Modelling dynamic systems

Transfer functions

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3 GOOD HEALTH AND WELL-BEING

SUSTAINABLE DEVELOPMENT GOAL 3:
Ensure healthy lives and promote well-being for all at all ages

https://sdgs.un.org/goals/goal3

QUESTION & ANSWERS

"Any" time on Telegram

Recap and goals

NOTE

5	April, FRIDAY
	REVISION ← No class on April 16 & April 20
9	April, TUE
	EXAM 1 ← RESCHEDULE NEEDED
11	April, THU

Brief recap

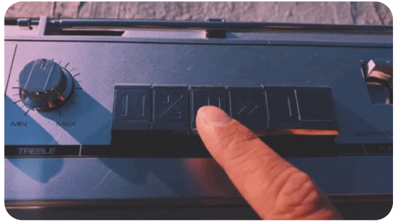
We did ...

- Classify dynamical models
  - Linear and nonlinear systems
  - Constant-parameter and time-varying-parameter systems
  - Static (memoryless) and dynamic (with memory) systems
  - Causal and noncausal systems
  - Lumped and distributed systems
  - Systems with time delay
  - Continuous-time and discrete-time systems

LTI

Goals

Today's lecture is about ...

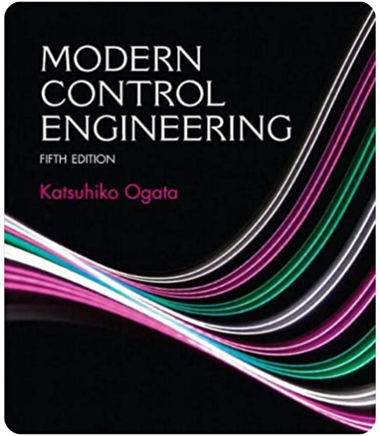


- Finalize Laplace transform of time functions.
  - Review on the Laplace transform of time functions.
  - Review on the inverse Laplace transform.
  - Review on the partial-fraction expansion.
- Introduce the concept of **transfer function**.
- Find the transfer function from a differential equation.
- Solve the differential equation using the transfer function.

Reading list

- Nise, *Control Systems Engineering* (6th Edition)[1]
- Ogata and Severo, *Engenharia de Controle Moderno* (3rd Edition)

[1] Today's lecture is mainly from Ch.2 of Nise.
Same concepts can be found in Ogata, Ch.3



Laplace transform

The LAPLACE TRANSFORM converts a function $f(t)$ in the TIME DOMAIN into the LAPLACE DOMAIN

$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty f(t)\, e^{-st}\, dt$$

is a complex number: $s = \sigma + j\omega$

↳ is designed to analyze a specific class of time domain signals impulse response that consists of sinusoidal and exponentials.

↳ investigate the time domain to identify the key features
→ the frequency of the sinusoids.
→ the decay constant of the exponential.

Laplace transform of time functions

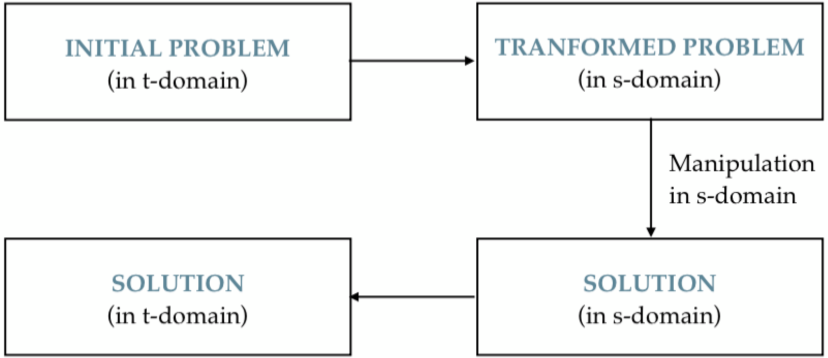


Table A–1 Laplace Transform Pairs

	$f(t)$	$F(s)$
1	Unit impulse $\delta(t)$	1
2	Unit step $1(t)$	$\frac{1}{s}$
3	t	$\frac{1}{s^2}$
4	$\frac{t^{n-1}}{(n-1)!}\quad (n=1,2,3,\dots)$	$\frac{1}{s^n}$
5	$t^n\quad (n=1,2,3,\dots)$	$\frac{n!}{s^{n+1}}$
6	e^{-at}	$\frac{1}{s+a}$
7	te^{-at}	$\frac{1}{(s+a)^2}$
8	$\frac{1}{(n-1)!}t^{n-1}e^{-at}\quad (n=1,2,3,\dots)$	$\frac{1}{(s+a)^n}$
9	$t^n e^{-at}\quad (n=1,2,3,\dots)$	$\frac{n!}{(s+a)^{n+1}}$
10	$\sin\omega t$	$\frac{\omega}{s^2+\omega^2}$
11	$\cos\omega t$	$\frac{s}{s^2+\omega^2}$
12	$\sinh\omega t$	$\frac{\omega}{s^2-\omega^2}$
13	$\cosh\omega t$	$\frac{s}{s^2-\omega^2}$
14	$\frac{1}{a}(1-e^{-at})$	$\frac{1}{s(s+a)}$
15	$\frac{1}{b-a}(e^{-at}-e^{-bt})$	$\frac{1}{(s+a)(s+b)}$
16	$\frac{1}{b-a}(be^{-bt}-ae^{-at})$	$\frac{s}{(s+a)(s+b)}$
17	$\frac{1}{ab}\left[1+\frac{1}{a-b}(be^{-at}-ae^{-bt})\right]$	$\frac{1}{s(s+a)(s+b)}$

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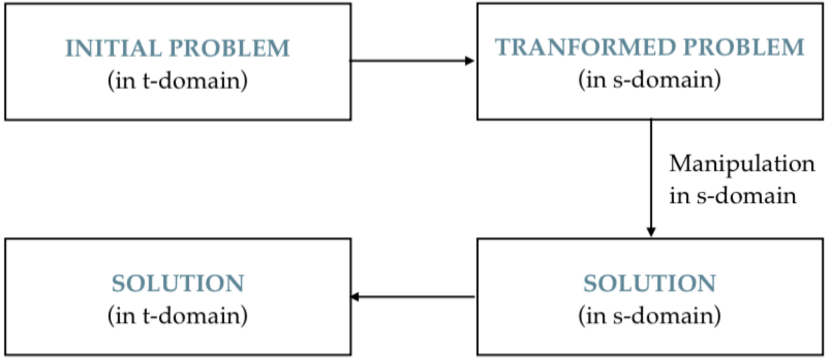
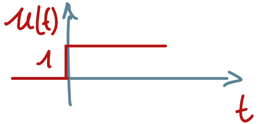
FULL TABLE on SIGAA !!

Laplace transform of time functions



Exercise L2E3: Using the Laplace transform, solve the following differential equation

- $y(t)$ if all initial conditions are zero
- $u(t)$ is a step function $\longrightarrow U(s) = 1/s$



$$\frac{d^2y(t)}{dt^2} + 12\frac{dy(t)}{dt} + 32y(t) = 32u(t)$$

$$s^2Y(s) + 12sY(s) + 32Y(s) = 32 \cdot \frac{1}{s} \qquad Y(s)\left(s^2+12s+32\right) = 32 \cdot \frac{1}{s}$$

$$Y(s) = \frac{32}{s(s^2+12s+32)} = \frac{32}{s(s+4)(s+8)} = \frac{K_1}{s} + \frac{K_2}{s+4} + \frac{K_3}{s+8}$$

$$K_1 = \cancel{s}\,\frac{32}{\cancel{s}(s+4)(s+8)}\bigg|_{s\to 0} = 1$$

$$K_2 = \cancel{(s+4)}\,\frac{32}{s\cancel{(s+4)}(s+8)}\bigg|_{s\to -4} = -2$$

$$K_3 = (s+8)\,\frac{32}{s(s+4)(s+8)}\bigg|_{s\to -8} = 1$$

$$Y(s) = \frac{1}{s} - \frac{2}{s+4} + \frac{1}{s+8} \qquad y(t) = 1 - 2e^{-4t} + e^{-8t}$$

Laplace transform of time functions

- We can apply partial-fraction expansion for solving differential equations
- We formulate the system block representation by
  - ⇝ establishing a **viable definition for a function that algebraically relates a system's output to its input**.
- This function will allow separation of the input, system, and output into three separate and distinct parts, unlike the differential equation.
- The function will also allow us to algebraically combine mathematical representations of subsystems to yield a total system representation.

Exercise L3E0: Find the inverse Laplace transform of

$$F(s) = \frac{3s^2+5s}{s^3+6s^2+11s+6} \quad (p_1 = -3)$$

$$\begin{array}{c|ccc|c} & 1 & 6 & 11 & 6 \\ -3 & & -3 & -9 & -6 \\ \hline & 1 & 3 & 2 & 0 \end{array}$$

$$(s+3)(s^2+3s+2)$$

$$F(s) = \frac{3s^2+5s}{s^3+6s^2+11s+6} = \frac{3s^2+5s}{(s+3)(s+2)(s+1)} = \frac{k_1}{s+3} + \frac{k_2}{s+2} + \frac{k_3}{s+1}$$

$$k_1 = (s+3)F(s)\Big|_{s\to-3} = \frac{3s^2+5s}{(s+2)(s+1)}\Big|_{s\to-3} = \frac{3\times(-3)^2+5(-3)}{(-3+2)(-3+1)} = 6$$

$$k_2 = (s+2)\cdot F(s)\Big|_{s\to-2} = \frac{3s^2+5s}{(s+3)(s+1)}\Big|_{s\to-2} = -2$$

$$k_3 = (s+1)\cdot F(s)\Big|_{s\to-1} = \frac{3s^2+5s}{(s+3)(s+2)}\Big|_{s\to-1} = -1$$

$$F(s) = \frac{6}{s+3} - \frac{2}{s+2} - \frac{1}{s+1}$$

$$f(t) = 6e^{-3t} - 2e^{-2t} - e^{-t}$$

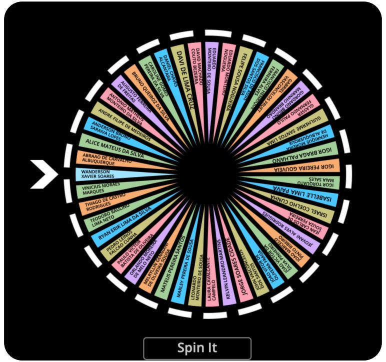
Laplace transform of time functions

Exercise L3E0: Find the inverse Laplace transform of

$$F(s) = \frac{s-6}{s^2(s+3)}$$

$$F(s) = \frac{20}{s(s^2+2s+5)}$$

$$F(s) = \frac{\text{[illegible]}}{s(s^2+2s+5)}$$



2. $F(s) = \frac{s-6}{s^2(s+3)} = \frac{K_1}{s^2} + \frac{K_2}{s} + \frac{K_3}{s+3}$

$K_1 = s^2 \cdot F(s)\Big|_{s\to 0} = \frac{s-6}{s+3}\Big|_{s\to 0} = -2$

$K_2 = \frac{d}{ds}\left(s^2 \cdot F(s)\right)\Big|_{s\to 0} = \frac{d}{ds}\left(\frac{s-6}{s+3}\right)\Big|_{s\to 0} = \frac{9}{(s+3)^2}\Big|_{s\to 0} = 1$

$K_3 = (s+3)\cdot F(s)\Big|_{s\to -3} = \frac{s-6}{s^2}\Big|_{s\to -3} = -1$

$F(s) = \frac{1}{s} - \frac{2}{s^2} - \frac{1}{s+3}$

$f(t) = 1 - 2t - e^{-3t}$

Laplace transform of time functions

Exercise L3E0: Find the inverse Laplace transform of

$$F(s) = \frac{20}{s(s^2+2s+5)}$$

$$F(s) = \frac{20}{s(s^2+2s+5)} = \frac{20}{s(s+1-j2)(s+1+j2)} = \frac{k_1}{s} + \frac{K}{s+1-j2} + \frac{K'}{s+1-j2}$$

$$K_1 = sF(s)\Big|_{s\to 0} = \frac{20}{s^2+2s+5}\Big|_{s\to 0} = 4$$

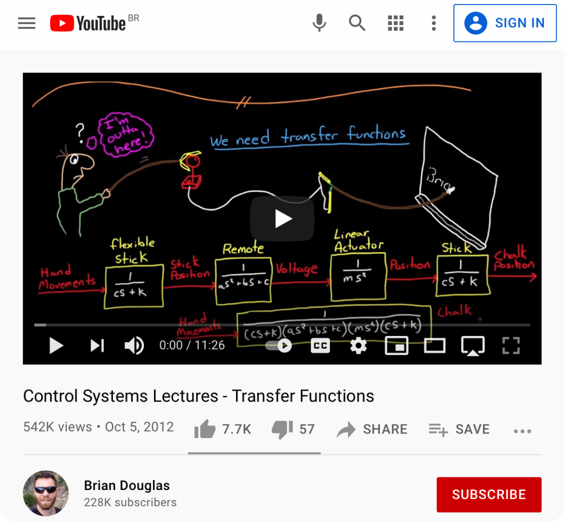
$$K = (s+1-j2)F(s)\Big|_{s\to -1+j2} = \frac{20}{s(s+1-j2)}\Big|_{s\to -1+j2} = \frac{20}{-8+j4} =$$

$$= \frac{20(-8+j4)}{(-8-j4)(-8+j4)} = \frac{1}{4}(-8+j4) = -2+j = u+jv$$

$$f(t) = 4 + Be^{\alpha t}\cos\omega t + C e^{\alpha t}\sin\omega t$$

$$B = 2u = -4 \qquad C = -2v = 2$$

Transfer functions



Transfer functions
Definition

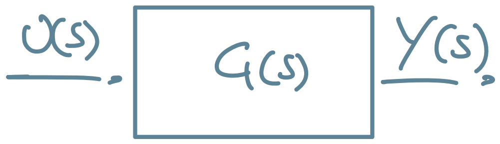
In control theory, functions called **transfer functions** are commonly used to

- Characterise the **input-output relationships** of systems that can be described by **linear, time-invariant, differential equations**.

The transfer function of a LTI system is defined as

- The ratio of the **Laplace transform of the output** (response function) to the **Laplace transform of the input** (driving function) under the assumption that **all initial conditions are zero**.

$$\text{Transfer function} = \frac{\text{Response function}}{\text{Input function}}$$



$Y(s) = G(s)\,U(s)$

LINEAR
TIME INVARIANT
SYSTEMS

LTI

Transfer functions
Definition

A general nth-order, linear, time-invariant differential equation is:

$$a_n \frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \ldots + a_0 y(t) = b_m \frac{d^m u(t)}{dt^m} + b_{m-1} \frac{d^{m-1} u(t)}{dt^{m-1}} + \ldots + b_0 u(t)$$

- $y(t)$ is the output and $u(t)$ is the input.
- The coefficients a_i and b_i and the form of the differential equation represent the system.

Taking the Laplace transform of both sides:

$$\begin{aligned} &a_n s^n Y(s) + a_{n-1} s^{n-1} Y(s) + \ldots + a_0 Y(s) + \text{initial conditions terms involving } y(t) \\ &= b_m s^m U(s) + b_{m-1} s^{m-1} U(s) + \ldots + b_0 U(s) + \text{initial conditions terms involving } u(t) \end{aligned}$$

we have a purely algebraic equation.

If we assume that **all initial conditions are zeros**:

$$\left(a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0\right) Y(s) = \left(b_m s^m + a_{m-1} s^{m-1} + \ldots + b_0\right) U(s)$$

Now form the ratio of the output transform, $Y(s)$, divided by the input transform, $U(s)$:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{\left(b_m s^m + a_{m-1} s^{m-1} + \ldots + b_0\right)}{\left(a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0\right)}$$

- We define the ratio $G(s)$ as the **transfer function** (TF) and evaluate it with **zero initial conditions**.

Transfer functions
Definition

The TF can be represented as a block diagram with the input on the left, the output on the right, and the system transfer function inside the block.

$$U(s) \longrightarrow \boxed{\dfrac{(b_m s^m + b_{m-1}s^{m-1} + \cdots + b_0)}{(a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0)}} \longrightarrow Y(s)$$

- Notice that the denominator of the transfer function is identical to the characteristic polynomial of the differential equation.

The output, $Y(s)$, can be found by using

$$Y(s) = G(s)U(s)$$

The **applicability** of the concept of the TF is limited to linear, time-invariant, differential equation systems. The transfer function approach, however, is extensively used in the analysis and design of such systems.

- The TF of a system is a **mathematical model**, expressing the relation between the output variable to the input variable.
- The TF is a property of a system itself, independent of the magnitude and nature of the input or driving function.
- The TF includes the units necessary to relate the input to the output; however, it does not provide **any information concerning the physical structure of the system**.
- If the TF of a system is known, **the output or response can be studied for various forms of inputs** with a view toward understanding the nature of the system.
- If the transfer function of a system is unknown, **it may be established experimentally by introducing known inputs and studying the output of the system**.

Transfer functions

Poles and zeros

It is often convenient to factor the polynomials in the numerator and denominator, and to write the transfer function in terms of those factors:

$$G(s) = \frac{N(s)}{D(s)} = K\frac{(s-z_1)(s-z_2)\dots(s-z_{m-1})(s-z_m)}{(s-p_1)(s-p_2)\dots(s-p_{n-1})(s-z_n)}$$

- $N(s)$ and $D(s)$ have real coefficients defined by the system's differential equation
  $\leadsto K = \dfrac{b_m}{a_n}$
- The z_i's are the **root of $N(s) = 0$** and are defined to be the system **zeros**.
- The p_i's are the **root of $D(s) = 0$** and are defined to be the system **poles**.

The poles and zeros are properties of the transfer function, and therefore of the differential equation describing the input-output system dynamics.

Together with the **gain constant** K, they completely characterise the differential equation, and provide a complete description of the system.

Exercises

Exercises

Exercise L3E1: It is required to

1. Find the transfer function represented by:

$$\frac{dy(t)}{dt} + 2y(t) = u(t)$$

2. Find the response, $y(t)$ to an unit step input, assuming zero initial conditions.

- In Matlab/Octave

```
num=1;
den=[1 2];

model=tf(num,den);
step(model);
```

- In Python

```
den = np.array([1.0, 2.0])
sys_1st = (1, den)
t, y = signal.step(sys_1st)
```

Exercises

Exercise L3E2: For a system with the following transfer function:

$$G(s) = \frac{s}{(s+4)(s+8)}$$

1. Find the impulse response of the system.
2. Find the ramp response of the system. $\rightsquigarrow U(s) = 1/s^2$

- In Matlab/Octave

```
% Construct the transfer function
num=[1 0];  den=[1 12 32]; G=tf(num,den);
% Impulse response
impulse(G)

% Construct the input ramp
t=0:0.1:10; alpha=1;
ramp=alpha*t;

% Simulate and plot the output
[y,t]=lsim(G,ramp,t);
figure; plot(t,y)
```

IMPULSE $U(s)=1$

STEP $U(s)=1/s$

RAMP $U(s)=1/s^2$

1. $Y(s) = G(s)U(s) = \frac{s}{(s+4)(s+8)} \cdot 1 = \frac{K_1}{s+4} + \frac{K_2}{s+8}$

$K_1 = (s+4)Y(s)\Big|_{s\to -4} = \frac{s}{s+8}\Big|_{s\to -4} = -1$

$K_2 = (s+8)Y(s)\Big|_{s\to -8} = \frac{s}{s+4}\Big|_{s\to -8} = 2$

$Y(s) = -\frac{1}{s+4} + \frac{2}{s+8}$

$y(t) = -e^{-4t} + 2e^{-8t}$

2. $Y(s) = \frac{s}{(s+4)(s+8)} \cdot \frac{1}{s^2} = \frac{K_1}{s} + \frac{K_2}{s+4} + \frac{K_3}{s+8}$

$y(t) = \frac{1}{32}e^{-t} - \frac{1}{16}e^{-4t} + \frac{1}{16}e^{-8t}$

Exercises

Exercise L3E2: For a system with the following transfer function:

- In Python

```
# Impulse response
num = np.array([1., 0.])
den = np.array([1.0, 12.0, 32.0])

system = (num, den)
t, y = signal.impulse(system)

# Plot
import matplotlib.pyplot as plt
plt.plot(t, y)

# Ramp response
import control as ctrl
import control.matlab as ctrlmatlab

t = np.arange(0,10,0.1)
alpha=1; ramp=alpha*t;

sys = ctrl.TransferFunction(num, den)
y_ramp = ctrlmatlab.lsim(sys,ramp,t)

plt.plot(t, y_ramp[0])
```

Exercises

Exercise L3E3: For each of the following transfer functions, write the corresponding differential equations.

a. $G(s) = \dfrac{7}{s^2+5s+10} = \dfrac{Y(s)}{X(s)}$

$Y(s)(s^2+5s+10) = 7X(s) \rightsquigarrow s^2Y(s)+5sY(s)+10Y(s) = 7X(s)$

$\ddot{y}(t)+5\dot{y}(t)+10y(t) = 7u(t)$

b. $G(s) = \dfrac{15}{(s+10)(s+11)}$

ToDo!!

c. $G(s) = \dfrac{s+3}{s^3+11s^2+12s+18}$

$$\boxed{\begin{aligned} &\mathcal{L}\left[\frac{df}{dt}\right] = sF(s) - f(0) \\ &\mathcal{L}\left[\frac{d^2f(t)}{dt}\right] = s^2F(s) - sF(0) - \frac{df(0)}{dt} \end{aligned}}$$

Exercises

Exercise L3E4: Consider the following model:

$$2\frac{d^2y(t)}{dt}+6\frac{dy(t)}{dt}+4y(t)=\frac{du(t)}{dt}+3u(t)$$

1. Find the free response of the system, assuming the following initial conditions:

$$y(t)\Big|_{t=0}=2, \quad \frac{dy(t)}{dt}\Big|_{t=0}=1$$

2. Find the transfer function for the system and its response to an input

$$u(t)=12e^{-4t}\delta(t)$$

$$2\left[s^2Y(s)-2sy_0-4\dot{y}_0\right]+$$

$$+6\left[sY(s)-y_0\right]+4Y(s)=sU(s)+3U(s)$$

$$(2s^2+6s+4)Y(s)-(2sy_0+2\dot{y}_0+6y_0)=(s+3)U(s)$$

$$Y(s)=\underbrace{\frac{2sy_0+6y_0+2\dot{y}_0}{2(s^2+3s+4)}}_{Y_{free}(s)}+\underbrace{\frac{s+3}{2(s^2+3s+4)}U(s)}_{Y_{forced}(s)}$$

[...]

Exercises

Exercise L3E5: Use Matlab (or Octave) and the Symbolic Math Toolbox to input and form LTI objects in polynomial and factored form for the following frequency functions:

a. $$G(s) = \frac{45(s+34.88)(s+26.83)(s+2.122)(s^2+1.17s+0.5964)}{(s+47)(s+39)(s+26.34)(s^2+0.6618s+0.5695)(s^2+2s+100)}$$

b. $$G(s) = \frac{56(s+47.72)(s+14)(s^2+1.276s+1.111)}{(s+87.62)(s+80.06)(s+54.59)(s+1.411)(s+0.3766)(s^2+0.9391s+0.8119)}$$

```
syms s
Ga=45*((s^2+37*s+74)*(s^3+28*s^2+32*s+16))...
    /((s+39)*(s+47)*(s^2+2*s+100)*(s^3+27*s^2+18*s+15));
[numga,denga]=numden(Ga);
numga=sym2poly(numga);
denga=sym2poly(denga);

Ga=tf(numga,denga)     % Ga polynomial
Ga=zpk(Ga)             % Ga factored

% Similar solution for (b)
```