote the final result as

$$|V_{cb}|\,\mathcal{F}(1) = (35.3 \pm 1.8) \times 10^{-3}\,. \tag{31}$$

In future analyses, the extrapolation to zero recoil should be performed including higher-order terms in the expansion (29). It can be shown in a model-independent way that the shape of the form factor is highly constrained by analyticity and unitarity requirements [48,49]. In particular, the curvature at $w = 1$ is strongly correlated with the slope of the form factor. For the value of $\widehat{\varrho}^2$ given in (30), one obtains a small positive curvature [49], in agreement with the assumption made in Ref. 47.

Table 1: Values for $|V_{cb}|\,\mathcal{F}(1)$ (in units of 10^{-3}) and $\widehat{\varrho}^2$ extracted from measurements of the recoil spectrum in $\bar{B} \to D^*\ell\,\bar{\nu}$ decays

	$\lvert V_{cb}\rvert\,\mathcal{F}(1)$ (10^{-3})	$\widehat{\varrho}^2$
ARGUS	$38.8 \pm 4.3 \pm 2.5$	$1.17 \pm 0.22 \pm 0.06$
CLEO	$35.1 \pm 1.9 \pm 2.0$	$0.84 \pm 0.12 \pm 0.08$
ALEPH	$31.4 \pm 2.3 \pm 2.5$	$0.39 \pm 0.21 \pm 0.12$
DELPHI	$35.0 \pm 1.9 \pm 2.3$	$0.81 \pm 0.16 \pm 0.10$

Heavy-quark symmetry implies that the general structure of the symmetry-breaking corrections to the form factor at zero recoil is [38]

$$\mathcal{F}(1) = \eta_A \left(1 + 0 \times \frac{\Lambda_{\rm QCD}}{m_Q} + \text{const} \times \frac{\Lambda_{\rm QCD}^2}{m_Q^2} + \dots \right) \equiv \eta_A \left(1 + \delta_{1/m^2}\right) , \quad (32)$$

where η_A is a short-distance correction arising from the (finite) renormalization of the flavour-changing axial current at zero recoil, and δ_{1/m^2} parametrizes second-order (and higher) power corrections. The absence of first-order power corrections at zero recoil is a consequence of Luke's theorem [39]. The one-loop expression for η_A has been known for a long time [22,25,50]:

$$\eta_A = 1 + \frac{\alpha_s(M)}{\pi} \left(\frac{m_b + m_c}{m_b - m_c} \ln \frac{m_b}{m_c} - \frac{8}{3} \right) \simeq 0.96 \, . \quad (33)$$

The scale M in the running coupling constant can be fixed by adopting the prescription of Brodsky, Lepage and Mackenzie (BLM) [51], according to which it is identified with the average virtuality of the gluon in the one-loop diagrams that contribute to η_A. If $\alpha_s(M)$ is defined in the modified minimal subtraction ($\overline{\rm MS}$) scheme, the result is [52] $M \simeq 0.51\sqrt{m_c m_b}$. Several estimates of higher-order corrections to η_A have been discussed. The next-to-leading order resummation of logarithms of the type $[\alpha_s \ln(m_b/m_c)]^n$ leads to [53,54] $\eta_A \simeq 0.985$. On the other hand, the resummation of "renormalon-chain" contributions of the form $\beta_0^{n-1}\alpha_s^n$, where β_0 is the first coefficient of the QCD β-function, gives [55] $\eta_A \simeq 0.945$. Using these partial resummations to estimate the uncertainty results in $\eta_A = 0.965 \pm 0.020$. Recently, Czarnecki has improved this estimate by calculating η_A at two-loop order [56]. His result,

$$\eta_A = 0.960 \pm 0.007 \, , \quad (34)$$

is in excellent agreement with the BLM-improved one-loop estimate (33). Here the error is taken to be the size of the two-loop correction.

The analysis of the power corrections δ_{1/m^2} is more difficult, since it cannot rely on perturbation theory. Three approaches have been discussed: in the "exclusive approach", all $1/m_Q^2$ operators in the HQET are classified and their matrix elements estimated, leading to [31,57] $\delta_{1/m^2} = -(3 \pm 2)\%$; the "inclusive approach" has been used to derive the bound $\delta_{1/m^2} < -3\%$, and to estimate that [58,c] $\delta_{1/m^2} = -(7 \pm 3)\%$; the "hybrid approach" combines the virtues of the former two to obtain a more restrictive lower bound on δ_{1/m^2}. This leads to [60]

$$\delta_{1/m^2} = -0.055 \pm 0.025 \, . \quad (35)$$

This bound has been criticised in Ref. 59.

Combining the above results, adding the theoretical errors linearly to be conservative, gives

$$\mathcal{F}(1) = 0.91 \pm 0.03 \tag{36}$$

for the normalization of the hadronic form factor at zero recoil. Thus, the corrections to the heavy-quark limit amount to a moderate decrease of the form factor of about 10%. This can be used to extract from the experimental result (31) the model-independent value

$$|V_{cb}| = (38.8 \pm 2.0_{\rm exp} \pm 1.2_{\rm th}) \times 10^{-3}\,. \tag{37}$$

3 Heavy-Quark Effective Theory

3.1 The Effective Lagrangian

The effects of a very heavy particle often become irrelevant at low energies. It is then useful to construct a low-energy effective theory, in which this heavy particle no longer appears. Eventually, this effective theory will be easier to deal with than the full theory. A familiar example is Fermi's theory of the weak interactions. For the description of weak decays of hadrons, the weak interactions can be approximated by point-like four-fermion couplings, governed by a dimensionful coupling constant G_F. Only at energies much larger than the masses of hadrons can the effects of the intermediate vector bosons, W and Z, be resolved.

The process of removing the degrees of freedom of a heavy particle involves the following steps [61−63]: one first identifies the heavy-particle fields and "integrates them out" in the generating functional of the Green functions of the theory. This is possible since at low energies the heavy particle does not appear as an external state. However, although the action of the full theory is usually a local one, what results after this first step is a non-local effective action. The non-locality is related to the fact that in the full theory the heavy particle with mass M can appear in virtual processes and propagate over a short but finite distance $\Delta x \sim 1/M$. Thus, a second step is required to obtain a local effective Lagrangian: the non-local effective action is rewritten as an infinite series of local terms in an Operator Product Expansion (OPE) [15,16]. Roughly speaking, this corresponds to an expansion in powers of $1/M$. It is in this step that the short- and long-distance physics is disentangled. The long-distance physics corresponds to interactions at low energies and is the same in the full and the effective theory. But short-distance effects arising from quantum corrections involving large virtual momenta (of order M) are not reproduced in the effective theory, once the heavy particle has been integrated out. In a third step, they have to be added in a perturbative way using renormalization-group techniques. These short-distance effects lead to a renormalization of the coefficients of the local operators in the effective Lagrangian. An example is the effective Lagrangian for non-leptonic weak decays, in which radiative corrections from hard gluons with virtual momenta in the range between m_W and some renormalization scale $\mu \sim 1$ GeV give rise to Wilson coefficients, which renormalize the local four-fermion interactions [64–66].

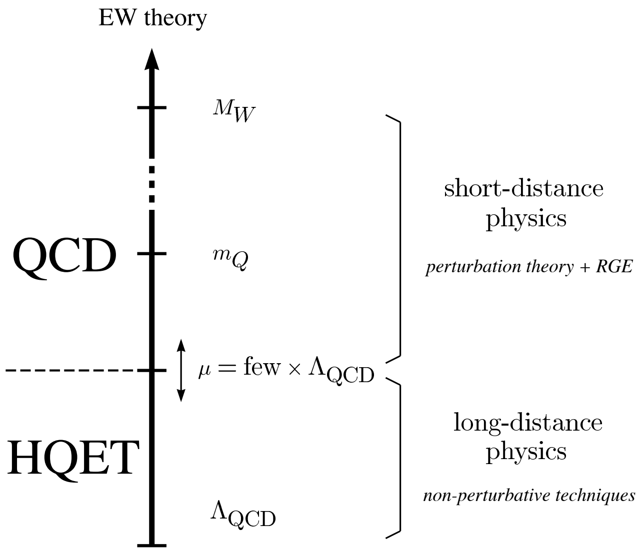

Figure 5: Philosophy of the heavy-quark effective theory.

The heavy-quark effective theory (HQET) is constructed to provide a simplified description of processes where a heavy quark interacts with light degrees of freedom predominantly by the exchange of soft gluons [67–77]. Clearly, m_Q is the high-energy scale in this case, and Λ_{QCD} is the scale of the hadronic physics we are interested in. The situation is illustrated in Fig. 5. At short distances, i.e. for energy scales larger than the heavy-quark mass, the physics is perturbative and described by ordinary QCD. For mass scales much below the heavy-quark mass, the physics is complicated and non-perturbative because of confinement. Our goal is to obtain a simplified description in this region using an effective field theory. To separate short- and long-distance effects, we introduce a separation scale μ such that $\Lambda_{\rm QCD} \ll \mu \ll m_Q$. The HQET will be constructed in such a way that it is identical to QCD in the long-distance region, i.e. for scales below μ. In the short-distance region, the effective theory is incomplete, however, since some high-momentum modes have been integrated out from the full theory. The fact that the physics must be independent of the arbitrary scale μ allows us to derive renormalization-group equations, which we shall employ to deal with the short-distance effects in an efficient way.

Compared with most effective theories, in which the degrees of freedom of a heavy particle are removed completely from the low-energy theory, the HQET is special in that its purpose is to describe the properties and decays of hadrons which do contain a heavy quark. Hence, it is not possible to remove the heavy quark completely from the effective theory. What is possible is to integrate out the "small components" in the full heavy-quark spinor, which describe the fluctuations around the mass shell.

The starting point in the construction of the low-energy effective theory is the observation that a very heavy quark bound inside a hadron moves more or less with the hadron's velocity v, and is almost on shell. Its momentum can be written as

$$p_Q^\mu = m_Q v^\mu + k^\mu \,, \tag{38}$$

where the components of the so-called residual momentum k are much smaller than m_Q. Note that v is a four-velocity, so that $v^2 = 1$. Interactions of the heavy quark with light degrees of freedom change the residual momentum by an amount of order $\Delta k \sim \Lambda_{\rm QCD}$, but the corresponding changes in the heavy-quark velocity vanish as $\Lambda_{\rm QCD}/m_Q \to 0$. In this situation, it is appropriate to introduce large- and small-component fields, h_v and H_v, by

$$h_v(x) = e^{i m_Q v\cdot x}\, P_+\, Q(x) \,, \qquad H_v(x) = e^{i m_Q v\cdot x}\, P_-\, Q(x) \,, \tag{39}$$

where P_+ and P_- are projection operators defined as

$$P_\pm = \frac{1 \pm \not{v}}{2} \,. \tag{40}$$

It follows that

$$Q(x) = e^{-i m_Q v\cdot x}\, [h_v(x) + H_v(x)] \,. \tag{41}$$

Because of the projection operators, the new fields satisfy $\not{v}\, h_v = h_v$ and $\not{v}\, H_v = -H_v$. In the rest frame, i.e. for $v^\mu = (1,0,0,0)$, h_v corresponds to the upper two components of Q, while H_v corresponds to the lower ones. Whereas h_v annihilates a heavy quark with velocity v, H_v creates a heavy antiquark with velocity v.

In terms of the new fields, the QCD Lagrangian for a heavy quark takes the form

$$\begin{aligned}
\mathcal{L}_Q &= \bar{Q}\,(i\,\not{D} - m_Q)\,Q \\
&= \bar{h}_v\,iv\!\cdot\! D\,h_v - \bar{H}_v\,(iv\!\cdot\! D + 2m_Q)\,H_v \\
&\quad + \bar{h}_v\,i\,\not{D}_\perp H_v + \bar{H}_v\,i\,\not{D}_\perp h_v\,,
\end{aligned} \tag{42}$$

where $D_\perp^\mu = D^\mu - v^\mu\, v\!\cdot\! D$ is orthogonal to the heavy-quark velocity: $v\!\cdot\! D_\perp = 0$. In the rest frame, $D_\perp^\mu = (0, \vec{D}\,)$ contains the spatial components of the covariant derivative. From (42), it is apparent that h_v describes massless degrees of freedom, whereas H_v corresponds to fluctuations with twice the heavy-quark mass. These are the heavy degrees of freedom that will be eliminated in the construction of the effective theory. The fields are mixed by the presence of the third and fourth terms, which describe pair creation or annihilation of heavy quarks and antiquarks. As shown in the first diagram in Fig. 6, in a virtual process a heavy quark propagating forward in time can turn into an antiquark propagating backward in time, and then turn back into a quark. The energy of the intermediate quantum state $hh\bar{H}$ is larger than the energy of the initial heavy quark by at least $2m_Q$. Because of this large energy gap, the virtual quantum fluctuation can only propagate over a short distance $\Delta x \sim 1/m_Q$. On hadronic scales set by $R_{\rm had} = 1/\Lambda_{\rm QCD}$, the process essentially looks like a local interaction of the form

$$\bar{h}_v\,i\,\not{D}_\perp\,\frac{1}{2m_Q}\,i\,\not{D}_\perp h_v\,, \tag{43}$$

where we have simply replaced the propagator for H_v by $1/2m_Q$. A more correct treatment is to integrate out the small-component field H_v, thereby deriving a non-local effective action for the large-component field h_v, which can then be expanded in terms of local operators. Before doing this, let us mention a second type of virtual corrections involving pair creation, namely heavy-quark loops. An example is shown in the second diagram in Fig. 6. Heavy-quark loops cannot be described in terms of the effective fields h_v and H_v, since the quark velocities inside a loop are not conserved and are in no way related to hadron velocities. However, such short-distance processes are proportional to the small coupling constant $\alpha_s(m_Q)$ and can be calculated in perturbation theory. They lead to corrections that are added onto the low-energy effective theory in the renormalization procedure to be discussed later.

On a classical level, the heavy degrees of freedom represented by H_v can be eliminated using the equation of motion. Taking the variation of the Lagrangian with respect to the field $\bar{H}_v$, we obtain

$$(iv\!\cdot\! D + 2m_Q)\,H_v = i\,\not{D}_\perp h_v\,. \tag{44}$$

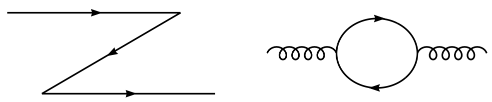

Figure 6: Virtual fluctuations involving pair creation of heavy quarks. In the first diagram, time flows to the right.

This equation can formally be solved to give

$$H_v = \frac{1}{2m_Q + iv\cdot D}\, i\,\not{D}_\perp h_v \,, \tag{45}$$

showing that the small-component field H_v is indeed of order $1/m_Q$. We can now insert this solution into (42) to obtain the "non-local effective Lagrangian"

$$\mathcal{L}_{\rm eff} = \bar h_v\, iv\cdot D\, h_v + \bar h_v\, i\,\not{D}_\perp \frac{1}{2m_Q + iv\cdot D}\, i\,\not{D}_\perp h_v \,. \tag{46}$$

Clearly, the second term corresponds to the first class of virtual processes shown in Fig. 6.

It is possible to derive this Lagrangian in a more elegant way by manipulating the generating functional for QCD Green's functions containing heavy-quark fields [77]. To this end, one starts from the field redefinition (41) and couples the large-component fields h_v to external sources ρ_v. Green's functions with an arbitrary number of h_v fields can be constructed by taking derivatives with respect to ρ_v. No sources are needed for the heavy degrees of freedom represented by H_v. The functional integral over these fields is Gaussian and can be performed explicitly, leading to the effective action

$$S_{\rm eff} = \int {\rm d}^4x\, \mathcal{L}_{\rm eff} - i\ln\Delta \,, \tag{47}$$

with $\mathcal{L}_{\rm eff}$ as given in (46). The appearance of the logarithm of the determinant

$$\Delta = \exp\left(\frac{1}{2}\,{\rm Tr}\,\ln\left[2m_Q + iv\cdot D - i\eta\right] \right) \tag{48}$$

is a quantum effect not present in the classical derivation presented above. However, in this case the determinant can be regulated in a gauge-invariant way, and by choosing the axial gauge $v\cdot A = 0$ one shows that $\ln\Delta$ is just an irrelevant constant [77,78].

Because of the phase factor in (41), the x dependence of the effective heavy-quark field h_v is weak. In momentum space, derivatives acting on h_v correspond to powers of the residual momentum k, which by construction is much smaller than m_Q. Hence, the non-local effective Lagrangian (46) allows for a derivative expansion in powers of iD/m_Q:

$$\mathcal{L}_{\text{eff}} = \bar{h}_v\, iv\!\cdot\! D\, h_v + \frac{1}{2m_Q} \sum_{n=0}^{\infty} \bar{h}_v\, i \not{D}_\perp \left(-\frac{iv\cdot D}{2m_Q} \right)^n i \not{D}_\perp h_v\,. \tag{49}$$

Taking into account that h_v contains a P_+ projection operator, and using the identity

$$P_+\, i\not{D}_\perp\, i\not{D}_\perp P_+ = P_+ \left[(iD_\perp)^2 + \frac{g_s}{2}\, \sigma_{\mu\nu}\, G^{\mu\nu} \right] P_+\,, \tag{50}$$

where $[iD^\mu, iD^\nu] = ig_s G^{\mu\nu}$ is the gluon field-strength tensor, one finds that [75,76]

$$\mathcal{L}_{\text{eff}} = \bar{h}_v\, iv\!\cdot\! D\, h_v + \frac{1}{2m_Q}\, \bar{h}_v\, (iD_\perp)^2\, h_v + \frac{g_s}{4m_Q}\, \bar{h}_v\, \sigma_{\mu\nu}\, G^{\mu\nu}\, h_v + O(1/m_Q^2)\,. \tag{51}$$

In the limit $m_Q \to \infty$, only the first terms remains:

$$\mathcal{L}_\infty = \bar{h}_v\, iv\!\cdot\! D\, h_v\,. \tag{52}$$

This is the effective Lagrangian of the HQET. It gives rise to the Feynman rules depicted in Fig. 7.

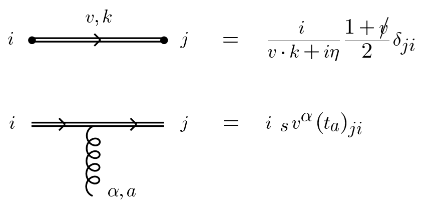

Figure 7: Feynman rules of the HQET (i, j and a are colour indices). A heavy quark is represented by a double line labelled by the velocity v and the residual momentum k. The velocity v is conserved by the strong interactions.

Let us take a moment to study the symmetries of this Lagrangian [27]. Since there appear no Dirac matrices, interactions of the heavy quark with gluons leave its spin unchanged. Associated with this is an SU(2) symmetry group, under which $\mathcal{L}_\infty$ is invariant. The action of this symmetry on the heavy-quark fields becomes most transparent in the rest frame, where the generators S^i of SU(2) can be chosen as

$$S^i = \frac{1}{2} \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}, \qquad [S^i, S^j] = i\epsilon^{ijk} S^k \,. \tag{53}$$

Here σ^i are the Pauli matrices. An infinitesimal SU(2) transformation $h_v \to (1 + i\vec{\epsilon}\cdot\vec{S}\,)\,h_v$ leaves the Lagrangian invariant:

$$\delta\mathcal{L}_\infty = \bar{h}_v\,[iv\cdot D, i\vec{\epsilon}\cdot\vec{S}\,]\,h_v = 0 \,. \tag{54}$$

Another symmetry of the HQET arises since the mass of the heavy quark does not appear in the effective Lagrangian. For N_h heavy quarks moving at the same velocity, eq. (52) can be extended by writing

$$\mathcal{L}_\infty = \sum_{i=1}^{N_h} \bar{h}_v^i\, iv\cdot D\, h_v^i \,. \tag{55}$$

This is invariant under rotations in flavour space. When combined with the spin symmetry, the symmetry group is promoted to SU($2N_h$). This is the heavy-quark spin–flavour symmetry [26,27]. Its physical content is that, in the limit $m_Q \to \infty$, the strong interactions of a heavy quark become independent of its mass and spin.

Consider now the operators appearing at order $1/m_Q$ in the effective Lagrangian (51). They are easiest to identify in the rest frame. The first operator,

$$\mathcal{O}_{\rm kin} = \frac{1}{2m_Q}\,\bar{h}_v\,(iD_\perp)^2\,h_v \to -\frac{1}{2m_Q}\,\bar{h}_v\,(i\vec{D}\,)^2\,h_v \,, \tag{56}$$

is the gauge-covariant extension of the kinetic energy arising from the off-shell residual motion of the heavy quark. The second operator is the non-abelian analogue of the Pauli interaction, which describes the chromo-magnetic coupling of the heavy-quark spin to the gluon field:

$$\mathcal{O}_{\rm mag} = \frac{g_s}{4m_Q}\,\bar{h}_v\,\sigma_{\mu\nu}\,G^{\mu\nu}\,h_v \to -\frac{g_s}{m_Q}\,\bar{h}_v\,\vec{S}\cdot\vec{B}_c\,h_v \,. \tag{57}$$

Here $\vec{S}$ is the spin operator defined in (53), and $B_c^i = -\frac{1}{2}\epsilon^{ijk} G^{jk}$ are the components of the chromo-magnetic field. The chromo-magnetic interaction is a relativistic effect, which scales like $1/m_Q$. This is the origin of the heavy-quark spin symmetry.

3.2 Wave-Function Renormalization of the Heavy-Quark Field in the HQET

Besides being an effective theory for the strong interactions of heavy quarks with light degrees of freedom, the HQET is a consistent, renormalizable (order by order in $1/m_Q$) quantum field theory in its own right. In particular, it provides a framework for calculating radiative corrections. We shall discuss as an illustration the wave-function renormalization of the heavy-quark field h_v.

In quantum field theory, the parameters and fields of the Lagrangian have no direct physical significance. They have to be renormalized before they can be related to observable quantities. In an intermediate step the theory has to be regularized. The most convenient regularization scheme in QCD is dimensional regularization [79−81], in which the dimension of space-time is analytically continued to $D = 4-2\epsilon$, with ϵ being infinitesimal. Loop integrals that are logarithmically divergent in four dimensions become finite for $\epsilon > 0$. From the fact that the action $S = \int \mathrm{d}^D x\, \mathcal{L}(x)$ is dimensionless, one can derive the mass dimensions of the fields and parameters of the theory. For instance, one finds that the "bare" coupling constant α_s^{bare} is no longer dimensionless if $D \neq 4$: $\dim[\,\alpha_s^{\mathrm{bare}}\,] = 2\epsilon$. In a renormalizable theory, it is possible to rewrite the Lagrangian in terms of renormalized quantities in such a way that Green's functions of the renormalized fields remain finite as $\epsilon \to 0$. For QCD, one introduces renormalized quantities by $Q^{\mathrm{bare}} = Z_Q^{1/2}\, Q^{\mathrm{ren}}$, $A^{\mathrm{bare}} = Z_A^{1/2} A^{\mathrm{ren}}$, $\alpha_s^{\mathrm{bare}} = \mu^{2\epsilon} Z_\alpha\, \alpha_s^{\mathrm{ren}}$, etc., where μ is an arbitrary mass scale introduced to render the renormalized coupling constant dimensionless. Similarly, in the HQET one defines the renormalized heavy-quark field by $h_v^{\mathrm{bare}} = Z_h^{1/2}\, h_v^{\mathrm{ren}}$. From now on, the superscript "ren" will be omitted.

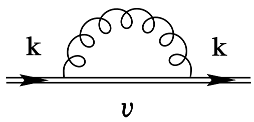

Figure 8: One-loop self-energy $-i\Sigma(v\cdot k)$ of a heavy quark in the HQET.

In the minimal subtraction (MS) scheme, Z_h can be computed from the $1/\epsilon$ pole in the heavy-quark self-energy using

$$1 - Z_h^{-1} = \frac{1}{\epsilon}\text{pole of } \frac{\partial\Sigma(v\cdot k)}{\partial v\cdot k}\,. \tag{58}$$

As long as $v\cdot k < 0$, the self-energy is infrared finite and real. The result is gauge-dependent, however. Evaluating the diagram shown in Fig. 8 in the

Feynman gauge, we obtain at one-loop order

$$
\begin{aligned}
\Sigma(v\cdot k) &= -ig_s^2\,t_a t_a \int\frac{\mathrm{d}^D t}{(2\pi)^D}\,\frac{1}{(t^2+i\eta)\big[v\cdot(t+k)+i\eta\big]} \\
&= -2iC_F g_s^2 \int\limits_0^\infty \mathrm{d}\lambda \int\frac{\mathrm{d}^D t}{(2\pi)^D}\,\frac{1}{\big[t^2+2\lambda\,v\cdot(t+k)+i\eta\big]^2} \\
&= \frac{C_F\alpha_s}{2\pi}\,\Gamma(\epsilon)\int\limits_0^\infty \mathrm{d}\lambda\left(\frac{\lambda^2+\lambda\omega}{4\pi\mu^2}\right)^{-\epsilon}, \qquad (59)
\end{aligned}
$$

where $C_F=4/3$ is a colour factor, λ is a dimensionful Feynman parameter, and $\omega=-2v\cdot k>0$ acts as an infrared cutoff. A straightforward calculation leads to

$$
\begin{aligned}
\frac{\partial\Sigma(v\cdot k)}{\partial v\cdot k} &= \frac{C_F\alpha_s}{\pi}\,\Gamma(1+\epsilon)\left(\frac{\omega^2}{4\pi\mu^2}\right)^{-\epsilon}\int\limits_0^1 \mathrm{d}z\,z^{-1+2\epsilon}\,(1-z)^{-\epsilon} \\
&= \frac{C_F\alpha_s}{\pi}\,\Gamma(2\epsilon)\,\Gamma(1-\epsilon)\left(\frac{\omega^2}{4\pi\mu^2}\right)^{-\epsilon}, \qquad (60)
\end{aligned}
$$

where we have substituted $\lambda=\omega\,(1-z)/z$. From an expansion around $\epsilon=0$, we obtain

$$
Z_h = 1+\frac{C_F\alpha_s}{2\pi\epsilon}\,. \qquad (61)
$$

This result was first derived by Politzer and Wise [71]. In the meantime, the calculation was also done at the two-loop order [82−85].

3.3 The Residual Mass Term and the Definition of the Heavy-Quark Mass

The choice of the expansion parameter in the HQET, i.e. the definition of the heavy-quark mass m_Q, deserves some comments. In the derivation presented earlier in this section, we chose m_Q to be the "mass in the Lagrangian", and using this parameter in the phase redefinition in (41) we obtained the effective Lagrangian (52), in which the heavy-quark mass no longer appears. However, this treatment has its subtleties. The symmetries of the HQET allow a "residual mass term" δm for the heavy quark, provided that δm is of order $\Lambda_{\rm QCD}$ and is the same for all heavy-quark flavours. Even if we arrange that such a term is not present at the tree level, it will in general be induced by quantum corrections. (This is unavoidable if the theory is regulated with a dimensionful cutoff.) Therefore, instead of (52) we should write the effective Lagrangian in the more general form [30]:

$$\begin{aligned} h_v(x) &= e^{im_Q v\cdot x}\, P_+\, Q(x) \\ \Rightarrow \qquad \mathcal{L}_\infty &= \bar h_v\, iv\!\cdot\! D\, h_v - \delta m\, \bar h_v h_v \,. \end{aligned} \tag{62}$$

If we redefine the expansion parameter according to $m_Q \to m_Q + \Delta m$, the residual mass changes in the opposite way: $\delta m \to \delta m - \Delta m$. This implies that there is a unique choice of the expansion parameter such that $\delta m = 0$. Requiring $\delta m = 0$, as it is usually done implicitly in the HQET, defines a heavy-quark mass, which in perturbation theory coincides with the pole mass [86]. This, in turn, defines for each heavy hadron a parameter $\bar\Lambda$ (sometimes called the "binding energy") through

$$\bar\Lambda = (m_H - m_Q)\Big|_{m_Q\to\infty} \,. \tag{63}$$

If one prefers to work with another choice of the expansion parameter, the values of non-perturbative parameters such as $\bar\Lambda$ change, but at the same time one has to include the residual mass term in the HQET Lagrangian. It can be shown that the various parameters that depend on the definition of m_Q enter the predictions for all physical observables in such a way that the results are independent of which particular choice one adopts [30].

There is one more subtlety hidden in the above discussion. The quantities m_Q, $\bar\Lambda$ and δm are non-perturbative parameters of the HQET, which have a similar status as the vacuum condensates in QCD phenomenology [87]. These parameters cannot be defined unambiguously in perturbation theory. The reason lies in the divergent behaviour of perturbative expansions in large orders, which is associated with the existence of singularities along the real axis in the Borel plane, the so-called renormalons [88–96]. For instance, the perturbation series which relates the pole mass m_Q of a heavy quark to its bare mass,

$$m_Q = m_Q^{\rm bare} \Big\{ 1 + c_1\, \alpha_s(m_Q) + c_2\, \alpha_s^2(m_Q) + \ldots + c_n\, \alpha_s^n(m_Q) + \ldots \Big\} \,, \tag{64}$$

contains numerical coefficients c_n that grow as $n!$ for large n, rendering the series divergent and not Borel summable [97,98]. The best one can achieve is to truncate the perturbation series at the minimal term, but this leads to an unavoidable arbitrariness of order $\Delta m_Q \sim \Lambda_{\rm QCD}$ (the size of the minimal term). This observation, which at first sight seems a serious problem for QCD phenomenology, should actually not come as a surprise. We know that because of confinement quarks do not appear as physical states in nature. Hence, there is no way to define their on-shell properties such as a pole mass. In view of this, it is actually remarkable that QCD perturbation theory "knows" about its incompleteness and indicates, through the appearance of renormalon singularities, the presence of non-perturbative effects. We must first specify a scheme how to truncate the QCD perturbation series before non-perturbative statements such as $\delta m = 0$ become meaningful, and hence before non-perturbative parameters such as m_Q and $\bar{\Lambda}$ become well-defined quantities. The actual values of these parameters will depend on this scheme.

We stress that the "renormalon ambiguities" are not a conceptual problem for the heavy-quark expansion. In fact, it can be shown quite generally that these ambiguities cancel in all predictions for physical observables [99]. The way the cancellations occur is intricate, however. The generic structure of the heavy-quark expansion for an observable is of the form:

$$\text{observable} \sim C[\alpha_s(m_Q)] \left(1 + \frac{\Lambda}{m_Q} + \dots \right) . \qquad (65)$$

Here $C[\alpha_s(m_Q)]$ represents a perturbative coefficient function, and Λ is a dimensionful non-perturbative parameter. The truncation of the perturbation series defining the coefficient function leads to an arbitrariness of order $\Lambda_{\mathrm{QCD}}/m_Q$, which precisely cancels against a corresponding arbitrariness of order Λ_{QCD} in the definition of the non-perturbative parameter Λ.

The renormalon problem poses itself when one imagines to apply perturbation theory in very high orders. In practise, the perturbative coefficients are known to finite order in α_s (at best to two-loop accuracy), and to be consistent one should use them in connection with the pole mass (and $\bar{\Lambda}$ etc.) defined to the same order.

4 Matching and Running

In section 2, we have discussed the first two steps in the construction of the HQET. Integrating out the small components in the heavy-quark fields, a non-local effective action was derived, which was then expanded in a series of local operators. The effective Lagrangian derived that way correctly reproduces the long-distance physics of the full theory. It does not contain the short-distance physics correctly, however. The reason is obvious: A heavy quark participates in strong interactions through its coupling to gluons. These gluons can be soft or hard, i.e. their virtual momenta can be small, of the order of the confinement scale, or large, of the order of the heavy-quark mass. But hard gluons can resolve the spin and flavour quantum numbers of a heavy quark. Their effects lead to a renormalization of the coefficients of the operators in the HQET.

Consider, as an example, matrix elements of the vector current $V = \bar{q}\,\gamma^\mu Q$. In QCD this current is (partially) conserved and needs no renormalization [100]. Its matrix elements are free of ultraviolet divergences. Still, these matrix elements have a logarithmic dependence on m_Q from the exchange of hard gluons with virtual momenta of the order of the heavy-quark mass. If one goes over to the effective theory by taking the limit $m_Q \to \infty$, these logarithms diverge. Consequently, the vector current in the effective theory does require a renormalization [71]. Its matrix elements depend on an arbitrary renormalization scale μ, which separates the regions of short- and long-distance physics. If μ is chosen such that $\Lambda_{\rm QCD} \ll \mu \ll m_Q$, the effective coupling constant in the region between μ and m_Q is small, and perturbation theory can be used to compute the short-distance corrections. These corrections have to be added to the matrix elements of the effective theory, which contain the long-distance physics below the scale μ. Schematically, then, the relation between matrix elements in the full and in the effective theory is

$$\langle V(m_Q)\rangle_{\rm QCD} = C_0(m_Q,\mu)\,\langle V_0(\mu)\rangle_{\rm HQET} + \frac{C_1(m_Q,\mu)}{m_Q}\,\langle V_1(\mu)\rangle_{\rm HQET} + \dots\,, \tag{66}$$

where we have indicated that matrix elements in the full theory depend on m_Q, whereas matrix elements in the effective theory are mass-independent, but do depend on the renormalization scale. The Wilson coefficients $C_i(m_Q,\mu)$ are defined by this relation. Order by order in perturbation theory, they can be computed from a comparison of the matrix elements in the two theories. Since the effective theory is constructed to reproduce correctly the low-energy behaviour of the full theory, this "matching" procedure is independent of any long-distance physics, such as infrared singularities, non-perturbative effects, the nature of the external states used in the matrix elements, etc.

The calculation of the coefficient functions in perturbation theory uses the powerful methods of the renormalization group. It is in principle straightforward, yet in practice rather tedious. A comprehensive discussion of most of the existing calculations of short-distance corrections in the HQET can be found in Ref. 8. Here, we shall discuss as an illustration the renormalization of the $1/m_Q$-suppressed operators in the effective Lagrangian (51). At the tree level, there appear two operators at order $1/m_Q$, which have been given in (56) and (57). Beyond the tree level, the coefficients of these operators may be modified, and other operators not present at the classical level may be induced. In general, we thus expect

$$\mathcal{L}_{1/m} = C_{\rm kin}(\mu)\,\mathcal{O}_{\rm kin}(\mu) + C_{\rm mag}(\mu)\,\mathcal{O}_{\rm mag}(\mu) + \text{new operators}\,, \tag{67}$$

where μ is the renormalization scale. But how do we calculate the Wilson coefficient functions, and what are the possible new operators? To extend the classical construction of Sec. 3.1 to include quantum corrections would be cumbersome. Fortunately, there is a systematic procedure which allows us to derive the result in the presence of quantum effects in a rather simple and straightforward way. It consists of three steps: construction of the operator basis, calculation of the "matching conditions" at $\mu = m_Q$, and renormalization-group improvement ("running"). Below, we shall first explain these steps in general and then illustrate them with the particular example of $\mathcal{L}_{1/m}$.

4.1 Construction of the Operator Basis

Similar to the fields and coupling constants, in a quantum field theory any composite operator built from quark and gluon fields may require a renormalization beyond that of its component fields. Such operators can be divided into three classes: gauge-invariant operators that do not vanish by the equations of motion (class-I), gauge-invariant operators that vanish by the equations of motion (class-II), and operators which are not gauge-invariant (class-III). In general, operators with the same dimension and quantum numbers mix under renormalization. However, things simplify if one works with the background field technique [101−104], which is an elegant method for quantizing gauge theories, preserving explicit gauge invariance. This offers the advantage that a class-I operator cannot mix with class-III operators, so that only gauge-invariant operators need to be considered [105]. Furthermore, class-II operators are irrelevant since their matrix elements vanish by the equations of motion. It it thus sufficient to consider class-I operators only.

Thus, we must find a complete set of class-I operators of the right dimension, carrying the quantum numbers allowed by the symmetries of the problem. In the case at hand, we are dealing with operators appearing at order $1/m_Q$ in a strong-interaction Lagrangian, and we thus have to find dimension-five operators containing two heavy-quark fields of the same velocity. Moreover, these operators must transform as scalars under the Lorentz group. The most general form of such operators is

$$\bar{h}_v\, \Gamma_{\mu\nu}\, iD^\mu iD^\nu\, h_v\,; \qquad \Gamma_{\mu\nu} \in \Big\{ g_{\mu\nu}, v_\mu v_\nu, \gamma_\mu v_\nu, \gamma_\nu v_\mu, \tfrac{1}{2}\, [\gamma_\mu, \gamma_\nu] \Big\}\,. \tag{68}$$

Note that the velocity is not a dynamical quantity in the HQET and thus can be used to construct the basis operators. Using that $\bar{h}_v \gamma_\mu\, h_v = \bar{h}_v\, v_\mu\, h_v$, we find that there are only three possible operators:

$$\bar{h}_v\, (iD)^2 h_v\,, \qquad \bar{h}_v\, (iv\!\cdot\! D)^2 h_v\,, \qquad \tfrac{1}{2}\, \bar{h}_v\, \sigma_{\mu\nu} g_s G^{\mu\nu} h_v\,. \tag{69}$$

Since the equation of motion of the HQET is $iv\cdot D\,h_v = 0$, it follows that there are two class-I and one class-II operators, which we choose in the form:

$$\begin{aligned}
\text{class-I:} \quad & \bar h_v\,(iD_\perp)^2 h_v\,, \qquad \tfrac{1}{2}\,\bar h_v\,\sigma_{\mu\nu} g_s G^{\mu\nu} h_v\,, \\
\text{class-II:} \quad & \bar h_v\,(iv\cdot D)^2 h_v\,.
\end{aligned} \tag{70}$$

Besides a class-II operator, which has vanishing matrix elements between physical states, the kinetic and chromo-magnetic operators already present at the tree level are thus the only operators which can appear in $\mathcal{L}_{1/m}$, even in the presence of quantum corrections. Once we have found a complete basis of class-I operators, our next goal is to calculate their coefficient functions in perturbation theory.

4.2 Matching Conditions at $\mu = m_Q$

The Wilson coefficient functions $C_{\rm kin}(\mu)$ and $C_{\rm mag}(\mu)$ in (67) can be obtained from the comparison ("matching") of Green's functions in QCD with those in the effective theory. It is crucial that, by construction, the Wilson coefficients receive only short-distance contributions (see Fig. 5) and are thus insensitive to the properties of the external states. This ensures that once the coefficients have been determined by requiring that some particular Green's function(s) be the same in the two theories, all other Green's functions will be the same. Moreover, since the Wilson coefficients are infrared insensitive they are calculable in perturbation theory, and we can perform their calculation using quark and gluon states rather than physical hadron states.

In the example at hand, the coefficients $C_{\rm kin}(\mu)$ and $C_{\rm mag}(\mu)$ can be obtained from a calculation of the Green's function of two heavy quarks and a background gluon field, to one-loop order in the full and in the effective theory. The relevant vertex diagrams in QCD are shown in Fig. 9. They have to be supplemented by the wave-function renormalization of the external quark lines. The background field is not renormalized. The momentum assignments are such that p is the outgoing momentum of the background field, and k and $(k-p)$ are the residual momenta of the heavy quarks. To order $1/m_Q$, it is sufficient to keep terms linear in k or p. The quarks can be taken on shell, in which case $v\cdot k = v\cdot p = 0$. A subtlety which has to be taken into account is that the QCD spinor $u_Q(P_Q,s)$ is related to the spinor $u_h(v,s)$ of the effective theory by

$$u_Q(P_Q,s) = \left(1 + \frac{\not k}{2m_Q} + \dots \right) u_h(v,s)\,, \tag{71}$$

where $P_Q = m_Q v + k$. In the matching calculation one has to use the same spinors in both theories. We thus define a vertex function Γ^μ by writing the amplitude as $i\mu^\epsilon g_s A_{\mu,a}(p)\,\bar u_h \Gamma^\mu t_a u_h$, so that at the tree level in QCD

$$\begin{aligned}\Gamma^\mu_{\mathrm{QCD},0} &= \left(1+\frac{\not k-\not p}{2m_Q}\right)\gamma^\mu\left(1+\frac{\not k}{2m_Q}\right)+\dots\\ &= v^\mu+\frac{(2k-p)^\mu}{2m_Q}+\frac{\big[\gamma^\mu,\not p\big]}{4m_Q}+\dots\,. \end{aligned}\tag{72}$$

Here the ellipses represent terms of higher order in k or p, and we have used that between the heavy-quark spinors γ^μ can be replaced by v^μ.

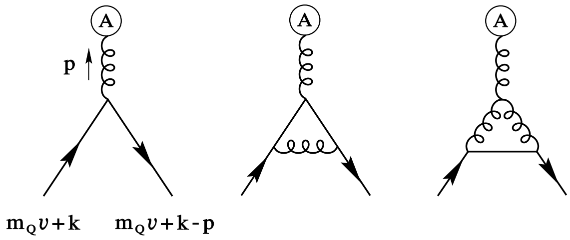

Figure 9: Diagrams for the calculation of the heavy quark-gluon vertex function in QCD. The background field is denoted by A.

The contributions to the vertex function arising at the one-loop level are also shown in Fig. 9. They contain both abelian and non-abelian vertices. Since the matching calculation is insensitive to any long-distance properties such as the nature of the infrared regulator, it is legitimate to work with any infrared regularization scheme that is convenient. Following Eichten and Hill [75,106], we choose to regulate both ultraviolet and infrared divergences using dimensional regularization. Moreover, we expand the resulting expressions for the Feynman amplitudes to linear order in the external momenta and then set the external momenta to zero inside the loop integrals. Then the only mass scale remaining is the heavy-quark mass. In the $\overline{\mathrm{MS}}$ scheme, the result for the one-loop contribution to the QCD vertex function is [75]

$$\Gamma^\mu_{\mathrm{QCD},1} = \frac{\big[\gamma^\mu,\not p\big]}{4m_Q}\,\frac{\alpha_s}{2\pi}\left(-C_A\ln\frac{m_Q}{\mu}+C_A+C_F\right),\tag{73}$$

where $C_F=\frac{1}{2}(N_c^2-1)/N_c=4/3$ and $C_A=N_c=3$ are the eigenvalues of the quadratic Casimir operator in the fundamental and the adjoint representations.

Now comes a clue: if dimensional regularization is used to regulate both ultraviolet and infrared singularities, all loop integrals in the HQET are no-scale integrals (after a power of the external momenta has been factored out) and vanish! So only the tree-level matrix elements of the HQET operators in the effective Lagrangian multiplied by their Wilson coefficient functions remain. This is why dimensional regularization is superb for matching calculations. The result is

$$\Gamma^\mu_{\rm HQET} = v^\mu + C_{\rm kin}(\mu)\,\frac{(2k-p)^\mu}{2m_Q} + C_{\rm mag}(\mu)\,\frac{[\gamma^\mu,\not p]}{4m_Q} + \dots\,. \qquad (74)$$

Requiring that the vertex functions be the same in the full and in the effective theory, we find (in the $\overline{\rm MS}$ scheme)

$$C_{\rm kin}(\mu) = 1\,, \qquad C_{\rm mag}(\mu) = 1 + \frac{\alpha_s}{2\pi}\left(- C_A \ln\frac{m_Q}{\mu} + C_A + C_F \right). \qquad (75)$$

The fact that the kinetic operator is not renormalized is not an accident, but follows from an invariance of the HQET under small redefinitions of the velocity used in the construction of the effective Lagrangian. Clearly, the predictions of the HQET should not depend on whether v is taken to be the velocity of the hadron containing the heavy quark, the velocity of the heavy quark itself, or some other velocity differing from the hadron velocity by an amount of order $\Lambda_{\rm QCD}/m_Q$. This so-called reparametrization invariance implies that $C_{\rm kin}(\mu)=1$ must hold to all orders in perturbation theory [107,108].

In the next paragraph, we will see that the scale dependence predicted by the one-loop result quoted above cannot be trusted if $\mu\ll m_Q$; however, what can be obtained from the matching calculation are the values of the coefficient functions at the matching scale $\mu=m_Q$ as well as their logarithmic derivatives. For the coefficient of the chromo-magnetic operator, we find:

$$\begin{aligned} C_{\rm mag}(m_Q) &= 1 + (C_A + C_F)\,\frac{\alpha_s(m_Q)}{2\pi}\,, \\ \frac{{\rm d}\ln C_{\rm mag}(\mu)}{{\rm d}\ln\mu} &= C_A\,\frac{\alpha_s}{2\pi}\,. \end{aligned} \qquad (76)$$

4.3 Renormalization-Group Evolution

The one-loop calculation presented above allows us to derive expressions for the Wilson coefficient functions provided that $m_Q/\mu = O(1)$. In practical applications of effective field theories, one is however often interested in the case where there is a large ratio of mass scales. After all, an effective theory is constructed to separate the physics on two very different energy scales. In such a situation, the coefficient functions contain large logarithms of the type

$[\alpha_s \ln(m_Q/\mu)]^n$, which must be summed to all orders in perturbation theory. This is achieved by using the powerful machinery of the renormalization group.

For a set $\{\mathcal{O}_i\}$ of n class-I operators that mix under renormalization, one defines an $n \times n$ matrix of renormalization factors Z_{ij} by $\mathcal{O}_i^{\rm bare} = Z_{ij}\,\mathcal{O}_j(\mu)$, such that the matrix elements of the renormalized operators $\mathcal{O}_j(\mu)$ remain finite as $\epsilon \to 0$. In contrast to the bare operators, the renormalized ones depend on the subtraction scale via the μ dependence of Z_{ij}:

$$\mu \frac{\rm d}{{\rm d}\mu}\,\mathcal{O}_i(\mu) = \left(\mu \frac{\rm d}{{\rm d}\mu} Z_{ij}^{-1} \right) \mathcal{O}_j^{\rm bare} = -\gamma_{ik}\,\mathcal{O}_k(\mu)\,, \tag{77}$$

where

$$\gamma_{ik} = -\left(\mu \frac{\rm d}{{\rm d}\mu} Z_{ij}^{-1} \right) Z_{jk} = Z_{ij}^{-1}\,\mu \frac{\rm d}{{\rm d}\mu}\,Z_{jk} \tag{78}$$

are called the anomalous dimensions. That under a change of the renormalization scale the operators mix among themselves follows from the fact that the basis of operators is complete. It is convenient to introduce a compact matrix notation, in which $\vec{\mathcal{O}}(\mu)$ is the vector of renormalized operators, $\hat Z$ is the matrix of renormalization factors, and $\hat\gamma$ denotes the anomalous dimension matrix. Then the scale dependence of the renormalized operators is controlled by the renormalization-group equation (RGE)

$$\left(\mu \frac{\rm d}{{\rm d}\mu} + \hat\gamma \right) \vec{\mathcal{O}}(\mu) = 0\,. \tag{79}$$

In the MS scheme, the matrix $\hat Z$ obeys an expansion of the form

$$\hat Z = 1 + \sum_{k=1}^\infty \frac{1}{\epsilon^k}\,\hat Z_k(\alpha_s)\,, \tag{80}$$

and by requiring that the anomalous dimensions in (78) be finite as $\epsilon \to 0$ one finds that $\hat\gamma$ can be computed in terms of the coefficient of the $1/\epsilon$ pole [109]:

$$\hat\gamma = -2\alpha_s\,\frac{\partial \hat Z_1(\alpha_s)}{\partial\alpha_s}\,. \tag{81}$$

The same relation holds in the $\overline{\rm MS}$ scheme.

From (79) and the fact that the product $C_i(\mu)\,\mathcal{O}_i(\mu)$ must be μ independent, we derive the RGE satisfied by the coefficient functions. It reads

$$\left(\mu \frac{\rm d}{{\rm d}\mu} - \hat\gamma^T \right) \vec C(\mu) = 0\,, \tag{82}$$

where we have collected the coefficients into a vector $\vec{C}(\mu)$. In general, the Wilson coefficients can depend on μ both explicitly or implicitly through the running coupling. We thus have

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu} = \mu\frac{\partial}{\partial\mu} + \beta(\alpha_s)\,\frac{\partial}{\partial\alpha_s(\mu)}\,, \tag{83}$$

where the β function

$$\beta(\alpha_s) = \mu\,\frac{\partial\alpha_s(\mu)}{\partial\mu} = -2\alpha_s\left[\beta_0\,\frac{\alpha_s}{4\pi} + \beta_1\left(\frac{\alpha_s}{4\pi}\right)^2 + \dots\right] \tag{84}$$

describes the scale dependence of the renormalized coupling constant. The one- and two-loop coefficients are scheme independent and are given by [18,19,110]

$$\begin{aligned} \beta_0 &= \frac{11}{3}\,C_A - \frac{4}{3}\,T_F\,n_f\,, \\ \beta_1 &= \frac{34}{3}\,C_A^2 - \left(\frac{20}{3}\,C_A + 4C_F\right)T_F\,n_f\,, \end{aligned} \tag{85}$$

where n_f is the number of light quark flavours, and $T_F = 1/2$ is the normalization of the SU(3) generators in the fundamental representation: $\mathrm{tr}(t_a t_b) = T_F\,\delta_{ab}$. It is now straightforward to obtain a formal solution of the RGE. It reads

$$\vec{C}(\mu) = \hat{U}(\mu, m_Q)\,\vec{C}(m_Q)\,, \tag{86}$$

with the evolution matrix [111–113]

$$\hat{U}(\mu, m_Q) = T_\alpha\,\exp\int\limits_{\alpha_s(m_Q)}^{\alpha_s(\mu)}\mathrm{d}\alpha\,\frac{\hat{\gamma}^T(\alpha)}{\beta(\alpha)}\,. \tag{87}$$

Here "T_α" means an ordering in the coupling constant such that the couplings increase from right to left (for $\mu < m_Q$). This is necessary since, in general, the anomalous dimension matrices at different values of α_s do not commute. Eq. (87) can be solved perturbatively by expanding the β function (84) and the anomalous dimension matrix in powers of the renormalized coupling constant:

$$\hat{\gamma}(\alpha_s) = \hat{\gamma}_0\,\frac{\alpha_s}{4\pi} + \hat{\gamma}_1\left(\frac{\alpha_s}{4\pi}\right)^2 + \dots\,. \tag{88}$$

Here we shall only discuss the important case of a single coefficient function, or equivalently, when there is no operator mixing. Then the matrix $\hat{\gamma}$ reduces to a number, and the evolution is described by a function $U(\mu, m_Q)$, for which the perturbative solution of (87) at next-to-leading order yields

$$U_{\rm NLO}(\mu, m_Q) = \left(\frac{\alpha_s(m_Q)}{\alpha_s(\mu)} \right)^a \left\{ 1 + \frac{\alpha_s(m_Q) - \alpha_s(\mu)}{4\pi}\, S + \dots \right\} , \tag{89}$$

with

$$a = \frac{\gamma_0}{2\beta_0} \,, \qquad S = \frac{\gamma_1}{2\beta_0} - \frac{\gamma_0\beta_1}{2\beta_0^2} \,. \tag{90}$$

The theoretical framework discussed here is called "renormalization-group (RG) improved perturbation theory". In the expression for the evolution function $U(\mu, m_Q)$, there are no large logarithms of the form $\alpha_s \ln(m_Q/\mu)$ left. They are all contained in the ratio of the running couplings evaluated at the scales m_Q and μ. Thus, RG-improved perturbation theory provides the optimal method to bridge wide energy intervals. (As a side remark, we note that the same technique is used to control the evolution of gauge couplings and running mass parameters from low energies up to very high energy scales characteristic of grand unified theories.) The terms shown explicitly in (89) correspond to the so-called next-to-leading order (NLO) in RG-improved perturbation theory. In this approximation, the leading and subleading logarithms $[\alpha_s \ln(m_Q/\mu)]^n$ and $\alpha_s[\alpha_s \ln(m_Q/\mu)]^n$ are summed correctly to all orders in perturbation theory. To achieve this, it is necessary to calculate the two-loop coefficient γ_1 of the anomalous dimension. When γ_1 is not known, it is only possible to evaluate the evolution function in the so-called leading logarithmic order (LO), in which

$$U_{\rm LO}(\mu, m_Q) = \left(\frac{\alpha_s(m_Q)}{\alpha_s(\mu)} \right)^a \,. \tag{91}$$

This still sums the leading logarithms to all orders, but does not contain the non-logarithmic terms of order α_s.

To complete the calculation of the RG-improved coefficient function, the evolution function $U(\mu, m_Q)$ must be combined with the initial condition for the Wilson coefficient at the high energy scale $\mu = m_Q$, as shown in (86). If the operator under consideration is present at the tree level, the matching condition can be written in the form

$$C(m_Q) = 1 + c_1\, \frac{\alpha_s(m_Q)}{4\pi} + \dots \,, \tag{92}$$

where c_1 is obtained from a one-loop calculation. To obtain a consistent (i.e. renormalization-scheme independent) result at next-to-leading order, we have to combine the one-loop matching condition with the expression for $U(\mu, m_Q)$

given in (89). This requires the calculation of the two-loop anomalous dimension. The result is

$$C_{\rm NLO}(\mu) = \left(\frac{\alpha_s(m_Q)}{\alpha_s(\mu)} \right)^a \left\{ 1 + \frac{\alpha_s(m_Q)}{4\pi}\,(S + c_1) - \frac{\alpha_s(\mu)}{4\pi}\,S \right\} . \tag{93}$$

In this expression, the terms involving the coupling constant $\alpha_s(m_Q)$ are renormalization-scheme independent [111,112]. The exponent a involves only the one-loop coefficients γ_0 and β_0 and is scheme independent by itself. For the coefficient $(S + c_1)$ of the next-to-leading term things are more complicated, however. The one-loop matching coefficient c_1, the two-loop anomalous dimension γ_1, and the QCD scale parameter $\Lambda_{\rm QCD}$ in the expression for the running coupling constant are all scheme dependent, but they conspire to give $\alpha_s(m_Q)$ a scheme-independent coefficient. On the other hand, the coefficient S of $\alpha_s(\mu)$ does depend on the renormalization procedure. This is not a surprise; only when the μ-dependent terms in the Wilson coefficient are combined with the μ-dependent matrix elements of the renormalized operator one can expect to obtain a scheme-independent result. For this reason, it is sometimes useful to factorize the solution (93) in the form $C(\mu) \equiv \widehat{C}(m_Q)\,K(\mu)$, where $\widehat{C}(m_Q)$ is RG-invariant and contains all dependence on the large mass scale m_Q. The scheme-dependent function $K(\mu)$ can be used to define a RG-invariant renormalized operator: $\widehat{\mathcal{O}} \equiv K(\mu)\,\mathcal{O}(\mu)$. At next-to-leading order, we obtain:

$$\begin{aligned} \widehat{C}(m_Q) &= [\alpha_s(m_Q)]^a \left\{ 1 + \frac{\alpha_s(m_Q)}{4\pi}\,(S + c_1) \right\} , \\ \widehat{\mathcal{O}} &= [\alpha_s(\mu)]^{-a} \left\{ 1 - \frac{\alpha_s(\mu)}{4\pi}\,S \right\} \mathcal{O}(\mu) . \end{aligned} \tag{94}$$

Let us finally apply this formalism to the operators appearing at order $1/m_Q$ in the effective Lagrangian of the HQET. The fact that reparametrization invariance ensures that the kinetic operator is not renormalized implies that the 2×2 anomalous dimension matrix for the operators $\mathcal{O}_{\rm kin}$ and $\mathcal{O}_{\rm mag}$ is diagonal and of the form

$$\hat\gamma = \begin{pmatrix} 0 & 0 \\ 0 & \gamma^{\rm mag} \end{pmatrix} . \tag{95}$$

The one-loop coefficient of the anomalous dimension of the chromo-magnetic operator, together with the one-loop matching coefficient, can be obtained from (76):

$$\gamma_0^{\rm mag} = 2 C_A \,, \qquad c_1^{\rm mag} = 2 (C_A + C_F) \,. \tag{96}$$

The result for $\gamma_0^{\rm mag}$ can also be obtained in a simpler way by computing only the $1/\epsilon$ poles in the matrix elements of the bare operators [76]. Unfortunately, the two-loop coefficient $\gamma_1^{\rm mag}$ is not yet known.[d] This means that the coefficient $S_{\rm mag}$ in the next-to-leading order solution (93) is still unknown.

4.4 Renormalization of Heavy-Quark Currents

As a final example, we discuss the renormalization of local current operators involving two heavy-quark fields. This case is of particular importance for phenomenology, as the weak current for $b \to c\,\ell\,\bar\nu$ transitions is of this form. In the HQET, the relevant current operators contain two heavy-quark fields at different velocity and are thus of the form $\bar h_{v'}\Gamma\,h_v$ (it does not matter whether the two fields have the same flavour), where Γ is some Dirac matrix, whose structure is irrelevant to our discussion. We have discussed in Sec. 2.3 that the matrix elements of such operators between meson states are proportional to the universal Isgur–Wise form factor $\xi(v\cdot v')$. We shall now derive, in leading logarithmic order, the Wilson coefficient function that relates the QCD current operators with their HQET counterparts renormalized at the scale $\mu \ll m_Q$.

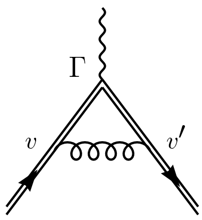

Figure 10: One-loop vertex diagram arising in the calculation of the anomalous dimension of heavy-quark currents. The external current changes the heavy-quark velocity from v to v'.

To this end, we need to calculate, using dimensional regularization, the $1/\epsilon$ pole in the matrix element of the bare current operator between quark states. The relevant vertex diagram is shown in Fig. 10. Since in the effective theory the coupling of a heavy quark to a gluon does not involve a γ matrix, it is easy to see that to all orders in perturbation theory the operator $\bar h_{v'}\Gamma\,h_v$ is renormalized multiplicatively and irrespective of its Dirac structure. The extraction of the one-loop ultraviolet divergence can be done in a few lines.

[d] This is one of the few cases of interest where an anomalous dimension in the HQET is not yet known to two-loop order. If you feel strong enough, you are invited to try the calculation of $\gamma_1^{\rm mag}$!

In the Feynman gauge,[e] the value of the vertex diagram is (omitting the quark spinors):

$$
\begin{aligned}
&-4ig_s^2 t_a t_a\, v\cdot v'\,\Gamma \int \frac{\mathrm{d}^D t}{(2\pi)^D}\,\frac{1}{(t^2+i\eta)(v\cdot t+i\eta)(v'\cdot t+i\eta)} \\
&= -4ig_s^2 C_F\, v\cdot v'\,\Gamma \int\limits_0^\infty \mathrm{d}\lambda \int\limits_0^\infty \mathrm{d}\rho \int \frac{\mathrm{d}^D t}{(2\pi)^D}\,\frac{1}{\big[t^2+2(\rho v+\lambda v')\cdot t+i\eta\big]^3} \\
&= -\frac{C_F\alpha_s}{\pi}\,\Gamma(1+\epsilon)\, v\cdot v'\,\Gamma\,(4\pi\mu^2)^\epsilon \int\limits_0^\infty \mathrm{d}\lambda \int\limits_0^\infty \mathrm{d}\rho\,\frac{1}{(\rho^2+\lambda^2+2v\cdot v'\,\rho\lambda)^{1+\epsilon}}\,.
\end{aligned}
\tag{97}
$$

Defining a new variable $z=\rho/\lambda$, and introducing an arbitrary infrared cutoff δ, we can rewrite the double integral in the form:

$$
\int\limits_0^\infty \mathrm{d}\lambda\,\frac{\lambda}{(\lambda^2+\delta^2)^{1+\epsilon}} \int\limits_0^\infty \mathrm{d}z\,\frac{1}{(1+z^2+2wz)^{1+\epsilon}} = \frac{\delta^{-2\epsilon}}{2\epsilon}\,r(w) + \text{finite terms}, \tag{98}
$$

where $w=v\cdot v'$, and

$$
r(w) = \int\limits_0^\infty \mathrm{d}z\,\frac{1}{1+z^2+2wz} = \frac{1}{\sqrt{w^2-1}}\,\ln\left(w+\sqrt{w^2-1}\right)\,. \tag{99}
$$

Hence, the $1/\epsilon$ pole of the vertex diagram is given by

$$
-\frac{C_F\alpha_s}{2\pi\epsilon}\,\Gamma\,w\,r(w)\,. \tag{100}
$$

To obtain the renormalization constant Z_{hh} of the bare current operator, we have to add a contribution

$$
Z_h\,\Gamma = \left(1+\frac{C_F\alpha_s}{2\pi\epsilon}\right)\Gamma \tag{101}
$$

from the wave-function renormalization of the heavy-quark fields, where Z_h has been given in (61). The result is

$$
Z_{hh} = 1 - \frac{C_F\alpha_s}{2\pi\epsilon}\,[w\,r(w)-1] + \text{finite terms}. \tag{102}
$$

[e] The final result for the anomalous dimension is gauge independent.

By means of the relation (81), we derive from this the one-loop coefficient of the anomalous dimension of heavy-quark currents in the HQET. This is the famous velocity-dependent anomalous dimension obtained by Falk et al. [74]:

$$\gamma_0^{\rm hh}(w) = 4C_F\,[w\,r(w) - 1]\ , \tag{103}$$

In the zero-recoil limit, i.e. for $w = 1$, the heavy-quark currents are the symmetry currents of the spin–flavour symmetry, and as such they are not renormalized, since the associated charges are conserved. This implies that

$$\gamma^{\rm hh}(1) = 0 \tag{104}$$

to all orders in perturbation theory. This constraint is satisfied by the one-loop result in (103), since $r(1) = 1$.

Since in the effective theory the velocity of a heavy quark is conserved by the strong interactions, the heavy quark can be described by a Wilson line [67]. An external current can instantaneously change the velocity, resulting in a kink of that line. It is well-known that such cusps lead to singular behaviour. The renormalization of cusp singularities of Wilson lines was investigated in detail by Korchemsky and Radyushkin [114] already in 1987, prior to the development of the HQET. In particular, they calculated the one- and two-loop coefficients of the so-called cusp anomalous dimension $\gamma^{\rm cusp}(\varphi)$ as a function of the hyperbolic cusp angle φ. But this anomalous dimension is precisely that of heavy-quark currents [115], with the identification $\cosh\varphi = w$. Later, the result for $\gamma_1^{\rm hh}(w)$, which we will not present here, has been confirmed in the context of the HQET [116].

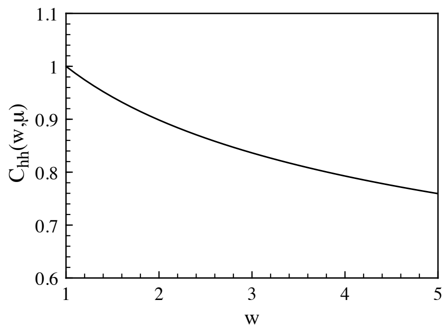

Figure 11: Velocity dependence of the Wilson coefficient $C_{\rm hh}(w,\mu)$, evaluated for $\alpha_s(\mu)/\alpha_s(m_Q) = 2$ and $n_f = 3$.

In leading logarithmic order, the expansion of heavy-quark currents in the HQET takes the form

$$\bar{Q}\,\Gamma\,Q \to C_{\rm hh}(w,\mu)\,\bar{h}_{v'}\Gamma\,h_v + O(1/m_Q)\,, \qquad (105)$$

where

$$C_{\rm hh}(w,\mu) = \left(\frac{\alpha_s(m_Q)}{\alpha_s(\mu)}\right)^{a_{\rm hh}(w)}\,, \qquad a_{\rm hh}(w) = \frac{2C_F}{\beta_0}\,\left[w\,r(w) - 1\right]\,. \qquad (106)$$

The velocity dependence of the Wilson coefficient is illustrated in Fig. 11. Physical decay or scattering amplitudes are proportional to the product

$$C_{\rm hh}(w,\mu)\,\xi(w,\mu)\,, \qquad (107)$$

where $\xi(w,\mu)$ is the renormalized Isgur–Wise function, which satisfies $\xi(1,\mu) = 1$. The fact that $a_{\rm hh}(1) = 0$ implies that RG effects respect the normalization of form factors at zero recoil.

We like to finish this discussion with a curious remark, which illustrates the power of RG methods. For values $w = O(1)$, the leading-order expression for the coefficient $C_{\rm hh}(w,\mu)$ in (106) contains all large logarithms of the type $[\alpha_s \ln(m_Q/\mu)]^n$. For very large values of w, however, we have $w\,r(w) \to \ln(2w) - 1$, so that in the RG improvement of $C_{\rm hh}(w,\mu)$ we have resummed terms of the form $[\alpha_s \ln w \ln(m_Q/\mu)]^n$. These are the well-known Sudakov double logarithms, which arise from the emission of gluon bremsstrahlungs during the scattering of the heavy quarks. This effect leads to a fractional power-like damping of the transition form factors at large recoil, which adds to the "soft" suppression contained in the Isgur–Wise form factor itself [117]. Explicitly, we obtain

$$C_{\rm hh}(w,\mu) \to \left(\frac{e}{2w}\right)^{\eta}\,, \qquad \eta = \frac{2C_F}{\beta_0}\ln\frac{\alpha_s(\mu)}{\alpha_s(m_Q)}\,. \qquad (108)$$

5 Concluding Remarks

We have presented an introduction to heavy-quark symmetry, the heavy-quark effective theory and the $1/m_Q$ expansion, which provide the modern theoretical tools to perform quantitative calculations in heavy-flavour physics. Our hope was to convince the reader that heavy-flavour physics is a rich and diverse area of research, which is at present characterized by a fruitful interplay between theory and experiments. This has led to many significant discoveries and developments on both sides. Heavy-quark physics has the potential to determine many important parameters of the electroweak theory and to test the Standard Model at low energies. At the same time, it provides an ideal laboratory to study the nature of non-perturbative phenomena in QCD, still one of the least understood properties of the Standard Model.

Let us finish with a somewhat philosophical remark: At this school, we have heard a lot about exciting new developments related to dualities, which relate apparently very different theories to each other. So are electric, weak-coupling phenomena in one theory dual to magnetic, strong-coupling phenomena in another theory. Some people argue quite convincingly that duality seems to be everywhere in nature, and consequently there are no really difficult questions in physics; very difficult problems become trivial when approached from a different, dual point of view. There are, however, "moderately difficult" problems in physics, which are "self-dual". It is the author's opinion that real-world (i.e. non-supersymmetric) QCD at hadronic energies belongs to this category. Having said this, we conclude that heavy-quark effective theory provides a powerful tool to tackle the "moderately difficult" problems of heavy-flavour physics.

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