

DATA-DRIVEN CONTROL OF CONTINUOUS-TIME SYSTEMS: A SYNTHESIS-OPERATOR APPROACH

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ABSTRACT. This paper addresses data-driven control of continuous-time systems. We develop a framework based on synthesis operators associated with input and state trajectories. A key advantage of the proposed method is that it does not require the state derivative and uses continuous-time data directly without sampling or filtering. First, systems compatible with given data are described by the synthesis operators into which data trajectories are embedded. Next, we characterize data informativity properties for system identification and for stabilization. Finally, we establish a necessary and sufficient condition for informativity for quadratic stabilization in the presence of process noise. This condition is formulated as linear matrix inequalities by exploiting the finite-rank structure of the synthesis operators.

1. INTRODUCTION

Motivation and literature review. Data-driven control has gained attention as an alternative paradigm to model-based control. Instead of identifying an explicit system model, data-driven approaches analyze an unknown system and construct controllers directly from measured trajectory data. One of the cornerstones in this field is the fundamental lemma by Willems *et. al.* [30], which essentially states that for linear time-invariant systems, all possible trajectories can be represented by finitely many trajectories if the input data are persistently exciting. This lemma has spurred various data-driven techniques for system analysis and controller design via system representations based on Hankel matrices of data trajectories; see, e.g., [1, 10, 15].

The notion of data informativity, introduced in [28], provides another theoretical framework for data-driven control. This framework examines whether the available data suffice to guarantee a specified property for every system compatible with the data, and whether controllers satisfying desired properties can be constructed from the data. Informativity for stabilization was characterized in [28], and subsequent studies [5, 14, 26] addressed noisy data. See the survey [27] for further references on data informativity. Much of the literature focused on discrete-time systems, reflecting the sampled nature of measurements. However, many physical systems evolve in continuous time. This paper investigates informativity for stabilization of continuous-time systems.

For data-driven control of continuous-time systems, it is assumed, e.g., in [4, 10, 12] that the state derivative is either measured directly or estimated from sampled state trajectories. In practice, however, accurate derivative information is often unavailable because measurements are corrupted by noise. To overcome this limitation, several derivative-free data-driven control methods were proposed. In [9, 24], discrete sequences were obtained from the integral version of the state equation. In [22], the input and state trajectories were transformed into discrete sequences using a polynomial orthogonal basis. The approach proposed in [19, 20] is based on sampled data obtained via linear functionals. Another line of work applied filtering to measured signals to avoid the state derivative; see [6, 7, 13, 21].

Contributions and comparisons. We develop a derivative-free approach for continuous-time systems by embedding data trajectories into synthesis operators. Synthesis operators have been extensively investigated in frame theory (see, e.g., [8]), and discrete synthesis operators were employed to study data-driven control of discrete-time infinite-dimensional systems in [29]. While synthesis operators are typically defined on L^2 -spaces, in this work we consider synthesis operators acting on H_0^1 -spaces. By applying integration by parts to the operator representation, we obtain a derivative-free formulation of continuous-time dynamics. The main contributions of this synthesis-operator approach are as follows.

- (i) We give a necessary and sufficient condition for a system to be compatible with given data, where synthesis operators play the same role as data matrices in the discrete-time setting. This condition allows for system analysis and controller design directly from data via synthesis operators.

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- (ii) We characterize data informativity properties for identification and for stabilization in terms of synthesis operators. We also provide equivalent statements involving matrix approximations of the synthesis operators.
- (iii) We establish a necessary and sufficient condition for data corrupted by process noise to be informative for quadratic stabilization. This condition is in the form of linear matrix inequalities (LMIs) without approximation, since products of synthesis operators and their adjoints have matrix representations.

The main advantage of this synthesis-operator approach, compared to the existing derivative-free methods mentioned above, is that it fully exploits continuous-time data without preprocessing such as sampling or filtering. Once irreversible processing that causes information loss is applied to continuous-time data, we can only address sufficient conditions for informativity of those data. In contrast, the synthesis-operator approach handles continuous-time data directly. Therefore, it provides a framework that allows us to study not only sufficiency but also necessity of informativity, in parallel with the discrete-time setting. Moreover, system analysis and controller design can be developed without parameters for sampling or filtering.

Organization. In Section 2, we introduce the notion of synthesis operators and use them to characterize systems that are compatible with the given data. In Sections 3 and 4, we investigate data informativity properties for system identification and for stabilization within the synthesis-operator framework, respectively. Section 5 establishes a necessary and sufficient LMI condition for noisy data to be informative for quadratic stabilization. Section 6 provides concluding remarks.

Notation. The inner product on the Euclidean space \mathbb{R}^n is denoted by $\langle v_1, v_2 \rangle_{\mathbb{R}^n}$ for $v_1, v_2 \in \mathbb{R}^n$. We denote by \mathbb{S}^n the set of $n \times n$ real symmetric matrices and by I_n the $n \times n$ identity matrix. The transpose and the Moore–Penrose pseudoinverse of a matrix A are denoted by A^\top and A^+ , respectively. If $A \in \mathbb{S}^n$ is positive definite (resp. nonnegative definite), then we write $A \succ 0$ (resp. $A \succeq 0$). Analogous notation applies to negative definite and nonpositive definite matrices.

Let $\tau > 0$. We denote by $L^2([0, \tau]; \mathbb{R}^n)$ the space of measurable functions $f: [0, \tau] \rightarrow \mathbb{R}^n$ satisfying $\int_0^\tau \|f(t)\|^2 dt < \infty$. The Sobolev space $H^1([0, \tau]; \mathbb{R}^n)$ consists of all absolutely continuous functions $\phi: [0, \tau] \rightarrow \mathbb{R}^n$ satisfying $\phi' \in L^2([0, \tau]; \mathbb{R}^n)$. We write $L^2[0, \tau] := L^2([0, \tau]; \mathbb{R})$ and $H^1[0, \tau] := H^1([0, \tau]; \mathbb{R})$. The space $H_0^1[0, \tau]$ consists of those functions in $H^1[0, \tau]$ that vanish at the endpoints 0 and τ . The inner product on $L^2[0, \tau]$ is

$$\langle f, g \rangle_{L^2} = \int_0^\tau f(t)g(t)dt$$

for $f, g \in L^2[0, \tau]$, and the inner product on $H_0^1[0, \tau]$ is

$$\langle \phi, \psi \rangle_{H_0^1} = \int_0^\tau \phi'(t)\psi'(t)dt$$

for $\phi, \psi \in H_0^1[0, \tau]$. For $\ell \in \mathbb{N}$, $PL_\ell[0, \tau]$ denotes the space of piecewise linear functions on $[0, \tau]$ that vanish at the endpoints 0 and τ , defined with respect to the uniform partition of $[0, \tau]$ into 2^ℓ subintervals of length $2^{-\ell}\tau$.

Let X and Y be Hilbert spaces. We denote by $\mathcal{L}(X, Y)$ the space of bounded linear operators from X to Y . Let $T \in \mathcal{L}(X, Y)$. The range and kernel of T are denoted by $\text{Ran } T$ and $\text{Ker } T$, respectively. The Hilbert space adjoint of T is denoted by T^* . Let E be a subset of X . We denote by $T|_E$ the restriction of T to E . The closure and the orthogonal complement of E are denoted by \overline{E} and E^\perp , respectively.

2. DATA AND SYNTHESIS OPERATORS

Fix $\tau > 0$ and assume that the state data $x \in H^1([0, \tau]; \mathbb{R}^n)$ and the input data $u \in L^2([0, \tau]; \mathbb{R}^m)$ are available. These data are denoted by (x, u, τ) . We assume that (x, u, τ) are generated by some continuous-time linear system.

Assumption 2.1. *Given data (x, u, τ) , there exist $A_s \in \mathbb{R}^{n \times n}$ and $B_s \in \mathbb{R}^{n \times m}$ such that*

$$(1) \quad x'(t) = A_s x(t) + B_s u(t)$$

for a.e. $t \in [0, \tau]$.

We can regard (1) as the true system generating the data (x, u, τ) . This paper considers the situation where A_s and B_s in Assumption 2.1 are unknown.

In the proposed approach, we embed the data into bounded linear operators on $H_0^1[0, \tau]$. These operators serve as continuous-time counterparts to the matrices whose columns are the data vectors in the discrete-time setting.

Definition 2.1. Let $f \in L^2([0, \tau]; \mathbb{R}^n)$. Define the operators $T, T_d \in \mathcal{L}(H_0^1[0, \tau], \mathbb{R}^n)$ by

$$T\phi := \int_0^\tau \phi(t)f(t)dt \quad \text{and} \quad T_d\phi := -\int_0^\tau \phi'(t)f(t)dt$$

for $\phi \in H_0^1[0, \tau]$. We call T the *synthesis operator associated with f* and T_d the *differentiated synthesis operator associated with f* .

Throughout this paper, we denote by Ξ and Υ the synthesis operators associated with the state data x and the input data u , respectively. We also denote by Ξ_d the differentiated synthesis operator associated with x .

Given data $\mathfrak{D} = (x, u, \tau)$, we define the set $\Sigma_{\mathfrak{D}}$ of systems by

$$\Sigma_{\mathfrak{D}} := \{(A, B) : x' = Ax + Bu \text{ a.e. on } [0, \tau]\}.$$

The set $\Sigma_{\mathfrak{D}}$ consists of systems compatible with the data \mathfrak{D} . In the following lemma, we characterize $\Sigma_{\mathfrak{D}}$ in terms of the synthesis operators associated with x and u .

Lemma 2.2. Let Ξ and Ξ_d be the synthesis operator and the differentiated synthesis operator associated with $x \in H^1[0, \tau]$, and let Υ be the synthesis operator associated with $u \in L^2[0, \tau]$. Let $\mathfrak{D} := (x, u, \tau)$. Then for all $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$, one has $(A, B) \in \Sigma_{\mathfrak{D}}$ if and only if $\Xi_d\phi = A\Xi\phi + B\Upsilon\phi$.

Proof. Let $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, and $\phi \in H_0^1[0, \tau]$. By integration by parts, $\Xi_d\phi = A\Xi\phi + B\Upsilon\phi$ if and only if

$$\int_0^\tau \phi(t)(x'(t) - Ax(t) - Bu(t))dt = 0.$$

The assertion follows from a standard property of test functions (see, e.g., [25, Proposition 13.2.2]). \square

3. DATA INFORMATIVITY FOR SYSTEM IDENTIFICATION

In this section, we investigate the following data property for system identification.

Definition 3.1. Under Assumption 2.1, the data $\mathfrak{D} = (x, u, \tau)$ are called *informative for system identification* if $\Sigma_{\mathfrak{D}} = \{(A_s, B_s)\}$.

The following result called Douglas' lemma relates range inclusion and factorization for operators on Hilbert spaces; see [11, Theorem 1] and [25, Proposition 12.1.2] for the proof.

Lemma 3.1. Let X , Y , and Z be Hilbert spaces. Let $T \in \mathcal{L}(Y, X)$ and $S \in \mathcal{L}(Z, X)$. Then $\text{Ran } T \subseteq \text{Ran } S$ if and only if there exists an operator $R \in \mathcal{L}(Y, Z)$ such that $T = SR$.

Using the piecewise linear approximation of smooth functions (see, e.g., [16, Proposition 1.2]), one can easily show that $\bigcup_{\ell \in \mathbb{N}} \text{PL}_\ell[0, \tau]$ is dense in $H_0^1[0, \tau]$. By definition, we also have that

$$(2) \quad \text{PL}_{\ell_1}[0, \tau] \subseteq \text{PL}_{\ell_2}[0, \tau] \quad \text{for all } \ell_1, \ell_2 \in \mathbb{N} \text{ with } \ell_1 \leq \ell_2.$$

Hence, for $T \in \mathcal{L}(H_0^1[0, \tau], \mathbb{R}^n)$, we expect the restriction $T|_{\text{PL}_\ell[0, \tau]}$ to provide a good approximation of T when ℓ is sufficiently large. The next lemma presents an approximation property in terms of operator ranges.

Lemma 3.2. For all $T \in \mathcal{L}(H_0^1[0, \tau], \mathbb{R}^n)$, there exists $\ell \in \mathbb{N}$ such that $\text{Ran } T = \text{Ran } T|_{\text{PL}_\ell[0, \tau]}$.

Proof. To show that $\text{Ran } T \subseteq \text{Ran } T|_{\text{PL}_\ell[0, \tau]}$, let $\{e_k : k = 1, \dots, m\}$ be an orthonormal basis of $\text{Ran } T$. For each $k = 1, \dots, m$, there exists $g_k \in H_0^1[0, \tau]$ such that $e_k = Tg_k$. By the density of $\bigcup_{\ell \in \mathbb{N}} \text{PL}_\ell[0, \tau]$ in $H_0^1[0, \tau]$ and (2), there exists $\ell \in \mathbb{N}$ such that, for every $k = 1, \dots, m$, $\text{PL}_\ell[0, \tau]$ contains an element h_k satisfying

$$\|g_k - h_k\|_{H_0^1} < \frac{1}{\sqrt{m}\|T\|}.$$

Then $\|e_k - Th_k\| < 1/\sqrt{m}$ for all $k = 1, \dots, m$. This implies that $\{Th_k : k = 1, \dots, m\}$ is also a basis of $\text{Ran } T$; see, e.g., [2, Exercise 6.B.6]. Hence, $\text{Ran } T \subseteq \text{Ran } T|_{\text{PL}_\ell[0, \tau]}$. Since the converse inclusion is trivial, we conclude that $\text{Ran } T = \text{Ran } T|_{\text{PL}_\ell[0, \tau]}$. \square

Let $\ell \in \mathbb{N}$. For $p = 1, \dots, 2^\ell - 1$, we define the hat function $\phi_p \in \text{PL}_\ell[0, \tau]$ by

$$\phi_p(t) := \begin{cases} \frac{2^\ell}{\tau}t - (p-1), & \frac{(p-1)\tau}{2^\ell} \leq t \leq \frac{p\tau}{2^\ell}, \\ -\frac{2^\ell}{\tau}t + (p+1), & \frac{p\tau}{2^\ell} < t \leq \frac{(p+1)\tau}{2^\ell}, \\ 0, & \text{otherwise.} \end{cases}$$

Then $(\phi_p)_{p=1}^{2^\ell-1}$ is a basis of $\text{PL}_\ell[0, \tau]$, and hence every $\phi \in \text{PL}_\ell[0, \tau]$ can be written as

$$(3) \quad \phi = \sum_{p=1}^{2^\ell-1} \mu_p \phi_p \quad \text{for some } \mu = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_{2^\ell-1} \end{bmatrix} \in \mathbb{R}^{2^\ell-1}.$$

Define the matrices $\Phi_\ell, \Phi_{d,\ell} \in \mathbb{R}^{n \times (2^\ell-1)}$ and $\Psi_\ell \in \mathbb{R}^{m \times (2^\ell-1)}$ by

$$(4) \quad \Phi_\ell := [\Xi \phi_1 \dots \Xi \phi_{2^\ell-1}],$$

$$(5) \quad \Phi_{d,\ell} := [\Xi_d \phi_1 \dots \Xi_d \phi_{2^\ell-1}],$$

$$(6) \quad \Psi_\ell := [\Upsilon \phi_1 \dots \Upsilon \phi_{2^\ell-1}].$$

Now we are in the position to characterize informativity for system identification. This result is a continuous-time analogue of [28, Proposition 6].

Proposition 3.3. *Suppose that the data (x, u, τ) satisfy Assumption 2.1. Let Ξ and Υ be the synthesis operators associated with x and u , respectively. Define the matrices Φ_ℓ and Ψ_ℓ by (4) and (6), respectively. Then the following statements are equivalent:*

(i) *The data (x, u, τ) are informative for system identification.*

$$\text{(ii)} \quad \text{Ran} \begin{bmatrix} \Xi \\ \Upsilon \end{bmatrix} = \mathbb{R}^{n+m}.$$

$$\text{(iii)} \quad \text{There exists } \ell \in \mathbb{N} \text{ such that } \text{Ran} \begin{bmatrix} \Phi_\ell \\ \Psi_\ell \end{bmatrix} = \mathbb{R}^{n+m}.$$

Proof. For simplicity, we write

$$\mathfrak{D} := (x, u, \tau), \quad V := \begin{bmatrix} \Xi \\ \Upsilon \end{bmatrix}, \quad \text{and} \quad M_\ell := \begin{bmatrix} \Phi_\ell \\ \Psi_\ell \end{bmatrix}, \quad \ell \in \mathbb{N}.$$

First, we prove the implication (i) \Rightarrow (ii). Assume, to get a contradiction, that $\text{Ran } V \neq \mathbb{R}^{n+m}$. Then there exist $\xi \in \mathbb{R}^n$ and $v \in \mathbb{R}^m$ such that

$$\begin{bmatrix} \xi \\ v \end{bmatrix} \in (\text{Ran } V)^\perp \setminus \{0\}.$$

Choosing $\zeta \in \mathbb{R}^n \setminus \{0\}$ arbitrarily, we define $A_0 := \zeta \xi^\top \in \mathbb{R}^{n \times n}$ and $B_0 := \zeta v^\top \in \mathbb{R}^{n \times m}$. Then $(A_s + A_0, B_s + B_0) \neq (A_s, B_s)$. Since

$$A_0 \Xi \phi + B_0 \Upsilon \phi = \zeta [\xi^\top \ v^\top] V \phi = 0$$

for all $\phi \in H_0^1[0, \tau]$, we obtain $(A_s + A_0, B_s + B_0) \in \Sigma_{\mathfrak{D}}$ by Lemma 2.2. This is a contradiction.

Next, we prove the implication (ii) \Rightarrow (i). By Lemma 3.1 (Douglas' lemma) with $T = I_{n+m}$, there exists a right inverse V_R^{-1} of V in $\mathcal{L}(\mathbb{R}^{n+m}, H_0^1[0, \tau])$. Therefore, Lemma 2.2 shows that for all $(A, B) \in \Sigma_{\mathfrak{D}}$,

$$[A \ B] = [A \ B] V V_R^{-1} = \Xi_d V_R^{-1}.$$

Since $(A, B) \in \Sigma_{\mathfrak{D}}$ is uniquely determined, we conclude that (x, u, τ) are informative for system identification.

Finally, we prove the equivalence (ii) \Leftrightarrow (iii). Let $\ell \in \mathbb{N}$ and let $\phi \in \text{PL}_\ell[0, \tau]$ be as in (3). Then $V\phi = M_\ell \mu$. This implies that the ranges of $V|_{\text{PL}_\ell[0, \tau]}$ and M_ℓ coincide. The equivalence (ii) \Leftrightarrow (iii) follows from Lemma 3.2. \square

4. DATA INFORMATIVITY FOR STABILIZATION

We introduce a data property, which implies that all systems compatible with the given data can be stabilized by a single feedback gain.

Definition 4.1. The data $\mathfrak{D} = (x, u, \tau)$ are called *informative for stabilization* if there exists $K \in \mathbb{R}^{m \times n}$ such that $A + BK$ is Hurwitz for all $(A, B) \in \Sigma_{\mathfrak{D}}$.

The aim of this section is to provide the following characterization of informativity for stabilization. The discrete-time case was established in [28, Theorem 16].

Theorem 4.1. Suppose that the data (x, u, τ) satisfy Assumption 2.1. Let Ξ and Ξ_d be the synthesis operator and the differentiated synthesis operator associated with x , respectively. Let Υ be the synthesis operator associated with u . Define the matrices Φ_ℓ and $\Phi_{d,\ell}$ by (4) and (5), respectively. Then the following statements are equivalent:

- (i) The data (x, u, τ) are informative for stabilization.
- (ii) $\text{Ran } \Xi = \mathbb{R}^n$, and there exists a right inverse Ξ_R^{-1} of Ξ in $\mathcal{L}(\mathbb{R}^n, H_0^1[0, \tau])$ such that $\Xi_d \Xi_R^{-1} \in \mathbb{R}^{n \times n}$ is Hurwitz.
- (iii) There exists $\ell \in \mathbb{N}$ such that $\text{Ran } \Phi_\ell = \mathbb{R}^n$ and Φ_ℓ has a right inverse $(\Phi_\ell)_R^{-1} \in \mathbb{R}^{(2^\ell-1) \times n}$ for which $\Phi_{d,\ell} (\Phi_\ell)_R^{-1} \in \mathbb{R}^{n \times n}$ is Hurwitz.

An LMI-based characterization of informativity for stabilization will also be obtained. See Proposition 5.4 and Theorem 5.5 for details.

The following lemma is a continuous-time analogue of [28, Lemma 15]; see also [22, Lemma 1]. This result will be used in the proof of the equivalence (i) \Leftrightarrow (ii) in Theorem 4.1. For a square matrix M , we denote by $s_{\max}(M)$ and $s_{\min}(M)$ the maximum and minimum of the real parts of the eigenvalues of M .

Lemma 4.2. Suppose that the data $\mathfrak{D} = (x, u, \tau)$ satisfy Assumption 2.1. Let Ξ and Υ be the synthesis operators associated with x and u , respectively. If $K \in \mathbb{R}^{m \times n}$ satisfies

$$(7) \quad \sup_{(A,B) \in \Sigma_{\mathfrak{D}}} s_{\max}(A + BK) < \infty,$$

then

$$\text{Ran} \begin{bmatrix} I_n \\ K \end{bmatrix} \subseteq \text{Ran} \begin{bmatrix} \Xi \\ \Upsilon \end{bmatrix}.$$

Proof. Define the set $\Sigma_{\mathfrak{D}}^0$ of systems by

$$\Sigma_{\mathfrak{D}}^0 := \{(A, B) : 0 = Ax + Bu \text{ a.e. on } [0, \tau]\}.$$

Using the same argument as in Lemma 2.2, we have that $(A, B) \in \Sigma_{\mathfrak{D}}^0$ is equivalent to $0 = A\Xi + B\Upsilon$.

We will prove that

$$(8) \quad A_0 + B_0 K = 0 \quad \text{for all } (A_0, B_0) \in \Sigma_{\mathfrak{D}}^0.$$

Let $(A, B) \in \Sigma_{\mathfrak{D}}$ and $(A_0, B_0) \in \Sigma_{\mathfrak{D}}^0$. Define $F := A + BK$ and $F_0 := A_0 + B_0K$. For all $\rho \in \mathbb{R}$, we obtain

$$(A + \rho A_0, B + \rho B_0) \in \Sigma_{\mathfrak{D}}.$$

Hence, by (7), there exists $C > 0$ such that for all $\rho \in \mathbb{R}$,

$$s_{\max}(F + \rho F_0) \leq C.$$

Therefore, $s_{\max}(F/\rho + F_0) \leq C/\rho$ if $\rho > 0$. Taking $\rho \rightarrow \infty$, we obtain $s_{\max}(F_0) \leq 0$. Similarly, since $s_{\min}(F/\rho + F_0) \geq C/\rho$ if $\rho < 0$, we obtain $s_{\min}(F_0) \geq 0$. Hence, F_0 has the eigenvalues only on $i\mathbb{R}$. Since $(F_0^\top A_0, F_0^\top B_0) \in \Sigma_{\mathfrak{D}}^0$, the eigenvalues of $F_0^\top F_0$ are also only on $i\mathbb{R}$. The spectral radius of $F_0^\top F_0$ is zero, and consequently $F_0 = 0$. Thus, (8) is proved.

Next, we will prove that

$$(9) \quad \text{Ker} \begin{bmatrix} \Xi^* & \Upsilon^* \end{bmatrix} \subseteq \text{Ker} \begin{bmatrix} I_n & K^\top \end{bmatrix}.$$

Take

$$\begin{bmatrix} \xi_0 \\ v_0 \end{bmatrix} \in \text{Ker} \begin{bmatrix} \Xi^* & \Upsilon^* \end{bmatrix}.$$

Then $\Xi^* \xi_0 + \Upsilon^* v_0 = 0$. Let $\zeta \in \mathbb{R}^n \setminus \{0\}$, and define $A_0 := \zeta \xi_0^\top \in \mathbb{R}^{n \times n}$ and $B_0 := \zeta v_0^\top \in \mathbb{R}^{n \times m}$. For all $\phi \in H_0^1[0, \tau]$,

$$A_0 \Xi \phi + B_0 \Upsilon \phi = \langle \Xi^* \xi_0 + \Upsilon^* v_0, \phi \rangle_{H_0^1} \zeta = 0.$$

Combining this with (8) implies that $A_0 + B_0 K = 0$. Then

$$0 = \zeta^\top (\zeta \xi_0^\top + \zeta v_0^\top K) = \|\zeta\|^2 \left(\begin{bmatrix} I_n & K^\top \end{bmatrix} \begin{bmatrix} \xi_0 \\ v_0 \end{bmatrix} \right)^\top.$$

Since $\zeta \neq 0$, it follows that

$$\begin{bmatrix} \xi_0 \\ v_0 \end{bmatrix} \in \text{Ker} \begin{bmatrix} I_n & K^\top \end{bmatrix}.$$

Hence, the inclusion (9) is obtained.

By (9), we obtain

$$\text{Ran} \begin{bmatrix} I_n \\ K \end{bmatrix} = ([I_n \ K^\top])^\perp \subseteq (\text{Ker} [\Xi^* \ \Upsilon^*])^\perp = \overline{\text{Ran} \begin{bmatrix} \Xi \\ \Upsilon \end{bmatrix}}.$$

Since $\begin{bmatrix} \Xi \\ \Upsilon \end{bmatrix}$ is a finite-rank operator, its range is closed. Thus, the assertion is proved. \square

The next lemma relates right inverses of the matrix Φ_ℓ and the restriction of Ξ to $\text{PL}_\ell[0, \tau]$. It is instrumental in establishing the equivalence (ii) \Leftrightarrow (iii) in Theorem 4.1.

Lemma 4.3. *Let Ξ and Ξ_d be the synthesis operator and the differentiated synthesis operator associated with $x \in L^2([0, \tau]; \mathbb{R}^n)$, respectively. Let $\ell \in \mathbb{N}$ and $\Xi_\ell := \Xi|_{\text{PL}_\ell[0, \tau]}$. Define the matrices Φ_ℓ and $\Phi_{d,\ell}$ by (4) and (5), respectively. Then the following statements are equivalent:*

- (i) $\text{Ran } \Phi_\ell = \mathbb{R}^n$, and there exists a right inverse $(\Phi_\ell)_R^{-1}$ of Φ_ℓ in $\mathbb{R}^{(2^\ell-1) \times n}$ such that $\Phi_{d,\ell}(\Phi_\ell)_R^{-1} \in \mathbb{R}^{n \times n}$ is Hurwitz.
- (ii) $\text{Ran } \Xi_\ell = \mathbb{R}^n$, and there exists a right inverse $(\Xi_\ell)_R^{-1}$ of Ξ_ℓ in $\mathcal{L}(\mathbb{R}^n, \text{PL}_\ell[0, \tau])$ such that $\Xi_d(\Xi_\ell)_R^{-1} \in \mathbb{R}^{n \times n}$ is Hurwitz.

Proof. Let $\phi \in \text{PL}_\ell[0, \tau]$ be as in (3). Then $\Xi_\ell \phi = \Phi_\ell \mu$. Therefore, the ranges of Ξ_ℓ and Φ_ℓ coincide.

First, suppose that $(\Phi_\ell)_R^{-1} \in \mathbb{R}^{(2^\ell-1) \times n}$ is a right inverse of Φ_ℓ such that $\Phi_{d,\ell}(\Phi_\ell)_R^{-1}$ is Hurwitz. Then

$$\Xi_\ell \sum_{p=1}^{2^\ell-1} ((\Phi_\ell)_R^{-1} v)_p \phi_p = \Phi_\ell (\Phi_\ell)_R^{-1} v = v$$

for all $v \in \mathbb{R}^n$, where $((\Phi_\ell)_R^{-1} v)_p$ be the p -th element of $(\Phi_\ell)_R^{-1} v$. This implies that the linear operator from \mathbb{R}^n to $\text{PL}_\ell[0, \tau]$ defined by

$$(10) \quad v \mapsto \sum_{p=1}^{2^\ell-1} ((\Phi_\ell)_R^{-1} v)_p \phi_p$$

is a bounded right inverse of Ξ_ℓ , which is denoted by $(\Xi_\ell)_R^{-1}$. Since

$$(11) \quad \Xi_d(\Xi_\ell)_R^{-1} v = \Phi_{d,\ell}(\Phi_\ell)_R^{-1} v$$

for all $v \in \mathbb{R}^n$, we conclude that $\Xi_d(\Xi_\ell)_R^{-1}$ is Hurwitz.

Conversely, suppose that $(\Xi_\ell)_R^{-1} \in \mathcal{L}(\mathbb{R}^n, \text{PL}_\ell[0, \tau])$ is a right inverse of Ξ_ℓ such that $\Xi_d(\Xi_\ell)_R^{-1}$ is Hurwitz. Since the range of $(\Xi_\ell)_R^{-1}$ is contained in $\text{PL}_\ell[0, \tau]$, each unit vector e_k in \mathbb{R}^n satisfies

$$(12) \quad (\Xi_\ell)_R^{-1} e_k = \sum_{p=1}^{2^\ell-1} \mu_{p,k} \phi_p$$

for some $\mu_k = [\mu_{1,k} \ \cdots \ \mu_{2^\ell-1,k}]^\top \in \mathbb{R}^{2^\ell-1}$. Then

$$e_k = \Xi_\ell(\Xi_\ell)_R^{-1} e_k = \sum_{p=1}^{2^\ell-1} \mu_{p,k} \Xi_\ell \phi_p = \Phi_\ell \mu_k$$

for all $k = 1, \dots, n$. This implies that the $(2^\ell - 1) \times n$ matrix with (p, k) -entry $\mu_{p,k}$ is a right inverse of Φ_ℓ , and we denote it by $(\Phi_\ell)_R^{-1}$. For all $v = [v_1 \ \cdots \ v_n]^\top \in \mathbb{R}^n$, we obtain

$$\Phi_{d,\ell}(\Phi_\ell)_R^{-1} v = \Xi_d \sum_{k=1}^n v_k \sum_{p=1}^{2^\ell-1} \mu_{p,k} \phi_p.$$

Combining this with (12), we derive $\Phi_{d,\ell}(\Phi_\ell)_R^{-1} = \Xi_d(\Xi_\ell)_R^{-1}$. Thus, $\Phi_{d,\ell}(\Phi_\ell)_R^{-1}$ is Hurwitz. \square

We are now ready to prove the main result of this section.

Proof of Theorem 4.1. Let $\mathfrak{D} := (x, u, \tau)$. First, we prove the implication (i) \Rightarrow (ii). Let $K \in \mathbb{R}^{m \times n}$ be such that $A + BK$ is Hurwitz for all $(A, B) \in \Sigma_{\mathfrak{D}}$. Combining Lemma 4.2 with Lemma 3.1 (Douglas' lemma), we obtain

$$\begin{bmatrix} I_n \\ K \end{bmatrix} = \begin{bmatrix} \Xi \\ \Upsilon \end{bmatrix} \Xi_0$$

for some $\Xi_0 \in \mathcal{L}(\mathbb{R}^n, H_0^1[0, \tau])$. Then $\text{Ran } \Xi = \mathbb{R}^n$, and Ξ_0 is a right inverse of Ξ . Since Lemma 2.2 yields $\Xi_d \Xi_0 = A + BK$ for all $(A, B) \in \Sigma_{\mathfrak{D}}$, it follows that $\Xi_d \Xi_0$ is also Hurwitz.

Next, we prove the implication (ii) \Rightarrow (i). Define $K := \Upsilon \Xi_R^{-1}$. By Lemma 2.2, we have that $\Xi_d \Xi_R^{-1} = A + BK$ for all $(A, B) \in \Sigma_{\mathfrak{D}}$. Therefore, the data (x, u, τ) are informative for stabilization.

The implication (iii) \Rightarrow (ii) follows immediately from Lemma 4.3. Finally, we prove the implication (ii) \Rightarrow (iii). By Lemma 4.3, it is enough to show that there exists $\ell \in \mathbb{N}$ such that $\text{Ran } \Xi_\ell = \mathbb{R}^n$ and Ξ_ℓ has a right inverse $(\Xi_\ell)_R^{-1} \in \mathcal{L}(\mathbb{R}^n, PL_\ell[0, \tau])$ for which $\Xi_d(\Xi_\ell)_R^{-1} \in \mathbb{R}^{n \times n}$ is Hurwitz.

Let $\varepsilon > 0$. For $k = 1, \dots, n$, we define $g_k := \Xi_R^{-1} e_k \in H_0^1[0, \tau]$, where e_k is the k -th unit vector in \mathbb{R}^n . Since $\bigcup_{\ell \in \mathbb{N}} PL_\ell[0, \tau]$ is dense in $H_0^1[0, \tau]$ and since $PL_\ell[0, \tau]$ has the monotone property (2) with respect to $\ell \in \mathbb{N}$, there exists $\ell \in \mathbb{N}$ such that, for every $k = 1, \dots, n$, $PL_\ell[0, \tau]$ contains an element h_k satisfying $\|g_k - h_k\|_{H_0^1} \leq \varepsilon$. Define $Q \in \mathcal{L}(\mathbb{R}^n, PL_\ell[0, \tau])$ by

$$Qv := \sum_{k=1}^n v_k h_k$$

for $v = [v_1 \ \cdots \ v_n]^\top \in \mathbb{R}^n$. Then $\Xi Q \in \mathbb{R}^{n \times n}$ satisfies

$$e_k - \Xi Q e_k = \Xi(g_k - h_k)$$

for all $k = 1, \dots, n$. This implies that

$$(13) \quad \|I_n - \Xi Q\| \leq \varepsilon \sqrt{n} \|\Xi\|.$$

Assume that $\varepsilon \leq 1/(2\sqrt{n}\|\Xi\|)$. By (13), ΞQ is invertible, and

$$(14) \quad \|(\Xi Q)^{-1}\| \leq 2.$$

Since $\Xi_\ell Q(\Xi Q)^{-1} = I_n$, it follows that $\text{Ran } \Xi_\ell = \mathbb{R}^n$. We also have that $(\Xi_\ell)_R^{-1} := Q(\Xi Q)^{-1}$ is a right inverse of Ξ_ℓ in $\mathcal{L}(\mathbb{R}^n, PL_\ell[0, \tau])$.

It remains to show that $\Xi_d(\Xi_\ell)_R^{-1}$ is Hurwitz whenever $\varepsilon > 0$ is sufficiently small. We obtain

$$(15) \quad \Xi_R^{-1} - (\Xi_\ell)_R^{-1} = (\Xi_R^{-1} - Q) + Q(\Xi Q - I_n)(\Xi Q)^{-1}.$$

Since $(\Xi_R^{-1} - Q)e_k = g_k - h_k$, it follows that

$$(16) \quad \|\Xi_R^{-1} - Q\| \leq \varepsilon \sqrt{n}.$$

Combining (15) with the estimates (13), (14), and (16), we derive

$$(17) \quad \|\Xi_R^{-1} - (\Xi_\ell)_R^{-1}\| \leq \varepsilon \sqrt{n} + 2\varepsilon \sqrt{n} \|\Xi\| (\|\Xi_R^{-1}\| + \varepsilon \sqrt{n}).$$

Since $\Xi_d \Xi_R^{-1}$ is Hurwitz, (17) implies that $\Xi_d(\Xi_\ell)_R^{-1}$ is also Hurwitz whenever $\varepsilon > 0$ is sufficiently small. ■

The proofs of Theorem 4.1 and Lemma 4.3 also provide the following design method for stabilizing gains.

Remark 4.2 (Design of stabilizing gains). Choose a right inverse Ξ_R^{-1} of Ξ in $\mathcal{L}(\mathbb{R}^n, H_0^1[0, \tau])$ such that $\Xi_d \Xi_R^{-1}$ is Hurwitz, and define $K := \Upsilon \Xi_R^{-1} \in \mathbb{R}^{m \times n}$. Then $A + BK$ is also Hurwitz for all $(A, B) \in \Sigma_{\mathfrak{D}}$. Conversely, if $K \in \mathbb{R}^{m \times n}$ is such that $A + BK$ is Hurwitz for all $(A, B) \in \Sigma_{\mathfrak{D}}$, then there exists a right inverse Ξ_R^{-1} of Ξ in $\mathcal{L}(\mathbb{R}^n, H_0^1[0, \tau])$ such that $K = \Upsilon \Xi_R^{-1}$. Next, suppose that there exists a right inverse $(\Phi_\ell)_R^{-1} \in \mathbb{R}^{(2^\ell-1) \times n}$ of Φ_ℓ such that $\Phi_{d,\ell}(\Phi_\ell)_R^{-1}$ is Hurwitz. Define $(\Xi_\ell)_R^{-1}$ by (10), which is a right inverse of Ξ_ℓ in $\mathcal{L}(\mathbb{R}^n, PL_\ell[0, \tau])$. Then (11) implies that $\Xi_d(\Xi_\ell)_R^{-1}$ is also Hurwitz. Hence, defining $K \in \mathbb{R}^{m \times n}$ by

$$K := \Upsilon(\Xi_\ell)_R^{-1} = \Psi_\ell(\Phi_\ell)_R^{-1},$$

where $\Psi_\ell \in \mathbb{R}^{m \times (2^\ell-1)}$ is as in (6), we have that $A + BK$ is Hurwitz for all $(A, B) \in \Sigma_{\mathfrak{D}}$. △

5. DATA INFORMATIVITY FOR QUADRATIC STABILIZATION IN THE PRESENCE OF NOISE

This section addresses data-driven stabilization in the presence of process noise. We begin by deriving integral representations for products of synthesis operators and their adjoints. We also give an interpretation of the norm of the adjoint. Next, we introduce the class of noise we consider, and then present a necessary and sufficient condition for noisy data to be informative for quadratic stabilization. Finally, we illustrate the result with a numerical example.

5.1. Adjoints of synthesis operators.

5.1.1. Integral representations. Since the synthesis operators T, T_d introduced in Definition 2.1 are of finite rank, products such as $T_d T_d^*$ and $T_d T^*$ have matrix representations. Here we give explicit formulae for these products. For $\tau > 0$, define $G: [0, \tau] \times [0, \tau] \rightarrow \mathbb{R}$ by

$$(18) \quad G(t, s) := \begin{cases} \frac{t(\tau-s)}{\tau}, & t \leq s, \\ \frac{s(\tau-t)}{\tau}, & t > s. \end{cases}$$

Lemma 5.1. *Let T and T_d be the synthesis operator and the differentiated synthesis operator associated with $f \in L^2([0, \tau]; \mathbb{R}^n)$, respectively. Define $G: [0, \tau] \times [0, \tau] \rightarrow \mathbb{R}$ by (18). Then the following statements hold:*

- a) *The adjoint operators $T^*, T_d^* \in \mathcal{L}(\mathbb{R}^n, H_0^1[0, \tau])$ are given by*

$$(19) \quad (T^* v)(t) = \left(\int_0^\tau G(t, s) f(s) ds \right)^\top v,$$

$$(20) \quad (T_d^* v)(t) = \left(- \int_0^t f(s) ds + \frac{t}{\tau} \int_0^\tau f(s) ds \right)^\top v$$

for all $v \in \mathbb{R}^n$ and $t \in [0, \tau]$.

- b) *The matrix $T_d T_d^* \in \mathbb{R}^{n \times n}$ is given by*

$$(21) \quad T_d T_d^* = \int_0^\tau f(t) f(t)^\top dt - \frac{1}{\tau} \int_0^\tau f(t) dt \int_0^\tau f(t)^\top dt.$$

- c) *Let S be the synthesis operator associated with $g \in L^2([0, \tau]; \mathbb{R}^m)$. Then the matrices $ST^*, ST_d^* \in \mathbb{R}^{m \times n}$ are given by*

$$(22) \quad ST^* = \int_0^\tau \int_0^\tau G(t, s) g(t) f(s)^\top ds dt,$$

$$(23) \quad ST_d^* = - \int_0^\tau \int_0^\tau \frac{\partial G}{\partial s}(t, s) g(t) f(s)^\top ds dt.$$

Proof. a) Let $\phi \in H_0^1[0, \tau]$ and $v \in \mathbb{R}^n$. Integration by parts yields

$$\langle T\phi, v \rangle_{\mathbb{R}^n} = - \int_0^\tau \phi'(t) \int_0^t f(s)^\top v ds dt.$$

We also have that by the definition of adjoints,

$$\langle T\phi, v \rangle_{\mathbb{R}^n} = \langle \phi, T^* v \rangle_{H_0^1} = \int_0^\tau \phi'(t) (T^* v)'(t) dt.$$

Therefore,

$$\int_0^\tau \phi'(t) \left((T^* v)'(t) + \int_0^t f(s)^\top v ds \right) dt = 0.$$

Since

$$\{\phi' : \phi \in H_0^1[0, \tau]\}^\perp = \{\psi \in L^2[0, \tau] : \psi \equiv C \text{ for some } C \in \mathbb{R}\},$$

there exists $C \in \mathbb{R}$ such that

$$(24) \quad (T^* v)'(t) = - \int_0^t f(s)^\top v ds + C$$

for a.e. $t \in [0, \tau]$. The fundamental theorem of calculus and Fubini's theorem show that

$$(T^*v)(t) - (T^*v)(0) = - \int_0^t (t-s)f(s)^\top v ds + Ct$$

for all $t \in [0, \tau]$. Recalling that $T^*v \in H_0^1[0, \tau]$, we obtain

$$C = \frac{1}{\tau} \int_0^\tau (\tau-s)f(s)^\top v ds.$$

Since

$$-(t-s)\chi_{[0,t]}(s) + \frac{t(\tau-s)}{\tau} = G(t,s)$$

for all $t, s \in [0, \tau]$, where $\chi_{[0,t]}$ denotes the characteristic function of the interval $[0, t]$, we conclude that (19) holds for all $t \in [0, \tau]$. The same argument shows that

$$(25) \quad (T_d^*v)'(t) = -f(t)^\top v + \frac{1}{\tau} \int_0^\tau f(s)^\top v ds$$

for a.e. $t \in [0, \tau]$. Thus, (20) also holds for all $t \in [0, \tau]$.

b) By (25), we obtain

$$T_d T_d^* v = - \int_0^\tau \left(-f(t)^\top v + \frac{1}{\tau} \int_0^\tau f(s)^\top v ds \right) f(t) dt$$

for all $v \in \mathbb{R}^n$. Therefore, (21) holds.

c) The first formula (22) follows immediately from (19). By (20),

$$(26) \quad ST_d^* v = \int_0^\tau \left(- \int_0^t f(s)^\top v ds + \frac{t}{\tau} \int_0^\tau f(s)^\top v ds \right) g(t) dt$$

for all $v \in \mathbb{R}^n$. Using the characteristic function $\chi_{[0,t]}$, we have that

$$-\chi_{[0,t]}(s) + \frac{t}{\tau} = -\frac{\partial G}{\partial s}(t, s)$$

for all $t, s \in [0, \tau]$. Hence, the right-hand integral of (26) can be written as

$$-\left(\int_0^\tau \int_0^\tau \frac{\partial G}{\partial s}(t, s) g(t) f(s)^\top ds dt \right) v.$$

Thus, we derive (23). \square

5.1.2. Norm. In Section 5.2, we consider the constraint of the form $TT^* \preceq cI_n$ for some constant $c \geq 0$, where T is the synthesis operator associated with $f \in L^2([0, \tau]; \mathbb{R}^n)$. This constraint is equivalent to $\|T^*\| \leq \sqrt{c}$. To see what this constraint implies, consider $h := T^*v$ for some $v \in \mathbb{R}^n$. By (24), h is the solution of the boundary value problem

$$(27) \quad h''(t) = -f(t)^\top v, \quad t \in (0, \tau); \quad h(0) = 0 \text{ and } h(\tau) = 0.$$

Then we can regard $\|T^*v\|_{H_0^1} = \|h'\|_{L^2}$ as the (potential) energy of the response h driven by $-f(\cdot)^\top v$.

To derive the norm $\|T^*\|$ numerically, we define the orthonormal basis $(\psi_j)_{j \in \mathbb{N}}$ in $L^2[0, \tau]$ by

$$\psi_j(t) := \sqrt{\frac{2}{\tau}} \sin\left(\frac{j\pi}{\tau}t\right)$$

for $t \in [0, \tau]$ and $j \in \mathbb{N}$. Let the (j, k) -entry of $\Gamma_N \in \mathbb{R}^{N \times n}$ be defined by $\langle f_k, \psi_j \rangle_{L^2}/j$, where f_k is the k -th element of f . Since the j -th row of Γ_N is multiplied by $1/j$, we see from the following result that low-frequency components of f have a larger effect on $\|T^*\|$ than high-frequency ones.

Proposition 5.2. *Let T be the synthesis operator associated with $f \in L^2([0, \tau]; \mathbb{R}^n)$. For $N \in \mathbb{N}$, define the matrix $\Gamma_N \in \mathbb{R}^{N \times n}$ as above. Then*

$$(28) \quad \|T^*\| = \frac{\tau}{\pi} \lim_{N \rightarrow \infty} \|\Gamma_N\|.$$

Proof. Let $v \in \mathbb{R}^n$ be arbitrary, and let $\ell^2(\mathbb{N})$ be the space of square-summable sequences $(\eta_j)_{j \in \mathbb{N}}$ of real numbers, equipped with the standard inner product $\langle \cdot, \cdot \rangle_{\ell^2}$. A simple calculation shows that the solution h of the boundary value problem (27) is given by

$$h(t) = \frac{\tau^2}{\pi^2} \sum_{j=1}^{\infty} \frac{\langle f(\cdot)^\top v, \psi_j \rangle_{L^2}}{j^2} \psi_j(t)$$

for $t \in [0, \tau]$. Define $\Gamma: \mathbb{R}^n \rightarrow \ell^2(\mathbb{N})$ by

$$\Gamma v := \left(\frac{1}{j} \sum_{k=1}^n \langle f_k, \psi_j \rangle_{L^2} v_k \right)_{j \in \mathbb{N}}$$

for $v = [v_1 \ \cdots \ v_n]^\top \in \mathbb{R}^n$. By the orthogonal property of the cosine family, we obtain

$$(29) \quad \|T^*v\| = \|h'\|_{L^2} = \frac{\tau}{\pi} \|\Gamma v\|_{\ell^2}.$$

For $N \in \mathbb{N}$, define the orthogonal projection Π_N on $\ell^2(\mathbb{N})$ by

$$\Pi_N \eta := (\eta_1, \eta_2, \dots, \eta_N, 0, 0, \dots), \quad \eta = (\eta_j)_{j \in \mathbb{N}} \in \ell^2(\mathbb{N}).$$

Since Γ is a finite-rank operator, it follows that $\|\Gamma - \Pi_N \Gamma\| \rightarrow 0$ as $N \rightarrow \infty$. This and (29) yield (28). \square

5.2. Noisy data. In addition to the state data $x \in H^1([0, \tau]; \mathbb{R}^n)$ and the input data $u \in L^2([0, \tau]; \mathbb{R}^m)$, we consider the process noise $w \in L^2([0, \tau]; \mathbb{R}^n)$. Throughout this section, we denote by W the synthesis operator associated with w . By Lemma 2.2, we obtain the following equivalence for fixed $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$:

$$(30) \quad x' = Ax + Bu + w \text{ a.e. on } [0, \tau] \Leftrightarrow \Xi_d = A\Xi + B\Upsilon + W.$$

By (30), the effect of the noise w to the data can be evaluated by using the norm $\|W\|$ rather than $\|w\|_{L^2}$; see Section 5.1.2 for the discussion on the norm $\|W\| = \|W^*\|$. For $c \geq 0$, noting that $\|W\| \leq \sqrt{c}$ is equivalent to $WW^* \preceq cI_n$, we define the noise class $\Delta_c[0, \tau]$ by

$$\Delta_c[0, \tau] := \{w \in L^2([0, \tau]; \mathbb{R}^n) : WW^* \preceq cI_n\}.$$

We assume that the available data (x, u, τ) are generated by some continuous-time linear system in the presence of process noise in $\Delta_c[0, \tau]$.

Assumption 5.3. Given data (x, u, τ) and a constant $c \geq 0$, there exist $A_s \in \mathbb{R}^{n \times n}$, $B_s \in \mathbb{R}^{n \times m}$, and $w_s \in \Delta_c[0, \tau]$ such that

$$(31) \quad x'(t) = A_s x(t) + B_s u(t) + w_s(t)$$

for a.e. $t \in [0, \tau]$.

The input-state data $\mathfrak{D} = (x, u, \tau)$ and the noise-intensity parameter c are available, but we consider the situation where the matrices A_s, B_s and the noise w_s in Assumption 5.3 are unknown. Given data \mathfrak{D} and a constant $c \geq 0$, we define the set $\Sigma_{\mathfrak{D}, c}$ of systems by

$$\Sigma_{\mathfrak{D}, c} := \{(A, B) : \text{there exists } w \in \Delta_c[0, \tau] \text{ such that } x' = Ax + Bu + w \text{ a.e. on } [0, \tau]\}.$$

Systems in $\Sigma_{\mathfrak{D}, c}$ are compatible with the data \mathfrak{D} in the presence of noise in $\Delta_c[0, \tau]$. In addition, $(A_s, B_s) \in \Sigma_{\mathfrak{D}, c}$ under Assumption 5.3.

We introduce the Lyapunov-based notion of informativity for stabilization.

Definition 5.1. Let $c \geq 0$. The data $\mathfrak{D} = (x, u, \tau)$ are called *informative for quadartic stabilization under the noise class $\Delta_c[0, \tau]$* if there exist $P \in \mathbb{S}^n$ and $K \in \mathbb{R}^{m \times n}$ such that $P \succ 0$ and

$$(32) \quad (A + BK)P + P(A + BK)^\top \succ 0$$

for all $(A, B) \in \Sigma_{\mathfrak{D}, c}$. In the case $c = 0$, the data \mathfrak{D} are simply called *informative for quadartic stabilization*.

In Definition 5.1, the matrices P and K are common for all $(A, B) \in \Sigma_{\mathfrak{D}}$. In contrast, in the definition of informativity for stabilization (Definition 4.1), the stabilizing gain K depends on $(A, B) \in \Sigma_{\mathfrak{D}}$. However, the following proposition shows that these informativity properties for quadratic stabilization and for stabilization are equivalent.

Proposition 5.4. Suppose that the data (x, u, τ) satisfy Assumption 2.1. Then (x, u, τ) are informative for stabilization if and only if (x, u, τ) are informative for quadratic stabilization.

Proof. Suppose that $\mathfrak{D} = (x, u, \tau)$ are informative for stabilization. By Theorem 4.1, there exists a right inverse Ξ_R^{-1} of Ξ in $\mathcal{L}(\mathbb{R}^n, H_0^1[0, \tau])$ such that $\Xi_d \Xi_R^{-1} \in \mathbb{R}^{n \times n}$ is Hurwitz. There exists $P \in \mathbb{S}^n$ such that $P \succ 0$ and

$$(\Xi_d \Xi_R^{-1})P + P(\Xi_d \Xi_R^{-1})^\top \succ 0.$$

On the other hand, if we define $K := \Upsilon \Xi_R^{-1}$, then $\Xi_d \Xi_R^{-1} = A + BK$ for all $(A, B) \in \Sigma_{\mathfrak{D}}$ by Lemma 2.2. Hence, (x, u, τ) are informative for quadratic stabilization. The converse implication follows immediately from Definitions 4.1 and 5.1. \square

5.3. Characterization via the matrix S -lemma. We characterize data informativity for quadratic stabilization in the presence of noise. This characterization is a continuous-time analogue of [26, Theorem 5.1]. A similar result with some approximation errors was obtained for discrete sequences transformed from continuous-time data via polynomial orthogonal bases in [22, Theorem 5]. Note that the products of the synthesis operators and their adjoints, such as $\Xi_d \Xi_d^*$ and $\Xi \Xi_d^* = (\Xi_d \Xi^*)^\top$, are matrices that can be computed by using Lemma 5.1.

Theorem 5.5. Suppose that the data $\mathfrak{D} = (x, u, \tau)$ and the constant $c \geq 0$ satisfy Assumption 5.3. Let Ξ and Ξ_d be the synthesis operator and the differentiated synthesis operator associated with x , respectively. Let Υ be the synthesis operator associated with u . Then the following statements are equivalent:

- (i) The data (x, u, τ) are informative for quadratic stabilization under the noise class $\Delta_c[0, \tau]$.
- (ii) There exist matrices $P \in \mathbb{S}^n$, $L \in \mathbb{R}^{m \times n}$ and a scalar $\alpha \geq 0$ such that $P \succ 0$ and

$$(33) \quad \begin{bmatrix} \alpha \Xi_d \Xi_d^* - (\alpha c + 1)I_n & -P - \alpha \Xi_d \Xi^* & -L^\top - \alpha \Xi_d \Upsilon^* \\ -P - \alpha \Xi \Xi_d^* & \alpha \Xi \Xi^* & \alpha \Xi \Upsilon^* \\ -L - \alpha \Upsilon \Xi_d^* & \alpha \Upsilon \Xi^* & \alpha \Upsilon \Upsilon^* \end{bmatrix} \succeq 0.$$

Furthermore, if statement (ii) holds, then $K := LP^{-1}$ is such that $A + BK$ is Hurwitz for all $(A, B) \in \Sigma_{\mathfrak{D}, c}$.

For the proof of Theorem 5.5, we consider $\mathcal{M}, \mathcal{N} \in \mathbb{S}^{q+r}$ partitioned as

$$(34) \quad \mathcal{M} = \begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} \\ \mathcal{M}_{12}^\top & \mathcal{M}_{22} \end{bmatrix} \quad \text{and} \quad \mathcal{N} = \begin{bmatrix} \mathcal{N}_{11} & \mathcal{N}_{12} \\ \mathcal{N}_{12}^\top & \mathcal{N}_{22} \end{bmatrix},$$

where the $(1, 1)$ -blocks have size $q \times q$ and the $(2, 2)$ -blocks have size $r \times r$. Define the matrix sets $\mathcal{Z}_{q,r}(\mathcal{N})$ and $\mathcal{Z}_{q,r}^+(\mathcal{M})$ by

$$(35) \quad \mathcal{Z}_{q,r}(\mathcal{N}) := \left\{ Z \in \mathbb{R}^{r \times q} : \begin{bmatrix} I_q \\ Z \end{bmatrix}^\top \mathcal{N} \begin{bmatrix} I_q \\ Z \end{bmatrix} \succeq 0 \right\},$$

$$(36) \quad \mathcal{Z}_{q,r}^+(\mathcal{M}) := \left\{ Z \in \mathbb{R}^{r \times q} : \begin{bmatrix} I_q \\ Z \end{bmatrix}^\top \mathcal{M} \begin{bmatrix} I_q \\ Z \end{bmatrix} \succ 0 \right\}.$$

The following lemma called the *matrix S -lemma* is useful to study data informativity in the presence of noise.

Lemma 5.6. Let $\mathcal{M}, \mathcal{N} \in \mathbb{S}^{q+r}$ be partitioned as in (34). Assume that

$$(37) \quad \mathcal{M}_{22} \preceq 0, \quad \mathcal{N}_{22} \preceq 0, \quad \text{Ker } \mathcal{N}_{22} \subseteq \text{Ker } \mathcal{M}_{12},$$

and that

$$(38) \quad \mathcal{N}_{11} - \mathcal{N}_{12} \mathcal{N}_{22}^+ \mathcal{N}_{12}^\top \succeq 0.$$

Then the following statements are equivalent:

- (i) $\mathcal{Z}_{q,r}(\mathcal{N}) \subseteq \mathcal{Z}_{q,r}^+(\mathcal{M})$.
- (ii) There exist scalars $\alpha \geq 0$ and $\beta > 0$ such that

$$(39) \quad \mathcal{M} - \alpha \mathcal{N} \succeq \begin{bmatrix} \beta I_q & 0 \\ 0 & 0 \end{bmatrix}.$$

Next, we present a preliminary result on the system set $\Sigma_{\mathfrak{D},c}$. For a constant $c \geq 0$ and operators $\Xi_d, \Xi \in \mathcal{L}(H_0^1[0, \tau], \mathbb{R}^n)$ and $\Upsilon \in \mathcal{L}(H_0^1[0, \tau], \mathbb{R}^m)$, we define $\mathcal{N} \in \mathbb{S}^{n+(n+m)}$ by

$$(40) \quad \mathcal{N} := \begin{bmatrix} cI_n - \Xi_d \Xi_d^* & \Xi_d \Xi^* & \Xi_d \Upsilon^* \\ \Xi \Xi_d^* & -\Xi \Xi^* & -\Xi \Upsilon^* \\ \Upsilon \Xi_d^* & -\Upsilon \Xi^* & -\Upsilon \Upsilon^* \end{bmatrix}.$$

The following lemma gives a characterization of $\Sigma_{\mathfrak{D},c}$ in terms of $\mathcal{Z}_{n,n+m}(\mathcal{N})$.

Lemma 5.7. *Assume that there exist $A_s \in \mathbb{R}^{n \times n}$, $B_s \in \mathbb{R}^{n \times m}$, and $w_s \in L^2([0, \tau]; \mathbb{R}^n)$ such that the data $\mathfrak{D} = (x, u, \tau)$ satisfy (31) for a.e. $t \in [0, \tau]$. Let Ξ and Ξ_d be the synthesis operator and the differentiated synthesis operator associated with x , respectively, and let Υ be the synthesis operator associated with u . For a constant $c \geq 0$, define $\mathcal{N} \in \mathbb{S}^{n+(n+m)}$ by (40) and $\mathcal{Z}_{n,n+m}(\mathcal{N})$ by (35). Then the following statements are equivalent for all $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$:*

- (i) $(A, B) \in \Sigma_{\mathfrak{D},c}$.
- (ii) $[A \ B]^\top \in \mathcal{Z}_{n,n+m}(\mathcal{N})$.

Proof. By the definition (40) of \mathcal{N} , we have that for all $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$,

$$(41) \quad [I_n \ A \ B] \mathcal{N} \begin{bmatrix} I_n \\ A^\top \\ B^\top \end{bmatrix} = cI_n - (\Xi_d - A\Xi - B\Upsilon)(\Xi_d - A\Xi - B\Upsilon)^*.$$

First, we prove the implication (i) \Rightarrow (ii). Let $(A, B) \in \Sigma_{\mathfrak{D},c}$. By (30), there exists $w \in \Delta_c[0, \tau]$ such that the synthesis operator W associated with w satisfies $\Xi_d - A\Xi - B\Upsilon = W$. Combining (41) with $WW^* \preceq cI_n$, we obtain $[A \ B]^\top \in \mathcal{Z}_{n,n+m}(\mathcal{N})$.

Next, we prove the implication (ii) \Rightarrow (i). Let $[A \ B]^\top \in \mathcal{Z}_{n,n+m}(\mathcal{N})$. The definition (35) of $\mathcal{Z}_{n,n+m}(\mathcal{N})$ and (41) give

$$(42) \quad cI_n - (\Xi_d - A\Xi - B\Upsilon)(\Xi_d - A\Xi - B\Upsilon)^* \succeq 0.$$

Let W_s be the synthesis operator associated with w_s . By (30) and (31),

$$\Xi_d - A\Xi - B\Upsilon = (A_s - A)\Xi + (B_s - B)\Upsilon + W_s.$$

This implies that $\Xi_d - A\Xi - B\Upsilon$ is the synthesis operator associated with

$$w := (A_s - A)x + (B_s - B)u + w_s \in L^2([0, \tau]; \mathbb{R}^n).$$

Since (42) yields $w \in \Delta_c[0, \tau]$, it follows from (30) that $(A, B) \in \Sigma_{\mathfrak{D},c}$. \square

Now we are in the position to prove Theorem 5.5.

Proof of Theorem 5.5. First, we prove the implication (i) \Rightarrow (ii). By assumption, there exist $P \in \mathbb{S}^n$ and $K \in \mathbb{R}^{m \times n}$ such that $P \succ 0$ and the Lyapunov inequality (32) holds for all $(A, B) \in \Sigma_{\mathfrak{D},c}$. Define $\mathcal{M} \in \mathbb{S}^{n+(n+m)}$ by

$$(43) \quad \mathcal{M} := \begin{bmatrix} 0 & -P & -PK^\top \\ -P & 0 & 0 \\ -KP & 0 & 0 \end{bmatrix}.$$

Let $\mathcal{N} \in \mathbb{S}^{n+(n+m)}$ be defined as in (40). To apply Lemma 5.6 (the matrix S -lemma), we will show that the assumptions given in (37) and (38) are satisfied.

The $(2, 2)$ -block $\mathcal{M}_{22} \in \mathbb{S}^{n+m}$ of \mathcal{M} is given by $\mathcal{M}_{22} = 0$, and hence, it is clear that $\mathcal{M}_{22} \preceq 0$. The $(2, 2)$ -block $\mathcal{N}_{22} \in \mathbb{S}^{n+m}$ of \mathcal{N} satisfies

$$\mathcal{N}_{22} = - \begin{bmatrix} \Xi \\ \Upsilon \end{bmatrix} [\Xi^* \ \Upsilon^*] \preceq 0.$$

Since $\text{Ker } \mathcal{N}_{22} = \text{Ker } [\Xi^* \ \Upsilon^*]$, the $(1, 2)$ -block $\mathcal{N}_{12} = \Xi_d [\Xi^* \ \Upsilon^*]$ of \mathcal{N} satisfies $\text{Ker } \mathcal{N}_{22} \subseteq \text{Ker } \mathcal{N}_{12}$. Therefore, the assumptions given in (37) are satisfied.

To prove that the assumption given in (38) is satisfied, we define $\mathcal{F}_\mathcal{N}: \mathbb{R}^{(n+m) \times n} \rightarrow \mathbb{R}^{n \times n}$ by

$$\mathcal{F}_\mathcal{N}(Z) := \begin{bmatrix} I_n \\ Z \end{bmatrix}^\top \mathcal{N} \begin{bmatrix} I_n \\ Z \end{bmatrix}.$$

Since $\text{Ker } \mathcal{N}_{22} \subseteq \text{Ker } \mathcal{N}_{12}$, it follows from [3, Fact 8.9.7] that

$$\mathcal{F}_{\mathcal{N}}(Z) = \mathcal{N}_{11} - \mathcal{N}_{12}\mathcal{N}_{22}^+\mathcal{N}_{12}^\top + (Z + \mathcal{N}_{22}^+\mathcal{N}_{12}^\top)^\top \mathcal{N}_{22}(Z + \mathcal{N}_{22}^+\mathcal{N}_{12}^\top)$$

for all $Z \in \mathbb{R}^{(n+m) \times n}$. By Assumption 5.3 and Lemma 5.7, we obtain $[A_s \ B_s]^\top \in \mathcal{Z}_{n,n+m}(\mathcal{N})$. Combining this with $\mathcal{N}_{22} \preceq 0$, we derive

$$\mathcal{N}_{11} - \mathcal{N}_{12}\mathcal{N}_{22}^+\mathcal{N}_{12}^\top \succeq \mathcal{F}_{\mathcal{N}}([A_s \ B_s]^\top) \succeq 0.$$

By the Lyapunov inequality (32), we obtain $[A \ B]^\top \in \mathcal{Z}_{n,n+m}(\mathcal{M})$ for all $(A, B) \in \Sigma_{\mathfrak{D},c}$. Therefore, Lemma 5.7 yields $\mathcal{Z}_{n,n+m}(\mathcal{N}) \subseteq \mathcal{Z}_{n,n+m}(\mathcal{M})$. By Lemma 5.6 (the matrix S -lemma), there exist scalars $\alpha \geq 0$ and $\beta > 0$ such that the inequality (39) with $q = n$ holds. Replacing P , KP , and α by βP , βL , and $\alpha\beta$, respectively, we conclude that the LMI (33) is satisfied.

Next, we prove the implication (ii) \Rightarrow (i). Define $K := LP^{-1}$ and let $\mathcal{M}, \mathcal{N} \in \mathbb{S}^{n+(n+m)}$ be defined by (43) and (40), respectively. Let $(A, B) \in \Sigma_{\mathfrak{D},c}$. By Lemma 5.7 and the LMI (33), we obtain

$$\begin{aligned} 0 &\leq [I_n \ A \ B] \left(\mathcal{M} - \alpha\mathcal{N} - \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} I_n \\ A^\top \\ B^\top \end{bmatrix} \\ &\preceq -(A + BK)P - P(A + BK)^\top - I_n \\ (44) \quad &\prec -(A + BK)P - P(A + BK)^\top. \end{aligned}$$

Thus, the data (x, u, τ) are informative for quadratic stabilization under the noise class $\Delta_c[0, \tau]$. The last assertion on the Hurwitz property of $A + BK$ follows immediately from (44). ■

5.4. Numerical example. We consider the linearized model of a batch reactor given in [23, p. 213], which was also used as a numerical example in the data-driven control literature, e.g., [7, 10, 21]. The system matrices A_s and B_s are given by

$$\begin{aligned} A_s &= \begin{bmatrix} 1.38 & -0.2077 & 6.715 & -5.676 \\ -0.5814 & -4.29 & 0 & 0.675 \\ 1.067 & 4.273 & -6.654 & 5.893 \\ 0.048 & 4.273 & 1.343 & -2.104 \end{bmatrix}, \\ B_s &= \begin{bmatrix} 0 & 0 \\ 5.679 & 0 \\ 1.136 & -3.146 \\ 1.136 & 0 \end{bmatrix}. \end{aligned}$$

The matrix A_s has two unstable eigenvalues, 1.9910 and 0.0635. Input and state trajectories were generated on the interval $[0, 1]$, where the initial state x_0 and the input u were chosen as $x_0 = [1 \ -1 \ 0 \ 1]^\top$ and

$$u(t) = 5 \begin{bmatrix} \sin(2\pi t) + \sin(4\pi t) \\ \sin(3\pi t) + \sin(6\pi t) \end{bmatrix}, \quad t \in [0, 1].$$

As the process noise w , zero-mean Gaussian white-noise stochastic process with covariance matrix $E[w(t)w(s)^\top] = \delta(t-s)10^{-2}I_n$ was applied, where $E[\cdot]$ is the expectation operator and δ is the Dirac delta function. By Proposition 5.2, the approximate value of the norm of the synthesis operator associated with the generated noise is 9.598×10^{-2} . Theorem 5.5 shows that the input-state data are informative for quadratic stabilization under noise class $\Delta_c[0, \tau]$ with $c = 0.1164$. For $c = 0.1$, we derived the stabilizing gain

$$K = \begin{bmatrix} -0.2351 & -1.618 & 1.343 & -1.445 \\ 8.389 & -0.9873 & 8.104 & -9.686 \end{bmatrix}.$$

To construct a stabilizing gain $K = LP^{-1}$ with small norm, we imposed the constraints $\|L\| \leq \gamma$ and $P \succ \delta I_n$ in addition to the LMI (33), and then minimized the objective $\gamma - \lambda\delta$ under $\gamma \geq 0$ and $0 < \delta < \delta_{\max}$, where $\lambda \geq 0$ is a weighting parameter and $\delta_{\max} > 0$ is a sufficiently large constant. We set $\lambda = 10^2$ and $\delta_{\max} = 10^6$. The LMIs and the minimization problem were solved by using MATLAB R2025b with YALMIP [17] and MOSEK [18].

6. CONCLUSION

We developed a synthesis-operator framework for derivative-free data-driven control of continuous-time systems. By embedding input and state trajectories into synthesis operators, we provided an operator-based representation of systems that are compatible with the given data. By this representation, we characterized the informativity properties for system identification and for stabilization. Moreover, we derived a necessary and sufficient LMI condition for noisy data to be informative for quadratic stabilization. Future work will extend the synthesis-operator approach to settings where only input-output data are available.

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