Chapter 1

Introduction and Overview

The course covers two main topics: *game theory* and *mechanism design*. We begin by introducing the fundamental concepts and topics in game theory and will then apply these to topics in mechanism design.

1.1 Game Theory

Game theory is a method for analyzing *strategic interactions*. We consider a situation to be strategic whenever the consequences of a choice by one individual depend not only on the individual's behavior alone but also on the behavior of other individuals. For example, suppose that Apple introduces a new iPhone and considers the price it will charge for it. Clearly, the demand for the new iPhone will depend on the price chosen by Apple—however, if we reasonably assume that the latest Samsung Galaxy is, at least to some extent, comparable to the new iPhone, then the two are substitutes, and the demand for the iPhone will also depend on the price chosen by Samsung for the latest Samsung Galaxy. Pricing a product in a market with few competitors (a so-called oligopoly) is a situation of strategic interaction that we can study with the tools of game theory. Notice that, in contrast, the pricing decision of a monopolist is *not* a strategic situation: A monopolist does not face competition and therefore solves a decision-theoretic rather than a game-theoretic problem.

1.1.1 Examples

There are many applications of game theory from a wide range of fields and topics.

1. How should two presidential candidates design their political agenda to maximize their chances of being elected?

2. A pedestrian wants to cross a street and a driver wants to drive through the crossing. Should the pedestrian wait for the car to pass or should the pedestrian cross the street? How should the driver of the car behave?

3. How should a football player decide where to kick the penalty? How does the goalie decide in which direction to jump?

4. Generative adversarial networks are inspired by the theory of zero-sum games: a generator generates new artificial examples (for example, fake photographs of celebrities) with the aim of fooling a discriminator. The discriminator attempts to distinguish generator-generated examples from real examples (for example, actual photographs of celebrities).

In the chapters on game theory, we will learn to describe such real-world situations as games using mathematical models. Based on this modeling, we will introduce concepts that allow us to understand individuals' behavior in games *given an economic environment*; that is, given the rules of the games.

1.1.2 Different Classes of Games

The tools of game theory have been applied to a variety of different strategic interactions. Depending on the context of the environment to be analyzed, different features of the strategic situation must be taken into account. The following is a broad overview of distinctions in the modeling framework.

Cooperative and Noncooperative Games. Game theory broadly consists of two main branches: cooperative and non-cooperative game theory. As the name suggests, the distinction is in the nature of the decision-making process of the players. In cooperative game theory, players can engage in *binding* pre-game agreements. That is, players can "sit together" before taking their actions in the actual game to be played and write a contract that they can enforce in court, specifying how they will behave in the game. As that contract is enforceable, the players will follow their agreement.

Why would players want to write such agreements? Such agreements can be beneficial because they may prevent bad outcomes that arise due to *individual incentives* to take different actions. For example, consider the decision of two countries whether to build nuclear weapons. If the presidents of the two countries can sign a binding agreement that they will never build such a nuclear weapon, there will be a nuclear-weapon-free world. If, however, such an agreement is not possible, it may be optimal, in a non-cooperative game, for each of the countries individually to build a nuclear weapon: If the other country has none, having one may help exert power over that country. If the other country has one, having one may help defend your interests viz-a-viz that country. Both countries might end up having nuclear weapons—although they would have preferred not to build one, all else equal.

The focus of our class and these notes will be on non-cooperative games. However, even in non-cooperative games, players might engage in similar cooperative agreements. To include this possibility, the negotiation process leading up to such agreements also has to be explicitly modeled.

Static and Multistage Games. One important decision when modeling a strategic situation is the timeline of the interaction and the sequence in which participants act. For example, when visiting a local bar as a tourist, it is likely that the barkeeper recognizes you as such. Therefore, the mutual understanding will be that your interaction is only of short duration (and probably even a one-time interaction only). In contrast, if you are a local visiting the same bar, the barkeeper may recognize you as a local. In this case, it is more likely that you will come back. In particular, you can condition your choice of returning to the bar based on the quality of the drinks and the service.

The preceding example illustrates the distinction between one-time interactions and more dynamic, repeated strategic situations. However, beyond this distinction, even a one-time interaction may be of a more dynamic nature. For example, as a tourist visiting a bar only once, you can decide how much to tip depending on the quality of the service in the bar. Such a setting is not static: the barkeeper first decides how friendly to be or how much effort to put into mixing a high-quality drink for you. Only after learning about the barkeeper's behavior do you decide on the tip, and, in particular, you may tip differently depending on the barkeeper's behavior.

In other settings, the interaction may be simultaneous in nature. For example, consider two innovative pharmaceutical companies. Research on novel drugs is usually highly secretive, as firms want to ensure that competitors cannot free-ride on insights previous internal research has delivered. Therefore, when deciding on the focus of R&D spending in the current year, managers of different firms act without knowing what the other firms focus on. Hence, the firms act simultaneously: they cannot condition their choice on other firms' choices. Note that this is even true if the decisions do not happen literally simultaneously. The crucial point is that these decisions occur *without knowing* what other firms decided.

Is the choice of the R&D focus a static or a multistage game? That depends. If we think of pharmaceutical firms as long-lived entities competing with a similar set of firms year after year, then we should model their choices as a multistage game, where, in each stage (e.g., year), the firms *simultaneously* choose the focus of their research efforts. If we think of them as short-lived or as competing with different firms year after year, then we can model their choices as a static game (that is, a one-time interaction with simultaneous moves).

A simpler example of a static game can be the occasional Rock-Paper-Scissors game that you play with your roommates to decide who does the dishes or at the beginning of a friendly soccer match to decide on the kickoff. Usually, there is a negligible dynamic component to such games, and by definition, players act simultaneously in that game.

Perfect and Imperfect Information Games. Another crucial feature of strategic situations is the information available to players when making their decisions. In a game of perfect information, a player can observe the past moves of all other players. Chess, for example, is a perfect information game. Scotland Yard, in contrast, is a game of imperfect information, as some players cannot observe the moves of other players.

Complete and Incomplete Information Games. In a game of complete information, players have access to all the relevant information about the game. For example, in chess, a player knows which actions the other player has available and that the other player does not want to lose the game. However, in many other real-world strategic situations, a player might not know what other players aim for or other relevant pieces of information.

Poker is an example of an incomplete information game. Players do not know their opponent's hand. Hence, they are uncertain about the consequences of their actions. Similarly, a buyer of a good might not know whether the good is of high or low quality.

We will see later in the class that there is an elegant way to establish a close conceptual connection between imperfect and incomplete information games, which allows us to study them with the same tools.

1.2 Mechanism Design

Mechanism design, on the contrary, uses game theory as input and assumes that individuals behave in strategic situations according to the concepts of game theory. Based on this hypothesis, mechanism design aims to answer questions about the optimal design of a game to achieve a particular objective with limited information. For example, suppose that a seller wishes to sell an object for the highest possible revenue. Should the seller put the object with a price tag into a store and the first buyer willing to pay the price receives the object? Or should the seller rather run an auction to sell the object? In this sense, mechanism design can be viewed as reverse game theory: there is an institution that wants to achieve an objective (for example, sell an object at the highest possible revenue), and to achieve this objective, the institution designs a game played among individual agents (for example, potential buyers of an object). Then, given the behavior of the agents in the games, the institution optimizes the rules of the game so that it achieves its objective.

1.2.1 Examples

Similar to game theory, mechanism design has been applied in many different contexts and fields.

1. How should a country design its presidential election? Should citizens vote for a single candidate, or should citizens be asked for a ranking of the candidates?

2. How does the traffic law optimally punish pedestrians and car drivers after a crash? Should there be pedestrian crossings and traffic lights?

3. Should the consequences of illegally preventing a last-minute goal in soccer be more severe than a penalty and a red card (for example, by assigning a goal to the team and a red card)?

4. What is the optimal way to reduce carbon emissions? By introducing a carbon tax or by designing a market for emissions?

5. How should organ donations be allocated to patients to maximize the number of lives saved?

In this part of the course, we will introduce the fundamental concepts of mechanism design required for studying such questions about the optimal design of environments to achieve the desired objectives.

Part I

Game Theory

Chapter 2

Representation of Static Games

The simplest class of games are static games, that is, strategic situations in which all players of the game act simultaneously and only once. Consequently, in a static game, a player cannot observe the actions of other players and can, therefore, not react to others' actions or influence others with their own actions. In this chapter, we will narrow our focus even further and assume that players have access to all information that is relevant to the strategic situation—that is, we focus on static games of complete information. We will be more precise about what we mean by this as we go along.

In the first step, we introduce a formal representation of static games. In particular, we want to understand which components of the strategic situation we have to incorporate into our model to be able to discuss with the tools of game theory. In the second step, we will treat the strategic situation as a *decision problem* to understand which outcomes of the game can be expected with minimal restrictions on the players' decision-making process beyond rationality.

Example 1. Consider the following simple example of a strategic interaction (the *grade game*) between two classmates: Ann and Bob. Ann and Bob individually contemplate whether to collaborate in the preparation for a game theory exam. Whenever both cooperate, they share their class notes and knowledge with each other and will write a good exam; say, their final grade will be a 29. However, if they both decide not to cooperate, they will be less well prepared and each receives a grade of 28. In the case that one of them shares his/her notes and knowledge, but the other one does not, then the one who did not cooperate will have a lot of knowledge and will do relatively better than the other one leading to a grade of 30L. The player who cooperated does not benefit from the other one's knowledge and only gets a grade of 27.

How can we describe this situation as a game? First, a formal description of a game has to include the set of players that are involved in the game. In this case, the set of players is $I = \{Ann,\ Bob\}$. Second, the description of a game has to state the set of available actions that each player can take. In this case, the set of available actions to both players is $A_{Ann} = A_{Bob} = \{Cooperate\ (C), Don't\ Cooperate\ (D)\}$. Third, we need to describe the possible outcomes of the game. In this case, the possible outcomes are a grade for each of the classmates, $y \in Y = \{(29,29),(28,28),(30L,27),(27,30L)\}$.[1] Fourth, a game description requires a mapping from action profiles—that is, the vector of actions taken by each player—into outcomes. In the grade game, the outcome function is

$$g(a_{Ann}, a_{Bob}) = \begin{cases} (29,29) & \text{, if } (a_{Ann}, a_{Bob}) = (C,C) \\ (28,28) & \text{, if } (a_{Ann}, a_{Bob}) = (D,D) \\ (27,30L) & \text{, if } (a_{Ann}, a_{Bob}) = (C,D) \\ (30L,27) & \text{, if } (a_{Ann}, a_{Bob}) = (D,C). \end{cases}$$

[1] We let the first entry in each outcome denote Ann's and the second entry Bob's grade.

Fifth, we need to assign preferences over outcomes for each player. We assume for now that players care only about their own grades and that they prefer better grades. For example, we could assume

$$u_{Ann}(y) = u_{Bob}(y) = \begin{cases} 1 & \text{, if } y = (29, 29) \\ 0 & \text{, if } y = (28, 28) \\ 3 & \text{, if } y = (30L, 27) \\ -1 & \text{, if } y = (27, 30L). \end{cases}$$

We often represent games with few players and few actions in a matrix as in Table 3.3.[2] Rows correspond to actions by the "row player"; that is, Ann in this case. Columns correspond to actions by the "column player"; that is, Bob in this case. Each entry of the matrix then corresponds to a realized action profile. The first number indicates the payoff of the row player given the realized action profile, the second number indicates the payoff of the column player given the realized action profile.

		Bob	
		Cooperate	Don't Cooperate
Ann	Cooperate	$1, 1$	$-1, 3$
	Don't Cooperate	$3, -1$	$0, 0$

Table 2.1: *The grade game represented in a game matrix.*

□

Example 1 suggests that a complete description of a static game with complete information requires: A description of

1. the participants in the game;
2. what each participant can do;
3. what are the possible outcomes of the strategic situation;
4. how the actions affect the potential outcomes;
5. how the participants evaluate the different outcomes.

The next section formalizes this description.

2.1 Formal Representation of Static Games

We now define the first formal representation of static games with complete information.[3] We will simplify this definition in the next step.

Definition 2.1. *A static game is a list $G = \langle I, (A_i)_{i\in I}, Y, g, (v_i)_{i\in I}\rangle$, where*

- *I is the set of players,*
- *A_i is the set of possible actions for player i,*
- *Y is the set of possible outcomes,*

[2] This game is more commonly known as *Prisoner's Dilemma*. We will introduce it as such later on and return to it frequently during the class.

[3] Note that this representation is often denoted as the *normal form* or *strategic form* representation of a game. We will come back to this notion later.

- *$g : \times_{i \in I} A_i \to Y$ is the outcome function,* [4]

- *$v_i : Y \to \mathbb{R}$ is the von Neumann-Morgenstern utility function of player i.*

We denote a player's action by $a_i \in A_i$ and define an action profile $a = (a_i)_{i \in I} \in A = \times_{i \in I} A_i$ as the vector of all players' actions. In addition, we use the notation $-i$ whenever we refer to the set of all players *except* player i. For example, $a_{-i} \in A_{-i}$ is a vector of actions chosen by the set players $j \in I \setminus \{i\}$.

Note that the outcome function g can be stochastic. That is, an action profile $a = (a_i)_{i \in I}$ can lead to different outcomes with corresponding probabilities.

We can simplify Definition 2.1. Observe that we can represent the players' preferences directly over action profiles instead of over outcomes that are determined by the action profiles. For stochastic outcome functions, preferences are defined over lotteries. For a definition of lotteries and probability measures over finite domains, see Appendix A.

Second, define the payoff function $u_i : A \to \mathbb{R}$ as a function representing the players' preferences as a function from action profiles directly into the reals. To see that such a payoff function can capture both the outcome function $g : A \to Y$ and the player's von Neumann-Morgenstern utility function v_i, denote by μ a given distribution over action profiles. As $v_i : Y \to \mathbb{R}$ represents player i's preferences $\succeq_i$ over lotteries over outcomes, $\lambda_1, \lambda_2 \in \Delta(Y)$, the following obtains

$$\lambda_1 \succeq_i \lambda_2 \Leftrightarrow \sum_{y \in Y} \lambda_1(y) v_i(y) \geq \sum_{y \in Y} \lambda_2(y) v_i(y). \tag{2.1}$$

Denote by $\hat{g} : \Delta(A) \to \Delta(Y)$ the pushforward function that describes the lottery over the set of outcomes induced by a distribution over the set of action profiles, μ, as

$$\hat{g}(\mu)(y) = \mu\left(g^{-1}(y)\right) = \sum_{a \in g^{-1}(y)} \mu(a). \tag{2.2}$$

In words, the likelihood of event y happening is determined by adding up the probabilities of actions within the action profile that result in event y where these probabilities are induced by μ.

Defining the payoff function $u_i : A \to \mathbb{R}$ as the composition $u_i = v_i \circ g$, we obtain for any two distributions over action profiles $\mu_1, \mu_2 \in \Delta(A)$

$$\begin{aligned} \mu_1 \succeq_i \mu_2 \Leftrightarrow \sum_{y \in Y} \hat{g}(\mu_1)(y) v_i(y) &\geq \sum_{y \in Y} \hat{g}(\mu_2)(y) v_i(y) \\ \sum_{y \in Y} \sum_{a \in g^{-1}(y)} \mu_1(a) v_i(g(a)) &\geq \sum_{y \in Y} \sum_{a \in g^{-1}(y)} \mu_2(a) v_i(g(a)) \\ \sum_{a \in A} \mu_1(a) u_i(a) &\geq \sum_{a \in A} \mu_2(a) u_i(a). \end{aligned}$$

Hence, the payoff functions $(u_i)_{i \in I}$ achieve the goal we wanted to achieve: it represents player i's preferences over action profiles. Using these payoff functions, we obtain the classic more concise representation of static games in the following definition.

Definition 2.2. *A static game is a list $G = \langle I, (A_i, u_i)_{i \in I} \rangle$ where*

- *I is the set of players,*

- *A_i is a nonempty set of possible actions for player i,*

- *$u_i : A \to \mathbb{R}$ is player i's payoff function.*

[4] $\times_{i \in I} A_i$ denotes the Cartesian product of the players' set of actions. That is, for $I = \{1, \ldots, n\}$, $\times_{i \in I} A_i = A_1 \times A_2 \times \cdots \times A_n$.

In this class, most of our examples fall into one of two classes of games, finite and compact-continuous games. We define these classes in the following two definitions.

Definition 2.3. *A static game G is finite if, for all players $i \in I$, the action set A_i is finite.*

Example 1 is a simple definition of a finite game. Each player can choose from two actions (cooperating and not cooperating).

Definition 2.4. *A static game G is compact-continuous if, for all players $i \in I$, A_i is a compact subset of a Euclidean space $\mathbb{R}^{k_i}$ where $k_i \in \mathbb{N}$ and $u_i : A \to \mathbb{R}$ is continuous.*

The following is an example of a compact-continuous game that is a classical economic example, which we will return to frequently. Notably, this game is one of the earliest games that have been studied with game-theoretic concepts by Augustin Cournot in Cournot (1838) long before game theory had been established as a research field.

Example 2. Consider the following game of two firms, $I = \{1, 2\}$, competing in a market for a homogenous good (that is, consumers do not care which firm they purchase the good from). Each firm i chooses a quantity q_i to produce; that is, $q_i \in A_i = [0, \infty)$. The cost of producing q_i units is $c_i(q_i) = c\, q_i$; that is, there is a constant marginal cost of production, $c \geq 0$. The inverse demand function determines the price at which the goods sell and is given by $p(q_1, q_2) = \max\{0, 1 - (q_1 + q_2)\}$. Thus, the payoff functions are $u_i(q_1, q_2) = q_i(1 - (q_i + q_{-i}) - c)$. □

Chapter 3

Justifiability and Dominance

In the next step, we aim to understand which actions a rational player may choose in a game. Towards this, we first take on the perspective of decision theory; that is, we ignore the potentially complicated strategic reasoning a player might engage in. We present the results and concepts in this section almost exclusively for finite games. However, with one exception (Lemma 3.4) all concepts and results also apply to compact-continuous games and you can use them in those instances as well.

We first introduce some basic definitions and then illustrate how we can apply them to reason about behavior in games in the simplest possible form.

Conjectures. We represent with a *conjecture* the behavior that a player expects from other players. For now, we will not discuss how conjectures are obtained and instead take them as given. Later on, we will take a closer look at the formation of conjectures. Formally, we define a conjecture as follows.

Definition 3.1. *A conjecture of player i is a probability measure $\mu^i \in \Delta(A_{-i})$, where we denote by $A_{-i} = \times_{j \in I \setminus \{i\}} A_j$ the set of action profiles by all players but player i.*

Note that this definition allows players to hold probabilistic conjectures; that is, a player can expect other players to randomize over their actions. Further, note that we can define the set of deterministic conjectures as those conjectures that assign probability one to a particular action profile (that is, $\mu^i = \delta_{a_{-i}}$ for some $a_{-i} \in A_{-i}$).[1] Hence, we can think of the set of other players' action profiles as a subset of conjectures, $A_{-i} \subseteq \Delta(A_{-i})$, which will simplify our notation in future instances.

Mixed Actions. A player can randomize over her available actions. We call such a randomization a *mixed action*. We assume that the random draws from players' mixed actions are statistically independent. That is, in a simultaneous move game, there is no way in which players can correlate their draws. We will discuss correlated behavior in a separate chapter on *correlated equilibrium*.

Definition 3.2. *A mixed action $\alpha_i \in \Delta(A_i)$ by player i is a probability distribution over the set of actions of player i. For every action $a_i \in A_i$, $\alpha_i(a_i)$ denotes the probability that a_i is taken. A mixed action is degenerate if it is such that it assigns probability one to a particular action $a_i \in A_i$ and we call such a mixed action a pure action.*

Example 3. Consider the game of Rock, Paper, Scissors as depicted in Chapter 3. In this game, suppose that Ann chooses to play all actions with an equal probability, while Bob randomizes only over Rock and Paper with equal probabilities but plays Scissors. Thus, $\alpha_A(\mathit{Rock}) = \alpha_A(\mathit{Paper}) = \alpha_A(\mathit{Scissors}) = 1/3$ and $\alpha_B(\mathit{Rock}) = \alpha_B(\mathit{Paper}) = 1/2$ and $\alpha_B(\mathit{Scissors}) = 0$. This yields for Ann's

[1] Here, we denote by $\delta_{a_{-i}}$ the Dirac measure that assigns measure one on the action profile $a_{-i} \in A_{-i}$ and measure zero on all other action profiles $a'_{-i} \neq a_{-i}$.

		Bob Rock	Bob Paper	Bob Scissors
Ann	Rock	$0,0$	$-1,1$	$1,-1$
Ann	Paper	$1,-1$	$0,0$	$-1,1$
Ann	Scissors	$-1,1$	$1,-1$	$0,0$

Table 3.1: *Rock Paper Scissors*

payoff, assuming that the players hold *correct conjectures*; that is, their conjectures coincide with the actual mixed actions of their opponent,

$$\begin{aligned}
u_A(\alpha_A, \mu^A = \alpha_B) = &\underbrace{\frac{1}{3}}_{\text{Pr(Ann plays Rock)}}\left(\underbrace{\frac{1}{2}}_{\text{Pr(Bob plays Rock)}} \cdot 0 + \underbrace{\frac{1}{2}}_{\text{Pr(Bob plays Paper)}} \cdot (-1) + \underbrace{0}_{\text{Pr(Bob plays Scissors)}} \cdot 1\right) \\
&+ \underbrace{\frac{1}{3}}_{\text{Pr(Ann plays Paper)}}\left(\underbrace{\frac{1}{2}}_{\text{Pr(Bob plays Rock)}} \cdot 1 + \underbrace{\frac{1}{2}}_{\text{Pr(Bob plays Paper)}} \cdot 0 + \underbrace{0}_{\text{Pr(Bob plays Scissors)}} \cdot (-1)\right) \\
&+ \underbrace{\frac{1}{3}}_{\text{Pr(Ann plays Scissors)}}\left(\underbrace{\frac{1}{2}}_{\text{Pr(Bob plays Rock)}} \cdot (-1) + \underbrace{\frac{1}{2}}_{\text{Pr(Bob plays Paper)}} \cdot 1 + \underbrace{0}_{\text{Pr(Bob plays Scissors)}} \cdot 0\right) \\
&= 0
\end{aligned}$$

and for Bob's payoffs, we obtain:

$$\begin{aligned}
u_B(\alpha_B, \mu^B = \alpha_A) = &\underbrace{\frac{1}{2}}_{\text{Pr(Bob plays Rock)}}\left(\underbrace{\frac{1}{3}}_{\text{Pr(Ann plays Rock)}} \cdot 0 + \underbrace{\frac{1}{3}}_{\text{Pr(Ann plays Paper)}} \cdot (-1) + \underbrace{\frac{1}{3}}_{\text{Pr(Ann plays Scissors)}} \cdot 1\right) \\
&+ \underbrace{\frac{1}{2}}_{\text{Pr(Bob plays Paper)}}\left(\underbrace{\frac{1}{3}}_{\text{Pr(Ann plays Rock)}} \cdot 1 + \underbrace{\frac{1}{3}}_{\text{Pr(Ann plays Paper)}} \cdot 0 + \underbrace{\frac{1}{3}}_{\text{Pr(Ann plays Scissors)}} \cdot (-1)\right) \\
&+ \underbrace{0}_{\text{Pr(Bob plays Scissors)}}\left(\underbrace{\frac{1}{3}}_{\text{Pr(Ann plays Rock)}} \cdot (-1) + \underbrace{\frac{1}{3}}_{\text{Pr(Ann plays Paper)}} \cdot 1 + \underbrace{\frac{1}{3}}_{\text{Pr(Ann plays Scissors)}} \cdot 0\right) \\
&= 0.
\end{aligned}$$

□

Rationality We impose that players are rational. Formally, we define rationality as follows.

Definition 3.3 (Rationality.)**.** *We say that player i is rational if, given a conjecture μ^i, player i maximizes her expected utility. A player i's expected utility given conjecture $\mu^i \in \Delta(A_{-i})$ (as usual* *defined over a finite support) and mixed action* $\alpha_i \in \Delta(A_i)$ *is*

$$u_i(\alpha_i, \mu^i) = \mathbb{E}_{\alpha_i,\mu^i}[u_i(a_i, a_{-i})] \tag{3.1}$$
$$= \sum_{a_i \in supp\ \alpha_i} \sum_{a_{-i} \in supp\ \mu^i} \alpha_i(a_i)\mu^i(a_{-i})u_i(a_i, a_{-i}). \tag{3.2}$$

3.1 Best Replies and Justifiable Actions

		Goalie	
		Left	Right
Striker	Left	$4, -4$	$9, -9$
	Middle	$6, -6$	$6, -6$
	Right	$9, -9$	$4, -4$

Table 3.2: *Penalty shooting. We can think of the striker's payoffs as the probability that she scores a goal given the realized action profile.*

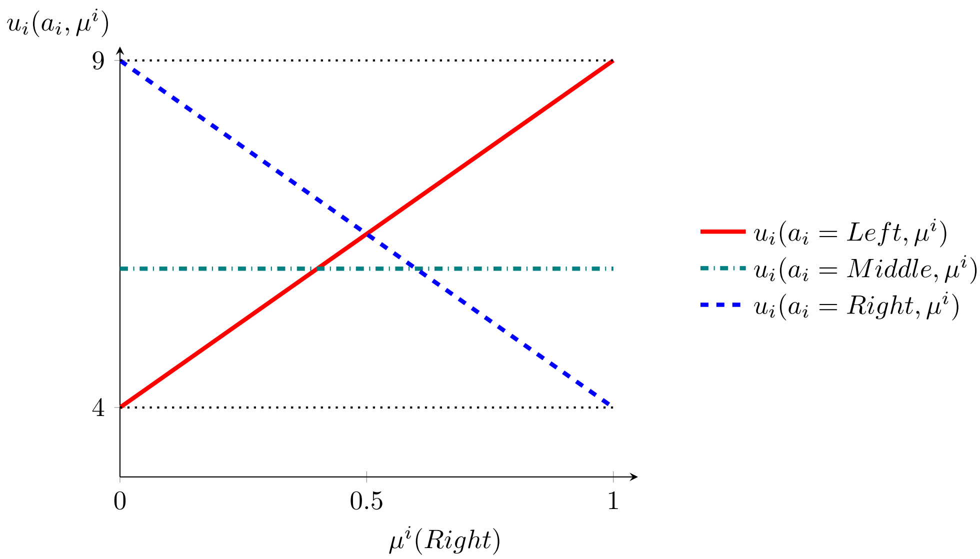

Figure 3.1: *Representation of payoffs in penalty shooting game.*

Our assumption of rationality—in the sense of expected utility maximization—entails that a player will only consider playing those actions that are a *best reply* against *some conjecture* that the player may hold. We define formally what it means for an action to be a best reply.

Definition 3.4. *A mixed action* $\alpha_i^* \in \Delta(A_i)$ *is a best reply to conjecture* $\mu^i \in \Delta(A_{-i})$ *if*

$$\forall \alpha_i \in \Delta(A_i),\ u_i(\alpha_i^*, \mu^i) \geq u_i(\alpha_i, \mu^i); \tag{3.3}$$

that is,

$$\alpha_i^* \in \arg\max_{\alpha_i \in \Delta(A_i)} u_i(\alpha_i, \mu^i). \tag{3.4}$$

We denote the set of pure actions that are best replies to conjecture μ^i *by*[2]

$$r_i(\mu^i) = A_i \cap \arg max_{\alpha_i \in \Delta(A_i)} u_i(\alpha_i, \mu^i). \tag{3.5}$$

We call the correspondence $r_i : \Delta(A_{-i}) \rightrightarrows A_i$ *the (pure-action) best-reply correspondence.*

With the definition of best replies at hand, we can identify the set of actions that a rational player might choose given *some conjecture*. In particular, we consider those actions to be the set of those actions for which there exists a conjecture about other players' behavior such that the action is a best reply to this conjecture. We call these actions to be *justifiable*.

Definition 3.5. *An action* a_i *is justifiable if there exists a conjecture* $\mu^i \in \Delta(A_{-i})$ *such that* $a_i \in r_i(\mu^i)$.

Illustration: Penalty Shooting. Suppose that a goalkeeper and a striker play the game represented in Table 3.2. In such a game with only two actions for the other player, it is straightforward to represent the payoffs graphically as illustrated in Figure 3.1. The horizontal axis represents the striker's conjectures and each line in the graph the expected payoff from an action given the conjecture. Shooting the penalty to the left is a best reply to conjecture μ^i whenever $\mu^i(Left) \leq 1/2$; that is, whenever the striker expects the goalie to jump to the left with no more than 50% probability, shooting to the left maximizes the striker's expected utility. Similarly, whenever the striker expects the goalie to jump to the right with no more than 50% probability, shooting to the right is a best reply to such conjectures. Thus, both Left and Right are justifiable actions for the striker. However, Middle is not justifiable because it is never a best reply for the striker for any conjecture μ^i.

In the following lemma, we show that if a player i has a mixed best reply, then she can always maximize her expected payoff with a pure action as well. This result is useful to keep in mind. There are instances, in which we might consider mixing by players unrealistic; in those instances, it is helpful to remember that randomization is not necessary to maximize expected utility.

Lemma 3.1. *Consider player* $i \in I$ *with conjecture* $\mu^i \in \Delta(A_{-i})$ *and a mixed action* $\alpha_i^* \in \Delta(A_i)$. *The mixed action* α_i^* *is a best reply to conjecture* μ^i *if and only if every pure action in the support of* α_i^* *is a best reply to* μ^i*; that is,*

$$\alpha_i^* \in \arg \max_{\alpha_i \in \Delta(A_i)} u_i(\alpha_i, \mu^i) \iff \textit{supp } \alpha_i^* \in r_i(\mu^i). \tag{3.6}$$

Proof. We first show that if the mixed action is a best reply, then every pure action in its support must be a best reply as well. We prove this statement using its contrapositive:[3] If some pure action in the support of a mixed action is not a best reply, then the mixed action cannot be a best reply.

Let a_i^* be a pure action chosen with positive probability in the mixed action α_i^* and suppose that there is a mixed action α_i for player i inducing a higher expected utility for player i; that is, $u_i(\alpha_i, \mu^i) > u_i(a_i^*, \mu^i)$. Hence, a_i^* is a pure action in the support of α_i^* and not a best reply.

Given that the expected utility of any mixed action is a weighted average of the expected utility from all of i's pure actions, $\left(u_i(a_i, \mu^i)\right)_{a_i \in A_i}$, there must be a pure action $a_i' \in A_i$ such that $u_i(a_i', \mu^i) > u_i(a_i^*, \mu^i)$. We can now construct a mixed action α_i' that yields a higher expected utility than α_i^* showing that α_i^* is not a best reply to conjecture μ^i.

Define the alternative mixed action $\alpha_i' \in \Delta(A_i)$ as

$$\alpha_i'(a_i) = \begin{cases} 0 & \text{, if } a_i = a_i^* \\ \alpha_i^*(a_i') + \alpha_i^*(a_i^*) & \text{, if } a_i = a_i' \\ \alpha_i^*(a_i) & \text{, if } a_i \neq a_i^*, a_i'. \end{cases} \tag{3.7}$$

[2] Recall that we can think of the set of pure actions as the set of degenerated pure actions; that is, $A_i \subset \Delta(A_i)$.

[3] The contrapositive of a statement $p \Rightarrow q$ is $\neg q \Rightarrow \neg p$.

Note that the difference in expected payoffs between α_i^* and α_i' is

$$u_i(\alpha_i^*, \mu^i) - u_i(\alpha_i', \mu^i) = \alpha_i^*(a_i^*)\left(u_i(a_i^*, \mu^i) - u_i(a_i', \mu^i)\right) < 0 \tag{3.8}$$

implying that α_i^* is not a best reply to μ^i.

Second, we show that if any pure action in the support of a mixed action is a best reply to conjecture μ^i, then the mixed action itself must be a best reply to conjecture μ^i.

Define the maximum value that player i can attain with her best reply as $\bar{u}_i(\mu^i) := \max_{\alpha_i \in \Delta(A_i)} u_i(\alpha_i, \mu^i)$. It follows from the definition of the best-reply correspondence that

$$\forall a_i \in r_i(\mu^i), \quad u_i(a_i, \mu^i) = \bar{u}_i(\mu^i) \tag{3.9}$$

$$\forall a_i \in A_i, \qquad u_i(a_i, \mu^i) \leq \bar{u}_i(\mu^i). \tag{3.10}$$

Suppose that *supp* $\alpha_i^* \subseteq r_i(\mu^i)$. Then, for every $\alpha_i \in \Delta(A_i)$, we obtain

$$\begin{aligned}
u_i(\alpha_i^*, \mu^i) &= \sum_{a_i \in supp\ \alpha_i^*} \alpha_i^*(a_i) u_i(a_i, \mu^i) && (3.11)\\
&= \sum_{a_i \in r_i(\mu^i)} \alpha_i^*(a_i) u_i(a_i, \mu^i) && (3.12)\\
&= \bar{u}_i(\mu^i) && (3.13)\\
&= \bar{u}_i(\mu^i) \sum_{a_i \in supp\ \alpha_i} \alpha_i(a_i) && (3.14)\\
&= \sum_{a_i \in supp\ \alpha_i} \alpha_i(a_i) \bar{u}_i(\mu^i) && (3.15)\\
&\geq \sum_{a_i \in supp\ \alpha_i} \alpha_i(a_i) u_i(a_i, \mu^i) && (3.16)\\
&= u_i(\alpha_i, \mu^i) && (3.17)
\end{aligned}$$

showing that α_i^* is a best reply to μ^i. The sequence of equalities and inequalities holds because (i) each pure action in the support of α_i^* is a best reply, (ii) each best reply generates $\bar{u}_i(\mu^i)$, (iii) the sum of mixing probabilities must be equal to one, (iv) and the expected utility of an arbitrary mixed action is at most $\bar{u}_i(\mu^i)$. □

Based on the previous lemma, we show next that, for any conjecture μ^i, a best reply exists for player i to any conjecture μ^i and that there always exists a pure-action best reply to any conjecture μ^i.

Lemma 3.2. *Consider a player $i \in I$ and a conjecture $\mu^i \in \Delta(A_{-i})$. Then,*

$$r_i(\mu^i) = \arg\max_{a_i \in A_i} u_i(a_i, \mu^i) \tag{3.18}$$

and mixed actions are not necessary to maximize the expected utility of player i and the best-reply correspondence r_i is non-empty valued for any conjecture μ^i.

Proof. Observe that the expected utility of player i given conjecture $\mu^i \in \Delta(A_{-i})$, $u_i(\cdot, \mu^i) : \Delta(A_i) \to \mathbb{R}$ is continuous in α_i and defined on a compact domain, $\Delta(A_i)$. Thus, by the Extreme Value Theorem (see Theorem A.1), there is at least one (possibly mixed) best reply $\alpha_i^* \in \Delta(A_i)$.

It follows then from Appendix B that the support of α_i^* must be a subset of the pure-action best replies to μ^i; that is, *supp* $\alpha_i^* \subseteq r_i(\mu^i)$. Thus, there is at least one pure-action best reply implying that $r_i(\mu^i) \neq \emptyset$.

In addition, given that there is at least one pure-action best reply, restricting player i to choosing only pure actions will not make player i worse off; that is,

$$\max_{a_i \in A_i} u_i(a_i, \mu^i) = \max_{\alpha_i \in \Delta(A_i)} u_i(\alpha_i, \mu^i). \tag{3.19}$$

It is immediate that $r_i(\mu^i) = \arg\max_{a_i \in A_i} u_i(a_i, \mu^i)$. □

Making use of the best-reply correspondence, we can now define the set of justifiable actions for player i; that is, the set of actions that a rational player might choose for some conjecture. In particular, the set of justifiable actions is the *image set of the best-reply correspondence*,

$$r_i(\Delta(A_{-i})) = \left\{a_i \in A_i : \exists \mu^i \in \Delta(A_{-i}), a_i \in r_i(\mu^i)\right\}. \tag{3.20}$$

Going back to the grade game in Example 1, we see that Cooperate is not justifiable for both players. There is no conjecture about the other player's actions: Don't Cooperate yields a higher payoff for any choice of the other classmate. Using justifiability, we can conclude that a rational player should only play Don't Cooperate in this game.

For the penalty shooting game in Table 3.2, we see that both Left and Right are justifiable for the striker. Middle, however, is not justifiable as is evident from Figure 3.1. For any conjecture, the striker can do better by choosing Left or Right than by choosing Middle.

3.2 Dominated Actions

We saw that to determine the set of justifiable actions for a player, we had to check for each action whether there exists a conjecture such that it becomes a best reply to this conjecture. However, it turns out that there is a simpler approach to determining justifiable actions. Towards this, we define the concept of *dominance*.

Definition 3.6. *A mixed action $\alpha_i \in \Delta(A_i)$ dominates a (pure or mixed) action $\beta_i \in \Delta(A_i)$ if it yields a strictly higher payoff* irrespective of the choices of the other players*; that is, for all $a_{-i} \in A_{-i}$,*

$$u_i(\alpha_i, a_{-i}) > u_i(\beta_i, a_{-i}). \tag{3.21}$$

Denote the set of pure actions that are not dominated by ND_i,

$$\begin{aligned}&ND_i = \\ &A_i \setminus \{a_i \in A_i : \ \exists \alpha_i \in \Delta(A_i), \ \forall a_{-i} \in A_{-i}, u_i(\alpha_i, a_{-i}) > u_i(a_i, a_{-i})\}.\end{aligned} \tag{3.22}$$

We call an action a_i^* dominant if all other pure actions are dominated by it.

It turns out that we can prove that the set of actions that are not dominated coincides with the set of justifiable actions as we show in the next lemma. This result establishes that a rational player will not play a dominated action.

Lemma 3.3. *Consider a finite game G, player i and an action $a_i^* \in A_i$. There exists a conjecture $\mu^i \in \Delta(A_{-i})$ such that a_i^* is a best reply to μ^i if and only if a_i^* is not dominated by any mixed action $\alpha_i \in \Delta(A_i)$. Thus, the set of not dominated actions and justifiable actions coincide*

$$ND_i = r_i(\Delta(A_{-i})). \tag{3.23}$$

Proof. Proving that if $a_i^* \in A_i$ is justifiable, then it is not dominated is left as an exercise.

We prove here that if an action is not dominated, then it must be justifiable. Again, we prove the contrapositive—that is, we prove that if an action is not justifiable, it must be dominated.

As we are considering a finite game, define by $k = |A_i|$ and $m = |A_{-i}|$ the cardinality of i's action set and of the set of other players' action profiles, respectively. Denote by $w \in \{1, 2, \ldots, k\}$ and $z \in \{1, 2, \ldots, m\}$ an element in those sets.

Next, define a $(k \times m)$-matrix U with elements

$$U_{wz} = u_i(a_i^*, z) - u_i(w, z) \tag{3.24}$$

for an action $a_i^* \in A_i$. Note that, by definition of best replies and justifiability, a_i^* is justifiable if and only if there is a conjecture $\mu^i \in \Delta(A_{-i}) \subset \mathbb{R}^m$ such that $U\,\mu_i \geq 0$ where $U\,\mu_i$ denotes the multiplication of the matrix U with the vector μ_i.

An equivalent way of stating the justifiability of a_i^* is by (i) defining a $((k+m+1) \times m)$-matrix M as

$$M = \begin{pmatrix} U \\ I_m \\ 1_m^T \end{pmatrix}, \tag{3.25}$$

where I_m denotes the $(m \times m)$-identity matrix and 1_m^T a $(1 \times m)$-vector of ones,[4] (ii) defining a $((k+m+1) \times 1)$-dimensional vector c as

$$c = \begin{pmatrix} 0_k \\ 0_m \\ 1 \end{pmatrix}, \tag{3.26}$$

where 0_l denotes an l-dimensional column vector of zeros, and (iii) noting that a_i^* is justifiable if and only if there is a vector $x \in \mathbb{R}^m$ such that $M\,x \geq c$.

Observe that $M\,x \geq c$ corresponds to the system

$$\begin{pmatrix} U\,x \geq 0 \\ x_1 \geq 0, \ldots, x_m \geq 0 \\ \sum_{z=1}^m x_z \geq 1 \end{pmatrix}, \tag{3.27}$$

which implies that if such an $x \in \mathbb{R}^m$ exists, it is a conjecture[5] and it justifies a_i^*.

It follows that if a_i^* is not justifiable, then there does not exist a vector $x \in \mathbb{R}^m$ such that $M\,x \geq c$. Therefore, by Farkas Lemma (see Theorem A.5), there is a vector $y \in \mathbb{R}^{m+k+1}$ such that $y \geq 0$, $y^T c > 0$, and $y^T M = 0^T$.

From the construction of c it follows that $y^T c = y_{k+m+1} > 0$.

$y^T M = 0^T$ implies that, for every $z \in A_{-i}$,

$$\sum_{w=1}^{k} y_w \left(u_i(a_i^*, z) - u_i(w, z)\right) = -(y_{k+z} + y_{k+m+1}) < 0, \tag{3.28}$$

where the inequality follows from y being nonnegative and $y^T c > 0$. Hence, we can construct a mixed action

$$\bar{y} := \left(\frac{y_1}{\sum_{w=1}^k y_w}, \ldots, \frac{y_k}{\sum_{w=1}^k y_w}\right) \tag{3.29}$$

such that, for every $z \in A_{-i}$, $u_i(a_i^*, z) < \sum_{w=1}^k \bar{y}_w u_i(w, z)$. Hence, the mixed action $\bar{y}$ dominates the action a_i^* proving the result. □

[4] We denote by v^T the transpose of a vector/matrix.

[5] Or x can be normalized to be a conjecture whenever $\sum_{z=1}^m x_z > 1$ by dividing its components by $\sum_{z=1}^m x_z$.

The two main takeaways are that a rational player will never choose a dominated action and that each undominated action is justifiable as a best reply to some belief. Going back to Example 1, it can be deduced that Cooperate is dominated by Don't Cooperate, so neither Ann nor Bob will choose to Cooperate in the game. There is a unique action justifiable action profile: $(Don't\,Cooperate, Don't\,Cooperate)$. However, if we slightly change the payoffs induced by the action profile (D, C) from (3,-1) to (-3,-1) and the payoffs induced by the action profile (C, D) from (-1, 3) to (-1, -3), none of the actions is dominated since when one of the two players chooses Cooperate, the best reply for the other player is Cooperate while when one of the two players chooses Don't Cooperate, the best reply for the other player is Don't Cooperate.

		Bob	
		Cooperate	Don't Cooperate
Ann	Cooperate	$1, 1$	$-1, -3$
	Don't Cooperate	$-3, -1$	$0, 0$

Table 3.3: *The Cooperative Grade Game.*

To illustrate how the concept of non-dominated actions helps to identify the set of justifiable actions, we consider the following (compact-continuous example).

Example 4 (Linear public good game.)**.** Consider a village with n citizens. The mayor of the village can build a park using contributions by the citizens using the following production function $y(x) = kx$ with $k \in (1/n, 1)$—both the value of the park, y, and the contributions, x, are measured in euros. That is, every additional euro contributed translates into an improvement of the park by $k < 1$.[6] Each citizen i has a finite initial wealth of $w_i > 0$ and decides on a monetary contribution $a_i \in A_i = [0, w_i]$. An action profile a translates into contribution $x(a) = \sum_{i=1}^{n} a_i$. Citizen's payoff functions are $u_i(a) = y(x(a)) - a_i = k \sum_{j=1}^{n} a_j - a_i$.

Rewriting the payoff functions as $u_i(a) = k \sum_{j \neq i} a_j - (1-k)a_i$ it becomes apparent that there is a unique profile of not dominated actions $a^* = (0, \ldots, 0)$ leading to a payoff profile of $(0, \ldots, 0)$. In fact, the dominant action for each individual is to contribute nothing since $k \sum_{j \neq i} a_j - (1-k)a_i > k \sum_{j \neq i} a_j$.

However, as in the grade game, the unique rational action profile is socially inefficient. To see this, note that the alternative action profile $a' = (\epsilon, \ldots, \epsilon)$ leads to a payoff profile $((nk-1)\epsilon, \ldots,$ $(nk-1)\epsilon) > (0, \ldots, 0)$ where the inequality holds as $k \in (1/n, 1)$. □

3.3 Cautious Players and Weakly Dominated Actions

In the previous section, we have seen that a rational player will only choose actions that are not dominated. Sometimes, an action for such a rational player may be justified by a conjecture that rules out a particular action profile of other players entirely. This may be too knife-edge for a cautious player who worries that the other players might make mistakes when choosing their actions or that there is some other uncertainty involving the other players' choices. This leads us to the following definition.

Definition 3.7. *A player i is cautious if her conjecture does not rule out any action profile $a_{-i} \in A_{-i}$. Let the set of cautious conjectures be defined as*

$$\Delta^c(A_{-i}) = \left\{\mu^i \in \Delta(A_{-i}) : supp\ \mu^i = A_{-i}\right\} \tag{3.30}$$
$$= \left\{\mu^i \in \Delta(A_{-i}) : \forall a_{-i} \in A_{-i}, \mu^i(a_{-i}) > 0\right\}. \tag{3.31}$$

Then, a rational and cautious player i chooses actions in $r_i(\Delta^c(A_-)$; that is, in the image set of the best-reply correspondence over cautious conjectures.

[6] For example, the mayor has to follow some bureaucratic procedures that do not directly translate into improvements of the park.

It will turn out that the set of justifiable actions for a cautious player is a subset of the set of justifiable actions for a non-cautious player. The reason is that a cautious player can rationalize fewer actions as she can hold strictly fewer conjectures than a non-cautious player. Thus, she can justify actions with fewer conjectures. However, we can define a slightly modified concept of domination, to obtain a similar insight about the set of justifiable actions.

Definition 3.8. *A mixed action $\alpha_i \in \Delta(A_i)$ weakly dominates another mixed action $\beta_i \in \Delta(A_i)$ if it yields at least the same expected payoff for player i, for every action profile $a_{-i} \in A_{-i}$ of other players, and a strictly higher expected payoff for at least on action profile $\hat{a}_{-i} \in A_{-i}$; that is,*

$$\forall a_{-i} \in A_{-i}, \quad u_i(\alpha_i, a_{-i}) \geq u_i(\beta_i, a_{-i}) \tag{3.32}$$

$$\exists \hat{a}_{-i} \in A_{-i}, \quad u_i(\alpha_i, \hat{a}_{-i}) > u_i(\beta_i, \hat{a}_{-i}). \tag{3.33}$$

Denote the set of actions that are not weakly dominated by NWD_i,

$$NWD_i = \left\{ a_i \in A_i : \forall \alpha_i \in \Delta(A_i), \quad \begin{matrix} (\exists \hat{a}_{-i} \in A_{-i},\ u_i(\alpha_i, \hat{a}_{-i}) < u_i(a_i, \hat{a}_{-i})) \\ \vee \\ (\forall a_{-i} \in A_{-i},\ u_i(\alpha_i, a_{-i}) \leq u_i(a_i, a_{-i})) \end{matrix} \right\}. \tag{3.34}$$

We call an action a_i^* weakly dominant if all other pure actions $a_i \in A_i \setminus \{a_i^*\}$ are weakly dominated by it.

We state the following lemma without proof and note that this is an instance, where the analogous result does not hold for compact-continuous games.

Lemma 3.4. *Consider a finite game G, a player $i \in I$, and an action $a_i^* \in A_i$. There exists a full-support conjecture $\mu^i \in \Delta^c(A_{-i})$ that justifies a_i^* if and only if a_i^* is not weakly dominated; that is,*[7]

$$NWD_i = r_i(\Delta^c(A_{-i})). \tag{3.35}$$

Example 5. Second-Price Auction. To illustrate that weak dominance can be a useful concept, we consider a second-price auction. Suppose that a piece of art is being sold in an auction house in the following way. Interested buyers have a valuation $v_i > 0$ for the artwork and write a bid into a sealed envelope and hand it to the auctioneer. The auctioneer opens the envelopes and the buyer who wrote down the highest bid receives the artwork and has to pay the highest bid submitted by all other interested buyers. In case of a tie, the auctioneer randomizes uniformly among the buyers who submitted the highest bid.[8]

We can describe this auction game as a game played among the players $i \in \{1, \dots, n\}$, actions are chosen from $A_i \in [0, \infty)$, and payoff functions are

$$u_i(v_i, a) = \begin{cases} v_i - \max_{j \neq i} a_j & \text{, if } a_i > \max_{j \neq i} a_j \\ \frac{1}{1 + |\arg\max_{j \neq i} a_j|}(v_i - \max_{j \neq i} a_j) & \text{, if } a_i = \max_{j \neq i} a_j \\ 0 & \text{, if } a_i < \max_{j \neq i} a_j. \end{cases} \tag{3.36}$$

It is straightforward to show that it is a weakly dominant action for player i to bid exactly her valuation, $a_i^* = v_i$. To see this, consider separately the cases $a_i < v_i$ and $a_i > v_i$. The case $a_i < v_i$ is left as an exercise.

Consider the case $a_i > v_i$.

[7] For compact-continuous games, it holds that $r_i(\Delta^c(A_{-i})) \subseteq NWD_i$; that is, there are cases in which the action set is infinite and there exist not weakly dominated actions for which there is no cautious conjecture to which they are a best reply. For an example, see Battigalli et al. (2023), Example 4 on page 51.

[8] We will discuss this auction format in more detail during the mechanism design part of the class.

1. Suppose that $a_{-i} \in A_{-i}$ is such that $a_i < \max_{j \neq i} a_j$. Then, i loses the auction with both bids a_i and v_i. Thus, her payoff is the same in both cases.
2. Suppose that $a_{-i} \in A_{-i}$ is such that $v_i > \max_{j \neq i} a_j$. Then, i wins the auction in both cases and pays the same price $\max_{j \neq i} a_j$ leading to the same (strictly positive) payoff.
3. Suppose that $a_{-i} \in A_{-i}$ is such that $\max_{j \neq i} a_j \in (v_i, a_i)$. Then, i loses the auction with bid v_i and receives a payoff of zero and i wins the auction with bid a_i, receives the object and pays $\max_{j \neq i} a_j > v_i$ leading to a negative payoff. Thus, bidding v_i generates a strictly higher payoff in this case.
4. Suppose that $a_{-i} \in A_{-i}$ is such that $v_i = \max_{j \neq i} a_j$. Then, i wins the auction with certainty when bidding a_i and pays $\max_{j \neq i} a_j = v_i$ leading to a payoff of zero. If i bids v_i, she (depending on the outcome of the randomization) either wins the auction paying a price equal to her valuation or she loses the auction both leading to a payoff of zero. Hence, both actions yield the same payoff.
5. Suppose that $a_{-i} \in A_{-i}$ is such that $a_i = \max_{j \neq i} a_j$. Then, i loses the auction with bid v_i leading to a payoff of zero. If i bids a_i, she (depending on the outcome of the randomization), she either loses the auction leading to a payoff of zero or she wins the auction and pays $a_i > v_i$ leading to a strictly negative payoff. Hence, i's expected payoff is strictly higher when bidding v_i instead of a_i.

It follows that by bidding v_i, i receives a payoff that is always at least as high as when bidding $a_i > v_i$ and in some instances her payoff is strictly higher.

□

3.4 Exercises

Exercise 1. Consider the following game and show that there is a mixed strategy that is strictly dominated although all pure strategies in its support are not dominated.

		Bob L	Bob R
Ann	U	$1,3$	$-2,0$
Ann	M	$-2,0$	$1,3$
Ann	D	$0,1$	$0,1$

Exercise 2. Show that if an action a_i^* is justifiable, then it must not be dominated.

Exercise 3. Consider a finite game $G = \langle I, (A_i, u_i)_{i \in I} \rangle$ and an action a_i^*. Show that the following conditions are equivalent:

(i) a_i^* is dominant (i.e., action a_i^* dominates every action $a_i \in A_i \setminus \{a_i^*\}$),

(ii) a_i^* dominates every mixed action $\alpha_i \neq a_i^*$,

(iii) a_i^* is the *unique* best reply to every conjecture.

Exercise 4. Show that bidding $a_i < v_i$ in a second-price auction is a weakly dominated action.