

Phase retrieval via overparametrized nonconvex optimization: nonsmooth amplitude loss landscapes

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Abstract

We study nonconvex optimization for phase retrieval and the more general problem of semidefinite low-rank matrix sensing; in particular, we focus on the global nonconvex landscape of overparametrized versions of the nonsmooth amplitude least-squares loss as well as a smooth reformulation of this loss based on the PhaseCut approach. We first give a general, deterministic result on properties of second-order critical points for a general class of loss functions; we then specialize this result to the nonsmooth amplitude loss and, additionally, prove nearly identical results for a smooth reformulation (similar to PhaseCut) as a synchronization problem over spheres. Finally, we show the usefulness of these tools by proving high-probability landscape guarantees in two settings: (1) phase retrieval with isotropic sub-Gaussian measurements, and (2) phase retrieval in a general (possibly infinite-dimensional) Hilbert space with Gaussian measurements. In both cases, our results give state-of-the-art and statistically optimal guarantees with only a constant amount of overparametrization (in the well-studied case of isotropic sub-Gaussian measurements, such statistical guarantees had previously required greater degrees of overparametrization/relaxation); this demonstrates the potential of overparametrized nonconvex optimization as a principled and scalable algorithmic approach to phase retrieval.

1 Introduction

In phase retrieval, we want to estimate a vector x_* from (possibly noisy) measurements of the form $y_i \approx |\langle f_i, x_* \rangle|$ for $i = 1, \dots, n$, where x_* and f_1, \dots, f_n are vectors in \mathbf{F}^d ; \mathbf{F} is the field of either real numbers ($\mathbf{F} = \mathbf{R}$) or complex numbers ($\mathbf{F} = \mathbf{C}$), and $\langle \cdot, \cdot \rangle$ denotes the standard Euclidean inner (dot) product on \mathbf{F}^d . This is a critical problem in, for example, computational imaging, where the linear part $\langle f_i, x_* \rangle$ often represents the (complex) amplitude of an electromagnetic wave at a point after some linear diffraction process, but we can only physically measure the strength (amplitude) of the wave and not its phase.

We can write this formally as the generalized linear model

$$y_i = |\langle f_i, x_* \rangle| + \varepsilon_i, \quad i = 1, \dots, n, \quad \text{or} \quad y = |Fx_*| + \varepsilon \in \mathbf{R}^n, \quad (1)$$

where $\varepsilon_1, \dots, \varepsilon_n \in \mathbf{R}$ represent measurement noise, we denote

$$F := \begin{bmatrix} f_1^* \\ \vdots \\ f_n^* \end{bmatrix} \in \mathbf{F}^{n \times d}, \quad y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}, \quad \text{and} \quad \varepsilon = \begin{bmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_n \end{bmatrix},$$

and the absolute value of a vector is understood elementwise.¹

An equivalent version of this is

$$y_i^2 = |\langle f_i, x_* \rangle|^2 + \xi_i, \quad i = 1, \dots, n, \quad \text{or} \quad y^2 = |FX|^2 + \xi, \quad (2)$$

where the square of a vector is understood elementwise, and ξ again represents noise (note that $\xi_i = 2|\langle f_i, x_* \rangle|\varepsilon_i + \varepsilon_i^2$).

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¹In fact, by the usual convention on the complex inner product, $(Fx_*)_i = \langle x_*, f_i \rangle = \overline{\langle f_i, x_* \rangle}$, but we find the ordering $\langle x_*, f_i \rangle$ unnatural. We will endeavor to be precise in the rare cases where this complex conjugation matters.

1.1 General semidefinite matrix sensing

A convenient analysis framework for phase retrieval is low-rank matrix sensing. This arises from the fact that $|\langle f_i, x_* \rangle|^2 = \langle f_i f_i^*, x_* x_*^* \rangle$, where $\langle \cdot, \cdot \rangle$ applied to matrices is the elementwise Euclidean (or Frobenius, or trace) inner product; this reformulation is sometimes called the “lifting” trick. Our problem is thus equivalent to that of recovering a rank-1 matrix $Z_* = x_* x_*^*$ from (noisy) linear measurements of the form $\langle A_i, Z_* \rangle$, where $A_i = f_i f_i^*$.

As in the paper [1], of which the present work is a continuation, much of our theoretical analysis of phase retrieval generalizes to the problem of estimating a rank- r positive semidefinite (PSD) matrix Z_* from (possibly noisy) measurements $\approx \langle A_i, Z_* \rangle$ for matrices A_1, \dots, A_n that are also PSD. We can write $Z_* = X_* X_*^*$ for some matrix $X_* \in \mathbf{F}^{d \times r}$. This generalization includes the important case of ordinary phase retrieval when the signal x_* is real ($\mathbf{F} = \mathbf{R}$), but the measurement vectors f_i are complex; we can then take $A_i = \text{real}(f_i f_i^*)$.

With this setup, following the notation of [2, 3], we denote, for arbitrary $r' \geq 1$, the maps $\alpha, \beta: \mathbf{F}^{d \times r'} \rightarrow \mathbf{R}^n$ defined by

$$\alpha(X) := \begin{bmatrix} \langle A_1, X X^* \rangle^{1/2} \\ \vdots \\ \langle A_n, X X^* \rangle^{1/2} \end{bmatrix} \quad \text{and} \\ \beta(X) := \begin{bmatrix} \langle A_1, X X^* \rangle \\ \vdots \\ \langle A_n, X X^* \rangle \end{bmatrix} = \alpha^2(X).$$

In the phase retrieval case where $A_i = f_i f_i^*$ for vectors $f_1, \dots, f_n \in \mathbf{F}^d$, we have

$$\alpha(X) = |FX|, \quad \text{and} \quad \beta(X) = |FX|^2,$$

where the absolute value is interpreted as the vector of row norms.

The model (1) thus generalizes to

$$y_i = \langle A_i, X_* X_*^* \rangle^{1/2} + \varepsilon_i, \quad i = 1, \dots, n, \quad \text{or} \quad y = \alpha(X_*) + \varepsilon. \quad (3)$$

Equivalently, we can write, similarly to the quadratic model (2),

$$y_i^2 = \langle A_i, X_* X_*^* \rangle + \xi_i, \quad i = 1, \dots, n, \quad \text{or} \quad y^2 = \beta(X_*) + \xi. \quad (4)$$

1.2 Recovery via optimization

Given phaseless measurements $y_i \approx |\langle f_i, x_* \rangle|$ for $i = 1, \dots, n$, an important question is how to estimate the vector of interest x_* . There are many ways to do this; see Section 2 for some pointers toward the vast literature on this topic. Many approaches use optimization to fit a variable $x \in \mathbf{F}^d$ to the data. For example, one simple method is the following least-squares problem based on the quadratic model (2), is the problem

$$\min_{x \in \mathbf{F}^d} \frac{1}{2n} \sum_{i=1}^n (|\langle f_i, x \rangle|^2 - y_i^2)^2. \quad (5)$$

The generalized version of this, based on (4), is

$$\min_{X \in \mathbf{F}^{d \times p}} \frac{1}{2n} \sum_{i=1}^n (\underbrace{\langle A_i, X X^* \rangle - y_i^2}_{= \|\beta(X) - y^2\|^2})^2, \quad (6)$$

where p represents the (maximum) rank of the matrices over which we are searching (typically, we would set $p = r$ if this is known). This approach is widely used and studied in large part because it is smooth (the objective is a quartic polynomial) and because it arises naturally from the low-rank matrix sensing framework described in Section 1.1.

Another least-squares approach, which will be the main focus of this paper, is based on the model (1):

$$\min_{x \in \mathbf{F}^d} \frac{1}{2n} \sum_{i=1}^n (|\langle f_i, x \rangle| - y_i)^2. \quad (7)$$

This is often known as an “amplitude” formulation, as, in imaging problems (as discussed above), $|\langle f_i, x_* \rangle|$ is often the amplitude of an electromagnetic wave at a point; in contrast, the problem (5) is an “intensity” formulation, as the energy in the wave is proportional to $|\langle f_i, x \rangle|^2$.

Although the objective function in (7) grows at most quadratically (similarly to linear least squares), it is *nonsmooth* due to the (unsquared) absolute value. Generalized to low-rank semidefinite matrix sensing, this becomes

$$\min_{X \in \mathbf{F}^{d \times p}} \frac{1}{2n} \sum_{i=1}^n \underbrace{(\langle A_i, X X^* \rangle^{1/2} - y_i)^2}_{= \|\alpha(X) - y\|^2}. \quad (8)$$

There are several reasons to prefer this nonsmooth loss function over the quartic one of (5) and (6):

- This loss function empirically works better in many situations. See, for example, [4, 5, 6] for some numerical evidence and further discussion.
- This loss function is a good (local) approximation to the Poisson (negative) log-likelihood function, which is a natural choice in many imaging problems; see, for example, [7]. It furthermore is finite everywhere, unlike the Poisson log-likelihood function.
- For obtaining theoretical guarantees for ordinary phase retrieval, the fact that the objective function in (5) and (6) is quartic in the measurement vectors $\{f_i\}_i$ makes the landscape analysis quite sensitive to large measurements. For example, even with Gaussian measurements (that is, the vectors f_1, \dots, f_n are independent standard Gaussian vectors), heavy tails lead to considerable difficulty in the analysis and result in some logarithmic factors in the results we can obtain (the paper [8] suggests this difficulty is inevitable for this loss function, at least without using completely different analysis tools). Our statistical guarantees for (8) in Section 5 will indeed avoid this issue.
- The nonsmooth problems (7) and (8) are amenable to reformulation as smooth *synchronization* problems over the unknown measurement phases/directions, as in the PhaseCut approach of [9] (a special case of this approach also appeared in [10]). Indeed, we explore this in Section 4 of the present paper.

Regardless of which formulation as an optimization problem we choose, each has the issue of being *nonconvex* and thus possibly having spurious local optima; thus the question remains of how well we can solve these problems in practice.

1.3 Summary of main contributions

In this paper, we are interested in theoretical guarantees for practical algorithms; can we prove that, under reasonable statistical assumptions on the measurements, there is a computationally practical algorithm that will (e.g., with high probability) return a good estimate of the ground truth? Most existing theoretical results fall into one of the following categories (see Section 2 for more details):

- Many study direct optimization approaches to nonconvex problems similar to those above. The results that give statistically optimal results in general only apply to the ordinary phase retrieval problem and make quite idealistic assumptions like Gaussian measurements. The results applying to more general problems and measurements are typically statistically suboptimal in terms of how large they require n , the number of measurements, to be.
- Most state-of-the-art statistical guarantees, outside of ordinary phase retrieval and Gaussian measurements, are for semidefinite relaxations (replacing the matrix XX^* by an arbitrary PSD matrix in (6) or a similar problem, thus making the problem convex). However, this greatly increases the computational complexity compared to direct nonconvex approaches, as this requires optimization over $\approx d^2$ variables.

The recent paper [1], of which the present work is a continuation, attempted to address these issues via nonconvex partial relaxations of the quartic problem (6): we relax (or overparametrize) the problem by setting the optimization rank p to be larger than the ground truth rank r (in ordinary phase retrieval, $r = 1$). That paper showed that, in a variety of settings, this rank overparametrization allows the nonconvex problem to have a benign landscape, that is, every local optimum is global (or at least is a good estimator in a statistical sense). Since many local optimization algorithms can find such

local optima (or, more technically, second-order critical points; see Section 3 for further discussion), this gives theoretical guarantees for computationally practical algorithms. When p is a logarithmic multiple of the true rank r , we can obtain statistically optimal results that had previously only been shown for semidefinite programming approaches; this can greatly decrease the computational burden for theoretically principled algorithms when the dimension d is large.

This present work seeks to expand on [1] by considering alternative loss functions to (6), primarily the nonsmooth loss in (8). As discussed above in Section 1.2, there are many practical and principled reasons to prefer this formulation. In Section 3, we extend and refine the deterministic landscape analysis techniques of [1] to more general loss functions (Lemma 1) and then apply this to several commonly-used loss functions to obtain Theorems 1 to 3 (Theorem 1, which applies to the quartic problem (6), is a refinement of [1, Lem. 1]). These results are deterministic (in terms of the problem data $(A_i, y_i)_i$) and are stepping stones to further statistical guarantees.

We then focus on the nonsmooth loss function from (8), for which Theorem 2 is our deterministic result. In Section 4, we explore an alternative smooth formulation of (8) as a synchronization problem over spheres based on the PhaseCut approach of [9]. We show (Theorem 4) deterministic landscape guarantees nearly identical to those of Theorem 2, which was for the original nonsmooth problem (8).

In Section 5, we show the usefulness of these deterministic landscape results in proving high-probability statistical guarantees for the ordinary phase retrieval problem. In particular, in Section 5.1, we give a result, Theorem 5 (parallel to [1, Thm. 3]), for phase retrieval with general sub-Gaussian measurements; only a constant (rather than logarithmic as in [1]) level of overparametrization is necessary to ensure statistically optimal recovery (in terms of sample complexity). This confirms the benefits of choosing the nonsmooth loss function in (8).

Finally, we show that, with some additional regularization, we can obtain theoretical guarantees for infinite-dimensional phase retrieval; our Theorem 6 in Section 5.2 gives statistically optimal (with respect to sample complexity and bias) recovery results for phase retrieval with Gaussian measurements in a general Hilbert space. Although currently limited to Gaussian measurements, this result could be a step toward guarantees for nonparametric phase retrieval in, for example, a reproducing kernel Hilbert space framework.

1.4 Notation

For reference, we compile here some notation that will be used throughout the paper. Certain other more specialized notation will be defined later as needed.

For an integer $m \geq 1$, we denote by I_m the $m \times m$ identity matrix. For matrices A, B of equal size (including ordinary vectors), we denote by $\langle A, B \rangle = \text{tr}(B^* A)$ their elementwise Euclidean (Frobenius) inner product. For a matrix A , we denote its Frobenius (elementwise ℓ_2), operator, and nuclear norms respectively by $\|A\|$, $\|A\|_{\text{op}}$, and $\|A\|_*$.

For a vector $v \in \mathbf{F}^m$ (with $\mathbf{F} = \mathbf{R}$ or $\mathbf{F} = \mathbf{C}$), we denote its Euclidean (ℓ_2), ℓ_∞ , and ℓ_1 norms respectively by $\|v\|$, $\|v\|_\infty$, and $\|v\|_1$. We will also write $v^2 \in \mathbf{F}^m$ to mean the elementwise square of v . Its elementwise absolute value is denoted by $|v| \in \mathbf{R}^m$. We will occasionally overload this notation to apply to a matrix $V \in \mathbf{F}^{m \times p}$: $|V| \in \mathbf{R}^m$ is the vector whose elements are the (ℓ_2) norms of the rows of V (which are vectors in \mathbf{F}^p).

We overload the notation diag as follows: for a vector $v \in \mathbf{F}^m$, $\text{diag}(v) \in \mathbf{F}^{m \times m}$ is the diagonal matrix whose diagonal elements are the elements of v . For $A \in \mathbf{F}^{m \times m}$, $\text{diag}(A) \in \mathbf{F}^m$ is the vector of the diagonal elements of A .

We write the real and imaginary parts of $a \in \mathbf{C}$ as $\text{real}(a)$ and $\text{imag}(a)$ respectively, and we write its complex conjugate as \bar{a} . We also overload this notation to apply elementwise if a is a vector or matrix.

We will often informally write $a \lesssim b$ to mean that $a \leq Cb$ for some unspecified constant $C > 0$ that does not depend on problem parameters such as the dimension or number of measurements. $a \gtrsim b$ means $b \lesssim a$. $a \approx b$ means $a \lesssim b$ and $a \gtrsim b$ simultaneously.

2 Related work

In this section, we review the most relevant literature on phase retrieval and the various optimization approaches to these problems. For further reading on the importance (particularly in imaging) of the phase retrieval problem and an overview of the many algorithms and theoretical analyses developed for it, see the surveys [11, 12, 13].

There are relatively few results on global nonconvex optimization landscapes for phase retrieval. For the quartic problem (5), the first global landscape guarantee was given in [14], which was then improved in [15]; this result (also recovered in [1]) ensures a globally benign landscape with $n \gtrsim d \log d$ Gaussian measurements. It is unlikely that we can do any better for this quartic loss function without fundamentally different arguments; the paper [8] shows that, in the regime $n \lesssim d \log d$, the landscape becomes more delicate, so standard analysis methods fail (nevertheless, the paper [16] provides numerical evidence and heuristic arguments that, indeed, the landscape is benign whenever $n \gtrsim d$). To avoid this logarithmic factor (which is statistically suboptimal), a different approach is needed. In [1], we showed that overparametrization (by at most a logarithmic factor, i.e., $p \approx \log d$) of the quartic formulation gets around this statistical suboptimality. Another option is to consider a different loss function, which we do in this paper.

To our knowledge, there are no existing works which study the global landscape of nonsmooth problems of the form (7) or (8). The paper [17] studies a different nonsmooth formulation, but its results only characterize (first-order) critical points rather than local optima or second-order critical points. To get around the issue of nonsmoothness, a recent series of papers [18, 19, 20, 21] has studied the landscape of *smoothed* versions of (7). These works indeed show that, with these alternative loss functions, one obtains a benign landscape with $n \gtrsim d$ Gaussian measurements. Interestingly, this applies to the non-overparametrized case, whereas our positive landscape results for the non-smoothed version do require some overparametrization ($p > 1$). It is unclear if this is a fundamental drawback of the unsmoothed loss function or merely of our analysis.

There has been far more work on understanding how well we can solve problems like (7) by a two-stage algorithm consisting of an initialization (e.g., by spectral methods) and then refinement by local optimization (e.g., by some form of gradient descent) [22, 23, 5, 24, 25, 26]. Some of these works ([22, 26]) consider smoothed versions of (7) similar to those mentioned in the previous paragraph. In the case of Gaussian measurements, these papers show that $n \gtrsim d$ measurements suffice. Furthermore, the recent paper [23] provides guarantees for a physically-inspired coded diffraction pattern measurement model with optimal sample complexity.

Two-stage algorithms and analyses for the quartic formulation in (5) have also been well-studied; see, for example, [27, 28, 29, 30] for state-of-the-art results and further reading. The Poisson maximum likelihood loss function which we briefly consider in Section 3.3 is also a popular practical choice (again, see, for example, [4, 7]), but the loss function is difficult to analyze theoretically, so guarantees are few; a recent work in this direction is [31].

The sub-Gaussian measurement model described in Section 5.1 has seen much recent work, as it is one of the most general models for which statistically (near-)optimal recovery guarantees have been proved. Several algorithms and theoretical analyses under variants of the assumptions of our Theorem 5 have been developed in the papers [32, 33, 34, 29, 28]. In particular, the paper [34] shows, under conditions nearly identical to our Theorem 5, that a semidefinite programming approach can recover the ground truth with optimal sample complexity (i.e., requiring only $n \gtrsim d$). The best results we are aware of for nonconvex approaches (prior to [1]) are those of [29, 28], which show that, under similar sub-Gaussian measurement assumptions, a two-stage initialization + gradient descent algorithm can recover the ground truth if $n \gtrsim d \log^2 d$.

Overparametrization and benign landscapes have also been extensively studied in synchronization problems, of which our PhaseCut-based formulation in Section 4 is an example. These problems have much broader application in areas such as robotics, dynamical systems, signal processing, and graph clustering. For an introduction and further references, see, for example, [35, 36, 37, 38]. The basic geometric tools we use in Section 4 are taken directly from this literature. However, our problem instances have considerably different properties than those studied previously, so new methods are also needed.

These themes of benign landscapes and overparametrization have also been important in low-rank matrix sensing, of which the semidefinite matrix sensing problem described in Section 1.1 is an instance. This literature primarily considers quartic formulations like (6). However, most existing results assume a type of *restricted isometry property* which does not, in general, hold for the measurements that we consider in this paper (getting around this difficulty was a major part of the contribution of [1]). See, for example, [39, 40], for state-of-the-art results and further references.

3 Direct optimization: deterministic results

We consider nonconvex optimization problems of the form

$$\min_{X \in \mathbf{F}^{d \times p}} L_\lambda(XX^*), \quad \text{where} \quad L_\lambda(Z) := \frac{1}{n} \sum_{i=1}^n \ell(\langle A_i, Z \rangle, y_i) + \lambda \operatorname{tr} Z, \quad (9)$$

$\ell(b, v)$ is a loss function² that we will specify, and $\lambda \geq 0$ is a regularization parameter.

A semidefinite program (SDP) version of (9) is

$$\min_{Z \succeq 0} L_\lambda(Z). \quad (10)$$

If $\ell(b, v)$ is convex in its first argument, this is a convex SDP.

In this paper, we will assume $\ell(b, v)$ is convex and twice differentiable in b , with the first and second derivatives in b denoted by $\ell'(b, v)$ and $\ell''(b, v)$ respectively, for all $v \geq 0$ and $b > 0$. To allow the possibility of nonsmoothness at $b = 0$, we will overload notation by setting

$$\ell'(0, v) = \lim_{b \downarrow 0} \ell'(b, v),$$

which could be $-\infty$.

The gradient³ of $L_\lambda(Z)$ with respect to the $d \times d$ matrix variable Z is (when it is well-defined)

$$\nabla L_\lambda(Z) = \frac{1}{n} \sum_{i=1}^n \ell'(\langle A_i, Z \rangle, y_i) A_i + \lambda I_d. \quad (11)$$

In this paper, we are interested in understanding *second-order critical points* of nonconvex problems like (9). For a twice-differentiable objective, these are points where the gradient with respect to X is zero and the Hessian quadratic form is positive semidefinite. However, as our objective is potentially nonsmooth, we need a slightly generalized notion of second-order criticality. We say that for a function $f: \mathbf{F}^D \rightarrow \mathbf{R}$, $x \in \mathbf{F}^D$ is a second-order critical point if

$$\limsup_{x' \rightarrow x} \frac{f(x') - f(x)}{\|x' - x\|^2} \geq 0.$$

In other words, there is no direction starting from x along which f decreases at least quadratically. In the smooth case, there are theoretical results guaranteeing that local search methods such as gradient descent or trust-region algorithms will converge to a second-order critical point; see, for example, [41, 42]. Although finding local minima of nonsmooth problems is NP-hard in general [43], the specific problems described in this section have some nice properties (see, e.g., the proof of Lemma 2 below), so we think it likely that algorithmic guarantees could be proved. However, this is beyond the scope of the present paper.

In practice, to avoid questions of nonsmoothness and to ensure numerical stability of gradient-based algorithms, we can replace $\ell(b, v)$ by

$$\ell_\delta(b, v) := \ell(b + \delta, \sqrt{v^2 + \delta})$$

for some smoothing parameter $\delta \geq 0$. If $\delta > 0$, our assumptions on ℓ ensure that $\ell_\delta(b, v)$ is twice differentiable in b on $[0, \infty)$ for all $v \geq 0$. We could explicitly adapt our theoretical results to account for general $\delta \geq 0$, but for brevity and simplicity we omit this; we would typically choose a very small $\delta > 0$, which will have little effect on the results.

The main result of this section is the following lemma:

Lemma 1. *Let $A_1, \dots, A_n \succeq 0$ and $y_1, \dots, y_n \geq 0$, and furthermore assume that $A_i \neq 0$ for all $i = 1, \dots, n$. Then every second-order critical point X of (9) satisfies the following:*

- *For all i , $\ell'(\langle A_i, XX^* \rangle, y_i)$ is finite, so $\nabla L_\lambda(XX^*)$ as given by (11) is well-defined.*

²We write the first argument as $\langle A_i, Z \rangle$ rather than $\langle A_i, Z \rangle^{1/2}$ to make derivatives cleaner.

³In the complex case, we can compactly define the gradient of a real-valued function $f: \mathbf{C}^D \rightarrow \mathbf{R}$ at $x \in \mathbf{C}^D$ as the vector $v \in \mathbf{C}^D$ such that $x' \mapsto f(x) + \operatorname{real}(\langle v, x' - x \rangle)$ is the best local linear approximation of f around x .

- $\nabla L_\lambda(XX^*)X = 0$.
- For all $Z' \succeq 0$,

$$0 \leq \langle \nabla L_\lambda(XX^*), Z' \rangle + \frac{2}{c_{\mathbf{F}}pn} \sum_{\substack{i=1 \\ \langle A_i, XX^* \rangle > 0}}^n \ell''(\langle A_i, XX^* \rangle, y_i) \langle A_i, XX^* \rangle \langle A_i, Z' \rangle,$$

where $c_{\mathbf{R}} = 1$, and $c_{\mathbf{C}} = 2$.

We defer the proof to Section 3.4. The assumption that each A_i is nonzero is simply to avoid additional edge cases; we can clearly discard any zero measurement matrices because the corresponding terms in (9) will not depend on X .

We can compare this result to what we would expect for the convex problem (10). Optimality of $Z \succeq 0$ for (10) is equivalent to $\nabla L_\lambda(Z)Z = 0$ and $\nabla L_\lambda(Z) \succeq 0$. The criticality property $\nabla L_\lambda(XX^*)X = 0$ clearly implies $\nabla L_\lambda(Z)Z = 0$ for $Z = XX^*$. However, we do not quite obtain the second condition $\nabla L_\lambda(Z) \succeq 0$; note that this would be equivalent to $\langle \nabla L_\lambda(XX^*), Z' \rangle \geq 0$ for all $Z' \succeq 0$. The additional nonnegative terms involving ℓ'' in the inequality of Lemma 1 are thus the price we pay for the nonconvexity of (9). Increasing the optimization rank p reduces this penalty (although, as discussed in [1], this does not immediately imply that this is a good thing to do).

For this result to be useful, we must consider specific examples of loss functions ℓ , which we do in the following subsections.

3.1 Example: quartic loss

We first consider the simple example of the smooth least-squares loss

$$\ell(b, v) = \frac{1}{2}(b - v^2)^2.$$

The problem (9) then becomes (noting that $\text{tr}(XX^*) = \|X\|^2$)

$$\min_{X \in \mathbf{F}^{d \times p}} \frac{1}{2n} \|\beta(X) - y^2\|^2 + \lambda \|X\|^2. \quad (12)$$

Note that this is simply a regularized version of (6); it is quartic because $\beta(X)$ is quadratic in X . In this case, our Lemma 1 recovers the recent result of [1]:

Theorem 1. *Under the model (4), every second-order critical point X of (12) satisfies, for any matrix $R \in \mathbf{F}^{p \times r}$,*

$$\begin{aligned} \frac{1}{n} \|\beta(X) - \beta(X_*)\|^2 &\leq \frac{1}{n} \langle \xi, \beta(X) - \beta(X_*) \rangle + \lambda (\|X_*\|^2 - \|X\|^2) \\ &\quad + \frac{2}{c_{\mathbf{F}}p + 2} \left(\frac{1}{n} \langle y^2, \beta(X_* - XR) \rangle - \lambda \|X_* - XR\|^2 \right). \end{aligned}$$

This is a slight generalization of [1, Lem. 1], which only considered the unregularized case $\lambda = 0$. In addition, in the complex case where $c_{\mathbf{C}} = 2$, we have improved the constant. As the properties of this loss function are thoroughly studied in that paper, we do not develop them further in the present paper.

Proof of Theorem 1. The derivatives of ℓ with respect to the first argument are

$$\begin{aligned} \ell'(b, v) &= b - v^2, \\ \ell''(b, v) &= 1. \end{aligned}$$

The identity $b\ell''(b, v) = \ell'(b, v) + v^2$ implies, for any $Z' \succeq 0$ and for each i ,

$$\ell''(\langle A_i, XX^* \rangle, y_i) \langle A_i, XX^* \rangle \langle A_i, Z' \rangle = \ell'(\langle A_i, XX^* \rangle, y_i) \langle A_i, Z' \rangle + y_i^2 \langle A_i, Z' \rangle.$$

As $\ell''(b, v)$ exists for all b and v , there is no need to separate into cases based on whether $\langle A_i, XX^* \rangle = 0$. We then obtain

$$\begin{aligned} & \frac{1}{n} \sum_{i=1}^n \ell''(\langle A_i, XX^* \rangle, y_i) \langle A_i, XX^* \rangle \langle A_i, Z' \rangle \\ &= \frac{1}{n} \sum_{i=1}^n (\ell'(\langle A_i, XX^* \rangle, y_i) \langle A_i, Z' \rangle + y_i^2 \langle A_i, Z' \rangle) \\ &= \langle \nabla L_\lambda(XX^*), Z' \rangle - \lambda \operatorname{tr}(Z') + \frac{1}{n} \left\langle \sum_{i=1}^n y_i^2 A_i, Z' \right\rangle. \end{aligned}$$

From Lemma 1 and some algebra, we then obtain

$$0 \leq \langle \nabla L_\lambda(XX^*), Z' \rangle + \frac{2}{c_{\mathbf{F}}p + 2} \left(\frac{1}{n} \left\langle \sum_{i=1}^n y_i^2 A_i, Z' \right\rangle - \lambda \operatorname{tr}(Z') \right).$$

Finally, we take $Z' = (X_* - XR)(X_* - XR)^*$, in which case

$$\left\langle \sum_{i=1}^n y_i^2 A_i, Z' \right\rangle = \langle y^2, \beta(X_* - XR) \rangle.$$

By the fact (also from Lemma 1) that $\nabla L_\lambda(XX^*)X = 0$, we have

$$\begin{aligned} \langle \nabla L_\lambda(XX^*), Z' \rangle &= \langle \nabla L_\lambda(XX^*), X_*X_*^* - XX^* \rangle \\ &= \frac{1}{n} \sum_{i=1}^n (\langle A_i, XX^* \rangle - y_i^2) \langle A_i, X_*X_*^* - XX^* \rangle + \lambda \operatorname{tr}(X_*X_*^* - XX^*) \\ &= -\frac{1}{n} \sum_{i=1}^n \langle A_i, XX^* - X_*X_*^* \rangle^2 + \frac{1}{n} \sum_{i=1}^n \xi_i \langle A_i, XX^* - X_*X_*^* \rangle + \lambda(\|X_*\|^2 - \|X\|^2) \\ &= -\frac{1}{n} \|\beta(X) - \beta(X_*)\|^2 + \frac{1}{n} \langle \xi, \beta(X) - \beta(X_*) \rangle + \lambda(\|X_*\|^2 - \|X\|^2). \end{aligned}$$

The result easily follows. \square

3.2 Example: nonsmooth “amplitude” loss

The main choice of loss function we study in this paper is

$$\ell(b, v) := (\sqrt{b} - v)^2,$$

which is nonsmooth at $b = 0$ whenever $v > 0$. The problem (9) becomes

$$\min_{X \in \mathbf{F}^{d \times p}} \frac{1}{n} \|\alpha(X) - y\|^2 + \lambda \|X\|^2, \quad (13)$$

which is a regularized version of (8). In this case, Lemma 1 implies the following:

Theorem 2. *In the case $\mathbf{F} = \mathbf{R}$, assume $p \geq 2$. Under the model (3), every second-order critical point X of (13) satisfies, for any $R \in \mathbf{F}^{p \times r}$,*

$$\begin{aligned} \frac{1}{n} \|\alpha(X) - \alpha(X_*)\|^2 &\leq \frac{2}{n} \langle \varepsilon, \alpha(X) - \alpha(X_*) \rangle + \lambda(\|X_*\|^2 - \|X\|^2) \\ &\quad + \frac{1}{c_{\mathbf{F}}p - 1} \left(\frac{1}{n} \|\alpha(X_* - XR)\|^2 + \lambda \|X_* - XR\|^2 \right). \end{aligned}$$

We further develop the consequences of this result in Section 5. If we can find a global optimum (for example, by setting $p = n$ and using convex semidefinite programming), that optimum will satisfy

$$\frac{1}{n} \|\alpha(X) - \alpha(X_*)\|^2 \leq \frac{2}{n} \langle \varepsilon, \alpha(X) - \alpha(X_*) \rangle + \lambda(\|X_*\|^2 - \|X\|^2).$$

The additional term in the bound of Theorem 2 is, similarly to the general result Lemma 1, the price we pay for only finding a second-order critical point.

Proof of Theorem 2. We have, for $\ell(b, v) = (\sqrt{b} - v)^2$,

$$\begin{aligned}\ell'(b, v) &= 1 - \frac{v}{\sqrt{b}}, \quad \text{and} \\ \ell''(b, v) &= \frac{v}{2b^{3/2}}\end{aligned}$$

for $b > 0$. The identity $2b\ell''(b, v) = 1 - \ell'(b, v)$ implies, for any $Z' \succeq 0$ and for each i such that $\langle A_i, XX^* \rangle > 0$,

$$\ell''(\langle A_i, XX^* \rangle, y_i) \langle A_i, XX^* \rangle \langle A_i, Z' \rangle = \frac{1}{2} [1 - \ell'(\langle A_i, XX^* \rangle, y_i)] \langle A_i, Z' \rangle.$$

Lemma 1 ensures that, for all i , $\ell'(\langle A_i, XX^* \rangle, y_i)$ is finite. Furthermore, note that $1 - \ell'(\langle A_i, XX^* \rangle, y_i) \geq 0$. We then have

$$\begin{aligned}\frac{1}{n} \sum_{\substack{i=1 \\ \langle A_i, XX^* \rangle > 0}}^n \ell''(\langle A_i, XX^* \rangle, y_i) \langle A_i, XX^* \rangle \langle A_i, Z' \rangle \\ \leq \frac{1}{2} \left(\frac{1}{n} \left\langle \sum_{i=1}^n A_i, Z' \right\rangle - \frac{1}{n} \sum_{i=1}^n \ell'(\langle A_i, XX^* \rangle, y_i) \langle A_i, Z' \rangle \right) \\ = \frac{1}{2} \left(\frac{1}{n} \left\langle \sum_{i=1}^n A_i, Z' \right\rangle - \langle \nabla L_\lambda(XX^*), Z' \rangle + \lambda \operatorname{tr} Z' \right).\end{aligned}$$

The inequality of Lemma 1 (multiplying by $c_{\mathbf{F}p}$) then implies

$$0 \leq (c_{\mathbf{F}p} - 1) \langle \nabla L_\lambda(XX^*), Z' \rangle + \left\langle \sum_{i=1}^n A_i, Z' \right\rangle + \lambda \operatorname{tr} Z'.$$

Finally, similarly to the proof of Theorem 1, we take $Z' = (X_* - XR)(X_* - XR)^*$, in which case

$$\left\langle \sum_{i=1}^n A_i, Z' \right\rangle = \|\alpha(X_* - XR)\|^2,$$

and we use $\nabla L_\lambda(XX^*)X = 0$ to obtain

$$\begin{aligned}\langle \nabla L_\lambda(XX^*), Z' \rangle &= \langle \nabla L_\lambda(XX^*), X_*X_*^* - XX^* \rangle \\ &= \frac{1}{n} \sum_{i=1}^n \ell'(\langle A_i, XX^* \rangle, y_i) \langle A_i, X_*X_*^* - XX^* \rangle + \lambda(\|X_*\|^2 - \|X\|^2).\end{aligned}$$

For i such that $\langle A_i, XX^* \rangle > 0$, we have, as $y_i \geq 0$,

$$\begin{aligned}\ell'(\langle A_i, XX^* \rangle, y_i) \langle A_i, X_*X_*^* - XX^* \rangle \\ = \left(1 - \frac{y_i}{\langle A_i, XX^* \rangle^{1/2}} \right) \langle A_i, X_*X_*^* - XX^* \rangle \\ \leq \left(1 - \frac{2y_i}{\langle A_i, XX^* \rangle^{1/2} + \langle A_i, X_*X_*^* \rangle^{1/2}} \right) \langle A_i, X_*X_*^* - XX^* \rangle \\ = (\langle A_i, XX^* \rangle^{1/2} - \langle A_i, X_*X_*^* \rangle^{1/2} - 2\varepsilon_i) (\langle A_i, X_*X_*^* \rangle^{1/2} - \langle A_i, XX^* \rangle^{1/2}) \\ = -(\langle A_i, XX^* \rangle^{1/2} - \langle A_i, X_*X_*^* \rangle^{1/2})^2 + 2\varepsilon_i (\langle A_i, XX^* \rangle^{1/2} - \langle A_i, X_*X_*^* \rangle^{1/2}).\end{aligned}$$

For i such that $\langle A_i, XX^* \rangle = 0$, we must also have $y_i = 0$ for $\ell'(\langle A_i, XX^* \rangle, y_i)$ to be finite (in which case its value is 1). In this case, the first and last expressions in the above inequality are both equal to $\langle A_i, X_*X_*^* \rangle$. We thus obtain

$$\langle \nabla L_\lambda(XX^*), Z' \rangle \leq -\frac{1}{n} \|\alpha(X) - \alpha(X_*)\|^2 + \frac{2}{n} \langle \varepsilon, \alpha(X) - \alpha(X_*) \rangle + \lambda(\|X_*\|^2 - \|X\|^2).$$

Some rearrangement completes the proof. \square

3.3 Example: Poisson loss

In this section, we consider the loss function

$$\ell(b, v) := b - v^2 \log b,$$

with which the general nonconvex problem (9) becomes

$$\min_{X \in \mathbf{F}^{d \times p}} \frac{1}{n} \sum_{i=1}^n (\langle A_i, XX^* \rangle - y_i^2 \log \langle A_i, XX^* \rangle) + \lambda \|X\|^2. \quad (14)$$

Under a Poisson noise model for (4), that is, $y_i^2 \sim \text{Poisson}(\langle A_i, Z_* \rangle)$, the problem (14) is a regularized, nonconvex, and possibly overparametrized maximum likelihood problem. For this loss, we have the following landscape result:

Theorem 3. *In the case $\mathbf{F} = \mathbf{R}$, assume $p \geq 3$. In the case $\mathbf{F} = \mathbf{C}$, assume $p \geq 2$. Under the model (4), every second-order critical point X of (14) satisfies, for any $R \in \mathbf{F}^{p \times r}$,*

$$\begin{aligned} & \frac{1}{n} \sum_{i=1}^n \left(1 - \frac{y_i^2}{\langle A_i, XX^* \rangle} \right) \langle A_i, XX^* - X_* X_*^* \rangle \\ & \leq \lambda (\|X_*\|^2 - \|X\|^2) + \frac{2}{c_{\mathbf{F}} p - 2} \left(\frac{1}{n} \|\alpha(X_* - XR)\|^2 + \lambda \|X_* - XR\|^2 \right). \end{aligned}$$

This bound is less clean and developed than Theorem 1 or Theorem 2. There are many interesting directions we could take it (for example, the terms in the left-hand side sum resemble or can be lower bounded by various measures of distance between Poisson distributions), but we leave that to future work.

Proof of Theorem 3. For this choice of loss function ℓ , for $b > 0$,

$$\ell'(b, v) = 1 - \frac{v^2}{b}, \quad \text{and} \quad \ell''(w, v) = \frac{v^2}{b^2}.$$

We now have the identity $b\ell''(b, v) = 1 - \ell'(b, v)$. Similar arguments to those in the proof of Theorem 2 (again based on Lemma 1) then give, for any $Z' \succeq 0$.

$$0 \leq (c_{\mathbf{F}} p - 2) \langle \nabla L_\lambda(XX^*), Z' \rangle + \frac{2}{n} \left\langle \sum_{i=1}^n A_i, Z' \right\rangle + 2\lambda \text{tr } Z'.$$

Similarly to the proof of Theorem 2, we choose $Z' = (X_* - XR)(X_* - XR)^*$, and we have, again using the fact that $\nabla L_\lambda(XX^*)X = 0$ and handling the case $\langle A_i, XX^* \rangle = 0$ similarly,

$$\begin{aligned} \langle \nabla L_\lambda(XX^*), Z' \rangle &= \langle \nabla L_\lambda(XX^*), X_* X_*^* - XX^* \rangle \\ &= \frac{1}{n} \sum_{i=1}^n \left(1 - \frac{y_i^2}{\langle A_i, XX^* \rangle} \right) \langle A_i, X_* X_*^* - XX^* \rangle + \lambda (\|X_*\|^2 - \|X\|^2). \end{aligned}$$

The result again follows by some rearrangement. \square

3.4 Proof of general landscape result

In this section, we provide a proof of the general result Lemma 1. First, we need the following characterization of second-order critical points, which is a generalization of standard gradient and Hessian calculations for factored matrix optimization (see, for example, [44, Sec. 3.3]).

Lemma 2. *Under the conditions of Lemma 1, for every second-order critical point X of (9), $\ell'(\langle A_i, XX^* \rangle, y_i)$ is finite for all i . Thus $\nabla L_\lambda(XX^*)$ is well-defined. Furthermore,*

$$\nabla L_\lambda(XX^*)X = 0, \quad (15)$$

and, for all $\dot{X} \in \mathbf{F}^{d \times p}$,

$$0 \leq \langle \nabla L_\lambda(XX^*), \dot{X} \dot{X}^* \rangle + \frac{1}{2n} \sum_{\substack{i=1 \\ \langle A_i, XX^* \rangle > 0}}^n \ell''(\langle A_i, XX^* \rangle) \langle A_i, X \dot{X}^* + \dot{X} X^* \rangle^2. \quad (16)$$

Proof. In this proof, we will use standard big-O and little-o notation: for functions $f(t) \geq 0$ and $g(t)$, we write $g(t) = O(f(t))$ to mean that $|g(t)| \leq C f(t)$ for some constant $C > 0$ and all sufficiently small $t \geq 0$, and we write $g(t) = o(f(t))$ to mean that $\lim_{t \downarrow 0} \frac{|g(t)|}{f(t)} = 0$.

To alleviate notation, denote $\ell'_i := \ell'(\langle A_i, XX^* \rangle, y_i)$ for all i , and $\ell''_i := \ell''(\langle A_i, XX^* \rangle, y_i)$ for i such that $\langle A_i, XX^* \rangle > 0$. We furthermore denote

$$I := \{i : \langle A_i, XX^* \rangle = 0\}, \quad \text{and} \\ J := \{i : \ell'_i = -\infty\}.$$

Note that, by our assumptions on ℓ , we have $J \subseteq I$.

Furthermore, note that, as $A_i \succeq 0$,

$$\langle A_i, XX^* \rangle = 0 \iff A_i X = 0.$$

For all $i \notin I$, $\ell(b, y_i)$ is twice differentiable at $b = \langle A_i, XX^* \rangle > 0$. Using a Taylor expansion on these terms, for any unit-norm $\dot{X} \in \mathbf{F}^{d \times p}$ and $t \geq 0$, we have

$$\begin{aligned} & L_\lambda((X + t\dot{X})(X + t\dot{X})^*) - L_\lambda(XX^*) \\ &= L_\lambda(XX^* + t(X\dot{X}^* + \dot{X}X^*) + t^2\dot{X}\dot{X}^*) - L_\lambda(XX^*) \\ &= \frac{1}{n} \sum_{i=1}^n (\ell(\langle A_i, XX^* + t(X\dot{X}^* + \dot{X}X^*) + t^2\dot{X}\dot{X}^* \rangle, y_i) - \ell(\langle A_i, XX^* \rangle)) \\ &\quad + \lambda \operatorname{tr}(t(X\dot{X}^* + \dot{X}X^*) + t^2\dot{X}\dot{X}^*) \\ &= \frac{1}{n} \sum_{i \in I} [\ell(t^2\langle A_i, \dot{X}\dot{X}^* \rangle, y_i) - \ell(0, y_i)] \\ &\quad + t \left\langle \frac{1}{n} \sum_{i \notin I} \ell'_i A_i + \lambda I_d, X\dot{X}^* + \dot{X}X^* \right\rangle \\ &\quad + t^2 \left(\frac{1}{n} \sum_{i \notin I} \left[\ell'_i \langle A_i, \dot{X}\dot{X}^* \rangle + \frac{1}{2} \ell''_i \langle A_i, X\dot{X}^* + \dot{X}X^* \rangle^2 \right] + \lambda \|\dot{X}\|^2 \right) + o(t^2). \end{aligned}$$

Second-order criticality of X means $L_\lambda((X + t\dot{X})(X + t\dot{X})^*) - L_\lambda(XX^*) \geq o(t^2)$. Furthermore, for all $i \in I \setminus J$,

$$\ell(t^2\langle A_i, \dot{X}\dot{X}^* \rangle, y_i) - \ell(0, y_i) = t^2 \ell'_i \langle A_i, \dot{X}\dot{X}^* \rangle + o(t^2) = O(t^2). \quad (17)$$

Together with the previous Taylor expansion, these facts imply

$$\frac{1}{n} \sum_{i \in J} [\ell(t^2\langle A_i, \dot{X}\dot{X}^* \rangle, y_i) - \ell(0, y_i)] + t \left\langle \frac{1}{n} \sum_{i \notin I} \ell'_i A_i + \lambda I_d, X\dot{X}^* + \dot{X}X^* \right\rangle \geq O(t^2). \quad (18)$$

Now, the fact that $\ell'(0, y_i) = -\infty$ for all $i \in J$ implies that, for sufficiently small $t \geq 0$, $\ell(t^2\langle A_i, \dot{X}\dot{X}^* \rangle, y_i) \leq \ell(0, y_i)$ for all $i \in J$. We then must have

$$\begin{aligned} & t \left\langle \frac{1}{n} \sum_{i \notin I} \ell'_i A_i + \lambda I_d, X\dot{X}^* + \dot{X}X^* \right\rangle \geq O(t^2) \\ & \Downarrow \\ & \left\langle \frac{1}{n} \sum_{i \notin I} \ell'_i A_i + \lambda I_d, X\dot{X}^* + \dot{X}X^* \right\rangle = 0. \end{aligned} \quad (19)$$

We now show that $J = \emptyset$. Suppose, by way of contradiction, that there exists some $i \in J$. There is a corresponding term on the left-hand side of (18); any terms corresponding to other elements of J will be nonpositive for sufficiently small t , so we obtain

$$\ell(t^2\langle A_i, \dot{X}\dot{X}^* \rangle, y_i) - \ell(0, y_i) \geq O(t^2).$$

As $A_i \succeq 0$ and $A_i \neq 0$, we can choose unit-norm $\dot{X} \in \mathbf{F}^{d \times p}$ such that $\gamma := \langle A_i, \dot{X} \dot{X}^* \rangle > 0$. Then, for some $C \geq 0$ and all sufficiently small $s > 0$, we have

$$\ell(\gamma s, y_i) - \ell(0, y_i) \geq -Cs.$$

This is, however, incompatible with the assumption that $\lim_{b \downarrow 0} \ell'(b, y_i) = -\infty$. Thus we obtain a contradiction. Hence, we conclude that $J = \emptyset$. This proves the claim that ℓ'_i is finite for all i , so $\nabla L_\lambda(X X^*)$ is well-defined.

As $A_i X = 0$ for all $i \in I$, the equality (19) implies

$$\begin{aligned} 0 &= \left\langle \frac{1}{n} \sum_{i=1}^n \ell'_i A_i + \lambda I_d, X \dot{X}^* + \dot{X} X^* \right\rangle \\ &= \langle \nabla L_\lambda(X X^*), X \dot{X}^* + \dot{X} X^* \rangle \\ &= 2 \operatorname{real} \langle \nabla L_\lambda(X X^*) X, \dot{X} \rangle. \end{aligned}$$

As \dot{X} can be any unit-norm matrix in $\mathbf{F}^{d \times p}$, this implies (15).

Finally, because $J = \emptyset$, (17) holds for all $i \in I$. With this and (19), we can then further develop the Taylor expansion and second-order criticality condition as

$$\begin{aligned} o(t^2) &\leq L_\lambda((X + t\dot{X})(X + t\dot{X})^*) - L_\lambda(X X^*) \\ &= t^2 \left(\frac{1}{n} \sum_{i=1}^n \ell'_i \langle A_i, \dot{X} \dot{X}^* \rangle + \frac{1}{2n} \sum_{i \notin I} \ell''_i \langle A_i, X \dot{X}^* + \dot{X} X^* \rangle^2 + \lambda \|\dot{X}\|^2 \right) + o(t^2) \\ &= t^2 \left(\langle \nabla L_\lambda(X X^*), \dot{X} \dot{X}^* \rangle + \frac{1}{2n} \sum_{i \notin I} \ell''_i \langle A_i, X \dot{X}^* + \dot{X} X^* \rangle^2 \right) + o(t^2). \end{aligned}$$

This implies (16). \square

With this, we can prove the main landscape result:

Proof of Lemma 1. Let X be a second-order critical point of (9). Denote, as in the proof of Lemma 2,

$$I = \{i : \langle A_i, X X^* \rangle = 0\}.$$

First, we use the inequality (16) from Lemma 2 with the choice of rank-1 $\dot{X} = uv^*$ for some $u \in \mathbf{F}^d$, $v \in \mathbf{F}^p$. This gives

$$0 \leq \|v\|^2 \langle \nabla L_\lambda(X X^*), uu^* \rangle + \frac{1}{2n} \sum_{i \notin I} \ell''_i (\langle A_i, X X^* \rangle) \langle A_i, X v u^* + u (X v)^* \rangle^2. \quad (20)$$

As noted in [1, Sec. 3], we have, because $A_i \succeq 0$,

$$\langle A_i, X v u^* + u (X v)^* \rangle^2 \leq 4 \langle A_i, X v v^* X^* \rangle \langle A_i, uu^* \rangle.$$

Plugging this into (20) and summing over v in an orthonormal basis of \mathbf{F}^p gives

$$0 \leq p \langle \nabla L_\lambda(X X^*), uu^* \rangle + \frac{2}{n} \sum_{i=1}^n \ell''_i (\langle A_i, X X^* \rangle) \langle A_i, X X^* \rangle \langle A_i, uu^* \rangle. \quad (21)$$

We cannot, in general, do better in the real case ($\mathbf{F} = \mathbf{R}$). In the complex case ($\mathbf{F} = \mathbf{C}$), we can improve the constant. Write $A_i = B_i B_i^*$ for some matrix B_i . Then, by the fact that for $z \in \mathbf{C}$, $2 \operatorname{real}(z)^2 = |z|^2 + \operatorname{real}(z^2)$, we have

$$\begin{aligned} \langle A_i, X v z^* + z (X v)^* \rangle^2 &= 4 (\operatorname{real}(\langle B_i^* z, B_i^* X v \rangle))^2 \\ &= 2 |\langle B_i^* z, B_i^* X v \rangle|^2 + 2 \operatorname{real}(\langle B_i^* z, B_i^* X v \rangle^2) \\ &\leq 2 \langle A_i, X v v^* X^* \rangle \langle A_i, z z^* \rangle + 2 \operatorname{real}(\langle B_i^* z, B_i^* X v \rangle^2). \end{aligned}$$

Multiplying v by the imaginary unit i negates the second term in this last expression but leaves the first term unchanged. Then, if $\{v_k\}_{k=1}^p$ is an orthonormal basis of \mathbf{C}^p , plugging the above inequality into (20) and summing over all $v \in \{v_k, iv_k\}_{k=1}^p$ gives

$$0 \leq 2p\langle L_\lambda(XX^*), uu^* \rangle + \frac{2}{n} \sum_{i=1}^n \ell''(\langle A_i, XX^* \rangle) \langle A_i, XX^* \rangle \langle A_i, uu^* \rangle. \quad (22)$$

If we write $Z' = \sum_m u_m u_m^*$, summing the inequality (21) (in the real case) or (22) (in the complex case) with $u = u_m$ over m gives

$$0 \leq c_{\mathbf{FP}} \langle L_\lambda(XX^*), Z' \rangle + \frac{2}{n} \sum_{i=1}^n \ell''(\langle A_i, XX^* \rangle) \langle A_i, XX^* \rangle \langle A_i, Z' \rangle.$$

The result immediately follows. \square

4 Nonconvex PhaseCut and its landscape

In this section, we develop and analyze a nonconvex variant of the PhaseCut approach of [9]. We only consider the case of rank-1 measurements, that is, $A_i = f_i f_i^*$ for $f_1, \dots, f_n \in \mathbf{F}^d$. The concepts can be extended to more general (higher-rank) PSD measurement matrices, but the notation and analysis become far more complicated, so we omit this generalization.

Recall the notation for this case from Section 1. In the case of ordinary phase retrieval where $Z_* = x_* x_*^*$ for some vector $x_* \in \mathbf{F}^d$, we have

$$y = \alpha(x_*) + \varepsilon = |Fx_*| + \varepsilon,$$

where, again,

$$F = \begin{bmatrix} f_1^* \\ \vdots \\ f_n^* \end{bmatrix}.$$

The least-squares estimator is (7) (which is also (13) with $p = 1$ and $\lambda = 0$), which we can write compactly as

$$\min_{x \in \mathbf{F}^d} \frac{1}{n} \| |Fx| - y \|^2. \quad (23)$$

The PhaseCut formulation comes from the observation that, for any $a \in \mathbf{F}$ and $b \geq 0$,

$$(|a| - b)^2 = \min_{s \in \mathbf{F}_1} |a - sb|^2,$$

where \mathbf{F}_1 is the unit sphere in \mathbf{F} ($\{\pm 1\}$ in \mathbf{R} or the unit circle in \mathbf{C}). The optimum value of $s \in \mathbf{F}_1$ is simply $s = \frac{a}{|a|}$ if $a \neq 0$ (s can be arbitrary if $a = 0$).

Thus we can rewrite the objective in the least-squares problem (23) as

$$\frac{1}{n} \| |Fx| - y \|^2 = \min_{u \in \mathbf{F}_1^n} \frac{1}{n} \| Fx - \text{diag}(y)u \|^2$$

(recall that $\text{diag}(y) \in \mathbf{R}^{n \times n}$ is the diagonal matrix whose diagonal elements are the elements of $y \in \mathbf{R}^n$). For fixed $u \in \mathbf{F}_1^n$, a closed-form minimizer of $\| Fx - \text{diag}(y)u \|^2$ over $x \in \mathbf{F}^d$ is $x = F^\dagger \text{diag}(y)u$, where F^\dagger is the Moore-Penrose pseudoinverse of F . We then have, using the fact that FF^\dagger and $I_n - FF^\dagger$ are orthogonal projection matrices,

$$\begin{aligned} \min_{x \in \mathbf{F}^d} \frac{1}{n} \| Fx - \text{diag}(y)u \|^2 &= \frac{1}{n} \| (FF^\dagger \text{diag}(y)u - \text{diag}(y)u) \|^2 \\ &= \frac{1}{n} \| (I_n - FF^\dagger) \text{diag}(y)u \|^2 \\ &= \frac{1}{n} \langle \text{diag}(y)(I_n - FF^\dagger) \text{diag}(y), uu^* \rangle. \end{aligned}$$

The resulting minimization problem over $u \in \mathbf{F}_1^n$ has a structure resembling max-cut (from which the name “PhaseCut” comes) or \mathbf{Z}_2 /angular synchronization. The paper [9] studies a semidefinite relaxation of this optimization problem.

We generalize this to include regularization and larger ranks of the ground truth matrices and optimization variable. The same reformulation (without regularization) was used in the work [10] on sensor network localization, which is a particular instance of our problem.

For $X_* \in \mathbf{F}^{d \times r}$, the model (3) becomes

$$y = |FX_*| + \varepsilon. \quad (24)$$

We will consider a PhaseCut-like reformulation of the nonconvex problem (13). Note that, similarly to the scalar case, we have, for all $v \in \mathbf{F}^p$ and $b \geq 0$,

$$(\|v\| - b)^2 = \min_{u \in \mathbf{F}_1^p} \|v - bu\|^2,$$

where \mathbf{F}_1^p is the unit sphere in \mathbf{F}^p . For any $X \in \mathbf{F}^{d \times p}$, we can apply this to each row of $FX \in \mathbf{F}^{n \times p}$ to obtain

$$\|\alpha(X) - y\|^2 = \| |FX| - y \|^2 = \min_{U \in \mathbf{F}^{n \times p}} \|FX - \text{diag}(y)U\|^2 \text{ s.t. } \text{diag}(UU^*) = \mathbf{1}.$$

The diagonal constraint on UU^* compactly represents the requirement that each row of U have unit norm.

We can plug this into the problem (13) and reverse the order of optimization. For fixed U , the problem

$$\min_{X \in \mathbf{F}^{d \times p}} \frac{1}{n} \|FX - \text{diag}(y)U\|^2 + \lambda \|X\|^2 \quad (25)$$

is simply multivariate ridge regression. This has the closed-form solution (unique if $\lambda > 0$ or F has full row rank)

$$X = (n\lambda I_d + F^*F)^{-1} F^* \text{diag}(y)U = F^*(n\lambda I_n + FF^*)^{-1} \text{diag}(y)U.$$

If $\lambda = 0$, we can take, in a limiting sense, $X = F^\dagger \text{diag}(y)U$ similarly to before.

As the optimal X is linear in U , the minimum value of (25) is quadratic in U . We show below in Section 4.1 that

$$\min_{X \in \mathbf{F}^{d \times p}} \frac{1}{n} \|FX - \text{diag}(y)U\|^2 + \lambda \|X\|^2 = \langle M_\lambda, UU^* \rangle,$$

where

$$M_\lambda := \lambda \text{diag}(y)(n\lambda I_n + FF^*)^{-1} \text{diag}(y).$$

We can thus write the reformulated, constrained problem over U as

$$\min_{U \in \mathbf{F}^{n \times p}} \langle M_\lambda, UU^* \rangle \text{ s.t. } \text{diag}(UU^*) = \mathbf{1}, \quad \text{where } M_\lambda = \lambda \text{diag}(y)(n\lambda I_n + FF^*)^{-1} \text{diag}(y). \quad (26)$$

Similarly to before, in the unregularized case $\lambda = 0$, we can take

$$M_0 = \lim_{\lambda \downarrow 0} M_\lambda = \frac{1}{n} \text{diag}(y)(I_n - FF^\dagger) \text{diag}(y).$$

We now consider the nonconvex landscape of (26). The feasible set of (26) is a Riemannian manifold (a product of spheres); apart from the (discrete) case $p = 1$ and $\mathbf{F} = \mathbf{R}$, the problem is smooth. A feasible point U is a second-order critical point if the Riemannian gradient of the objective is zero and the Riemannian Hessian is positive semidefinite at U (see Section 4.2 for more details). Our landscape result for such second-order critical points looks similar to Theorem 2:

Theorem 4. *In the case $\mathbf{F} = \mathbf{R}$, assume $p \geq 2$. Under the model (24), every second-order critical point U of (26), satisfies, with $X = (n\lambda I_d + F^*F)^{-1} \text{diag}(y)U$, for all $R \in \mathbf{F}^{p \times r}$,*

$$\begin{aligned} \frac{1}{n} \|\alpha(X) - \alpha(X_*)\|^2 &\leq \frac{2}{n} \langle \varepsilon, \alpha(X) - \alpha(X_*) \rangle + \lambda (\|X_*\|^2 - \|X\|^2) \\ &\quad + \frac{1}{c_{\mathbf{F}}p - 1} \left(\frac{1}{n} \|F(X_\lambda - XR)\|^2 + \lambda \|X_\lambda - XR\|^2 \right), \end{aligned} \quad (27)$$

where $c_{\mathbf{F}}$ is as defined in Lemma 1, and $X_\lambda \in \mathbf{F}^{d \times r}$ satisfies

$$\frac{1}{n} \|F(X_\lambda - X_*)\|^2 + \lambda \|X_\lambda - X_*\|^2 \leq \left(\sqrt{\lambda} \|x_*\| + \frac{\|\varepsilon\|}{\sqrt{n}} \right)^2. \quad (28)$$

The guarantee is identical to that of Theorem 2 apart from the presence of X_λ . X_λ is the solution to (25) given perfect knowledge of the “true” measurement directions; see (30) below. The bound on $X_\lambda - X_*$ in (28) is sufficient for our purposes in this paper but is not necessarily tight.

We explore some statistical consequences of Theorem 4 in Section 5. As discussed in Section 4.1 below, if U were a global optimum of (26), we could drop the last term in (27) and thus obtain an identical guarantee as for global optima of (13) (see Theorem 2 and following discussion).

In the following subsections, we fill in some important details about this problem, finishing with a full proof of Theorem 4.

4.1 Preliminary calculations

First, we review some basic properties of ridge regression that are used above and/or will be useful for the proof of Theorem 4.

The objective of the multivariate ridge regression problem (25) is convex and quadratic, and the global minimum satisfies

$$\begin{aligned} 0 &= \nabla_X \left(\frac{1}{2n} \|FX - \text{diag}(y)U\|^2 + \frac{\lambda}{2} \|X\|^2 \right) \\ &= \frac{1}{n} F^*(FX - \text{diag}(y)U) + \lambda X. \end{aligned}$$

We will write the closed-form solution as

$$X = T_{y,\lambda}U, \quad \text{where} \quad T_{y,\lambda} := (n\lambda I_d + F^*F)^{-1}F^* \text{diag}(y) = F^*(n\lambda I_n + FF^*)^{-1} \text{diag}(y).$$

Note that, by construction of $T_{y,\lambda}$, the zero-gradient-like condition

$$0 = \frac{1}{n} F^*(FT_{y,\lambda}W - \text{diag}(y)W) + \lambda T_{y,\lambda}W$$

holds for any $W \in \mathbf{F}^{n \times p}$. This gives the identity

$$\begin{aligned} &\frac{1}{n} \|FT_{y,\lambda}W - \text{diag}(y)W\|^2 + \lambda \|T_{y,\lambda}W\|^2 \\ &= \frac{1}{n} (\|\text{diag}(y)W\|^2 + 2 \underbrace{\langle FT_{y,\lambda}W, FT_{y,\lambda}W - \text{diag}(y)W \rangle}_{=-n\lambda \|T_{y,\lambda}W\|^2} - \|FT_{y,\lambda}W\|^2) + \lambda \|T_{y,\lambda}W\|^2 \\ &= \frac{1}{n} (\|\text{diag}(y)W\|^2 - \|FT_{y,\lambda}W\|^2) - \lambda \|T_{y,\lambda}W\|^2 \\ &= \langle M_\lambda W, W \rangle, \end{aligned}$$

where

$$\begin{aligned} M_\lambda &:= \frac{1}{n} (\text{diag}(y)^2 - T_{y,\lambda}^* F^* F T_{y,\lambda}) - \lambda T_{y,\lambda}^* T_{y,\lambda} \\ &= \frac{1}{n} \text{diag}(y) (I_n - F(n\lambda I_d + F^*F)^{-1} F^* F (n\lambda I_d + F^*F)^{-1} F^* - n\lambda F(n\lambda I_d + F^*F)^{-2} F^*) \text{diag}(y) \\ &= \frac{1}{n} \text{diag}(y) (I_n - F(n\lambda I_d + F^*F)^{-1} F^*) \text{diag}(y) \\ &= \frac{1}{n} \text{diag}(y) (I_n - FF^*(n\lambda I_n + FF^*)^{-1}) \text{diag}(y) \\ &= \lambda \text{diag}(y) (n\lambda I_n + FF^*)^{-1} \text{diag}(y). \end{aligned}$$

The intermediate identity

$$\langle M_\lambda W, W \rangle = \frac{1}{n} \|\text{diag}(y)W\|^2 - \frac{1}{n} \|FT_{y,\lambda}W\|^2 - \lambda \|T_{y,\lambda}W\|^2, \quad (29)$$

which holds for all $W \in \mathbf{F}^{n \times p}$, will later prove useful.

Let $U_* \in \mathbf{F}^{n \times r}$ be a matrix of “ground truth” directions of FX_* , that is, a⁴ matrix with unit-norm rows satisfying

$$(FX_*)_i = \|(FX_*)_i\| (U_*)_i \quad \forall i \quad \Longleftrightarrow \quad FX_* = \text{diag}(\|FX_*\|) U_*.$$

⁴The choice is not necessarily unique if $(FX_*)_i = 0$ for some i .

The matrix X_λ in Theorem 4 is the optimum of (25) (with $p = r$) given $U = U_*$:

$$\begin{aligned} X_\lambda &:= T_{y,\lambda} U_* \\ &= \frac{1}{n} \left(\lambda I_d + \frac{1}{n} F^* F \right)^{-1} F^* (F X_* + \text{diag}(\varepsilon) U_*) \\ &= X_* + \left(\lambda I_d + \frac{1}{n} F^* F \right)^{-1} \left(-\lambda X_* + \frac{1}{n} F^* \text{diag}(\varepsilon) U_* \right). \end{aligned} \quad (30)$$

Our analysis is somewhat complicated by the fact that we do not obtain X_* exactly. The error term in this last expression is standard from (multivariate) ridge regression. As U_* is not necessarily uniquely defined, neither is X_λ . However, this makes no difference to our analysis in this paper.

We now consider some properties of optima of (26). We do this in several steps. First, we can estimate

$$\begin{aligned} \langle M_\lambda, U_* U_*^* \rangle &= \frac{1}{n} \|F X_\lambda - \text{diag}(y) U_*\|^2 + \lambda \|X_\lambda\|^2 \\ &\leq \frac{1}{n} \|F X_* - \text{diag}(y) U_*\|^2 + \lambda \|X_*\|^2 \\ &= \frac{\|\varepsilon\|^2}{n} + \lambda \|X_*\|^2. \end{aligned}$$

This upper bound (which follows from the fact that X_λ minimizes the previous expression over all $X \in \mathbf{F}^{d \times r}$) will prove simpler to use than the exact expression.

Next, for any feasible U of (26), for $X = T_{y,\lambda} U$, we have

$$\begin{aligned} \langle M_\lambda, U U^* \rangle &= \frac{1}{n} \|F X - \text{diag}(y) U\|^2 + \lambda \|X\|^2 \\ &\geq \frac{1}{n} \|F X - y\|^2 + \lambda \|X\|^2 \\ &= \frac{1}{n} \|\alpha(X) - \alpha(X_*) - \varepsilon\|^2 + \lambda \|X\|^2 \\ &= \frac{1}{n} (\|\varepsilon\|^2 + \|\alpha(X) - \alpha(X_*)\|^2 - 2\langle \varepsilon, \alpha(X) - \alpha(X_*) \rangle) + \lambda \|X\|^2. \end{aligned}$$

Combining these previous two inequalities, we obtain

$$\langle M_\lambda, U U^* - U_* U_*^* \rangle \geq \frac{1}{n} \|\alpha(X) - \alpha(X_*)\|^2 - \frac{2}{n} \langle \varepsilon, \alpha(X) - \alpha(X_*) \rangle + \lambda (\|X\|^2 - \|X_*\|^2). \quad (31)$$

However, if U is globally optimal⁵ for (26) (with $p \geq r$), the left-hand-side of (31) must be ≤ 0 , so we obtain

$$\frac{1}{n} \|\alpha(X) - \alpha(X_*)\|^2 \leq \frac{2}{n} \langle \varepsilon, \alpha(X) - \alpha(X_*) \rangle + \lambda (\|X_*\|^2 - \|X\|^2).$$

Comparing to Theorem 2, this is precisely the same inequality we would obtain from a global optimum (or semidefinite relaxation) of (13). The additional term in Theorem 4 is similarly the price we pay for only having a second-order critical point of the nonconvex problem (26).

4.2 Nonconvex landscape proof

Finally, we analyze the nonconvex landscape of (26). As quadratically constrained quadratic problems of this particular form are by now well-studied and are not the main focus of this paper, we omit much of the context and details. See, for example, [37, 38, 45] for further details and references. The following lemma provides the properties we will need of second-order critical points of (26). The operator ddiag on $n \times n$ matrices extracts the diagonal, setting off-diagonal elements to zero. We write the Hadamard (elementwise) product between two matrices A and B of equal size as $A \circ B$, and we denote the Hadamard square by $A^{\circ 2} = A \circ A$.

⁵We could find a global optimum, for example, by taking $p = n$ and using semidefinite programming; the original PhaseCut paper [9] only considered such semidefinite relaxations.

Lemma 3. Let $M \in \mathbf{F}^{n \times n}$ be Hermitian. Every second-order critical point U of the problem

$$\min_{U \in \mathbf{F}^{n \times p}} \langle M, UU^* \rangle \text{ s.t. } \text{diag}(UU^*) = \mathbf{1}$$

satisfies the following properties, where

$$S(U) := M - \text{real}(\text{ddiag}(MUU^*)).$$

- $S(U)U = 0$, and,
- For all $z \in \mathbf{F}^n$,

$$\langle S(U), (c_{\mathbf{F}p} - 2)zz^* + \text{real}(D_z^*UU^*D_z) \circ (UU^*) \rangle \geq 0, \quad (32)$$

where $D_z = \text{diag}(z)$.

This is a slight generalization of intermediate results in [38]; we provide a proof at the end of this section. With this and the calculations developed in Section 4.1 above, we can prove the main landscape result.

Proof of Theorem 4. Note that, for $z \in \mathbf{F}^n$, we can rewrite the inequality (32) from Lemma 2 (with $M = M_\lambda$) as

$$\begin{aligned} 0 &\leq (c_{\mathbf{F}p} - 1)\langle S(U), zz^* \rangle + \langle S(U), \text{real}(D_z^*UU^*D_z) \circ (UU^*) - zz^* \rangle \\ &= (c_{\mathbf{F}p} - 1)\langle S(U), zz^* \rangle + \langle M_\lambda, \text{real}(D_z^*UU^*D_z) \circ (UU^*) - zz^* \rangle \\ &\quad - \underbrace{\langle \text{real}(\text{ddiag}(M_\lambda UU^*)), \text{real}(D_z^*UU^*D_z) \circ (UU^*) - zz^* \rangle}_{=0} \\ &= (c_{\mathbf{F}p} - 1)\langle S(U), zz^* \rangle + \langle M_\lambda, \text{real}(D_z^*UU^*D_z) \circ (UU^*) - zz^* \rangle. \end{aligned}$$

The third inner product on the middle line is zero because the matrix on the right-hand side has zero diagonal.

As $M_\lambda \preceq \frac{1}{n} \text{diag}(y)^2$,

$$\begin{aligned} \langle M_\lambda, \text{real}(D_z^*UU^*D_z) \circ (UU^*) \rangle &\leq \frac{1}{n} \text{tr}(\text{diag}(y)(\text{real}(D_z^*UU^*D_z) \circ (UU^*)) \text{diag}(y)) \\ &= \frac{1}{n} \|\text{diag}(y)z\|^2. \end{aligned}$$

Furthermore, by (29),

$$\langle M_\lambda, zz^* \rangle = \frac{1}{n} (\|\text{diag}(y)z\|^2 - \|FT_{y,\lambda}z\|^2) - \lambda \|T_{y,\lambda}z\|^2.$$

Combining these previous three displays, we obtain

$$0 \leq (c_{\mathbf{F}p} - 1)\langle S(U), zz^* \rangle + \frac{1}{n} \|FT_{y,\lambda}z\|^2 + \lambda \|T_{y,\lambda}z\|^2.$$

Now, choose $z = z_k = (U_* - UR)v_k$, where $\{v_k\}_{k=1}^p$ is an orthonormal basis for \mathbf{F}^p , and sum up the resulting inequalities to obtain

$$\begin{aligned} 0 &\leq (c_{\mathbf{F}p} - 1)\langle S(U), (U_* - UR)(U_* - UR)^* \rangle + \frac{1}{n} \|FT_{y,\lambda}(U_* - UR)\|^2 + \lambda \|T_{y,\lambda}(U_* - UR)\|^2 \\ &= (c_{\mathbf{F}p} - 1)\langle S(U), (U_* - UR)(U_* - UR)^* \rangle + \frac{1}{n} \|F(X_\lambda - XR)\|^2 + \lambda \|X_\lambda - XR\|^2. \end{aligned}$$

The fact that $S(U)U = 0$ from Lemma 2 implies

$$\langle S(U), (U_* - UR)(U_* - UR)^* \rangle = \langle S(U), U_*U_*^* - UU^* \rangle = \langle M_\lambda, U_*U_*^* - UU^* \rangle.$$

Thus we obtain

$$\langle M_\lambda, UU^* - U_*U_*^* \rangle \leq \frac{1}{c_{\mathbf{F}p} - 1} \left(\frac{1}{n} \|F(X_\lambda - XR)\|^2 + \lambda \|X_\lambda - XR\|^2 \right).$$

Together with (31), this proves the main inequality (27).

Finally, using (30), we can bound

$$\begin{aligned}
& \left(\frac{1}{n} \|F(X_\lambda - X_*)\|^2 + \lambda \|X_\lambda - X_*\|^2 \right)^{1/2} \\
&= \left\langle \left(\frac{1}{n} F^* F + \lambda I_d \right), (X_\lambda - X_*)(X_\lambda - X_*)^* \right\rangle^{1/2} \\
&= \left\langle \left(\lambda I_d + \frac{1}{n} F^* F \right)^{-1}, \left(-\lambda X_* + \frac{1}{n} F^* \text{diag}(\varepsilon) U_* \right) \left(-\lambda X_* + \frac{1}{n} F^* \text{diag}(\varepsilon) U_* \right)^* \right\rangle^{1/2} \\
&\leq \left\langle \left(\lambda I_d + \frac{1}{n} F^* F \right)^{-1}, \lambda^2 X_* X_*^* \right\rangle^{1/2} \\
&\quad + \left\langle \frac{1}{n} F \left(\lambda I_d + \frac{1}{n} F^* F \right)^{-1} F^*, \frac{1}{n} (\text{diag}(\varepsilon) U_*) (\text{diag}(\varepsilon) U_*)^* \right\rangle^{1/2} \\
&\leq (\lambda \text{tr}(X_* X_*^*))^{1/2} + \left(\frac{1}{n} \text{tr}(\text{diag}(\varepsilon) U_* U_*^* \text{diag}(\varepsilon)) \right)^{1/2} \\
&= \sqrt{\lambda} \|X_*\| + \frac{\|\varepsilon\|}{\sqrt{n}},
\end{aligned}$$

which gives (28). \square

To finish, we sketch the proof of our auxiliary landscape lemma based on arguments from [38, 37].

Proof of Lemma 3. The optimization problem is that of a smooth function on a smooth Riemannian manifold; in particular, the constraint set is a product of n unit spheres in \mathbf{F}^p . First-order criticality of a feasible point U is equivalent to $S(U)U = 0$, as this quantity is proportional to the Riemannian gradient of the objective at U .

Second-order criticality requires, in addition to first-order criticality, that the Riemannian Hessian be positive semidefinite at U . For this problem, that means

$$\langle S(U), \dot{U} \dot{U}^* \rangle \geq 0 \quad \text{for all } \dot{U} \in T_U := \{\dot{U} \in \mathbf{F}^{n \times p} : \text{diag}(U \dot{U}^* + \dot{U} U^*) = 0\}.$$

T_U is the tangent space to the constraint manifold at U . We denote by \mathcal{P}_{T_U} the orthogonal projection (in $\mathbf{F}^{n \times p}$) to T_U . To obtain a useful inequality, we consider, for unit-norm $v \in \mathbf{F}^p$, the tangent vector

$$\begin{aligned}
\dot{U}_v &= \mathcal{P}_{T_U}(z v^*) \\
&= z v^* - \text{ddiag}(\text{real}(z v^* U^*)) U \\
&= z v^* - \text{real}(D_z^* \text{diag}(U v)) U.
\end{aligned}$$

Then

$$\begin{aligned}
\dot{U}_v \dot{U}_v^* &= z z^* - \text{real}(D_z^* \text{diag}(U v)) U v z^* - z (U v)^* \text{real}(D_z^* \text{diag}(U v)) \\
&\quad + \text{real}(D_z^* \text{diag}(U v)) U U^* \text{real}(D_z^* \text{diag}(U v)).
\end{aligned}$$

We now consider separately the real and complex cases. In the real case ($\mathbf{F} = \mathbf{R}$), we have

$$\begin{aligned}
\dot{U}_v \dot{U}_v^* &= z z^* - D_z |U v|^2 z^* - z (|U v|^2)^* D_z \\
&\quad + D_z \text{diag}(U v) U U^* \text{diag}(U v) D_z,
\end{aligned}$$

where, as before, $|U v|^2$ is the elementwise squared absolute value of the vector $U v$. Note that, if $\{v_k\}_{k=1}^p$ is an orthonormal basis for \mathbf{R}^p , we have, for all $i, j \in \{1, \dots, n\}$,

$$\sum_{k=1}^p (U v_k)_i (U v_k)_j = (U U^*)_{ij}.$$

Therefore,

$$\sum_{k=1}^p D_z |U v_k|^2 z^* = D_z \mathbf{1} z^* = z z^*,$$

and

$$\sum_{k=1}^p D_z \text{diag}(Uv_k) U U^* \text{diag}(Uv_k) D_z = D_z (U U^*)^{\circ 2} D_z.$$

This implies

$$\sum_{k=1}^p \dot{U}_{v_k} \dot{U}_{v_k}^* = (p-2) z z^* + D_z (U U^*)^{\circ 2} D_z,$$

so, finally,

$$\begin{aligned} 0 &\leq \sum_{k=1}^p \langle S(U), \dot{U}_{v_k} \dot{U}_{v_k}^* \rangle \\ &= \left\langle S(U), \sum_{k=1}^p \dot{U}_{v_k} \dot{U}_{v_k}^* \right\rangle \\ &= \langle S(U), (p-2) z z^* + D_z (U U^*)^{\circ 2} D_z \rangle. \end{aligned}$$

This completes the proof in the real case.

In the complex case ($\mathbf{F} = \mathbf{C}$), the argument is similar but messier. We now have

$$\begin{aligned} \dot{U}_v \dot{U}_v^* &= z z^* - \frac{1}{2} (D_z^* \text{diag}(Uv) + D_z \text{diag}(Uv)^*) U v z^* - \frac{1}{2} z (Uv)^* (\text{diag}(Uv)^* D_z + \text{diag}(Uv) D_z^*) \\ &\quad + \frac{1}{4} (D_z^* \text{diag}(Uv) + D_z \text{diag}(Uv)^*) U U^* (\text{diag}(Uv)^* D_z + \text{diag}(Uv) D_z^*). \end{aligned}$$

To simplify this, first, one can easily verify that

$$\begin{aligned} \dot{U}_v \dot{U}_v^* + \dot{U}_{iv} \dot{U}_{iv}^* &= 2 z z^* - D_z |Uv|^2 z^* - z (|Uv|^2)^* D_z^* \\ &\quad + \frac{1}{2} D_z^* \text{diag}(Uv) U U^* \text{diag}(Uv)^* D_z + \frac{1}{2} D_z \text{diag}(Uv)^* U U^* \text{diag}(Uv) D_z^*. \end{aligned}$$

If $\{v_k\}_{k=1}^p$ is an orthonormal basis of \mathbf{C}^p , we have

$$\sum_{k=1}^p (Uv_k)_i (Uv_k)_j^* = (U U^*)_{ij},$$

so

$$\begin{aligned} \sum_{k=1}^p (\dot{U}_{v_k} \dot{U}_{v_k}^* + \dot{U}_{iv_k} \dot{U}_{iv_k}^*) &= (2p-2) z z^* + \frac{1}{2} (D_z^* (U U^*)^{\circ 2} D_z + D_z |U U^*|^2 D_z^*) \\ &= (2p-2) z z^* + \text{real}(D_z^* U U^* D_z) \circ (U U^*). \end{aligned}$$

The result follows similarly to the real case. □

5 Some statistical consequences for phase retrieval

In this section, we give some statistical results for phase retrieval, that is, recovering a vector x_* from (noisy) measurements of the form $y_i \approx |\langle f_i, x_* \rangle|$. The results are not exhaustive but are intended to illustrate briefly the usefulness of the previous deterministic landscape results. Further implications are left to future work.

5.1 Finite dimension and isotropic sub-Gaussian measurements

First, we consider the case of estimating $x_* \in \mathbf{F}^d$ from measurements of the form $y_i \approx |\langle f_i, x_* \rangle|$, where f_1, \dots, f_n are isotropic and sub-Gaussian. We adopt the assumptions of [33, 34], presented as stated in [1]; see those papers for further context and references. These assumptions are, to the best of our knowledge, among the most general under which one can obtain known-optimal statistical guarantees.

Specifically, we assume that the measurement vectors f_i are independent and identically distributed (i.i.d.), and, furthermore, that their entries are i.i.d. copies of a sub-Gaussian zero-mean random variable w . We say⁶ that a zero-mean random variable w is K -sub-Gaussian for $K > 0$ if $\mathbf{E} e^{|w|^2/K^2} \leq 2$. We additionally need some moment conditions on w to ensure that the phase retrieval map $x_* \mapsto \alpha(x_*)$ is injective (modulo trivial symmetries).

The following statistical landscape result for the nonconvex problems (8) (equivalently, (13) with $\lambda = 0$) and (26) is a counterpart to [38, Thm. 3], which instead studied the landscape of (6):

Theorem 5. *Consider the model (3) with rank-one $Z_* = x_* x_*^*$ for nonzero $x_* \in \mathbf{F}^d$. Suppose $A_i = f_i f_i^*$, where f_1, \dots, f_n are i.i.d. random vectors whose entries are i.i.d. copies of a zero-mean random variable w . If $\mathbf{F} = \mathbf{R}$ but w is complex, we can take $A_i = \text{real}(f_i f_i^*)$. Suppose the following properties are true about w and x_* :*

- $\mathbf{E}|w|^2 = 1$.
- w is K -sub-Gaussian for some $K > 0$.
- At least one of the following two statements holds:
 1. $\mathbf{E}|w|^4 > 1$, or
 2. $\|x_*\|_\infty \leq \mu \|x_*\|$ for a sufficiently small universal constant $\mu > 0$.
- If $\mathbf{F} = \mathbf{C}$, $|\mathbf{E} w^2| < 1$.

Then there exist $c_1, c_2, c_3, c_4, c_5 > 0$ depending only on the properties of w (not on the dimension d) such that, if $n \geq c_1 d$, with probability at least $1 - c_2 n^{-2}$, for all

$$p \geq c_3,$$

for every second-order critical point X of (8) or (in the case $A_i = f_i f_i^*$) $X = F^\dagger \text{diag}(y)U$ where U is any second-order critical point of (26) with $\lambda = 0$, we have

$$\|X X^* - x_* x_*^*\|_* \leq c_4 \left(\|x_*\| \frac{\|\varepsilon\|}{\sqrt{n}} + \frac{\|\varepsilon\|^2}{n} \right).$$

Furthermore, if a closest rank-1 approximation to X in $\|\cdot\|$ is $\hat{x} v^*$ for some $\hat{x} \in \mathbf{F}^d$ and unit-norm $v \in \mathbf{F}^p$, we have

$$\min_{\|s\|=1} \|\hat{x} - s x_*\| \leq c_5 \frac{\|\varepsilon\|}{\sqrt{n}}.$$

We prove this in Section 5.3. A version of this result for more general measurement covariance and $\lambda \geq 0$ (but assuming Gaussian measurement vectors) appears as Theorem 6 below.

The error bounds of Theorem 5 with respect to noise are optimal within constants. Indeed, if $\varepsilon = \delta \alpha(x_*)$ for some $\delta \geq 0$, we have, with high probability, $\|\varepsilon\| \approx \delta \sqrt{n} \|x_*\|$, and, as $y = (1 + \delta) \alpha(x_*)$, any exact recovery algorithm will return $x_\delta := (1 + \delta) x_*$, which has vector error

$$\|x_\delta - x_*\| = \delta \|x_*\| \approx \frac{\|\varepsilon\|}{\sqrt{n}}$$

and matrix error

$$\begin{aligned} \|x_\delta x_\delta^* - x_* x_*^*\|_* &= [(1 + \delta)^2 - 1] \|x_*\|^2 \\ &= (2\delta + \delta^2) \|x_*\|^2 \\ &\approx \|x_*\| \frac{\|\varepsilon\|}{\sqrt{n}} + \frac{\|\varepsilon\|^2}{n}. \end{aligned}$$

Compared to [38, Thm. 3], which considers the problem (6), we only need the optimization rank p to be constant rather than (potentially) logarithmic in the dimension d . On the other hand, if the noise is non-adversarial (e.g., i.i.d. Gaussian), Theorem 5 gives a poorer error bound scaling (by a factor of $\sqrt{\frac{n}{d}}$ for large n) than [38, Thm. 3]. We believe this is an artifact of our analysis, as initial numerical experiments suggest that (8) has similar or better statistical performance with noise than (6), but it is unclear how to improve our analysis. The biggest technical obstacle seems to be the fact, noted in [3], that for $p > 1$ we cannot in general lower bound $\|\alpha(X) - \alpha(x_*)\|$ by some constant multiple of $\min_{\|v\|=1} \|X - x_* v^*\|$.

⁶This is one of several equivalent conditions common in the literature.

5.2 Infinite-dimensional, Gaussian measurements

Next, we consider the case where the dimension of the problem is potentially large (even infinite) compared to the number of measurements n . We will assume, for simplicity, that the measurements are Gaussian, though this can likely be relaxed; an interesting potential further development would be recovery results for phase retrieval in a general reproducing kernel Hilbert space.

Let \mathcal{H} be a (real or complex) Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$. We say $f \in \mathcal{H}$ is a zero-mean Gaussian random variable on \mathcal{H} with covariance Σ if, for every $u \in \mathcal{H}$, $\langle u, f \rangle$ is zero-mean Gaussian, and, for every $u, v \in \mathcal{H}$, $\mathbf{E}\langle u, f \rangle \overline{\langle v, f \rangle} = \langle \Sigma u, v \rangle$. We assume Σ is *trace-class*, that is, $\text{tr } \Sigma = \mathbf{E}\|f\|^2 < \infty$.

We denote, for $u, v \in \mathcal{H}$, the L_2 inner product

$$\langle u, v \rangle_{L_2} := \mathbf{E}\langle u, f \rangle \overline{\langle v, f \rangle} = \langle \Sigma u, v \rangle$$

with induced norm

$$\|u\|_{L_2}^2 := \mathbf{E}|\langle u, f \rangle|^2 = \|\Sigma^{1/2}u\|^2.$$

With this, we can define various norms on operators on \mathcal{H} . In particular, we will use the L_2 nuclear and operator norms

$$\begin{aligned} \|T\|_{*, L_2} &:= \|\Sigma^{1/2}T\Sigma^{1/2}\|_* \quad \text{and} \\ \|T\|_{\text{op}, L_2} &:= \|\Sigma^{1/2}T\Sigma^{1/2}\|_{\text{op}} \end{aligned}$$

defined for trace-class operators T on \mathcal{H} , where $\|\cdot\|_{\text{op}}$ and $\|\cdot\|_*$ are the operator and nuclear norms as defined on operators on \mathcal{H} .

If \mathcal{H} is infinite-dimensional, we cannot hope to recover arbitrary $x_* \in \mathcal{H}$ exactly with finite samples. Instead, we try to estimate x_* accurately in an L_2 or regression sense. Thus we aim not to “invert” the measurement process but rather to “learn” the mapping $f \mapsto \langle x_*, f \rangle$. In the linear regression case, where we observe i.i.d copies of $(f, \langle x_*, f \rangle)$ without the nonlinear absolute value, the difficulty is determined by the eigenvalues of Σ . Indeed, these will feature prominently in the phase retrieval result below.

To estimate x_* from measurements of the form $y_i \approx |\langle f_i, x_* \rangle|$, we can still use the algorithms and theory of Sections 3 and 4. However, we must make several adaptations and clarifications:

- We replace $\mathbf{F}^{d \times p}$ by \mathcal{H}^p . We overload our previous notation to write, for $X = (x_1, \dots, x_p) \in \mathcal{H}^p$,

$$\|X\|^2 = \sum_{k=1}^p \|x_k\|^2, \quad \|X\|_{L_2}^2 = \sum_{k=1}^p \|x_k\|_{L_2}^2,$$

and

$$XX^* = \sum_{k=1}^p x_k^{\otimes 2},$$

where $x^{\otimes 2} = x \otimes x$ denotes the tensor product of $x \in \mathcal{H}$ with itself. Furthermore, for any linear operator T with domain \mathcal{H} , we write $TX = (Tx_1, \dots, Tx_n)$.

- For measurement vectors $f_1, \dots, f_n \in \mathcal{H}$, we now denote by $F: \mathcal{H} \rightarrow \mathbf{F}^n$ the linear operator defined by

$$Fx = \begin{bmatrix} \langle x, f_1 \rangle \\ \vdots \\ \langle x, f_n \rangle \end{bmatrix}.$$

With the linear map notation overloading from the previous bullet point, we can then write, for $X \in \mathcal{H}^p$,

$$\alpha(X) = |FX|$$

similarly to the finite-dimensional case.

- With this notation, the nonconvex problem (13) becomes

$$\min_{X \in \mathcal{H}^p} \frac{1}{n} \|\alpha(X) - y\|^2 + \lambda \|X\|^2, \quad (33)$$

while the PhaseCut-reformulated problem (26) remains unchanged, noting that the matrix FF^* is given by $(FF^*)_{ij} = \langle f_j, f_i \rangle$.

- We do not consider computation in detail, but, for example, this can be done by the standard “kernel trick” if we have access to inner products $\langle u, v \rangle$ for arbitrary $u, v \in \mathcal{H}$.

With these adaptations, we have the following nonconvex landscape and recovery result:

Theorem 6. *Consider the model $y = \alpha(x_*) + \varepsilon$ for nonzero $x_* \in \mathcal{H}$, where \mathcal{H} is a Hilbert space over \mathbf{F} , and f_1, \dots, f_n are i.i.d. Gaussian random vectors in \mathcal{H} (circularly symmetric if $\mathbf{F} = \mathbf{C}$) with trace-class covariance Σ . Let the eigenvalues of Σ be*

$$\sigma_1 \geq \sigma_2 \geq \dots \geq 0.$$

There exist universal constants $c_1, c_2, c_3, c_4, c_5 > 0$ such that the following holds.

Fix a positive integer $d \leq \text{rank}(\Sigma)$. If $n \geq c_1 d$, with probability at least $1 - c_2 n^{-2}$, for all

$$\lambda \geq c_3 \left(\sigma_{d+1} + \frac{1}{n} \sum_{m>d} \sigma_m \right)$$

(with the convention that, if $d = \text{rank}(\Sigma) < \infty$, $\sigma_{d+1} = 0$), every second-order critical point X of (33) or, alternatively,

$$X = F^*(n\lambda I_n + FF^*)^{-1} \text{diag}(y)U,$$

where U is any second-order critical point of (26), satisfies

$$\|XX^* - x_*x_*^*\|_{*,L_2} \leq c_4 \left(\frac{\|\varepsilon\|^2}{n} + \|x_*\|_{L_2} \left(\frac{\|\varepsilon\|}{\sqrt{n}} + \sqrt{\lambda}\|x_*\| \right) \right).$$

Furthermore, if the closest rank-1 approximation to X in $\|\cdot\|_{L_2}$ is $\hat{x}v^ = (\bar{v}_1\hat{x}, \dots, \bar{v}_p\hat{x})$ for some $\hat{x} \in \mathcal{H}$ and unit-norm $v \in \mathbf{F}^p$, we have*

$$\min_{|s|=1} \|\hat{x} - sx_*\|_{L_2} \leq c_5 \left(\frac{\|\varepsilon\|}{\sqrt{n}} + \sqrt{\lambda}\|x_*\| \right).$$

If $\text{rank}(\Sigma) < \infty$ and we choose $d = \text{rank}(\Sigma)$, we recover a version (more general in terms of Σ and λ) of Theorem 5. The error terms involving ε are identical to those in Theorem 5 and hence are optimal (within constants) as discussed after that result.

The error term involving λ is also optimal within constants. To isolate this term, consider $\varepsilon = 0$; we then have, for some unit-magnitude $s \in \mathbf{F}$,

$$\|\hat{x} - x_*s\|_{L_2} \lesssim \sqrt{\lambda}\|x_*\|.$$

An error term of this form is standard even in ordinary linear regression in a Hilbert space, where we know the signs/phases of the measurements, and it is optimal in some cases. The requirement on λ in terms of the eigenvalues is also similar to conditions in existing results for infinite-dimensional linear regression. See, for example, [46, 47] for some relevant results and further reading.

5.3 Proofs

We first prove the somewhat simpler Theorem 5. Many of the techniques and necessary intermediate results will carry over to the proof of Theorem 6.

First, we state a standard result on the concentration of the empirical covariance of isotropic sub-Gaussian random vectors. We say a zero-mean random vector $f \in \mathbf{C}^d$ (this covers the real case as well) is isotropic and K -sub-Gaussian (for some $K > 0$) if, for all unit-norm $x \in \mathbf{C}^d$, $\langle f, x \rangle$ is unit-variance (i.e., $\mathbf{E}|\langle f, x \rangle|^2 = 1$) and K -sub-Gaussian, that is, $\mathbf{E}e^{|\langle f, x \rangle|^2/K^2} \leq 2$. Note that this is, in particular, true under the conditions of Theorem 5.

Lemma 4 (e.g., [48, Thm. 1]). *Let f_1, \dots, f_n be independent zero-mean isotropic K -sub-Gaussian vectors in \mathbf{C}^d . There exists a constant $c > 0$ depending only on K such that, for all $t \geq 1$, with probability at least $1 - e^{-t}$,*

$$\left\| \frac{1}{n} \sum_{i=1}^n f_i f_i^* - I_d \right\|_{\text{op}} \leq c \left(\sqrt{\frac{d}{n}} + \frac{d}{n} + \sqrt{\frac{t}{n}} + \frac{t}{n} \right).$$

In particular, choosing $t = 2 \log n$, for constants $c_1, c_2 > 0$ depending only on K , if $n \geq c_1 d$, with probability at least $1 - n^{-2}$,

$$\left\| \frac{1}{n} \sum_{i=1}^n \text{real}(f_i f_i^*) - I_d \right\|_{\text{op}} \leq \left\| \frac{1}{n} \sum_{i=1}^n f_i f_i^* - I_d \right\|_{\text{op}} \leq c_2 \sqrt{\frac{d + \log n}{n}}.$$

Next, the following result will be essential:

Lemma 5 ([34], as stated in [1, Lem. 4]). *Under the conditions of Theorem 5, there exist constants $c_1, c_2, c_3, c_4 > 0$ depending only on the properties of w such that, for $n \geq c_1 d$, with probability at least $1 - c_2 e^{-c_3 n}$, for all $Z \succeq 0$,*

$$\frac{1}{n} \sum_{i=1}^n |\langle A_i, Z - Z_* \rangle| \geq c_4 \|Z - x_* x_*^*\|_*.$$

The following two technical lemmas allow us to use the previous lemma in our framework:

Lemma 6. *For all $X_1 \in \mathbf{F}^{d \times r_1}$, $X_2 \in \mathbf{F}^{d \times r_2}$,*

$$\frac{1}{n} \|\alpha(X_1) - \alpha(X_2)\|^2 \geq \frac{\|\beta(X_1) - \beta(X_2)\|_1^2}{n \|\alpha(X_1) + \alpha(X_2)\|^2}.$$

Proof. Convexity of the function $(x, y) \mapsto \frac{x^2}{y}$ and Jensen's inequality (Cauchy-Schwartz would also work) imply

$$\begin{aligned} \frac{1}{n} \|\alpha(X_1) - \alpha(X_2)\|^2 &= \frac{1}{n} \sum_{i=1}^n (\langle A_i, X_1 X_1^* \rangle^{1/2} - \langle A_i, X_2 X_2^* \rangle^{1/2})^2 \\ &= \frac{1}{n} \sum_{i=1}^n \frac{|\langle A_i, X_1 X_1^* \rangle - \langle A_i, X_2 X_2^* \rangle|^2}{(\langle A_i, X_1 X_1^* \rangle^{1/2} + \langle A_i, X_2 X_2^* \rangle^{1/2})^2} \\ &\geq \frac{(\frac{1}{n} \sum_{i=1}^n |\langle A_i, X_1 X_1^* \rangle - \langle A_i, X_2 X_2^* \rangle|)^2}{\frac{1}{n} \sum_{i=1}^n (\langle A_i, X_1 X_1^* \rangle^{1/2} + \langle A_i, X_2 X_2^* \rangle^{1/2})^2} \\ &= \frac{\|\beta(X_1) - \beta(X_2)\|_1^2}{n \|\alpha(X_1) + \alpha(X_2)\|^2}. \end{aligned}$$

□

Lemma 7. *Let $x_* \in \mathbf{F}^d$ and $X \in \mathbf{F}^{d \times p}$. There is a unit-norm $v \in \mathbf{F}^p$ such that*

$$\|X X^* - x_* x_*^*\|_* \geq \|X X^* - x_* x_*^*\|_{\text{F}} \geq \frac{1}{2\sqrt{2}} \|x_*\| \|x_* - X v\|.$$

In particular, $(X v) v^$ can be any best rank-1 approximation to X in $\|\cdot\|$.*

Proof. Let $v \in \mathbf{F}^p$ be a unit-norm leading right singular vector of X , which is equivalent to $(X v) v^*$ being a rank-1 projection of X in $\|\cdot\|$. We can furthermore choose v (multiplying by a unit-magnitude number if needed) such that $\langle X v, x_* \rangle \geq 0$. As $X v v^* X^*$ is an optimal rank-1 approximation to $X X^*$,

$$\begin{aligned} 0 &\leq \|X X^* - x_* x_*^*\|_{\text{F}}^2 - \|X X^* - X v v^* X^*\|_{\text{F}}^2 \\ &= -\|X v v^* X^* - x_* x_*^*\|_{\text{F}}^2 + 2 \langle X v v^* X^* - x_* x_*^*, X X^* - x_* x_*^* \rangle \\ &\leq -\|X v v^* X^* - x_* x_*^*\|_{\text{F}}^2 + 2 \|X v v^* X^* - x_* x_*^*\|_{\text{F}} \|X X^* - x_* x_*^*\|_{\text{F}}. \end{aligned}$$

Hence

$$\|X v v^* X^* - x_* x_*^*\|_{\text{F}} \leq 2 \|X X^* - x_* x_*^*\|_{\text{F}}.$$

Furthermore,

$$\begin{aligned} \|X v v^* X^* - x_* x_*^*\|_{\text{F}}^2 &= \|X v\|^4 + \|x_*\|^4 - 2 \langle X v, x_* \rangle^2 \\ &\geq \frac{(\|X v\|^2 + \|x_*\|^2)^2}{2} - 2 \langle X v, x_* \rangle^2 \\ &= \frac{1}{2} \|X v + x_*\|^2 \|X v - x_*\|^2 \\ &\geq \frac{1}{2} \|x_*\|^2 \|x_* - X v\|^2. \end{aligned}$$

Combining this with the previous bound completes the proof. □

Next, the following bound will prove useful:

Lemma 8. *Consider the model (3). For any $\lambda \geq 0$, any X that is a first-order critical point of (13), or $X = T_{y,\lambda}U$ for any feasible U of (26), satisfies*

$$\frac{1}{n}\|\alpha(X)\|^2 + \lambda\|X\|^2 \leq \frac{1}{n}\|y\|^2 \leq \frac{1}{n}(\|\alpha(X_*)\| + \|\varepsilon\|)^2.$$

Proof. First, we consider a first-order critical point X of (13) (see Lemma 2). The condition $\nabla L_\lambda(XX^*)X = 0$ implies

$$\begin{aligned} 0 &= \langle \nabla L_\lambda(XX^*), XX^* \rangle \\ &= \frac{1}{n} \sum_{i=1}^n (\langle A_i, XX^* \rangle - y_i \langle A_i, XX^* \rangle^{1/2}) + \lambda\|X\|^2 \\ &= \frac{1}{n}\|\alpha(X)\|^2 - \frac{1}{n}\langle y, \alpha(X) \rangle + \lambda\|X\|^2. \end{aligned}$$

This implies

$$\frac{1}{n}\|\alpha(X)\|^2 + \lambda\|X\|^2 \leq \frac{1}{n}\|y\|\|\alpha(X)\| \leq \frac{1}{\sqrt{n}}\|y\|\sqrt{\frac{1}{n}\|\alpha(X)\|^2 + \lambda\|X\|^2},$$

from which the claimed bound follows.

Second, we consider $X = T_{y,\lambda}U$, where U is feasible for (26). The identity (29) implies

$$\begin{aligned} \frac{1}{n}\|\alpha(X)\|^2 + \lambda\|X\|^2 &= \frac{1}{n}\|FX\|^2 + \lambda\|X\|^2 \\ &= \frac{1}{n}\|\text{diag}(y)U\|^2 - \langle M_\lambda U, U \rangle \\ &\leq \frac{1}{n}\|y\|^2, \end{aligned}$$

where the inequality is due to $M_\lambda \succeq 0$. □

With these, we can prove the main finite-dimensional statistical result:

Proof of Theorem 5. Throughout the proof, we use c, c' , etc. to denote positive constants which may change from one usage to another but which do not depend on the problem parameters n and d (though, as discussed in the theorem statement, they may depend on properties of w).

Assuming $n \geq cd$, we can use Lemmas 4 and 5 (combining and simplifying the failure probabilities) to obtain, with probability at least $1 - cn^{-2}$,

$$c\|X'\|^2 \leq \frac{1}{n}\|\alpha(X')\|^2 \leq c'\|X'\|^2 \quad \text{for all } X' \in \mathbf{F}^{d \times r'}, \quad r' \geq 1, \quad (34)$$

and

$$\frac{1}{n}\|\beta(X) - \beta(x_*)\|_1 \geq c\|XX^* - x_*x_*^*\|_* \quad \text{for all } X \in \mathbf{F}^{d \times p}. \quad (35)$$

From now on, we assume this event holds.

We divide the rest of the proof into two cases depending on the norm of ε . First, in the (trivial) case that $\|\varepsilon\|^2 \geq n\|x_*\|^2$, for X as in the theorem statement, Lemma 8 and (34) give

$$\begin{aligned} \|XX^* - x_*x_*^*\|_* &\leq \|X\|^2 + \|x_*\|^2 \\ &\leq \frac{c}{n}\|\alpha(X)\|^2 + \|x_*\|^2 \\ &\leq \frac{c}{n}(\|\varepsilon\|^2 + \|\alpha(x_*)\|^2) + \|x_*\|^2 \\ &\leq c\left(\frac{\|\varepsilon\|^2}{n} + \|x_*\|^2\right) \\ &\leq c\frac{\|\varepsilon\|^2}{n}, \end{aligned}$$

and, for \hat{x} as in the theorem statement,

$$\begin{aligned}\|\hat{x} - x_*\| &\leq \|\hat{x}\| + \|x_*\| \\ &\leq \|X\| + \|x_*\| \\ &\leq c \frac{\|\varepsilon\|}{\sqrt{n}}.\end{aligned}$$

Thus, from now on, assume $\|\varepsilon\|^2 \leq n\|x_*\|^2 \leq c\|\alpha(x_*)\|^2$. Together with (again) Lemma 8 and (34), we obtain

$$\begin{aligned}\|\alpha(X) + \alpha(x_*)\| &\leq \|\alpha(X)\| + \|\alpha(x_*)\| \\ &\leq \|\varepsilon\| + 2\|\alpha(x_*)\| \\ &\leq c\sqrt{n}\|x_*\|.\end{aligned}$$

Together with this last bound, Lemma 6 and (35) give

$$\begin{aligned}\frac{1}{n}\|\alpha(X) - \alpha(x_*)\|^2 &\geq \frac{\|\beta(X) - \beta(x_*)\|_1^2}{n\|\alpha(X) + \alpha(x_*)\|^2} \\ &\geq \frac{c}{\|x_*\|^2} \|XX^* - x_*x_*^*\|_*^2.\end{aligned}$$

By Lemma 7, there is $v \in \mathbf{F}^p$ such that

$$\|XX^* - x_*x_*^*\|_*^2 \geq c\|x_*\|^2\|x_* - Xv\|^2.$$

We therefore have

$$\frac{1}{n}\|\alpha(X) - \alpha(x_*)\|^2 \geq \frac{c}{\|x_*\|^2} \|XX^* - x_*x_*^*\|_*^2 + c\|x_* - Xv\|^2. \quad (36)$$

To finish, we first consider the problem (13) and then make suitable modifications for the PhaseCut approach of (26). If X is a second-order critical point of (13), Theorem 2 gives

$$\begin{aligned}\frac{1}{n}\|\alpha(X) - \alpha(x_*)\|^2 &\leq \frac{2}{n}\langle \varepsilon, \alpha(X) - \alpha(x_*) \rangle + \frac{c_{\mathbf{F}}}{2p - c_{\mathbf{F}}} \cdot \frac{1}{n}\|\alpha(x_* - Xv)\|^2 \\ &\leq \frac{2}{n}\|\varepsilon\|\|\alpha(X) - \alpha(x_*)\| + \frac{c}{pn}\|\alpha(x_* - Xv)\|^2.\end{aligned}$$

Some algebra together with (34) then implies

$$\frac{1}{n}\|\alpha(X) - \alpha(x_*)\|^2 \leq c \frac{\|\varepsilon\|^2}{n} + \frac{c'}{pn}\|\alpha(x_* - Xv)\|^2 \leq c \frac{\|\varepsilon\|^2}{n} + \frac{c'}{p}\|x_* - Xv\|^2. \quad (37)$$

If $p \geq c$, combining (36) with (37) gives

$$\frac{\|XX^* - x_*x_*^*\|_*^2}{\|x_*\|^2} \leq c \frac{\|\varepsilon\|^2}{n},$$

which implies the claimed bound on $\|XX^* - x_*x_*^*\|_*$. The bound on $\min_{|s|=1} \|\hat{x} - sx_*\|$ also follows by Lemma 7.

Now, suppose $X = T_{y,\lambda}U$, where U is a second-order critical point of (26). Theorem 4 and some algebra give, similarly to (37),

$$\begin{aligned}\frac{1}{n}\|\alpha(X) - \alpha(x_*)\|^2 &\leq c \frac{\|\varepsilon\|^2}{n} + \frac{c'}{pn}\|F(X_\lambda - Xv)\|^2 \\ &\leq c \frac{\|\varepsilon\|^2}{n} + \frac{c'}{pn}\|F(x_* - Xv)\|^2 \\ &\leq c \frac{\|\varepsilon\|^2}{n} + \frac{c'}{p}\|x_* - Xv\|^2,\end{aligned}$$

where the second inequality uses (28) (with $\lambda = 0$), and v is chosen to be the same as in the previous case. This is identical (within constants) to (37); the result follows similarly. \square

We now turn to the proof of the infinite-dimensional result Theorem 6. Theorems 2 and 4 and the intermediate technical results Lemmas 6 and 8 adapt in obvious ways with \mathbf{F}^d replaced by \mathcal{H} . We will also use Lemmas 4, 5 and 7 again but only on a finite-dimensional subspace of \mathcal{H} , so no adaptation is necessary.

In addition, the following infinite-dimensional Gaussian counterpart to Lemma 4 will be useful:

Lemma 9 ([48, Cor. 2]). *Let z_1, \dots, z_n be i.i.d. Gaussian random vectors with covariance Σ in a Hilbert space \mathcal{H} . Let*

$$r(\Sigma) := \frac{\text{tr } \Sigma}{\|\Sigma\|_{\text{op}}}.$$

Let

$$\widehat{\Sigma} := \frac{1}{n} \sum_{i=1}^n z_i^{\otimes 2}$$

be the sample covariance. There is a universal constant $c > 0$ such that, for any $t \geq 1$, with probability at least $1 - e^{-t}$,

$$\|\widehat{\Sigma} - \Sigma\|_{\text{op}} \leq c\|\Sigma\|_{\text{op}} \left(\sqrt{\frac{r(\Sigma)}{n}} + \frac{r(\Sigma)}{n} + \sqrt{\frac{t}{n}} + \frac{t}{n} \right).$$

In particular, choosing $t = 2 \log n$, this implies, with probability at least $1 - n^{-2}$, for another universal constant $c' > 0$,

$$\|\widehat{\Sigma}\|_{\text{op}} \leq c' \left(\|\Sigma\|_{\text{op}} + \frac{\text{tr } \Sigma}{n} \right).$$

With this and the tools already presented for the proof of Theorem 5, we can proceed to the proof of our infinite-dimensional result:

Proof of Theorem 6. Let $u_1, \dots, u_d \in \mathcal{H}$ be the eigenvectors of Σ corresponding to eigenvalues $\sigma_1 \geq \dots \geq \sigma_d$. We set

$$G := \text{span}\{x_*, u_1, \dots, u_d\}.$$

Note that $d_G := \dim(G)$ is either d or $d + 1$. Let G^\perp be the orthogonal complement of G in \mathcal{H} . Denote by P_G and P_{G^\perp} the orthogonal projection operators onto (respectively) G and G^\perp .

Note that $\Sigma_G := P_G \Sigma P_G$ has rank at most d_G , and $P_G f \in \text{range}(\Sigma_G)$ almost surely. We then have that $\{\Sigma_G^{-1/2} P_G f_i\}_{i=1}^n$ are i.i.d. standard Gaussian (standard complex Gaussian if $\mathbf{F} = \mathbf{C}$ —recall that we required f to be circularly symmetric in this case) vectors on $\text{range}(\Sigma_G)$ with respect to any orthonormal (in L_2) basis. Assuming $n \geq c d_G$, Lemmas 4 and 5 imply that

$$c \Sigma_G \preceq \frac{1}{n} \sum_{i=1}^n (P_G f_i)^{\otimes 2} \preceq c' \Sigma_G$$

and, for all $X \in \mathcal{H}^p$,

$$\frac{1}{n} \|\beta(P_G X) - \beta(x_*)\|_1 \geq c \|P_G X X^* P_G - x_* x_*^*\|_{*, L_2}. \quad (38)$$

Furthermore, Lemma 9 (with covariance Σ_{G^\perp} ; see also the discussion following that lemma) implies

$$\left\| \frac{1}{n} \sum_{i=1}^n (P_{G^\perp} f_i)^{\otimes 2} \right\|_{\text{op}} \leq c \left(\|\Sigma_{G^\perp}\|_{\text{op}} + \frac{\text{tr } \Sigma_{G^\perp}}{n} \right) =: \delta. \quad (39)$$

Combining the failure probabilities of these events with a union bound, these inequalities hold with probability at least $1 - cn^{-2}$. From now on, assume these hold. In particular, for all $X' \in \mathcal{H}^{r'}$ (for any $r' \geq 1$), we have

$$c \|P_G X'\|_{L_2}^2 \leq \frac{1}{n} \|\alpha(P_G X')\|^2 \leq c' \|P_G X'\|_{L_2}^2, \quad (40)$$

and

$$\frac{1}{n} \|\alpha(P_{G^\perp} X')\|^2 \leq \delta \|P_{G^\perp}(X')\|^2 \leq \delta \|X'\|^2. \quad (41)$$

Noting that $\|\Sigma_{G^\perp}\|_{\text{op}} \leq \sigma_{d+1}$ and $\text{tr} \Sigma_{G^\perp} \leq \sum_{m>d} \sigma_m$, (39) allows us to have $\lambda \geq c\delta$. We can then derive the following simple inequality for any X satisfying the conditions of the theorem (we defer the proof to later):

$$\|X\|_{L_2}^2 \leq c \left(\|x_*\|_{L_2}^2 + \frac{\|\varepsilon\|^2}{n} \right). \quad (42)$$

We first consider the (trivial) case where $\|\varepsilon\|^2 \geq n\|x_*\|^2$. From (42) and the triangle inequality, we have

$$\begin{aligned} \|XX^* - x_*x_*^*\|_{*,L_2} &\leq \|X\|_{L_2}^2 + \|x_*\|_{L_2}^2 \\ &\leq c \frac{\|\varepsilon\|^2}{n}, \end{aligned}$$

and

$$\begin{aligned} \|\hat{x} - x_*\|_{L_2} &\leq \|X\|_{L_2} + \|x_*\|_{L_2} \\ &\leq c \frac{\|\varepsilon\|}{\sqrt{n}}. \end{aligned}$$

Thus from now on assume $\|\varepsilon\|^2 \leq n\|x_*\|_{L_2}^2$.

By (42) and related calculations (again, we defer these to later), we can then derive a number of useful inequalities:

$$\|X\|_{L_2} \leq c\|x_*\|_{L_2} \quad (43)$$

$$\|P_G X\|_{L_2} \leq c\|x_*\|_{L_2} \quad (44)$$

$$\|\alpha(P_G X) + \alpha(x_*)\| \leq c\sqrt{n}\|x_*\|_{L_2}. \quad (45)$$

Lemma 6, (38), (41), (45), and the triangle inequality imply

$$\begin{aligned} \frac{1}{\sqrt{n}}\|\alpha(X) - \alpha(x_*)\| &\geq \frac{1}{\sqrt{n}}\|\alpha(P_G X) - \alpha(x_*)\| - \frac{1}{\sqrt{n}}\|\alpha(X) - \alpha(P_G X)\| \\ &\geq \frac{\|\beta(P_G X) - \beta(x_*)\|_1}{\sqrt{n}\|\alpha(P_G X) + \alpha(x_*)\|} - \frac{1}{\sqrt{n}}\|\alpha(P_{G^\perp} X)\| \\ &\geq c \frac{\|P_G X X^* P_G - x_* x_*^*\|_{*,L_2}}{\|x_*\|_{L_2}} - \delta^{1/2}\|X\|. \end{aligned}$$

Furthermore, using (43), (44), we have

$$\begin{aligned} \|P_G X X^* P_G - x_* x_*^*\|_{*,L_2} &\geq \|X X^* - x_* x_*^*\|_{*,L_2} - \|P_{G^\perp} X X^*\|_{*,L_2} - \|P_G X X^* P_{G^\perp}\|_{*,L_2} \\ &\geq \|X X^* - x_* x_*^*\|_{*,L_2} - \|P_{G^\perp} X\|_{L_2} \|X\|_{L_2} - \|P_G X\|_{L_2} \|P_{G^\perp} X\|_{L_2} \\ &\geq \|X X^* - x_* x_*^*\|_{*,L_2} - c\|x_*\|_{L_2} \|\Sigma_{G^\perp}\|_{\text{op}}^{1/2} \|X\|. \end{aligned}$$

Next, Lemma 7 implies that, for some unit-norm $v \in \mathbf{F}^p$,

$$\|P_G X X^* P_G - x_* x_*^*\|_{*,L_2} \geq c\|x_*\|_{L_2} \|x_* - P_G X v\|_{L_2}.$$

Combining the previous three displays and the inequality $\delta \geq c\|\Sigma_{G^\perp}\|_{\text{op}}$ from (39), we have

$$\frac{1}{\sqrt{n}}\|\alpha(X) - \alpha(x_*)\| \geq \frac{c}{\|x_*\|_{L_2}} \|X X^* - x_* x_*^*\|_{*,L_2} + c'\|x_* - P_G X v\|_{L_2} - c''\delta^{1/2}\|X\|,$$

which implies

$$\frac{1}{n}\|\alpha(X) - \alpha(x_*)\|^2 \geq \frac{c}{\|x_*\|_{L_2}^2} \|X X^* - x_* x_*^*\|_{*,L_2}^2 + c'\|x_* - P_G X v\|_{L_2}^2 - c''\delta\|X\|^2. \quad (46)$$

We now turn to upper bounding $\|\alpha(X) - \alpha(x_*)\|^2$. We first consider the case where X is a second-order critical point of (13). Theorem 2 implies, for the same (unit-norm) v as above,

$$\begin{aligned} \frac{1}{n}\|\alpha(X) - \alpha(x_*)\|^2 &\leq \frac{2}{n} \langle \varepsilon, \alpha(X) - \alpha(x_*) \rangle + \lambda(\|x_*\|^2 - \|X\|^2) \\ &\quad + \frac{1}{c_{\mathbf{F}^p} - 1} \left(\frac{1}{n} \|\alpha(x_* - Xv)\|^2 + \lambda\|x_* - Xv\|^2 \right) \\ &\leq \frac{2}{n} \|\varepsilon\| \|\alpha(X) - \alpha(x_*)\| + \lambda(\|x_*\|^2 - \|X\|^2) \\ &\quad + \frac{c}{p} (\|x_* - P_G X v\|_{L_2}^2 + \delta\|Xv\|^2 + \lambda(\|x_*\|^2 + \|Xv\|^2)), \end{aligned}$$

where the second inequality used (40) and (41). This implies, recalling $\lambda \geq c\delta$,

$$\begin{aligned} \frac{1}{n} \|\alpha(X) - \alpha(x_*)\|^2 &\leq c \frac{\|\varepsilon\|^2}{n} + c' \lambda (\|x_*\|^2 - \|X\|^2) \\ &\quad + \frac{c''}{p} (\|x_* - P_G X\|_{L_2}^2 + \lambda (\|x_*\|^2 + \|X\|^2)). \end{aligned} \quad (47)$$

Together with (46), with $p \geq c$ and $\lambda \geq c\delta$, we obtain

$$\frac{1}{\|x_*\|_{L_2}^2} \|X X^* - x_* x_*^*\|_{*, L_2}^2 \leq c \frac{\|\varepsilon\|^2}{n} + c' \lambda \|x_*\|^2.$$

The result immediately follows (again Lemma 7 to bound $\min_{|s|=1} \|\hat{x} - s x_*\|_{L_2}$).

In the case where $X = T_{y,\lambda} U$ for a second-order critical point U of (26), similarly to the proof of Theorem 5 above, Theorem 4 and similar arguments as above give an inequality identical (within constants) to (47). The result follows similarly.

Finally, we prove the inequalities (42)–(45). By (40) and (41),

$$\begin{aligned} \frac{1}{\sqrt{n}} \|\alpha(X)\| &\geq \frac{1}{\sqrt{n}} \|\alpha(P_G X)\| - \frac{1}{\sqrt{n}} \|\alpha(P_{G^\perp} X)\| \\ &\geq c \|P_G X\|_{L_2} - \sqrt{\delta} \|X\| \\ &\geq c \|X\|_{L_2} - c' \sqrt{\delta} \|X\|, \end{aligned}$$

where the last inequality uses the fact that $\delta \geq c \|\Sigma_G\|_{\text{op}}$. Some algebra implies

$$\frac{1}{n} \|\alpha(X)\|^2 \geq c \|X\|_{L_2}^2 - c' \delta \|X\|^2 \quad \text{and} \quad \frac{1}{n} \|\alpha(X)\|^2 \geq c \|P_G X\|_{L_2}^2 - c' \delta \|X\|^2.$$

Together with Lemma 8 and $\lambda \geq c\delta$, this gives, for X as in the theorem statement,

$$\begin{aligned} \max\{\|X\|_{L_2}^2, \|P_G X\|_{L_2}^2\} &\leq \frac{c}{n} \|\alpha(X)\|^2 + c' \delta \|X\|^2 \\ &\leq c \left(\frac{1}{n} \|\alpha(X)\|^2 + \lambda \|X\|^2 \right) \\ &\leq \frac{c}{n} (\|\alpha(x_*)\|^2 + \|\varepsilon\|^2) \\ &\leq c \|x_*\|_{L_2}^2 + c' \frac{\|\varepsilon\|^2}{n}, \end{aligned}$$

where the last inequality again used (40). We thus have (42). In the case $\|\varepsilon\|^2 \leq n \|x_*\|_{L_2}^2$, (43) and (44) also follow immediately. Combining (44) with (40) gives (45). \square

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