

# On the utility problem in a market where price impact is transient

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## Abstract

We consider a discrete-time model of a financial market where a risky asset is bought and sold with transactions having a transient price impact. It is shown that the corresponding utility maximization problem admits a solution. We manage to remove some unnatural restrictions on the market depth and resilience processes that were present in earlier work. A non-standard feature of the problem is that the set of attainable portfolio values may fail the convexity property.

**Keywords:** price impact; market friction; optimal investment; dynamic programming; nonconvex domain of optimization

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## 1 Introduction

Investors' actions move market prices and make large position changes costly. More or less realistic models for this *price impact* phenomenon have been worked out in the mathematical finance literature. One of the principal questions is the study of optimal investment in the presence of these (nonlinear) transaction costs.

Price impact may be assumed *instantaneous* if it affects the investor only at the moment of his/her portfolio rebalancing. This assumption leads to a relatively simple market dynamics, see [5, 2]. At the other extreme, price impact may be *permanent*, in which case the investor's action pushes the price in a direction and this effect pertains to the whole future. The most accurate description of reality is probably in between: price impact should be *transient*, with a gradually fading effect for the future, see [1, 3]. The speed at which these effects disappear is characterized by *market resilience*: if  $r$  is resilience then the bid-ask spread is diminished by a factor of  $e^{-r}$  in one unit of time. The size of the disturbance caused by trading a unit amount of the asset is described by *market depth*: if  $\delta$  is market depth then  $1/\delta$  is the effect on the bid-ask spread of trading one unit of the risky asset.

In the present article we prove that the utility maximization problem in discrete time for an agent trading with transient price impact is well-posed: it admits a solution. The problem under consideration has a complex, non-linear dynamics involving all previous strategies at a given time. Moreover, the domain of optimization is *non-convex*, which is a highly unusual feature.

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A continuous-time model with *instantaneous* price impact was considered in [5]: they proved in their Theorem 5.1 that the utility maximization problem (with a concave utility function) admits a solution under mild conditions.

In [3] a more advanced model with *transient* price impact was treated where the markets' resilience and depth were both assumed constant. Theorem 3.3 of [3] proved the existence of an optimal investment strategy in such a setting.

In [1] market resilience and depth were both allowed to be stochastic processes but a related monotonicity condition was imposed in their Assumption 2.4 which implies that the set of attainable portfolio values is convex. That condition holds for constant resilience and depth but otherwise it is rather restrictive. The paper [1] did not provide a general existence theorem for optimizers but a superreplication result (Theorem 3.2 in [1]) with a dual characterization of contingent claims that can be superhedged from a given initial position. They also proved a verification result (Corollary 3.5 in [1]): a sufficient condition implying that a given strategy is optimal.

In the present work we are dealing with the discrete-time version of the model of [1]. Our purpose is to prove the existence of an optimal strategy for a utility maximizer while removing the stringent monotonicity assumption of [1] on market resilience and market depth, see Theorem 2.4 and Remark 2.6 below.

Section 2 presents the technical details of our model, the main result (Theorem 2.4) and some discussions about the lack of convexity and its implications. Proofs will then be provided starting with Section 3. Section 4 deals with the one-step case while Section 5 provides the dynamic programming argument. Further reflections are given in Section 6.

## 2 Setup and results

For  $x \in \mathbb{R}$  we denote by  $x^+, x^-$  the positive and negative parts of  $x$ . Fix a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  once and for all, together with a discrete-time filtration  $\mathcal{F}_t$ ,  $t = 0, \dots, T$  where  $\mathcal{F}_0$  coincides with  $\mathbb{P}$ -null sets. Mathematical expectation with respect to  $\mathbb{P}$  will be denoted by  $\mathbb{E}[\cdot]$ ,  $\mathbb{E}_t[\cdot]$  stands for  $\mathcal{F}_t$ -conditional expectation.  $L^0$  is the set of all real-valued random variables.  $\mathbf{1}_A$  denotes the indicator function of a set  $A$ .

We now present the discrete-time version of the model in [1]. The time horizon of the investor will be some  $T \in \mathbb{N}$ . In the  $T$ th step the investor must liquidate her position in the risky asset hence genuine decisions are made only at the previous times  $t = 1, \dots, T - 1$ . To have a nontrivial problem we thus need to assume  $T \geq 2$ .

The risky asset's midprice (that is, the middle point of the bid-ask spread) is described by an adapted real-valued process  $P_t$ ,  $t = 0, \dots, T$ . This is the price followed when there is no trading from the part of the investor in consideration.

Position in the risky asset at time  $t$  is denoted by  $X_t$ ,  $t = 0, \dots, T$ , we assume  $X_0 := 0$ . At each time  $t$  the investor makes a portfolio adjustment based on previous information (before the new price  $P_t$  is revealed) hence  $X_t$  is assumed  $\mathcal{F}_{t-1}$  measurable, that is, the strategy process  $X_t$ ,  $1 \leq t \leq T$  is predictable. We follow the convention  $X_{-1} := 0$ . The set of all strategies is denoted by  $\mathcal{A}$ . We define

$$\mathcal{A}_0 := \{X \in \mathcal{A} : X_T = 0\},$$

the set of strategies that liquidate the position in the risky asset by the end of the time horizon. We note here, that due to the dynamics utilized in the paper, presented below, maximization of the utility of the terminal wealth  $\xi_T^X$  in  $X$  is economically

meaningful only over the set of strategies  $\mathcal{A}_0$ . Outside of  $\mathcal{A}_0$ , an investor could attain a position in the bank account with favourable expected utility while having large negative positions, and we would need to deal with liquidation value: such scenarios will be excluded.

We model liquidity with two primitives, resilience rate, and market depth: market resilience is described by a non-negative adapted process denoted by  $r_t \geq 0$ ,  $t = 0, \dots, T-1$ , and market depth is a positive adapted process  $\delta_t > 0$ ,  $t = 1, \dots, T$ . The *half-spread* follows a linear dynamics, namely

$$\zeta_{t+1}^X = e^{-r_t} \zeta_t^X + \frac{1}{\delta_{t+1}} |X_{t+1} - X_t|, \quad 0 \leq t \leq T-1, \quad (1)$$

starting from an initial value  $\zeta_0^X := \zeta_0 \geq 0$ . Finally, the cash account at time  $t = 1, \dots, T$ , considering that the investor pays the spread when trading, is calculated as

$$\xi_{t+1}^X - \xi_t^X = -P_{t+1}(X_{t+1} - X_t) - \zeta_{t+1}^X |X_{t+1} - X_t|, \quad 0 \leq t \leq T-1. \quad (2)$$

Setting  $\xi_0^X := 0$  for simplicity, with initial capital  $z \in \mathbb{R}$ , the investor has a wealth of  $z + \xi_T^X$  at time  $T$ . The above model is the discrete-time counterpart of the model introduced in [1]: except that in our setup only transient impact is modeled.

We further assume that the investor may possibly receive a random endowment during the trading period, described by an  $\mathcal{F}_T$ -measurable  $\mathbb{R}$ -valued random variable  $B$ . Negative  $B$  means that the investor has certain payment obligations during the period considered.

The investor aims to maximize her expected utility from terminal wealth, hence we fix a *utility function*  $u : \mathbb{R} \rightarrow \mathbb{R}$ .

**Assumption 2.1.** *The function  $u$  is non-decreasing, continuous,  $\lim_{x \rightarrow -\infty} u(x) = -\infty$ , and  $u$  is bounded from above. We furthermore assume that for all  $x, y, z \in \mathbb{R}$  and for all  $t = 1, \dots, T-1$ ,*

$$\mathbb{E}[u(x + yP_t + zP_T - B)] > -\infty \quad (3)$$

*holds.*

**Remark 2.2.** If  $u$  is concave then (3) is implied by the simpler condition

$$\mathbb{E}[u(x + yP_t - B)] > -\infty, \quad t = 1, \dots, T.$$

Indeed, by concavity of the mapping  $v \rightarrow u(x + v - B)$ ,

$$\mathbb{E}[u(x + yP_t + zP_T - B)] \geq \frac{\mathbb{E}[u(x + 2yP_t - B)] + \mathbb{E}[u(x + 2zP_T - B)]}{2}.$$

Our next hypothesis stipulates that market depth is always above a fixed threshold.

**Assumption 2.3.** *There is a constant  $\delta > 0$  such that  $\delta_t \geq \delta$  almost surely, for all  $t = 1, \dots, T$ .*

Our main result asserts that an investor with an arbitrary initial capital  $z \in \mathbb{R}$  may find an optimal portfolio strategy  $X^*(z)$ .

**Theorem 2.4.** *Let Assumption 2.1 and Assumption 2.3 be in force. Then, for each  $z \in \mathbb{R}$  there exists  $X^*(z) \in \mathcal{A}_0$  such that*

$$\bar{u}(z) := \mathbb{E} \left[ u \left( z + \xi_T^{X^*(z)} - B \right) \right] = \sup_{X \in \mathcal{A}_0} \mathbb{E} \left[ u \left( z + \xi_T^X - B \right) \right].$$

Theorem 2.4 shows that, despite the possible lack of convexity for the set of attainable values (see Example 2.5 below), the utility maximization problem admits an optimal strategy. We will present the proof of Theorem 2.4 in the coming sections, using a customized dynamic programming procedure.

From now on, for any stochastic process  $Z_t$ , we denote its increments by  $\Delta Z_t := Z_t - Z_{t-1}$ ,  $0 \leq t \leq T$  with the convention  $Z_{-1} := 0$ . It is convenient to use another parametrization for strategies: for a given real-valued process  $H_t$ ,  $1 \leq t \leq T$  such that  $H_t$  is  $\mathcal{F}_{t-1}$ -measurable, we may assign a unique strategy  $X_t$  such that  $X_0 = 0$ ,  $\Delta X_t = H_t$ ,  $1 \leq t \leq T$ . For such strategies we will also use the alternative notations  $\zeta^H, \xi^H$  for the corresponding half-spread and portfolio value processes. With a slight abuse of notation we will also write  $H \in \mathcal{A}_0$  when we really mean that the corresponding  $X$  is in  $\mathcal{A}_0$ . Note that  $H \in \mathcal{A}_0$  implies that  $H_T = -\sum_{j=1}^{T-1} H_j$ , in particular,  $H_T$  is  $\mathcal{F}_{T-2}$ -measurable.

Introduce the notation  $\rho_{j,t} := \exp \left[ -\sum_{i=j}^{t-1} r_i \right]$ ,  $1 \leq t \leq T$ ,  $0 \leq j \leq t$ . Note that  $\rho_{t,t} = 1$ . From (1) and (2) we derive the explicit formula

$$\xi_T^X = -\sum_{t=1}^T P_t H_t - \sum_{t=1}^T |H_t| \left( \rho_{0,t} \zeta_0 + \sum_{j=1}^t \frac{\rho_{j,t}}{\delta_j} |H_j| \right). \quad (4)$$

**Example 2.5.** Let  $T = 3$ ,  $r_t = 0$  for all  $0 \leq t \leq 3$ ,  $P_0 = \zeta_0 = 0$  and let  $P_1, P_2, P_3$  be independent standard Gaussian. Let  $\mathcal{F}_t$ ,  $0 \leq t \leq 3$  be the natural filtration of the process  $P$ . Let  $\delta_1 = 1$ ,  $\delta_2 = \delta_3 = 10$ . We claim that the set

$$\mathcal{S} := \{Y \in L^0 : \exists X \in \mathcal{A}_0 \text{ such that } \xi_T^X \geq Y\}$$

fails convexity. We argue by contradiction. Convexity would imply, in particular, that for arbitrary *non-negative deterministic* strategies  $H_i, G_i$ ,  $i = 1, 2$  such that  $H_3 := -H_1 - H_2$ ,  $G_3 = -G_1 - G_2$ , there would exist some  $\bar{H}_i$ ,  $i = 1, 2, 3$  such that

$$\xi_T^{\bar{H}} \geq \frac{\xi_T^H + \xi_T^G}{2}$$

almost surely. In view of (4), this inequality can be rewritten as

$$\begin{aligned} & \frac{1}{2} \left[ H_1^2 + H_2 H_1 + \frac{1}{10} H_2^2 + (H_1 + H_2) H_1 + \frac{1}{10} (H_1 + H_2) H_2 + \frac{1}{10} (H_1 + H_2)^2 \right] \\ & + \frac{1}{2} \left[ G_1^2 + G_2 G_1 + \frac{1}{10} G_2^2 + (G_1 + G_2) G_1 + \frac{1}{10} (G_1 + G_2) G_2 + \frac{1}{10} (G_1 + G_2)^2 \right] \\ & - \left[ \bar{H}_1^2 + \bar{H}_2 \bar{H}_1 + \frac{1}{10} \bar{H}_1^2 + (\bar{H}_1 + \bar{H}_2) \bar{H}_1 + \frac{1}{10} (\bar{H}_1 + \bar{H}_2) \bar{H}_2 + \frac{1}{10} (\bar{H}_1 + \bar{H}_2)^2 \right] \\ & \geq \left( \bar{H}_3 - \frac{H_3}{2} - \frac{G_3}{2} \right) P_3 + \left( \bar{H}_2 - \frac{H_2}{2} - \frac{G_2}{2} \right) P_2 + \left( \bar{H}_1 - \frac{H_1}{2} - \frac{G_1}{2} \right) P_1. \end{aligned}$$

Notice that the  $\mathcal{F}_2$ -conditional law of  $(\bar{H}_3 - H_3/2 - G_3/2)P_3$  is nondegenerate Gaussian on the set  $\{\bar{H}_3 - H_3/2 - G_3/2 \neq 0\}$ . Also, the left-hand side and the last two

terms of the right-hand side are  $\mathcal{F}_2$  measurable. Hence the above inequality necessarily implies that  $\mathbb{P}(\bar{H}_3 - H_3/2 - G_3/2 \neq 0) = 0$ . By an analogous argument, also  $\bar{H}_i - H_i/2 - G_i/2 = 0, i = 1, 2$ . But then

$$\begin{aligned} & \frac{1}{2} \left[ \frac{21}{10} H_1^2 + \frac{3}{10} H_2^2 + \frac{23}{10} H_1 H_2 + \frac{21}{10} G_1^2 + \frac{3}{10} G_2^2 + \frac{23}{10} G_1 G_2 \right] \\ & \geq \frac{21}{10} \bar{H}_1^2 + \frac{3}{10} \bar{H}_2^2 + \frac{23}{10} \bar{H}_1 \bar{H}_2. \end{aligned}$$

The latter property, however, badly fails. Take, for instance,  $H_1 = H_2 = 1$  and  $G_1 = 1.5, G_2 = 0$ . We conclude that  $\mathcal{S}$  is *not* a convex set.

**Remark 2.6.** The paper [1] made a monotonicity assumption. In the current discrete-time context it would require that the process  $\rho_{0,t}^2 \delta_t$  is a.s. strictly decreasing in  $t$ . Under this hypothesis, [1] showed that  $\mathcal{S}$  is convex. The novel contribution of our work is to drop such monotonicity assumptions and nevertheless to prove the existence of optimal strategies.

We finally point out in a simple example why non-convexity of the domain of optimization may cause trouble in optimal investment problems.

**Example 2.7.** We consider a one-step frictionless market model. Let the family of permitted strategies be  $\Phi := \{0, 1\}$ . That is, the investor may take either a unit position in the risky asset or no position at all. Let us consider the utility function  $u(x) := \ln(x)$  for  $x > 0$ . Let the investor have initial capital  $z > 1$ . Assume that the return on his investment is a random variable  $R$  with  $\mathbb{P}(R = 2) = 1/2 = \mathbb{P}(R = -1)$ . His indirect utility is then

$$\bar{u}(z) := \sup_{\phi \in \Phi} \mathbb{E}[u(z + \phi R)] = \max \left\{ \ln(z), \frac{\ln(z+2) + \ln(z-1)}{2} \right\}, \quad z > 1.$$

This function fails concavity: it is non-differentiable at 2 with the right-hand derivative being strictly larger than the left-hand derivative.

We conclude that even if the investor's utility is risk-averse (concave), his/her *indirect* utility may well fail this property when the family of permitted strategies is non-convex. Hence in related multistep optimization problems one needs to deal with a dynamic programming procedure involving *non-concave* functions, as in [4].

### 3 Preparation for the proof

Recall from (2) and (1) that

$$\Delta \xi_t^H = -P_t H_t - |H_t| \left( \rho_{0,t} \zeta_0 + \sum_{j=1}^t \frac{\rho_{j,t}}{\delta_j} |H_j| \right).$$

Inspired by this formula, for  $1 \leq t \leq T$  and  $h = (h_1, \dots, h_t)$  we introduce the random functions

$$\kappa_t(h) := -P_t h_t - |h_t| \left( \rho_{0,t} \zeta_0 + \sum_{j=1}^t \frac{\rho_{j,t}}{\delta_j} |h_j| \right) \quad (5)$$

for all  $h_1, \dots, h_t \in \mathbb{R}$ . Note that the mapping, describing the innovation corresponding to a deterministic action of the trader,

$$h_t \rightarrow \kappa_t((h_1, \dots, h_{t-1}, h_t)),$$

is *concave* for every fixed  $(h_1, \dots, h_{t-1})$ , but  $\kappa_t$ , as a function of  $t$  variables, has no reason to be concave. Note also that innovation has an "action-independent" market bound, namely the quantity

$$\kappa_t((h_1, \dots, h_t)) \leq \lambda_t(h_t) := -P_t h_t - \frac{h_t^2}{\delta_t} \leq \frac{P_t^2 \delta_t}{4}. \quad (6)$$

We recall Lemma 6.8 of [4].

**Lemma 3.1.** *Let  $(\Omega, \mathcal{H}, P)$  be a complete probability space. Let  $\Xi^d$  be the set of  $\mathcal{H}$ -measurable  $d$ -dimensional random variables. Let  $F : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}$  be a function such that for almost all  $\omega \in \Omega$ ,  $F(\omega, \cdot)$  is continuous and for each  $y \in \mathbb{R}^d$ ,  $F(\cdot, y)$  is  $\mathcal{H}$ -measurable. Let  $K > 0$  be an  $\mathcal{H}$ -measurable random variable.*

*Set  $f(\omega) = \text{ess. sup}_{\xi \in \Xi^d, |\xi| \leq K} F(\omega, \xi(\omega))$ . Then, for almost all  $\omega$ ,*

$$f(\omega) = \sup_{y \in \mathbb{R}^d, |y| \leq K(\omega)} F(\omega, y). \quad (7)$$

□

A compactness result involving random subsequences comes next, this is Lemma 2 of [6].

**Lemma 3.2.** *Let  $\mathcal{H} \subset \mathcal{F}$  be a sigma-algebra. Let  $X_n$  be a sequence of  $\mathcal{H}$ -measurable  $d$ -dimensional random variables such that*

$$\liminf_{n \rightarrow \infty} |X_n| < \infty \quad (8)$$

*almost surely. Then there exist  $\mathcal{H}$ -measurable random variables  $n_k : \Omega \rightarrow \mathbb{N}$ ,  $k \in \mathbb{N}$  with  $n_k(\omega) < n_{k+1}(\omega)$  for all  $\omega \in \Omega$  and  $k \in \mathbb{N}$  and an  $\mathcal{H}$ -measurable random variable  $X$  such that  $X_{n_k} \rightarrow X$  a.s. In such a case we write that there exists an  $\mathcal{H}$ -measurable random subsequence.*

□

The following lemma uses the same idea as Lemma A.3 of [8]; it provides continuous versions for certain random fields.

**Lemma 3.3.** *Let  $\mathcal{H} \subset \mathcal{F}$  be a sigma-algebra. Define  $\mathcal{K} := [-N, N]^n$ . Let  $L : \Omega \times \mathcal{K} \rightarrow \mathbb{R}$  be such that for a.e.  $\omega \in \Omega$ ,  $L(\omega, \cdot)$  is continuous and for all  $x \in \mathcal{K}$ ,  $L(\cdot, x)$  is measurable such that  $\sup_{z \in \mathcal{K}} |L(\omega, z)|$  is integrable. Then there is  $l : \Omega \times \mathcal{K} \rightarrow \mathbb{R}$  such that for a.e.  $\omega \in \Omega$ ,  $l(\omega, \cdot)$  is continuous and for all  $k \in \mathcal{K}$ ,  $E(L(k)|\mathcal{H}) = l(k)$  a.s.*

*Proof.* Consider the separable Banach space  $\mathbb{B} := C([-N, N]^n)$  of continuous functions on  $[-N, N]^n$  with the supremum norm. Clearly,  $L : \Omega \rightarrow \mathbb{B}$  and for all  $\mu$  in the dual space  $\mathbb{B}'$  (which can be represented as a Borel signed measure),  $\mu(L) = \int_{\mathcal{K}} L(\omega, x) \mu(dx)$  is a measurable function on  $(\Omega, \mathcal{F})$ : indeed, this is clear for  $\mu$  with finite support and then follows for general  $\mu$  by approximation. Note that, for each  $k \in \mathcal{K}$ , the linear functional  $f_k(x) := x(k)$ ,  $x \in \mathbb{B}$  is continuous (w.r.t. the norm of  $\mathbb{B}$ ) so  $f_k \in \mathbb{B}'$ . Now it follows from Proposition V.2.5. of [7] that there is a measurable  $l : \Omega \rightarrow \mathbb{B}$  such that

$$l(k) = f_k(l) = E(f_k(L)|\mathcal{H}) = E(L(k)|\mathcal{H}),$$

for each  $k \in \mathcal{K}$ , as claimed.

□

Now we turn to a set of Lemmas that ensure that we can perform a backward iteration, and produce a series of actions that forms a candidate strategy of optimal execution.

## 4 Single step case

Let  $t \geq 1$  be an integer and let  $\mathcal{G}, \mathcal{H}$  be  $\mathbb{P}$ -complete sigma-algebras over  $\Omega$  such that  $\mathcal{H} \subset \mathcal{G} \subset \mathcal{F}$  holds, and denote the set of  $\mathcal{H}$ -measurable  $\mathbb{R}$ -valued random variables by  $\Xi$ . We will consider functions

$$G_0 : \Omega \times \mathbb{R} \times \mathbb{R}^t \rightarrow \mathbb{R}, (x, v) \mapsto G_0(x, v),$$

and

$$V : \Omega \times \mathbb{R} \times \mathbb{R}^{t-1} \times \mathbb{R} \rightarrow \mathbb{R}, (x, j, h) \mapsto G_0(x + \kappa_t((j, h)), (j, h)),$$

where  $(j, h) = (j_1, \dots, j_{t-1}, h)$ , and the  $\kappa_t$  is as in (5). Below we introduce assumptions that serve as a basis for the iterative arguments later.

**Assumption 4.1.** *The function  $G_0$  is  $\mathcal{G} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^t)$ -measurable, the mapping  $G_0(\omega, \cdot, \cdot)$ , is jointly continuous and non-decreasing in its first variable,  $\mathbb{P}$ -almost surely.*

**Assumption 4.2.** *There exists a function  $\bar{G}_0 : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ , and a constant  $C > 0$ , such that for all  $x \in \mathbb{R}$  there exists a zero measure set outside of which*

$$\bar{G}_0(x) \rightarrow -\infty$$

*holds as  $x \rightarrow -\infty$ . Furthermore,  $G_0(\omega, \cdot)$  is non-decreasing almost surely, and for all pairs  $(x, v) \in \mathbb{R} \times \mathbb{R}^t$  we have that the inequalities*

$$G_0(x, v) \leq \bar{G}_0(x) \leq C$$

*hold, again outside a set of measure zero.*

**Assumption 4.3.** *Assume that for any  $m \in \mathbb{N}$  and for all  $1 \leq t' \leq t$ , there exists an integrable random variable  $M = M(m, t')$  so that*

$$M \leq G_0(x + \kappa_{t'}((v_1, \dots, v_{t'})), v)$$

*holds for every  $x \in [-m, m]$ ,  $v \in [-m, m]^{t'}$ , and for almost every  $\omega \in \Omega$ .*

Throughout Section 4 we postulate that the conditions prescribed by Assumption 4.1, Assumption 4.2, and Assumption 4.3 hold.

**Lemma 4.4.** *There exists  $L : \Omega \times \mathbb{R} \times \mathbb{R}^{t-1} \times \mathbb{R} \rightarrow \mathbb{R}$  so that for all  $(x, j, h) \in \mathbb{R} \times \mathbb{R}^{t-1} \times \mathbb{R}$  we have*

$$E[V(x, j, h) | \mathcal{H}] = L(x, j, h)$$

*almost surely, and furthermore,  $L(\cdot, \cdot, \cdot)$  is continuous in all its variables for almost all  $\omega \in \Omega$ .*

*Proof.* Under Assumption 4.1, Assumption 4.2, and Assumption 4.3 the proof is a direct consequence of Lemma 3.3. □

**Lemma 4.5.** Let  $L$  be as in Lemma 4.4,  $\bar{G}_0$  be as in Assumption 4.2,  $\lambda_t$  be as in (6), and let  $x \in \mathbb{R}$ . As  $|h| \rightarrow -\infty$  we have that

$$\sup_{j \in \mathbb{R}^{t-1}} L(x, j, h) \rightarrow -\infty \quad (9)$$

almost surely. (In the case  $t = 1$  we mean that  $L$  does not depend on  $j$  and there is no supremum.)

*Proof.* Without loss of generality we assume  $t = 2$ , the case of  $t > 2$  being only notationally more cumbersome. By Assumption 4.2, the definition of  $L$  and using (6), we have for all  $x, j, h \in \mathbb{R}$  that

$$\begin{aligned} L(x, j, h) &= E[V(x, j, h)|\mathcal{H}] \\ &\leq E[\bar{G}_0(x + \kappa_2(j, h))|\mathcal{H}] \\ &\leq E[\bar{G}_0(x + \lambda_2(h))|\mathcal{H}] \end{aligned}$$

almost surely. Apply Fatou's reverse lemma to the inequalities above. Considering (6) and the absence of dependence on the variable  $j$ , continuity of  $\bar{G}_0$  shows (9).  $\square$

**Lemma 4.6.** The inequality

$$L(x_1, j, h) \leq L(x_2, j, h), \quad (10)$$

holds for all  $x_1, x_2, h \in \mathbb{R}$  with  $x_1 < x_2$ , for all  $j \in \mathbb{R}^{t-1}$ , and for almost every  $\omega \in \Omega$ .

*Proof.* Without loss of generality we can assume  $t = 2$ . let  $\Omega'$  be a  $\mathbb{P}$ -full measure set such that (10) holds for all  $x_1, x_2, j, h \in \mathbb{Q}$  on  $\Omega'$ . Let  $\Omega''$  be the full measure set on which  $L$  is continuous. Then on  $\Omega' \cap \Omega''$  (10) holds for all  $x_1, x_2, j, h \in \mathbb{R}$ , by continuity.  $\square$

**Lemma 4.7.** Fix  $l \in \mathbb{Z}$  and  $m \in \mathbb{N}$ . There exists an  $\mathcal{H}$ -measurable  $K = K(l, m)$  such that

$$L(x, j, h) \leq L(x, j, h \mathbf{1}_{\{|h| \leq K\}}), \quad (11)$$

for all  $x \in [l, l+1]$ ,  $j \in [-m, m]^{t-1} \subset \mathbb{R}^{t-1}$ ,  $h \in \mathbb{R}$ , and for almost every  $\omega \in \Omega$ .

*Proof.* Without loss of generality assume  $t = 2$ , and let  $\bar{\Omega}$  denote the full measure set where the conclusion of Lemma 4.4 and Lemma 4.6 hold. For every  $\omega \in \bar{\Omega}$  choose a (measurable)  $m^+(h) = m^+(\omega, l, h)$  such that

$$L(l+1, j, h)(\omega) \leq L(l+1, m^+(h), h)(\omega)$$

holds for all  $j \in [-m, m]$  and  $h \in \mathbb{R}$ , this is possible by continuity of  $L$ . Likewise, for every  $\omega \in \bar{\Omega}$  choose  $m^- = m^-(\omega, l)$  such that

$$L(l, m^-, 0)(\omega) \leq L(l, j, 0)(\omega)$$

holds for all  $j \in [-m, m]$ . Fix  $l \in \mathbb{Z}$  and  $m \in \mathbb{N}$ . Using Lemma 4.5 for all  $\omega \in \bar{\Omega}$  there exists  $K(\omega) = K(\omega, l, m)$  so that for all  $h \in \mathbb{R}$  it holds that

$$|h| > K(\omega) \implies L(l+1, m^+(h), h)(\omega) \leq L(l, m^-, 0)(\omega). \quad (12)$$



Now note, that  $\omega \rightarrow K(\omega)$  can be chosen in a way that it is  $\mathcal{H}$ -measurable as a random variable. On the event  $\{|h| > K\} \cap \bar{\Omega}$ , using Lemma 4.6, and the statement in (12), we have

$$\begin{aligned} L(x, j, h) &\leq L(l+1, j, h) \leq L(l+1, m^+(h), h), \\ &\leq L(l, m^-, 0) \leq L(l, j, 0) \leq L(x, j, 0), \end{aligned} \quad (13)$$

for every  $x \in [l, l+1]$ ,  $j \in [-m, m]$ ,  $h \in \mathbb{R}$ , and for all  $\omega \in \bar{\Omega}$ : completing the argument.  $\square$

**Lemma 4.8.** *There exists an  $\mathcal{H} \otimes \mathcal{B}(\mathbb{R})$ -measurable function*

$$G : \Omega \times \mathbb{R} \times \mathbb{R}^{t-1} \mapsto \mathbb{R},$$

*such that  $G$  is continuous almost surely,  $G$  is non-decreasing in its first variable for almost all  $\omega \in \Omega$ , furthermore, for all  $x$ , and for all  $j \in \mathbb{R}^{t-1}$  we have*

$$G(x, j) = \text{ess. sup}_{H \in \Xi} L(x, j, H), \quad (14)$$

*almost surely.*

*Proof.* We follow arguments of Lemma 3.17 in [4]. Let  $x, j \in \mathbb{R}$ , and without loss of generality assume  $t = 2$  and set  $l \in \mathbb{Z}$  and  $m \in \mathbb{N}$  so that  $x \in [l, l+1]$  and  $j \in [-m, m]$  holds. We will work out the statement in consideration elementwise on  $\Omega$ , and to this end – out of the usual – we do not omit to display dependence on  $\omega \in \Omega$  throughout the proof, until further notice. Denote with  $\bar{\Omega}$  the full measure set for which the conclusions of Lemma 4.4, 4.6, 4.7 hold, and let  $\omega \in \bar{\Omega}$ . Let us define

$$B(\omega, x, j) = \sup_{h \in \mathbb{Q}, |h| \leq K(\omega)} L(\omega, x, j, h) = \sup_{h \in \mathbb{R}, |h| \leq K(\omega)} L(\omega, x, j, h),$$

where  $K(\omega) = K(\omega, l, m) = K(l, m)$  is as in Lemma 4.7. This is measurable, being the supremum of countably many measurable functions. Let us fix furthermore a sequence  $(x_n, j_n)_{n \in \mathbb{N}} \subset ([l, l+1] \times [-m, m]) \cap \mathbb{R}^2$  for which  $(x_n, j_n) \rightarrow (x, j)$ . Observe, that by definition of  $B$ , for every  $k \in \mathbb{N}$  there exists  $h_k(\omega, x, j)$ , with  $h_k(\omega, x, j) \leq K(\omega)$ , so that  $B(\omega, x, j) - 1/k \leq L(\omega, x, j, h_k(\omega, x, j))$ . The fact that for all  $n \in \mathbb{N}$  we have  $B(\omega, x_n, j_n) \geq L(\omega, x_n, j_n, h_k(\omega, x, j))$ , along with continuity of  $L$  yields

$$\begin{aligned} \liminf_n B(\omega, x_n, j_n) &\geq L(\omega, x, j, h_k(\omega, x, j)) \\ &\geq B(\omega, x, j) - 1/k, \end{aligned}$$

which in the limiting case of  $k \rightarrow \infty$  means  $\liminf_n B(\omega, x_n, j_n) \geq B(\omega, x, j)$ .

Take a sequence  $\{n_k, k \in \mathbb{N}\} \subset \mathbb{N}$  so that

$$\limsup_n B(\omega, x_n, j_n) = \lim_k B(\omega, x_{n_k}, j_{n_k}). \quad (15)$$

Since  $\{h : h \in \mathbb{Q}, |h| \leq K(\omega)\}$  is a precompact set in  $\mathbb{R}$ , for every  $k \in \mathbb{N}$  there exists  $\mathbb{R} \ni h_{n_k}^*(\omega) \leq K(\omega)$  so that  $B(\omega, x_{n_k}, j_{n_k}) = L(\omega, x_{n_k}, j_{n_k}, h_{n_k}^*(\omega))$ . Using the compactness of the closure there exists  $\mathbb{R} \ni h^*(\omega) \leq K(\omega)$  and a subsequence  $(a_k)_{k \in \mathbb{N}}$  of  $\{n_k : k \in \mathbb{N}\}$  so that  $h_{a_k}^*(\omega) \rightarrow h^*(\omega)$ ,  $k \rightarrow \infty$ . These, and (15) imply

$$\begin{aligned} \limsup_n B(\omega, x_n, j_n) &= \lim_k B(\omega, x_{n_k}, j_{n_k}) = \lim_k B(\omega, x_{a_k}, j_{a_k}) \\ &= \lim_k L(\omega, x_{a_k}, j_{a_k}, h_{a_k}^*(\omega)) = L(\omega, x, j, h^*(\omega)) \leq B(\omega, x, j), \end{aligned}$$

establishing the continuity of  $B$ .

As far as monotonicity is concerned, the mapping  $x \rightarrow B(\omega, x, j)$  inherits the non-decreasing property from  $L$  (stated in Lemma 4.6) naturally.

From Lemma 3.1 it follows that

$$B(\omega, x, j) = \text{ess.sup}_{H(\omega) \leq K(\omega)} L(\omega, x, j, H(\omega)).$$

In the discussion above  $\omega \in \bar{\Omega}$  was arbitrary, and returning to the usual practice of not displaying the dependence on it, Lemma 4.7 shows that

$$\begin{aligned} \text{ess.sup}_{H \in \Xi} L(x, j, H) &\leq \text{ess.sup}_{\Xi \ni H \leq K} L(x, j, H) = B(x, j) \\ &\leq \text{ess.sup}_{H \in \Xi} L(x, j, H) \end{aligned}$$

holds almost surely, finishing the argument.  $\square$

**Lemma 4.9.** *Let  $X, H_1, \dots, H_{t-1}$  be  $\mathcal{H}$ -measurable random variables, and let  $G$  be as in Lemma 4.7. Then, the quantity  $G(X, (H_1, \dots, H_{t-1}))$  is a version of the essential supremum*

$$\text{ess. sup}_{H \in \Xi_{t-1}} L(X, (H_1, \dots, H_{t-1}), H).$$

*Proof.* Without loss of generality we assume that  $t = 2$ , and that  $X$  and  $H_1$  take values in  $[l, l+1]$  and  $[-m, m]$  respectively. Let  $X_n$  and  $H_1^{(n)}$  be  $\mathcal{H}$ -measurable random variables, taking values in  $[l, l+1] \cap \mathbb{Q}$  and  $[-m, m] \cap \mathbb{Q}$ , respectively, for all  $n \in \mathbb{N}$ , and possessing also the limiting properties  $X_n \rightarrow X$  and  $H_1^{(n)} \rightarrow H_1$ .

Observe that for all  $n \in \mathbb{N}$ , on a full measure set, we have that

$$\text{ess.sup}_{H \in \Xi_{t-1}} L(X_n, H_1^{(n)}, H) = G(X_n, H_1^{(n)}),$$

and furthermore, as a consequence, for all  $n \in \mathbb{N}$  there exists  $H_n$  so that

$$L(X_n, H_1^{(n)}, H_n) \geq G(X_n, H_1^{(n)}) - 1/n \quad (16)$$

almost surely. According to Lemma 4.7 there exists an  $\mathcal{H}$ -measurable  $K = K(l, m) = K(\omega, l, m)$  so that

$$L(X_n, H_1^{(n)}, H_n \mathbf{1}_{\{|H_n| \leq K\}}) \geq L(X_n, H_1^{(n)}, H_n). \quad (17)$$

Note that the  $K$  does not depend on the integer  $n$  in any way. Putting together (16) and (17) gives, for all  $n \in \mathbb{N}$  the almost sure inequality

$$L(X_n, H_1^{(n)}, H_n \mathbf{1}_{\{|H_n| \leq K\}}) \geq G(X_n, H_1^{(n)}) - 1/n. \quad (18)$$

Now using Lemma 3.2, there exists an  $\mathcal{H}$ -measurable subsequence  $k_n = k_n(\omega)$ ,  $k \in \mathbb{N}$  and an  $\mathcal{H}$ -measurable  $\bar{H}$  so that  $H_{k_n} \mathbf{1}_{\{|H_{k_n}| \leq K\}} \rightarrow \bar{H}$  holds almost surely. In (18) taking the limit as  $n \rightarrow \infty$  along the sequence  $k_n$ ,  $n \in \mathbb{N}$ , and utilizing continuity of  $L$  and  $G$  yields

$$L(X, H_1, \bar{H}) \geq G(X, H_1),$$

which in return implies  $\text{ess.sup}_{H \in \Xi_{t-1}} L(X, H_1, H) \geq G(X, H_1)$ .

On the other hand, by definition of  $G$ , for every  $\mathcal{H}$ -measurable  $H$  we have

$$\begin{aligned} L(X, H_1, H) &= \lim_n L(X_n, H_1^{(n)}, H) \leq \lim_n \text{ess.sup}_H L(X_n, H_1^{(n)}, H) \\ &= \lim_n G(X_n, H_1^{(n)}) = G(X, H_1). \end{aligned}$$

Taking the essential supremum of both sides yields the inequality

$$\text{ess.sup}_{H \in \Xi_{t-1}} L(X, H_1, H) \leq G(X, H_1)$$

on a full measure set: finishing the proof.  $\square$

**Lemma 4.10.** *Let  $G$  be as in Lemma 4.8. Let  $X, H_1, \dots, H_{t-1}$  be  $\mathcal{H}$ -measurable random variables. Then, there exists an  $\mathcal{H}$ -measurable  $H^*$  so that*

$$G(X, (H_1, \dots, H_{t-1})) = L(X, (H_1, \dots, H_{t-1}), H^*)$$

*holds almost surely.*

*Proof.* Without loss of generality assume that  $t = 2$ ,  $X$  almost surely takes values in the closed interval  $[l, l+1]$ ,  $H_1$  takes values in the closed interval  $[-m, m]$  for some  $l \in \mathbb{Z}$  and for some  $m \in \mathbb{N}$ . Let us define

$$\mathcal{O} = \left\{ L(X, H_1, H), \ H \in \Xi(\mathcal{H}) \right\}.$$

This set is upward directed in terms of the almost sure sense of the usual "less than or equal" relation. Thus, using Proposition VI.1.1. of [7] we have that there exists a sequence  $\{H_n, n \in \mathbb{N}\} \subset \Xi(\mathcal{H})$  for which the limiting property

$$L(X, H_1, H_n) \rightarrow G(X, H_1) \tag{19}$$

holds almost surely as  $n \rightarrow \infty$ . Utilizing Lemma 4.7, and the  $K = K(l, m)$  within, we have for each  $n \in \mathbb{N}$  that

$$L(X, H_1, H_n) \leq L(X, H_1, H_n \mathbf{1}_{\{|H_n| \leq K\}}), \tag{20}$$

almost surely. Then, again using Lemma 3.2, there exists an  $\mathcal{H}$ -measurable subsequence, say  $k_n$ ,  $n \in \mathbb{N}$ , and there exists an  $\mathcal{H}$ -measurable  $H^*$  so that

$$H_{k_n} \mathbf{1}_{\{|H_{k_n}| \leq K\}} \rightarrow H^*$$

in the almost sure sense. Continuity of  $L$ , (19), and (20) together guarantees

$$\begin{aligned} G(X, H_1) &= \lim_{n \rightarrow \infty} L(X, H_1, H_{k_n}) \\ &\leq \lim_{n \rightarrow \infty} L(X, H_{k_n} \mathbf{1}_{\{|H_{k_n}| \leq K\}}) = L(X, H_1, H^*). \end{aligned}$$

The proof is complete.  $\square$

**Lemma 4.11.** *Let  $G$  be as in Lemma 4.8. There exists  $\bar{G} : \mathbb{R} \rightarrow \mathbb{R}$ , and  $C > 0$  such that the following requirements are met: as  $x \rightarrow -\infty$  we have*

$$\bar{G}(x) \rightarrow -\infty,$$

*almost surely and, for all  $x \in \mathbb{R}, j \in \mathbb{R}^{t-1}$  we have*

$$G(x, j) \leq \bar{G}(x) \leq C$$

*in the almost sure sense.*

*Proof.* Let  $G$  and  $H^*$  be as in Lemma 4.8 and Lemma 4.10 respectively. Without loss of generality we can assume  $t = 2$ . Let  $H^*$  be as in Lemma 4.10, and let us choose a sequence of rational-valued random variables  $H_n^*$  increasing to it. Using the market bound for the innovations  $\kappa$ . in (6), Assumption 4.3, and Fatous' reverse Lemma, we have for  $x, j \in \mathbb{R}$  that

$$\begin{aligned} G(x, j) &= L(x, j, H^*) = \lim_n L(x, j, H_n^*) \leq \limsup_n E[V(x, j, H_n^*) | \mathcal{H}] \\ &\leq \limsup_n E[\bar{G}_0(x + \kappa_2(j, H_n^*)) | \mathcal{H}] \leq E[\bar{G}_0(x + \kappa_2(j, H^*)) | \mathcal{H}] \\ &\leq E[\bar{G}_0(x + P_t^2 \delta_t / 4) | \mathcal{H}] =: G'(x) \end{aligned}$$

almost surely.

Fix some  $v_0 \in \mathbb{R}$  and note that we have  $G_0(x, v_0) \leq \bar{G}_0(x)$ . With Assumption 4.3 in mind, take a continuous version of  $G'$  using Lemma 3.3, and denote it by  $G''$ . By construction for a.e.  $\omega \in \Omega$  we have  $G(\cdot, \cdot) \leq C$  and  $G''(\cdot) < C$ , and observe that by the reverse Fatou lemma we have  $G''(x) \rightarrow -\infty$  and  $x \rightarrow -\infty$ . The function  $G''(\cdot)$  inherits monotonicity from  $\bar{G}_0$ . The former fact can be seen with similar reasoning given in the proof of Lemma 4.6. The choice  $\bar{G} = G''$  completes the proof.  $\square$

**Lemma 4.12.** *Let  $G$  be as in Lemma 4.8, and let  $t' \in \{1, \dots, t-1\}$ . For any  $m \in \mathbb{N}$  there exists an integrable random variable  $M_1 = M_1(m, t')$  so that*

$$M_1 \leq G(x + \kappa_{t'}((j_1, \dots, j_{t'})), j)$$

for every  $x \in \mathbb{R}$ ,  $j \in [-m, m]^{t-1}$ , and for almost every  $\omega \in \Omega$ .

*Proof.* Let  $m \in \mathbb{N}$  and  $t' \in \{1, \dots, t-1\}$ . Using Assumption 4.3 there exists and integrable,  $\mathcal{G}$ -measurable  $M_0 = M_0(m, t')$  so that with the notation  $(j, 0) = (j_1, \dots, j_{t-1}, 0)$  we have

$$G_0(x + \kappa_{t'}((j_1, \dots, j_{t'})), (j, 0)) \geq M_0$$

for all  $x \in [-m, m]$ , for all  $j \in [-m, m]^{t-1}$ , and for almost every  $\omega \in \Omega$ . For fixed  $j = (j_1, \dots, j_{t-1}) \in [-m, m]^{t-1}$  and  $x \in [-m, m]$ , we have

$$\begin{aligned} G(x + \kappa_{t'}((j_1, \dots, j_{t'})), j) &\geq L(x + \kappa_{t'}((j_1, \dots, j_{t'})), j, 0) \\ &= E[V(x + \kappa_{t'}((j_1, \dots, j_{t'})), j, 0) | \mathcal{H}] \\ &= E[G_0(x + \kappa_{t'}((j_1, \dots, j_{t'})) + \kappa_t((j, 0)), (j, 0)) | \mathcal{H}] \\ &= E[G_0(x + \kappa_{t'}((j_1, \dots, j_{t'})), (j, 0)) | \mathcal{H}] = E[M_0 | \mathcal{H}] \end{aligned} \tag{21}$$

almost surely. Since  $G$  is continuous the relation established in (21) above holds for all  $x \in [-m, m]$ , for all  $j \in [-m, m]^{t-1}$ , and for almost every  $\omega \in \Omega$ . The choice  $M_1 = E[M_0 | \mathcal{H}]$  gives a desired lower bound.  $\square$

## 5 The generic step, dynamic programming

First, in a phase of *backward induction* we construct actions that – depending parametrically on previous decisions and accumulated wealth – are optimal in an instantaneous sense. These actions however would only be optimal in one-step markets.

Then, we use these actions to build a strategy for the entire interval of trading, and this strategy will serve as a *candidate strategy* for optimal trading.

Second, with a *forward iteration* we show that the *candidate* indeed represents a series of actions that dominates all admissible executions in terms of expected utility: arriving to the conclusion of the paper.

Assumptions 2.1 and 2.3 will be in force from now on. Fix  $C_u \geq 0$  such that  $u(x) \leq C_u$  for all  $x \in \mathbb{R}$ . We denote with  $\Xi_t$  the  $\mathcal{F}_t$ -measurable random variables.

*Proof of Theorem 2.4.* Define  $\tilde{G}_T : \Omega \times \mathbb{R} \times \mathbb{R}^T \rightarrow \mathbb{R}$  as

$$\tilde{G}_T(\omega, x, h_1, \dots, h_T) := u(x - B(\omega)), \quad \omega \in \Omega, \quad x, h_1, \dots, h_T \in \mathbb{R}. \quad (22)$$

Note that  $h_1, \dots, h_T$  are dummy variables here, and  $\tilde{G}_T$  is continuous and nondecreasing in  $x$  almost surely.

The first step of the *backward induction* is different from the other steps since  $h_T = -h_1 - \dots - h_{T-1}$  due to the constraint on liquidation. We thus consider  $\tilde{G}_{T-1} : \Omega \times \mathbb{R} \times \mathbb{R}^{T-1} \rightarrow \mathbb{R}$  with the definition

$$\begin{aligned} & \tilde{G}_{T-1}(\omega, x, h_1, \dots, h_{T-1}) \\ &= E \left[ \tilde{G}_T \left( x + \kappa_T \left( h_1, \dots, h_{T-1}, -\sum_{k=1}^{T-1} h_k \right), h_1, \dots, h_{T-1}, -\sum_{k=1}^{T-1} h_k \right) \middle| \mathcal{F}_{T-1} \right]. \end{aligned} \quad (23)$$

To start the *backward induction* one has to examine whether the conditions prescribed by Assumption 4.1, Assumption 4.2, and Assumption 4.3 hold with the choice  $G_0 = \tilde{G}_{T-1}$ . To this end, we note that the function  $\tilde{G}_{T-1}$  is jointly continuous in its real variables, it is non-decreasing in the first real variable almost surely. Using the bound in (6) we define  $\hat{G}_{T-1} : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  as

$$\hat{G}_{T-1}(\omega, x) = E \left[ u \left( x + \frac{P_T^2(\omega) \delta_T(\omega)}{4} \right) \middle| \mathcal{F}_{T-1} \right],$$

and we note that – due to Assumption 2.1 – for all  $x \in \mathbb{R}$  it holds that  $\hat{G}_{T-1}(x) \leq C_u$ , and as  $x \rightarrow -\infty$  the quantity  $\hat{G}_{T-1}(x)$  tends to  $-\infty$  as  $x \rightarrow -\infty$ , in the  $\mathbb{P}$ -almost sure sense:  $u$  does so by assumption and due to boundedness from above the reverse Fatou Lemma is applicable. Moreover, for all  $x, h_1, \dots, h_{T-1}$ , we have

$$\tilde{G}_{T-1}(\omega, x, h_1, \dots, h_{T-1}) \leq \hat{G}_{T-1}(\omega, x)$$

almost surely, leading us to the choice  $\bar{G}_0 = \hat{G}_{T-1}$  (again following notation of Section 4).

We will establish the following claim after the present proof.

**Claim 5.1.** *For any  $m > 0$  there exists an  $\mathcal{F}_{T-1}$ -measurable and integrable  $M = M(m)$  such that for all  $1 \leq t \leq T-1$  and for all  $(x, h_1, \dots, h_{T-1}) \in [-m, m]^T$  we have*

$$M(m) \leq G_{T-1}(x + \kappa_t(h_1, \dots, h_t), h_1, \dots, h_{T-1})$$

*almost surely.*

One can thus conclude that Assumption 4.1, Assumption 4.2, and Assumption 4.3 of Section 4 are satisfied and we are ready to perform the first step of the *backward induction*: the lemmas of Section 4, for the first step, will be utilized with the choice  $t = T - 1$  and  $G_0 = \tilde{G}_{T-1}$ ,  $\bar{G}_0 := \hat{G}_{T-1}$ .

Lemmas 4.4, 4.5, 4.6, 4.7, and 4.8 produce a mapping – denoted by  $G$  in their own context – which, in our notation will take the form  $\tilde{G}_{T-2} : \Omega \times \mathbb{R}^{T-1} \rightarrow \mathbb{R}$  with the following properties. The  $\tilde{G}_{T-2}$  is  $\mathcal{F}_{T-2} \otimes \mathcal{B}(\mathbb{R}^{T-1})$ -measurable, non-decreasing in its first real variable, jointly continuous in all its real variables, and for all  $x, h_1, \dots, h_{T-2} \in \mathbb{R}$  it almost surely satisfies

$$\begin{aligned} & \tilde{G}_{T-2}(x, h_1, \dots, h_{T-2}) \\ &= \text{ess.sup}_{H \in \Xi_{T-2}} E \left[ \tilde{G}_{T-1}(x + \kappa_{T-1}(h_1, \dots, h_{T-2}, H), h_1, \dots, h_{T-2}, H) | \mathcal{F}_{T-2} \right]. \end{aligned} \quad (24)$$

Furthermore, Lemmas 4.9, and 4.10 ensure the existence of a mapping  $H_{T-1}^* : \Omega \times \mathbb{R}^{T-1} \rightarrow \mathbb{R}$  that is  $\Omega \otimes \mathcal{B}(\mathbb{R}^{T-1})$ -measurable, and is such that for all random variables  $X, H_1, \dots, H_{T-2}$  that are measurable with respect to  $\mathcal{F}_{T-2}$ , with the notation  $H_{T-1}^* = H_{T-1}^*(X, H_1, \dots, H_{T-2})$ , we have

$$\begin{aligned} & \tilde{G}_{T-2}(X, H_1, \dots, H_{T-2}) \\ &= E \left[ \tilde{G}_{T-1}(X + \kappa_{T-1}(H_1, \dots, H_{T-2}, H_{T-1}^*), H_1, \dots, H_{T-2}, H_{T-1}^*) | \mathcal{F}_{T-1} \right], \end{aligned} \quad (25)$$

$\mathbb{P}$ -almost surely.

Lemmas 4.11, and 4.12 imply that there exists an action-independent bound  $\hat{G}_{T-2}$ , with properties analogous to the  $\hat{G}_{T-1}$  presented above, and thus finally, we arrive to the conclusion that the quantity  $\tilde{G}_{T-2}$  is such that it again satisfies Assumption 4.1, Assumption 4.2, and Assumption 4.3. That is, in the next step of iteration, the choice  $G_0 = \tilde{G}_{T-2}$ ,  $\bar{G}_0 := \hat{G}_{T-2}$  can be made.

Iterating backwards in this manner goes with the usual mechanics of induction. Take  $\tilde{G}_{T-2}$  as a starting point.

When  $\tilde{G}_{t'}, \dots, \tilde{G}_T$  (and  $H_{t'-1}^*, \dots, H_{T-1}^*$ ) are given for some  $t' \leq T - 2$ , applying the lemmas of Section 4 with the choice  $G_0 = \tilde{G}_{t'}$ ,  $\bar{G}_0 := \hat{G}_{t'}$  and  $t = t'$  yield  $\tilde{G}_{t'-1}$ , and with this procedure we construct the pairs

$$(\tilde{G}_{T-1}^*, \tilde{G}_{T-2}), (\tilde{G}_{T-2}^*, \tilde{G}_{T-3}), \dots, (\tilde{G}_1^*, \tilde{G}_0) \quad (26)$$

with the properties shown below.

For  $t \in \{0, 1, \dots, T - 2\}$ ,  $\tilde{G}_t : \Omega \times \mathbb{R}^{t+1} \rightarrow \mathbb{R}$  is  $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^{t+1})$ -measurable, non-decreasing in its first real variable, jointly continuous in its real variables, and for all  $x, h_1, \dots, h_t \in \mathbb{R}$ , in the almost sure sense we have

$$\begin{aligned} & \tilde{G}_t(x, h_1, \dots, h_t) \\ &= \text{ess.sup}_{H \in \Xi_t} E \left[ \tilde{G}_{t+1}(x + \kappa_{t+1}(h_1, \dots, h_t, H), h_1, \dots, h_t, H) | \mathcal{F}_t \right]. \end{aligned}$$

The mapping  $H_{t+1}^* : \Omega \times \mathbb{R}^t \rightarrow \mathbb{R}$  is  $\Omega \otimes \mathcal{B}(\mathbb{R}^t)$ -measurable and for all  $\mathcal{F}_t$ -measurable random variables  $X, H_1, \dots, H_t$  we have for  $\tilde{H}_{t+1}^* = H_{t+1}^*(X, H_1, \dots, H_t)$  that

$$\begin{aligned} & \tilde{G}_t(X, H_1, \dots, H_t) \\ &= E \left[ \tilde{G}_{t+1}(X + \kappa_{t+1}(H_1, \dots, H_t, \tilde{H}_{t+1}^*), H_1, \dots, H_t, \tilde{H}_{t+1}^*) | \mathcal{F}_t \right], \end{aligned} \quad (27)$$

holds in the  $\mathbb{P}$ -almost sure sense.

Introducing notation, for any admissible trading strategy  $H = (H_t)_{t \in \{1, \dots, T\}}$ , we denote by  $\Gamma_t(H)$  the strategy  $(H_s)_{s \in \{1, \dots, t\}}$ , the same trading strategy as  $H$ , but without liquidation, and corresponding to the trading interval up to  $t$ .

Now we construct the *candidate strategy* using the mapping in (26). Let  $\hat{H}_1 = H_1^*(z)$ . We define the optimal steps using a forward recursion. That is, when the  $\hat{H}_t$  is constructed for some  $T - 1 \geq t \geq 1$ , with accumulated wealth  $\hat{X}_t = \xi_t^{\Gamma_t(\hat{H})}$ , the next action  $\hat{H}_{t+1}$  is given as

$$\hat{H}_{t+1} = H_{t+1}^*(\hat{X}_t, \hat{H}_1, \dots, \hat{H}_t). \quad (28)$$

Then, we set  $\hat{H}_T = -\hat{H}_{T-1} - \dots - \hat{H}_1$ .

We proceed with the *forward iteration*, giving an upper bound for all expected pay-offs which – as we shall see – equals the expected payoff associated with the *candidate strategy*  $\hat{H}$ .

Letting  $H_1, \dots, H_{T-1}, H_T$  denote an arbitrary series of admissible actions. Iterating with the rule in (27) yields

$$\begin{aligned} & E[u(z + \xi_T^H + B)] \\ &= E[E[u(z + \xi_{T-1}^{\Gamma_{T-1}(H)} + \kappa_T(H_1, \dots, H_{T-1}, -\sum_{k=1}^{T-1} H_k) + B)] | \mathcal{F}_{T-1}]] \\ &= E[\tilde{G}_{T-1}(z + \xi_{T-2}^{\Gamma_{T-2}(H)} + \kappa_{T-1}(H_1, \dots, H_{T-1}), H_1, \dots, H_{T-1})] \\ &\leq E[E[\tilde{G}_{T-2}(z + \xi_{T-3}^{\Gamma_{T-3}(H)} + \kappa_{T-2}(H_1, \dots, H_{T-2}), H_1, \dots, H_{T-2}) | \mathcal{F}_{T-2}]] \\ &\vdots \\ &\leq E[\tilde{G}_0(z)]. \end{aligned} \quad (29)$$

Furthermore, as a result of (22), (23), (24), and (25) we have that

$$\begin{aligned} \tilde{G}_0(z) &= E[\tilde{G}_1(z + \kappa_1(\hat{H}_1), \hat{H}_1)] \\ &= E[\tilde{G}_2(z + \kappa_1(\hat{H}_1) + \kappa_2(\hat{H}_1, \hat{H}_2), \hat{H}_1, \hat{H}_2)] \\ &\vdots \\ &= E[\tilde{G}_{T-1}(z + \sum_{i=1}^{T-1} \kappa_i(\hat{H}_1, \dots, \hat{H}_i), \hat{H}_1, \dots, \hat{H}_{T-1})] \\ &= E[u(z + \xi_T^{\hat{H}} + B)]. \end{aligned} \quad (30)$$

Thus, we can conclude, due to (30) and (29), that the candidate strategy given with trading actions  $(\hat{H}_t)_{t=1, \dots, T}$ , defined in (28), is indeed optimal. Theorem 2.4 is now shown.  $\square$

*Proof of Claim 5.1.* For  $h_1, \dots, h_{T-1} \in [-m, m]$  we clearly have

$$\kappa_t(h_1, \dots, h_t) \geq -m\zeta_0 - tm^2/\delta + \min\{mP_t, -mP_t\}.$$

Similarly,

$$\begin{aligned} & \kappa_T \left( h_1, \dots, h_{T-1}, -\sum_{k=1}^{T-1} h_k \right) \\ & \geq -(T-1)m\zeta_0 - (T-1)m^2(2T-2)/\delta + \min\{m(T-1)P_T, -m(T-1)P_T\} \end{aligned}$$

Define the constant

$$D_m := -(2T-2)\zeta_0 - (T-1)m^2(2T-1)/\delta.$$

We can thus see that

$$\tilde{G}_{T-1}(x + \kappa_t(h_1, \dots, h_t), h_1, \dots, h_{T-1}) \geq \mathbb{E}[\min\{J_1, J_2, J_3, J_4\} | \mathcal{F}_{T-1}]$$

where

$$\begin{aligned} J_1 &= u(x + D_m + mP_t + m(T-1)P_T - B), \\ J_2 &= u(x + D_m - mP_t + m(T-1)P_T - B), \\ J_3 &= u(x + D_m + mP_t - m(T-1)P_T - B), \\ J_4 &= u(x + D_m - mP_t - m(T-1)P_T - B). \end{aligned}$$

Now (3) implies our statement.  $\square$

## 6 Conclusion

One could add a solvency constraint (that is,  $z + \xi_t^X \geq 0$  for all  $t$ ) with minimal modifications in the arguments. Setting  $B := 0$ , utility maximization for  $u$  defined on the positive real axis could be treated in this way.

It is unclear whether (and how) results of the present paper could be transferred to continuous-time models. We rely, in an essential way, on the fact that the treated portfolio optimization problem can be broken down into one-step problems. In continuous time such an approach is out of question.

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