Atividade 06 - Pedro José Garcia 11846943

1)

2-a) Seja $f(x)=(x^3-1)^2$, com $x \in [0,1]$. Vamos aproximar $f(x)$ por uma reta, ou seja, $f(x) \approx \alpha_0^* + \alpha_1^* x = P_1(x)$; com $P_1 = 0$, temos que $\alpha_0^* + \alpha_1^* \cdot 1 = 0$, $\alpha_0^* = -\alpha_1^*$, logo $P_1(x) = -\alpha_1^* + \alpha_1^* x = \alpha_1^*(x-1)$, então $f(x) \approx \alpha_1^*(x-1) = P_1(x)$, considerando o produto escalar $\langle f,g \rangle = \int_0^1 f(x)g(x)dx$, logo, para determinar P_1 que mais se aproxima de f no sentido de mínimos quadrados, vamos calcular a matriz A e o vetor B, na sequência, resolveremos $A\alpha^* = b$, então $A = (\langle \varphi_0, \varphi_0 \rangle) = (\langle (x-1),(x-1) \rangle)$, $\langle (x-1),(x-1) \rangle = \int_0^1 (x-1)(x-1)dx = \int_0^1 (x^2-2x+1)dx = \int_0^1 x^2 dx - 2\int_0^1 x\,dx + 1 = \frac{x^3}{3}\Big|_0^1 - 2\cdot\frac{x^2}{2}\Big|_0^1 + 1 = \frac{1}{3} - 1 + 1 = \frac{1}{3}$, e $b = [\langle f, \varphi_0 \rangle] = [\langle f,(x-1) \rangle]$, $\langle f,(x-1) \rangle = \langle (x^3-1)^2,(x-1) \rangle = \int_0^1 (x^3-1)^2 \cdot (x-1)dx = \int_0^1 (x^6-2x^3+1)(x-1)dx = \int_0^1 (x^7-2x^4+x-x^6+2x^3-1)dx = \int_0^1 x^7 dx - \int_0^1 x^6 dx - 2\int_0^1 x^4 dx + 2\int_0^1 x^3 dx + \int_0^1 x\,dx - \int_0^1 1\,dx = \frac{x^8}{8}\Big|_0^1 - \frac{x^7}{7}\Big|_0^1 - 2\cdot\frac{x^5}{5}\Big|_0^1 + 2\frac{x^4}{4}\Big|_0^1 + \frac{x^2}{2}\Big|_0^1 - 1 = \frac{1}{8} - \frac{1}{7} - \frac{2}{5} + \frac{1}{2} + \frac{1}{2} - 1 = \frac{1}{8} - \frac{1}{7} - \frac{2}{5} + 1 - 1 = \frac{1}{8} - \frac{1}{7} - \frac{2}{5} = \frac{35-40-112}{280} = \frac{117}{280}$, logo $\left(\frac{1}{3}\right) \cdot [\alpha_1^*] = \frac{117}{280}$, logo $\frac{1}{3} \cdot \alpha_1^* = \frac{117}{280}$, logo $\alpha_1^* = \frac{351}{280}$, portanto, $P_1(x) = \frac{351}{280}(x-1)$.

2-b) Seja $f(x)=(x^3-1)^2$, com $x \in [0,1]$. Vamos aproximar $f(x)$ por uma parábola, isto é, $f(x) \approx \alpha_0^* + \alpha_1^* x + \alpha_2^* x^2 = P_2(x)$, considerando o produto escalar $\langle f,g\rangle = \int_0^1 f(x)g(x)\,dx$, logo, para determinar P_2 que mais se aproxima de f no sentido de mínimos quadrados, vamos calcular a matriz A e o vetor b, na sequência, resolveremos $A\alpha^*=b$, então

$$A=\begin{pmatrix} \langle \phi_0,\phi_0\rangle & \langle \phi_0,\phi_1\rangle & \langle \phi_0,\phi_2\rangle \\ \langle \phi_1,\phi_0\rangle & \langle \phi_1,\phi_1\rangle & \langle \phi_1,\phi_2\rangle \\ \langle \phi_2,\phi_0\rangle & \langle \phi_2,\phi_1\rangle & \langle \phi_2,\phi_2\rangle \end{pmatrix} =$$

$$\begin{pmatrix} \langle 1,1\rangle & \langle 1,x\rangle & \langle 1,x^2\rangle \\ \langle x,1\rangle & \langle x,x\rangle & \langle x,x^2\rangle \\ \langle x^2,1\rangle & \langle x^2,x\rangle & \langle x^2,x^2\rangle \end{pmatrix}$$

, sabemos que $\langle 1,x\rangle=\langle x,1\rangle$, $\langle 1,x^2\rangle=\langle x^2,1\rangle$ e $\langle x,x^2\rangle=\langle x^2,x\rangle$, então $\langle 1,1\rangle=\int_0^1 1.1\,dx=\int_0^1 1\,dx= x\Big|_0^1=1$, $\langle 1,x\rangle=\int_0^1 1.x\,dx=\int_0^1 x\,dx=\frac{x^2}{2}\Big|_0^1=\frac{1}{2}$, $\langle 1,x^2\rangle=\int_0^1 1.x^2\,dx=\int_0^1 x^2\,dx=\frac{x^3}{3}\Big|_0^1=\frac{1}{3}$, $\langle x,x\rangle=\int_0^1 x.x\,dx=\int_0^1 x^2\,dx=\frac{x^3}{3}\Big|_0^1=\frac{1}{3}$, $\langle x,x^2\rangle=\int_0^1 x.x^2\,dx=\int_0^1 x^3\,dx=\frac{x^4}{4}\Big|_0^1=\frac{1}{4}$ e $\langle x^2,x^2\rangle=\int_0^1 x^2.x^2\,dx=\int_0^1 x^4\,dx=\frac{x^5}{5}\Big|_0^1=\frac{1}{5}$ e

$b=\begin{bmatrix} \langle 1,f\rangle \\ \langle x,f\rangle \\ \langle x^2,f\rangle \end{bmatrix}$, $\langle 1,f\rangle=\langle 1,(x^3-1)^2\rangle=\int_0^1 1.(x^3-1)^2\,dx=\int_0^1 (x^3-1)^2\,dx=\int_0^1 (x^6-2x^3+1)\,dx=\int_0^1 x^6\,dx-2\int_0^1 x^3\,dx+1=\frac{x^7}{7}\Big|_0^1-2\frac{x^4}{4}\Big|_0^1+1=\frac{1}{7}-\frac{1}{2}+1=\frac{2-7+14}{14}=\frac{9}{14}$, $\langle x,f\rangle=\langle x,(x^3-1)^2\rangle=\int_0^1 x.(x^3-1)^2\,dx=\int_0^1 x(x^6-2x^3+1)\,dx=\int_0^1 (x^7-2x^4+x)\,dx=\int_0^1 x^7\,dx-2\int_0^1 x^4\,dx+\int_0^1 x\,dx=\frac{x^8}{8}\Big|_0^1-2.\frac{x^5}{5}\Big|_0^1+\frac{x^2}{2}\Big|_0^1=\frac{1}{8}-\frac{2}{5}+\frac{1}{2}=\frac{5-16+20}{40}=\frac{9}{40}$ e $\langle x^2,f\rangle=\langle x^2,(x^3-1)^2\rangle=\int_0^1 x^2.(x^3-1)^2\,dx=\int_0^1 x^2(x^6-2x^3+1)\,dx=\int_0^1 (x^8-2x^5+x^2)\,dx=\int_0^1 x^8\,dx-2\int_0^1 x^5\,dx+\int_0^1 x^2\,dx=\frac{x^9}{9}\Big|_0^1-2\frac{x^6}{6}\Big|_0^1+\frac{x^3}{3}\Big|_0^1=\frac{1}{9}-\frac{1}{3}+\frac{1}{3}=\frac{1}{9}$, logo

$$\begin{pmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{pmatrix} \cdot \begin{bmatrix} \alpha_0^* \\ \alpha_1^* \\ \alpha_2^* \end{bmatrix} = \begin{bmatrix} 9/14 \\ 9/40 \\ 1/9 \end{bmatrix}$$

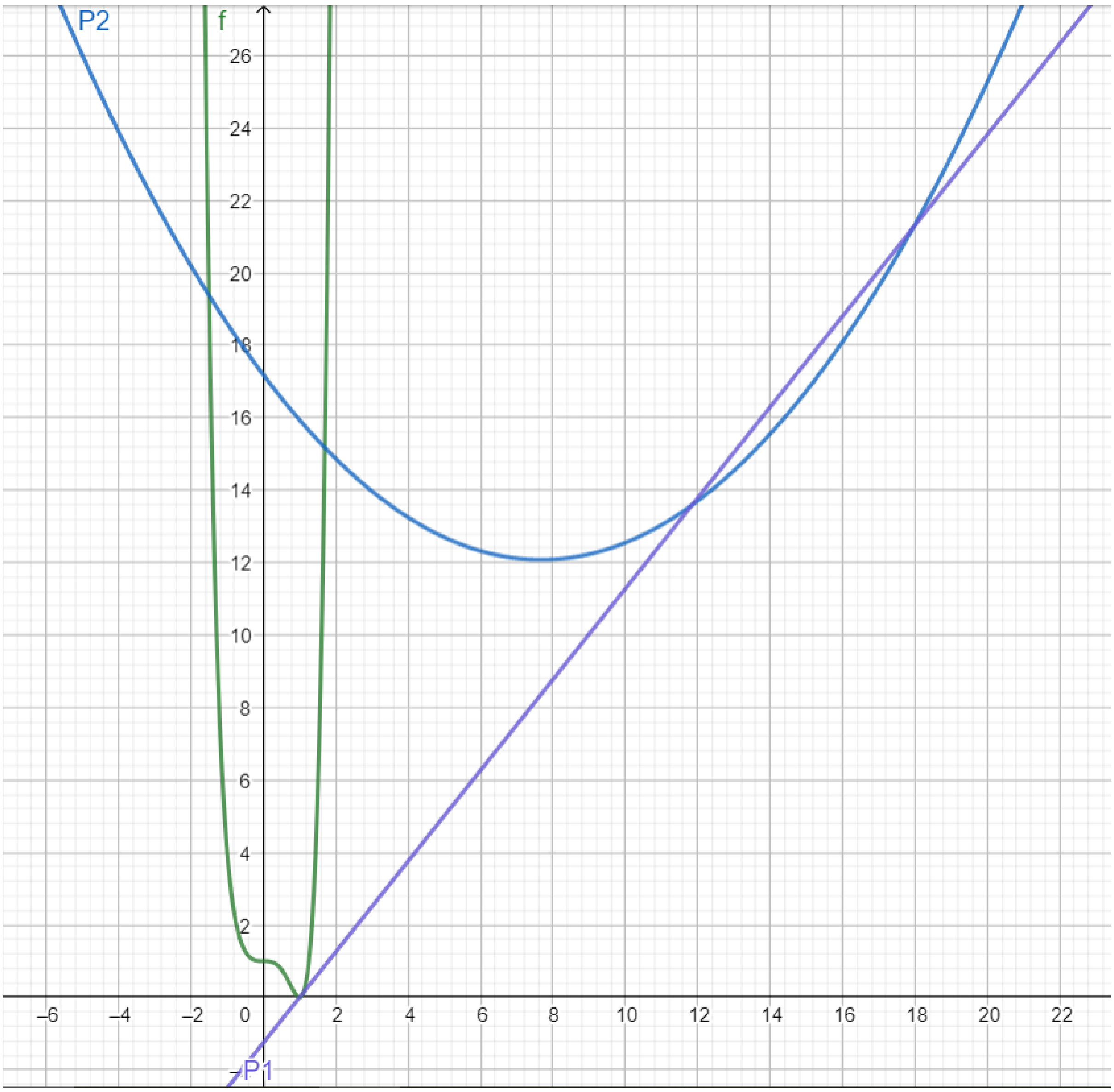
, portanto, $\alpha_0^*+\frac{\alpha_1^*}{2}+\frac{\alpha_2^*}{3}=\frac{9}{14}$, $\frac{\alpha_0^*}{2}+\frac{\alpha_1^*}{3}+\frac{\alpha_2^*}{4}=\frac{9}{40}$ e $\frac{\alpha_0^*}{3}+\frac{\alpha_1^*}{4}+\frac{\alpha_2^*}{5}=\frac{1}{9}$, então temos o sistema:

$$\begin{cases} \alpha_0^*+\frac{\alpha_1^*}{2}+\frac{\alpha_2^*}{3}=\frac{9}{14} \\ \frac{\alpha_0^*}{2}+\frac{\alpha_1^*}{3}+\frac{\alpha_2^*}{4}=\frac{9}{40} \\ \frac{\alpha_0^*}{3}+\frac{\alpha_1^*}{4}+\frac{\alpha_2^*}{5}=\frac{1}{9} \end{cases}$$

, resolvendo ele, temos que $\alpha_0^*=\frac{9}{14}-\frac{\alpha_1^*}{2}-\frac{\alpha_2^*}{3}=\frac{27-21\alpha_1^*-14\alpha_2^*}{42}$, substituindo nas equações, temos que $\frac{27-21\alpha_1^*-14\alpha_2^*}{84}+\frac{\alpha_1^*}{3}+\frac{\alpha_2^*}{4}=\frac{9}{40}$, $\frac{27-21\alpha_1^*-14\alpha_2^*+28\alpha_1^*+21\alpha_2^*}{84}=\frac{9}{40}$, $\frac{27+7\alpha_1^*+14\alpha_2^*}{84}=\frac{9}{40}$, $7\alpha_1^*+14\alpha_2^*=\frac{-81}{10}$ e $\frac{27-21\alpha_1^*-14\alpha_2^*}{126}+\frac{\alpha_1^*}{4}+\frac{\alpha_2^*}{5}=\frac{1}{9}$, $\frac{270-210\alpha_1^*-140\alpha_2^*+315\alpha_1^*+252\alpha_2^*}{1260}=\frac{1}{9}$, $105\alpha_1^*+112\alpha_2^*=-130$, logo temos o sistema

$$\begin{cases} 7\alpha_1^*+14\alpha_2^*=-\frac{81}{10} \\ 105\alpha_1^*+112\alpha_2^*=-130 \end{cases}$$

, resolvendo ele, temos que $7\alpha_1^*=-\frac{81}{10}-14\alpha_2^*$, $7\alpha_1^*=\frac{-81-140\alpha_2^*}{10}$, $\alpha_1^*=\frac{-81-140\alpha_2^*}{70}$, substituindo, temos $105.\left(\frac{-81-140\alpha_2^*}{70}\right)+112\alpha_2^*=-130$, $\frac{-243-420\alpha_2^*}{2}+112\alpha_2^*=-130$, $-243-420\alpha_2^*+224\alpha_2^*=-260=-196\alpha_2^*=-17$, $\alpha_2^*=\frac{17}{196}$, então $7\alpha_1^* + 14\left(\frac{17}{196}\right) = -\frac{81}{10}$, $7\alpha_1^* = -\frac{81}{10} - \frac{17}{14}$, $7\alpha_1^* = \frac{-567-85}{70}$, $\alpha_1^* = \frac{-652}{490} = -\frac{326}{245}$, então $\alpha_0^* = \left(\frac{27}{42}\right)\cdot\left[-21\left(\frac{-326}{245}\right) - 14\left(\frac{17}{196}\right)\right] = \left(\frac{27}{42}\right)\cdot\left(\frac{6846}{245} - \frac{17}{14}\right) = \left(\frac{27}{42}\right)\left(\frac{95844-4165}{3430}\right) = \left(\frac{27}{42}\right)\left(\frac{91679}{3430}\right) = \left(\frac{3.3.3}{3.2.7}\right)\left(\frac{7.7.1871}{7.7.7.10}\right) = \left(\frac{9}{14}\right)\left(\frac{1871}{70}\right) = \frac{16839}{980}$, $\alpha_0^* = \frac{16839}{980}$, portanto, $P_2(x) = \frac{16839}{980} - \frac{326x}{245} + \frac{17x^2}{196}$. Observando o gráfico, podemos concluir que $P_1(x)$ melhor aproxima $f(x)$ no intervalo dado, uma vez que, o seu gráfico está mais próximo do gráfico de $f(x)$ nesse intervalo.



2)

4-a) Seja $f(x)$ uma função discreta dada pela tabela:

x	-2	-1	1	2
$f(x)$	1	-3	1	9

, vamos aproximar no sentido de mínimos quadrados, a função $f(x)$ pela função $g_1(x)$, ou seja, $f(x) \approx ax^2+bx = g_1(x)$, então $m=3$ e $n=2$, logo $u_0=\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$, $u_1=\begin{bmatrix}-2\\-1\\1\\2\end{bmatrix}$ e $u_2=\begin{bmatrix}4\\1\\1\\4\end{bmatrix}$,

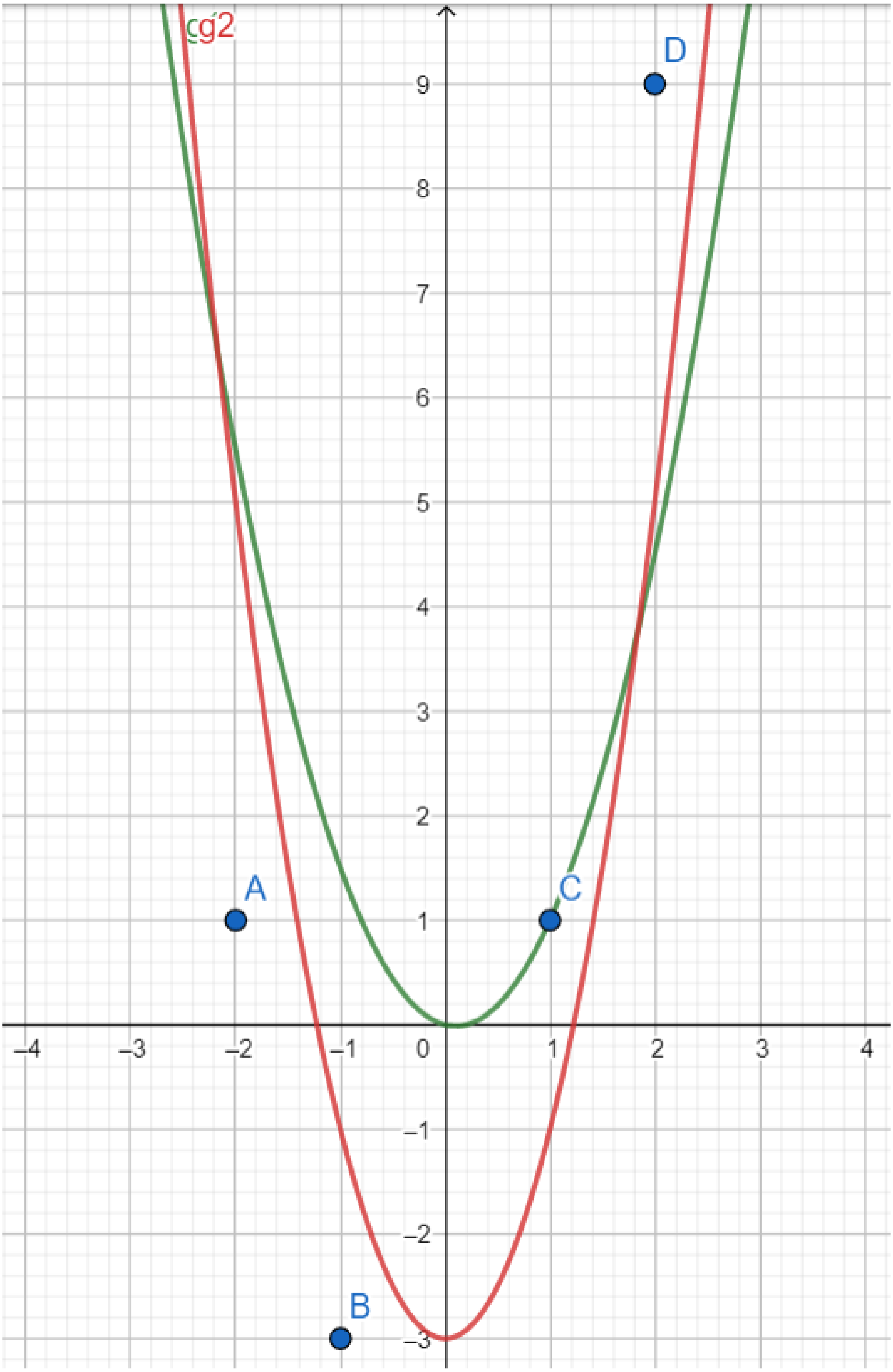
agora vamos calcular a projeção ortogonal de $y=(1,-3,1,9)^T$, no subespaço gerado por u, para isso devemos montar e resolver o sistema, porém, antes precisamos calcular os elementos da matriz A e do vetor b, na sequência, calcular $A\alpha^*=b$, vamos calcular a matriz $A=\begin{pmatrix}\langle u_1,u_1\rangle & \langle u_1,u_2\rangle \\ \langle u_2,u_1\rangle & \langle u_2,u_2\rangle\end{pmatrix}$, $\langle u_1,u_1\rangle = \sum_{k=0}^{3} x_k x_k = \sum_{k=0}^{3} x_k^2 = 4+1+1+4=10$, $\langle u_1,u_2\rangle = \langle u_2,u_1\rangle = \sum_{k=0}^{3} x_k x_k^2 = \sum_{k=0}^{3} x_k^3 = 8+1+1+8=18$, $\langle u_2,u_2\rangle = \sum_{k=0}^{3} x_k^2 x_k^2 = \sum_{k=0}^{3} x_k^4 = 16+1+1+16=34$, e o vetor $b=\begin{bmatrix}\langle u_1,y\rangle \\ \langle u_2,y\rangle\end{bmatrix}$ $\langle u_1,y\rangle = \sum_{k=0}^{3} x_k y_k = (-2)(1)+(-1)(-3)+(1)(1)+(2)(9) = -2+3+1+18=20$ e $\langle u_2,y\rangle = \sum_{k=0}^{3} x_k^2 y_k = (4)(1)+(1)(-3)+(1)(1)+(4)(9) = 4-3+1+36=38$, logo $\begin{pmatrix}10 & 18\\18 & 34\end{pmatrix}\cdot\begin{bmatrix}\alpha_1^*\\ \alpha_2^*\end{bmatrix}=\begin{bmatrix}20\\38\end{bmatrix}$, logo

$10\alpha_1^*+18\alpha_2^*=20$, $5\alpha_1^*+9\alpha_2^*=10$ e $18\alpha_1^*+34\alpha_2^*=38$, $9\alpha_1^*+17\alpha_2^*=19$, então temos o sistema: $\begin{cases}5\alpha_1^*+9\alpha_2^*=10\\ 9\alpha_1^*+17\alpha_2^*=19\end{cases}$, resolvendo ele, temos que $\alpha_1^*=\frac{10-9\alpha_2^*}{5}$, substituindo, $9\left(\frac{10-9\alpha_2^*}{5}\right)+17\alpha_2^*=19$, $\frac{90-81\alpha_2^*}{5}+17\alpha_2^*=19$, $90-81\alpha_2^*+85\alpha_2^*=95$, $4\alpha_2^*=5$, $\alpha_2^*=\frac{5}{4}$, logo $\alpha_1^*=\frac{10-9\left(\frac{5}{4}\right)}{5}=\frac{40-45}{20}=-\frac{5}{20}=-\frac{1}{4}$,

portanto, $g_1(x)=\frac{5}{4}x^2-\frac{x}{4}$. Agora, vamos aproximar no sentido de mínimos quadrados, a função $f(x)$ pela função $g_2(x)$, ou seja, $f(x) \simeq cx^2+d = g_2(x)$, então $n=3$ e $m=2$, logo $u_0=\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$, $u_1=\begin{bmatrix}-2\\-1\\1\\2\end{bmatrix}$ e $u_2=\begin{bmatrix}4\\1\\1\\4\end{bmatrix}$,

agora vamos calcular a projeção ortogonal de $y=(1,-3,1,9)^T$, no subespaço gerado por u, para isso devemos montar e resolver o sistema, porém, antes precisamos calcular os elementos da matriz A e do vetor b, na sequência, calcular $Ax^*=b$, vamos calcular a matriz $A=\begin{pmatrix}\langle u_0,u_0\rangle & \langle u_2,u_0\rangle\\ \langle u_0,u_2\rangle & \langle u_2,u_2\rangle\end{pmatrix}$, $\langle u_0,u_0\rangle=\sum_{k=0}^{3}1=4$, $\langle u_2,u_0\rangle=\langle u_0,u_2\rangle=\sum_{k=0}^{3}1.x_k^2=4+1+1+4=10$ e $\langle u_2,u_2\rangle=\sum_{k=0}^{3}x_k^2.x_k^2=\sum_{k=0}^{3}x_k^4=(4)^2+1+1+(4)^2=16+1+1+16=34$, e o vetor $b=\begin{bmatrix}\langle u_0,y\rangle\\ \langle u_2,y\rangle\end{bmatrix}$, $\langle u_0,y\rangle=\sum_{k=0}^{3}1.y_k=1-3+1+9=8$ e $\langle u_2,y\rangle=\sum_{k=0}^{3}x_k^2y_k=(4)(1)+(1)(-3)+(1)(1)+(4)(9)=4-3+1+36=38$, logo $\begin{pmatrix}4 & 10\\10 & 34\end{pmatrix}\cdot\begin{bmatrix}\alpha_3^*\\ \alpha_4^*\end{bmatrix}=\begin{pmatrix}8\\38\end{pmatrix}$, logo $4\alpha_3^*+10\alpha_4^*=8$, $2\alpha_3^*+5\alpha_4^*=4$ e $10\alpha_3^*+34\alpha_4^*=38$, $5\alpha_3^*+17\alpha_4^*=19$, então temos o sistema: $\begin{cases}2\alpha_3^*+5\alpha_4^*=4\\5\alpha_3^*+17\alpha_4^*=19\end{cases}$, resolvendo ele, temos que: $\alpha_3^*=\frac{4-5\alpha_4^*}{2}$, substituindo: $5\left(\frac{4-5\alpha_4^*}{2}\right)+17\alpha_4^*=19$, $\frac{20-25\alpha_4^*}{2}+17\alpha_4^*=19$, $20-25\alpha_4^*+34\alpha_4^*=38$, $9\alpha_4^*=18$, $\alpha_4^*=2$, logo $\alpha_3^*=\frac{4-10}{2}=\frac{-6}{2}=-3$, portanto, $g_2(x)=2x^2-3$.

4-b) Vamos calcular o erro de truncamento para $g_1(x)$, então $Q_1=\|f-g_1\|^2=\langle f-g_1,f-g_1\rangle=\sum_{k=0}^{3}(y_k-g_1(x_k))^2=\left(1-\left(\frac{9}{2}\right)\right)^2+(-3-(16))^2+(1-(1))^2+(9-(99))^2=\left(-\frac{7}{2}\right)^2+(-19)^2+0+(-90)^2=\frac{49}{4}+361+8100=8.473,25$, agora, vamos calcular o erro de truncamento para $g_2(x)$, então $Q_2=\|f-g_2\|^2=\langle f-g_2,f-g_2\rangle=\sum_{k=0}^{3}(y_k-g_2(x_k))^2=(1-(5))^2+(-3-(-1))^2+(1-(-1))^2+(9-(5))^2=16+4+4+16=40$, portanto, como $Q_2<Q_1$, a função que fornece o melhor ajuste segundo o critério dos mínimos quadrados é a função $g_2(x)$.



Verde - g1
Vermelho - g2
Azul - Pontos da tabela