

# Dynamic Mechanism Collapse: A Boundary Characterization

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## Abstract

When are dynamics valuable? In Bayesian environments with public signals and no intertemporal commitment, we study a seller who allocates an economically single-shot resource over time. We provide necessary and sufficient conditions under which the optimal dynamic mechanism collapses to a simple terminal design: a single public experiment at date 0 followed by a posterior-dependent static mechanism executed at a deterministic date, with no further disclosure. The key condition is the existence of a global affine shadow value that supports the posterior-based revenue frontier and uniformly bounds all history-dependent revenues. When this condition fails, a collapse statistic pinpoints the dates and public state variables that generate genuine dynamic value. The characterization combines martingale concavification on the belief space with an affine-support duality for concave envelopes.

*Keywords:* dynamic mechanism design; concavification; martingales; terminal mechanisms; boundary characterization. *JEL:* D44, D82, D83, C73.

## 1 Introduction

Dynamic mechanisms let a designer interact with agents over time, disclose information gradually, and condition allocations on public histories. Sometimes this dynamic flexibility strictly raises the attainable value; in other cases, the best dynamic mechanism is ex-ante equivalent to a much simpler static (e.g. [Bolton and Dewatripont, 2005](#)). This contrast motivates a central question: when are dynamics valuable? More precisely, in a general Bayesian environment with public histories, when does conditioning mechanisms and information policies on

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the path of public signals strictly expand the designer's ex-ante value, and when can the full dynamic mechanism be collapsed into a static design?

A simple intuition comes from a seller facing a fluctuating cost of supplying a single unit over time. The seller can post a price in each period but only wishes to trade when the cost is low. In an optimal policy, trade is concentrated in low-cost states, while high-cost states are made effectively irrelevant by prohibitively high prices. From an ex-ante perspective, the rich dynamic environment then behaves as if the seller simply waited for low costs and ran a single static mechanism.

We study this question by working in a dynamic Bayesian environment with public histories. Time is discrete, and in each period the designer (seller) faces a set of potential buyers for a economically single-shot resource: it can be allocated at most once and no further surplus can be generated after allocation. Uncertainty is represented by a probability space, and at each date  $t$ , the seller announces a public information policy of public histories  $(\mathcal{F}_t)_{t \in \mathcal{T}}$  and a submechanism  $\Gamma_t$  that could be executed and that is drawn from a given submechanism space  $\mathcal{H}_t$ . The collection of information and submechanisms  $\mathcal{M} = \{M_t = (\mathcal{F}_t, \Gamma_t)\}_{t \in \mathcal{T}}$  is called the mechanism calendar. Buyers play a Bayesian game in each submechanism, and the seller's payoff is summarized by an analytic revenue correspondence  $\mathcal{R}_t[\mathcal{F}_t, \Gamma_t]$  for each  $t$ , which is solely determined by the past history and the current feasible space of the mechanism.

Our standing assumptions isolate the source of dynamic value we study. We work without intertemporal commitment ([Assumption 1](#)); past announcements about future submechanisms are not binding. The resource is economically single-shot ([Assumption 2](#)): it can be allocated at most once and no further surplus can be generated thereafter. These restrictions remove dynamic gains from long-term commitment or repeated trade and leave information and timing as the only intertemporal margins. Technically, to handle the infinite-dimensional belief space, we adopt the standard assumption of value regularity ([Assumption 3](#)). Otherwise, the so-called equivalent conditions degenerate into purely sufficient conditions.

We approach the problem by working on the belief space. At each date  $t$ , the public history can be written as  $\mathcal{F}_t = \sigma(S_t, Y_t)$ , where  $S_t \in E = \Delta(\Theta)$  is the public posterior and  $Y_t$  is a public state. For any public  $\sigma$ -algebra  $\mathcal{G}_t$  with  $\sigma(S_t) \subseteq \mathcal{G}_t \subseteq \mathcal{F}_t$ , we define a reduced-form value  $U_t^{\mathcal{G}_t}(s)$  as the maximal revenue attainable at belief  $s$  with  $\mathcal{G}_t$ -measurable submechanisms. The posterior-based benchmark  $g(s) := \sup_{t \in \mathcal{T}} U_t^{\sigma(S_t)}(s)$  and the history-based benchmark  $\hat{g}(s) := \sup_{t \in \mathcal{T}} U_t^{\mathcal{F}_t}(s)$  then summarize, belief by belief, the best values when mechanisms may condition only on posteriors or on full public histories. We show that, under our assumptions, the ex-ante value of any mechanism calendar is given by the concave envelope of  $g$  or  $\hat{g}$  on the belief space, in the spirit of concavification arguments in Bayesian persuasion and information design (e.g. [Kamenica and Gentzkow, 2011](#); [Bergemann and Morris, 2016](#)), and that the incremental value of dynamics is fully captured by whether

$(\text{conc } \widehat{g})(S_0)$  strictly exceeds  $(\text{conc } g)(S_0)$  at the prior  $S_0$ .

Our main results link the value comparison between dynamic and static designs, and the structural form of optimal mechanisms, to a single condition on the belief space. The first is an ex-ante no-gain result ([Theorem 1](#)). For a given mechanism calendar and associated information histories  $\{\mathcal{G}_t\}$ , we show that richer public histories generate no additional value at the prior if and only if there exists, for every  $\varepsilon > 0$ , a continuous affine “shadow value” on the belief space that ( $\varepsilon$ -approximately) supports the posterior-based benchmark  $g$  at  $S_0$  and uniformly dominates all date-wise values  $U_t^{\mathcal{G}_t}$ . Equivalently, dynamic gains arise only when history-dependent values force the concave envelope of  $\widehat{g}$  to lie strictly above that of  $g$  at the prior. The second result is a structural collapse theorem ([Theorem 2](#)), which shows that a mechanism calendar collapses if and only if the global affine-support condition ( $A''_\varepsilon$ ) holds. Under this condition, any seller-optimal mechanism calendar is ex-ante equivalent to a *terminal* design that conducts a single information experiment on the posterior space at date 0 and, at each realized posterior, chooses a static submechanism and a deterministic execution date with no further public disclosure or strategic interaction.

These characterizations admit a straightforward economic interpretation. When the global affine-support condition holds, all non-posterior public histories are redundant: from the designer’s perspective, the mechanism calendar can be replaced by a terminal, belief-based design that conditions only on posteriors. Dynamic choices of labels, promised utilities, or reputations may raise date-wise revenues at some beliefs, but an optimal information policy from  $S_0$  never finds it worthwhile to steer beliefs into those regions. Conversely, whenever  $(\text{conc } \widehat{g})(S_0) > (\text{conc } g)(S_0)$ , our diagnostic identifies beliefs and dates at which the full public state  $(S_t, Y_t)$  attains belief–revenue pairs in a non-posterior gain region: these are exactly the histories at which continuation states beyond the posterior are essential for reaching the dynamic revenue frontier.

There is a long tradition in dynamic mechanism and contract theory of asking when dynamic interactions add content beyond what can be captured by a suitable static model. Survey treatments such as [Bolton and Dewatripont \(2005\)](#) emphasize that in some environments dynamics change the feasible allocations or values, while in others an appropriate choice of state variables makes the problem effectively static. This strands obtain “no value of dynamics” results in specific settings. In robust mechanism design with minimax objectives and i.i.d. rounds, in certain environments, optimal dynamic mechanisms can perform no better than repeating an optimal static mechanism (e.g. [Balseiro et al., 2021](#)); here the conclusion is driven by worst-case reasoning under independence. In sequential screening, dynamic contracts can often be reduced to a static screening problem with an enlarged type space (e.g. [Krähmer and Strausz, 2017](#)), by encoding the payoff-relevant evolution of information in an augmented state that serves as the type in the corresponding static problem. Methodologically, our anal-

ysis is close in spirit to this state-compression perspective: we also separate payoff-relevant beliefs from other public contingencies and ask when a reduced state description suffices. Our approach differs in scope and in the object of study. We work in a Bayesian environment with general type spaces, and for each date  $t$  we take as given a set of feasible submechanisms and an associated revenue correspondence; the mechanism technology at  $t$  is arbitrary but fixed, and dynamic value comes from how this technology is conditioned on public histories. We decompose each public history as  $\mathcal{F}_t = \sigma(S_t, Y_t)$  into a posterior  $S_t$  and a non-posterior public state  $Y_t$ , and compare posterior-only and history-dependent values through the functions  $g$  and  $\hat{g}$ . Our collapse result shows that the global affine-support condition  $(A''_\epsilon)$  is equivalent to the following property: from the prior  $S_0$ , any mechanism calendar can be replaced, without ex-ante loss, by a terminal posterior-based design.

In addition to these global characterizations, we introduce a collapse statistic that localizes dynamic value at the level of public variables and dates. This statistic ([Proposition 1](#)) generates a lattice that is downward- and meet-closed but not generally join-closed, highlighting that public information labels that are individually harmless may become value-relevant in combination. [Proposition 2](#) characterizes date-wise collapse: a mechanism collapses on date  $t$  if and only if every non-posterior public variable is collapsible at  $t$ , and [Proposition 3](#) shows that calendar-wide collapse is sufficient for a global no-gain condition on all information histories. To detect strictly positive non-posterior dynamic value, we define a non-posterior gain region  $\mathcal{B}_\epsilon(S_0)$ , then [Proposition 4](#) implies that collapsed calendars never hit this region, whereas any other calendar must hit it at some belief and date. These auxiliary results provide variable-level and date-level diagnostics.

On the technical side, the main challenge is to work on a general posterior space  $E = \Delta(\Theta)$ , where  $\Theta$  may be infinite and the belief-based value functions  $g$  and  $\hat{g}$  need not be continuous. Standard finite-state constructions are therefore not available. We handle this in three steps. First, starting from the public histories  $(\mathcal{F}_t)_{t \in \mathcal{T}}$  and the analytic revenue correspondence  $\mathcal{R}_t[\mathcal{F}_t, \Gamma_t]$ , we represent any mechanism calendar by a family of belief-based value functions  $\{U_t^\mathcal{G}\}$  on  $E$ , indexed by intermediate public  $\sigma$ -algebras  $\mathcal{G}$  with  $\sigma(S_t) \subseteq \mathcal{G} \subseteq \mathcal{F}_t$ . Under the analytic-graph assumption, these reduced-form values  $U_t^\mathcal{G}$  admit versions that are bounded and upper semianalytic; this follows from standard results on analytic sets, measurable correspondences, and upper semianalytic functions. (see, for example, [Castaing and Valadier, 1977](#); [Bertsekas and Shreve, 1978](#); [Kechris, 1995](#)) Second, we apply a martingale concavification for the posterior process  $\{S_t\}$  and obtain a value representation: the ex-ante value of any calendar equals the concave envelope of  $g$  or  $\hat{g}$  on  $E$ , in the spirit of concavification arguments in Bayesian persuasion and information design. (e.g. [Kamenica and Gentzkow, 2011](#); [Bergemann and Morris, 2016](#)) Third, we use an affine-support duality on  $E$  to translate the condition  $(\text{conc } \hat{g})(S_0) = (\text{conc } g)(S_0)$  into the existence of continuous affine ‘shadow values’

that dominate all date-wise values, and to construct the non-posterior gain regions used in our diagnostic, drawing on classical results on concave envelopes over compact convex subsets of locally convex spaces. (see, [Rockafellar, 1970](#); [Aliprantis and Border, 2006](#))

Further related work treats dynamic interaction entirely through the evolution of beliefs. In dynamic persuasion and dynamic information design, optimal policies are characterized by the law of the posterior process and by concavification on the belief space; see, for example, [Ely \(2017\)](#), [Ball \(2023\)](#), and the broader discussion in [Forges \(2020\)](#). These papers share with the present analysis the use of posterior martingales and belief-space geometry. The primitives and the questions, however, are different. The question here is whether allowing these technologies to be conditioned on rich public histories changes the concavified revenue relative to a terminal benchmark.

Our decomposition of public histories into  $(S_t, Y_t)$  language is also related to state-reduction techniques in dynamic mechanism design. In long-term contracting and dynamic screening, [Battaglini \(2005\)](#) and [Garrett and Pavan \(2015\)](#) show how histories can be summarized by belief and promised-utility type variables, and in dynamic revenue-management and auction models, [Board and Skrzypacz \(2016\)](#) show that, under suitable conditions, simple state variables and pricing rules are optimal. The split of  $\mathcal{F}_t$  can be viewed as an abstract version of such state compression. In contrast to these model-specific reductions, the present paper keeps the static technology at each date arbitrary and asks on the belief space.

The remainder of the paper is organized as follows. Section 2 introduces the dynamic environment and basic assumptions. Section 3 defines date-wise collapse and develops the value-invariance lattice and the ex-ante no-gain theorem. Section 4 presents the structural collapse result. Section 5 show our applications.

## 2 Preliminaries

In discrete time, at each date  $t \in \mathcal{T} := \{0, 1, 2, \dots\}$ , there is a seller and some arbitrary potential buyers for the resource. Uncertainty is defined on a Borel probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . At date  $t$ , the seller specifies a public information policy given by an increasing filtration of public history  $\sigma$ -algebras  $\mathcal{F}_t$  with  $\mathcal{F}_0$  trivial. Alongside  $\{\mathcal{F}_t\}$ , the seller posts a  $\mathcal{F}_t$ -measurable mechanism  $\Gamma_t$  from a given  $\mathcal{H}_t$  where  $\mathcal{H}_t$  is a Polish space of mechanisms, and *could* be executed in date  $t$ .<sup>1</sup> The posterior space  $E = \Delta(\Theta)$ , with  $\Theta$  Polish and equipped with the Borel  $\sigma$ -algebra under the weak topology, is Polish. The set

$$\mathcal{M} := \{M_t = (\mathcal{F}_t, \Gamma_t) \mid t \in \mathcal{T}\}$$

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<sup>1</sup>For example: post a price; offer a menu; run an auction; or specify a timing lottery.

A submechanism  $\Gamma_t$  is said to be  $\mathcal{F}_t$ -measurable if its realization depends only on the public histories information available at date  $t$ .

is the seller's *mechanism calendar*. Let  $G(\mathcal{M})$  be the seller's revenue. All players observe  $\mathcal{M}$  about payoff-relevant primitives are updated via Bayes' rule. Given the buyers' equilibrium behavior, the resource may or may not be sold. Let  $\tau \in \mathcal{T} \cup \{\emptyset\}$  denote the date in which the resource is actually allocated to buyers. If  $\tau = \emptyset$ , no sale ever occurs.

**Assumption 1** (No intertemporal commitment). At each date  $t$  and after any public history  $\mathcal{F}_t$ , the seller chooses a submechanism  $\Gamma_t \in \mathcal{H}_t$  that is  $\mathcal{F}_t$ -measurable; past announcements about future submechanisms are not binding.

[Assumption 1](#) does not require credibility for future submechanisms that are never executed, and relies only on the credibility of the date- $t$  submechanism. given  $(\mathcal{F}_t, \Gamma_t)$  all agents play the induced Bayesian game in date  $t$ , and play of the overall dynamic game follows a Bayesian equilibrium.

**Assumption 2** (Single-shot resource). The resource is economically single-shot: it can be allocated at most once. If the resource is assigned to some buyers, the transfers are made, then no later date  $t' > t$  can generate additional surplus through any  $M_{t'}$ .

[Assumption 2](#) restricts *how many times* the resource can be exercised but not *what occurs before* that exercise. Without intertemporal commitment, the seller may design public experiments prior to trade, observe the evolution of beliefs, and choose the execution date; the resulting program delivers an optimal ex-ante value. Thus dynamic information design and timing are genuine decision margins.<sup>2</sup>

**Observation 1** (Local cash-out). *At any  $t$ , the seller always has an absorbing action: starting at  $t$ , release no additional public information and make no further attempts to trade until the seller chooses to execute some static submechanism.*

[Observation 1](#) is an availability claim, not a value-preserving transformation: the seller may enter an absorbing 'no further information' mode at any history; we do not assert that doing so is without loss.

For each date  $t$ , public  $\sigma$ -algebra  $\mathcal{F}_t$ , and submechanism  $\Gamma_t \in \mathcal{H}_t$ , we represent the primitive mechanism technology by a correspondence

$$\mathcal{R}_t[\mathcal{F}_t, \Gamma_t] : \Omega \rightrightarrows \mathbb{R},$$

where  $\mathcal{R}_t[\mathcal{F}_t, \Gamma_t](\omega)$  is the set of seller revenues that can be implemented at date  $t$  in state  $\omega$  when agents play the Bayesian game induced by  $(\mathcal{F}_t, \Gamma_t)$ .<sup>3</sup> In other words,  $\mathcal{R}_t$  is a  $\mathcal{F}_t$ -

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<sup>2</sup>If [Assumption 2](#) does not hold, the problem might degenerate into a simple situation of finding the suboptimal solution at each  $t$ , for instance, if trading is allowed once for every  $t$ .

<sup>3</sup>We exclude the meaningless case in which  $\mathcal{R}_t(\mathcal{F}_t, \Gamma_t)$  is constant.

measurable random feasible–revenue correspondence:

$$\mathcal{R}_t[\mathcal{F}_t, \Gamma_t](\omega) := \{r \in \mathbb{R} : r \text{ is implementable at } t \text{ given } \mathcal{F}_t(\omega) \text{ and } \Gamma_t\}.$$

$\mathcal{R}_t$  is nonempty and bounded above. Its graph

$$\text{Gr}(\mathcal{R}_t) := \{(\omega, \Gamma_t, r) \in \Omega \times \mathcal{H}_t \times \mathbb{R} : r \in \mathcal{R}_t\}$$

is an analytic (Suslin) subset of  $\Omega \times \mathcal{H}_t \times \mathbb{R}$ . Hence  $\sup \mathcal{R}_t \mathcal{F}_t, \Gamma_t$  is upper semianalytic and admits a universally measurable version.

The standard setting is satisfied in familiar Bayesian mechanism environments, for instance when  $\mathcal{R}_t[\mathcal{F}_t, \Gamma_t]$  is the set of seller revenues generated by Bayesian Nash equilibria, or by Bayes–correlated outcomes, of the stage subgame induced by  $(\mathcal{F}_t, \Gamma_t)$ . Furthermore, we do not generally require an explicit solution for all equilibrium strategies of the stage game induced by  $(\mathcal{F}_t, \Gamma_t)$ . The revenue correspondence  $\mathcal{R}_t[\mathcal{F}_t, \Gamma_t]$ , is just induced by the primitive  $\mathcal{H}_t$  and policy  $\mathcal{F}_t$  to summarize the seller revenues that are attainable at date  $t$  under some admissible behavior of the agents.<sup>45</sup>

### 3 Collapse statistic

#### 3.1 Date-wise collapse

We now formalize the notion of a *collapse variable* and give several equivalent characterizations that are convenient for verification in economics. Retain analytic graphs and boundedness, let  $S_t$  be a Borel version of the posterior kernel given  $\mathcal{F}_t$ . It is  $\mathcal{F}_t$ –measurable and  $\{S_t\}$  is a bounded  $(\mathcal{F}_t)$ –martingale.

**Lemma 1.** *Under the Bayesian primitives  $(\Theta, E = \Delta(\Theta))$  and public filtration  $(\mathcal{F}_t)_{t \in \mathcal{T}}$ , there exists an  $\mathcal{F}_t$ –measurable posterior kernel  $S_t : \Omega \rightarrow E$  such that, for all bounded Borel  $\varphi$  on  $\Theta$ ,*

$$\int_{\Theta} \varphi dS_t = \mathbb{E}[\varphi(\theta) \mid \mathcal{F}_t] \quad a.s.$$

*In particular,  $\{S_t\}_{t \geq 0}$  is a bounded  $(\mathcal{F}_t)$ –martingale:  $\mathbb{E}[S_{t+1} \mid \mathcal{F}_t] = S_t \quad a.s.$*

The [Lemma 1](#) states that public learning follows Bayes’ rule and exhibits no predictable drift in expectation: given the current public history  $\mathcal{F}_t$ , future public signals only deliver

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<sup>4</sup>For example, under a given equilibrium notion and selection rule, or under a conservative lower bound on equilibrium payoffs.

<sup>5</sup>All subsequent value functions and comparative statements are therefore understood relative to the primitive: if, in application,  $\mathcal{H}_t$  is specified as a strict inner approximation of the true feasible submechanism, our results continue to apply.



a mean-preserving refinement of beliefs about the state  $\theta$ , and thus do not systematically raise or lower its mean. Consequently, all  $\theta$ -relevant content in  $\mathcal{F}_t$  is summarized by the “public posterior”  $S_t$  (a sufficient statistic in the Doob–Dynkin sense). The boundedness ensures the integrability required to legitimize technical steps such as optional sampling and concavification.<sup>6</sup>

Consider the date- $t$  function under a public information  $\sigma$ -algebra  $\mathcal{G}$ , let  $U_t^{\mathcal{G}} : E \rightarrow \mathbb{R}$  be Borel maps:

$$U_t^{\mathcal{G}}(s) := \sup_{\Gamma: \mathcal{G}\text{-meas.}} \text{ess sup} \left\{ \mathcal{R}_t[\mathcal{F}_t, \Gamma] \mid S_t = s \right\}, \quad U_t(s) := U_t^{\sigma(S_t)}(s),$$

and for any non-posterior public variable  $Y_t$  inside  $\mathcal{F}_t$ , let  $U_t^{(S_t, Y_t)}(s) := U_t^{\sigma(S_t, Y_t)}(s)$ .

**Lemma 2.**  $U_t^{\mathcal{G}}(s)$  admits a version that is bounded and upper semianalytic on  $E$ .

Until now, public histories information  $\mathcal{F}_t$  admits a clean decomposition into two parts: posterior information  $S_t$ , which determines “what we believe about the state,” and non-posterior public historical variable  $\{Y_t^{(1)}, \dots\}$ , which determine “what can be done today under the same beliefs.” Contrast with strands of the literature, we explicitly separate posterior information, which changes beliefs about the state, from non-posterior public features, which alter the date- $t$  implementable set, holding the posterior fixed. Thus, when the buyer’s time preferences are unknown to the designer and modeled as random, they are embedded in the  $S_t$  component. By saying “embedded”, we mean that the primitive state  $\theta$  is allowed to include any payoff-relevant uncertainty that is subject to Bayesian updating—values, costs, discount factors, long-run types, and so forth. All of these can be treated as part of  $\theta$  and therefore as part of the posterior  $S_t$ . By contrast, waiting and other realized aspects of the trading history are embedded in the  $Y_t$  component. Substantively, this construction recasts standard dynamic ingredients within the  $(S_t, Y_t)$  analytical framework.<sup>7</sup>

Under the  $(S_t, Y_t)$  decomposition, the remaining primitive and non-random objects are the underlying technological and preference parameters, such as common-knowledge cost processes, the seller’s discount factor, and institutional constraints on transfers. These primitives are embedded in the per- $t$  submechanism space  $\mathcal{H}_t$  and are taken as *given* when we evaluate the dynamic value.

**Definition 1.** A public variable  $Y$  is *collapsible* at date  $t$  to  $\mathcal{G}$  iff

$$U_t^{(\mathcal{G}, Y)}(s) = U_t^{\mathcal{G}}(s) \quad \text{for all } s \in E.$$

<sup>6</sup> $\mathcal{F}_t$  still contains additional public history details, that do not alter  $S_t$ ; while they do not update beliefs about  $\theta$ , they can affect the period- $t$  feasible set and value through constraints or feasibility.

<sup>7</sup>This complements the information-structure perspective in Forges (2020), given a representation of dynamic interaction in terms of information structures and posterior processes, they characterize which public states can be dropped, in an ex-ante sense, without changing the attainable revenue frontier.



**Proposition 1** (Basic algebra). *Fix  $t$ , then:*

- (a) Function closure. *If  $Y_t$  is collapse, then  $\phi(Y_t)$  is collapse for any Borel  $\phi$ .*
- (b) Independent labels. *If  $Z_t$  is public and conditionally independent of primitives given  $S_t$ , then  $Z_t$  is collapse. Moreover, any  $(Y_t, Z_t)$  with  $Y_t$  collapse remains collapse.*
- (c) Meet-closure and downward-closure.  *$\mathfrak{D}_t^S$  is closed under arbitrary intersections and is downward closed: if  $\mathcal{Q} \subseteq \mathcal{G} \in \mathfrak{D}_t^S$  then  $\mathcal{Q} \in \mathfrak{D}_t^S$ .*
- (d) No general join-closure. *In general,  $\mathfrak{D}_t^S$  is not closed under joins: it may happen that  $\sigma(S_t, Y_t^{(1)}) \in \mathfrak{D}_t^S$  and  $\sigma(S_t, Y_t^{(2)}) \in \mathfrak{D}_t^S$  while  $\sigma(S_t, Y_t^{(1)}, Y_t^{(2)}) \notin \mathfrak{D}_t^S$ .<sup>8</sup>*  
*The value-invariance lattice at date  $t$ :  $\mathfrak{D}_t^S := \{ \mathcal{G} : \sigma(S_t) \subseteq \mathcal{G} \subseteq \mathcal{F}_t, U_t^{\mathcal{G}} = U_t^S \}$  is nonempty, as it contains  $\sigma(S_t)$ .*

[Definition 1](#) organizes “date- $t$  deletable public information” into a value-equivalence framework: the notion of collapse replaces [Blackwell \(1953\)](#) statistical sufficiency with objective-function sufficiency. In particular, the non-join-closure (d) of [Proposition 1](#) reveals complementarities among public details and a key risk: details that are each “non-improving” in isolation may complement one another when combined, thereby allowing the mechanism to raise the value upper bound.

**Proposition 2.** *Any non-posterior public variable  $Y_t$  is collapsible at date  $t$  to  $\mathcal{G}$  such that  $\sigma(S_t) \subseteq \mathcal{G} \subseteq \mathcal{F}_t$ , iff*

$$U_t^{\mathcal{F}_t}(s) = U_t(s) \quad \text{for all } s \in E.$$

*We say that the mechanism collapses on date  $t$ .*

Intuitively, observing  $Y_t$  as “promised utility”, “eligibility”, “inventory”, or “reputation” can be suggestive, but it does not determine whether  $Y_t$  is collapsible. Even when  $Y_t$  is policy-dependent, it may still be collapsible. To avoid perceptual guesswork, we provide three verifiable certificates that certify when  $Y_t$  can be safely deleted; see [Section A.3](#).

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<sup>8</sup>The appendix provides a simple [example 1](#).

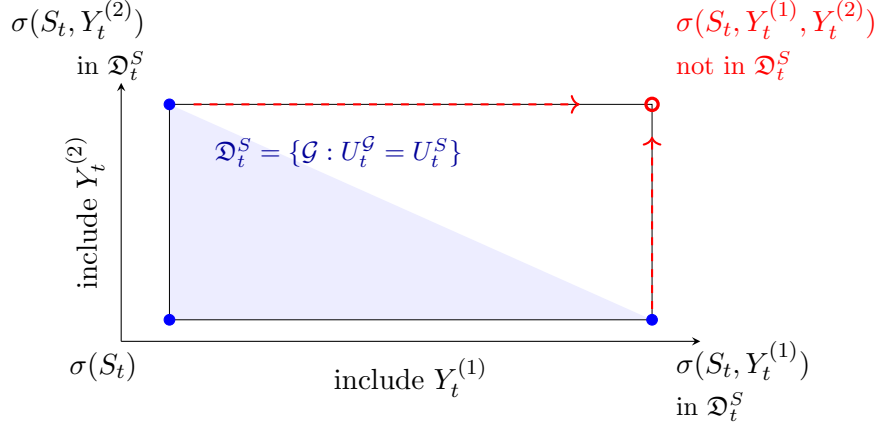


Figure 1: Illustration of the value-invariance lattice  $\mathfrak{D}_t^S$ : the shaded region contains those public histories  $\mathcal{G}$  for which  $U_t^{\mathcal{G}} = U_t^S$  (downward- and meet-closed). Both  $\sigma(S_t, Y_t^{(1)})$  and  $\sigma(S_t, Y_t^{(2)})$  lie in  $\mathfrak{D}_t^S$ , but their join  $\sigma(S_t, Y_t^{(1)}, Y_t^{(2)})$  (top-right corner) may fall outside  $\mathfrak{D}_t^S$ , so  $\mathfrak{D}_t^S$  is not generally closed under joins.

### 3.2 From date-wise to calendar global

Given histories information  $\mathcal{G}_t = \sigma(S_t, Y_t^{(1)}, \dots) \subseteq \mathcal{F}_t$ , to record the best revenue the seller can obtain by choosing an execution date  $t$  for each posterior belief  $s$ , denote

$$g(s) := \sup_{t \in \mathbb{T}} U_t(s), \quad \widehat{g}(s) := \sup_{t \in \mathbb{T}} U_t^{\mathcal{G}_t}(s),$$

**Assumption 3** (Regularity of reduced-form values). For each  $t$  and each public  $\sigma$ -algebra  $\mathcal{G}$  with  $\sigma(S_t) \subseteq \mathcal{G} \subseteq \mathcal{F}_t$ , the reduced-form value  $U_t^{\mathcal{G}} : E \rightarrow \mathbb{R}$  admits a lower semicontinuous version.

[Assumption 3](#) is satisfied in familiar standard mechanism-design environments studied in the literature. In continuous-type settings with quasilinear utilities and uniformly bounded transfers, equilibrium payoffs are continuous in types and allocations, so the seller’s date- $t$  revenue as a function of beliefs is upper semicontinuous; see, for example, [Myerson \(1981\)](#) and [Pavan et al. \(2014\)](#). Taking suprema over incentive-compatible mechanisms preserves upper semicontinuity, and hence the negative of revenue is lower semicontinuous. More generally, under the usual compactness and continuity assumptions on primitives, the reduced-form revenue curves that arise in the literature, including implementations based on Bayes-correlated equilibria, dominant strategies, and Bayes-Nash equilibria—are lower semicontinuous on the belief space; see, e.g., [Bergemann and Morris \(2016\)](#). If we drop [Assumption 3](#), all of the subsequent “if and only if” results should be read as providing sufficient conditions but no longer necessary ones.

Its concave envelope  $\text{conc } g$  is the tight upper bound on ex-ante revenue over all dynamic

information structures. Write  $\text{Aff}_c(E)$  for the set of continuous affine maps  $\ell : E \rightarrow \mathbb{R}$ , and define the concave envelope of  $g : E \rightarrow \mathbb{R}$  by

$$(\text{conc } g)(s) := \inf \{ \ell(s) : \ell \in \text{Aff}_c(E), \ell \geq g \}.$$

For each  $\varepsilon > 0$ , define

$$\mathcal{L}_g^\varepsilon(S_0) := \left\{ \ell \in \text{Aff}_c(E) : \ell \geq g \text{ on } E, \ell(S_0) \leq (\text{conc } g)(S_0) + \varepsilon \right\}.$$

**Theorem 1** (Information  $\mathcal{G}_t$   $\varepsilon$ -no-gain). *The following are equivalent:*

$$(\text{conc } \widehat{g})(S_0) = (\text{conc } g)(S_0). \tag{a}$$

$$\forall \varepsilon > 0, \exists \ell_\varepsilon \in \mathcal{L}_g^\varepsilon(S_0) \text{ such that } \widehat{g}(s) \leq \ell_\varepsilon(s) \quad \forall s \in E. \tag{A_\varepsilon}$$

Moreover, to the global datewise condition

$$\forall \varepsilon > 0, \exists \ell_\varepsilon \in \mathcal{L}_g^\varepsilon(S_0) \text{ such that } U_t^{\mathcal{G}_t}(s) \leq \ell_\varepsilon(s) \quad \forall s \in E, \forall t \in \mathbb{T}. \tag{A'_\varepsilon}$$

Allowing richer public histories  $\{\mathcal{G}_t\}$  and a larger menu of submechanisms expands each date- $t$  value  $U_t^{\mathcal{G}_t}$  and thus  $\widehat{g}(s) = \sup_t U_t^{\mathcal{G}_t}(s)$ . Theorem 1 shows that the seller gains no additional ex-ante value at the prior  $S_0$ , if and only if there exists an affine 'shadow value'  $\ell$  on the belief space that simultaneously supports  $g$  at  $S_0$  and majorizes all date-wise values  $U_t^{\mathcal{G}_t}$ . In this case, all non-posterior public details and calendar elaborations are globally value-irrelevant for the seller: they may raise revenue at some beliefs and dates, but an optimal information structure from  $S_0$  never visits those beliefs, so the concavified revenue at  $S_0$  remains unchanged,<sup>9</sup> as the following proposition. We say that the information histories  $\mathcal{G}_t$  has no non-posterior gain if equation (a) holds under the histories  $\mathcal{G}_t$ , as the non-posterior public variable  $\{Y_t^{(1)}, \dots\}$  has no value. Conversely, this is not true.

**Proposition 3.** *If the mechanism calendar collapses on every  $t \in \mathcal{T}$ , then any information  $\mathcal{G}_t = \sigma(S_t, Y_t^{(1)}, \dots) \subseteq \mathcal{F}_t$  has no non-posterior gain.*

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<sup>9</sup>The appendix provides a simple [example 2](#).

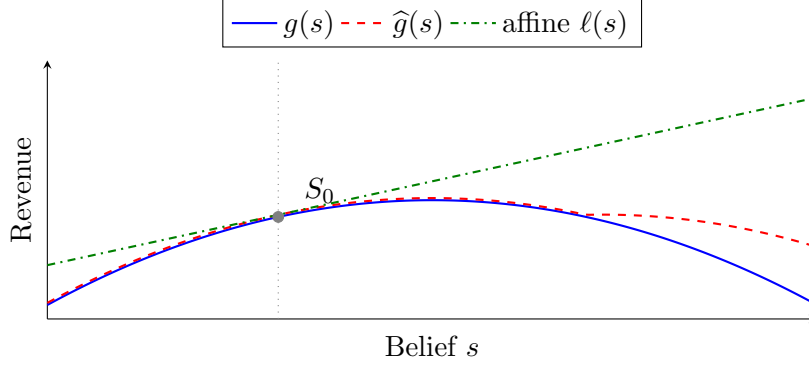


Figure 2: Richer public histories enlarge the date-wise values  $U_t^{\mathcal{G}_t}$  and hence  $\hat{g}(s) = \sup_t U_t^{\mathcal{G}_t}(s)$  (red dashed) relative to  $g(s) = \sup_t U_t(s)$  (blue solid). [Theorem 1](#) states that there is no additional ex-ante gain at  $S_0$  if and only if there exists an affine “shadow value”  $\ell$  (green dash-dotted) that supports  $g$  at  $S_0$  and uniformly dominates all date-wise values  $U_t^{\mathcal{G}_t}$ . In this case, non-posterior public details and calendar elaborations are globally value-irrelevant for the seller.

### 3.3 Discussion: the role of information and submechanisms

A natural concern is that  $\mathcal{F}_t$  is “only” an information  $\sigma$ -algebra, while the upper bound on feasible revenues is determined by the collection of  $\mathcal{F}_t$ -measurable submechanisms  $\Gamma_t \in \mathcal{H}_t$ . This subsection explains why we treat information as the source of dynamic value rather than the availability of more intricate submechanisms.

At date  $t$ , the given primitives are the space of feasible submechanisms  $\mathcal{H}_t$  and the prior  $S_0$ , which, combined with seller’s history information policy  $\sigma$ -algebra  $\mathcal{F}_t$ , induce the revenue correspondence  $\mathcal{R}_t[\mathcal{F}_t, \Gamma_t]$ . As we know, fixing  $t$  pins down the underlying trading technology. Thus public  $\sigma$ -algebra  $\mathcal{G}$  with  $\sigma(S_t) \subseteq \mathcal{G} \subseteq \mathcal{F}_t$  can be viewed as the collection of public events on which the contract is allowed to condition. A  $\mathcal{G}$ -measurable submechanism is a rule that may prescribe different allocations and transfers on different  $\mathcal{G}$ -events, but cannot distinguish histories that are indistinguishable under  $\mathcal{G}$ . Thus, once  $t$  and the primitive technology  $\mathcal{H}_t$  are fixed, the set of feasible history-contingent mechanisms at date  $t$  is completely determined by the information  $\sigma$ -algebra  $\mathcal{G}$ : enlarging  $\mathcal{G}$  enlarges the set of  $\mathcal{G}$ -measurable submechanisms and may raise the upper envelope of implementable revenues.

For a given information state  $\mathcal{G} = \sigma(S_t, Y_t^{(1)}, \dots)$  and belief  $s$ , a rational seller chooses a submechanism that attains this upper bound whenever possible. The effect of richer public histories is thus to expand, through  $\mathcal{R}_t[\mathcal{F}_t, \Gamma_t]$ , the implementable revenue frontier at  $(\mathcal{G}, s)$ ; taking the supremum over  $\Gamma$  makes this frontier explicit. In this sense, the static mechanism technology  $\{\mathcal{H}_t, \mathcal{R}_t\}$  is encapsulated in the indirect value  $U_t^{\mathcal{G}}$ , much as indirect utility encapsulates preferences over bundles in static consumer theory. Moreover, every unknown dynamic ingredient can be embedded in an algebraic analysis framework as part of  $\mathcal{G}$ . The remaining,

known primitives are embedded in the submechanism space  $\mathcal{H}_t$ , so the static submechanism technology itself should not be taken as the main object of analysis.

Once we work with the reduced forms  $\{U_t^g\}$ , different mechanism calendars matter only through the information  $\sigma$ -algebras they make available at each date. Because  $U_t^g$  already reflects the optimal use of the given primitive at each information state, any gap between  $g$  and  $\hat{g}$  at a belief  $s$  is entirely due to the additional public details contained in  $\mathcal{F}_t$  beyond  $\sigma(S_t)$ : richer public histories refine the partition of states and enlarge the implementable revenue frontier at some beliefs. Our notion of "dynamic value" is exactly this contribution of finer conditioning on public histories, rather than the sheer availability of more complicated static submechanisms.<sup>10</sup>

## 4 Mechanism calendar collapses

### 4.1 Terminal mechanism

**Definition 2** (Collapse). We say that the mechanism calendars collapse if, for every  $\varepsilon > 0$ , there exists a terminal calendar  $\widetilde{\mathcal{M}}_\varepsilon$  such that

$$\mathbb{E}[G(\widetilde{\mathcal{M}}_\varepsilon)] \geq \sup_{\mathcal{M}} \mathbb{E}[G(\mathcal{M})] - \varepsilon,$$

A mechanism is called *terminal* if it has the following structure:

- (i) At date 0, the seller conducts a single public experiment on the posterior space  $E$  with support  $\{S^1, \dots, S^K, \dots\} \subset E$  and probabilities  $\{\pi_k\}$  such that  $\sum \pi_k S^k = S_0$ . The realized posterior  $S^k$  becomes common knowledge.
- (ii) There exists a mapping  $t : E \rightarrow \mathcal{T}$  such that the seller selects the deterministic date  $t(S)$  and, only at that date, executes a *terminal* submechanism  $\Gamma_{t(S)}(S)$  that uses no additional public information beyond  $S$  and entails no further strategic interaction or public disclosure thereafter.

**Theorem 2** (Structural Collapse). *mechanism calendars collapse if and only if the global condition holds:*

$$\forall \varepsilon > 0, \exists \ell_\varepsilon \in \mathcal{L}_g^\varepsilon(S_0) \text{ such that } U_t^{\mathcal{F}_t}(s) \leq \ell_\varepsilon(s) \quad \forall s \in E, \forall t \in \mathbb{T}. \quad (\text{A}''_\varepsilon)$$

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<sup>10</sup>In spirit, this reduction is close to the state-compression ideas used to recast sequential screening problems as static screening problems on an augmented state space (e.g. [Krähmer and Strausz, 2017](#)), but here it is carried out in a general Bayesian mechanism environment and phrased in terms of belief-based value functions  $g$  and  $\hat{g}$ .

According to [Theorem 2](#), the entire dynamic mechanism calendar is ex-ante equivalent to a *terminal* design that (i) conducts a single information experiment on  $E$  at date 0, and (ii) at the realized posterior  $S$ , chooses a static submechanism and a deterministic execution date  $t(S)$ , with no further public disclosure or strategic interaction thereafter. All apparent dynamic richness in the sequence  $\{\Gamma_t\}$  can be compressed into a one-shot experiment and a posterior-dependent static choice.

In a terminal mechanism of [Definition 2](#), the designer conducts at date 0 a public information experiment with countable support  $\{S_k\}_{k \in K} \subset E$ , and then, in the underlying Bayesian environment, one may start from arbitrary signal structures and hence arbitrary laws  $\pi$  of posteriors on  $E$ . For any bounded belief-based payoff and any  $\varepsilon > 0$ , there exists a probability measure  $\pi^\varepsilon$  on  $E$  with *countable* support. Thus, the supremum ex-ante value over all information structures can be  $\varepsilon$ -approximated by terminal experiments with countable support.

Recall that in our setting the revenue correspondence  $\mathcal{R}_t \mathcal{F}_t, \Gamma_t$  being bounded above is not an operational requirement that the designer be able to compute the exact upper bound at each history. Instead, it could be a structural restriction on the given primitives: there exists some finite constant such that no admissible  $\mathcal{F}_t$ -measurable submechanism at date  $t$  can generate revenue above it in any state.<sup>11</sup> Our main results are formulated *relative to* these primitives. Given  $\mathcal{H}_t$ , the analyst is free to specify  $\mathcal{R}_t$  as coarsely or as tightly as convenient. In particular, testing whether a given environment exhibits collapse does not require solving explicitly for all equilibria or computing a closed-form expression for  $U_t^G$ ; it only requires that the modeller accept a primitive bounding of feasible revenues in the sense above. Given primitives, a *mechanism calendar* is then a choice of public histories  $\{\mathcal{F}_t\}_{t \in \mathcal{T}}$  and  $\mathcal{F}_t$ -measurable submechanisms  $\{\Gamma_t \in \mathcal{H}_t\}_{t \in \mathcal{T}}$ . When we say that the mechanism calendar “collapses”, we mean that, within these fixed primitives, every optimal dynamic calendar can be replaced, without ex-ante  $\varepsilon$ -loss, by a terminal mechanism.

**Corollary 1.** *Suppose the mechanism calendar  $\mathcal{M}$  collapses. Then the seller’s optimal ex-ante revenue is  $\varepsilon$ -achieved by a terminal mechanism  $\widetilde{\mathcal{M}}_\varepsilon$ .*

A natural concern is that the terminal calendars in [Definition 2](#) require the execution date to be a deterministic function  $t(S)$  of the date-0 posterior. This restriction excludes mechanisms in which the allocation timing itself depends on signals realized after date 0.<sup>12</sup> In such environments, [Theorem 2](#) should therefore be interpreted as providing an ex-ante upper bound, and the diagnostic tools developed in the paper remain valid without change. At a purely representational level, however, our state space  $\Theta$  is allowed to be infinite-dimensional,

<sup>11</sup>The familiar applications follow from standard economic bounds: for example, quasilinear utilities with bounded values and transfers, or exogenous constraints on maximal payments.

<sup>12</sup>For instance, dynamic search and optimal-stopping problems with recall in the spirit of [Weitzman \(1979\)](#), auction-timing and real-options models where the seller optimally chooses when to run the auction (e.g. [Cong, 2020](#)). Such models extending the collapse characterization to them is left for future work.

so in principle the model can embed an entire future signal path into the primitive state  $\theta$ , and let  $(\mathcal{F}_t)$  be the filtration generated by its coordinate projections. Under such an embedding any dynamic stopping policy can be viewed as the outcome of a single date-0 experiment on the posterior space  $E$ , followed by a deterministic execution date  $t(S)$ . In that sense, the collapse characterization in [Theorem 2](#) continues to apply.

## 4.2 Proof roadmap, tools, and technicals

This subsection sketches the proof of [Theorems 1](#) and [2](#) and indicates their scope. The arguments rely on three ingredients:

- (i) a reduction from dynamic mechanism calendars to belief-based value functions on the posterior space;
- (ii) a martingale concavification step, as in the Bayesian persuasion literature;
- (iii) a dual, affine-support characterization of concave envelopes in belief spaces.

**Dynamic reduction: from calendars to belief-based values.** Given [Assumptions 1–2](#) and buyers’ Bayesian behavior, the seller’s date- $t$  revenue possibilities are summarized by the correspondence  $\mathcal{R}_t[\mathcal{F}_t, \Gamma_t] \subseteq \mathbb{R}$ , whose graph is analytic and bounded above. For a public  $\sigma$ -algebra  $\mathcal{G}$  with  $\sigma(S_t) \subseteq \mathcal{G} \subseteq \mathcal{F}_t$ , we define, by [Lemma 2](#), the reduced-form value

$$U_t^{\mathcal{G}}(s) := \sup_{\Gamma: \mathcal{G}\text{-meas.}} \text{ess sup} \left\{ \mathcal{R}_t[\mathcal{F}_t, \Gamma] \mid S_t = s \right\}, \quad s \in E.$$

Conditional on a public state  $(\mathcal{G}, s)$ , we take the supremum over all feasible  $\mathcal{G}$ -measurable submechanisms. The mechanism technology  $\{\mathcal{H}_t, \mathcal{R}_t\}$  is therefore fully captured by the family of value functions  $\{U_t^{\mathcal{G}}\}$ ; the submechanisms  $\Gamma_t$  are “optimized away” inside these reduced forms.

We then introduce the two benchmark maps  $g(s)$  and  $\widehat{g}(s)$  which compare, belief by belief, the best values attainable when the seller can condition only on the posterior  $S_t$  versus on the richer public history  $\mathcal{G}_t$ .

**Martingale concavification on the posterior space.** As [Lemma 1](#) states that, the posterior process  $\{S_t\}_{t \geq 0}$  is a bounded  $(\mathcal{F}_t)$ -martingale with values in  $E = \Delta(\Theta)$ , endowed with the weak topology.<sup>13</sup> Any public information policy, possibly dynamic and history-dependent,

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<sup>13</sup>This is the standard martingale structure in information design and repeated games with incomplete information (e.g. [Aumann et al., 1995](#); [Bergemann and Morris, 2016](#)).



induces a law of  $(S_t)_{t \geq 0}$  that is a martingale with given  $S_0$ , and conversely such laws can be implemented by suitable public experiments on  $E$ .<sup>14</sup>

Given a bounded belief-based payoff  $\varphi : E \rightarrow \mathbb{R}$ , the seller's ex-ante problem over all such martingale laws is a pure concavification problem:

$$\sup_{\text{feasible laws of } (S_t)} \mathbb{E}[\varphi(S_\tau)] = (\text{conc } \varphi)(S_0),$$

where  $\tau$  is the stopping time at which the resource is allocated. This is the usual value–concavification equivalence from static Bayesian persuasion (Kamenica and Gentzkow, 2011), applied here to the dynamic value functions  $g$  and  $\widehat{g}$  that already embed optimal submechanism choice. In particular, the optimal ex-ante values under terminal calendars and under general calendars are

$$V^{\text{term}} = (\text{conc } g)(S_0), \quad V^{\text{cal}} = (\text{conc } \widehat{g})(S_0).$$

Assumptions 1–2 ensure that the seller never benefits from delaying trade once a given belief has been reached, so the trade date enters only through the concavification over belief paths.

**Affine supports in infinite-dimensional belief spaces.** The posterior space  $E = \Delta(\Theta)$  is typically infinite-dimensional whenever  $\Theta$  is non-finite. We work with the weak topology, under which  $E$  is a compact, convex, metrizable subset of a locally convex topological vector space. In this setting, the concave envelope of any bounded function  $f : E \rightarrow \mathbb{R}$  admits the classical dual representation

$$(\text{conc } f)(s) = \inf\{\ell(s) : \ell \in \text{Aff}_c(E), \ell \geq f\}, \quad s \in E,$$

where  $\text{Aff}_c(E)$  denotes the family of continuous affine functionals on  $E$ ; see, for instance, Tyrrell Rockafellar (1970, Thm. 5) and Aliprantis and Border (2006, Ch. 6). Moreover, for every  $s_0 \in E$  and  $\varepsilon > 0$  there exists an  $\varepsilon$ -supporting affine functional: some  $\ell \in \text{Aff}_c(E)$  such that  $\ell \geq f$  and  $\ell(s_0) \leq (\text{conc } f)(s_0) + \varepsilon$ .

We apply this convex-analytic machinery to the posterior-based benchmark  $g$ . The family  $\mathcal{L}_g^\varepsilon(S_0)$  collects precisely those affine  $\varepsilon$ -supports of  $g$  at  $S_0$ . The key observation behind Theorem 1 is that the ex-ante no-gain condition

$$(\text{conc } \widehat{g})(S_0) = (\text{conc } g)(S_0)$$

is equivalent to the existence, for each  $\varepsilon > 0$ , of an affine shadow value  $\ell_\varepsilon \in \mathcal{L}_g^\varepsilon(S_0)$  that uniformly dominates all date-wise values under the richer information,  $U_t^{\mathcal{G}_t} \leq \ell_\varepsilon$ . The implication

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<sup>14</sup>In the finite-type case this is immediate; in more general Polish settings one can argue via standard disintegration and measurable selection, as in Doval and Skreta (2024) and related work.

$(A''_\varepsilon) \Rightarrow (a)$  follows from the infimum representation of  $\text{conc } \widehat{g}$  and the inequalities  $g \leq \widehat{g} \leq \ell_\varepsilon$ . The converse  $(a) \Rightarrow (A''_\varepsilon)$  constructs such  $\ell_\varepsilon$  as  $\varepsilon$ -supports of  $\widehat{g}$ , which also majorize  $g$  and hence belong to  $\mathcal{L}_g^\varepsilon(S_0)$ . The equivalence between the pointwise condition  $(A_\varepsilon)$  and the date-wise condition  $(A''_\varepsilon)$  is immediate from  $\widehat{g}(s) = \sup_t U_t^{\mathcal{G}}(s)$ .

A technical subtlety is that the belief space is infinite-dimensional and the functions  $g, \widehat{g}$  need not be continuous. We use boundedness and analytic-graph assumptions on the revenue correspondences  $\mathcal{R}_t$  to ensure that the date-wise values  $U_t^{\mathcal{G}}$  are upper semianalytic and hence universally measurable (Lemma 2). This permits us to apply the convex-analytic results in a pointwise fashion at the prior  $S_0$ , while keeping all value functions measurable enough for the martingale arguments.

**From concavification equality to structural collapse.** Theorem 2 identifies structural collapse with the global affine-support condition  $(A''_\varepsilon)$ . The value representation

$$V^{\text{term}} = (\text{conc } g)(S_0), \quad V^{\text{cal}} = (\text{conc } \widehat{g})(S_0),$$

combined with Theorem 1, implies that  $(A''_\varepsilon)$  holds if and only if  $V^{\text{cal}} = V^{\text{term}}$ . Since terminal calendars form a subset of all calendars, this equality is equivalent to the existence, for every  $\varepsilon > 0$ , of a terminal calendar whose ex-ante revenue is within  $\varepsilon$  of the optimal calendar value, that is, to collapse in the sense of Definition 2.

In summary, the proofs combine posterior martingales, concavification, and affine supports in infinite-dimensional spaces to convert the dynamic mechanism calendar into a geometric condition on the belief space. These tools remain valid in the general Polish type spaces we consider.

### 4.3 A diagnostic for non-posterior dynamic value

We next introduce a simple geometric diagnostic, based on  $\mathcal{L}_g^\varepsilon(S_0)$ , for detecting when non-posterior public histories create genuine dynamic value beyond terminal calendars.

Fix a prior  $S_0 \in E$  and  $\varepsilon > 0$ . Let

$$H_\varepsilon(s) := \sup_{\ell \in \mathcal{L}_g^\varepsilon(S_0)} \ell(s), \quad s \in E,$$

be the pointwise envelope of affine functions in  $\mathcal{L}_g^\varepsilon(S_0)$ . The *non-posterior gain test region* at  $S_0$  is

$$\mathcal{B}_\varepsilon(S_0) := \{(s, r) \in E \times \mathbb{R} : r > H_\varepsilon(s)\}.$$

Given a mechanism calendar  $\mathcal{M} = \{(\mathcal{F}_t, \Gamma_t)\}_{t \in \mathcal{T}}$ , we say that  $\mathcal{M}$  *hits* the test region if there

exist  $t \in \mathcal{T}$  and a belief  $s \in E$  such that

$$(s, U_t^{\mathcal{F}_t}(s)) \in \mathcal{B}_\varepsilon(S_0). \quad (\text{test})$$

$H_\varepsilon$  collects all affine  $\varepsilon$ -supports of the posterior-based benchmark  $g(s) = \sup_t U_t(s)$  at  $S_0$ ; the region  $\mathcal{B}_\varepsilon(S_0)$  consists of belief–revenue pairs that lie strictly above every such support. Hitting this region means that the calendar attains, at some belief  $s$ , a revenue that cannot be bounded by any posterior-based affine shadow value compatible with  $g$  at  $S_0$ .

**Proposition 4.** *If the mechanism calendar  $\mathcal{M}$  collapses, then for every  $\varepsilon > 0$ , the calendar never hits the non-posterior gain test region  $\mathcal{B}_\varepsilon(S_0)$ .*

The family  $\mathcal{L}_g^\varepsilon(S_0)$  is entirely pinned down by the posterior-based benchmark  $g(s) = \sup_t U_t(s)$  and the prior  $S_0$ : it is the set of affine  $\varepsilon$ -supports of the concave envelope  $\text{conc } g$  at  $S_0$ . In particular,  $\mathcal{L}_g^\varepsilon(S_0)$  does not depend on non-posterior public histories or the detailed dynamic calendar structure. Given a specific calendar  $\mathcal{M} = \{(\mathcal{F}_t, \Gamma_t)\}_{t \in \mathcal{T}}$ , the test region  $\mathcal{B}_\varepsilon(S_0)$  becomes operational through the date-wise values  $U_t^{\mathcal{F}_t}$ . hitting this region is a sufficient certificate that non-posterior public histories  $\mathcal{F}_t$  generate genuine dynamic value beyond terminal calendars, whereas avoiding it is a necessary condition for structural collapse.

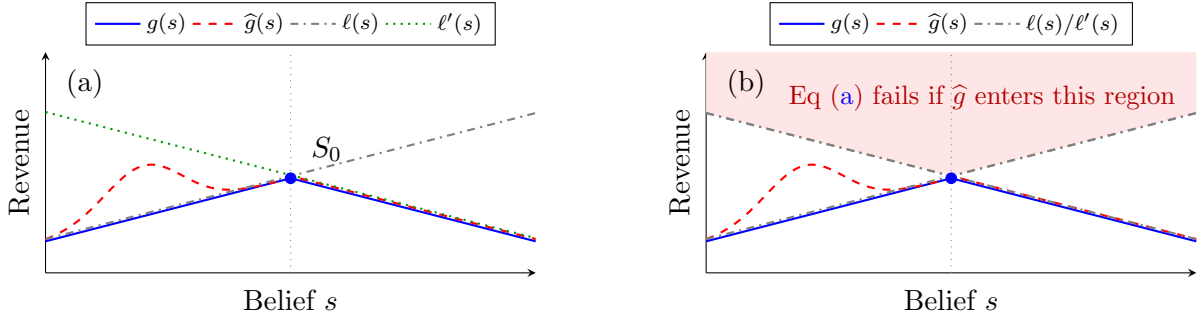


Figure 3: Panel (a) illustrates  $\mathcal{L}_g^\varepsilon(S_0)$  in Theorem 1: a concave date-wise revenue  $g$  (blue), an expanded revenue  $\hat{g} \geq g$  (red dashed), and two affine functions  $\ell, \ell'$  in  $\mathcal{L}_g^\varepsilon(S_0)$ . Here  $\hat{g}$  exceeds  $\ell$  at some beliefs, so this particular  $\ell$  cannot certify condition  $(A_\varepsilon)$ , but there exists another affine support  $\ell'$  that still dominates  $\hat{g}$  pointwise. Panel (b) shows the pointwise envelope  $\sup_{\ell \in \mathcal{L}_g^\varepsilon(S_0)} \ell(s)$  (grey) induced by the family  $\mathcal{L}_g^\varepsilon(S_0)$ . Any  $\hat{g}$  that lies strictly above this envelope at some belief  $s$  must violate condition  $(A_\varepsilon)$ , and hence  $(\text{conc } \hat{g})(S_0) > (\text{conc } g)(S_0)$ . Thus the shaded region in panel (b) is a sufficient “bad” region for the failure of Eq (a).

## 5 Applications and Diagnostics

### 5.1 A loan–probation example

This subsection presents a one-dimensional example that illustrates the shadow-value condition and separates local date-wise improvements from gains in the concavified ex-ante value.

Consider a pool of borrowers with credit type  $\theta \in \{H, L\}$ . Let  $p := \Pr(\theta = H) \in [0, 1]$  denote the public posterior for a randomly drawn borrower and let  $S_0 = p_0 = 0.8$  be the prior. When submechanisms may condition only on  $p$ , the date-1 reduced-form revenue is

$$g_{ab}(p) := 0.4 + 0.2 \min\{p, 1 - p\}, \quad p \in [0, 1]. \quad (5.1)$$

One may interpret  $g_{ab}$  as the envelope of two loan products. On  $[0, \frac{1}{2}]$ ,  $g_{ab}(p) = 0.4 + 0.2p$ , and on  $[\frac{1}{2}, 1]$ ,  $g_{ab}(p) = 0.4 + 0.2(1 - p) = 0.6 - 0.2p$ . Thus  $\text{conc } g_{ab} = g_{ab}$ , and  $(\text{conc } g_{ab})(S_0) = g_{ab}(0.8) = 0.44$ .

At  $S_0 = 0.8$ , the left and right derivatives of  $g_{ab}$  coincide and are equal to  $-0.2$ , so the subdifferential at  $S_0$  is the singleton  $\{-0.2\}$ . The unique affine support of  $g_{ab}$  at  $S_0$  is

$$\ell(p) := g_{ab}(0.8) - 0.2(p - 0.8) = 0.6 - 0.2p.$$

Hence the envelope of all affine supports at  $S_0$  is

$$H_0(p) := \sup_{\ell \in \mathcal{L}_{g_{ab}}^0(S_0)} \ell(p) = \ell(p) = 0.6 - 0.2p.$$

On  $A := [0, \frac{1}{2}]$ ,

$$g_{ab}(p) = 0.4 + 0.2p, \quad H_0(p) - g_{ab}(p) = 0.2 - 0.4p.$$

Now introduce a non-posterior public label  $Y^a$  that is available only when  $p \in A$ . One can interpret  $Y^a$  as the repayment outcome of a small probationary loan. Conditional on  $p$ , the distribution of  $Y^a$  is already summarized by  $p$  itself; the role of  $Y^a$  is to expand the set of feasible date-1 submechanisms and hence the reduced-form frontier  $U_1^{\mathcal{F}_1}(p)$ .<sup>15</sup>

Allowing submechanisms to depend on  $(p, Y^a)$  rather than on  $p$  alone yields a new reduced-form frontier of the form

$$\widehat{g}(p) := \begin{cases} g_{ab}(p) + \delta, & p \in [p_L, p_U], \\ g_{ab}(p), & \text{otherwise,} \end{cases}$$

for some interval  $[p_L, p_U] \subset (0, \frac{1}{2})$  and  $\delta := \inf_{p \in [p_L, p_U]} (\widehat{g}(p) - g_{ab}(p)) > 0$ . The scalar  $\delta$  indexes the size of the local improvement generated by the additional label  $Y^a$  within the reduced-form

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<sup>15</sup>Formally,  $Y^a$  refines the revenue correspondence  $\mathcal{R}_1[\mathcal{F}_1, \Gamma_1]$  at beliefs  $p \in A$ , and thereby enlarges the reduced-form value  $U_1^{\mathcal{F}_1}(p)$ . In a fully specified model,  $Y^a \in \{0, 1\}$  could be the repayment outcome of a small test loan, and the bank could offer continuation contracts that condition on  $Y^a$ . For suitable parameters, the resulting incentive-compatible Bayesian Nash equilibrium generates a local improvement of expected revenue on an interval  $[p_L, p_U] \subset (0, \frac{1}{2})$  and thus induces a frontier  $\widehat{g}$  of the form displayed below. Constructing such a microfoundation is straightforward but model-specific: it requires choosing functional forms, a particular signal structure, and an equilibrium selection rule, and leads to several pages of algebra that do not affect the geometric arguments of this section. Since the main results of the paper are formulated entirely at the reduced-form level, in terms of the belief-based frontiers  $g$  and  $\widehat{g}$  generated by  $\{\mathcal{R}_t\}_{t \in \mathcal{T}}$ , we keep the example at this abstract level and let the scalar  $\delta$  below parameterize the strength of the local improvement in  $U_1^{\mathcal{F}_1}(p)$ .

revenue frontier.

Fix  $[p_L, p_U] = [0.1, 0.3]$ . On  $A$ ,  $H_0(p) - g_{ab}(p) = 0.2 - 0.4p$ , which is strictly decreasing in  $p$  on  $[0, \frac{1}{2}]$ . Hence  $\inf_{p \in [0.1, 0.3]} (H_0(p) - g_{ab}(p)) = H_0(0.3) - g_{ab}(0.3) = 0.2 - 0.4 \cdot 0.3 = 0.08$ . Let

$$\delta_{\max} := 0.08.$$

If  $0 < \delta \leq \delta_{\max}$ , then

$$\widehat{g}(p) \leq H_0(p) \quad \forall p \in [0, 1].$$

In particular,  $\widehat{g}(p) > g_{ab}(p)$  on  $[0.1, 0.3]$ : the additional label  $Y^a$  strictly improves the date-1 revenue frontier on a non-trivial set of beliefs. However, the frontier  $\widehat{g}$  still lies below the affine support  $H_0$  at  $S_0$ . By Theorem 1,

$$(\text{conc } \widehat{g})(S_0) = (\text{conc } g_{ab})(S_0) = 0.44.$$

Thus, although  $Y^a$  raises revenues locally in  $p$ , it has no additional ex-ante value at the prior and the optimal mechanism calendar remains ex-ante equivalent to a terminal, posterior-based design. This parameter region therefore illustrates the case in which introducing a richer label enlarges  $U_t^{\mathcal{F}_t}$  at some beliefs but does not affect collapse.

If  $\delta > \delta_{\max}$ , then there exists  $p^* \in [0.1, 0.3]$  with

$$\widehat{g}(p^*) > H_0(p^*),$$

so  $(p^*, \widehat{g}(p^*)) \in \mathcal{B}_0(S_0)$  and

$$(\text{conc } \widehat{g})(S_0) > (\text{conc } g_{ab})(S_0).$$

In this case the non-posterior label  $Y^a$  generates strictly positive dynamic value.

In summary, the loan-probation example separates two phenomena. For  $\delta \leq \delta_{\max}$ , the label  $Y^a$  expands the date-wise frontier but remains dominated by the affine shadow value at  $S_0$ , so concavification and collapse are unchanged. Once  $\delta$  is large enough that  $\widehat{g}$  lies above  $H_0$  at some belief, the calendar ceases to collapse and non-posterior public labels become genuinely value-relevant.

## 5.2 Diagnosing history-dependence in canonical models

We now apply the collapse statistic and the non-posterior gain region to two benchmark models that have shaped the modern literature on dynamic mechanism design. In each case, the diagnostic yields a clean, reduced-form statement about whether the additional state variables used in the standard formulation are ex-ante redundant from the prior  $S_0$ .

### 5.2.1 Dynamic pricing with fluctuating supply cost

Consider a simple dynamic pricing environment in which the seller faces i.i.d. cost shocks  $c_t \in \{c_L, c_H\}$  and sells a single unit to buyers with known values, in the spirit of dynamic revenue-management models such as [Board and Skrzypacz, 2016](#).

For clarity, suppose that at each date  $t$  the public history decomposes as  $\mathcal{F}_t = \sigma(S_t, Y_t)$ , where  $S_t$  is the posterior over all payoff-relevant primitives and  $Y_t$  collects non-posterior public labels generated by past behavior. Under this convention, cost shocks  $c_t$  are part of the primitive state and hence of  $S_t$ , rather than of  $Y_t$ . By contrast,  $Y_t$  is naturally interpreted as the history of buyers' actions, bids, accept decisions, participation and reputation variables.

When realized costs are never directly observed or learned by the buyers: for instance, because buyers care only about posted prices and product attributes, not about the seller's internal cost fluctuations. In that case, whether the mechanism calendar collapses is entirely driven by the collapsibility of the buyers' action history  $Y_t$ . If  $Y_t$  is collapsible at every date, then the calendar is equivalent to a terminal mechanism ([Proposition 3](#)): costs enter only through the posterior path  $S$  and the associated date mapping  $t(S)$ , which selects the execution date of the static submechanism. In particular, when the cost process  $c_t$  is the main driver of  $t(S)$ , the terminal description makes precise the usual intuition that optimal trade is concentrated in relatively low-cost periods.

When realized costs are learned by the buyers, and let  $Y_t^c$  denote the enlarged public history that records both buyers' actions and whatever cost information. Whether this additional cost disclosure creates genuine dynamic value is then captured exactly by the global condition in [Theorem 2](#) applied to the histories  $\mathcal{F}_t = \sigma(S_t, Y_t^c)$ : if, for every  $\varepsilon > 0$ , there exists  $\ell_\varepsilon \in \mathcal{L}_g^\varepsilon(S_0)$  such that

$$U_t^{(S_t, Y_t^c)}(s) \leq \ell_\varepsilon(s) \quad \forall s \in E, \quad \forall t \in \mathbb{T},$$

then public learning about costs is redundant for the seller's objective, and the optimal calendar still collapses to a terminal. Conversely, if this inequality fails for some  $\varepsilon > 0$ , buyers' information about the historical path of costs strictly enlarges the concavified revenue relative to the terminal benchmark, so the realization of cost shocks becomes a genuinely value-relevant component of the public history.

### 5.2.2 Dynamic screening and promised utilities

Consider a standard dynamic screening environment with a long-lived agent whose type  $\theta \in \Theta$  is fixed forever (see, e.g., [Courty and Li, 2000](#); [Pavan et al., 2014](#)). In the usual recursive formulation, the public state at date  $t$  is summarized by two variables: the current belief  $S_t \in \Delta(\Theta)$  and the promised-utility state  $w_t \in \mathbb{R}$ .

The natural question, in our language, is whether conditioning on  $w_t$  creates dynamic

value beyond what can be achieved by conditioning only on  $S_t$ . In many benchmark dynamic screening models, including those studied by [Pavan et al. \(2014\)](#), the optimal dynamic contract can deliver strictly higher ex-ante value than natural static benchmark mechanisms that ignore promised-utility states. Within our framework this corresponds to

$$(\text{conc } \widehat{g})(S_0) > (\text{conc } g)(S_0),$$

so  $\widehat{g}$  must lie strictly above  $g$  at some beliefs, and  $w_t$  is not collapsible at those beliefs in the sense of [Proposition 2](#). Equivalently, there exist dates and beliefs at which  $(S_t, w_t)$  attains belief-revenue pairs in a non-posterior gain region  $\mathcal{B}_\varepsilon(S_0)$ , and any seller-optimal calendar that attains the dynamic benchmark value has to exploit dependence on the promised-utility state.

By contrast, the diagnostic also points toward situations in which collapse is a good approximation. For example, in certain quadratic environments with uniform types (in the spirit of [Battaglini, 2005](#)) the reduced-form value function can be linear in  $w_t$ , so  $U_t^{\mathcal{F}_t}(s)$  is affine in the non-posterior state and admits affine supports that depend only on  $S_t$ . Similarly, in simple two-type models with very impatient agents, where the optimal policy separates immediately and the continuation promised utility becomes constant, the additional state  $w_t$  is effectively frozen after the initial period and need not expand the concavified frontier relative to a posterior-only design.

In this sense, promised utility typically plays a substantive dynamic role in those dynamic screening environments that exhibit strict dynamic gains in the literature, while the affine-support condition provides a simple way to identify special parameter regions in which history-dependence collapses and the recursive state can be compressed back to beliefs alone.

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## A Additional supplements

### A.1 Additional characterizations of collapse

**Proposition 5.** *The following are equivalent:*

(i)  $U_t^{(S,Y)}(s) = U_t(s).$

(ii) **Replication:** For every  $\varepsilon > 0$  and every  $\sigma(S_t, Y_t)$ -measurable family  $y \mapsto \Gamma^{(y)}$ , there exists  $\bar{\Gamma} \in \mathcal{A}_t^S$  such that

$$\text{ess sup} \{ \mathcal{R}_t[\mathcal{F}_t, \bar{\Gamma}] \mid S_t = s \} \geq \text{ess sup} \{ \mathcal{R}_t[\mathcal{F}_t, \Gamma^{(Y_t)}] \mid S_t = s \} - \varepsilon.$$

(iii) **Kernel folding:** For every  $\sigma(S_t, Y_t)$ -measurable stochastic kernel  $K(d\gamma \mid s, y)$  on  $\Gamma_t$ , there exists a  $\sigma(S_t)$ -measurable kernel  $\bar{K}(d\gamma \mid s)$  such that

$$\text{ess sup} \left\{ \int \mathcal{R}_t[\mathcal{F}_t, \gamma] \bar{K}(d\gamma \mid s) \mid S_t = s \right\} \geq \text{ess sup} \left\{ \int \mathcal{R}_t[\mathcal{F}_t, \gamma] K(d\gamma \mid s, Y_t) \mid S_t = s \right\}.$$

Recall  $g(s) := \sup_{t \in \mathcal{T}} U_t(s)$ , and  $\{S_t\}$  is bounded<sup>16</sup>, so optional sampling applies to concave transforms of posteriors.

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<sup>16</sup> $\{\text{conc } g(S_t)\}$  is uniformly integrable.

## A.2 Finite-dimensional case

**Corollary 2.** *Suppose in addition that  $E$  is a compact convex subset of a finite-dimensional normed vector space (e.g.,  $\Theta$  is finite so that  $E = \Delta(\Theta) \subset \mathbb{R}^d$  in the usual simplex embedding). Then the following are equivalent:*

$$(\text{conc } \widehat{g})(S_0) = (\text{conc } g)(S_0). \quad (\text{a})$$

$$\exists \ell \in \mathcal{L}_g(S_0) \text{ such that } \widehat{g}(s) \leq \ell(s) \quad \forall s \in E, \quad (\text{A}')$$

where

$$\mathcal{L}_g(S_0) := \{ \ell \in \text{Aff}_c(E) : \ell \geq g \text{ on } E, \ell(S_0) = (\text{conc } g)(S_0) \}.$$

Moreover, (A') is equivalent to the datewise condition

$$\exists \ell \in \mathcal{L}_g(S_0) \text{ such that } U_t^{\mathcal{G}_t}(s) \leq \ell(s) \quad \forall s \in E, \forall t \in \mathbb{T}. \quad (\text{A}'')$$

*Remark A.1* (Representation of supporting affine maps on  $\Delta(\Theta)$ ). Every continuous affine  $\ell : E \rightarrow \mathbb{R}$  is of the form  $\ell(\mu) = a + \int_{\Theta} \varphi d\mu$  with  $a \in \mathbb{R}$  and  $\varphi \in C_b(\Theta)$ . Hence (A'') is equivalent to the existence of  $a \in \mathbb{R}$  and  $\varphi \in C_b(\Theta)$  such that

$$U_t^{\mathcal{G}_t}(\mu) \leq a + \int_{\Theta} \varphi d\mu \quad \forall \mu \in \Delta(\Theta), \forall t, \quad a + \int_{\Theta} \varphi dS_0 = (\text{conc } g)(S_0).$$

## A.3 Join-stable certificates

We record three structural criteria that are *stable under finite joins* and hence deliver safe deletion.

**Assumption 4** (Join-stable primitives). Fix date  $t$ . For each posterior state  $s \in E$ :

- (a) *(BCE/LP primitives and exposed faces)* The feasible set of Bayes-correlated outcomes is a nonempty polytope  $\mathcal{P}_t(s) \subset \mathbb{R}^d$ , refined under  $Y_t = y$  to a polytope  $\mathcal{P}_t(s, y) \subseteq \mathcal{P}_t(s)$ . The profit is linear  $p \mapsto \langle c_t(s), p \rangle$  and the graphs of  $(s, y) \mapsto \mathcal{P}_t(s, y)$ ,  $s \mapsto \mathcal{P}_t(s)$  are analytic. Define the optimal face at  $(t, s)$  by

$$F_t(s) := \arg \max_{p \in \mathcal{P}_t(s)} \langle c_t(s), p \rangle \quad (\text{a nonempty exposed face of } \mathcal{P}_t(s)).$$

- (b) *(Strong duality and KKT regularity)* The date- $t$  problem admits a convex representation

$$\max_{x \in X_t(s, y)} \langle c_t(s), x \rangle \quad \text{with } X_t(s, y) = \{x : A_t(s, y)x \leq b_t(s, y)\},$$

strong duality holds, and optimal primal/dual sets admit Borel-measurable selections in  $(s, y)$ . Let  $\Lambda_t(s, y)$  be the optimal dual set and  $\mathcal{A}_t(s, y) \subseteq \{1, \dots, m\}$  be the index set of active (tight) inequalities at some primal maximizer.

- (c) (*Directional regularity and  $Y$ -tangent cone*) The map  $U_t(\cdot)$  admits finite one-sided Gateaux derivatives  $DU_t(s; h)$  in all directions  $h \in T_Y(s)$  for a closed cone  $T_Y(s) \subseteq \mathbb{T}_s E$  (the  $Y$ -tangent cone), consisting of mean-preserving perturbations reachable by conditioning on  $Y_t$  at posterior  $s$  (so  $0 \in T_Y(s)$  and  $\langle \mathbf{1}, h \rangle = 0$ ).

**Proposition 6** (Exposed faces and join-stable collapse). *Under [Assumption 4](#), fix  $t$ . For each  $s \in E$ , let  $N_{\mathcal{P}_t(s)}(F_t(s))$  be the normal cone of  $\mathcal{P}_t(s)$  along  $F_t(s)$ .*

- (i) (*Optimal-face invariance*)  $Y_t$  collapses at date  $t$  at posterior  $s$  if and only if for  $\mathbb{P}(Y_t \in \cdot \mid S_t = s)$ -a.e.  $y$  and every maximizer  $p^* \in \arg \max_{p \in \mathcal{P}_t(s, y)} \langle c_t(s), p \rangle$  one has

$$p^* \in F_t(s) \quad \text{and} \quad c_t(s) \in N_{\mathcal{P}_t(s)}(p^*) \subseteq N_{\mathcal{P}_t(s)}(F_t(s)).$$

- (ii) (*Join-stable sufficient condition*) *If, for a statistic  $Y_t$  and every  $s \in E$ ,*

$$\arg \max_{p \in \mathcal{P}_t(s, y)} \langle c_t(s), p \rangle \subseteq F_t(s) \quad \text{for } \mathbb{P}(Y_t \in \cdot \mid S_t = s)\text{-a.e. } y,$$

*then  $Y_t$  collapses at date  $t$ . Moreover, if  $Y_t^{(1)}, \dots, Y_t^{(N)}$  each satisfy this face stability with respect to the same  $F_t(s)$ , then  $\sigma(S_t, Y_t^{(1)}, \dots, Y_t^{(N)})$  also collapses at date  $t$ .*

**Proposition 7** (Common duals, active-set stability, and joins). *Under [Assumption 4](#), fix  $t$  and  $s \in E$ .*

- (i) (*Certificates for collapse*) *The following are equivalent:*

- (a)  $Y_t$  collapses at date  $t$  at posterior  $s$ , i.e.  $U_t^{(S, Y)}(s) = U_t(s)$ .
- (b) (*Common certificate*) *There exists a  $\sigma(S_t)$ -measurable dual multiplier  $\lambda^*(s) \geq 0$  such that for  $\mathbb{P}(Y_t \in \cdot \mid S_t = s)$ -a.e.  $y$ :*
  - $\lambda^*(s)$  is dual-feasible for  $(t, s, y)$  and attains the dual optimum;
  - there exists a  $\sigma(S_t)$ -measurable primal optimizer  $\bar{x}_t(s) \in X_t(s)$  with

$$\langle c_t(s), \bar{x}_t(s) \rangle = \langle b_t(s, y), \lambda^*(s) \rangle, \quad \lambda_i^*(s) (A_{t,i}(s, y) \bar{x}_t(s) - b_{t,i}(s, y)) = 0 \quad \forall i.$$

- (c) (*Active-set stability*) *There exists an index set  $\mathcal{A}^*(s)$  (depending only on  $s$ ) such that for  $\mathbb{P}(Y_t \in \cdot \mid S_t = s)$ -a.e.  $y$  one can choose primal/dual optimizers  $(x^*, \lambda^*)$  with  $\mathcal{A}_t(s, y) = \mathcal{A}^*(s)$  and  $A_{t, \mathcal{A}^*}(s, y)$  having constant rank.*

(ii) (Join-stable sufficient condition) *If there exists a  $\sigma(S_t)$ -measurable  $\lambda^*(s)$  such that  $\lambda^*(s) \in \Lambda_t(s, y)$  for  $\mathbb{P}(Y_t \in \cdot \mid S_t = s)$ -a.e.  $y$  and one can choose primal optimizers with a common active set  $\mathcal{A}^*(s)$  across those  $y$ , then  $Y_t$  collapses at date  $t$  at posterior  $s$ . If several statistics admit the same  $(\lambda^*, \mathcal{A}^*)$  certificate, any finite join of them collapses at date  $t$ .*

**Proposition 8** (Directional no-gain and join-stable collapse). *Under [Assumption 4](#), fix  $t$  and  $s \in E$ .*

(i) (Necessary condition) *If  $Y_t$  collapses at date  $t$  at posterior  $s$ , then for every  $h \in T_Y(s)$ ,*

$$DU_t(s; h) \leq 0 \quad \text{and} \quad DU_t(s; -h) \leq 0,$$

*hence  $DU_t(s; h) = 0$  for all  $h \in T_Y(s)$ . Equivalently, every subgradient  $g \in \partial U_t(s)$  satisfies  $\sup_{h \in T_Y(s)} \langle g, h \rangle = 0$ .*

(ii) (Local sufficiency) *If  $U_t$  is convex in a neighborhood of  $s$  and*

$$\sup_{g \in \partial U_t(s)} \sup_{h \in T_Y(s)} \langle g, h \rangle = 0,$$

*then  $U_t^{(S, Y)}(s) = U_t(s)$  (no local value from  $Y$ -refinement at  $s$ ).*

(iii) (Join-stable sufficient condition) *If, for a statistic  $Y_t$  and all  $h \in T_Y(s)$ , one has  $DU_t(s; h) = 0$  (equivalently,  $\sup_{g \in \partial U_t(s)} \langle g, h \rangle = 0$ ), then  $Y_t$  collapses at date  $t$  at posterior  $s$ . If  $Y_t^{(1)}, \dots, Y_t^{(N)}$  each satisfy this with cones  $T_{Y^{(i)}}(s)$ , then the finite conic hull  $T_\vee(s) := \text{cone}\{T_{Y^{(i)}}(s)\}$  also has zero gain, hence the join collapses at date  $t$  at posterior  $s$ .*

[Proposition 6](#) provides a geometric criterion stated in terms of exposed faces and requires that the optimal face remain unchanged as  $y$  varies. [Proposition 7](#) provides a criterion in terms of dual multipliers and active sets and characterizes collapse through the existence of a dual multiplier and an active set that are valid for all realizations  $y$ . [Proposition 8](#) provides a local criterion based on directional derivatives and requires that every direction that is reachable by conditioning on  $Y_t$  and has mean zero yield zero directional gain.

## B Example

### B.1 Each variable collapses alone, their join does not

**Example 1** (Entry fee  $\times$  opening threshold). Two bidders  $i = 1, 2$  have i.i.d. values  $v_i \sim U[0, 1]$ . Fix a date- $t$  posterior  $s$ . The baseline is a second-price auction (SPA) without

reserve, with expected revenue  $\mathbb{E}[\text{rev}^{\text{SPA}}] = \mathbb{E}[V_{(2)}] = \frac{1}{3}$ .

There are two *public* tags:

- $Y_1 \in \{0, 1\}$  (fee authorization): if  $Y_1 = 1$ , the seller may charge a common entry fee  $f \geq 0$ ; else fees are banned.
- $Y_2 \in \{0, 1\}$  (opening threshold): if  $Y_2 = 1$ , the auction opens whenever at least two register; if  $Y_2 = 0$ , it does not open. Entry fees, if charged, are non-refundable.

*Visibility and optimal date- $t$  revenue.*

- (i) *No tags observed.* Any  $f > 0$  is infeasible/violates IR in the branch  $Y_1 = 0$  or  $Y_2 = 0$ ; hence  $f = 0$  and  $U_t^{(S)}(s) = \frac{1}{3}$ .
- (ii) *Only  $Y_1$  observed.* The seller must remain feasible when  $Y_2 = 0$  (no opening, non-refundable fee), forcing  $f = 0$ ; thus  $U_t^{(S, Y_1)}(s) = \frac{1}{3}$ .
- (iii) *Only  $Y_2$  observed.* Fees may be banned when  $Y_1 = 0$ ; again SPA is optimal,  $U_t^{(S, Y_2)}(s) = \frac{1}{3}$ .
- (iv) *Both tags observed with  $(Y_1, Y_2) = (1, 1)$ .* In the SPA with two symmetric bidders, expected bidder surplus per bidder equals  $1/6$ . Set  $f^* = 1/6$  so IR binds and both register. Then

$$\text{entry fees} = 2f^* = \frac{1}{3}, \quad \text{SPA revenue} = \mathbb{E}[V_{(2)}] = \frac{1}{3} \Rightarrow U_t^{(S, Y_1, Y_2)}(s) = \frac{2}{3}.$$

Hence

$$U_t^{(S)}(s) = U_t^{(S, Y_1)}(s) = U_t^{(S, Y_2)}(s) = \frac{1}{3} < U_t^{(S, Y_1, Y_2)}(s) = \frac{2}{3},$$

then each tag is collapsible alone, but their join is not.

## B.2 Calendar no-gain without every date-wise collapse

**Example 2.** Let  $\Theta = \{0, 1\}$  and  $E = \Delta(\Theta) \cong [0, 1]$  with coordinate  $p = \mathbb{P}(\theta = 1)$ . Consider two dates  $t = 1, 2$  and fix any prior  $S_0 = p_0 \in (0, 1)$ .

Define date-wise value functions (ignoring their mechanism origin) as follows:

$$\begin{aligned} U_1(p) &:= 0 & \forall p \in [0, 1], \\ U_1^{\mathcal{F}_1}(p) &:= \frac{1}{2}p & \forall p \in [0, 1], \\ U_2(p) &:= p, \quad U_2^{\mathcal{F}_2}(p) := U_2(p) & \forall p \in [0, 1]. \end{aligned}$$

Interpretation: at date 1, if the seller can only condition on the posterior  $S_1 = p$ , she gets zero revenue ( $U_1 \equiv 0$ ); with additional public detail  $Y_1$  contained in  $\mathcal{F}_1$ , she can improve date-1 revenue to  $U_1^{\mathcal{F}_1}(p) = \frac{1}{2}p$ . At date 2, additional public details do not help, so  $U_2^{\mathcal{F}_2} = U_2$ .

Now take  $\mathcal{G}_1 = \mathcal{F}_1$  and  $\mathcal{G}_2 = \mathcal{F}_2$ . Then

$$g(p) := \sup_t U_t(p) = \max\{U_1(p), U_2(p)\} = U_2(p) = p,$$

and

$$\hat{g}(p) := \sup_t U_t^{\mathcal{G}_t}(p) = \max\{U_1^{\mathcal{F}_1}(p), U_2^{\mathcal{F}_2}(p)\} = \max\{\frac{1}{2}p, p\} = p$$

for all  $p \in [0, 1]$ . Hence

$$\hat{g} = g \quad \text{on } E,$$

so  $g$  is already concave (affine) and

$$(\text{conc } \hat{g})(S_0) = (\text{conc } g)(S_0) = g(S_0) = p_0.$$

Thus the calendar no-gain condition of [Theorem 1](#) holds at  $S_0$ .

However, at date 1 we have a strict inequality

$$U_1^{\mathcal{F}_1}(p) = \frac{1}{2}p > 0 = U_1(p) \quad \text{for all } p > 0.$$

Therefore

$$U_1^{\mathcal{F}_1} \neq U_1,$$

so the mechanism *does not collapse at date 1* in the sense of [Proposition 2](#). In particular, the condition

$$U_1^{\mathcal{F}_1}(s) = U_1(s) \quad \forall s \in E$$

fails, and some non-posterior public feature in  $\mathcal{F}_1$  is non-collapsible at date 1.

This example shows that calendar no-gain at  $S_0$  in the sense of [Theorem 1](#) can hold even though [Proposition 2](#) fails at some dates.

## C Proof

### C.1 Proof of [Lemma 1](#)

*Proof.* Fix  $t \in \mathcal{T}$ . Since  $\Theta$  is a Polish (hence standard Borel) space and  $\theta : \Omega \rightarrow \Theta$  is Borel measurable, there exists a regular conditional distribution of  $\theta$  given  $\mathcal{F}_t$ , that is, a probability kernel

$$K_t : \Omega \times \mathcal{B}(\Theta) \rightarrow [0, 1], \quad (\omega, B) \mapsto K_t(\omega, B),$$

such that:

- for each  $B \in \mathcal{B}(\Theta)$ , the map  $\omega \mapsto K_t(\omega, B)$  is  $\mathcal{F}_t$ -measurable;



- for each  $A \in \mathcal{F}_t$  and  $B \in \mathcal{B}(\Theta)$ ,  $\mathbb{P}(\theta \in B, A) = \int_A K_t(\omega, B) d\mathbb{P}(\omega)$ .

(See, for example, Dellacherie and Meyer (1978, Ch. VI, Thm. 2) or Kallenberg (2002, Thm. 6.3).)

Define  $S_t : \Omega \rightarrow E = \Delta(\Theta)$  by

$$S_t(\omega)(B) := K_t(\omega, B), \quad B \in \mathcal{B}(\Theta).$$

Since  $K_t(\omega, \cdot)$  is a probability measure on  $\Theta$  for  $\mathbb{P}$ -a.e.  $\omega$ ,  $S_t(\omega) \in E$  almost surely. The Borel  $\sigma$ -algebra on  $E$  is generated by the evaluation maps  $\mu \mapsto \mu(B)$  for  $B$  in a countable generating class of  $\mathcal{B}(\Theta)$ , and each composition  $\omega \mapsto S_t(\omega)(B) = K_t(\omega, B)$  is  $\mathcal{F}_t$ -measurable. It follows that  $S_t$  is  $\mathcal{F}_t$ -measurable as an  $E$ -valued random element (see, e.g., Kallenberg, 2002, Lem. 1.36).

Let  $\varphi : \Theta \rightarrow \mathbb{R}$  be bounded and Borel. Define the  $\mathcal{F}_t$ -measurable random variable

$$M_t(\omega) := \int_{\Theta} \varphi(\theta') S_t(\omega)(d\theta') = \int_{\Theta} \varphi(\theta') K_t(\omega, d\theta').$$

By the defining property of the regular conditional distribution, for every  $A \in \mathcal{F}_t$ ,

$$\int_A M_t(\omega) d\mathbb{P}(\omega) = \int_A \int_{\Theta} \varphi(\theta') K_t(\omega, d\theta') d\mathbb{P}(\omega) = \int_A \varphi(\theta(\omega)) d\mathbb{P}(\omega).$$

Thus  $M_t$  is a version of the conditional expectation  $\mathbb{E}[\varphi(\theta) \mid \mathcal{F}_t]$ , and we have

$$\int_{\Theta} \varphi dS_t = M_t = \mathbb{E}[\varphi(\theta) \mid \mathcal{F}_t] \quad \text{a.s.},$$

which proves the first claim.

For the martingale property, fix again any bounded Borel  $\varphi : \Theta \rightarrow \mathbb{R}$ . By the first part, for each  $t$ ,

$$\int_{\Theta} \varphi dS_t = \mathbb{E}[\varphi(\theta) \mid \mathcal{F}_t] \quad \text{a.s.}$$

Applying the tower property of conditional expectation,

$$\mathbb{E} \left[ \int_{\Theta} \varphi dS_{t+1} \mid \mathcal{F}_t \right] = \mathbb{E} \left[ \mathbb{E}[\varphi(\theta) \mid \mathcal{F}_{t+1}] \mid \mathcal{F}_t \right] = \mathbb{E}[\varphi(\theta) \mid \mathcal{F}_t] = \int_{\Theta} \varphi dS_t \quad \text{a.s.}$$

Hence  $\{S_t\}$  is a martingale with respect to  $(\mathcal{F}_t)$  in the usual weak sense: for every bounded Borel test function  $\varphi$  on  $\Theta$ , the real-valued process  $\left\{ \int_{\Theta} \varphi dS_t \right\}_{t \geq 0}$  is an  $(\mathcal{F}_t)$ -martingale. Since each  $S_t(\omega)$  is a probability measure, the process is bounded in total variation norm, which completes the proof.  $\square$

## C.2 Proof of Lemma 2

*Proof.* We proceed in three steps. Throughout,  $\Omega$  and  $\mathcal{H}_t$  are standard Borel (indeed Polish),  $\mathbb{P}$  is a Borel probability on  $(\Omega, \mathcal{F})$ , and  $\mathcal{R}_t$  has nonempty, bounded values and analytic graph.

**Step 1: From an analytic graph to an upper semianalytic payoff.** Define

$$r_t : \Omega \times \mathcal{H}_t \rightarrow \mathbb{R}, \quad r_t(\omega, \gamma) := \sup \mathcal{R}_t[\mathcal{F}_t, \gamma](\omega).$$

Nonemptiness and boundedness of  $\mathcal{R}_t[\mathcal{F}_t, \gamma](\omega)$  imply that  $r_t$  is finite and bounded on  $\Omega \times \mathcal{H}_t$ . Since  $\text{Gr}(\mathcal{R}_t) \subseteq \Omega \times \mathcal{H}_t \times \mathbb{R}$  is analytic with nonempty sections, the epigraph of  $r_t$  is analytic, hence  $r_t$  is upper semianalytic on  $\Omega \times \mathcal{H}_t$ ; see, for example, Castaing and Valadier (1977, Thm. III.38) or Bertsekas and Shreve (1978, Prop. 7.33). In particular,  $r_t$  is universally measurable and bounded.

**Step 2: Conditioning on the posterior.** Fix a public  $\sigma$ -algebra  $\mathcal{G}$  with  $\sigma(S_t) \subseteq \mathcal{G} \subseteq \mathcal{F}_t$ . For any  $\mathcal{G}$ -measurable submechanism  $\Gamma : \Omega \rightarrow \mathcal{H}_t$ , define

$$X_\Gamma(\omega) := r_t(\omega, \Gamma(\omega)), \quad \omega \in \Omega.$$

Because  $r_t$  is upper semianalytic and  $\Gamma$  is  $\mathcal{G}$ -measurable,  $X_\Gamma$  is bounded and universally measurable; see Bertsekas and Shreve (1978, Prop. 7.40).

Let  $S_t : \Omega \rightarrow E$  be the posterior kernel from Lemma 1, where  $E = \Delta(\Theta)$  is endowed with the weak topology. Since  $\sigma(S_t)$  is generated by the Borel map  $S_t : \Omega \rightarrow E$ , there exists a bounded  $\sigma(S_t)$ -measurable version of the conditional essential supremum

$$\widehat{X}_\Gamma = \text{ess sup}\{X_\Gamma \mid \sigma(S_t)\} \quad \text{a.s.},$$

which is the smallest  $\sigma(S_t)$ -measurable function dominating  $X_\Gamma$  almost surely; see Dellacherie and Meyer (1978, Ch. V, Sect. 57). By standard disintegration, there is a bounded Borel map  $u_t^{\mathcal{G}, \Gamma} : E \rightarrow \mathbb{R}$  such that

$$u_t^{\mathcal{G}, \Gamma}(S_t(\omega)) = \widehat{X}_\Gamma(\omega) = \text{ess sup}\{X_\Gamma \mid \sigma(S_t)\}(\omega) \quad \text{for } \mathbb{P}\text{-a.e. } \omega.$$

Moreover, one may choose  $u_t^{\mathcal{G}, \Gamma}$  upper semianalytic on  $E$  by applying the conditional version of the upper semianalyticity results above to a regular conditional distribution of  $(\omega, \Gamma(\omega))$  given  $S_t = s$ , pointwise in  $s \in E$ . Thus each  $u_t^{\mathcal{G}, \Gamma}$  is bounded and upper semianalytic on  $E$ , and

$$u_t^{\mathcal{G}, \Gamma}(s) = \text{ess sup}\{r_t(\omega, \Gamma(\omega)) \mid S_t = s\} \quad \text{for } S_t\text{-a.e. } s \in E,$$

where the conditional essential supremum is taken with respect to the regular conditional law

of  $\omega$  given  $S_t = s$ .

**Step 3: Taking the supremum over all  $\mathcal{G}$ -measurable submechanisms.** Define

$$U_t^{\mathcal{G}}(s) := \sup_{\Gamma: \mathcal{G}\text{-measurable}} u_t^{\mathcal{G}, \Gamma}(s), \quad s \in E.$$

We claim that  $U_t^{\mathcal{G}}$  is bounded and upper semianalytic on  $E$ .

Boundedness follows from Step 1: all values  $u_t^{\mathcal{G}, \Gamma}(s)$  lie in the bounded range of  $r_t$ , so their supremum is bounded as well.

To establish upper semianalyticity, we work directly with the epigraph of  $U_t^{\mathcal{G}}$ . For  $x \in \mathbb{R}$ , consider the strict upper level set

$$A_x := \{(s, \Gamma) \in E \times \mathcal{K} : u_t^{\mathcal{G}, \Gamma}(s) > x\},$$

where  $\mathcal{K}$  denotes the (possibly large) collection of  $\mathcal{G}$ -measurable maps  $\Gamma : \Omega \rightarrow \mathcal{H}_t$ . For each fixed  $\Gamma$ , the map  $s \mapsto u_t^{\mathcal{G}, \Gamma}(s)$  is upper semianalytic by Step 2, hence

$$\{(s, \Gamma) : u_t^{\mathcal{G}, \Gamma}(s) > x\}$$

is analytic in  $E \times \mathcal{K}$  as a pointwise union over  $\Gamma$  of upper level sets of upper semianalytic functions; see Castaing and Valadier (1977, Prop. III.40) or Bertsekas and Shreve (1978, Prop. 7.33 and 7.40). Thus  $A_x$  is analytic for each  $x$ .

By definition of  $U_t^{\mathcal{G}}$ ,

$$\{s \in E : U_t^{\mathcal{G}}(s) > x\} = \text{proj}_E(A_x),$$

the projection of an analytic set in  $E \times \mathcal{K}$ . Projections of analytic sets are analytic, so each strict upper level set  $\{s : U_t^{\mathcal{G}}(s) > x\}$  is analytic. This is exactly the definition of upper semianalyticity of  $U_t^{\mathcal{G}}$  on  $E$ .

Finally,  $E$  is a standard Borel space. Any bounded upper semianalytic function on a standard Borel space admits a Borel version, obtained by modification on a universally null set; see Bertsekas and Shreve (1978, Prop. 7.30). We therefore identify  $U_t^{\mathcal{G}}$  with such a bounded Borel version on  $E$ , which proves the lemma.  $\square$

### C.3 Proof of Proposition 2

*Proof.* (a)  $\Rightarrow$  (b). Assume  $U_t^{\mathcal{F}_t}(s) = U_t(s)$  for all  $s \in E$ . Fix an arbitrary public variable  $Y_t$  with  $\sigma(S_t) \subseteq \sigma(S_t, Y_t) \subseteq \mathcal{F}_t$  and any  $\mathcal{G}$  satisfying  $\sigma(S_t) \subseteq \mathcal{G} \subseteq \mathcal{F}_t$ . By monotonicity of the value with respect to information,

$$U_t(s) = U_t^{\sigma(S_t)}(s) \leq U_t^{\mathcal{G}}(s) \leq U_t^{\sigma(\mathcal{G}, Y_t)}(s) \leq U_t^{\mathcal{F}_t}(s) = U_t(s) \quad \forall s \in E.$$

Hence all inequalities must bind pointwise in  $s$ , and in particular

$$U_t^{\sigma(\mathcal{G}, Y_t)}(s) = U_t^{\mathcal{G}}(s) \quad \forall s \in E.$$

Since  $Y_t$  and  $\mathcal{G}$  were arbitrary, every non-posterior public variable is collapsible at date  $t$  to any intermediate  $\mathcal{G}$  between  $\sigma(S_t)$  and  $\mathcal{F}_t$ .

(b)  $\Rightarrow$  (a). Assume (b). Because  $(\Omega, \mathcal{F})$  is standard Borel and  $\sigma(S_t) \subseteq \mathcal{F}_t$ , there exists a public variable  $Y_t^\dagger$  such that

$$\mathcal{F}_t = \sigma(S_t, Y_t^\dagger).$$

Take  $\mathcal{G} = \sigma(S_t)$  and  $Y_t = Y_t^\dagger$  in (b). The collapsibility condition then gives

$$U_t^{\mathcal{F}_t}(s) = U_t^{\sigma(S_t, Y_t^\dagger)}(s) = U_t^{\sigma(S_t)}(s) = U_t(s) \quad \forall s \in E,$$

which is exactly (a). □

## C.4 Proof of Proposition 3

*Proof.* By Proposition 2, “collapse at date  $t$ ” is equivalent to

$$U_t^{\mathcal{F}_t}(s) = U_t(s) \quad \text{for all } s \in E$$

and, moreover, to the statement that any non-posterior public variable  $Y_t$  is collapsible at date  $t$  to any  $\mathcal{G}$  with  $\sigma(S_t) \subseteq \mathcal{G} \subseteq \mathcal{F}_t$ . Hence, for every  $t \in \mathcal{T}$  and every such  $\mathcal{G}_t$ , we must have

$$U_t^{\mathcal{G}_t}(s) = U_t(s) \quad \text{for all } s \in E.$$

Taking pointwise suprema over  $t$  yields

$$\widehat{g}(s) = \sup_t U_t^{\mathcal{G}_t}(s) = \sup_t U_t(s) = g(s) \quad \forall s \in E.$$

Since  $\widehat{g} = g$ , their concave envelopes coincide everywhere, so in particular  $(\text{conc } \widehat{g})(S_0) = (\text{conc } g)(S_0)$  for any prior  $S_0$ . By Theorem 1, this is equivalent to conditions  $(A_\varepsilon)$  and  $(A''_\varepsilon)$  therein. □

## C.5 Proof of Theorem 1

*Proof.* We first recall a standard representation of the concave envelope. For any bounded function  $f : E \rightarrow \mathbb{R}$ ,

$$(\text{conc } f)(s) = \inf\{\ell(s) : \ell \in \text{Aff}_c(E), \ell \geq f\}, \quad s \in E. \quad (\text{C.1})$$

In particular, for every  $\varepsilon > 0$  there exists an affine  $\varepsilon$ -support of  $\text{conc } g$  at  $S_0$ : an affine  $\ell \in \text{Aff}_c(E)$  with  $\ell \geq \text{conc } g$  and  $\ell(S_0) \leq (\text{conc } g)(S_0) + \varepsilon$ . Since  $\text{conc } g \geq g$ , such an  $\ell$  belongs to  $\mathcal{L}_g^\varepsilon(S_0)$ , so  $\mathcal{L}_g^\varepsilon(S_0) \neq \emptyset$ .

$(A_\varepsilon \Rightarrow (a))$ . Since  $\widehat{g} \geq g$ , we always have  $\text{conc } g \leq \text{conc } \widehat{g}$  and hence  $(\text{conc } g)(S_0) \leq (\text{conc } \widehat{g})(S_0)$ . Now suppose  $(A_\varepsilon)$  holds. Then for each  $\varepsilon > 0$  there exists  $\ell_\varepsilon \in \mathcal{L}_g^\varepsilon(S_0)$  such that  $\widehat{g} \leq \ell_\varepsilon$  on  $E$ . By (C.1),

$$(\text{conc } \widehat{g})(S_0) \leq \ell_\varepsilon(S_0) \leq (\text{conc } g)(S_0) + \varepsilon \quad \forall \varepsilon > 0.$$

Combining these inequalities yields

$$(\text{conc } g)(S_0) \leq (\text{conc } \widehat{g})(S_0) \leq (\text{conc } g)(S_0) + \varepsilon \quad \forall \varepsilon > 0,$$

and letting  $\varepsilon \downarrow 0$  proves  $(\text{conc } \widehat{g})(S_0) = (\text{conc } g)(S_0)$ .

$((a) \Rightarrow (A_\varepsilon))$ . Assume  $(\text{conc } \widehat{g})(S_0) = (\text{conc } g)(S_0)$ . Since  $\widehat{g} \geq g$ , we have  $\text{conc } \widehat{g} \geq \text{conc } g$ , so equality at  $S_0$  implies

$$(\text{conc } \widehat{g})(S_0) = (\text{conc } g)(S_0) = \inf\{\ell(S_0) : \ell \in \text{Aff}_c(E), \ell \geq g\}.$$

Fix  $\varepsilon > 0$ . By (C.1) with  $f = \widehat{g}$ , there exists an affine  $\ell_\varepsilon$  such that  $\ell_\varepsilon \geq \widehat{g}$  on  $E$  and

$$\ell_\varepsilon(S_0) \leq (\text{conc } \widehat{g})(S_0) + \varepsilon = (\text{conc } g)(S_0) + \varepsilon.$$

Since  $\widehat{g} \geq g$ , this  $\ell_\varepsilon$  also satisfies  $\ell_\varepsilon \geq g$ , hence  $\ell_\varepsilon \in \mathcal{L}_g^\varepsilon(S_0)$  and  $\widehat{g} \leq \ell_\varepsilon$  on  $E$ . This is precisely condition  $(A_\varepsilon)$ .

$(A_\varepsilon \Leftrightarrow (A'_\varepsilon))$ . Recall that  $\widehat{g}(s) = \sup_{t \in \mathcal{T}} U_t^{\mathcal{G}_t}(s)$ . If  $(A''_\varepsilon)$  holds, then for each  $\varepsilon > 0$  there exists  $\ell_\varepsilon \in \mathcal{L}_g^\varepsilon(S_0)$  such that  $U_t^{\mathcal{G}_t}(s) \leq \ell_\varepsilon(s)$  for all  $t$  and  $s$ . Taking the supremum over  $t$  yields  $\widehat{g}(s) = \sup_t U_t^{\mathcal{G}_t}(s) \leq \ell_\varepsilon(s)$  for all  $s$ , so  $(A_\varepsilon)$  holds.

Conversely, if  $(A_\varepsilon)$  holds, then for each  $\varepsilon > 0$  there exists  $\ell_\varepsilon \in \mathcal{L}_g^\varepsilon(S_0)$  such that  $\widehat{g}(s) \leq \ell_\varepsilon(s)$  for all  $s$ . Since  $U_t^{\mathcal{G}_t}(s) \leq \sup_t U_t^{\mathcal{G}_t}(s) = \widehat{g}(s)$  for every  $t$  and  $s$ , we obtain  $U_t^{\mathcal{G}_t}(s) \leq \ell_\varepsilon(s)$  for all  $t$  and  $s$ , which is  $(A''_\varepsilon)$ . This proves the equivalence.  $\square$

## C.6 Proof of Theorem 2

*Proof.* Recall that  $g(s) := \sup_{t \in \mathcal{T}} U_t(s)$  and  $\hat{g}(s) := \sup_{t \in \mathcal{T}} U_t^{\mathcal{F}_t}(s)$ , where  $U_t$  uses only the posterior  $S_t$  as state and  $U_t^{\mathcal{F}_t}$  allows full conditioning on the public history  $\mathcal{F}_t$ . By construction,  $\hat{g} \geq g$  pointwise on  $E$ .

Let

$$V^{\text{cal}} := \sup_{\mathcal{M}} \mathbb{E}[G(\mathcal{M})] \quad \text{and} \quad V^{\text{term}} := \sup_{\widetilde{\mathcal{M}} \text{ terminal}} \mathbb{E}[G(\widetilde{\mathcal{M}})]$$

denote, respectively, the optimal ex-ante values over all mechanism calendars and over terminal calendars as in Definition 2. The dynamic concavification argument developed above implies the value representation

$$V^{\text{cal}} = (\text{conc } \hat{g})(S_0), \quad V^{\text{term}} = (\text{conc } g)(S_0), \quad (\text{C.2})$$

and clearly  $V^{\text{term}} \leq V^{\text{cal}}$  since terminal calendars are a subset of all calendars.

(*Global condition*  $\Rightarrow$  *collapse*). Assume the global affine-support condition

$$\forall \varepsilon > 0, \exists \ell_\varepsilon \in \mathcal{L}_g^\varepsilon(S_0) \text{ such that } U_t^{\mathcal{F}_t}(s) \leq \ell_\varepsilon(s) \quad \forall s \in E, \forall t \in \mathcal{T} \quad (\text{A}_\varepsilon'')$$

holds. By Theorem 1, this is equivalent to

$$(\text{conc } \hat{g})(S_0) = (\text{conc } g)(S_0).$$

Using (C.2), we obtain

$$V^{\text{cal}} = V^{\text{term}}. \quad (\text{C.3})$$

By the definition of  $V^{\text{term}}$ , for every  $\varepsilon > 0$  there exists a terminal calendar  $\widetilde{\mathcal{M}}_\varepsilon$  such that

$$\mathbb{E}[G(\widetilde{\mathcal{M}}_\varepsilon)] \geq V^{\text{term}} - \varepsilon.$$

Let  $\mathcal{M}^*$  be a seller-optimal calendar, so that  $\mathbb{E}[G(\mathcal{M}^*)] = V^{\text{cal}}$ . Combining with (C.3) yields

$$\mathbb{E}[G(\widetilde{\mathcal{M}}_\varepsilon)] \geq V^{\text{term}} - \varepsilon = V^{\text{cal}} - \varepsilon = \mathbb{E}[G(\mathcal{M}^*)] - \varepsilon.$$

This is exactly the definition of collapse in Definition 2. Thus the global condition implies that the mechanism calendar collapses.

(*Collapse*  $\Rightarrow$  *global condition*). Conversely, suppose the calendar collapses. By Definition 2, for every  $\varepsilon > 0$  there exists a terminal calendar  $\widetilde{\mathcal{M}}_\varepsilon$  such that

$$\mathbb{E}[G(\widetilde{\mathcal{M}}_\varepsilon)] \geq \mathbb{E}[G(\mathcal{M}^*)] - \varepsilon = V^{\text{cal}} - \varepsilon.$$

Taking the supremum over terminal calendars on the left-hand side yields

$$V^{\text{term}} \geq V^{\text{cal}} - \varepsilon \quad \forall \varepsilon > 0.$$

Letting  $\varepsilon \downarrow 0$  and recalling that  $V^{\text{term}} \leq V^{\text{cal}}$ , we obtain

$$V^{\text{term}} = V^{\text{cal}}.$$

Using (C.2) again, this is equivalent to

$$(\text{conc } \widehat{g})(S_0) = (\text{conc } g)(S_0).$$

Now Theorem 1 shows that this equality holds if and only if the global affine-support condition  $(A''_\varepsilon)$  is satisfied. Hence collapse implies the global condition.

Combining the two directions proves the theorem.  $\square$

## C.7 Proof of Proposition 4

*Proof.* By Theorem 2, collapse is equivalent to the global affine-support condition: for every  $\varepsilon > 0$ , there exists  $\ell_\varepsilon \in \mathcal{L}_g^\varepsilon(S_0)$  such that

$$U_t^{\mathcal{F}^t}(s) \leq \ell_\varepsilon(s) \quad \forall s \in E, \forall t \in \mathcal{T}.$$

Fix  $\varepsilon > 0$  and such an  $\ell_\varepsilon$ . For any  $t$  and  $s$ , we then have  $(s, U_t^{\mathcal{F}^t}(s)) \notin \mathcal{B}_\varepsilon(S_0)$ , because  $U_t^{\mathcal{F}^t}(s) \leq \ell_\varepsilon(s) \leq H_\varepsilon(s)$  by definition of  $H_\varepsilon$ . Thus the calendar does not hit  $\mathcal{B}_\varepsilon(S_0)$  for any  $\varepsilon > 0$ . The equivalent contrapositive statement follows immediately.  $\square$