

# Optimal Dividend, Reinsurance and Capital Injection Strategies for Collaborating Business Lines: The Case of Excess-of-Loss Reinsurance

Tim J. Boonen\*

Engel John C. Dela Vega†

## Abstract

This paper considers an insurer with two collaborating business lines that must make three critical decisions: (1) dividend payout, (2) a combination of proportional and excess-of-loss reinsurance coverage, and (3) capital injection between the lines. The reserve level of each line is modeled using a diffusion approximation, with the insurer's objective being to maximize the weighted total discounted dividends paid until the first ruin time. We obtain the value function and the optimal strategies in closed form. We then prove that the optimal dividend payout strategy for bounded dividend rates is of threshold type, while for unbounded dividend rates it is of barrier type. The optimal combination of proportional and excess-of-loss reinsurance is shown to be pure excess-of-loss reinsurance. We also show that the optimal level of risk ceded to the reinsurer decreases as the aggregate reserve level increases. The optimal capital injection strategy involves transferring reserves to prevent the ruin of one line. Finally, numerical examples are presented to illustrate these optimal strategies.

## 1 Introduction

Optimal dividend payout problems have been a long-standing topic in the field of actuarial science, as dividends serve as a classical metric for assessing a company's value. In the seminal paper of [de Finetti \(1957\)](#), it is shown that the optimal strategy to distribute dividends is the one that maximizes the total discounted dividends paid to shareholders until bankruptcy time (i.e., the ruin time). This strategy is known as a *barrier strategy*, where dividends are paid only when the reserve level exceeds a certain level, called the barrier. In this framework, the amount of dividends to be paid is the excess of the reserve above the barrier, effectively capping the reserves at this level. Since [de Finetti \(1957\)](#), numerous extensions and variations have been explored (see, e.g., [Albrecher and Thonhauser, 2009](#); [Avanzi, 2009](#), for comprehensive reviews). However, it is important to note that most of these papers focus on the univariate case, where the company manages reserves for a single line of business.

We consider an extension of the classical problem studied by [de Finetti \(1957\)](#), involving a company with multiple lines of business, each with its own reserves. This scenario introduces a level of complexity that requires a multivariate approach to understanding the dividend distribution. In this context, defining the ruin time is more complicated. In the univariate case, the ruin time is simply defined as the first time when a company's reserves fall below zero. However, in the multivariate case, there are several definitions, including: (i) *first* ruin time: the first time when at least one reserve level falls below zero; (ii) *sum* ruin time: the first time when the sum of the reserves across all lines falls below zero; and (iii) *simultaneous* ruin time: the first time when all reserve levels fall below zero simultaneously. Consequently, much of the research in the multivariate case has focused on minimizing the probabilities of ruin (see, e.g., [Asmussen and Albrecher, 2010](#), Chapter XIII, and references therein).

Although many studies focus on minimizing the probabilities of ruin in the multivariate case, fewer have specifically aimed to maximize the total discounted dividends paid. To our knowledge, [Czarna and Palmowski \(2011\)](#) are the first to study the maximization of total discounted dividends for two business lines, using reserve processes that follow the Cramér-Lundberg (CL) model (i.e., a compound Poisson process). Subsequent studies using the CL model include [Liu and Cheung \(2014\)](#), [Albrecher et al. \(2017\)](#),

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\*Department of Statistics and Actuarial Science, School of Computing and Data Science, University of Hong Kong, China. Email: [tjboonen@hku.hk](mailto:tjboonen@hku.hk).

†Department of Statistics and Actuarial Science, School of Computing and Data Science, University of Hong Kong, China. Email: [ejdv@hku.hk](mailto:ejdv@hku.hk).

Azcue et al. (2019), Azcue and Muler (2021), and Strietzel and Heinrich (2022). Reserve levels modeled as diffusion processes in the multivariate setting have been explored in Gu et al. (2018), Grandits (2019), Yang et al. (2025), and Boonen et al. (2025).

In addition to maximizing the total discounted dividends, managing risks across multiple lines of business is essential to ensure the sustainability of dividend payouts. Risk control in the form of reinsurance has also been studied in the multivariate framework of optimal dividend problems. Proportional reinsurance has been explored in the multivariate case, as presented in Czarna and Palmowski (2011), Liu and Cheung (2014), Azcue et al. (2019), Strietzel and Heinrich (2022), Yang et al. (2025), and Boonen et al. (2025).

Given the complexities associated with the management of multiple business lines, effective risk management strategies are essential. One common approach is capital injection, which involves transferring reserves between lines to support a line that is at risk of its reserves falling below zero. This mechanism, sometimes referred to as *collaboration*, not only mitigates the risk of insolvency for individual lines but also improves the company's capacity to distribute dividends. In Albrecher et al. (2017), Gu et al. (2018), and Boonen et al. (2025), the rule of collaboration states that capital must be transferred from one line to another if a line is at risk of ruin, provided the transfer does not endanger the solvent line. This rule is modified in Grandits (2019), in which the solvent line is not obliged to transfer capital to the insolvent line.

The research agenda of this paper is to determine the optimal dividend payout, reinsurance, and capital injection strategies of an insurer with two business lines, subject to the following constraints: (1) the dividend rate may be bounded or unbounded; (2) the reinsurance contract combines proportional and excess-of-loss reinsurance; and (3) capital injections are used primarily to save one line from ruin (in the univariate sense). The goal is to maximize the weighted total discounted dividends paid until the first ruin time. The related work of Boonen et al. (2025) imposes stricter limitations, allowing only a bounded dividend rate and using only proportional reinsurance. We incorporate excess-of-loss reinsurance because it serves as a common alternative risk control mechanism in the univariate context of optimal dividend payout problems. Moreover, Asmussen et al. (2000) shows that optimal excess-of-loss reinsurance outperforms optimal proportional reinsurance. When studying risk measures of terminal losses without dividends, Aboagye et al. (2025) derive the optimal design of excess-of-loss reinsurance.

We summarize the main contributions of this paper:

1. We show that the optimal combination of proportional and excess-of-loss reinsurance is pure excess-of-loss reinsurance. This finding aligns with the results of Centeno (1985), Zhang et al. (2007), and Liang and Guo (2011) in the univariate setting.
2. We show that the optimal dividend payout strategy for bounded dividend rates is a *threshold* strategy in which dividends are continuously paid at a fixed rate whenever the aggregate reserve level exceeds a threshold. We also show that for unbounded dividend rates, the optimal strategy is a barrier strategy. These findings are consistent with the results presented in Asmussen et al. (2000) within the univariate setting.
3. In the case of bounded dividend rates, we identify three scenarios based on the following conditions: (a) the sum of the maximum dividend rates of the two lines is “large enough”, (b) the maximum dividend rate of the more important line is “large enough”, and (c) the support of the claim size distribution is finite. This leads to three configurations for the reinsurance threshold level  $w_0$  and the dividend threshold levels  $u_1$  and  $u_2$ : (i)  $w_0 \leq u_1 \leq u_2$ ; (ii)  $u_1 < w_0 \leq u_2$ ; and (iii)  $u_1 \leq u_2 < w_0$ . These configurations are similar to those found in Boonen et al. (2025). We also find that the reinsurance threshold level can exceed exactly one of the dividend threshold levels, which is a possibility ruled out in Asmussen et al. (2000). For unbounded dividend rates, we identify two scenarios based on whether the support of the claim size distribution is finite.
4. We show that the (excess-of-loss) reinsurance level decreases as the aggregate reserve level increases. Moreover, in the bounded support case, the reinsurance levels of both lines remain constant simultaneously when the aggregate reserve level is sufficiently large.

The remainder of the paper is organized as follows. Section 2 introduces the model and the formulation of the problem. The gain of pure excess-of-loss reinsurance is discussed in Section 3. Section 4 presents

the main results for the case of bounded dividend rates, while Section 5 presents the case of unbounded dividend rates. Numerical examples are provided in Section 6. The proofs of the main results are detailed in Section 7. Section 8 concludes.

## 2 Model

Consider a complete probability space  $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ , where  $\mathbb{F} := \{\mathcal{F}_t\}_{t \geq 0}$  is a right-continuous,  $\mathbb{P}$ -completed filtration to which all processes defined below, including the Brownian motions and the Poisson processes, are adapted.

We consider an insurer with two collaborating lines of business and model each line's reserve using the classical Cramér-Lundberg model:

$$P_i(t) = x_i + p_i t - \sum_{k=1}^{\Lambda_i(t)} Y_{i,k},$$

where, for  $i = 1, 2$ ,  $\Lambda_i := \{\Lambda_i(t)\}_{t \geq 0}$  is a Poisson process with intensity  $\lambda_i > 0$  that represents claim arrivals, and  $\{Y_{i,k}\}_{k \geq 1}$  are independent and identically distributed (i.i.d.), nonnegative claim sizes with common distribution  $F_i$  and finite first and second moments  $\tilde{\mu}_i$  and  $\tilde{\sigma}_i^2$ . The premium rate  $p_i$  is assumed to be calculated under the expected value principle, i.e.,

$$p_i = (1 + \kappa_i) \lambda_i \tilde{\mu}_i,$$

where  $\kappa_i > 0$  is the relative safety loading of Line  $i$ .

The insurer has to make three decisions regarding the operation of each line:

1. Reinsurance decision. The insurer arranges a combination of proportional and excess-of-loss reinsurance in the manner of Centeno (1985): For each line, the insurer first chooses a proportional retention level  $\theta_i(t) \in (0, 1]$ , and then chooses an excess-of-loss retention level  $\pi_i(t) \in [0, \infty]$  such that the insurer's  $k$ th claim, net of proportional and excess-of-loss reinsurance, can be represented by  $\min\{\theta_i(t)Y_{i,k}, \pi_i(t)\} = \theta_i(t)Y_{i,k} \wedge \pi_i(t)$ . We assume that the reinsurer applies the same expected value principle and relative safety loading  $\kappa_i$  as the insurer; this is sometimes referred to as "cheap reinsurance".
2. Dividend payout decision. The insurer chooses a dividend strategy to distribute profits to the shareholders of each line. Let  $C_i(t) \geq 0$  be the cumulative dividends paid by Line  $i$  at time  $t$ . The insurer can consider two types of dividend strategies: (1) an unbounded dividend rate:  $C_i(t)$  is an arbitrary nonnegative and nondecreasing function; and (2) a bounded dividend rate:  $C_i(t)$  satisfies  $C_i(t) = \int_0^t c_i(s)ds$ , where  $c_i(s) \in [0, \bar{c}_i]$  and  $\bar{c}_i > 0$  is the maximum possible dividend rate. For the unbounded dividend rates, we treat  $C_1$  and  $C_2$  as singular controls.
3. Capital injection decision. We assume that the insurer can inject capital from one line to the other without incurring any additional costs. The capital injection allows the insurer to prevent a line from bankruptcy by using the available resources within the company. Let  $L_i(t)$  be the cumulative amount of capital transferred to Line  $i$  from Line  $3 - i$ . Moreover, we treat  $L_1$  and  $L_2$  as singular controls.

After the purchase of reinsurance contracts, the reserve level of Line  $i$  is given by

$$R_i^{\theta, \pi}(t) = x_i + p_i^{\theta, \pi} t - \sum_{k=1}^{\Lambda_i(t)} (\theta_i(t)Y_{i,k} \wedge \pi_i(t)),$$

where

$$p_i^{\theta, \pi} = p_i - (1 + \kappa_i) \mathbb{E} \left[ \sum_{k=1}^{\Lambda_i(t)} (Y_{i,k} - (\theta_i(t)Y_{i,k} \wedge \pi_i(t))) \right] = (1 + \kappa_i) \lambda_i \mathbb{E}(\theta_i(t)Y_{i,k} \wedge \pi_i(t)).$$

Following Grandell (1977), the diffusion approximation of  $\{R_i^{\theta, \pi}(t)\}_{t \geq 0}$  is given by

$$dR_i(t) = \lambda_i \kappa_i \theta_i(t) \mu_i \left( \frac{\pi_i(t)}{\theta_i(t)} \right) dt + \sqrt{\lambda_i} \theta_i(t) \sigma_i \left( \frac{\pi_i(t)}{\theta_i(t)} \right) dW_i(t),$$

where  $W_1 := \{W_1(t)\}_{t \geq 0}$  and  $W_2 := \{W_2(t)\}_{t \geq 0}$  are independent Brownian motions and are independent of the Poisson processes  $\Lambda_1$  and  $\Lambda_2$ , and

$$\mu_i(s) := \int_0^s [1 - F_i(y)] dy \quad \text{and} \quad \sigma_i^2(s) := \int_0^s 2y[1 - F_i(y)] dy.$$

Without loss of generality, we assume  $\lambda_i = 1$ . Write  $\bar{F}_i(y) := 1 - F_i(y)$  and define

$$M_i := \inf\{y \geq 0 : \bar{F}_i(y) = 0\} \leq \infty.$$

By definition,  $\mu_i(M_i) = \tilde{\mu}_i$  and  $\sigma_i^2(M_i) = \tilde{\sigma}_i^2$ . Moreover, we have  $\pi_i(t) \in [0, M_i]$ .

Let  $\theta_i := \{\theta_i(t)\}_{t \geq 0}$ ,  $\pi_i := \{\pi_i(t)\}_{t \geq 0}$ ,  $C_i := \{C_i(t)\}_{t \geq 0}$ , and  $L_i := \{L_i(t)\}_{t \geq 0}$  be the proportional reinsurance, excess-of-loss reinsurance, dividend payout, and capital injection strategies, respectively, of Line  $i$ , for  $i = 1, 2$ . Given an admissible control  $u := (\theta_1, \theta_2, \pi_1, \pi_2, C_1, C_2, L_1, L_2)$ , the controlled reserve process of Line  $i$ , denoted by  $X_i := X_i^u$ , follows the dynamics

$$dX_i(t) = \kappa_i \theta_i(t) \mu_i \left( \frac{\pi_i(t)}{\theta_i(t)} \right) dt + \theta_i(t) \sigma_i \left( \frac{\pi_i(t)}{\theta_i(t)} \right) dW_i(t) - dC_i(t) + dL_i(t) - dL_{3-i}(t), \quad (2.1)$$

where  $X_i(0) = x_i \geq 0$  is the initial reserve of Line  $i$ . We define the corresponding ruin time as the first ruin time defined as

$$\tau := \inf\{t \geq 0 : \min\{X_1(t), X_2(t)\} < 0\}.$$

We formally define admissible strategies as follows:

**Definition 2.1.** A strategy  $u$  is said to be admissible if  $u$  is adapted to the filtration  $\mathbb{F}$  and satisfies the following conditions:

- (i)  $\theta_i(t) \in (0, 1]$  and  $\pi_i(t) \in [0, M_i]$  for  $i = 1, 2$  and  $t \geq 0$ ;
- (ii)  $C_i$  and  $L_i$  are nonnegative, nondecreasing, and right continuous with left limits, for  $i = 1, 2$ .

Denote by  $\mathcal{U}$  the set of all admissible strategies.

**Remark 2.2.** The constraint on the proportional reinsurance variables,  $\theta_i$ , implies that the business lines cannot cede all risk to the reinsurer; that is, full reinsurance is not allowed. On the other hand, the excess-of-loss retention can take any nonnegative value, including infinity. We say that  $F_i$  has a bounded support if  $M_i < \infty$ , and has unbounded support otherwise. We can interpret  $\pi_i(t) = M_i$  as Line  $i$  retaining all risk; that is, Line  $i$  does not engage in any reinsurance.

The goal of the insurer is to determine optimal dividend, (proportional and excess-of-loss) reinsurance, and capital injection strategies that maximize the weighted average of the dividend payouts from both lines up to the ruin time. That is, the insurer is faced with the following maximization problem:

$$\begin{aligned} V(x_1, x_2) &:= \sup_{u \in \mathcal{U}} J(x_1, x_2; u), \\ J(x_1, x_2; u) &:= \mathbb{E} \left[ a \int_0^\tau e^{-\delta t} dC_1(t) + (1-a) \int_0^\tau e^{-\delta t} dC_2(t) \right], \end{aligned} \quad (2.2)$$

where  $a \in [0, 1]$  is a weighting factor that reflects the relative importance of Line 1 for the company,  $\delta > 0$  is the discount factor, and the expectation  $\mathbb{E}$  is taken under  $X_1(0) = x_1$  and  $X_2(0) = x_2$ . We denote the objective function by  $J$  and the value function by  $V$ .

**Remark 2.3.** From the definition of the value function  $V$ , we can see that  $V$  is increasing in both arguments  $x_1$  and  $x_2$ ; that is,  $V$  increases as the initial reserves increase.

### 3 The Gain of Pure Excess-of-Loss Reinsurance

In this section, we will show that an optimal reinsurance, dividend payout, and capital injection strategy is of the form  $u_0 := (1, 1, \pi_1^0, \pi_2^0, C_1^0, C_2^0, L_1^0, L_2^0) \in \mathcal{U}$ ; that is, the optimal combination of proportional and excess-of-loss reinsurance strategies is the pure excess-of-loss reinsurance.

**Lemma 3.1.** Define

$$h_i(s) := \frac{\sigma_i^2(s)}{\mu_i^2(s)}.$$

Then,  $h_i(s)$  is an increasing function for  $s > 0$ .

*Proof.* Taking the derivative yields

$$h'_i(s) = \frac{2\bar{F}_i(s)[s\mu_i(s) - \sigma_i^2(s)]}{\mu_i^3(s)}.$$

Since  $0 \leq Y_{i,k} \wedge s \leq s$ , we have  $\mathbb{E}(Y_{i,k} \wedge s)^2 \leq \mathbb{E}(s(Y_{i,k} \wedge s)) = s\mathbb{E}(Y_{i,k} \wedge s)$ , and thus

$$\sigma_i^2(s) = \mathbb{E}(Y_{i,k} \wedge s)^2 \leq s\mathbb{E}(Y_{i,k} \wedge s) = s\mu_i(s).$$

This implies that  $h'_i(s) \geq 0$ , which completes the proof.  $\square$

**Proposition 3.2.** For any fixed admissible control  $u = (\theta_1, \theta_2, \pi_1, \pi_2, C_1, C_2, L_1, L_2)$ , where  $\theta_i(t) \in (0, 1)$ , there exists an admissible control  $u_0 = (1, 1, \pi_1^0, \pi_2^0, C_1^0, C_2^0, L_1^0, L_2^0)$  such that

$$J(x_1, x_2; u) \leq J(x_1, x_2; u_0).$$

*Proof.* For any fixed  $\theta_i(t) \in (0, 1)$ ,  $\pi_i(t)$ ,  $C_i(t)$ , and  $L_i(t)$ , there exists a unique  $\pi_i^0(t)$  such that

$$\theta_i(t)\sigma_i\left(\frac{\pi_i(t)}{\theta_i(t)}\right) = \sigma_i(\pi_i^0(t)), \quad i = 1, 2.$$

Since  $\sigma_i$  is a strictly increasing function and  $\theta_i(t) \in (0, 1)$ , it follows that  $\frac{\pi_i(t)}{\theta_i(t)} > \pi_i^0(t)$ . By Lemma 3.1,

$$h(\pi_i^0(t)) \leq h\left(\frac{\pi_i(t)}{\theta_i(t)}\right).$$

It follows that

$$\theta_i^2(t) = \frac{\sigma_i^2(\pi_i^0(t))}{\sigma_i^2\left(\frac{\pi_i(t)}{\theta_i(t)}\right)} \leq \frac{\mu_i^2(\pi_i^0(t))}{\mu_i^2\left(\frac{\pi_i(t)}{\theta_i(t)}\right)},$$

which implies that

$$\theta_i(t)\mu_i\left(\frac{\pi_i(t)}{\theta_i(t)}\right) \leq \mu_i(\pi_i^0(t)).$$

Let

$$C_i^0(t) := C_i(t) + \kappa_i \int_0^t \left[ \mu_i(\pi_i^0(s)) - \theta_i(s)\mu_i\left(\frac{\pi_i(s)}{\theta_i(s)}\right) \right] ds \geq C_i(t),$$

$$L_i^0(t) := L_i(t).$$

Then  $u_0$  is an admissible control and

$$\begin{aligned} dX_i^{u_0}(t) &= \kappa_i \mu_i(\pi_i^0(t))dt + \sigma_i(\pi_i^0(t))dW_i(t) - dC_i^0(t) + dL_i^0(t) - dL_{3-i}^0(t) \\ &= \kappa_i \theta_i(t)\mu_i\left(\frac{\pi_i(t)}{\theta_i(t)}\right)dt + \theta_i(t)\sigma_i\left(\frac{\pi_i(t)}{\theta_i(t)}\right)dW_i(t) - dC_i(t) + dL_i(t) - dL_{3-i}(t). \end{aligned}$$

Hence,  $X_i^{u_0}(t) \stackrel{\mathcal{D}}{=} X_i^u(t)$  while  $C_i^0(t) \geq C_i(t)$  and  $L_i^0(t) = L_i(t)$  for all  $t$ . The result then follows.  $\square$

Write  $u_1 := (1, 1, \pi_1, \pi_2, C_1, C_2, L_1, L_2) \in \mathcal{U}$ . The above proposition implies the following:

$$V(x_1, x_2) = \sup_{u_1 \in \mathcal{U}} J(x_1, x_2; u_1).$$

Hence, we only need to consider pure excess-of-loss reinsurance strategies to solve the maximization problem in (2.2).

## 4 Bounded Dividend Rates

In this section, we consider the case in which the dividend rate of Line  $i$  is bounded above by some constant  $\bar{c}_i \in (0, \infty)$ . In this case, we can represent  $C_i(t)$  as

$$C_i(t) = \int_0^t c_i(s) ds, \quad c_i(s) \in [0, \bar{c}_i].$$

We henceforth represent the dividend control by the rates  $c_1$  and  $c_2$ . Using the results of the previous section, we can then rewrite the reserve process of Line  $i$  given in (2.1) as

$$dX_i(t) = [\kappa_i \mu_i(\pi_i(t)) - c_i(t)] dt + \sigma_i(\pi_i(t)) dW_i(t) + dL_i(t) - dL_{3-i}(t),$$

and the value function defined in (2.2) as

$$V(x_1, x_2) = \sup_{u_1 \in \mathcal{U}} \mathbb{E} \left[ a \int_0^\tau e^{-\delta t} c_1(t) dt + (1-a) \int_0^\tau e^{-\delta t} c_2(t) dt \right]. \quad (4.1)$$

**Remark 4.1.** For the case of bounded dividend rates, we can immediately observe that as the initial reserves become sufficiently large, the lines pay the maximum dividend rates and  $V$  approaches the limit  $\frac{a\bar{c}_1 + (1-a)\bar{c}_2}{\delta}$ .

For all  $\pi_i \in [0, M_i]$  and  $c_i \in [0, \bar{c}_i]$ , define the generator  $\mathcal{L}^{\pi, c}(\phi)$  for some  $C^{2,2}$  function  $\phi$  by

$$\mathcal{L}^{\pi, c}(\phi) = \sum_{i=1}^2 \left[ [\kappa_i \mu_i(\pi_i) - c_i] \frac{\partial \phi}{\partial x_i} + \frac{1}{2} \sigma_i^2(\pi_i) \frac{\partial^2 \phi}{\partial x_i^2} \right] - \delta \phi + ac_1 + (1-a)c_2.$$

We solve the problem in (4.1) via the dynamic programming principle. Using arguments similar to those in Schmidli (2008, Chapter 2.5.1), we can characterize the value function  $V$  as a solution to the following Hamilton-Jacobi-Bellman (HJB) equation:

$$\sup \left\{ \sup_{\pi_i \in [0, M_i], c_i \in [0, \bar{c}_i]} \mathcal{L}^{\pi, c}(V), \frac{\partial V}{\partial x_1} - \frac{\partial V}{\partial x_2}, \frac{\partial V}{\partial x_2} - \frac{\partial V}{\partial x_1} \right\} = 0, \quad (4.2)$$

with boundary condition  $V(0, 0) = 0$ . If we can find a classical solution to the HJB equation (4.2), we can use a standard verification lemma, such as the one presented in Schmidli (2008, Theorem 2.51). This lemma essentially states that if a function satisfies the HJB equation and its boundary conditions, then the function is equal to the value function associated with the optimization problem. In this case, the classical solution we obtain will correspond to  $V$  defined in (4.1).

We begin the analysis by supposing that a classical solution  $V$  to the HJB equation in (4.2) exists. Since both  $\frac{\partial V}{\partial x_1} - \frac{\partial V}{\partial x_2} \leq 0$  and  $\frac{\partial V}{\partial x_2} - \frac{\partial V}{\partial x_1} \leq 0$  must hold by (4.2), it follows that  $\frac{\partial V}{\partial x_1} = \frac{\partial V}{\partial x_2}$ . Hence, there exists a univariate function  $g : x \in \mathbb{R}_+ \mapsto \mathbb{R}$  such that

$$g(x) = V(x_1, x_2), \text{ with } x := x_1 + x_2 \geq 0.$$

We then have the following

$$g'(x) = \frac{\partial V}{\partial x_i}(x_1, x_2) \quad \text{and} \quad g''(x) = \frac{\partial^2 V}{\partial x_i \partial x_j}(x_1, x_2), \quad i, j = 1, 2.$$

To solve the optimization problem  $\sup_{\pi_i \in [0, M_i], c_i \in [0, \bar{c}_i]} \mathcal{L}^{\pi, c}(V)$ , we first isolate the optimization over the dividend payout variable  $c_i$ :

$$\sup_{c_1 \in [0, \bar{c}_1], c_2 \in [0, \bar{c}_2]} \{(a - g'(x))c_1 + (1 - a - g'(x))c_2\}.$$

This yields the following candidate optimal dividend rates

$$\hat{c}_1(x) = \begin{cases} 0, & \text{if } g'(x) > a, \\ \bar{c}_1, & \text{if } g'(x) < a, \end{cases} \quad \text{and} \quad \hat{c}_2(x) = \begin{cases} 0, & \text{if } g'(x) > 1 - a, \\ \bar{c}_2, & \text{if } g'(x) < 1 - a. \end{cases}$$

Define the two constants  $u_1$  and  $u_2$  by

$$u_1 := \inf\{u : g'(u) = 1 - a\} \quad \text{and} \quad u_2 := \inf\{u : g'(u) = a\}. \quad (4.3)$$

We first hypothesize that  $g$  is a concave function; that is,  $g''(x) < 0$  for all  $x$ . By symmetry and without loss of generality, we assume  $a \leq \frac{1}{2}$ . Since  $g$  is concave and  $a \leq \frac{1}{2}$ , we have  $u_1 \leq u_2$ . We then have the following candidate for the optimal dividend strategies:

$$(\widehat{c}_1(x), \widehat{c}_2(x)) = \begin{cases} (0, 0), & \text{if } x < u_1, \\ (0, \bar{c}_2), & \text{if } u_1 < x < u_2, \\ (\bar{c}_1, \bar{c}_2), & \text{if } x > u_2. \end{cases}$$

Next, we solve the optimization over the reinsurance variable  $\pi_i$ :

$$\sup_{\pi_1 \in [0, M_1], \pi_2 \in [0, M_2]} \sum_{i=1}^2 \left[ \kappa_i \mu_i(\pi_i) g'(x) + \frac{1}{2} \sigma_i^2(\pi_i) g''(x) \right].$$

We then have the following candidate maximizer:

$$\widehat{\pi}_1(x) = -\kappa_1 \frac{g'(x)}{g''(x)} \quad \text{and} \quad \widehat{\pi}_2(x) = -\kappa_2 \frac{g'(x)}{g''(x)}. \quad (4.4)$$

We can express  $\widehat{\pi}_2$  as

$$\widehat{\pi}_2(x) = \frac{\kappa_2}{\kappa_1} \widehat{\pi}_1(x).$$

Since the capital injection decision  $(L_1, L_2)$  is represented as a singular control, the first-order conditions do not apply, unlike for the reinsurance and dividend controls. We will determine the optimal capital injection strategy later by examining the boundaries of specified regions.

We define  $w_0$  such that it satisfies

$$\widehat{\pi}_1(w_0) = M_1. \quad (4.5)$$

The existence of  $w_0$  will be studied in Theorems 4.2, 4.3, and 4.4. By definition, we can interpret  $w_0$  as the aggregate reserve level at which the insurer chooses *zero reinsurance* for Line 1. We refer to  $w_0$  as the *reinsurance threshold level*. Without loss of generality, we assume that  $\frac{M_2}{M_1} \geq \frac{\kappa_2}{\kappa_1}$ . This assumption, combined with (4.4), implies that the threshold for the insurer to retain all risk for Line 1 is not greater than that of Line 2.

We introduce the following notations that will be used in the discussion:

$$\begin{aligned} \overline{N}_1(y) &:= \kappa_1 \mu_1(y) + \kappa_2 \mu_2 \left( \frac{\kappa_2}{\kappa_1} y \right), & \overline{N}_2(y) &:= \sigma_1^2(y) + \sigma_2^2 \left( \frac{\kappa_2}{\kappa_1} y \right), \\ \overline{\gamma}_{2\pm}(y) &:= \frac{-\overline{N}_1(y) \pm \sqrt{\overline{N}_1(y)^2 + 2\delta \overline{N}_2(y)}}{\overline{N}_2(y)}, \\ \overline{\gamma}_{3\pm}(y) &:= \frac{-(\overline{N}_1(y) - \bar{c}_2) \pm \sqrt{(\overline{N}_1(y) - \bar{c}_2)^2 + 2\delta \overline{N}_2(y)}}{\overline{N}_2(y)}, \\ \overline{\gamma}_{4-}(y) &:= \frac{-(\overline{N}_1(y) - \bar{c}_1 - \bar{c}_2) - \sqrt{(\overline{N}_1(y) - \bar{c}_1 - \bar{c}_2)^2 + 2\delta \overline{N}_2(y)}}{\overline{N}_2(y)}. \end{aligned} \quad (4.6)$$

Write  $N_i := \overline{N}_i(M_1)$ ,  $\gamma_{2\pm} := \overline{\gamma}_{2\pm}(M_1)$ ,  $\gamma_{3\pm} := \overline{\gamma}_{3\pm}(M_1)$ , and  $\gamma_{4-} := \overline{\gamma}_{4-}(M_1)$ . In addition, we define two functions  $\psi, \zeta : (-\infty, 0) \mapsto \mathbb{R}$  by

$$\begin{aligned} \psi(z) &:= (1 - a - \gamma_{3-} z) e^{\gamma_{3+} \zeta(z)} + \gamma_{3-} z e^{\gamma_{3-} \zeta(z)} - a, \\ \zeta(z) &:= \frac{1}{\gamma_{3+} - \gamma_{3-}} \ln \left[ \frac{\gamma_{3-}(\gamma_{4-} - \gamma_{3-}) z}{(1 - a - \gamma_{3-} z)(\gamma_{3+} - \gamma_{4-})} \right]. \end{aligned} \quad (4.7)$$

We can now present the main results. The reinsurance threshold level  $w_0$  plays an important role in the results. Its position relative to the dividend threshold levels  $u_1$  and  $u_2$  yields three mutually exclusive cases: (i)  $w_0 \leq u_1 \leq u_2$  (Theorem 4.2), (ii)  $u_1 < w_0 \leq u_2$  (Theorem 4.3), and (iii)  $u_1 \leq u_2 < w_0$  (Theorem 4.4). Each case has its corresponding analytical form of the value function and a dividend-reinsurance strategy  $(\pi_1^*, \pi_2^*, c_1^*, c_2^*)$ . Since the capital injection strategies  $L_1^*$  and  $L_2^*$  are modeled as singular controls, we obtain a uniform optimal strategy as discussed in Theorem 4.5. The main results also cover whether  $F_1$  has bounded or unbounded support (i.e.,  $M_1 < \infty$  or  $M_1 = \infty$ ).

**Theorem 4.2.** Suppose (i)  $M_1 < \infty$ , (ii)  $\bar{c}_1 + \bar{c}_2 \geq N_1 - \frac{\kappa_1 N_2}{2M_1} + \frac{\delta M_1}{\kappa_1}$  and (iii)  $\psi(\alpha_{LB}) \leq 0$ , where  $\psi$  is defined by (4.7) and

$$\begin{aligned}\alpha_{LB} &:= \left[ \left( \frac{1-a}{\gamma_{3-}} - \alpha_0 \right) \mathbb{1}_{\{\bar{c}_2 \geq N_1 - \frac{\kappa_1 N_2}{2M_1} + \frac{\delta M_1}{\kappa_1}\}} + \alpha_0 \right] \mathbb{1}_{\{N_1 > \frac{\kappa_1 N_2}{2M_1}\}} \\ &\quad + \left[ \left( \frac{1-a}{\gamma_{3-}} - \underline{\alpha} \right) \mathbb{1}_{\{\bar{c}_2 \geq \frac{\delta N_2}{2N_1}\}} + \underline{\alpha} \right] \mathbb{1}_{\{N_1 \leq \frac{\kappa_1 N_2}{2M_1}\}}, \\ \alpha_0 &:= \frac{(1-a)\gamma_{3+}}{\gamma_{3+} - \gamma_{3-}} \left[ \frac{N_1}{\delta} - \frac{\kappa_1 N_2}{2\delta M_1} - \frac{1}{\gamma_{3+}} - \frac{\bar{c}_2}{\delta} \right], \\ \underline{\alpha} &:= -\frac{(1-a)\gamma_{3+}}{\gamma_{3+} - \gamma_{3-}} \left( \frac{1}{\gamma_{3+}} + \frac{\bar{c}_2}{\delta} \right).\end{aligned}\tag{4.8}$$

We have the following results:

1.  $w_0$  defined in (4.5) exists and is given by

$$w_0 = G(M_1),$$

where

$$G(y) := \int_0^y \frac{\sigma_1^2(z) + \sigma_2^2\left(\frac{\kappa_2}{\kappa_1}z\right)}{2\kappa_1 z \mu_1(z) + 2\kappa_2 z \mu_2\left(\frac{\kappa_2}{\kappa_1}z\right) + \frac{2\delta}{\kappa_1}z^2 - \kappa_1\left(\sigma_1^2(z) + \sigma_2^2\left(\frac{\kappa_2}{\kappa_1}z\right)\right)} dz,\tag{4.9}$$

and  $u_1$  and  $u_2$ , defined in (4.3), are explicitly given by

$$\begin{aligned}u_1 &= w_0 + \frac{1}{\gamma_{2+} - \gamma_{2-}} \ln \left( \frac{\alpha_{2-}(\gamma_{2-} - \alpha_3 - 1)}{\alpha_{2+}(1 - \gamma_{2+} + \alpha_3)} \right), \\ \text{and } u_2 &= u_1 + \frac{1}{\gamma_{3+} - \gamma_{3-}} \ln \left( \frac{K_{3-}\gamma_{3-}(\gamma_{4-} - \gamma_{3-})}{K_{3+}\gamma_{3+}(\gamma_{3+} - \gamma_{4-})} \right),\end{aligned}\tag{4.10}$$

where

$$\alpha_{2+} := \frac{-\gamma_{2-} - \frac{\kappa_1}{M_1}}{\gamma_{2+}(\gamma_{2+} - \gamma_{2-})}, \quad \alpha_{2-} := \frac{\gamma_{2+} + \frac{\kappa_1}{M_1}}{\gamma_{2-}(\gamma_{2+} - \gamma_{2-})},\tag{4.11}$$

$$\alpha_3 := \frac{1}{\gamma_{3+}} + \frac{\bar{c}_2}{\delta} + \left(1 - \frac{\gamma_{3-}}{\gamma_{3+}}\right) \frac{K_{3-}}{1-a}, \quad K_{3+} := \frac{1}{\gamma_{3+}} (1-a - K_{3-}\gamma_{3-}),\tag{4.12}$$

and  $K_{3-}$  is the unique solution to  $\psi(z) = 0$  on  $(\alpha_{LB}, \alpha_{UB})$  with

$$\alpha_{UB} := \frac{(1-a)(\gamma_{3+} - \gamma_{4-})}{\gamma_{3-}(\gamma_{3+} - \gamma_{3-})}.$$

The relation  $w_0 \leq u_1 \leq u_2$  holds.

2. The function  $g$ , defined by

$$g(x) = \begin{cases} K_1 \int_0^x \exp \left[ \int_z^{w_0} \frac{\kappa_1}{G^{-1}(y)} dy \right] dz & \text{if } x < w_0, \\ K_1 [\alpha_{2+} e^{\gamma_{2+}(x-w_0)} + \alpha_{2-} e^{\gamma_{2-}(x-w_0)}] & \text{if } w_0 \leq x < u_1, \\ K_{3+} e^{\gamma_{3+}(x-u_1)} + K_{3-} e^{\gamma_{3-}(x-u_1)} + \frac{(1-a)\bar{c}_2}{\delta} & \text{if } u_1 \leq x < u_2, \\ \frac{a}{\gamma_{4-}} e^{\gamma_{4-}(x-u_2)} + \frac{a\bar{c}_1 + (1-a)\bar{c}_2}{\delta} & \text{if } x \geq u_2,\end{cases}\tag{4.13}$$

where  $G^{-1}$  is the inverse of  $G$  defined in (4.9) and

$$K_1 = (1 - a) \left[ \alpha_{2+} \gamma_{2+} e^{\gamma_{2+}(u_1 - w_0)} + \alpha_{2-} \gamma_{2-} e^{\gamma_{2-}(u_1 - w_0)} \right]^{-1}, \quad (4.14)$$

is a classical solution to the HJB equation in (4.2) and thus equals the value function  $V$  of the optimization problem in (2.2). Moreover,  $g$  is strictly concave.

3. The optimal reinsurance and dividend strategies  $(\pi_1^*, \pi_2^*, c_1^*, c_2^*)$  are given by

$$(\pi_1^*, \pi_2^*, c_1^*, c_2^*)(x) = \begin{cases} \left( G^{-1}(x), \frac{\kappa_2}{\kappa_1} G^{-1}(x), 0, 0 \right) & \text{if } x < w_0, \\ \left( M_1, \frac{\kappa_2}{\kappa_1} M_1, 0, 0 \right) & \text{if } w_0 \leq x < u_1, \\ \left( M_1, \frac{\kappa_2}{\kappa_1} M_1, 0, \bar{c}_2 \right) & \text{if } u_1 \leq x < u_2, \\ \left( M_1, \frac{\kappa_2}{\kappa_1} M_1, \bar{c}_1, \bar{c}_2 \right) & \text{if } x \geq u_2. \end{cases}$$

We first discuss the optimal dividend strategies  $(c_1^*, c_2^*)$ . When the aggregate reserve level is low (i.e.,  $x < u_1$ ), the lines do not distribute dividends. Recall that we assume greater importance assigned to Line 2 (i.e.,  $a \leq \frac{1}{2}$ ). As soon as the first threshold  $u_1$  is crossed, Line 2 pays dividends at the maximum rate and Line 1 has to wait until the second threshold  $u_2$  is crossed.

For the optimal reinsurance strategy  $(\pi_1^*, \pi_2^*)$ , recall that we also assume that  $\frac{M_2}{M_1} \geq \frac{\kappa_2}{\kappa_1}$ . This ensures that Line 1 does not purchase excess-of-loss reinsurance while Line 2 retains a constant level of risk on incoming claims when the aggregate reserve level is sufficiently large (i.e.,  $x > w_0$ ).

Under condition (i), the distribution function  $F_1$  has bounded support, which implies that Line 1 has bounded claim sizes. A sufficient condition for conditions (ii) and (iii) is  $\bar{c}_2 > N_1 - \frac{\kappa_1 N_2}{2M_1} + \frac{\delta M_1}{\kappa_1}$ . This suggests that the maximum dividend rate of Line 1 can be relatively low while still satisfying the total maximum dividend rate condition. We also remark that the unique root of  $\psi$  ensures that  $g$  is differentiable at  $x = u_2$ .

The next theorem discusses the case when conditions (i) and (ii) of Theorem 4.2 are still satisfied, while condition (iii) is not.

**Theorem 4.3.** Suppose (i)  $M_1 < \infty$ , (ii)  $\bar{c}_1 + \bar{c}_2 \geq N_1 - \frac{\kappa_1 N_2}{2M_1} + \frac{\delta M_1}{\kappa_1}$  and (iii)  $\psi(\alpha_{LB}) > 0$ , where

$$\alpha_{LB} := (\alpha_0 - \underline{\alpha}) \mathbb{1}_{\{\bar{c}_2 > \frac{\delta N_2}{2N_1}\}} + \underline{\alpha}. \quad (4.15)$$

Here,  $\alpha_0$  and  $\underline{\alpha}$  are defined as in Theorem 4.2. We then have the following results:

1.  $w_0$  defined in (4.5) exists and is given by  $H(w_0) = M_1$ , where  $H$  satisfies the following differential equation

$$\frac{d}{dx} H(x) = \frac{2\kappa_1 z \mu_1(H(x)) + 2\kappa_2 z \mu_2\left(\frac{\kappa_2}{\kappa_1} H(x)\right) + \frac{2\delta}{\kappa_1} [H(x)]^2 - 2\bar{c}_2 H(x)}{\sigma_1^2(H(x)) + \sigma_2^2\left(\frac{\kappa_2}{\kappa_1} H(x)\right)} - \kappa_1, \quad (4.16)$$

and  $u_1$  and  $u_2$  defined in (4.3) are explicitly given by

$$\begin{aligned} u_1 &= w_0 - \frac{1}{\gamma_{3+} - \gamma_{3-}} \ln \left( \frac{K_{3-} \gamma_{3-} \left( \gamma_{3-} + \frac{\kappa_1}{M_1} \right)}{K_{3+} \gamma_{3+} \left( -\frac{\kappa_1}{M_1} - \gamma_{3+} \right)} \right), \text{ and} \\ u_2 &= w_0 + \frac{1}{\gamma_{3+} - \gamma_{3-}} \ln \left( \frac{(\gamma_{4-} - \gamma_{3-}) \left( -\frac{\kappa_1}{M_1} - \gamma_{3+} \right)}{(\gamma_{3+} - \gamma_{4-}) \left( \gamma_{3-} + \frac{\kappa_1}{M_1} \right)} \right), \end{aligned} \quad (4.17)$$

where  $K_{3+}$  is defined in (4.12) and  $K_{3-}$  is the unique solution to  $\psi(z) = 0$  on  $(\frac{1-a}{\gamma_{3-}}, \alpha_{LB})$ . Moreover, the relation  $u_1 < w_0 \leq u_2$  holds.

2. The function  $g$ , defined by

$$g(x) = \begin{cases} (1-a) \int_0^x \exp \left[ \int_z^{u_1} \frac{\kappa_1}{G^{-1}(y)} dy \right] dz & \text{if } x < u_1, \\ (1-a) \left[ \int_{u_1}^x \exp \left[ \int_z^{u_1} \frac{\kappa_1}{H(y)} dy \right] dz + \int_0^{u_1} \exp \left[ \int_z^{u_1} \frac{\kappa_1}{G^{-1}(y)} dy \right] dz \right] & \text{if } u_1 \leq x < w_0, \\ K_{3+} e^{\gamma_{3+}(x-u_1)} + K_{3-} e^{\gamma_{3-}(x-u_1)} + \frac{(1-a)\bar{c}_2}{\delta} & \text{if } w_0 \leq x < u_2, \\ \frac{a}{\gamma_{4-}} e^{\gamma_{4-}(x-u_2)} + \frac{a\bar{c}_1+(1-a)\bar{c}_2}{\delta} & \text{if } x \geq u_2, \end{cases}$$

where  $G^{-1}$  is the inverse of  $G$  defined in (4.9), is a classical solution to the HJB equation in (4.2) and thus equals the value function  $V$  of the optimization problem in (2.2). In addition,  $g$  is strictly concave.

3. The optimal reinsurance and dividend strategies  $(\pi_1^*, \pi_2^*, c_1^*, c_2^*)$  are given by

$$(\pi_1^*, \pi_2^*, c_1^*, c_2^*)(x) = \begin{cases} \left( G^{-1}(x), \frac{\kappa_2}{\kappa_1} G^{-1}(x), 0, 0 \right) & \text{if } x < u_1, \\ \left( H(x), \frac{\kappa_2}{\kappa_1} H(x), 0, \bar{c}_2 \right) & \text{if } u_1 \leq x < w_0, \\ \left( M_1, \frac{\kappa_2}{\kappa_1} M_1, 0, \bar{c}_2 \right) & \text{if } w_0 \leq x < u_2, \\ \left( M_1, \frac{\kappa_2}{\kappa_1} M_1, \bar{c}_1, \bar{c}_2 \right) & \text{if } x \geq u_2. \end{cases}$$

By comparing Theorems 4.2 and 4.3, we can observe notable similarities in the optimal reinsurance and dividend strategies in both cases. The main distinction is the form of the retention level when the aggregate reserve level is between  $w_0$  and  $u_1$ . Moreover, Theorem 4.2 provides an explicit expression for  $w_0$ , whereas in Theorem 4.3,  $w_0$  is presented implicitly.

In this scenario, a necessary condition is  $0 < \bar{c}_2 < N_1 - \frac{\kappa_1 N_2}{2M_1} + \frac{\delta M_1}{\kappa_1}$ , which means that the maximum dividend rate of Line 1,  $\bar{c}_1$ , must adequately compensate for the balance arising from the maximum dividend rate of Line 2 to achieve a minimum total of  $N_1 - \frac{\kappa_1 N_2}{2M_1} + \frac{\delta M_1}{\kappa_1}$ . This suggests that Line 1 may have a larger maximum dividend rate than Line 2, which could be strategically advantageous in improving overall financial performance by setting a higher maximum dividend rate.

It is important to note that Asmussen et al. (2000) show that having a reinsurance threshold level that exceeds the dividend threshold level is not possible in the univariate case. However, in our case, this scenario is possible.

The next theorem discusses the case in which either condition (i) or condition (ii) of Theorems 4.2 and 4.3 does not hold.

**Theorem 4.4.** Suppose that either (i)  $M_1 = \infty$  or (ii)  $\bar{c}_1 + \bar{c}_2 < N_1 - \frac{\kappa_1 N_2}{2M_1} + \frac{\delta M_1}{\kappa_1}$  holds. We have the following results:

1.  $w_0$  defined in (4.5) is infinite ( $w_0 = \infty$ ),  $u_1$  defined in (4.3) is the unique solution to

$$\int_{u_2}^x \frac{\kappa_1}{H(y)} dy = \ln \left( \frac{a}{1-a} \right) \quad (4.18)$$

in  $(0, u_2)$ , and  $u_2$  defined in (4.3) satisfies

$$H(u_2) = M_0,$$

where  $H$  satisfies (4.16) and  $M_0$  is a solution to

$$-\frac{\kappa_1}{y} = \bar{\gamma}_{4-}(y)$$

in  $(0, M_1)$ . Moreover, the relation  $u_1 \leq u_2 < w_0$  holds.

2. The function  $g$ , defined by

$$g(x) = \begin{cases} (1-a) \int_0^x \exp \left[ \int_z^{u_1} \frac{\kappa_1}{G^{-1}(y)} dy \right] dz & \text{if } x < u_1, \\ (1-a) \left[ \int_{u_1}^x \exp \left[ \int_z^{u_1} \frac{\kappa_1}{H(y)} dy \right] dz + \int_0^{u_1} \exp \left[ \int_z^{u_1} \frac{\kappa_1}{G^{-1}(y)} dy \right] dz \right] & \text{if } u_1 \leq x < u_2, \\ \frac{a}{\bar{\gamma}_{4-}(M_0)} e^{\bar{\gamma}_{4-}(M_0)(x-u_2)} + \frac{a\bar{c}_1+(1-a)\bar{c}_2}{\delta} & \text{if } x \geq u_2, \end{cases}$$

where  $G^{-1}$  is the inverse of  $G$  defined in (4.9), is a classical solution to the HJB equation in (4.2) and thus equals the value function  $V$  of the optimization problem in (2.2). In addition,  $g$  is strictly concave.

3. The optimal reinsurance and dividend strategies  $(\pi_1^*, \pi_2^*, c_1^*, c_2^*)$  are given by

$$(\pi_1^*, \pi_2^*, c_1^*, c_2^*)(x) = \begin{cases} \left(G^{-1}(x), \frac{\kappa_2}{\kappa_1}G^{-1}(x), 0, 0\right) & \text{if } x < u_1, \\ \left(H(x), \frac{\kappa_2}{\kappa_1}H(x), 0, \bar{c}_2\right) & \text{if } u_1 \leq x < u_2, \\ \left(M_0, \frac{\kappa_2}{\kappa_1}M_0, \bar{c}_1, \bar{c}_2\right) & \text{if } x \geq u_2. \end{cases}$$

Under condition (i), the support of the distribution function  $F_1$  is unbounded, which implies that  $w_0$  defined in (4.5) does not exist. Under condition (ii), the two lines can pay at most  $N_1 - \frac{\kappa_1 N_2}{2M_1} + \frac{\delta M_1}{\kappa_1}$  in total. This frees up some of the reserves, allowing the insurer to purchase more reinsurance. Similar to the results in [Asmussen et al. \(2000\)](#), a reinsurance threshold that exceeds all dividend thresholds is not possible.

We now discuss the optimal capital injection strategy. We partition the domain of the reserve level pair  $(x_1, x_2) \in \mathbb{R}_+^2$  into seven regions (see Figure 1). Define the constants  $\delta_i$ ,  $i = 0, 1, 2$ , corresponding to each of the three cases in Theorems 4.2, 4.3, and 4.4, by

$$(\delta_0, \delta_1, \delta_2) = \begin{cases} (w_0, u_1, u_2) & \text{if } w_0 \leq u_1 \leq u_2, \\ (u_1, w_0, u_2) & \text{if } u_1 < w_0 \leq u_2, \\ (u_1, u_1, u_2) & \text{if } u_1 \leq u_2 < w_0. \end{cases}$$

The seven regions  $A_i$ ,  $i = 1, 2, \dots, 7$ , are defined as follows (see Figure 1):

- $A_1 = \{(x_1, x_2) : x_1 \geq 0, x_2 > \delta_2\}$ ,
- $A_2 = \{(x_1, x_2) : x_1 > 0, x_2 \in [0, \delta_2], x_1 + x_2 > \delta_2\}$ ,
- $A_3 = \{(x_1, x_2) : x_1 \geq 0, x_2 \in (\delta_1, \delta_2], x_1 + x_2 \leq \delta_2\}$ ,
- $A_4 = \{(x_1, x_2) : x_1 > 0, x_2 \in [0, \delta_1], x_1 + x_2 \in (\delta_1, \delta_2]\}$ ,
- $A_5 = \{(x_1, x_2) : x_1 \geq 0, x_2 \in (\delta_0, \delta_1], x_1 + x_2 \leq \delta_1\}$ ,
- $A_6 = \{(x_1, x_2) : x_1 > 0, x_2 \in [0, \delta_0], x_1 + x_2 \in (\delta_0, \delta_1]\}$ ,
- $A_7 = \{(x_1, x_2) : x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \leq \delta_0\}$ .

**Theorem 4.5.** *The optimal capital injection strategy is given by one of the following cases:*

1. If  $x \in A_1$  and Line 1 reaches zero, Line 2 transfers an amount of  $x_2 - \delta_2$  to Line 1, and we proceed to region  $A_2$ . If Line 1 does not reach zero, we remain in  $A_1$  until we move to region  $A_2$  or  $A_3$ .
2. If  $x \in A_2$  and Line 2 reaches zero, Line 1 transfers an amount of  $x_1 - \delta_2$  to Line 2. We remain in  $A_2$  until we move to region  $A_1$ ,  $A_3$ , or  $A_4$ , regardless of whether Line 2 reaches zero.
3. If  $x \in A_3$  and Line 1 reaches zero, Line 2 transfers an amount of  $x_2 - \delta_1$  to Line 1, and we proceed to region  $A_4$ . If Line 1 does not reach zero, we remain in  $A_3$  until we move to region  $A_2$ ,  $A_4$ , or  $A_5$ .
4. If  $x \in A_4$  and Line 2 reaches zero, Line 1 transfers an amount of  $x_1 - \delta_1$  to Line 2. We remain in  $A_4$  until we move to region  $A_2$ ,  $A_3$ ,  $A_5$ , or  $A_6$ , regardless of whether Line 2 reaches zero.
5. If  $x \in A_5$  and Line 1 reaches zero, Line 2 transfers an amount of  $x_2 - \delta_0$  to Line 1, and we proceed to region  $A_6$ . If Line 1 does not reach zero, we stay in  $A_5$  until we move to region  $A_4$ ,  $A_6$ , or  $A_7$ .
6. If  $x \in A_6$  and Line 2 reaches zero, Line 1 transfers an amount of  $x_1 - \delta_0$  to Line 2. We remain in  $A_6$  until we move to region  $A_4$ ,  $A_5$ , or  $A_7$ , regardless of whether Line 2 reaches zero.
7. If  $x \in A_7$ , we remain in  $A_7$  until we move to region  $A_6$ . The problem ends when the reserves exit the nonnegative quadrant.

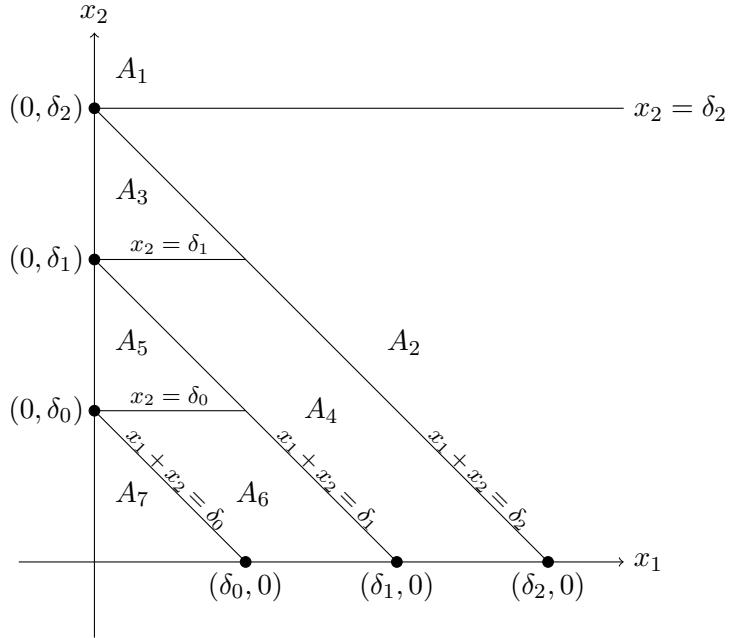


Figure 1: Regions for Capital Injection Decisions

## 5 Unbounded Dividend Rates

In this section, we consider the case in which there are no restrictions on the dividend rates of both lines. The corresponding HJB equation is given by:

$$\sup \left\{ \sup_{\pi_i \in [0, M_i]} \mathcal{L}^\pi(V), \quad \frac{\partial V}{\partial x_1} - \frac{\partial V}{\partial x_2}, \quad \frac{\partial V}{\partial x_2} - \frac{\partial V}{\partial x_1}, a - \frac{\partial V}{\partial x_1}, 1 - a - \frac{\partial V}{\partial x_2} \right\} = 0,$$

with boundary condition  $V(0, 0) = 0$  and

$$\mathcal{L}^\pi(\phi) := \sum_{i=1}^2 \left[ \kappa_i \mu_i(\pi_i) \frac{\partial \phi}{\partial x_i} + \frac{1}{2} \sigma_i^2(\pi_i) \frac{\partial^2 \phi}{\partial x_i^2} \right], \quad \phi \in C^{2,2}.$$

The first, fourth, and fifth terms within the outer supremum function can be handled using arguments similar to those in Schmidli (2008, Proposition 2.54). The second and third terms arise from the capital injection strategies modeled as singular controls.

As in the previous section, a standard verification lemma for unbounded dividend rates implies that a classical solution, if it exists, is the value function that solves the associated optimization problem.

We now present the main results for the unbounded dividend rates. Two cases arise depending on whether  $M_1$  is finite (Theorem 5.1) or infinite (Theorem 5.2). As in the previous section, each case has its corresponding analytical form of the value function and a reinsurance strategy  $(\pi_1^*, \pi_2^*)$ . Since the dividend payout strategies  $C_1$  and  $C_2$  are also modeled as singular controls, similar to the capital injection strategies, we obtain a uniform optimal strategy  $(C_1^*, C_2^*, L_1^*, L_2^*)$  in Theorem 5.3.

**Theorem 5.1.** *Suppose  $M_1 < \infty$ . We have the following results:*

1.  $w_0$  defined in (4.5) exists and is given by

$$w_0 = G(M_1),$$

where  $G$  is defined in (4.9), and  $u_1$  defined in (4.3) is explicitly given by

$$u_1 = w_0 + \frac{1}{\gamma_{2+} - \gamma_{2-}} \ln \left( \frac{\gamma_{2-}(\kappa_1 + \gamma_{2+}M_1)}{\gamma_{2+}(\kappa_1 + \gamma_{2-}M_1)} \right), \quad (5.1)$$

where  $\gamma_{2\pm}$  equals  $\gamma_{2\pm}(M_1)$  given in (4.6). The relation  $w_0 \leq u_1$  holds.

2. The function  $g$ , defined by

$$g(x) = \begin{cases} K_1 \int_0^x \exp \left[ \int_z^{w_0} \frac{\kappa_1}{G^{-1}(y)} dy \right] dz & \text{if } x < w_0, \\ \frac{1-a}{\gamma_{2+}\gamma_{2-}(\gamma_{2+}-\gamma_{2-})} [\gamma_{2+}^2 e^{\gamma_{2-}(x-u_1)} - \gamma_{2-}^2 e^{\gamma_{2+}(x-u_1)}] & \text{if } w_0 \leq x < u_1, \\ (1-a) [x - u_1 + \frac{N_1}{\delta}] & \text{if } x \geq u_1, \end{cases} \quad (5.2)$$

where  $G^{-1}$  is the inverse of  $G$  defined in (4.9),  $N_1$  equals  $\bar{N}_1(M_1)$  defined in (4.6), and

$$K_1 = \frac{1-a}{\gamma_{2+}-\gamma_{2-}} \left[ \gamma_{2+} e^{\gamma_{2-}(w_0-u_1)} - \gamma_{2-} e^{\gamma_{2+}(w_0-u_1)} \right],$$

is a classical solution to the HJB equation in (4.2) and thus equals the value function  $V$  of the optimization problem in (2.2). Moreover,  $g$  is concave.

3. The optimal reinsurance strategy  $(\pi_1^*, \pi_2^*)$  is given by

$$(\pi_1^*, \pi_2^*)(x) = \begin{cases} \left( G^{-1}(x), \frac{\kappa_2}{\kappa_1} G^{-1}(x) \right) & \text{if } x < w_0, \\ \left( M_1, \frac{\kappa_2}{\kappa_1} M_1 \right) & \text{if } x \geq w_0. \end{cases}$$

Similar to the bounded case where  $w_0 \leq u_1$ , Theorem 5.1 shows that the optimal reinsurance strategy remains constant for both lines when the aggregate reserve levels reach the threshold  $w_0$ .

**Theorem 5.2.** Suppose  $M_1 = \infty$ . We have the following results:

1.  $w_0$  defined in (4.5) is infinite ( $w_0 = \infty$ ) and  $u_1$  defined in (4.3) is explicitly given by

$$u_1 = G(\infty) := \lim_{y \rightarrow \infty} G(y),$$

where  $G$  is defined by (4.9). The relation  $u_1 < w_0$  holds.

2. The function  $g$ , defined by

$$g(x) = \begin{cases} (1-a) \int_0^x \exp \left[ \int_z^{u_1} \frac{\kappa_1}{G^{-1}(y)} dy \right] dz & \text{if } x < u_1, \\ (1-a) [x - u_1 + \frac{N_1}{\delta}] & \text{if } x \geq u_1, \end{cases} \quad (5.3)$$

where  $G^{-1}$  is the inverse of  $G$  defined in (4.9) and  $N_1$  equals  $\bar{N}_1(M_1)$  defined in (4.6), is a classical solution to the HJB equation in (4.2) and thus equals the value function  $V$  of the optimization problem in (2.2). Moreover,  $g$  is concave.

3. The optimal reinsurance strategy  $(\pi_1^*, \pi_2^*)$  is given by

$$(\pi_1^*, \pi_2^*)(x) = \left( G^{-1}(x), \frac{\kappa_2}{\kappa_1} G^{-1}(x) \right), \quad x < u_1.$$

Theorem 5.2, in conjunction with Theorem 5.3 below, implies that the aggregate reserve level must remain below  $x = u_1$  since the aggregate reserves above  $u_1$  are transferred between lines and are immediately paid as dividends.

For the optimal dividend payout and capital injection strategy, we partition the domain of the reserve level pair  $(x_1, x_2) \in \mathbb{R}_+^2$  into 3 regions (see Figure 2). The three regions  $A_i$ ,  $i = 1, 2, 3$ , are defined as follows:

- $A_1 = \{(x_1, x_2) : x_2 \geq 0, x_1 > u_1\}$ ,
- $A_2 = \{(x_1, x_2) : x_1 \in [0, u_1], x_2 > 0, x_1 + x_2 > u_1\}$ ,
- $A_3 = \{(x_1, x_2) : x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \leq u_1\}$ .

**Theorem 5.3.** The optimal dividend payout and capital injection strategy is given by one of the following cases:

1. If  $x \in A_1$ , Line 1 transfers an amount  $x_1 - u_1$  to Line 2, and we proceed to region  $A_2$ .
2. If  $x \in A_2$ , Line 2 pays  $x_1 + x_2 - u_1$  directly as dividends, and we proceed to region  $A_1$ .
3. If  $x \in A_3$ , no dividends are paid and we remain in  $A_3$  until we move to region  $A_2$ . The problem ends when the reserves exit the nonnegative quadrant.

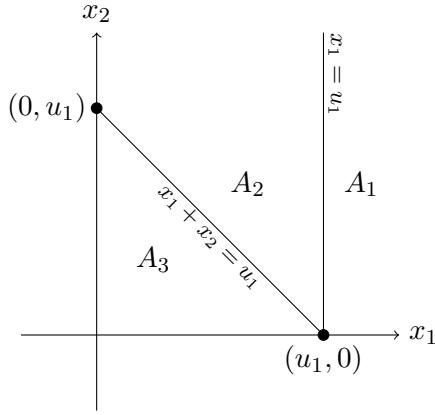


Figure 2: Regions for Dividend Payout and Capital Injection Decisions

## 6 Numerical Examples

In this section, we present several numerical examples to gain further insight into the main results in Sections 4 and 5. Since the capital injection strategies were presented in Theorems 4.5 and 5.3 and the dividend payout strategy is a threshold strategy in the bounded-dividend case and a singular control in the unbounded-dividend case, we focus on illustrating the optimal (excess-of-loss) reinsurance strategies for each of the main results, along with their corresponding value functions.

We fix the following parameters:  $\kappa_1 = 4$ ,  $\kappa_2 = 2$ ,  $\delta = 0.5$ , and  $a = 0.3$ . For the claim size distributions  $F_1$  and  $F_2$  in the bounded support case, we use uniform distributions on  $[0, 1]$  and  $[0, 1.5]$ , respectively, which correspond to  $M_1 = 1$  and  $M_2 = 1.5$ . In the unbounded support case ( $M_1 = M_2 = \infty$ ), we use exponential distributions with parameters 1 and 1.5. Figures 3 to 7 display the value functions on the left and the reinsurance strategies on the right. The vertical dotted lines represent the threshold levels.

With these parameters, the optimal reinsurance (retention) level for Line 2 is exactly half that of Line 1:  $\frac{\kappa_2}{\kappa_1} = \frac{1}{2}$ . For uniformly distributed claim sizes, those in Lines 1 and 2 are capped at  $M_1 = 1$  and  $M_2 = 1.5$ , respectively. This means that if the optimal excess-of-loss retention level for Line 1 is equal to 1, then the insurer retains all risk. For an exponential distribution, claim sizes have no upper bound and can take any value in  $[0, \infty)$ .

Figures 3 to 5 correspond to the main results for the bounded dividend rates in Section 4. The maximum dividend rates of the business lines vary, and their sum decreases noticeably across the three figures. Figures 6 and 7 correspond to the main results for the unbounded dividend rates in Section 5. The optimal excess-of-loss retention level increases as the aggregate reserve level increases across all examples. Equivalently, the amount of risk ceded to the reinsurer decreases as the aggregate reserve level increases.

By definition, the reinsurance threshold level  $w_0$  indicates that the insurer should retain all risks of Line 1 once the aggregate reserve level exceeds it. This is evident in Figures 3, 4, and 6, where the retention levels for both lines remain constant (i.e., flat) when the aggregate reserve level is greater than  $w_0$ . In the scenario where  $w_0$  does not exist, as stated in Theorem 4.4, the insurer must always transfer a portion of the risk to the reinsurer, regardless of the aggregate reserve level. Figure 5 illustrates this scenario, where the retained risk for Line 1 is capped at  $M_0 = 0.71$ .

In Figure 7, it is worth noting that the level of retained risk increases without bound, with  $x = u_1 = 1.25$  acting as a vertical asymptote for the reinsurance level. As stated in Theorem 5.3, the aggregate reserve level must remain below  $u_1$  since any reserve above  $u_1$  is allocated to dividends and capital transfers between lines.

The value functions are all increasing and concave. For the bounded dividend rates (Figures 3 to 5), they approach the limit  $\frac{a\bar{c}_1 + (1-a)\bar{c}_2}{\delta}$ . In particular, the value function in Figure 5 approaches this limit more quickly than in the other cases, attributed to the smaller values of  $u_1$  and  $u_2$ .

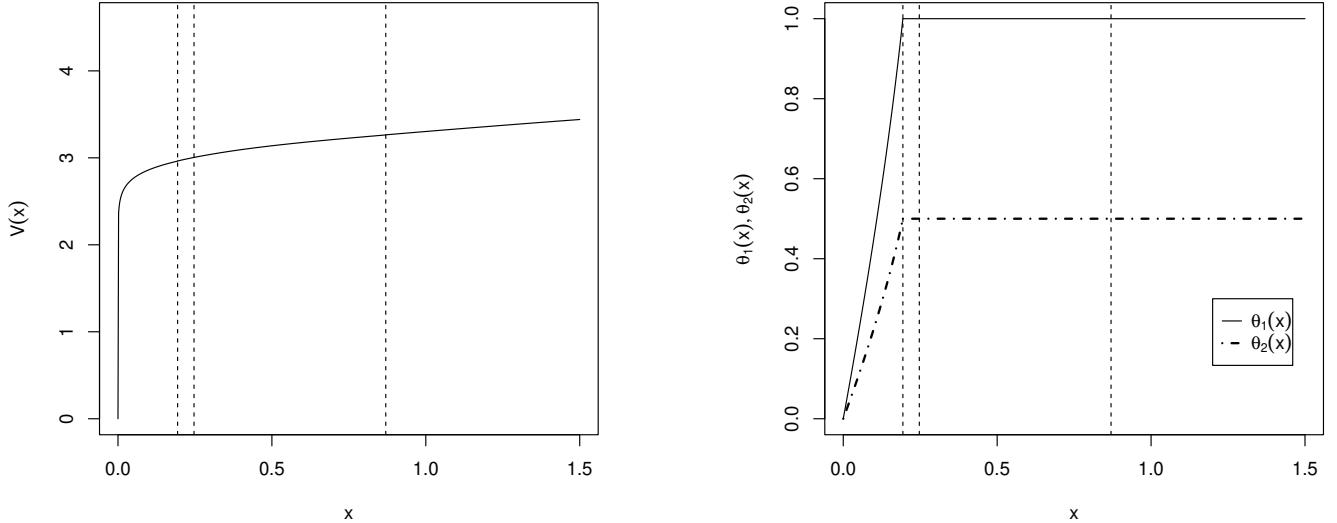


Figure 3: Uniformly distributed claims:  $\kappa_1 = 4$ ,  $\kappa_2 = 2$ ,  $\delta = 0.5$ ,  $a = 0.3$ ,  $\bar{c}_1 = 3$ ,  $\bar{c}_2 = 2$   
 $(M_1 = 1$ ,  $M_2 = 1.5$ ,  $w_0 = 0.19 < u_1 = 0.24 < u_2 = 0.87$ , corresponding to Theorem 4.2)

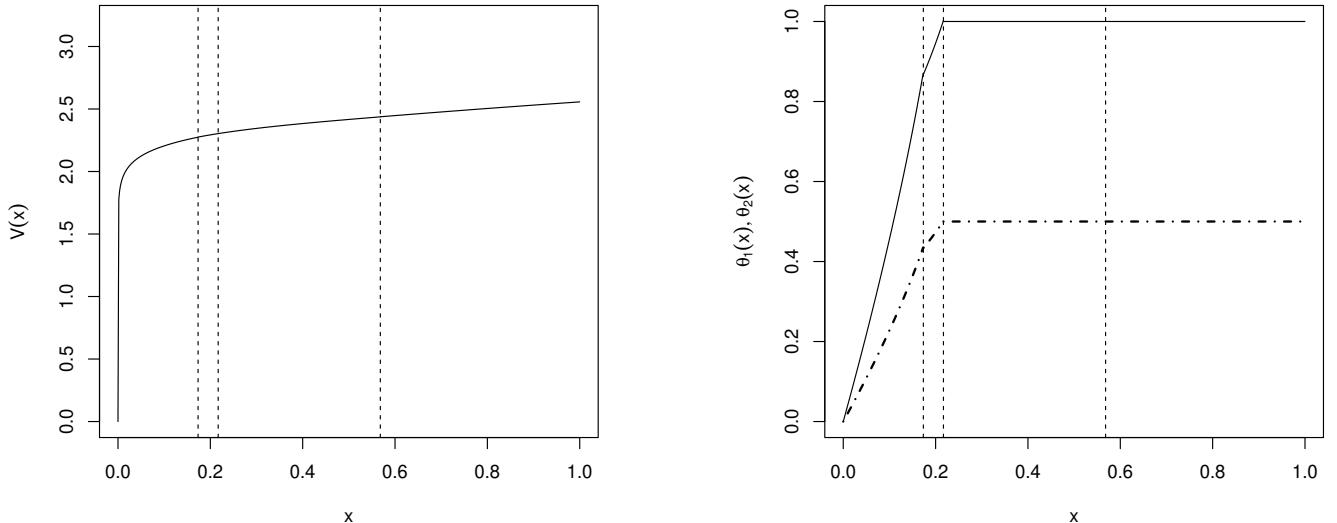


Figure 4: Uniformly distributed claims:  $\kappa_1 = 4$ ,  $\kappa_2 = 2$ ,  $\delta = 0.5$ ,  $a = 0.3$ ,  $\bar{c}_1 = 3$ ,  $\bar{c}_2 = 1$   
 $(M_1 = 1$ ,  $M_2 = 1.5$ ,  $u_1 = 0.17 < w_0 = 0.22 < u_2 = 0.57$ , corresponding to Theorem 4.3)

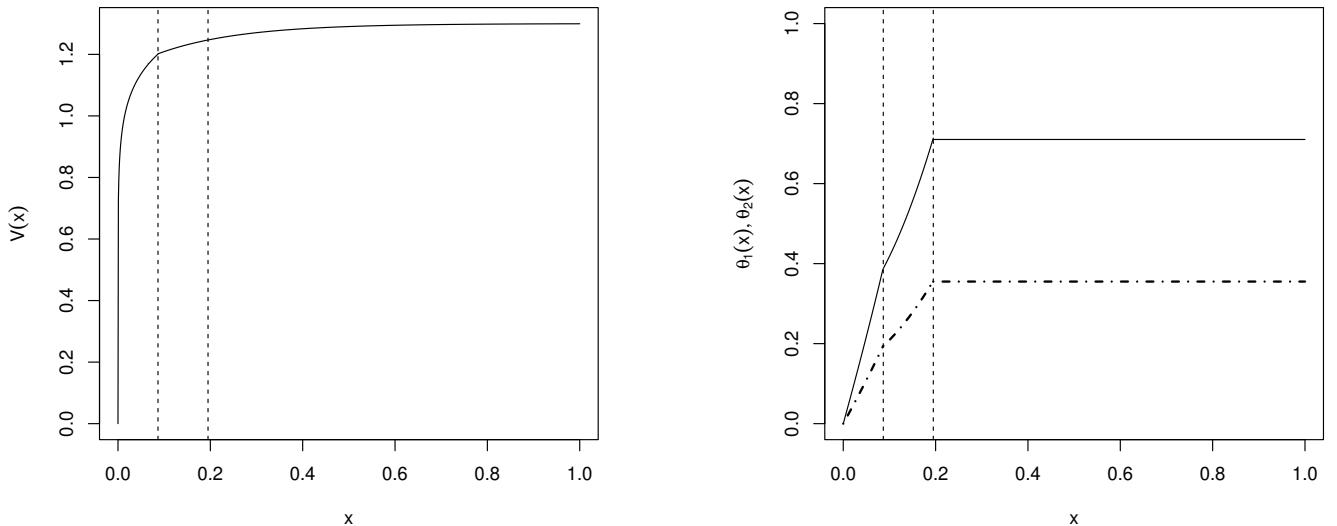


Figure 5: Uniformly distributed claims:  $\kappa_1 = 4$ ,  $\kappa_2 = 2$ ,  $\delta = 0.5$ ,  $a = 0.3$ ,  $\bar{c}_1 = 1$ ,  $\bar{c}_2 = 0.5$   
 $(M_0 = 0.71$ ,  $M_1 = 1$ ,  $M_2 = 1.5$ ,  $u_1 = 0.09 < u_2 = 0.19$ , corresponding to Theorem 4.4)

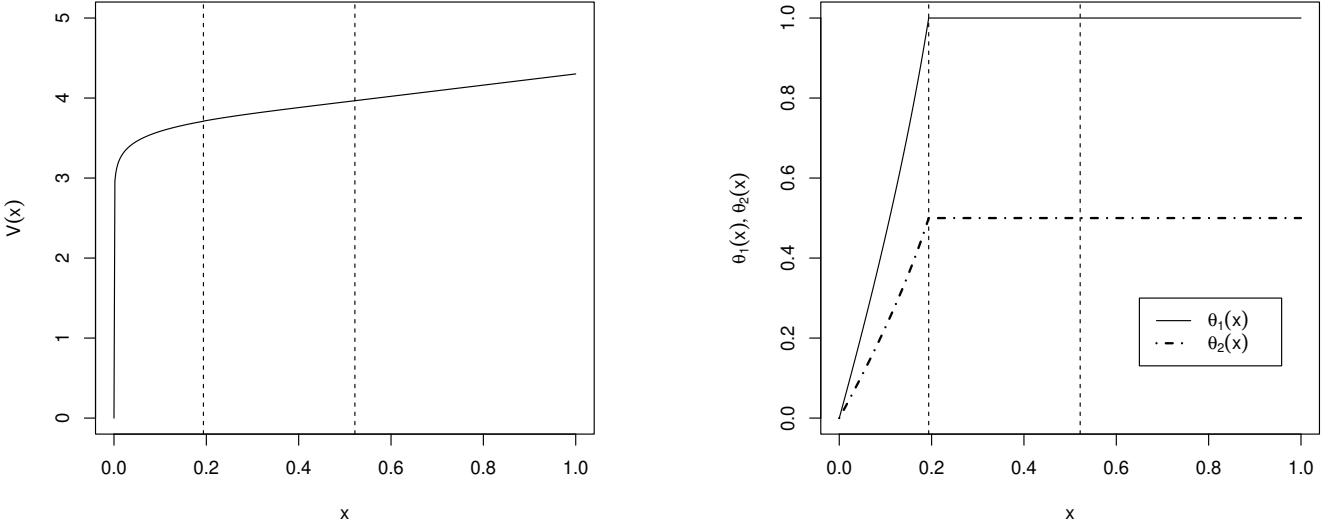


Figure 6: Uniformly distributed claims:  $\kappa_1 = 4$ ,  $\kappa_2 = 2$ ,  $\delta = 0.5$ ,  $a = 0.3$   
 $(M_1 = 1, M_2 = 1.5, w_0 = 0.19 < u_1 = 0.52$ , corresponding to Theorem 5.1)

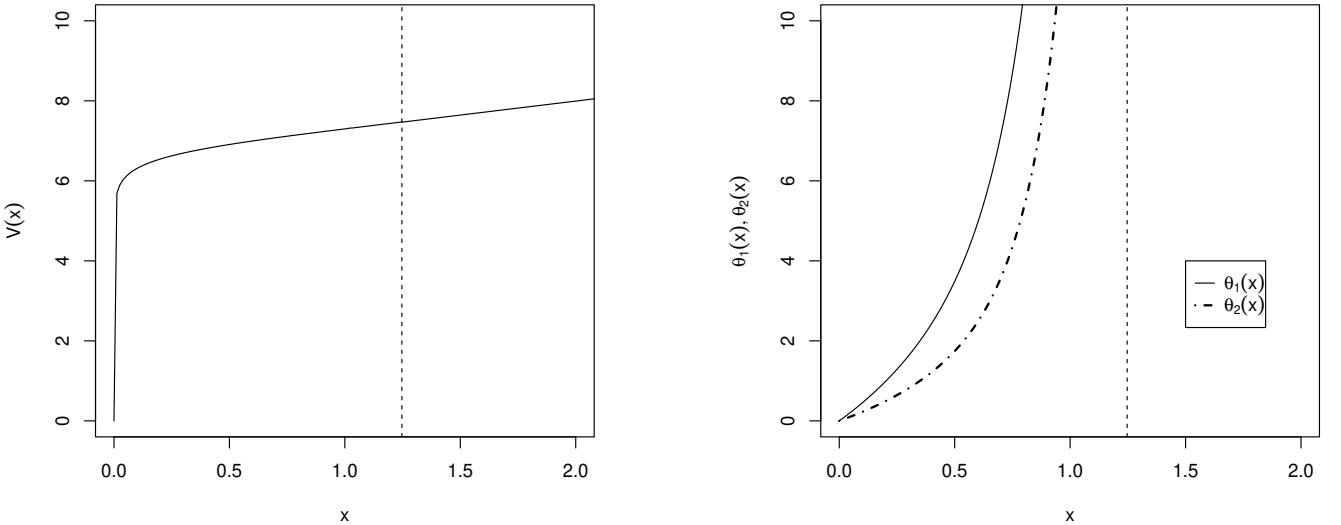


Figure 7: Exponentially distributed claims:  $\kappa_1 = 4$ ,  $\kappa_2 = 2$ ,  $\delta = 0.5$ ,  $a = 0.3$   
 $(M_1 = \infty, M_2 = \infty, u_1 = 1.25$ , corresponding to Theorem 5.2)

## 7 Proof of Main Results

In this section, we provide the proofs of Theorems 4.2, 4.3, 4.4, 5.1, and 5.2.

### 7.1 Proof of Theorem 4.2

#### Deriving the analytical solution

Suppose for now that  $w_0 \leq u_1 \leq u_2$  and that  $w_0$  exists. In the region  $\{x < w_0\}$ , we have  $\pi_1^*(x) = \hat{\pi}_1(x)$ , where  $\hat{\pi}_1$  is given in (4.4), and the corresponding HJB equation becomes

$$0 = \left[ \kappa_1 \mu_1(\pi_1^*(x)) + \kappa_2 \mu_2 \left( \frac{\kappa_2}{\kappa_1} \pi_1^*(x) \right) - \frac{\kappa_1}{2\pi_1^*(x)} \left( \sigma_1^2(\pi_1^*(x)) + \sigma_2^2 \left( \frac{\kappa_2}{\kappa_1} \pi_1^*(x) \right) \right) \right] g'(x) - \delta g(x). \quad (7.1)$$

Differentiating with respect to  $x$  leads to

$$\begin{aligned} 0 &= \left[ \frac{\kappa_1 \frac{d\pi_1^*(x)}{dx}}{2(\pi_1^*(x))^2} \left( \sigma_1^2(\pi_1^*(x)) + \sigma_2^2 \left( \frac{\kappa_2}{\kappa_1} \pi_1^*(x) \right) \right) - \delta \right] g'(x) \\ &\quad + \left[ \kappa_1 \mu_1(\pi_1^*(x)) + \kappa_2 \mu_2 \left( \frac{\kappa_2}{\kappa_1} \pi_1^*(x) \right) - \frac{\kappa_1}{2\pi_1^*(x)} \left( \sigma_1^2(\pi_1^*(x)) + \sigma_2^2 \left( \frac{\kappa_2}{\kappa_1} \pi_1^*(x) \right) \right) \right] g''(x). \end{aligned}$$

Using (4.4) once more yields

$$0 = \left[ \frac{\kappa_1^2 + \kappa_1 \frac{d\pi_1^*(x)}{dx}}{2(\pi_1^*(x))^2} \left( \sigma_1^2(\pi_1^*(x)) + \sigma_2^2 \left( \frac{\kappa_2}{\kappa_1} \pi_1^*(x) \right) \right) - \frac{\kappa_1^2}{\pi_1^*(x)} \mu_1(\pi_1^*(x)) - \frac{\kappa_1 \kappa_2}{\pi_1^*(x)} \mu_2 \left( \frac{\kappa_2}{\kappa_1} \pi_1^*(x) \right) - \delta \right] g'(x).$$

Since  $g$  is assumed to be strictly increasing and concave, we obtain the following differential equation for  $\pi_1^*$ :

$$\frac{d\pi_1^*(x)}{dx} = \frac{dG(y)}{dy} \Big|_{y=\pi_1^*(x)},$$

where  $G$  is defined by (4.9). It can be shown that  $G$  is continuous and strictly increasing; hence, its inverse, denoted by  $G^{-1}$ , exists. Since  $\pi_1^*(0) = 0$ , we then obtain

$$\pi_1^*(x) = G^{-1}(x).$$

Moreover, since  $\frac{d\pi_1^*(x)}{dx} > 0$  and  $\pi_1^*(0) = 0$ , there exists an  $x_{M_1} > 0$  such that  $\pi_1^*(x_{M_1}) = M_1$ . Choosing  $w_0 = x_{M_1}$  proves the existence of  $w_0$  defined in (4.5) and that  $w_0 = G(M_1)$ . From (4.4), we can see that  $-\frac{\kappa_1}{\pi_1^*(x)} = \frac{g''(x)}{g'(x)} = \frac{d}{dx} \ln g'(x)$ . Hence, a solution to the HJB equation (7.1) that satisfies  $g(0) = 0$  is given by

$$g(x) = K_1 \int_0^x \exp \left[ \int_z^{w_0} \frac{\kappa_1}{G^{-1}(y)} dy \right] dz,$$

where  $K_1 > 0$  is an unknown constant.

By the definition of  $w_0$  and the constraint  $\pi_1 \in [0, M_1]$ , it follows that  $\pi_1^*(x) = M_1$  for  $x \geq w_0$ . It is easy to see that  $(\pi_1^*, \pi_2^*)(x) = (M_1, \frac{\kappa_2}{\kappa_1} M_1)$  since  $\frac{M_2}{M_1} \geq \frac{\kappa_2}{\kappa_1}$  holds. Hence, in the region  $\{w_0 < x < u_1\}$ , the HJB equation becomes

$$0 = \frac{1}{2} N_2 g''(x) + N_1 g'(x) - \delta g(x), \quad (7.2)$$

where  $N_1 = \bar{N}_1(M_1)$  and  $N_2 = \bar{N}_2(M_1)$  are defined in (4.6). The solution is given by

$$g_2(x) = K_{2+} e^{\gamma_{2+}(x-w_0)} + K_{2-} e^{\gamma_{2-}(x-w_0)}, \quad (7.3)$$

where  $\gamma_{2\pm} := \bar{\gamma}_{2\pm}(M_1)$  are given by (4.6) and  $K_{2\pm}$  are unknown constants.

In the region  $\{u_1 < x < u_2\}$ , the HJB equation becomes

$$0 = \frac{1}{2} N_2 g''(x) + (N_1 - \bar{c}_2) g'(x) - \delta g(x) + (1-a)\bar{c}_2, \quad (7.4)$$

whose solution is given by

$$g_3(x) = K_{3+} e^{\gamma_{3+}(x-u_1)} + K_{3-} e^{\gamma_{3-}(x-u_1)} + \frac{(1-a)\bar{c}_2}{\delta}, \quad (7.5)$$

where  $\gamma_{3\pm} := \bar{\gamma}_{3\pm}(M_1)$  are given by (4.6) and  $K_{3\pm}$  are unknown constants.

In the region  $\{x > u_2\}$ , the HJB equation becomes

$$0 = \frac{1}{2} N_2 g''(x) + (N_1 - \bar{c}_1 - \bar{c}_2) g'(x) - \delta g(x) + a\bar{c}_1 + (1-a)\bar{c}_2, \quad (7.6)$$

whose solution is given by

$$g_4(x) = K_{4-} e^{\gamma_{4-}(x-u_2)} + \frac{a\bar{c}_1 + (1-a)\bar{c}_2}{\delta}, \quad (7.7)$$

where  $\gamma_{4-} := \bar{\gamma}_{4-}(M_1)$  is defined in (4.6). We conjecture the following solution:

$$g(x) = \begin{cases} K_1 \int_0^x \exp \left[ \int_z^{w_0} \frac{\kappa_1}{G^{-1}(y)} dy \right] dz & \text{if } x < w_0, \\ K_{2+} e^{\gamma_{2+}(x-w_0)} + K_{2-} e^{\gamma_{2-}(x-w_0)} & \text{if } w_0 < x < u_1, \\ K_{3+} e^{\gamma_{3+}(x-u_1)} + K_{3-} e^{\gamma_{3-}(x-u_1)} + \frac{(1-a)\bar{c}_2}{\delta} & \text{if } u_1 < x < u_2, \\ K_{4-} e^{\gamma_{4-}(x-u_2)} + \frac{a\bar{c}_1 + (1-a)\bar{c}_2}{\delta} & \text{if } x > u_2, \end{cases} \quad (7.8)$$

where  $w_0 = G(M_1)$  and  $K_1, K_{2\pm}, K_{3\pm}, K_{4-}, u_1, u_2$  are yet to be determined.

We now solve for the unknowns using the principle of smooth fit. At  $x = w_0$ , we have the following system of equations:

$$\begin{aligned} K_1 &= K_{2+\gamma_2+} + K_{2-\gamma_2-} \\ -K_1 \frac{\kappa_1}{M_1} &= K_{2+\gamma_2+}^2 + K_{2-\gamma_2-}^2, \end{aligned}$$

whose solution is given by

$$K_{2+} = K_1 \alpha_{2+} \quad \text{and} \quad K_{2-} = K_1 \alpha_{2-},$$

where  $\alpha_{2\pm}$  are given in (4.11). Recall that  $\gamma_{2-} < 0 < \gamma_{2+}$ . It is then easy to see that  $\alpha_{2-} < 0$ . To prove  $\alpha_{2+} > 0$ , it suffices to show that  $-\gamma_{2-} - \frac{\kappa_1}{M_1} > 0$ . This result is proved in the following lemma.

**Lemma 7.1.**  $-\frac{\kappa_1}{\gamma_{2-}} < M_1$ .

*Proof.* It holds that

$$\tilde{\sigma}_1^2 = \sigma_1^2(M_1) = \int_0^{M_1} x^2 dF_1(x) < M_1 \int_0^{M_1} x dF_1(x) = M_1 \mu_1(M_1) = M_1 \tilde{\mu}_1$$

and

$$\sigma_2^2 \left( \frac{M_1 \kappa_2}{\kappa_1} \right) = \int_0^{\frac{M_1 \kappa_2}{\kappa_1}} x^2 dF_2(x) < \frac{2M_1 \kappa_2}{\kappa_1} \int_0^{\frac{M_1 \kappa_2}{\kappa_1}} x dF_2(x) = \frac{2M_1 \kappa_2}{\kappa_1} \mu_2 \left( \frac{M_1 \kappa_2}{\kappa_1} \right),$$

Then,

$$-\frac{\kappa_1}{\gamma_{2-}} = \frac{\kappa_1 N_2}{N_1 + \sqrt{N_1^2 + 2\delta N_2}} < \frac{\kappa_1 N_2}{2N_1} < \frac{M_1 \left[ N_1 + \kappa_2 \mu_2 \left( \frac{M_1 \kappa_2}{\kappa_1} \right) \right]}{2N_1} < M_1,$$

which proves the result.  $\square$

We can then rewrite  $g_2$  as

$$g_2(x) = K_1 \left[ \alpha_{2+} e^{\gamma_{2+}(x-w_0)} + \alpha_{2-} e^{\gamma_{2-}(x-w_0)} \right],$$

where  $K_1 > 0$  is still unknown.

By the definition of  $u_1$  in (4.3), we have  $g'(u_1) = 1 - a$ . Hence,

$$1 - a = g'_3(u_1) = K_{3+\gamma_3+} + K_{3-\gamma_3-},$$

which is equivalent to  $K_{3+}$  defined in (4.12). It suffices to show that  $g_2(u_1) = g_3(u_1)$  and  $g'_2(u_1) = g'_3(u_1)$  to ensure that  $g$  is twice continuously differentiable at  $x = u_1$ . We then have the following system of equations:

$$\begin{aligned} K_1 \left[ \alpha_{2+} e^{\gamma_{2+}(u_1-w_0)} + \alpha_{2-} e^{\gamma_{2-}(u_1-w_0)} \right] &= (1-a)\alpha_3 \\ K_1 \left[ \alpha_{2+\gamma_2+} e^{\gamma_{2+}(u_1-w_0)} + \alpha_{2-\gamma_2-} e^{\gamma_{2-}(u_1-w_0)} \right] &= (1-a), \end{aligned} \tag{7.9}$$

where  $\alpha_3$  is defined in (4.12). From the second equation in (7.9), we obtain  $K_1$  defined in (4.14). Dividing the first equation in (7.9) by the second equation yields

$$\frac{\alpha_{2+} e^{\gamma_{2+}(u_1-w_0)} + \alpha_{2-} e^{\gamma_{2-}(u_1-w_0)}}{\alpha_{2+\gamma_2+} e^{\gamma_{2+}(u_1-w_0)} + \alpha_{2-\gamma_2-} e^{\gamma_{2-}(u_1-w_0)}} = \alpha_3,$$

which is equivalent to  $u_1$  defined in (4.10). We point out that we have yet to establish that  $u_1$  is well defined.

By the definition of  $u_2$ , we have  $g'(u_2) = a$ . Hence,

$$a = g'_4(u_2) = K_{4-} \alpha_{4-},$$

or, equivalently,

$$K_{4-} = \frac{a}{\gamma_{4-}}.$$

It suffices to show that  $g'_3(u_2) = g'_4(u_2)$  and  $g''_3(u_2) = g''_4(u_2)$  to ensure that  $g$  is twice continuously differentiable at  $x = u_2$ . We then have the following system of equations:

$$\begin{aligned} K_{3+}\gamma_{3+}e^{\gamma_{3+}(u_2-u_1)} + K_{3-}\gamma_{3-}e^{\gamma_{3-}(u_2-u_1)} &= a \\ K_{3+}\gamma_{3+}^2e^{\gamma_{3+}(u_2-u_1)} + K_{3-}\gamma_{3-}^2e^{\gamma_{3-}(u_2-u_1)} &= a\gamma_{4-}. \end{aligned} \quad (7.10)$$

Dividing the second equation by the first equation yields

$$\frac{K_{3+}\gamma_{3+}^2e^{\gamma_{3+}(u_2-u_1)} + K_{3-}\gamma_{3-}^2e^{\gamma_{3-}(u_2-u_1)}}{K_{3+}\gamma_{3+}e^{\gamma_{3+}(u_2-u_1)} + K_{3-}\gamma_{3-}e^{\gamma_{3-}(u_2-u_1)}} = \gamma_{4-},$$

which is equivalent to  $u_2$  defined in (4.10), where  $K_{3-}$  is still unknown. Thus, we have obtained the form of the value function in (4.13). The next steps are (i) to establish that the formulas for  $u_1$  and  $u_2$  in (4.10) are well defined, (ii) to guarantee that  $w_0 \leq u_1 \leq u_2$ , and (iii) to show that  $g$  is increasing and concave.

### Establishing the bounds for $K_{3-}$

We now establish the bounds for  $K_{3-}$ . Since the candidate value function  $g$  must be positive for  $x > 0$ , we must have  $\alpha_3 > 0$  from the first equation in (7.9). Since  $\alpha_{2-} < 0 < \alpha_{2+}$ , we must have  $1 - \gamma_{2+}\alpha_3 > 0$  for  $u_1$  in (4.10) to be well defined. Combining these inequalities for  $\alpha_3$  yields  $0 < \alpha_3 < \frac{1}{\gamma_{2+}}$ , which is equivalent to

$$\underline{\alpha} < K_{3-} < \frac{(1-a)\gamma_{3+}}{\gamma_{3+} - \gamma_{3-}} \left( \frac{1}{\gamma_{2+}} - \frac{1}{\gamma_{3+}} - \frac{\bar{c}_2}{\delta} \right) =: \bar{\alpha}, \quad (7.11)$$

where  $\underline{\alpha}$  is defined in (4.8). It is clear that  $\underline{\alpha} < \bar{\alpha}$  since  $\frac{1}{\gamma_{2+}} > 0$ .

We have now established conditions under which (4.10) is well defined; however, these conditions do not guarantee that  $w_0 \leq u_1$ . The following lemma provides necessary and sufficient conditions for  $\alpha_0$ , defined in (4.8), to be a lower bound of  $K_{3-}$  that is greater than  $\underline{\alpha}$ .

**Lemma 7.2.**  $\alpha_0 > \underline{\alpha}$  if and only if

$$N_1 - \frac{\kappa_1 N_2}{2M_1} > 0.$$

*Proof.* From (7.11) and (4.8), it follows that  $\alpha_0 > \underline{\alpha}$  if and only if  $\frac{N_1}{\delta} - \frac{\kappa_1 N_2}{2\delta M_1} > 0$ . The result follows directly.  $\square$

The following lemma proves that  $\alpha_0$  does not exceed the upper bound  $\bar{\alpha}$ .

**Lemma 7.3.**  $\alpha_0 < \bar{\alpha}$ .

*Proof.* We can rewrite  $\frac{N_1}{\delta} - \frac{\kappa_1 N_2}{2\delta M_1} < \frac{1}{\gamma_{2+}}$  as

$$\left( 2N_1 - \frac{\kappa_1}{M_1} N_2 \right) \left( \sqrt{N_1^2 + 2\delta N_2} - N_1 \right) < 2\delta N_2,$$

which immediately holds if  $N_1 < \frac{\kappa_1 N_2}{2M_1}$ . Suppose  $N_1 \geq \frac{\kappa_1 N_2}{2M_1}$ . We can then rewrite the above inequality as

$$\left( 2N_1 - \frac{\kappa_1}{M_1} N_2 \right) \sqrt{N_1^2 + 2\delta N_2} < 2\delta N_2 + 2N_1^2 - \frac{\kappa_1}{M_1} N_1 N_2.$$

Squaring both sides yields  $N_1 - \frac{\kappa_1 N_2}{2M_1} + \frac{\delta M_1}{\kappa_1} > 0$ , which proves the result.  $\square$

The following result gives a necessary and sufficient condition to ensure that  $w_0 \leq u_1$ .

**Lemma 7.4.**  $w_0 \leq u_1$  if and only if

$$\underline{\alpha} + (\alpha_0 - \underline{\alpha}) \cdot \mathbf{1}_{\{N_1 > \frac{\kappa_1 N_2}{2M_1}\}} < K_{3-} < \bar{\alpha}.$$

*Proof.* From (4.10),  $w_0 \leq u_1$  is equivalent to

$$\frac{\alpha_{2-}(\gamma_{2-}\alpha_3 - 1)}{\alpha_{2+}(1 - \gamma_{2+}\alpha_3)} \geq 1. \quad (7.12)$$

Moreover,  $K_{3-} < \bar{\alpha}$  is equivalent to  $1 - \gamma_{2+}\alpha_3 > 0$ , and  $K_{3-} > \underline{\alpha}$  is equivalent to  $\alpha_3 > 0$ . Hence, we can rewrite (7.12) as

$$\alpha_3 \geq \frac{N_1}{\delta} - \frac{\kappa_1 N_2}{2\delta M_1},$$

or, equivalently,

$$K_{3-} \geq \alpha_0.$$

Using Lemma 7.2 completes the proof.  $\square$

We now determine the signs of  $K_{3\pm}$ . First, we note that a sufficient condition such that  $g$  is increasing, particularly in the region  $\{u_1 < x < u_2\}$ , is  $K_{3-} \leq 0 \leq K_{3+}$ . The following lemma proves that  $K_{3-} < 0$ .

**Lemma 7.5.**  $K_{3-} < 0$ .

*Proof.* From (7.11), it suffices to prove that

$$\frac{\bar{c}_2}{\delta} \geq \frac{1}{\gamma_{2+}} - \frac{1}{\gamma_{3+}}.$$

Write  $k := 2\delta N_2 > 0$ . Suppose otherwise that  $\frac{\bar{c}_2}{\delta} < \frac{1}{\gamma_{2+}} - \frac{1}{\gamma_{3+}}$ . This is equivalent to

$$\begin{aligned} & 2\bar{c}_2 \sqrt{(N_1 - \bar{c}_2)^2 + k} \cdot \sqrt{N_1^2 + k} - \bar{c}_2(k - 2N_1(N_1 - \bar{c}_2)) \\ & < (2N_1\bar{c}_2 + k)\sqrt{(N_1 - \bar{c}_2)^2 + k} + (2\bar{c}_2(N_1 - \bar{c}_2) - k)\sqrt{N_1^2 + k}. \end{aligned} \quad (7.13)$$

The left-hand side of (7.13) can be shown to be always positive. If the right-hand side of (7.13) is negative, it results in a contradiction. Otherwise, we can square both sides and get

$$\bar{c}_2^2 k < 0,$$

which is also a contradiction. The proof is complete.  $\square$

We now show that  $K_{3+} \geq 0$ . Using (4.12), we must have the following:

$$K_{3-} \geq \frac{1-a}{\gamma_{3-}}.$$

The following lemma proves that  $\frac{1-a}{\gamma_{3-}}$  does not exceed the upper bound  $\bar{\alpha}$ .

**Lemma 7.6.**  $\frac{1-a}{\gamma_{3-}} < \bar{\alpha}$ .

*Proof.* It suffices to prove that

$$\frac{\bar{c}_2}{\delta} < \frac{1}{\gamma_{2+}} - \frac{1}{\gamma_{3-}}. \quad (7.14)$$

From the elementary inequality

$$\sqrt{A^2 + B} - A < \frac{B}{2A}, \quad A, B > 0,$$

we obtain the following:

$$\gamma_{2+} < \frac{\delta}{N_1} \Leftrightarrow \frac{1}{\gamma_{2+}} > \frac{N_1}{\delta}.$$

If  $N_1 - \bar{c}_2 \leq 0$ , then

$$-\gamma_{3-} < \frac{\delta}{\bar{c}_2 - N_1} \Leftrightarrow -\frac{1}{\gamma_{3-}} > \frac{\bar{c}_2 - N_1}{\delta},$$

and (7.14) follows. If  $N_1 - \bar{c}_2 > 0$ , then

$$\gamma_{2+} < \frac{\delta}{N_1} < \frac{\delta}{\bar{c}_2} \Leftrightarrow \frac{\bar{c}_2}{\delta} < \frac{1}{\gamma_{2+}},$$

and (7.14) follows since  $\gamma_{3-} < 0$ . The proof is complete.  $\square$

We now determine the conditions under which  $\frac{1-a}{\gamma_{3-}}$ ,  $\alpha_0$ , or  $\underline{\alpha}$  serves as the lower bound for  $K_{3-}$  via the lemma below.

**Lemma 7.7.** (i)  $\frac{1-a}{\gamma_{3-}} > \underline{\alpha}$  if and only if

$$\bar{c}_2 > \frac{\delta N_2}{2N_1}. \quad (7.15)$$

(ii)  $\frac{1-a}{\gamma_{3-}} > \alpha_0$  if and only if

$$\bar{c}_2 > N_1 - \frac{\kappa_1 N_2}{2M_1} + \frac{\delta M_1}{\kappa_1}. \quad (7.16)$$

*Proof.*  $\frac{1-a}{\gamma_{3-}} > \underline{\alpha}$  is equivalent to

$$\delta N_2 - \bar{c}_2(N_1 - \bar{c}_2) < \bar{c}_2\sqrt{(N_1 - \bar{c}_2)^2 + 2\delta N_2}. \quad (7.17)$$

The inequality holds when  $\delta N_2 - \bar{c}_2(N_1 - \bar{c}_2) < 0$ , or, equivalently,  $N_1 \bar{c}_2 > \delta N_2 + \bar{c}_2^2$ . Squaring both sides of (7.17) yields (7.15), which is equivalent to  $N_1 \bar{c}_2 > \frac{\delta N_2}{2}$ . Since  $\delta N_2 + \bar{c}_2^2 > \frac{\delta N_2}{2}$ , then (i) follows.

$\frac{1-a}{\gamma_{3-}} > \alpha_0$  is equivalent to

$$1 < \gamma_{3-} \left( \frac{N_1 - \bar{c}_2}{\delta} - \frac{\kappa_1 N_2}{2\delta M_1} \right).$$

A necessary condition for the above inequality to hold is that  $\bar{c}_2 > N_1 - \frac{\kappa_1 N_2}{2M_1}$ . Moreover, the inequality is equivalent to the following:

$$2\delta M_1 N_2 + 2M_1(N_1 - \bar{c}_2)^2 - \kappa_1 N_2(N_1 - \bar{c}_2) < (\kappa_1 N_2 - 2M_1(N_1 - \bar{c}_2))\sqrt{(N_1 - \bar{c}_2)^2 + 2\delta N_2}. \quad (7.18)$$

The above inequality is immediately satisfied when  $2M_1(N_1 - \bar{c}_2)^2 - \kappa_1 N_2(N_1 - \bar{c}_2) < -2\delta M_1 N_2$  (i.e., the left-hand side is negative) and  $\bar{c}_2 > N_1 - \frac{\kappa_1 N_2}{2M_1}$  (i.e., the right-hand side is positive). We call this scenario the “trivial” case.

Squaring both sides of (7.18) yields (7.16), which satisfies the necessary condition. It must be noted that squaring both sides of (7.18) is valid if  $N_1 - \bar{c}_2 < 0$ . Moreover, if  $N_1 - \bar{c}_2 > 0$ , we can rewrite (7.16) as  $2M_1(N_1 - \bar{c}_2)^2 - \kappa_1 N_2(N_1 - \bar{c}_2) < -\frac{2\delta M_1^2(N_1 - \bar{c}_2)}{\kappa_1}$ . Since  $-2\delta M_1 N_2 < -\frac{2\delta M_1^2(N_1 - \bar{c}_2)}{\kappa_1}$  if  $0 < N_1 - \bar{c}_2 < \frac{\kappa_1 N_2}{M_1}$ , then the trivial case is covered by (ii). The proof is complete.  $\square$

Before presenting the formula for the lower bound of  $K_{3-}$ , we state the following lemma which is useful in bridging the results of Lemmas 7.4 and 7.7:

**Lemma 7.8.**  $\frac{\delta N_2}{2N_1} < N_1 - \frac{\kappa_1 N_2}{2M_1} + \frac{\delta M_1}{\kappa_1}$  if and only if

$$N_1 > \frac{\kappa_1 N_2}{2M_1}.$$

*Proof.*  $\frac{\delta N_2}{2N_1} \leq N_1 - \frac{\kappa_1 N_2}{2M_1} + \frac{\delta M_1}{\kappa_1}$  is equivalent to

$$2\kappa_1 M_1 N_1^2 + (2\delta M_1^2 - \kappa_1^2 N_2)N_1 - \delta\kappa_1 M_1 N_2 > 0.$$

Solving the inequality with respect to  $N_1$  yields

$$N_1 > \frac{\kappa_1 N_2}{2M_1} \quad \text{or} \quad N_1 < -\frac{\delta M_1}{\kappa_1}.$$

Since  $N_1 > 0$ , the result follows.  $\square$

Following the results of Lemmas 7.4, 7.7, and 7.8, we now present the four cases that yield the expression for the lower bound of  $K_{3-}$  (denoted by  $\alpha_{LB}$ ):

(i) If  $N_1 > \frac{\kappa_1 N_2}{2M_1}$  and  $\bar{c}_2 \geq N_1 - \frac{\kappa_1 N_2}{2M_1} + \frac{\delta M_1}{\kappa_1}$ , then  $\alpha_{LB} = \frac{1-a}{\gamma_{3-}} > \alpha_0 > \underline{\alpha}$ .

(ii) If  $N_1 > \frac{\kappa_1 N_2}{2M_1}$  and  $\bar{c}_2 < N_1 - \frac{\kappa_1 N_2}{2M_1} + \frac{\delta M_1}{\kappa_1}$ , then  $\alpha_{LB} = \alpha_0 > (\underline{\alpha} \vee \frac{1-a}{\gamma_{3-}})$ .

(iii) If  $N_1 \leq \frac{\kappa_1 N_2}{2M_1}$  and  $\bar{c}_2 \geq \frac{\delta N_2}{2N_1}$ , then  $\alpha_{LB} = \frac{1-a}{\gamma_{3-}} > \underline{\alpha} > \alpha_0$ .

(iv) If  $N_1 \leq \frac{\kappa_1 N_2}{2M_1}$  and  $\bar{c}_2 < \frac{\delta N_2}{2N_1}$ , then  $\alpha_{LB} = \underline{\alpha} > (\alpha_0 \vee \frac{1-a}{\gamma_{3-}})$ .

For case (iii), we used Lemma 7.8 to combine the inequalities  $\bar{c}_2 \geq N_1 - \frac{\kappa_1 N_2}{2M_1} + \frac{\delta M_1}{\kappa_1}$  and  $\bar{c}_2 \geq \frac{\delta N_2}{2N_1}$ . For case (iv), we also used Lemma 7.8 to combine two subcases: (1)  $N_1 - \frac{\kappa_1 N_2}{2M_1} + \frac{\delta M_1}{\kappa_1} \leq \bar{c}_2 < \frac{\delta N_2}{2N_1}$  and (2)  $\bar{c}_2 < N_1 - \frac{\kappa_1 N_2}{2M_1} + \frac{\delta M_1}{\kappa_1}$ . Hence, we obtain the formula for  $\alpha_{LB}$  given in (4.8). Thus,  $K_{3-} < 0 < K_{3+}$  and  $w_0 \leq u_1$  if the following inequalities hold:

$$\alpha_{LB} < K_{3-} < \bar{\alpha}.$$

**Remark 7.9.** It follows that  $g$  is strictly increasing in the region  $\{u_1 < x < u_2\}$ . It can easily be shown that  $g$  is also strictly increasing in the other regions. By the continuity of the first derivative,  $g$  is strictly increasing for all  $x > 0$ .

It can also be easily shown that  $g$  is strictly concave in the regions  $\{x < w_0\}$  and  $\{x > u_2\}$ . Since  $g''' > 0$  in the regions  $\{w_0 < x < u_1\}$  and  $\{u_1 < x < u_2\}$ , then, by the continuity of the second derivative,  $g$  is strictly concave on  $[w_0, u_2]$ .

It can be shown that  $\gamma_{4-} - \gamma_{3-} > 0$ . Hence,  $u_2$  is well defined. We now require that  $u_1 \leq u_2$ . The following lemma gives a necessary and sufficient condition such that  $u_1 \leq u_2$ .

**Lemma 7.10.**  $u_1 \leq u_2$  if and only if

$$K_{3-} \leq \frac{(1-a)(\gamma_{3+} - \gamma_{4-})}{\gamma_{3-}(\gamma_{3+} - \gamma_{3-})} =: \alpha_{UB}.$$

*Proof.* From (4.10),  $u_1 \leq u_2$  is equivalent to

$$\frac{K_{3-}\gamma_{3-}(\gamma_{4-} - \gamma_{3-})}{K_{3+}\gamma_{3+}(\gamma_{3+} - \gamma_{4-})} \geq 1.$$

Using (4.12) yields the result.  $\square$

The following lemma proves that  $\alpha_{UB}$  is a lower upper bound of  $K_{3-}$ .

**Lemma 7.11.**  $\alpha_{UB} \leq \bar{\alpha}$ .

*Proof.*  $\alpha_{UB} \leq \bar{\alpha}$  is equivalent to

$$\sqrt{N_1^2 + 2\delta N_2} \sqrt{(N_1 - \bar{c}_1 - \bar{c}_2)^2 + 2\delta N_2} \geq -N_1(N_1 - \bar{c}_1 - \bar{c}_2) - 2\delta N_2. \quad (7.19)$$

The above inequality is always true if  $\bar{c}_1 + \bar{c}_2 < N_1 + \frac{2\delta N_2}{N_1}$ . Suppose  $\bar{c}_1 + \bar{c}_2 \geq N_1 + \frac{2\delta N_2}{N_1}$ . Squaring both sides of the inequality (7.19) yields

$$(\bar{c}_1 + \bar{c}_2)^2 \geq 0,$$

which is always true. The result is proved.  $\square$

The following lemma gives a necessary and sufficient condition such that  $\alpha_{LB} \leq \alpha_{UB}$ .

**Lemma 7.12.**  $\alpha_{LB} \leq \alpha_{UB}$  if and only if

$$\bar{c}_1 + \bar{c}_2 \geq N_1 - \frac{\kappa_1 N_2}{2M_1} + \frac{\delta M_1}{\kappa_1}. \quad (7.20)$$

*Proof.* Suppose  $N_1 > \frac{\kappa_1 N_2}{2M_1}$  and  $\bar{c}_2 \geq N_1 - \frac{\kappa_1 N_2}{2M_1} + \frac{\delta M_1}{\kappa_1}$ . Then,  $\alpha_{LB} = \frac{1-a}{\gamma_{3-}}$  and  $\alpha_{LB} \leq \alpha_{UB}$  are equivalent to

$$\gamma_{3-} - \gamma_{4-} \leq 0,$$

which can be shown to be always true. Since  $\bar{c}_1 > 0$ , then (7.20) follows. For the case  $N_1 \leq \frac{\kappa_1 N_2}{2M_1}$  and  $\bar{c}_2 \geq \frac{\delta N_2}{2N_1}$ , we also obtain the equivalence between  $\alpha_{LB} \leq \alpha_{UB}$  and  $\gamma_{3-} - \gamma_{4-} \leq 0$ . By Lemma 7.8,  $\frac{\delta N_2}{2N_1} \geq N_1 - \frac{\kappa_1 N_2}{2M_1} + \frac{\delta M_1}{\kappa_1}$ , which implies (7.20).

Suppose  $N_1 > \frac{\kappa_1 N_2}{2M_1}$  and  $\bar{c}_2 < N_1 - \frac{\kappa_1 N_2}{2M_1} + \frac{\delta M_1}{\kappa_1}$ . Then,  $\alpha_{LB} = \alpha_0$  and  $\alpha_{LB} \leq \alpha_{UB}$  are equivalent to

$$-(N_1 - \bar{c}_1 - \bar{c}_2) + \frac{\kappa_1 N_2}{M_1} \geq \sqrt{(N_1 - \bar{c}_1 - \bar{c}_2)^2 + 2\delta N_2},$$

which holds only if  $\bar{c}_1 + \bar{c}_2 \geq N_1 - \frac{\kappa_1 N_2}{M_1}$ . Squaring both sides of the above inequality yields

$$\bar{c}_1 + \bar{c}_2 \geq N_1 - \frac{\kappa_1 N_2}{2M_1} + \frac{\delta M_1}{\kappa_1} > N_1 - \frac{\kappa_1 N_2}{M_1},$$

which proves the result for when  $\alpha_{LB} = \alpha_0$ .

Suppose  $N_1 \leq \frac{\kappa_1 N_2}{2M_1}$  and  $\bar{c}_2 < \frac{\delta N_2}{2N_1}$ . Then,  $\alpha_{LB} = \underline{\alpha}$  and  $\alpha_{LB} \leq \alpha_{UB}$  are equivalent to

$$N_1 + \bar{c}_1 + \bar{c}_2 \geq \sqrt{(N_1 - \bar{c}_1 + \bar{c}_2)^2 + 2\delta N_2}.$$

Squaring both sides yields

$$\bar{c}_1 + \bar{c}_2 \geq \frac{\delta N_2}{2N_1} > N_1 - \frac{\kappa_1 N_2}{2M_1} + \frac{\delta M_1}{\kappa_1},$$

where the second inequality follows from Lemma 7.8. The proof is complete.  $\square$

We now state the following result which gives a necessary and sufficient condition such that  $w_0 \leq u_1 \leq u_2$ .

**Lemma 7.13.**  $w_0 \leq u_1 \leq u_2$  if and only if  $K_{3-} \in (\alpha_{LB}, \alpha_{UB})$ .

*Proof.* This is a direct consequence of Lemmas 7.4, 7.10, and 7.12.  $\square$

### Solving for $K_{3-}$

It remains to prove the existence of  $K_{3-}$ . We do this via the first equation in (7.10), which ensures that the first derivative at  $x = u_2$  is continuous. The following lemma gives a necessary and sufficient condition such that  $K_{3-}$  exists and is unique.

**Lemma 7.14.**  $K_{3-}$  is the unique solution to  $\psi(z) = 0$  in  $(\alpha_{LB}, \alpha_{UB})$  if and only if  $\psi(\alpha_{LB}) \leq 0$ , where  $\psi$  is defined in (4.7).

*Proof.* We can rewrite  $\psi$  as follows:

$$\begin{aligned} \psi(z) &= (1 - a - \gamma_{3-}z) \left[ \frac{\gamma_{3-}(\gamma_{4-} - \gamma_{3-})z}{(1 - a - \gamma_{3-}z)(\gamma_{3+} - \gamma_{4-})} \right]^{\frac{\gamma_{3+}}{\gamma_{3+} - \gamma_{3-}}} + \gamma_{3-}ze^{\gamma_{3-}\zeta(z)} - a \\ &= (1 - a - \gamma_{3-}z)^{-\frac{\gamma_{3-}}{\gamma_{3+} - \gamma_{3-}}} \left[ \frac{\gamma_{3-}(\gamma_{4-} - \gamma_{3-})z}{\gamma_{3+} - \gamma_{4-}} \right]^{\frac{\gamma_{3+}}{\gamma_{3+} - \gamma_{3-}}} + \gamma_{3-}ze^{\gamma_{3-}\zeta(z)} - a. \end{aligned}$$

Since  $\gamma_{3-} < 0$  and  $\lim_{z \downarrow \frac{1-a}{\gamma_{3-}}} \zeta(z) = +\infty$ , then  $\lim_{z \downarrow \frac{1-a}{\gamma_{3-}}} \psi(z) = -a < 0$ . From Lemma 7.10,  $\zeta(\alpha_{UB}) = 0$ .

Since  $a \leq \frac{1}{2}$ ,  $\psi(\alpha_{UB}) = 1 - 2a > 0$ . Hence, by the intermediate value theorem, there exists a  $z_0 \in \left(\frac{1-a}{\gamma_{3-}}, \alpha_{UB}\right)$  such that  $\psi(z_0) = 0$ . We now prove the uniqueness of  $z_0$ . From the definitions of  $\zeta$  and  $\psi$  in (4.7), we have

$$\begin{aligned} \zeta'(z) &= \frac{1-a}{(\gamma_{3+} - \gamma_{3-})(1 - a - \gamma_{3-}z)}, \\ \psi'(z) &= \gamma_{3-} \left( e^{\gamma_{3-}\zeta(z)} - e^{\gamma_{3+}\zeta(z)} \right) + \frac{1-a}{\gamma_{3+} - \gamma_{3-}} \left[ \frac{\gamma_{3+}}{z} e^{\gamma_{3+}\zeta(z)} + \frac{\gamma_{3-}^2}{1-a-\gamma_{3-}z} e^{\gamma_{3-}\zeta(z)} \right]. \end{aligned}$$

Since  $\gamma_{3+} > \gamma_{3-}$ , it holds that

$$(\gamma_{3+} - \gamma_{3-})\zeta(z) = \ln \left( \frac{\gamma_{3-}(\gamma_{4-} - \gamma_{3-})z}{(1 - a - \gamma_{3-}z)(\gamma_{3+} - \gamma_{4-})} \right) < \ln \left( \frac{-\gamma_{3-}^2 z}{\gamma_{3+}(1 - a - \gamma_{3-}z)} \right).$$

It follows that

$$\frac{\gamma_{3+}}{z} e^{\gamma_{3+}\zeta(z)} > \frac{-\gamma_{3-}^2}{1-a-\gamma_{3-}z} e^{\gamma_{3-}\zeta(z)}.$$

Now using  $\zeta(z) \geq 0$  for  $z \in (\frac{1-a}{\gamma_{3-}}, \alpha_{UB})$ , we obtain  $\psi'(z) > 0$  on  $(\frac{1-a}{\gamma_{3-}}, \alpha_{UB})$ . Hence,  $z_0$  is unique.

If  $\alpha_{LB} = \frac{1-a}{\gamma_{3-}}$ , the result immediately follows by choosing  $z_0 = \alpha_{3-}$ . Suppose  $\alpha_{LB} = \alpha_0$  or  $\alpha_{LB} = \underline{\alpha}$ . If  $\psi(\alpha_{LB}) \leq 0$ , then the result follows by choosing  $z_0 = \alpha_{3-}$ . Suppose  $\psi(\alpha_{LB}) > 0$ . By the intermediate value theorem and the strict monotonicity of  $\psi$ , the unique solution  $z_1$  of  $\psi$  is on the interval  $(\frac{1-a}{\gamma_{3-}}, \alpha_{LB})$ . Hence, there is no solution on the interval  $(\alpha_{LB}, \alpha_{UB})$ . The proof is complete.  $\square$

**Remark 7.15.** Lemmas 7.14 and 7.8 imply that if  $\bar{c}_2 \geq N_1 - \frac{\kappa_1 N_2}{2M_1} + \frac{\delta M_1}{\kappa_1}$  holds, then the existence (and uniqueness) of  $K_{3-}$  is guaranteed.

## 7.2 Proof of Theorem 4.3

### Deriving the analytical solution

Suppose for now that  $u_1 < w_0 \leq u_2$  and  $w_0$  exists. In the region  $\{x < u_1\}$ , the HJB equation becomes (7.1) and its solution is given by  $g_1(x) = K_1 \int_0^x \exp \left[ \int_z^{u_1} \frac{\kappa_1}{G^{-1}(y)} dy \right] dz$ , where  $K_1 > 0$  is a constant and  $G^{-1}$  is the inverse of  $G$  given by (4.9). We also obtain  $\pi_1^*(x) = G^{-1}(x)$ .

In the region  $\{u_1 < x < w_0\}$ , we have  $\pi_1^*(x) = \hat{\pi}_1(x)$  and the HJB equation becomes

$$0 = \left[ \kappa_1 \mu_1(\pi_1^*(x)) + \kappa_2 \mu_2 \left( \frac{\kappa_2}{\kappa_1} \pi_1^*(x) \right) - \bar{c}_2 \right] g'(x) + \left[ \frac{1}{2} \sigma_1^2(\pi_1^*(x)) + \frac{1}{2} \sigma_2^2 \left( \frac{\kappa_2}{\kappa_1} \pi_1^*(x) \right) \right] g''(x) - \delta g(x) + (1-a)\bar{c}_2. \quad (7.21)$$

Using (4.4), we obtain  $\pi_1^*(x) = H(x)$ , where  $H$  satisfies the differential equation in (4.16). Hence,  $w_0$  exists and satisfies  $H(w_0) = M_1$  via (4.5). Similar to solving (7.1), we obtain the following solution for (7.21):

$$g_2(x) = K_{2+} \int_{u_1}^x \exp \left[ \int_z^{w_0} \frac{\kappa_1}{H(y)} dy \right] dz + K_{2-},$$

where  $K_{2\pm}$  are unknown constants.

In the region  $\{w_0 < x < u_2\}$ , the HJB equation becomes (7.4), whose solution is of the form given in (7.5). In the region  $\{x > u_2\}$ , the HJB equation simplifies to (7.6), which has a solution described by (7.7). We conjecture the following solution:

$$g(x) = \begin{cases} K_1 \int_0^x \exp \left[ \int_z^{u_1} \frac{\kappa_1}{G^{-1}(y)} dy \right] dz & \text{if } x < u_1, \\ K_{2+} \int_{u_1}^x \exp \left[ \int_z^{w_0} \frac{\kappa_1}{H(y)} dy \right] dz + K_{2-} & \text{if } u_1 < x < w_0, \\ K_{3+} e^{\gamma_{3+}(x-u_1)} + K_{3-} e^{\gamma_{3-}(x-u_1)} + \frac{(1-a)\bar{c}_2}{\delta} & \text{if } w_0 < x < u_2, \\ K_{4-} e^{\gamma_{4-}(x-u_2)} + \frac{a\bar{c}_1 + (1-a)\bar{c}_2}{\delta} & \text{if } x > u_2, \end{cases} \quad (7.22)$$

where  $w_0 = H(M_1)$  and  $K_1, K_{2\pm}, K_{3\pm}, K_{4-}, u_1, u_2$  are yet to be determined.

To ensure twice continuous differentiability at  $x = u_1$ , we obtain the following equations:

$$\begin{aligned} K_1 \int_0^{u_1} \exp \left[ \int_z^{u_1} \frac{\kappa_1}{G^{-1}(y)} dy \right] dz &= K_{2-}, \\ K_1 = K_{2+} \exp \left[ \int_{u_1}^{w_0} \frac{\kappa_1}{H(y)} dy \right] &= 1-a, \end{aligned} \quad (7.23)$$

which imply the following:

$$\begin{aligned} K_1 &= 1-a, \\ K_{2-} &= (1-a) \int_0^{u_1} \exp \left[ \int_z^{u_1} \frac{\kappa_1}{G^{-1}(y)} dy \right] dz, \\ K_{2+} &= (1-a) \exp \left[ - \int_{u_1}^{w_0} \frac{\kappa_1}{H(y)} dy \right], \end{aligned} \quad (7.24)$$

where  $u_1$  is still unknown.

To ensure twice continuous differentiability at  $x = w_0$ , we have the following equations:

$$\begin{aligned} K_{2+} &= K_{3+}\gamma_{3+}e^{\gamma_{3+}(w_0-u_1)} + K_{3-}\gamma_{3-}e^{\gamma_{3-}(w_0-u_1)}, \\ -\frac{\kappa_1 K_{2+}}{M} &= K_{3+}\gamma_{3+}^2 e^{\gamma_{3+}(w_0-u_1)} + K_{3-}\gamma_{3-}^2 e^{\gamma_{3-}(w_0-u_1)}. \end{aligned}$$

Dividing the second equation by the first equation yields the formula for  $u_1$  in (4.17).

### Establishing the bounds for $K_{3-}$

We now establish the bounds for  $K_{3-}$ . Since the candidate value function  $g$  must be positive for  $x > 0$ , then, from (7.5), we must have  $\alpha_3 > 0$ , where  $\alpha_3$  is defined in (4.12). This implies that  $K_{3-} > \underline{\alpha}$ , where  $\underline{\alpha}$  is defined in (4.8).

Next, we establish the conditions such that the formula for  $u_1$  in (4.17) is well defined. Suppose for now that  $K_{3-} < 0 < K_{3+}$ . We then require the following result:

**Lemma 7.16.**  $\gamma_{3-} + \frac{\kappa_1}{M_1} < 0$  if and only if

$$\bar{c}_2 < N_1 - \frac{\kappa_1 N_2}{2M_1} + \frac{\delta M_1}{\kappa_1} \quad \text{and} \quad N_1 > \frac{\kappa_1 N_2}{2M_1}.$$

*Proof.*  $\gamma_{3-} + \frac{\kappa_1}{M_1} < 0$  is equivalent to

$$\sqrt{(N_1 - \bar{c}_2)^2} \geq \frac{\kappa_1 N_2}{M_1} - (N_1 - \bar{c}_2).$$

The inequality holds immediately if  $0 < \bar{c}_2 < N_1 - \frac{\kappa_1 N_2}{M_1}$ . Suppose that  $\bar{c}_2 \geq N_1 - \frac{\kappa_1 N_2}{M_1}$ . Squaring both sides of the above inequality yields the desired result.  $\square$

The above lemma implies that  $u_1$  in (4.17) is well defined if  $\bar{c}_2 < N_1 - \frac{\kappa_1 N_2}{2M_1} + \frac{\delta M_1}{\kappa_1}$ . We now require that  $w_0 > u_1$ . It can be shown that  $\frac{(1-a)\left(\frac{\kappa_1}{M_1} + \gamma_{3+}\right)}{\gamma_{3-}(\gamma_{3+} - \gamma_{3-})} = \alpha_0$ . Hence,  $w_0 > u_1$  is equivalent to  $K_{3-} \leq \alpha_0$ . This also proves that  $K_{3-} < 0$ . Since (4.12) still holds, we have  $K_{3-} \geq \frac{1-a}{\gamma_{3-}}$ , or, equivalently,  $K_{3+} > 0$ . Since  $\bar{c}_2 < N_1 - \frac{\kappa_1 N_2}{2M_1} + \frac{\delta M_1}{\kappa_1}$  and  $N_1 > \frac{\kappa_1 N_2}{2M_1}$ , then by Lemmas 7.7 and 7.8, we have  $(\underline{\alpha} \vee \frac{1-a}{\gamma_{3-}}) < \alpha_0$  and the following bounds for  $K_{3-}$ :

$$\frac{1-a}{\gamma_{3-}} < K_{3-} < \alpha_{LB},$$

where  $\alpha_{LB}$  is defined in (4.15).

For  $u_2$  defined in (4.17), it is well defined by Lemma 7.16. The following lemma gives a necessary and sufficient condition such that  $u_2 \geq w_0$ .

**Lemma 7.17.**  $u_2 \geq w_0$  if and only if

$$\bar{c}_1 + \bar{c}_2 \geq N_1 - \frac{\kappa_1 N_2}{2M_1} + \frac{\delta M_1}{\kappa_1}.$$

*Proof.*  $u_2 \geq w_0$  is equivalent to

$$\frac{(\gamma_{4-} - \gamma_{3-}) \left( -\frac{\kappa_1}{M_1} - \gamma_{3+} \right)}{(\gamma_{3+} - \gamma_{4-}) \left( \gamma_{3-} + \frac{\kappa_1}{M_1} \right)} \geq 1.$$

The above inequality is equivalent to  $\gamma_{4-} + \frac{\kappa_1}{M_1} \geq 0$ , which can be rewritten as

$$-(N_1 - \bar{c}_1 - \bar{c}_2) + \frac{\kappa_1 N_2}{M_1} \geq \sqrt{(N_1 - \bar{c}_1 - \bar{c}_2)^2 + 2\delta N_2}.$$

A necessary condition for the above inequality to hold is  $\bar{c}_1 + \bar{c}_2 \geq N_1 - \frac{\kappa_1 N_2}{M_1}$ . Suppose  $\bar{c}_1 + \bar{c}_2 \geq N_1 - \frac{\kappa_1 N_2}{M_1}$ . Squaring both sides of the above inequality yields the desired result.  $\square$

### Solving for $K_{3-}$

It remains to determine  $K_{3-}$ . Suppose  $\psi(\alpha_{LB}) > 0$ . By Lemma 7.14, it follows that  $\bar{c}_2 < N_1 - \frac{\kappa_1 N_2}{2M_1} + \frac{\delta M_1}{\kappa_1}$  and  $N_1 > \frac{\kappa_1 N_2}{2M_1}$ . Since  $\psi\left(\frac{1-a}{\gamma_{3-}}\right) < 0$ , we have that  $K_{3-}$  is the unique solution of  $\psi(z) = 0$  on  $\left(\frac{1-a}{\gamma_{3-}}, \alpha_{LB}\right)$  via Lemma 7.14.

## 7.3 Proof of Theorem 4.4

### Proving that $w_0$ is infinite when $M_1 < \infty$

Suppose  $\bar{c}_1 + \bar{c}_2 < N_1 - \frac{\kappa_1 N_2}{2M_1} + \frac{\delta M_1}{\kappa_1}$ . Suppose further that  $w_0 < \infty$  exists. In the region  $\{x > w_0\}$ , we get (7.6), whose solution is given by (7.7). From (4.5) and the assumption  $\bar{c}_1 + \bar{c}_2 < N_1 - \frac{\kappa_1 N_2}{2M_1} + \frac{\delta M_1}{\kappa_1}$ , we get

$$0 = -\kappa_1 \frac{g'_4(w_0)}{g''_4(w_0)} - M_1 = -\frac{\kappa_1}{\gamma_{4-}} - M_1 < 0,$$

which is a contradiction. Hence, no such  $w_0$  exists and we write  $w_0 = \infty$ .

### Deriving the analytical solution for $M_1 < \infty$

Suppose for now that  $u_1 \leq u_2 < w_0 = \infty$ . In the region  $\{x < u_1\}$ , the HJB equation becomes (7.1) whose solution that satisfies  $g(0) = 0$  is given by  $g_1(x) = K_1 \int_0^x \exp\left[\int_z^{u_1} \frac{\kappa_1}{G^{-1}(y)} dy\right] dz$ , where  $K_1 > 0$  is an unknown constant and  $G^{-1}$  is the inverse of the function  $G$  given by (4.9). In the region  $\{u_1 < x < u_2\}$ , the HJB equation becomes (7.21) whose solution is given by  $g_2(x) = K_{2+} \int_{u_1}^x \exp\left[\int_z^{u_2} \frac{\kappa_1}{H(y)} dy\right] dz + K_{2-}$ , where  $K_{2\pm}$  are unknown constants and  $H$  satisfies (4.16).

Since  $w_0 = \infty$ , we conjecture that  $\pi_1^*(x) = M_0$  for all  $x \geq u_2$  and some unknown constant  $M_0 \in (0, M_1)$ . In the region  $\{x > u_2\}$ , the HJB equation then becomes

$$0 = [\bar{N}_1(M_0) - \bar{c}_1 - \bar{c}_2] g'(x) + \frac{1}{2} \bar{N}_2(M_0) g''(x) - \delta g(x) + a\bar{c}_1 + (1-a)\bar{c}_2,$$

whose solution that satisfies  $\lim_{x \rightarrow \infty} g(x) = \frac{a\bar{c}_1 + (1-a)\bar{c}_2}{\delta}$  is given by

$$g_3(x) = K_3 e^{\bar{\gamma}_{4-}(M_0)x} + \frac{a\bar{c}_1 + (1-a)\bar{c}_2}{\delta},$$

where  $\bar{\gamma}_{4-}(y)$  is defined in (4.6). We conjecture the following solution:

$$g(x) = \begin{cases} K_1 \int_0^x \exp\left[\int_z^{u_1} \frac{\kappa_1}{G^{-1}(y)} dy\right] dz & \text{if } x < u_1, \\ K_{2+} \int_{u_1}^x \exp\left[\int_z^{u_2} \frac{\kappa_1}{H(y)} dy\right] dz + K_{2-} & \text{if } u_1 < x < u_2, \\ K_3 e^{\bar{\gamma}_{4-}(M_0)x} + \frac{a\bar{c}_1 + (1-a)\bar{c}_2}{\delta} & \text{if } x > u_2, \end{cases}$$

where  $K_1, K_{2\pm}, K_3, M_0, u_1, u_2$  are yet to be determined.

To ensure twice continuous differentiability at  $x = u_1$ , we obtain the equations in (7.23), which yield the formulas for  $K_1$  and  $K_{2-}$  in (7.24). Moreover, we have

$$K_{2+} = (1-a) \exp\left[-\int_{u_1}^{u_2} \frac{\kappa_1}{H(y)} dy\right].$$

To ensure twice continuous differentiability at  $x = u_2$ , we obtain the following equations:

$$\begin{aligned} (1-a) \exp\left[\int_{u_2}^{u_1} \frac{\kappa}{H(y)} dy\right] &= K_3 \bar{\gamma}_{4-}(M_0) e^{\bar{\gamma}_{4-}(M_0)u_2} = a, \\ -\frac{(1-a)\kappa_1}{M_0} \exp\left[\int_{u_2}^{u_1} \frac{\kappa}{H(y)} dy\right] &= K_3 \bar{\gamma}_{4-}^2(M_0) e^{\bar{\gamma}_{4-}(M_0)u_2}. \end{aligned} \tag{7.25}$$

From the first equation, we obtain

$$K_3 = \frac{a}{\bar{\gamma}_{4-}(M_0)} e^{-\bar{\gamma}_{4-}(M_0)u_2}.$$

Dividing the second equation in (7.25) by the first equation yields

$$-\frac{\kappa_1}{M_0} = \bar{\gamma}_{4-}(M_0). \quad (7.26)$$

We still have to prove that  $M_0$  exists.

**Lemma 7.18.** *Suppose  $\bar{c}_1 + \bar{c}_2 < N_1 - \frac{\kappa_1 N_2}{2M_1} + \frac{\delta M_1}{\kappa_1}$ . Then,  $M_0$  is a solution to*

$$-\frac{\kappa_1}{y} = \bar{\gamma}_{4-}(y), \quad y \in (0, M_1).$$

*Proof.* Since  $\bar{N}_1(0) = \bar{N}_2(0) = 0$ , we have

$$\bar{\gamma}_{4-}(0) := \lim_{y \rightarrow 0} \bar{\gamma}_{4-}(y) = \lim_{y \rightarrow 0} \frac{-2\delta}{-(\bar{N}_1(y) - \bar{c}_1 - \bar{c}_2) + \sqrt{(\bar{N}_1(y) - \bar{c}_1 - \bar{c}_2)^2 + 2\delta \bar{N}_2(y)}} = \frac{-\delta}{\bar{c}_1 + \bar{c}_2}.$$

Define  $f(y) := \bar{\gamma}_{4-}(y) + \frac{\kappa_1}{y}$ . We have the following:

$$\lim_{y \downarrow 0} f(y) = \infty \quad \text{and} \quad f(M_1) = \gamma_{4-} + \frac{\kappa_1}{M_1}.$$

By the intermediate value theorem,  $f$  has a solution in  $(0, M_1)$  if  $\gamma_{4-} + \frac{\kappa_1}{M_1} < 0$ . From the proof of Lemma 7.17,  $\gamma_{4-} + \frac{\kappa_1}{M_1} < 0$  is equivalent to  $\bar{c}_1 + \bar{c}_2 < N_1 - \frac{\kappa_1 N_2}{2M_1} + \frac{\delta M_1}{\kappa_1}$ , which proves the result.  $\square$

Similarly to the previous arguments, we have  $\pi_1^*(x) = H(x)$ . Hence, it follows that  $u_2$  satisfies

$$H(u_2) = M_0.$$

Finally, from the first equation in (7.25),  $u_1$  satisfies (4.18). The following lemma proves that  $u_1$  is the unique solution to the above equation.

**Lemma 7.19.**  *$u_1$  is the unique solution to (4.18) in  $(0, u_2)$ .*

*Proof.* Define  $f(x) := - \int_x^{u_2} \frac{\kappa_1}{H(y)} dy - \ln\left(\frac{a}{1-a}\right)$ . Since  $a \leq \frac{1}{2}$ , we have  $f(u_2) = -\ln\left(\frac{a}{1-a}\right) > 0$ . Since  $H(0) = 0$ , we have

$$\lim_{x \rightarrow 0} f(x) = -\infty.$$

Thus, by the intermediate value theorem, there exists a unique  $x_0 \in (0, u_2)$  such that  $f(x_0) = 0$ . Choosing  $x_0 = u_1$  proves the result.  $\square$

### Proving that $w_0$ is infinite if $M_1 = \infty$

We now consider the case where the support is unbounded (i.e.,  $M_1 = \infty$ ). We first prove that the configurations  $w_0 \leq u_1 \leq u_2$  and  $u_1 < w_0 \leq u_2$  do not hold. Suppose the configuration  $w_0 \leq u_1 \leq u_2$  holds and  $w_0$  exists. Similar to the bounded support case (i.e.,  $M_1 < \infty$ ), we obtain the form of the candidate solution in (7.8). The discussion follows with the bounded support case. However, from Lemma 7.12, we must have  $\bar{c}_1 + \bar{c}_2 \geq \infty$  for  $\alpha_{LB} \leq \alpha_{UB}$  to hold. This implies that  $K_{3-}$  does not exist. Thus, the configuration  $w_0 \leq u_1 \leq u_2$  is not possible.

Suppose the configuration  $u_1 < w_0 \leq u_2$  holds and  $w_0$  exists. Similar to the bounded support case, we also obtain the form of the candidate solution in (7.22), which leads to Lemma 7.17. The configuration will hold if and only if  $\bar{c}_1 + \bar{c}_2 \geq \infty$ , which implies that  $u_1 < w_0 \leq u_2$  does not hold.

## Deriving the analytical solution when $M_1 = \infty$

Suppose now that the configuration  $u_1 \leq u_2 < w_0$  holds. The discussion follows similarly as in the bounded support case. We obtain the equation (7.26). The following lemma states that  $M_0$  is indeed a solution to (7.26):

**Lemma 7.20.**  $M_0$  is a solution to

$$-\frac{\kappa_1}{y} = \bar{\gamma}_{4-}(y), \quad y \in (0, M_1).$$

*Proof.* Since  $\bar{N}_1(0) = \bar{N}_2(0) = 0$ , then we have

$$\bar{\gamma}_{4-}(0) := \lim_{y \rightarrow 0} \bar{\gamma}_{4-}(y) = \frac{-\delta}{\bar{c}_1 + \bar{c}_2}.$$

Define  $f(y) := \bar{\gamma}_{4-}(y) + \frac{\kappa_1}{y}$ . We have the following:

$$\lim_{y \downarrow 0} f(y) = \infty \quad \text{and} \quad f(M_1) = \gamma_{4-} < 0.$$

By the intermediate value theorem,  $f$  has a solution in  $(0, M_1)$ .  $\square$

## 7.4 Proof of Theorem 5.1

Suppose  $M_1 < \infty$ . The candidates for the optimal reinsurance strategies still satisfy (4.4). In the region  $\{x < u_1\}$ , the HJB equation becomes (7.1), whose solution is given by  $g_1(x) = K_1 \int_0^x \exp \left[ \int_z^{w_0} \frac{\kappa_1}{G^{-1}(y)} dy \right] dz$ , where  $K_1 > 0$  is an unknown constant and  $G^{-1}$  is the inverse of the function  $G$  given by (4.9). We also obtain  $\pi_1^*(x) = G^{-1}(x)$ . Since  $\pi_i'(x) > 0$  and  $\pi_i(0) = 0$ ,  $w_0$  exists and satisfies  $w_0 = G(M_1)$ .

In the region  $\{w_0 < x < u_1\}$ , the HJB equation becomes (7.2), whose solution is given by (7.3). In the region  $\{x > u_1\}$ , we must have  $g'(x) = 1 - a$ . Hence,

$$g_3(x) = (1 - a)[x - u_1 + K_3].$$

We conjecture the following solution:

$$g(x) = \begin{cases} K_1 \int_0^x \exp \left[ \int_z^{w_0} \frac{\kappa_1}{G^{-1}(y)} dy \right] dz & \text{if } x < w_0, \\ K_{2+} e^{\gamma_{2+}(x-w_0)} + K_{2-} e^{\gamma_{2-}(x-w_0)} & \text{if } w_0 < x < u_1, \\ (1 - a)[x - u_1 + K_3] & \text{if } x > u_1, \end{cases}$$

where  $w_0 = G(M_1)$ ,  $\gamma_{2\pm} := \bar{\gamma}_{2\pm}(M_1)$  are defined in (4.6), and  $K_1, K_{2\pm}, K_3, u_1$  are yet to be determined.

We now solve for the unknowns using the principle of smooth fit. At  $x = u_1$ , we have the following system of equations:

$$\begin{aligned} K_{2+} e^{\gamma_{2+} u_1} + K_{2-} e^{\gamma_{2-} u_1} &= (1 - a) K_3, \\ K_{2+} \gamma_{2+} e^{\gamma_{2+} u_1} + K_{2-} \gamma_{2-} e^{\gamma_{2-} u_1} &= 1 - a, \\ K_{2+} \gamma_{2+}^2 e^{\gamma_{2+} u_1} + K_{2-} \gamma_{2-}^2 e^{\gamma_{2-} u_1} &= 0. \end{aligned}$$

From the second and third equations, we obtain

$$K_{2+} = -\frac{\gamma_{2-}(1 - a)e^{-\gamma_{2+} u_1}}{\gamma_{2+}(\gamma_{2+} - \gamma_{2-})} > 0 \quad \text{and} \quad K_{2-} = \frac{\gamma_{2+}(1 - a)e^{-\gamma_{2-} u_1}}{\gamma_{2-}(\gamma_{2+} - \gamma_{2-})} < 0.$$

Substituting it into the first equation yields

$$K_3 = \frac{\gamma_{2+} + \gamma_{2-}}{\gamma_{2+}\gamma_{2-}} = \frac{N_1}{\delta},$$

where  $N_1 := \bar{N}_1(M_1)$  is defined in (4.6).

At  $x = w_0$ , it suffices to show that the derivatives are continuous. We have the following system of equations:

$$\begin{aligned} K_1 &= \frac{1-a}{\gamma_{2+} - \gamma_{2-}} \left[ \gamma_{2+} e^{\gamma_{2-}(w_0-u_1)} - \gamma_{2-} e^{\gamma_{2+}(w_0-u_1)} \right], \\ -\frac{\kappa_1}{M_1} K_1 &= \frac{(1-a)\gamma_{2+}\gamma_{2-}}{\gamma_{2+} - \gamma_{2-}} \left[ e^{\gamma_{2-}(w_0-u_1)} - e^{\gamma_{2+}(w_0-u_1)} \right]. \end{aligned}$$

Combining the two equations yields

$$e^{(\gamma_{2+}-\gamma_{2-})(w_0-u_1)} = \frac{\gamma_{2+}(\kappa_1 + \gamma_{2-}M_1)}{\gamma_{2-}(\kappa_1 + \gamma_{2+}M_1)} = \frac{\frac{\kappa_1}{\gamma_{2-}} + M_1}{\frac{\kappa_1}{\gamma_{2+}} + M_1}. \quad (7.27)$$

Since we assume that  $w_0 \leq u_1$ , it must be the case that  $e^{(\gamma_{2+}-\gamma_{2-})(w_0-u_1)} \in (0, 1)$ . Moreover, since  $\gamma_{2-} < 0$ , we have  $\frac{\frac{\kappa_1}{\gamma_{2-}} + M_1}{\frac{\kappa_1}{\gamma_{2+}} + M_1} < 1$ . From (7.27) and Lemma 7.1, the formula for  $u_1$  in (5.1) is well defined.

It must be noted that since  $\frac{\gamma_{2-}(\kappa_1 + \gamma_{2+}M_1)}{\gamma_{2+}(\kappa_1 + \gamma_{2-}M_1)} > 1$ , it holds that  $w_0 \leq u_1$ . Thus, we have obtained the form of the value function in (5.2). Similar arguments from the previous results yield that  $g$  is increasing and concave.

## 7.5 Proof of Theorem 5.2

In this scenario,  $M_1 = \infty$ . Hence, we only have one ‘‘switching’’ point, which is  $u_1$  defined in (4.3).

In the region  $\{x < u_1\}$ , we obtain the HJB equation in (7.1), whose solution that satisfies  $g(0) = 0$  and  $g'(u_1) = 1 - a$  is given by

$$g_1(x) = (1-a) \int_0^x \exp \left[ \int_z^{u_1} \frac{\kappa_1}{G^{-1}(y)} dy \right] dz,$$

where  $G^{-1}$  is the inverse of the function  $G$  defined in (4.9). In the region  $\{x > u_1\}$ , we must have  $g'(x) = 1 - a$ . We conjecture the following solution:

$$g(x) = \begin{cases} (1-a) \int_0^x \exp \left[ \int_z^{u_1} \frac{\kappa_1}{G^{-1}(y)} dy \right] dz & \text{if } x < u_1, \\ (1-a)[x - u_1 + K_3] & \text{if } x > u_1, \end{cases}$$

where  $K_3$  and  $u_1$  are yet to be determined.

By construction,  $g'(x)$  is continuous, so we only need to make  $g(x)$  and  $g''(x)$  be continuous at the switching point  $u_1$ . We first ensure that  $g''(x)$  is continuous, that is, we want

$$-\frac{(1-a)\kappa_1}{\pi_1(u_1-)} = g''(u_1-) = 0.$$

This implies that  $\pi_1(u_1-) = \infty$ . Define

$$G(\infty) := \lim_{y \rightarrow \infty} G(y).$$

We require the following result to ensure that  $G(\infty)$  is finite.

**Lemma 7.21.**  $G(\infty) < \infty$ .

*Proof.* Since  $\sigma_i^2(z) \rightarrow \tilde{\sigma}_i^2$  and  $\mu_i(z) \rightarrow \tilde{\mu}_i$  as  $z \rightarrow \infty$ , the integrand of  $G$  converges to zero at the rate of  $\frac{1}{z^2}$ , which proves the integrability of  $G(\infty)$ .  $\square$

Since  $g'(x)$  and  $g''(x)$  are continuous at  $x = u_1$ , then from the HJB equation, we have

$$\begin{aligned} 0 &= \frac{1}{2} \bar{N}_2(u_1) g''(u_1) + [\bar{N}_1(u_1)] g'(u_1) - \delta g(u_1) \\ &= [\kappa_1 \tilde{\mu}_1 + \kappa_2 \tilde{\mu}_2] (1-a) - \delta(1-a)K_3, \end{aligned}$$

where  $\bar{N}_1(y)$  and  $\bar{N}_2(y)$  are defined in (4.6). It implies that  $K_3 = \frac{N_1}{\delta}$ , where  $N_1 = \bar{N}_1(M_1)$ . Thus, we have obtained the form of the candidate value function in (5.3). Similar to the previous arguments, it can be shown that  $g$  is increasing and concave.

## 8 Conclusion

In this paper, we investigate optimal dividend payout, reinsurance, and capital injection strategies for insurers with two business lines, where reinsurance combines proportional and excess-of-loss coverage. We establish that the optimal reinsurance strategy is pure excess-of-loss and identify distinct, mutually exclusive dividend payout strategies for both bounded and unbounded dividend rates. The optimal capital injection strategy under both bounded and unbounded dividend rates is the same across all scenarios: capital transfers occur only to save one business line from ruin, provided that adequate reserves remain.

Future research could explore alternative types of dividend, such as periodic dividends and immediate dividends that incorporate transaction costs (see, e.g., [Kelbert and Moreno-Franco, 2025](#); [Avanzi et al., 2020, 2021](#)). Alternative processes that model the reserve level, such as Lévy processes, could also be considered (e.g., [Mata López et al., 2024](#)). The objective function could be modified to account for penalties for early ruin (e.g., [Strini and Thonhauser, 2023](#); [Xu and Woo, 2020](#)). It would also be of interest to investigate deep learning approaches (e.g., [Cheng et al., 2020](#)).

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