Modelling dynamic systems

Revision exercises

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EXAM 1
4 exercises
1 T/F
content :
LO1 - LO7

TODAY
REVISION
TOMORROW
EXAMS QUESTIONS

Exercises

Exercise L9E1: Identify the properties of the following input-output models, where α and β are constant real parameters:

$$\frac{d^3y(t)}{dt^3}+3\alpha t\frac{dy(t)}{dt}+7y(t)=3\frac{d^3u(t)}{dt^3}+\beta t$$

$$\frac{d^3y(t)}{dt^3}+3\left(\frac{dy(t)}{dt}\right)^{\beta}+7y(t)=3\frac{d^3u(t)}{dt^3}+(\alpha+t)u(t)$$

$$\beta\frac{d^3y(t)}{dt^3}+3\frac{dy(t)}{dt}+7t^{\alpha}y(t)=3\frac{d^3u(t)}{dt^3}$$

$$\frac{d^3y(t)}{dt^3}+3(1-\beta t)\frac{dy(t)}{dt}+7ty(t)=3\alpha\frac{d^3u(t)}{dt^3}$$

- How is the model modified by the parameters α and β?
- Discuss your answers.

Linear model if $\beta = 0$
Time Invariant if $\alpha = \beta = 0$
Without time delay
Proper (but not strictly)
Dynamic

....

ToDo!!

Complete the exercise

Exercises

Exercise L9E2: Find the transfer function $G(s) = Y(s)/X(s)$ corresponding to the following differential equations in $x(t)$ and $y(t)$:

$$5\dot{y} + 4y = 3x \qquad \ddot{y} + 3\dot{y} + 2y = 4\dot{x} + 5x$$

$$\dddot{y} + 4\ddot{y} + y = 6\dot{x} + 2x \qquad \alpha\,\dddot{y} + \beta\ddot{y} + \gamma\dot{y} + \delta y = a\ddot{x} + bx$$

Exercise L9E3: Find the differential equations in $x(t)$ and $y(t)$ corresponding to the following transfer function $G(s) = Y(s)/X(s)$:

$$G(s) = \frac{2s+3}{3s^2+5s+1} \qquad G(s) = \frac{s^2+2s+5}{s^3+3s^2+4s+6}$$

$$G(s) = \frac{s^2+3}{(s+2)(s+1)^2} \qquad G(s) = \frac{as^2+bs+c}{s(s^2+\alpha s+\beta)}$$

$$5\dot{y} + 4y = 3x \rightsquigarrow 5s\,Y(s) + 4Y(s) = 3X(s)$$

$$\frac{Y(s)}{X(s)} = \frac{3}{5s+4}$$

$$3\ddot{y} + 5\dot{y} + y = 2\dot{x} + 3x$$

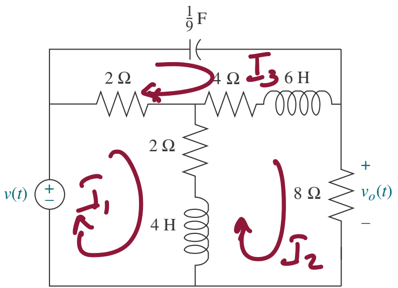
....

ToDo!!

Complete the exercise

Exercises

Exercise L9E4: Find the transfer function, $G(s) = V_o(s)/V(s)$ for the network:



Component	Voltage-current	Current-voltage	Voltage-charge	Impedance $Z(s) = V(s)/I(s)$	Admittance $Y(s) = I(s)/V(s)$
Capacitor	$v(t) = \frac{1}{C}\int_0^t i(\tau)d\tau$	$i(t) = C\frac{dv(t)}{dt}$	$v(t) = \frac{1}{C}q(t)$	$\frac{1}{Cs}$	Cs
Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R}v(t)$	$v(t) = R\frac{dq(t)}{dt}$	R	$\frac{1}{R} = G$
Inductor	$v(t) = L\frac{di(t)}{dt}$	$i(t) = \frac{1}{L}\int_0^t v(\tau)d\tau$	$v(t) = L\frac{d^2q(t)}{dt^2}$	Ls	$\frac{1}{Ls}$

MESH EQUATIONS

$$(4+4s)I_1(s) - (2+4s)I_2(s) - 2I_3(s) = V(s)$$

$$-(2+4s)I_1(s) + (14+10s)I_2(s) - (4+6s)I_3(s) = 0$$

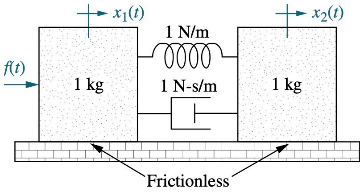
$$-2I_1(s) - (4+6s)I_2(s) + (6+6s+9/s) = 0$$

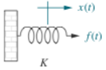
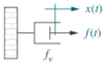
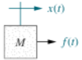
[...]

$$G(s) = \frac{48s^3+96s^2+112s+36}{48s^3+120s^2+220s+117}$$

Exercises

Exercise L9E5: Find the transfer function $G(s) = X_2(s)/F(s)$ for the system below.



Component	Force-velocity	Force-displacement	Impedence $Z_M(s) = F(s)/X(s)$
Spring 	$f(t) = K\int_0^t v(\tau)d\tau$	$f(t) = Kx(t)$	K
Viscous damper 	$f(t) = f_v v(t)$	$f(t) = f_v \frac{dx(t)}{dt}$	$f_v s$
Mass 	$f(t) = M\frac{dv(t)}{dt}$	$f(t) = M\frac{d^2x(t)}{dt^2}$	Ms^2

$$(s^2+s+1)X_1(s) - (s+1)X_2(s) = F(s) \quad \text{MASS1}$$
$$-(s+1)X_1(s) + (s^2+s+1)X_2(s) = 0 \quad \text{MASS2}$$

Solving for X_2

$$\frac{\begin{vmatrix} s^2+s+1 & F(s) \\ -(s+1) & 0 \end{vmatrix}}{\begin{vmatrix} s^2+s+1 & -(s+1) \\ -(s+1) & s^2+s+1 \end{vmatrix}} = \frac{F(s)(s+1)}{s^4+2s^3+2s^2}$$

$$G(s) = \frac{X_2}{F(s)} = \frac{s+1}{s^2(s^2+2s+2)}$$

Exercises

Exercise L9E6: For the system shown below, write the state equations and the output equation for the phase-variable representation.

$$R(s) \longrightarrow \boxed{\dfrac{100}{s^4 + 20s^3 + 10s^2 + 7s + 100}} \longrightarrow C(s)$$

(*a*)

$$R(s) \longrightarrow \boxed{\dfrac{30}{s^5 + 8s^4 + 9s^3 + 6s^2 + s + 30}} \longrightarrow C(s)$$

(*b*)

Check L07

Exercises

Exercise L9E7: Given the following state space system:

$$\begin{cases} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -10x_1(t) - 11x_2(t) + u(t) \\ y(t) &= x_1(t) - x_2(t) \end{cases}$$

- Find the transfer function of the system with input $u(t)$ and output $y(t)$. Assume $x_1(t_0) = x_2(t_0) = 0$.
- Find the analytical expression of the output response to a unit step input.

$$G(s) = \frac{Y(s)}{U(s)} = \left[C(sI-A)^{-1}B + D\right]$$

First method

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -10x_1(t) - 11x_2(t) + u(t) \\ y(t) = x_1(t) - x_2(t) \end{cases} \Rightarrow \begin{cases} sX_1(s) = X_2(s) \\ sX_2(s) = -10X_1(s) - 11X_2(s) + U(s) \\ Y(s) = X_1(s) - X_2(s) \end{cases}$$

Substituting X_2 in the second eq.

$$\begin{cases} sX_1(s) = X_2(s) \\ s(sX_1(s)) = -10X_1(s) - 11\, s\, X_1(s) + U(s) \\ Y(s) = X_1(s) - X_2(s) \end{cases} \Rightarrow s^2X_1(s) = -10X_1(s) - 11sX_1 + U$$

$$X_1(s) = \frac{1}{s^2+11s+10} U(s)$$

$$Y(s) = \left(\frac{1}{s^2+11s+10} - s \cdot \frac{1}{s^2+11s+10}\right) U(s) =$$

$$= \frac{1-s}{s^2+11s+10} U(s)$$

$$\boxed{\frac{Y(s)}{U(s)} = \frac{1-s}{s^2+11s+10}}$$

Exercises

Exercise L9E7: Given the following state space system:

$$\begin{cases} \dot{x}_1(t) & = x_2(t) \\ \dot{x}_2(t) & = -10x_1(t) - 11x_2(t) + u(t) \\ y(t) & = x_1(t) - x_2(t) \end{cases}$$

- Find the transfer function of the system with input $u(t)$ and output $y(t)$. Assume $x_1(t_0) = x_2(t_0) = 0$.
- Find the analytical expression of the output response to a unit step input.

Second method:

$$\boxed{G(s) = C(sI-A)^{-1}B + D}$$

$$A = \begin{bmatrix} 0 & 1 \\ -10 & -11 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & -1 \end{bmatrix} \quad D = 0$$

$$(sI - A) = \begin{bmatrix} s & -1 \\ 10 & s+11 \end{bmatrix} \Rightarrow (sI-A)^{-1} = \frac{1}{s^2+11s+10}\begin{bmatrix} s+11 & 1 \\ -10 & s \end{bmatrix}$$

$$G(s) = \frac{1}{s^2+11s+10}\begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} s+11 & 1 \\ -10 & s \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} =$$

$$= \frac{1}{s^2+11s+10}\begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ s \end{bmatrix} = \boxed{\frac{1-s}{s^2+11s+10}}$$

Exercises

Exercise L9E7: Given the following state space system:

$$\begin{cases} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -10x_1(t) - 11x_2(t) + u(t) \\ y(t) &= x_1(t) - x_2(t) \end{cases}$$

- Find the transfer function of the system with input $u(t)$ and output $y(t)$ Assume $x_1(t_0) = x_2(t_0) = 0$.
- Find the analytical expression of the output response to a unit step input.

$y(t) =$

$$Y(s) = \frac{1-s}{s^2+11s+10} \cdot \frac{1}{s} = \frac{1-s}{s(s+1)(s+10)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+10}$$

$$A = s \cdot Y(s)\Big|_{s\to 0} = \frac{1-s}{(s+1)(s+10)}\Big|_{s\to 0} = \frac{1}{10}$$

$$B = (s+1)\,Y(s)\Big|_{s\to -1} = \frac{1-s}{s(s+10)}\Big|_{s\to -1} = -\frac{2}{9}$$

$$C = (s+10)\,Y(s)\Big|_{s\to -10} = \frac{1-s}{s(s+1)}\Big|_{s\to -10} = \frac{11}{-10(-10+1)} = \frac{11}{90}$$

$$Y(s) = \frac{1}{10s} - \frac{2}{9} \cdot \frac{1}{s+1} + \frac{11}{90} \frac{1}{s+10}$$

$$\Downarrow$$

$$y(t) = \frac{1}{10} - \frac{2}{9} e^{-t} + \frac{11}{90} e^{-10t}$$

Exercises

Exercise L9E8: Given the following system represented in state space:

$$\dot{\mathbf{x}} = \begin{bmatrix} 2 & -3 & -8 \\ 0 & 5 & 3 \\ -3 & -5 & -4 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix} r$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathbf{x}$$

- Find the transfer function $G(s) = Y(s)/R(s)$.
- Find the input-output model corresponding to the model.

$$G(s) = \frac{s^2 - 61s + 293}{s^3 - 3s^2 - 27s + 103}$$

ToDo!!