Modelling dynamic systems

Systems and mathematical models

Michela Mulas

https://sdgs.un.org/goals/goal1

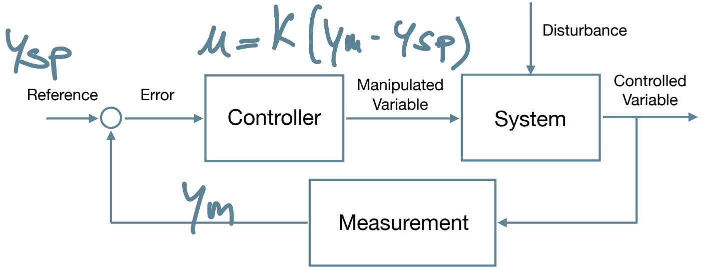
A SDG a day will keep the semester awake

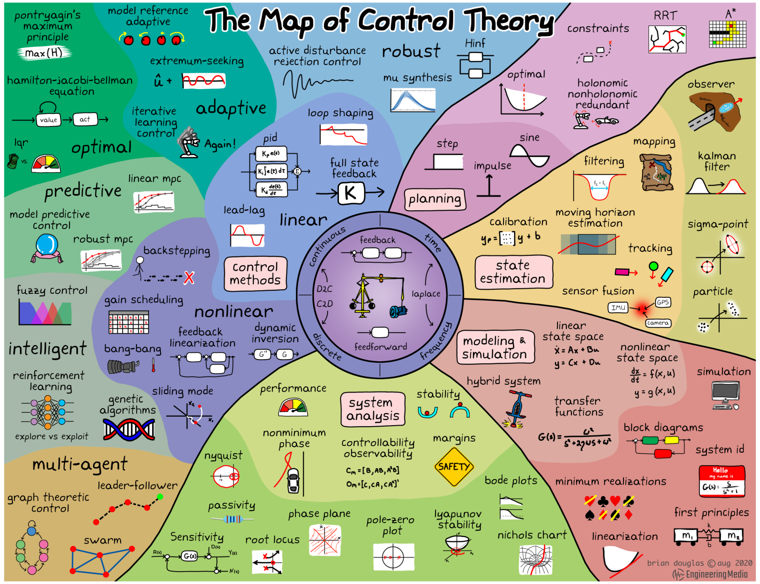
HW1 coming soon!!

Brief recap

We did...

- Define a control system and describe some applications
- Describe the basic features and configurations of control systems
- Describe control systems analysis and design objectives
- Describe a control system's design process





Source: https://engineeringmedia.com/map-of-control

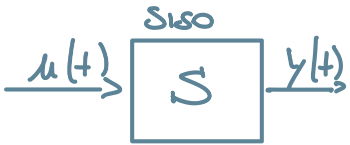
Available in different languages

Brief recap ⟶ HOW TO DESIGN A CONTROL SYSTEM

1. Understand the control problem:
   - Which variables can be manipulated by actuators?
   - What are the output variables of interest?
   - What should/can measure?
   - Which are the disturbances?

2. Get a reliable simulator
   ⟶ software/programming language

3. Get a simplified model of the main process dynamics



4. Use design techniques to synthetise the control algorithm

5. Test, simulate and validate

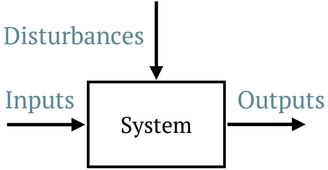
CLASSICAL CONTROL DESIGN requires a dynamical model of the open loop process

NOTE that the model is not explicitly entering the control loop but we use it to study/understand/investigate the system behaviour

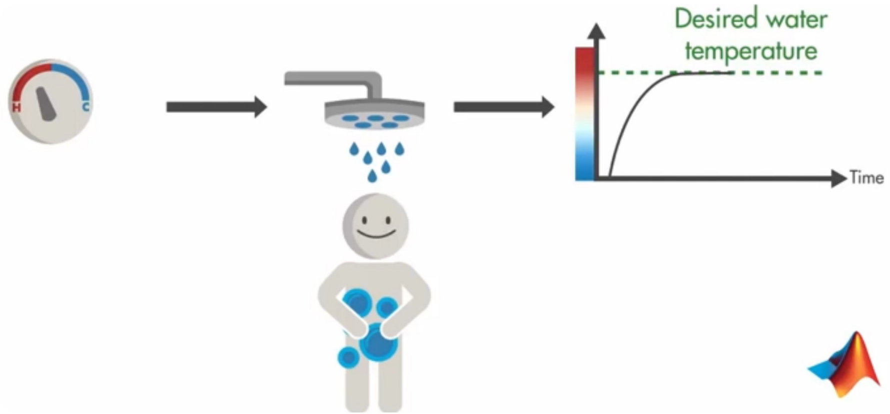
Brief recap

We did...

- A **system** is an "object" in which variables of different kinds interact and produce observable signals.
- If the system is **dynamic** these variables (or signals) evolve with time.



- **Outputs** are the observable signals of interest to us.
- **Inputs** are external signals that can be manipulated by the observer.
- **Disturbances** are external signals that can not be manipulated.
- **The choice between inputs and outputs depends on the control objectives.**



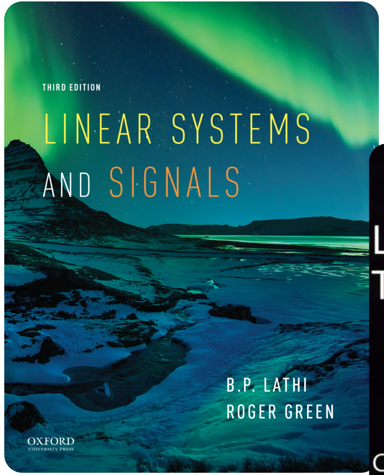
Goals

Today's lecture is about ...

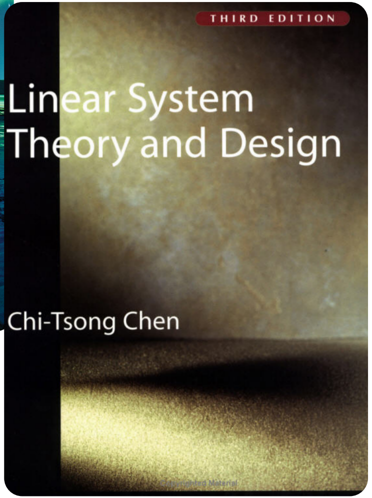
- Describe dynamic systems
- Describe and understand mathematical models
- Define the main properties of a mathematical model

Reading list

- Nise, *Control Systems Engineering* (6th Edition)
- Ogata and Severo *Engenharia de Controle Moderno* (3th Ed., or newer), in Portuguese.



Chap 1



Chap 2

Goals

Today's lecture is about ...

- Describe dynamic systems
- Describe and understand mathematical models
- Define the main properties of a mathematical model

Reading list

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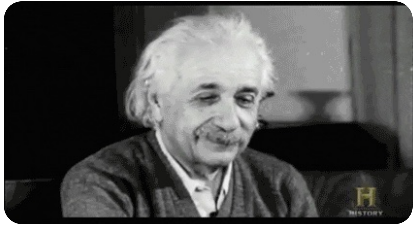
DYNAMICAL MODEL

↳ is an object (or a set of objects) that evolves over time possible under external excitations

The way that the system evolves is called the DYNAMIC of the system

A mathematical model of a system is a set of mathematical laws explaining it in a compact form



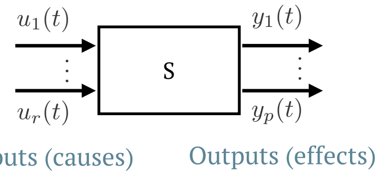
CONFLICTING MODELLING OBJECTIVES

Describe enough to capture the main behaviour of the system & simple enough for analysing it

MAKE EVERYTHING AS SIMPLE AS POSSIBLE BUT NOT SIMPLER !!

Dynamic systems

Input-output representation



- The **inputs** (external causes) are developed outside the system, their evolution influence the system behaviour.

  They are the **manipulated variables** and the **disturbances**.
- The **outputs** (effects) depend on the inputs and on the nature of the system itself.
- The **system** (S) can be considered as an operator that assigns a specific behaviour to the outputs given every possible behaviour of the inputs.

SYSTEM ANALYSIS OBJECTIVES

↳ study the relationship between the system input (CAUSE) and the system output (EFFECT)

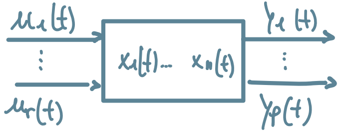
INPUT - OUTPUT representation

MIMO $u(t) \in \mathbb{R}^r$

$y(t) \in \mathbb{R}^p$

SISO $p = r = 1$

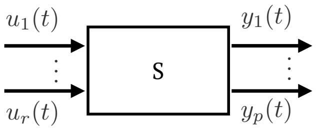
STATE SPACE representation



STATE of a system describes enough about the system to determine the future output behaviour of $y(t)$ given any external force $u(t)$ affecting the system

Dynamic systems

Input-output representation



Inputs (causes) Outputs (effects)

$$\mathbf{u}(t) = [u_1(t) \ldots u_r(t)] \in \mathbb{R}^r$$
$$\mathbf{y}(t) = [y_1(t) \ldots y_p(t)] \in \mathbb{R}^p$$

- A **Single-Input-Single-Output** (SISO) system has only 1-input ($r = 1$) and 1-output ($p = 1$).
- A **Multiple-Input-Multiple-Output** (MIMO) system has r-inputs and p-outputs.

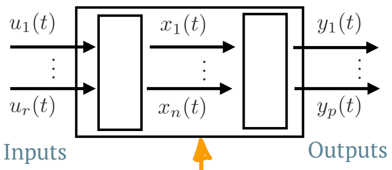
CLASSICAL CONTROL THEORY

The system analysis is carried out in time s-domain

It deals with techniques developed before ~1950
embodies methods as Root Locus, Bode, Nyquist ...
These methods have in common the use of
TRANSFER FUNCTIONS in the complex frequency domain
and emphasis is given on the use of graphical
techniques, the use of simplifying assumptions to
approximate the time response

Dynamic systems

State variable representation



x_0 initial conditions

- A state variable is one of the set of variables used to describe the mathematical "state" of a dynamical system.
- The **state of a system describes enough about the system to determine the future output behaviour** of $\mathbf{y}(t)$ given any external forces $\mathbf{u}(t)$ affecting the system.
- The state of a dynamic system is a set of physical quantities, the specification of which (in absence of external excitation) completely determine the evolution of the system.

"MODERN" CONTROL THEORY

The system analysis is carried out in the time domain

It refers to state space methods developed in the ~1960

The mathematical model is assumed to consist of ordinary differential equations which have an unique solution for all inputs and initial conditions.

STATE VARIABLES provide the minimum set of variables that fully describe the system that is they give enough information to predict the future behaviour of the system

Mathematical models

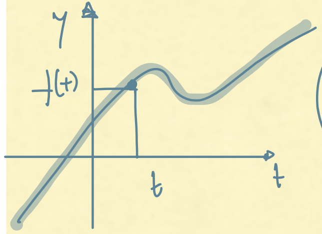
In order to analyse (and then control) the system, we need to have a mathematical model of the system to describe it in a **quantitative** manner.

The mathematical model gives an exact mathematical description of the link between inputs and outputs:

- An **input-output model** describes the link between $\mathbf{y}(t)$ and the inputs $\mathbf{u}(t)$ as differential equations.
- A **state-variable model** describes:
  - The evolution of the state variable $\dot{\mathbf{x}}(t) \in \mathbb{R}^n$ depends on the state $\mathbf{x}(t) \in \mathbb{R}^n$ and the input $\mathbf{u}(t) \in \mathbb{R}^r$ (**state equation**).
  - The output $\mathbf{y}(t) \in \mathbb{R}^p$ depends on the state $\mathbf{x}(t)$ and the input $\mathbf{u}(t)$ (**output transformation**).

We need to investigate how the system evolves with time ⇝ differential equations

A function $y=f(t)$ encodes the relation between two quantities y and t ⇝ variables



Consider the rate of change of quantity y corresponding to a change in t

⇝ it is the ratio between the differential change in y and the corresponding differential change in t

We conventionally call the RATIO of DIFFERENTIAL CHANGES the derivative of function f.

The rate of change is understood as the slope of the tangent line to the function @ a specific point t

Cont.

Mathematical models

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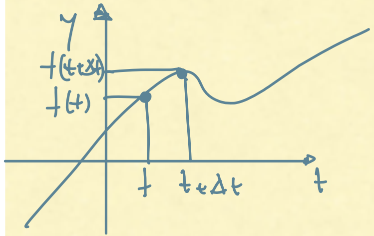


The value of the derivative can be approximated by using small changes in t and f(t)

$$\boxed{\frac{df(t)}{dt} \simeq \frac{f(t+\Delta t) - f(t)}{t+\Delta t - t}}$$

The tangent line will be approximated
- by the secant line to function
- its slope is the approximation

$$\frac{df(t)}{dt} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

Mathematical models

Input-output models

The IO model for a SISO system is given by one differential equation as:

$$h\Big(\underbrace{y(t),\dot{y}(t),\ldots,y^{n}(t)}_{\text{output}},\underbrace{u(t),\dot{u}(t),\ldots,u^{m}(t)}_{\text{input}}\Big)=0$$

- $\dot{y}(t)=\frac{d}{dt}y(t),\ldots,\dot{y}^{(n)}(t)=\frac{d}{dt}y^{n}(t)$;
- h is a function of parameter that depends on the system under study;
- n is the higher derivation order of the outputs and coincides with the system order.
- m is the higher derivation order of the inputs.

Example:

$$2\dot{y}(t)y(t)+2\sqrt{t}u(t)\ddot{u}(t)=0$$

n=1 m=2

I/O Model

Mathematical models

Input-output models

The IO model for a MIMO system with p output and r inputs is given by p differential equations as:

$$\begin{cases} h_1\Big(\underbrace{y_1(t),\dot{y}_1(t),\ldots,y_1^n(t)}_{\text{output 1}},\underbrace{u_1(t),\dot{u}_1(t),\ldots,u_1^m(t)}_{\text{input 1}},\ldots,\underbrace{u_r(t),\dot{u}_r(t),\ldots,u_r^m(t)}_{\text{input r}},t\Big)=0 \\ h_2\Big(\underbrace{y_2(t),\dot{y}_2(t),\ldots,y_2^n(t)}_{\text{output 2}},\underbrace{u_1(t),\dot{u}_1(t),\ldots,u_1^m(t)}_{\text{input 1}},\ldots,\underbrace{u_r(t),\dot{u}_r(t),\ldots,u_r^m(t)}_{\text{input r}},t\Big)=0 \\ \vdots \quad \vdots \\ h_p\Big(\underbrace{y_p(t),\dot{y}_p(t),\ldots,y_p^n(t)}_{\text{output p}},\underbrace{u_1(t),\dot{u}_1(t),\ldots,u_1^m(t)}_{\text{input 1}},\ldots,\underbrace{u_r(t),\dot{u}_r(t),\ldots,u_r^m(t)}_{\text{input r}},t\Big)=0 \end{cases}$$

- h_i, with $i=1,\ldots,p$ are functions that depend on the system.
- n_i is the maximum degree of differentiation of the output $y_i(t)$.
- m_i is the maximum degree of differentiation of the input $u_i(t)$.

Mathematical models

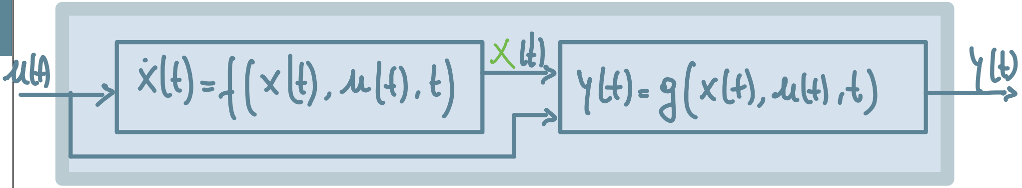
State-variable models

The state-variable model for a SISO system is given by n **ordinary differential equations** that link the derivative of each state variable to the input and the output to the state variables and the input:

$$\begin{cases} \dot{x}_1(t) = f_1(x_1(t), \dots, x_n(t), u(t), t) \\ \vdots \qquad \vdots \\ \dot{x}_n(t) = f_n(x_1(t), \dots, x_n(t), u(t), t) \\ y(t) = g(x_1(t), \dots, x_n(t), u(t), t) \end{cases}$$

- f_i, $i = 1, \dots, n$ and g are functions of different parameters given by the dynamics of the system under study.

NOTE: $\dot{x} = \frac{dx}{dt}$



Given $\dot{\mathbf{x}}$ as

$$\dot{\mathbf{x}}(t) = \frac{d}{dt}\mathbf{x}(t) = \begin{bmatrix} \dot{x}(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix}$$

we can simply write the system as:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), u(t), t) \\ y(t) = g(\mathbf{x}(t), u(t), t) \end{cases}$$

Mathematical models
State-variable models

The SV model for a MIMO system with r inputs and p outputs is given by:

$$\begin{cases} \dot{x}(t) = f_1(x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_r(t), t) \\ \dot{x}_n(t) = f_1(x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_r(t), t) \\ \vdots \quad\quad \vdots \\ y_1(t) = g_1(x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_r(t), t) \\ \vdots \quad\quad \vdots \\ y_p(t) = g_p(x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_r(t), t) \end{cases}$$

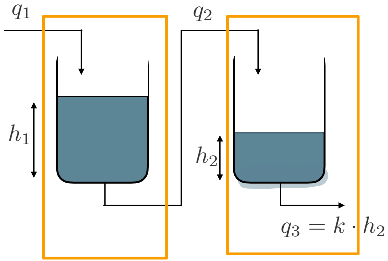
Equivalently, we can write:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) \\ \dot{\mathbf{y}}(t) = \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t), t) \end{cases}$$

Dynamic systems
Example

Consider a system of two identical tanks of cross-sectional area A [m^2].

- q_1 [m^3/s] is the flow-rate on the 1st tank
- q_2 [m^3/s] is the flow-rate on the 2nd tank
- q_3 [m^3/s] is the output flow-rate
- h_1 [m] is the liquid height in the 1st tank
- h_2 [m] is the liquid height in the 2nd tank



q_1 and q_2 can be set to a desired value by manipulating them with some pumps.

q_3 is a linear function of the liquid in the second tank, h_2.

INPUT : $u = q_i \quad (i = 1, 2)$

OUTPUT : $y = d = h_1 - h_2$

CONSERVATION OF MASS

$$\begin{cases} \dfrac{dV_1}{dt} = q_1(t) - q_2(t) \\ \dfrac{dV_2}{dt} = q_2(t) - q_3(t) \end{cases}$$

$V_1 = h_1 A \qquad h_1 = V_1 / A$

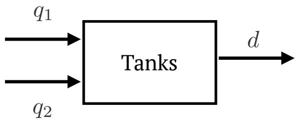
$V_2 = h_2 A \qquad h_2 = V_2 / A$

Dynamic systems

Example

Consider a system of two identical tanks of cross-sectional area A [m^2].

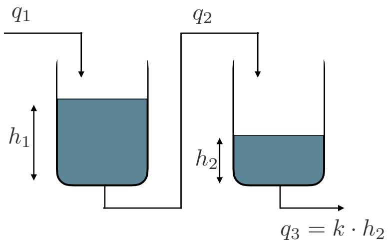
- q_1 and q_2 [m^3/s] are measurable and manipulable inputs
  They affect the liquid levels on the two tanks.
- $d = h_1 - h_2$ [m] is the output variable.
  It is measurable but manipulable only through q_1 and q_2.



This is a MISO (**multiple-input-single-output**) system with 2 inputs and 1 output.

Dynamic systems

State variable representation



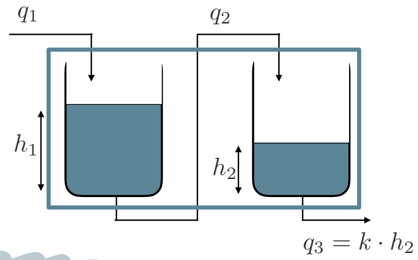
Again, consider the two-tank systems:

- $d_0 = h_{1,0} - h_{2,0}$ is the output value at the initial time $t = t_0$.
- $h_{1,0}$ and $h_{2,0}$ are the liquid level at the time t_0.
- Suppose that at t_0, the inputs are zero: $q_{1,0} = q_{2,0} = 0$.
- The output at any $t > t_0$ depends on the values of $q_1(t)$ and $q_2(t)$ during the interval $[t_0, t]$.
- The state variable is the link between the inputs and the output.

Mathematical models

Example

Define the IO and the SV models for the two-tank system we discussed previously.



For the conservation of mass for a incompressible fluid, we have:

$$\begin{cases} \dot{V}_1 = q_1(t) - q_2(t) \\ \dot{V}_2 = q_2(t) - q_3(t) = q_2(t) - k \cdot h_2(t) \end{cases}$$

Given that $h_1 = V_1/A$ and that $h_2 = V_2/A$, we can write:

$$\begin{cases} \dot{h}_1 = \frac{1}{A}\Big(q_1(t) - q_2(t)\Big) \\ \dot{h}_2 = \frac{1}{A}\Big(q_2(t) - q_3(t)\Big) = \frac{1}{A}\Big(q_2(t) - k \cdot h_2(t)\Big) \end{cases}$$

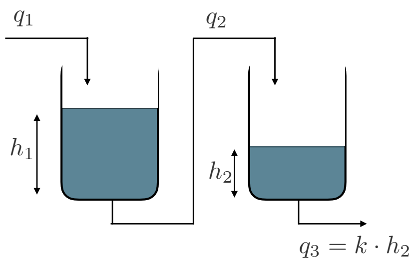
$\dot{h}_1 = \frac{dh}{dt} =$

- A is the cross-sectional area of the tanks.

SV model

Mathematical models
Example

Define the IO and the SV models for the two-tank system we discussed previously.



The SV model is given by:

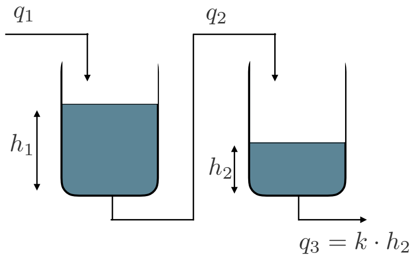
$$\begin{cases} \dot{x}_1(t) = \dfrac{1}{A}\Big(u_1(t) - u_2(t)\Big) \\ \dot{x}_2(t) = \dfrac{1}{A}\Big(-k \cdot x_2(t) + u_2(t)\Big) \\ y(t) = x_1(t) - x_2(t) \end{cases}$$

- $x_1(t) = h_1(t)$ and $x_2(t) = h_2(t)$ are the state-variables;
- $u_1(t) = q_1(t)$ and $u_2(t) = q_2(t)$ are the inputs;
- $y(t)$ is the output.

$$\begin{cases} \dot{x} = f(x, u, t \mid \theta_x) \\ y = g(x, u, t \mid \theta_y) \end{cases}$$

Mathematical models
Example

Define the IO and the SV models for the two-tank system we discussed previously.



We define, $y(t) = h_1(t) - h_2(t)$:

$$\dot{y}(t) = \dot{h}_1(t) - \dot{h}_2(t) = \frac{1}{A}\Big(q_1(t) - 2q_2(t) + k \cdot h_2(t)\Big)$$
$$= \frac{1}{A}\Big(u_1(t) - 2u_2(t) + k(h_1(t) - y(t))\Big)$$

Taking the second derivative and rearranging, the IO model is:

$$\ddot{y}(t) + \frac{k}{A}\dot{y}(t) = \frac{1}{A}\dot{u}_1(t) - \frac{2}{A}\dot{u}_2(t) + \frac{k}{A^2}u_1(t) - \frac{k}{A^2}u_2(t)$$

IO model

Dynamic and instantaneous systems

A system is said to be:

- **Instantaneous** (or **memoryless**) if the value $\mathbf{y}(t)$ of the output at time t only depends on the value of the input $\mathbf{u}(t)$ at the same time t.
- **Dynamic** (or **with memory**), otherwise.

A SISO system is said to be instantaneous if and only if the IO relation:

$$h(y(t), u(t), t) = 0$$

Analogously, for a MIMO system with r inputs and p outputs:

$$\begin{cases} h_1(y_1(t), u_1(t), \ldots, u_r(t), t) = 0 \\ h_2(y_2(t), u_1(t), \ldots, u_r(t), t) = 0 \\ \vdots \\ h_p(y_p(t), u_1(t), \ldots, u_r(t), t) = 0 \end{cases}$$

Linear and nonlinear systems

A system (model) is said to be:

- **Linear** if the system obeys the **superposition principle**.
  - ⇝ If the system reacts to a cause c_1 with the effect e_1 and to a cause c_2 with an effect e_2, then the system responds to the cause $ac_1 + bc_2$ with an effect $ae_1 + be_2$, for every constants a and b.
  - ⇝ The linear system exhibits property of **additivity** and satisfies the property of **homogeneity**.
- **Nonlinear**, otherwise.

An IO model is said to be **linear** if and only if the input-output relation is given by a linear differential equation:

$$a_0(t)y(t) + a_1\dot{y}(t) + \cdots + a_n(t)y^{(n)}(t) = b_0(t)u(t) + b_1\dot{y}(t) + \cdots + b_m(t)y^{(n)}(t)$$

where the linear combinations of the IO coefficients are functions of time.

CAUSE C1 ⇝ EFFECT e1
CAUSE C2 ⇝ EFFECT e2

CAUSE (C1+C2) ⇝ EFFECT (e1+e2)
Additivity

CAUSE KC1 ⇝ EFFECT Ke1
Homogeneity (or SCALING)
K is a real or imaginary number

Linear and nonlinear systems

A state-space model is said to be **linear** if and only if the equations of state equation and the output transformation are linear equations:

$$
\begin{cases}
\dot{x}_1(t) = a_{1,1}(t)x_1(t) + \cdots + a_{1,n}(t)x_n(t) + b_{1,1}(t)u_1(t) + \cdots + b_{1,r}(t)u_r(t) \\
\vdots \\
\dot{x}_n(t) = a_{n,1}(t)x_1(t) + \cdots + a_{n,n}(t)x_n(t) + b_{1,1}(t)u_1(t) + \cdots + b_{n,r}(t)u_r(t) \\
y_1(t) = c_{1,1}(t)x_1(t) + \cdots + c_{1,n}(t)x_n(t) + d_{1,1}(t)u_1(t) + \cdots + d_{1,r}(t)u(t)_r(t) \\
\vdots \\
y_n(t) = c_{n,1}(t)x_1(t) + \cdots + c_{n,n}(t)x_n(t) + d_{1,1}(t)u_1(t) + \cdots + d_{n,r}(t)u_r(t)
\end{cases}
$$

That is:

$$
\begin{cases}
\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \\
\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t)
\end{cases}
$$

where $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times r}$, $\mathbf{C} \in \mathbb{R}^{p\times n}$, $\mathbf{D} \in \mathbb{R}^{p\times r}$

tells how the state vector changes

linear combination of the state vector

$$
\begin{cases}
\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \\
\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t)
\end{cases}
$$

linear combination of the input vector

output of the system: what we are interested in knowing

A tells how the states are interconnected to each other

B tells how the inputs enter into the system ⤳ which states are affected

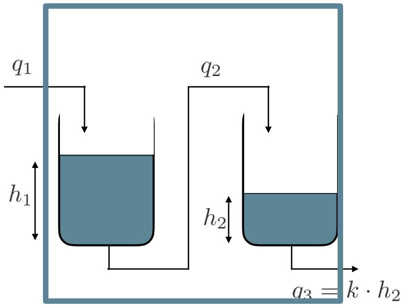
C tells how the states are combined to get the output

D allows the inputs to bypass the system and feedforward to y

Linear and nonlinear systems

The two-tank system model is linear.

- The function that links h to the output and its derivatives to the input and its derivatives is linear.



The SV model can be written as:

$$\mathbf{A}(t) = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{k}{A} \end{bmatrix} \qquad \mathbf{B}(t) = \begin{bmatrix} \frac{1}{A} & -\frac{1}{A} \\ 0 & \frac{1}{A} \end{bmatrix}$$

$$\mathbf{C}(t) = \begin{bmatrix} \frac{1}{A} & \frac{1}{A} \end{bmatrix} \qquad \mathbf{D}(t) = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$\begin{cases} \dot{x}_1(t) = \frac{1}{A}\left(u_1(t) - u_2(t)\right) \\ \dot{x}_2(t) = \frac{1}{A}\left(-k \cdot x_2(t) + u_2(t)\right) \\ y(t) = \left(x_1(t) - x_2(t)\right) / A \end{cases}$$

Linear and nonlinear systems

Let consider the system described by:

$$y(t) = u(t) + 1$$

Is this a linear or non-linear system?

It is a non-linear system because the equation is a non-linear algebraic equation where the non-linearity is given by the term +1.

$u_1 = 1 \rightsquigarrow y_1 = 1+1 = 2$

$u_2 = 2 \rightsquigarrow y_2 = 2+1 = 3$

$u_3 = u_1 + u_3 = 3 \rightsquigarrow y_3 = 3+1$

!!

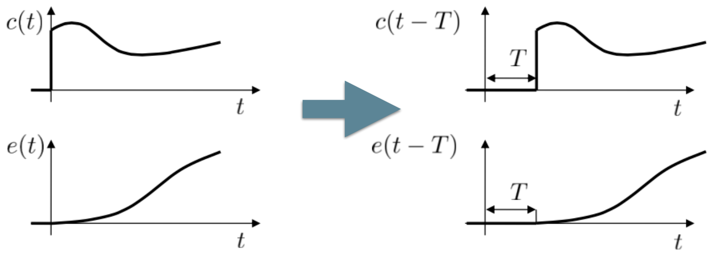
Time-invariant and time-variant systems

A system is said to be:

- **Stationary** (or **time-invariant**) if it obeys to the **cause-effect translation principle** in time.
  - ⇝ The system responds always with the same effect to the same cause, regardless of the time.
- **Non stationary** (or **time-variant**), otherwise

CAUSE $c(t)$ ⇝ EFFECT $e(t)$

CAUSE $c(t-T)$ ⇝ EFFECT $e(t-T)$



Time-invariant and time-variant systems

An IO system is said to be **time-invariant** if and only if the IO relation is (explicitly) time independent.

That is, for a SISO:

$$h\Big(y(t), \dot{y}(t), \dots, y^{(n)}, u(t), \dot{u}(t), \dots, u^{(m)}(t)\Big) = 0$$

and for a linear SISO:

$$a_0 y(t) + a_1 \dot{y}(t) + \dots + a_n y^{(n)}(t) = b_0 u(t) + b_1 \dot{u}(t) + \dots + b_n u^{(m)}(t)$$

A state-variable system is said to be **time-invariant** if and only if the state equation and the output transformation equations are time independent:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) \\ \mathbf{y}(t) = \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t)) \end{cases}$$

and for a linear system:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{cases}$$

Time-invariant and time-variant systems

Let consider an instantaneous linear system:

$$y(t) = tu(t)$$

Cleary, it is a non-stationary system.

This can be verified using the cause-effect translation principle considering the following input:

$$u(t) = \begin{cases} 1 \text{ if } t \in [0,1] \\ 0 \text{ otherwise} \end{cases}$$

Proper and improper systems

A system is said to be:

- **Proper** if the **causality principle** applies CAUSAL SYSTEM
  - ⇝ The effect does not anticipate in time the cause that generate it.
- **Improper**, otherwise

An IO model for a SISO system is proper if and only if in the IO relationship:

$$h\Big(\big(y(t), \dot{y}(t), \ldots, y^{(n)}, u(t), \dot{u}, \ldots, u^{m}(t), t\big) = 0$$

the output order is greater or equal to the input order ($n \geq m$).

If $n > m$ the system is said to be strictly proper.

A MIMO system is said to be proper if $n_i \geq \max_{j=1,\ldots,r} m_{i,j}$.

y(t) does not depend on future inputs
u(τ) ∀ τ > t
or
the value of the output depends only on the past and the present values of the input, not on its future values

Proper and improper systems

A system is said to be:

- **Proper** if the **causality principle** applies
  - $\rightsquigarrow$ The effect does not anticipate in time the cause that generate it.
- **Improper**, otherwise

A SV model described by:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) \\ \mathbf{y}(t) = \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t), t) \end{cases}$$

is always a proper system.

- The system is strictly proper if $\mathbf{y}(t) = \mathbf{g}(\mathbf{x}(t), t)$.
- The SV model for a linear and stationary strictly proper system becomes:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) \end{cases}$$

Lumped and distributed systems

A system is said:

- a **Lumped parameter** (or 'with finite dimension") if the state is described by a finite number of quantities (each associated to one component).
- a **Distributed parameter** (or "with infinite dimension"), otherwise.

For an IO model,

- a system with lumped parameters is described by a differential ordinary equation.
- a system with distributed parameters id described by a partial differential equation.

For a SV model,

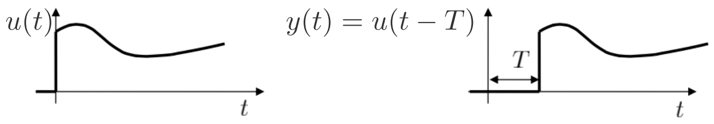
- the state vector of a lumped parameter system has a finite number of components;
- the state vector of a distributed parameter system has an infinite number of components.

All variables are function of time alone (ORDINARY DIFF. EQUATIONS)

All variables are function of time and one or more spatial variables (PARTIAL DIFFERENTIAL EQUATIONS)

Time delay systems

A **finite delay element** is a system whose output $y(t)$ at the time t is given by the input at the time $u(t-T)$ at the time $t-T$, where $T \in (0, +\infty)$ is the delay introduced by the element.



Exercise

Exercise L1E1: Given the following mathematical models of dynamic systems:

$$\ddot{y}(t) + y(t) = 5\dot{u}(t)u(t)$$

$$t^2\ddot{y}(t) + t\dot{y}(t) + y(t) = 5sin(t)\ddot{u}(t) - 1$$

$$\begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -2 & t^2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + 3u(t) \end{cases}$$

$$y(t) = \dot{u}(t - T)$$

1. Classify the models as IO or SV, defining the significant parameters such as derivation order, dimension of the state vector, inputs and outputs.
2. Define the properties of the models: linear or non-linear, stationary or time-variant, dynamic or instantaneous, with or without time delay, proper (strictly or not) or improper. Discuss your answers.

1. Nonlinear : due to the $\dot{u}(t)u(t)$
Time invariant : the coeff. are constant
Dynamic : in-out relationship is not algebraic
Lumped parameters : there aren't partial diff equations
Time-delay : none
Strictly proper : $n=2$ and $m=1$ $n>m$

2. Nonlinear : due to the term -1
Time variant : some coeff. vary with time
Dynamic : in-out relationship is not algebraic
Lumped parameters : there aren't partial diff equations
Time-delay : none
Proper : $n=m=2$

Exercise

Exercise L1E1: Given the following mathematical models of dynamic systems:

$$\ddot{y}(t) + y(t) = 5\dot{u}(t)u(t)$$

$$t^2\ddot{y}(t) + t\dot{y}(t) + y(t) = 5sin(t)\ddot{u}(t) - 1$$

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$$y(t) = \dot{u}(t - T)$$

1. Classify the models as IO or SV, defining the significant parameters such as derivation order, dimension of the state vector, inputs and outputs.
2. Define the properties of the models: linear or non-linear, stationary or time-variant, dynamic or instantaneous, with or without time delay, proper (strictly or not) or improper. Discuss your answers.

③ Linear in the standard (SV) representation
Time variant: coeff. $a_{2,1}$ is varying with time
Dynamic: $n = 2 > 0$
Lumped parameters: state vector has a finite number of comp.
Time-delay: none
Proper (but not strictly): $D = 0$

④ Linear: the I/O relationship is a diff. eq.
Time invariant: coeff. are constant
Dynamic: in-out relationship is not algebraic
Lumped parameters: there aren't partial diff equations
Time-delay: yes!
Not proper: $n = 0$, $m = 1$ $n < m$