

Joint learning of a network of linear dynamical systems via total variation penalization*

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Abstract

We consider the problem of joint estimation of the parameters of m linear dynamical systems, given access to single realizations of their respective trajectories, each of length T . The linear systems are assumed to reside on the nodes of an undirected and connected graph $G = ([m], \mathcal{E})$, and the system matrices are assumed to either vary smoothly or exhibit small number of “jumps” across the edges. We consider a total variation penalized least-squares estimator and derive non-asymptotic bounds on the mean squared error (MSE) which hold with high probability. In particular, the bounds imply for certain choices of well connected G that the MSE goes to zero as m increases, even when T is constant. The theoretical results are supported by extensive experiments on synthetic and real data.

Keywords: vector autoregression (VAR), total variation penalty, linear dynamical systems, federated learning, networks

1 Introduction

The problem of learning a linear dynamical system¹ (LDS) from samples of its trajectory has many applications, e.g., in control systems, time-series analysis and reinforcement learning. In its simplest form, the state $x_t \in \mathbb{R}^d$ of the (autonomous) system evolves over T time-steps as

$$x_{t+1} = A^* x_t + \eta_{t+1}; \quad t = 0, \dots, T$$

and the goal is to estimate the $d \times d$ matrix A^* from a realization $(x_t)_{t=0}^{T+1}$. Here, $(\eta_t)_{t=0}^{T+1}$ are unobserved, and typically centered and independent random vectors, often referred to as the process noise or the “excitation” of the system. The problem has a rich history in the control systems and statistics communities, with classical works typically consisting of either asymptotic results (e.g., [18, 16, 17]) or involve quantities that are exponential in the degree of the system (e.g. [6, 40]).

A recent line of work [28, 29, 27, 13] has focused on finite-time identification of LDSs where the goal is to provided non-asymptotic error guarantees for recovering A^* . Here, the ordinary-least-squares (OLS) is analyzed theoretically and error bounds are derived w.r.t the spectral norm

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¹Also referred to as a vector autoregressive (VAR) model in the literature.

$\|\hat{A} - A^*\|_2$ – the bounds holding provided T is large enough. A typical assumption made in these results is that A^* is stable, i.e., its spectral radius is either strictly less than one [13] or even more broadly, is less than or equal to one [29, 27]. In both these scenarios, the error rate is typically shown to go to zero at the near-parametric rate² $\tilde{O}(1/\sqrt{T})$ provided T is suitably large (typically polynomially large w.r.t d). The result in [13] implies that with probability at least $1 - \delta$,

$$\|\hat{A} - A^*\|_2 \lesssim \sqrt{\frac{\log(1/\delta) + d}{T}} \quad \text{if } T \gtrsim \Delta(\log(1/\delta) + d),$$

where Δ is finite if the spectral radius of A^* is less than one. Moreover, Δ is a constant if $\|A^*\|_2 < 1$. The above bound is also known to be optimal in terms of the dependence on δ, d and T [29].

1.1 Learning multiple LDSs via total variation (TV) penalization

In this paper, we consider the problem of joint identification of m LDSs. More precisely, for a given undirected (and connected) graph $G = ([m], \mathcal{E})$, we consider that at node $l \in [m]$, the state $x_{l,t} \in \mathbb{R}^d$ evolves as

$$x_{l,t+1} = A_l^* x_{l,t} + \eta_{l,t+1}; \quad t = 0, \dots, T, \quad x_{l,0} = 0, \quad (1.1)$$

where $\eta_{l,t}$ are assumed to be i.i.d centered standard Gaussian vectors. Furthermore, we assume that the matrices A_l^* 's either vary smoothly (i.e., $\sum_{\{l,l'\} \in \mathcal{E}} \|A_l^* - A_{l'}^*\|_{1,1}$, is³ small) or exhibit only a small number of jumps across edges (i.e., $(A_l^* - A_{l'}^*)_{l \neq l'}$, has a small number of non-zeroes). Given the data $(x_{l,t})_{t=0}^T$ for each $l \in [m]$, we then obtain estimates \hat{A}_l of A_l^* for each l , via the TV-penalized least squares estimator

$$(\hat{A}_1, \dots, \hat{A}_m) = \underset{A_1, \dots, A_m \in \mathbb{R}^{d \times d}}{\operatorname{argmin}} \left\{ \frac{1}{2m} \sum_{l=1}^m \sum_{t=1}^T \|x_{l,t+1} - A_l x_{l,t}\|_F^2 + \lambda \sum_{\{l,l'\} \in \mathcal{E}} \|A_l - A_{l'}\|_{1,1} \right\} \quad (1.2)$$

where $\lambda \geq 0$ is a regularization parameter.

Our aim is to establish precise *non-asymptotic* bounds⁴ on the mean squared error (MSE) which hold with high probability. In particular, we would consequently like to ensure that the MSE goes to zero as m increases, i.e.,

$$(\text{Weak consistency}) \quad \text{MSE} := \frac{1}{m} \sum_{l=1}^m \left\| \hat{A}_l - A_l^* \right\|_F^2 \longrightarrow 0 \quad \text{as } m \rightarrow \infty. \quad (1.3)$$

Note that (1.3) is particularly relevant in the “small- T ” regime where T is independent of m , and is small. In this case, we cannot hope to reliably estimate the matrices by estimating them via the OLS estimator at each node. More generally, even if T were to grow (mildly) with m , it is still meaningful to ask whether the MSE for the estimator (1.2) goes to zero at a faster rate w.r.t m as compared to the naive node-wise OLS estimator.

Contributions. At a high level, our main contribution is to show that when the parameters of the linear systems either vary smoothly or exhibit only a small number of changes across edges, then a joint least-squares estimator with a graph-TV penalty can pool information across nodes to

²Here, $\tilde{O}(\cdot)$ hides logarithmic factors

³Here, $\|\cdot\|_{1,1}$ is the entry-wise ℓ_1 norm; see Section 2.1 for notation.

⁴All of our results, including the conditions on T and m are non-asymptotic.

learn the parameters far more efficiently than estimating each system in isolation. Concretely, by leveraging the graph structure, our method attains tighter non-asymptotic error guarantees (and in well-connected graphs, consistency as the number of systems grows) even when each trajectory is short, thereby outperforming node-wise OLS in these regimes. Our main result is to provide non-asymptotic bounds on the MSE for the estimator (1.2), which hold with high probability (see Theorem 1). More interpretable versions of Theorem 1 are provided as Corollaries 1 and 2, when $\|A_l^*\|_2 < 1$ holds for each l . In this case, transparent dependencies between T, m , the smoothness of $(A_l^*)_{l=1}^m$ w.r.t G , and the connectivity of G are provided (see conditions (2.9) - (2.12)). To the best of our knowledge, these are the first theoretical results for graph-based TV-penalized linear regression when the data contains dependencies; see the end of Section 1.2 for comparison with related work in this regard.

- In Corollary 3, we address the case where T is a constant, and show that if the graph G is well connected and the matrices $(A_l^*)_{l=1}^m$ either have few jumps or vary smoothly across the edges, then weak consistency is guaranteed as m increases. This is shown for two examples of well connected graphs - the complete graph (Example 1) and the Erdős-Renyi graph (Example 2).
- For certain graphs such as the star graph (Example 3) and the 2D grid (Example 4), our results require T to grow with m . Nevertheless, we show that if $(A_l^*)_{l=1}^m$ are sufficiently smooth w.r.t G , and m is suitably large, then T is only required to grow poly-logarithmically with m, d for (weakly-) consistent recovery. In this case, the MSE for (1.2) is $o(1)$, while that for the naive node-wise OLS is $O(1)$ w.r.t m (see Remark 4).
- In terms of the proof techniques for Theorem 1, the bulk of the effort lies in establishing the restricted eigenvalue condition for block-diagonal design matrices where we now face difficulties on account of the *dependencies* within each block; see Lemmas 3 - 5. This is achieved via a range-nullspace decomposition tailored to the graph-TV operator. Our main probabilistic tool here is the concentration result of [15] for controlling the suprema of second order subgaussian chaos processes involving positive semidefinite (p.s.d) matrices. Other parts in the proof of Theorem 1 make use of tail bounds for controlling the norm of self-normalized vector-valued martingales, using ideas from the proof of [1, Theorem 1]; see Lemmas 1 and 2. Such tail bounds have been deployed recently for learning a single LDS [27], and also for learning multiple LDS's under a graph-based quadratic variation smoothness assumption [37]. However the above ingredients are – to the best of our knowledge – new for *graph-based TV-penalized regression with dependent observations*.
- Finally, we provide extensive empirical results on synthetic data, and also real datasets related to the U.S Environmental Protection Agency (EPA) national air-quality monitoring network, where we compare the performance of (1.2) (w.r.t parameter estimation and prediction error) with other baseline methods. For synthetic data, we find that our graph-TV estimator typically outperforms other methods on different graph topologies, especially when the system matrices exhibit a piecewise constant structure over the edges. When the parameters vary smoothly over the edges, the graph-TV estimator is found to be competitive with Laplacian smoothing [37]. For the U.S EPA datasets, we find that graph-TV has comparable performance, in terms of prediction error, w.r.t the best competing method.

1.2 Related work

The task of joint estimation of multiple LDSs from their trajectories has received considerable attention over the past few years, mainly due to various applications arising in the modeling of,

e.g., flight path dynamics at different altitudes [4], brain network dynamics [30, 10], and gene expressions in genomics [3], to name a few.

Recently, [23] considered the setting where the system matrices $(A_l^*)_{l=1}^m$ are unknown linear combinations of k unknown basis functions. They propose an estimator for estimating the system matrices along with bounds on the MSE, and show that reliable estimation can be performed provided $T > k$. This is particularly meaningful when $k \ll d^2$.

A recent line of work has also focused on federated learning of LDSs. For instance, [42] consider the l th system (or client) to be modeled as

$$x_{l,t+1} = A_l^* x_{l,t} + B_l^* u_{l,t} + \eta_{l,t+1}; \quad t = 0, \dots, T, \quad (1.4)$$

where $u_{l,t}$ represents the external input to the system. The clients are only allowed to communicate with a central server and N_l independent trajectories are observed for the l th client. The goal is to find a common estimate \hat{A}, \hat{B} to all the system matrices A_l^*, B_l^* . It is assumed that

$$\max_{l,l' \in [m]} \|A_l^* - A_{l'}^*\|_2 \leq \epsilon, \quad \max_{l,l' \in [m]} \|B_l^* - B_{l'}^*\|_2 \leq \epsilon. \quad (1.5)$$

Then, the estimates \hat{A}, \hat{B} are obtained via the OLS method, and error bounds are derived w.r.t the spectral norm, for each client $l \in [m]$. Their main result states that if N_l is suitably large w.r.t T and the system dimensions, and ϵ is suitably small, then the estimation error at each l is smaller than that obtained by individual client-wise estimation. Clearly, the edge-wise condition (1.5) is much stronger than requiring that the TV of the system matrices is suitably small. [7] considered jointly estimating systems of the form (1.4) under different types of structural assumptions on A_l^* 's (e.g., group sparsity; smoothness of the form (1.5) w.r.t the Frobenius norm between all pairs of matrices). Some theoretical results are provided (albeit not for parameter estimation) for a group-lasso type estimator, along with several experiments on synthetic and real data.

A closely related setting to the one above is where we are given the trajectory of the true system, and that of a “similar” system. This was studied by [45] – they proposed a weighted least-squares approach for estimating the true system parameters, and show that the estimation error can be reduced by using data from the two systems.

The recent work [37] considered a similar setting as in the present paper, but under a different smoothness assumption wherein $\sum_{\{l,l'\} \in \mathcal{E}} \|A_l^* - A_{l'}^*\|_F^2$ is small. For a smoothness penalized estimator (namely Laplacian smoothing), it was shown that weak consistency is obtained when T is at least as large as the condition required for identification of a single LDS, and also grows with m as $\log m$. It was also shown for a subspace constrained estimator that weak consistency is obtained for $T = 2$ provided the smoothness term is sufficiently small. It is by now well known that TV penalization is able to better preserve sharp edges and boundaries in signals as compared to the aforementioned quadratic penalization (e.g, [26, Figure 1]). Moreover the estimators considered in [37] admit a closed form solution and require a very different analysis than in our setting.

In the special case where $A_l^* = A^*$ for each l , i.e., the perfectly smooth setting, our problem setting reduces to learning a single LDS from m independent trajectories. In this case, there exist many results for estimating the system matrix, with error bounds w.r.t the spectral norm (see e.g., [44, 46]).

Note that by stacking the vectorized versions of the matrices A_1^*, \dots, A_l^* to form a tall vector $a^* \in \mathbb{R}^{md^2}$, the model in (1.1) can be recast as the linear model (2.2) with a design matrix Q . Then, the estimator (1.2) can be equivalently written as a TV-penalized least squares estimator of the form (2.3), which is referred to as the generalized lasso in the literature [34, 19]. Estimators of the form

(2.3) have been theoretically analyzed recently⁵ in [20, 35]. The main difference between our setting and these works is that [20, 35] consider the rows of the design matrix to be independent samples from a centered Gaussian (with covariance Σ) and also assume that the noise is independent of the design matrix. In our case, the design matrix Q is block-diagonal, where the diagonal blocks are independent (as they correspond to different systems) but the entries within a block are *dependent* (as they correspond to data generated by the same system). Secondly, the “noise” term in our setting in (2.2) shares dependencies with Q , while the setting in [20, 35] assumes that the noise is independent of the design matrix. These two points lead to highly non-trivial challenges in our setting, see discussion after Theorem 1, and also Remark 1 for further details.

Notice that (1.1) can be equivalently written as a single LDS

$$\begin{bmatrix} x_{1,t+1} \\ x_{2,t+1} \\ \vdots \\ x_{m,t+1} \end{bmatrix} = \underbrace{\begin{bmatrix} A_1^* & 0 & \dots & 0 \\ 0 & A_2^* & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & A_m^* \end{bmatrix}}_{=:A^*} \begin{bmatrix} x_{1,t} \\ x_{2,t} \\ \vdots \\ x_{m,t} \end{bmatrix} + \begin{bmatrix} \eta_{1,t+1} \\ \eta_{2,t+1} \\ \vdots \\ \eta_{m,t+1} \end{bmatrix}; \quad t = 0, \dots, T.$$

Hence the recovery problem is equivalent to that of estimating the block-diagonal matrix $A^* \in \mathbb{R}^{md \times md}$. Here A^* is structured as it is not only block-diagonal (i.e., linear equality constraints), but its diagonal blocks do not change too quickly, or have few jumps over \mathcal{E} . There is by now a substantial literature on estimating a LDS with a structured system matrix from its trajectory.

- These results are typically for penalized estimators, where the penalty consists of a suitable norm $R(\cdot)$. Here R is usually the ℓ_1 norm for sparse matrices (e.g., [21, 22, 2, 14, 9, 41]). Other examples include the nuclear norm for low-rank matrices (e.g., [41]), and the (sparse) group LASSO penalty [22]. Penalties imposing a smoothness constraint w.r.t a graph G are comparatively non-existent in the literature, barring the recent work [37] (discussed earlier).
- Alternatively, one could also replace (1.2) with a constrained least-squares estimator where the constraint is of the form $\left\{ \sum_{\{l,l'\} \in \mathcal{E}} \|A_l - A_{l'}\|_{1,1} \leq \lambda' \right\}$ which is a convex set. While such a constrained least-squares estimator has not been studied explicitly for structured LDS estimation, recently [38] studied this for general convex constraints \mathcal{K} and showed error bounds that depend on the local complexity⁶ of \mathcal{K} . The analysis therein also involves a restricted eigenvalue condition, which is shown using the concentration result of [15]. While it seems possible to bound the γ_2 term using similar ideas as in our analysis, the γ_1 term is usually more difficult to control. In general, it is unclear whether bounds of similar nature as in the present paper can be obtained from the result in [38].

Finally, we mention the related problem of signal denoising on networks where we are given $y = \beta^* + \eta$ with $\beta^* \in \mathbb{R}^m$ unknown, and η denoting noise (typically with centered and independent subgaussian entries). Assuming β^* has few jumps or varies smoothly across the edges of G , a well-studied estimator is

$$\widehat{\beta} = \underset{\beta \in \mathbb{R}^m}{\operatorname{argmin}} \left\{ \|y - \beta\|_2^2 + \lambda \sum_{\{l,l'\} \in \mathcal{E}} |\beta_l - \beta_{l'}| \right\},$$

⁵We remark that [20, 35] consider a more general graph-based penalization involving both the ℓ_1 and ℓ_2^2 penalties. This is referred to as the *generalized elastic net* in [35] and the *graph total-variation method* in [20].

⁶as measured by Talagrand’s γ_1, γ_2 functionals, see Appendix C.1 for definition.

see e.g., [12, 43, 24], for non-asymptotic ℓ_2 estimation error results. In particular, [12] derives an oracle inequality for the estimation error which is then instantiated for different choices of graphs G . Similar to [12], our results also depend on parameters such as the compatibility factor and the inverse scaling factor (see Section 2.3). However our results also depend on new quantities, such as an alternate inverse scaling factor (see section 2.3) and a dispersion functional capturing the smoothness of the controllability Grammian's of A_l^* 's w.r.t G (see (2.6)).

2 Problem setup and results

2.1 Notation

For $x \in \mathbb{R}^n$ and $p \in \mathbb{N}$, $\|x\|_p$ denotes the usual ℓ_p norm of x . For $X \in \mathbb{R}^{n \times m}$, the spectral and Frobenius norms of X are denoted by $\|X\|_2$ and $\|X\|_F$ respectively, while $\langle X, Y \rangle := \text{Tr}(X^\top Y)$ denotes the inner product between X and Y . Here $\text{Tr}(\cdot)$ denotes the trace operator. The vector $\text{vec}(X) \in \mathbb{R}^{nm}$ is formed by stacking the columns of X , and $\|X\|_{p,p} := \|\text{vec}(X)\|_p$ denotes the entry-wise ℓ_p norm of X . For integers $1 \leq p, q \leq \infty$ we define the norm $\|X\|_{p \rightarrow q}$ as

$$\|X\|_{p \rightarrow q} = \sup_{u \neq 0} \frac{\|Xu\|_q}{\|u\|_p}$$

with $\|X\|_{2 \rightarrow 2}$ corresponding to the spectral norm of X (denoted by $\|X\|_2$ for simplicity). For $n \times n$ matrices X with eigenvalues $\lambda_i \in \mathbb{C}$, we denote the spectral radius of X by $\rho(X) := \max_{i=1,\dots,n} |\lambda_i|$.

I_n denotes the $n \times n$ identity matrix. The symbol \otimes denotes the Kronecker product between matrices, and $\mathbf{1}_n \in \mathbb{R}^n$ denotes the all-ones vector. The canonical basis of \mathbb{R}^n is denoted by e_1, e_2, \dots, e_n . For $x \in \mathbb{R}^n$ and $\mathcal{S} \subseteq [n]$, the restriction of x on \mathcal{S} is denoted by $(x)_{\mathcal{S}}$, while $\text{supp}(x) \subseteq [n]$ refers to the support of x . The unit (ℓ_2 -norm) sphere in \mathbb{R}^n is denoted by \mathbb{S}^{n-1} , while $\mathbb{B}_p^n(r)$ denotes the ℓ_p ball of radius r in \mathbb{R}^n .

For $a, b > 0$, we say $a \lesssim b$ if there exists a constant $C > 0$ such that $a \leq Cb$. If $a \lesssim b$ and $a \gtrsim b$, then we write $a \asymp b$. The values of symbols used for denoting constants (e.g., c, C, c_1 etc.) may change from line to line. It will sometimes be convenient to use asymptotic notation using the standard symbols $O(\cdot), \Omega(\cdot), o(\cdot)$ and $\omega(\cdot)$; see for example [8].

2.2 Setup and preliminaries

Let $G = ([m], \mathcal{E})$ be a known undirected connected graph. At each node $l \in [m]$, the state $x_{l,t} \in \mathbb{R}^d$ at time t is described by a linear dynamical system (1.1) where $A_l^* \in \mathbb{R}^{d \times d}$ is unknown. We will assume throughout for convenience that $\eta_{l,t} \stackrel{iid}{\sim} \mathcal{N}(0, I_d)$ for each l, t . All our results continue to hold up to absolute constants, if $\eta_{l,t}$ were independent and centered subgaussian random variables, with their subgaussian norms bounded uniformly by some constant.

Given the data $(x_{l,t})_{l=1,t=0}^{m,T}$ our goal is to estimate A_1^*, \dots, A_m^* . Clearly, without additional assumptions on these matrices, we cannot do better than to estimate each A_l^* individually via the OLS estimator. Our interest is in doing better than this naive strategy, and we will see that this is possible when the matrices either vary smoothly, or do not exhibit too many jumps across \mathcal{E} .

The aim is to bound the mean-squared error (MSE) $\frac{1}{m} \sum_{l=1}^m \left\| \widehat{A}_l - A_l^* \right\|_F^2$ for the estimator (1.2), and establish conditions under which the MSE goes to zero as $m \rightarrow \infty$. Ideally, this should hold under weaker conditions than the aforementioned naive strategy, e.g., when T is small. To this

end, it will be useful to denote

$$\begin{aligned}\tilde{X}_l &= [x_{l,2} \cdots x_{l,T+1}] \in \mathbb{R}^{d \times T}, \quad X_l = [x_{l,1} \cdots x_{l,T}] \in \mathbb{R}^{d \times T} \\ \text{and } E_l &= [\eta_{l,2} \ \eta_{l,3} \ \cdots \ \eta_{l,T+1}] \in \mathbb{R}^{d \times T}.\end{aligned}$$

Furthermore, let

$$\begin{aligned}\tilde{x}_l &:= \text{vec}(\tilde{X}_l) \in \mathbb{R}^{dT} \quad \text{and denote} \\ Q &:= \text{blkdiag}(X_l^\top \otimes I_d)_{l=1}^m \quad \text{with} \\ a_l^* &:= \text{vec}(A_l^*) \quad \text{and} \quad \eta_l := \text{vec}(E_l).\end{aligned}\tag{2.1}$$

We form $a^* \in \mathbb{R}^{md^2}$, $\tilde{x} \in \mathbb{R}^{mdT}$ and $\eta \in \mathbb{R}^{mdT}$ by column-stacking a_l^* 's, \tilde{x}_l 's and η_l 's respectively. Then the model (1.1) can be rewritten as

$$\tilde{x} = Qa^* + \eta.\tag{2.2}$$

Notice that while the diagonal blocks of Q are respectively independent, the entries within each diagonal block are dependent. Moreover, each block of η , i.e. η_l , is dependent on the corresponding diagonal block $X_l^\top \otimes I_d$ of Q . These dependencies constitute the main challenges in our theoretical analysis.

Denote $D \in \{-1, 1, 0\}^{|\mathcal{E}| \times m}$ to be the incidence matrix of any edge-orientation of G , and set $\tilde{D} = D \otimes I_{d^2}$. Then for any $a \in \mathbb{R}^{md^2}$ formed by column-stacking $a_1, \dots, a_m \in \mathbb{R}^{d^2}$, we have

$$\tilde{D}a = \begin{pmatrix} \vdots \\ a_l - a_{l'} \\ \vdots \end{pmatrix} \in \mathbb{R}^{|\mathcal{E}|d^2}.$$

Consequently, (1.2) can be written as

$$(\hat{a}_1, \dots, \hat{a}_m) = \underset{a \in \mathbb{R}^{md^2}}{\operatorname{argmin}} \left\{ \frac{1}{2m} \|\tilde{x} - Qa\|_2^2 + \lambda \|\tilde{D}a\|_1 \right\}.\tag{2.3}$$

In order to analyze the performance of (2.3), the geometry of the underlying graph will play a central role. In the next section, we introduce a set of key quantities that capture this geometry and govern the statistical and computational behavior of our estimator.

2.3 Graph-geometry parameters

Throughout the sequel we shall repeatedly invoke a small collection of graph-dependent scalar quantities that capture key geometric properties of the underlying graph $G = ([m], \mathcal{E})$, and influence the difficulty of joint parameter recovery across the network.

Fiedler eigenvalue. Denote by

$$L := D^\top D \in \mathbb{R}^{m \times m},$$

the combinatorial Laplacian and let L^\dagger be the pseudoinverse of L . Denote the eigenvalues of $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_{m-1}(G) > \lambda_m(G) (= 0)$. Throughout, $\lambda_{m-1}(G)$ (or simply λ_{m-1}) stands for the *algebraic connectivity*—the smallest positive eigenvalue of L (also known as the Fiedler eigenvalue of G). It is well known that $\lambda_{m-1} > 0$ iff G is connected, and more generally, λ_{m-1} measures how well-knit the graph is. A small value indicates bottlenecks or sparse regions, making global information propagation harder.

Inverse scaling factors. The inverse scaling factors of D are defined as follows.

$$D^\dagger = [s_1 \ s_2 \ \dots \ s_{|\mathcal{E}|}], \quad \mu := \max_{j \in [|\mathcal{E}|]} \|s_j\|_2,$$

and

$$(D^\dagger)^\top = [s'_1 \ s'_2 \ \dots \ s'_m], \quad \mu' := \max_{l \in [m]} \|s'_l\|_2.$$

These quantify how “spread out” the rows and columns of the (pseudo) inverse incidence matrix are – small values of μ, μ' enable better estimates of the model parameters. Interestingly, while μ appeared in the work [12] for TV based denoising on graphs, we will additionally have the term μ' appearing in our analysis. It is unclear whether this is an artefact of the analysis, or is intrinsic to the problem. While μ is typically bounded via arguments specialized to G (see [12]), it is also possible to bound both μ and μ' using λ_{m-1} .

Proposition 1 (Bounds on inverse scaling factors). *Let D be the incidence matrix of a connected graph G . Then,*

$$\mu \leq \frac{\sqrt{2}}{\lambda_{m-1}} \wedge \frac{1}{\sqrt{\lambda_{m-1}}} \quad \text{and} \quad \mu' \leq \frac{1}{\sqrt{\lambda_{m-1}}}$$

Proof. To bound μ' , note that $\mu' \leq \|(D^\dagger)^\top\|_2 = \|D^\dagger\|_2$ and also $\|D^\dagger\|_2 = \frac{1}{\sqrt{\lambda_{m-1}}}$. The same bound is readily seen to apply for μ , while the bound $\mu \leq \frac{\sqrt{2}}{\lambda_{m-1}}$ is taken from [12, Prop. 14]. \square

Compatibility factors. These quantities characterize the relationship between sparsity in edge differences (TV) and energy in the parameter vector. Larger κ ensures that sparse total-variation structure leads to well-behaved solutions. The notion of the so-called compatibility factor (see [12]) will be crucial. We recall it below.

Definition 1. *Denoting $\tilde{D} = D \otimes I_{d^2}$, the compatibility factor of \tilde{D} for a set $\mathcal{T} \subseteq [d^2 |\mathcal{E}|]$ is defined as*

$$\kappa_\emptyset = 1, \quad \kappa_{\mathcal{T}} := \inf_{\theta \in \mathbb{R}^{md^2}} \frac{\sqrt{|\mathcal{T}|} \|\theta\|_2}{\|(\tilde{D}\theta)_{\mathcal{T}}\|_1} \text{ for } \mathcal{T} \neq \emptyset.$$

Moreover, we denote $\kappa := \min_{\mathcal{T} \subseteq [d^2 |\mathcal{E}|]} \kappa_{\mathcal{T}}$.

As a small remark, note that the compatibility factor is defined for \tilde{D} in our setup, as opposed to D in [12]. We refer the reader to [12] where upper bounds on μ were derived for different types of graphs. For the compatibility factor $\kappa_{\mathcal{T}}$, it is not difficult to obtain a lower bound for bounded degree graphs, along the lines of the proof of [12, Lemma 3]. We just note that given any $\emptyset \neq \mathcal{T} \subseteq [d^2 |\mathcal{E}|]$, we can denote the subset of $[\mathcal{E}]$ “appearing” in \mathcal{T} as

$$\mathcal{T}_{\mathcal{E}} := \{j \in [\mathcal{E}] : \exists i \in \mathcal{T} \text{ s.t. } (j-1)d^2 + 1 \leq i \leq jd^2\}. \quad (2.4)$$

Proposition 2 (Bounds on compatibility factor). *Let $D \in \{-1, 1, 0\}^{|\mathcal{E}| \times m}$ be the incidence matrix of a graph $G = ([m], \mathcal{E})$ with maximal degree Δ_{\deg} . For any $\emptyset \neq \mathcal{T} \subseteq [d^2 |\mathcal{E}|]$, consider $\mathcal{T}_{\mathcal{E}} \subseteq [\mathcal{E}]$ as defined in (2.4). Then,*

$$\kappa_{\mathcal{T}} \geq \frac{1}{2 \min \left\{ \sqrt{\Delta_{\deg}}, \sqrt{|\mathcal{T}_{\mathcal{E}}|} \right\}}.$$

The proof is outlined in Appendix A. The main difference from [12, Lemma 3] is the appearance of $|\mathcal{T}_{\mathcal{E}}|$ instead of $|\mathcal{T}|$ in the bound. In Table 1, we summarize the values of these graph-dependent scalar quantities for several commonly studied graph models. These will be invoked in the next section for instantiating our main result.

Graph Model	μ (Inverse Scaling)	μ' (Alt. Inverse Scaling)	$\kappa_{\mathcal{T}}$ (Compatibility Factor)
2D Grid	$\lesssim \sqrt{\log m}$ ([12])	$\lesssim \sqrt{\log m}$ (App. F.1)	$\gtrsim C$ (Prop. 2)
Complete Graph	$\lesssim 1/m$ (Prop. 1)	$\lesssim 1/\sqrt{m}$ (Prop. 1)	$\gtrsim 1/\sqrt{m}$ (Prop. 2)
Star Graph	≤ 1 (Prop. 1)	≤ 1 (Prop. 1)	$\gtrsim 1/\sqrt{ \mathcal{T}_{\mathcal{E}} }$ (Prop. 2)
Erdős–Rényi $G(m, p)$	$\lesssim 1/\sqrt{mp}$ (Prop. 1)	$\lesssim 1/\sqrt{mp}$ (Prop. 1)	$\gtrsim 1/\sqrt{mp}$ (Prop. 2)

Table 1: Summary of bounds for inverse scaling factors μ, μ' and compatibility factor $\kappa_{\mathcal{T}}$ for some graph models.

2.4 Main results

Our results will depend on the so-called controllability Grammian of each system $l \in [m]$,

$$\Gamma_t(A_l^*) := \sum_{k=0}^t (A_l^*)^k ((A_l^*)^k)^\top \in \mathbb{R}^{d \times d}$$

We will also require that the spectral norms of the following matrices (for each $l \in [m]$) are bounded.

$$\tilde{A}_l^* := \begin{bmatrix} I_d & 0 & \dots & 0 \\ A_l^* & I_d & \dots & 0 \\ \vdots & & \ddots & \vdots \\ (A_l^*)^{T-1} & \dots & A_l^* & I_d \end{bmatrix}, \quad \beta := \max_l \left\| \tilde{A}_l^* \right\|_2. \quad (2.5)$$

Finally, denoting

$$G_l = \sum_{t=1}^T \Gamma_{t-1}(A_l^*) \quad \text{and} \quad \bar{G} = \frac{1}{m} \sum_{l=1}^m G_l,$$

our results will depend on the quantity

$$\Delta_G := \max_{a,b \in [d]} \left(\sum_{l=1}^m \left[(G_l)_{b,a} - (\bar{G})_{b,a} \right]^2 \right)^{1/2}. \quad (2.6)$$

Conceptually, Δ_G denotes the maximum (over $a, b \in [d]$) standard deviation of the entries $((G_l)_{b,a})_{l=1}^m$. The following theorem is our main result in full generality, its proof is detailed in Section 3.

Theorem 1. *There exist constants $c < 1/6$, $c' > 0$, and $c_1 \in (0, 1)$ such that the following is true. Let $\delta \in (0, c_1)$, $v \geq 1$, and consider $\mathcal{S} \subseteq [|\mathcal{E}|d^2]$. For F_1, F_2, F_3 as defined in Lemmas 1 - 3 respectively, and Δ_G as in (2.6), suppose $\lambda \geq \frac{2}{m} \max \{F_1, F_2\}$, and the following conditions are satisfied.*

1. $F_3\sqrt{v} \leq cT$,
2. $\frac{\beta^2}{m}(\sqrt{mT} + d)(d + v) \leq cT$, and
3. $\frac{\mu}{\sqrt{m}} \left(1 + \frac{\sqrt{|\mathcal{S}|}}{\kappa_S} + \|(\tilde{D}a^*)_{S^c}\|_1 \right) \left[d\Delta_G + \beta^2 \sqrt{T} \log \left(\frac{|\mathcal{E}|d}{\delta} \right) \right] \leq cT$.

Then w.p at least $1 - 4 \exp(-c'\sqrt{v}) - \delta$,

$$\|\hat{a} - a^*\|_2 \leq \frac{2m\lambda}{T} \left(1 + 3 \frac{\sqrt{|\mathcal{S}|}}{\kappa_S} \right) + \sqrt{8 \frac{\lambda m}{T} \|(\tilde{D}a^*)_{S^c}\|_1}.$$

The statement of Theorem 1 captures the dependence on A_l^* via the terms β , $\text{Tr}(\Gamma_t(A_l^*))$ and Δ_G , as can be seen from the definitions of F_1, F_2 and F_3 . Conditions 1 - 3 arise in the course of ensuring that the matrix Q in (2.3) satisfies the Restricted Eigenvalue (RE) condition; see Lemma's 3, 4 and 5. The main tool used here is the concentration bound of [15] for controlling the suprema of second order subgaussian chaos processes involving positive semidefinite (p.s.d) matrices. The terms F_1 and F_2 arise on account of upper bounding the quantities on the RHS of (3.2), see Lemmas 1 and 2. Here, the dependencies inherent in our observations lead us to use tail bounds for the norm of self-normalized vector-valued martingales using ideas from the proof of [1, Theorem 1].

Remark 1. *As noted in Section 1.2, existing theoretical results for TV-penalized regression problems (e.g. [20, 35]) assume that the rows of the design matrix are i.i.d centered Gaussian's (with a covariance Σ), and establish the RE condition by adapting the technique of [25]. In our setting, due to the nature of the block-diagonal design matrix Q containing dependent rows within each block, we find it convenient to establish the RE condition by a different approach. By first showing that the error term $\hat{a} - a^*$ lies in \mathcal{C}_S with \mathcal{C}_S defined in (3.5), we decompose any $h \in \mathcal{C}_S$ as $h = h_1 + h_2$ where h_1 lies in the range space of \tilde{D} while h_2 lies in the null space of \tilde{D} . This allows us to ensure the RE condition by using the inequality (3.7), and suitably controlling each of the three quantities on the RHS therein (via Lemmas 3, 4 and 5).*

Furthermore, the results in [20, 35] also assume that the noise is independent of the design matrix, while in our case, the noise term η in (2.2) shares dependencies with the design Q at a block level. This leads us to use a different approach wherein we bound the quantities on the RHS of (3.2) using tail bounds for self-normalized vector valued martingales (see Lemmas 1 and 2).

In order to interpret Theorem 1, we now consider for convenience the particular case where there exists a uniform constant $\rho_{\max} < 1$ such that

$$\|A_l^*\|_2 \leq \rho_{\max}, \quad l = 1, \dots, m. \quad (2.7)$$

This condition ensures that each system matrix is Schur stable, i.e., $\rho(A_l^*) < \rho_{\max} < 1$ for each l . Under this condition, we can bound β , as well as F_1, F_2, F_3 , thus leading to useful corollaries of Theorem 1. To this end, set

$$\begin{aligned} \Delta &:= (1 - \rho_{\max})^2, & L_1 &:= \log\left(\frac{dT}{\delta\Delta}\right), & L_2 &:= \log\left(\frac{d|\mathcal{E}|}{\delta}\right), \\ \text{and } \Phi_S &= 1 + \frac{\sqrt{|S|}}{\kappa_S} + \|(\tilde{D}a^*)_{S^c}\|_1, \end{aligned}$$

with κ_S the compatibility factor from Def. 1 and $S \subseteq [|S|d^2]$ arbitrary. Under $\rho_{\max} < 1$, we have

$$\beta \leq (1 - \rho_{\max})^{-1} = \Delta^{-1/2} \quad \text{and} \quad \text{Tr}(\Gamma_t(A_l^*)) \leq d/\Delta$$

which then allows us to suitably bound the terms F_1, F_2 and F_3 ; see Section B for details.

Choice of λ . Now Theorem 1 requires λ to satisfy

$$\lambda \geq \frac{2}{m} \max\{F_1, F_2\}, \quad \text{with } F_1, F_2 \text{ from Lemmas 1 - 2.}$$

In the stable case, using the aforementioned bounds on F_1, F_2 , this yields the following valid choice for λ (for a suitably large constant $c_1 > 0$)

$$\lambda = \frac{c_1}{m} \sqrt{\frac{T}{\Delta}} \max\left\{ d^{3/2} L_1, \mu L_2 \right\}. \quad (2.8)$$

Sample-size conditions. The above considerations allow us to obtain a simplified set of sufficient conditions (for a suitably large constant $C > 0$) which ensure Conditions 1-3 in Theorem 1.

$$(C1) \quad T \geq C \frac{v}{\Delta^2} \left(1 + \mu' \Phi_S \sqrt{L_2}\right)^2 \quad (\text{from Condition 1 in Theorem 1}), \quad (2.9)$$

$$(C2) \quad T \geq C \frac{(d+v)^2}{m \Delta^2} \quad (\text{from Condition 2 in Theorem 1}), \quad (2.10)$$

$$(C3a) \quad T \geq C \frac{\mu^2 \Phi_S^2 L_2^2}{m \Delta^2} \quad (\text{from Condition 3 in Theorem 1}), \quad (2.11)$$

$$(C3b) \quad T \geq C \frac{\mu}{\sqrt{m}} \Phi_S d \Delta_G \quad (\text{from Condition 3 in Theorem 1}). \quad (2.12)$$

Remark 2 (Interpreting and controlling Δ_G). Note that the dispersion functional Δ_G defined in (2.6) can be bounded as

$$\Delta_G \leq \left(\sum_{l=1}^m \|G_l - \bar{G}\|_F^2 \right)^{1/2},$$

so any mechanism that suppresses edgewise variations of G_l (e.g., graph smoothness) controls Δ_G . Under Schur stability $\|A_l^*\|_2 \leq \rho_{\max} < 1$, we also have the crude bounds $\Gamma_{t-1}(A_l^*) \preceq \Delta^{-1} I_d$ with $\Delta := (1 - \rho_{\max})^2$, hence $\|G_l\|_2 \leq T/\Delta$ and $\|G_l\|_F \leq \sqrt{d}T/\Delta$. Therefore, in the worst case (no cross-node cancellations) one gets

$$\Delta_G \lesssim \frac{T}{\Delta} \sqrt{md},$$

while whenever the G_l 's vary smoothly across edges the control of Δ_G can be much tighter. In particular, in Appendix G we show that

$$\Delta_G \leq \frac{L_T(\rho_{\max})}{\sqrt{\lambda_{m-1}(G)}} \left(\sum_{\{l,l'\} \in E} \|A_l^* - A_{l'}^*\|_F^2 \right)^{1/2} \quad (\text{see Lemma 9}) \quad (2.13)$$

where $L_T(\rho_{\max}) \leq \frac{2\rho_{\max} T}{(1-\rho_{\max}^2)^2}$. Hence Δ_G is small when the system matrices $(A_l^*)_{l=1}^m$ vary little across the edges of G . A small Δ_G relaxes the condition (2.12), thus weakening the requirement on T . Finally, observe that the numerator of (2.13) can be upper bounded in terms of the total-variation $\sum_{\{l,l'\} \in E} \|A_l^* - A_{l'}^*\|_{1,1}$ in a straightforward manner.

We now have the following useful corollary of Theorem 1 when each A_l^* is stable. Its proof is outlined in Section B.

Corollary 1 (Stable A_l^*). Assume $\|A_l^*\|_2 \leq \rho_{\max} < 1$ for all l , choose λ as in (2.8), and suppose the sample-size side conditions (2.9)–(2.12) hold for some $\delta \in (0, c_2)$, $v \geq 1$ for a suitably small constant $c_2 < 1$. Then there exists a constant $c' > 0$ such that, w.p at least $1 - 4\exp(-c'\sqrt{v}) - \delta$,

$$\|\hat{a} - a^*\|_2 \leq \frac{2m\lambda}{T} \left(1 + 3 \frac{\sqrt{|\mathcal{S}|}}{\kappa_S}\right) + \sqrt{8 \frac{m\lambda}{T} \left\|(\tilde{D}a^*)_{\mathcal{S}^c}\right\|_1}. \quad (2.14)$$

Two natural choices for S are the empty set $S = \emptyset$ in the “smooth regime”, where the signal is smooth over the network, and $S = \text{supp}(\tilde{D}a^*)$ in the “few-changes regime”, wherein the signal is expected to have a small number of changes over the network. Next we specialize our results to these two cases. Define the geometry-log multiplier

$$\mathfrak{M} := \Delta^{-1/2} \max\{d^{3/2}L_1, \mu L_2\}.$$

Corollary 2 (Two canonical choices of S). *Under the assumptions of Corollary 1 and with λ as in (2.8), there exist absolute $C, c' > 0$ such that, with probability at least $1 - 4e^{-c'\sqrt{v}} - \delta$,*

$$\frac{1}{\sqrt{m}} \|\hat{a} - a^*\|_2 \leq C \left(\frac{\mathfrak{M}}{\sqrt{mT}} \left(1 + \frac{\sqrt{|S|}}{\kappa_S} \right) + \sqrt{\frac{\mathfrak{M}}{m\sqrt{T}} \|(\tilde{D}a^*)_{S^c}\|_1} \right). \quad (2.15)$$

The sample-size side conditions are (2.9)–(2.12). In particular, the following holds.

- **Smooth regime** ($S = \emptyset, \kappa_\emptyset = 1$):

$$\frac{1}{\sqrt{m}} \|\hat{a} - a^*\|_2 \lesssim \frac{\mathfrak{M}}{\sqrt{mT}} + \sqrt{\frac{\mathfrak{M}}{m\sqrt{T}} \|\tilde{D}a^*\|_1}.$$

Here $\Phi_S = \|\tilde{D}a^*\|_1 + 1$.

- **Few-changes regime** ($S = \text{supp}(\tilde{D}a^*), s = \|\tilde{D}a^*\|_0$):

$$\frac{1}{\sqrt{m}} \|\hat{a} - a^*\|_2 \lesssim \frac{\mathfrak{M}}{\sqrt{mT}} \left(1 + \frac{\sqrt{s}}{\kappa_S} \right), \quad \Phi_S = 1 + \frac{\sqrt{s}}{\kappa_S}.$$

Remark 3 (Regime selection). Equating the two bounds in Corollary 2 yields the switch criterion $\|\tilde{D}a^*\|_1 \lesssim \frac{\mathfrak{M}}{\sqrt{T}} (s/\kappa_S^2)$, favoring $S = \emptyset$ when many changes are tiny (small TV magnitude), and $S = \text{supp}(\tilde{D}a^*)$ when the number of changes s is small on a graph with large κ_S .

Notice from the sampling conditions (2.9)–(2.12) that $T \gtrsim \frac{v}{\Delta^2}$ is necessary for them to be satisfied. In case $T = c \frac{v}{\Delta^2}$ for a large enough constant c , then (2.9) would hold provided $\mu' = o(1)$ as m increases. More precisely, μ' would need to go to zero sufficiently fast w.r.t m .

Corollary 3 (Small- T). Assume $\|A_l^*\|_2 \leq \rho_{\max} < 1$ for all l . There exists $c_1 < 1$ such that for any $\delta \in (0, c_1)$, the following holds. Let $v \asymp \log^2(1/\delta)$, $T \asymp v/\Delta^2$ (so $T \asymp \frac{\log^2(1/\delta)}{\Delta^2}$). If in addition

$$(i) \text{ (Condition 2.10) } m \gtrsim \frac{d^2}{\log^2(1/\delta)} + \log^2(1/\delta),$$

- (ii) **Smooth regime** ($S = \emptyset$):

$$\mu' \left(1 + \|\tilde{D}a^*\|_1 \right) \lesssim 1/\sqrt{\log(|\mathcal{E}| d/\delta)}, \quad (\text{Condition 2.9})$$

$$\mu \left(1 + \|\tilde{D}a^*\|_1 \right) \lesssim \min \left\{ \frac{\sqrt{m} \log^2(1/\delta)}{d\Delta_G \Delta^2}, \frac{\sqrt{m} \log(1/\delta)}{\log(d|\mathcal{E}|/\delta)} \right\}, \quad (\text{Conditions 2.11, 2.12})$$

- (iii) **Few-changes regime** ($S = \text{supp}(\tilde{D}a^*), s = \|\tilde{D}a^*\|_0$):

$$\mu' \left(1 + \frac{\sqrt{s}}{\kappa_S} \right) \lesssim 1/\sqrt{\log(|\mathcal{E}| d/\delta)}, \quad (\text{Condition 2.9})$$

$$\mu \left(1 + \frac{\sqrt{s}}{\kappa_S} \right) \lesssim \min \left\{ \frac{\sqrt{m} \log^2(1/\delta)}{d\Delta_G \Delta^2}, \frac{\sqrt{m} \log(1/\delta)}{\log(d|\mathcal{E}|/\delta)} \right\}, \quad (\text{Conditions 2.11, 2.12})$$

then with λ as in (2.8) the corresponding bounds from Corollary 2 hold w.p at least $1 - \delta$.

Let us now instantiate Corollary 3 for two examples of well connected graphs G for which μ and μ' go to zero sufficiently fast w.r.t m . For simplicity, we focus on the smooth-regime ($S = \emptyset$).

Example 1 (Complete graph, $S = \emptyset$, small T). For the complete graph, $\lambda_{m-1} = m$, and Proposition 1 yields $\mu \lesssim 1/m$, $\mu' \lesssim 1/\sqrt{m}$. Since $|\mathcal{E}| \asymp m^2$, $T \asymp \frac{\log^2(1/\delta)}{\Delta^2}$ and $m \gtrsim \frac{d^2}{\log^2(1/\delta)} + \log^2(1/\delta)$, hence the conditions involving μ, μ' in Corollary 3 are satisfied provided

$$(1 + \|\tilde{D}a^*\|_1) \lesssim \min \left\{ \frac{m^{3/2} \log^2(1/\delta)}{d\Delta_G \Delta^2}, \frac{\sqrt{m}}{\sqrt{\log(md/\delta)}} \right\}.$$

Moreover, we have that $\mathfrak{M} \lesssim \frac{d^{3/2}}{\Delta^{1/2}} \log(\frac{dT}{\delta\Delta})$. Consequently, for λ as in (2.8),

$$\frac{1}{m} \|\hat{a} - a^*\|_2^2 \lesssim \left(\frac{d^3 \log^2(\frac{dT}{\delta\Delta})}{\Delta m T} + \frac{d^{3/2} \log(\frac{dT}{\delta\Delta}) \|\tilde{D}a^*\|_1}{\Delta^{1/2} m \sqrt{T}} \right)$$

holds w.p at least $1 - \delta$. Hence if $\|\tilde{D}a^*\|_1$ is small enough, then the MSE goes to zero w.r.t m , even when T is fixed.

Example 2 (Erdős–Rényi $G(m, p)$, $S = \emptyset$, small T). It is well known that for any $\delta \in (0, 1)$,

$$\mathbb{P}(\lambda_{m-1} \geq mp/2) \geq 1 - \delta$$

holds provided $p \geq c \log(m/\delta)/m$ for some suitably large constant $c > 0$ (see e.g., [36, Prop. 4]). Then, Proposition 1 gives $\mu, \mu' \lesssim (mp)^{-1/2}$. Now the quantity $\|\tilde{D}a^*\|_1$ is dependent on G , and hence also random, however it is possible to control it via Bernstein's inequality. Indeed, note that

$$\|\tilde{D}a^*\|_1 = \sum_{l < l'} \|a_l^* - a_{l'}^*\|_1 Y_{l,l'}$$

where $Y_{l,l'} \stackrel{i.i.d.}{\sim} \mathcal{B}(p)$ (i.i.d Bernoulli random variables) for each $l < l' \in [m]$. Then denoting

$$S_1 := \sum_{l < l'} \|a_l^* - a_{l'}^*\|_1, \quad S_2 := \sqrt{\sum_{l < l'} \|a_l^* - a_{l'}^*\|_1^2}, \quad S_3 := \max_{l < l'} \|a_l^* - a_{l'}^*\|_1$$

it follows readily from Bernstein's inequality (see e.g., [5]) that w.p $\geq 1 - 2n^{-c}$ (for constants $c, c_1 > 0$),

$$\|\tilde{D}a^*\|_1 \leq c_1 \left(pS_1 + \sqrt{p \log m} S_2 + S_3 \log m \right).$$

Since $|\mathcal{E}| \leq \frac{m(m-1)}{2}$ a.s, $T \asymp \frac{\log^2(1/\delta)}{\Delta^2}$ and $m \gtrsim \frac{d^2}{\log^2(1/\delta)} + \log^2(1/\delta)$, hence the conditions involving μ, μ' in Corollary 3 are satisfied provided

$$\frac{1 + pS_1 + \sqrt{p \log m} S_2 + S_3 \log m}{\sqrt{mp}} \lesssim \min \left\{ \frac{\sqrt{m} \log^2(1/\delta)}{d\Delta_G \Delta^2}, \frac{1}{\log(md/\delta)} \right\}. \quad (2.16)$$

Note that (2.16) is ensured, for instance, if $p \asymp 1$ and $S_1 + S_2 + S_3 = o(\sqrt{m}/\text{polylog}(m))$.

Now, we also have

$$\mathfrak{M} \lesssim \frac{1}{\sqrt{\Delta}} \max \left\{ d^{3/2} \log\left(\frac{dT}{\delta\Delta}\right), \frac{\log\left(\frac{dm}{\delta}\right)}{\sqrt{mp}} \right\} =: \mathfrak{M}^*.$$

Hence putting everything together, we have for $\lambda \asymp \mathfrak{M}^* \frac{\sqrt{T}}{m}$ that w.p at least $1 - \delta - 2n^{-c}$,

$$\frac{1}{m} \|\hat{a} - a^*\|_2^2 \lesssim \frac{(\mathfrak{M}^*)^2}{mT} + \frac{\mathfrak{M}^* (pS_1 + \sqrt{p \log m} S_2 + S_3 \log m)}{m\sqrt{T}}.$$

Hence if $(A_l^*)_{l=1}^m$ are sufficiently smooth w.r.t the complete graph (in the sense that S_1, S_2 and S_3 are suitably small), and p is suitably large, then the MSE goes to zero w.r.t m , even when T is fixed.

Next we instantiate Corollary 2 for examples of G where T is required to grow with m , albeit mildly, for consistency of the MSE (w.r.t m). For simplicity, we again focus on the smooth-regime ($\mathcal{S} = \emptyset$).

Example 3 (Star graph, $S = \emptyset$, moderate-sized T). For the star graph $\lambda_{m-1} = 1$, and Proposition 1 yields $\mu, \mu' \leq 1$. Also, $|\mathcal{E}| \asymp m$. Take $v \asymp \log^2(1/\delta)$ as before, then conditions 2.9-2.12 correspond to

$$T \gtrsim \max \left\{ \frac{\log^2(1/\delta)}{\Delta^2} \Phi_S^2 \log(dm/\delta), \frac{d^2 + \log^4(1/\delta)}{m \Delta^2}, \frac{\Phi_S d \Delta_G}{\sqrt{m}} \right\},$$

where we recall $\Phi_S = 1 + \|\tilde{D} a^*\|_1$.

To get a feel for the above condition on T , consider the very smooth regime where $\|\tilde{D} a^*\|_1, \Delta_G \leq c$ for some constant $c \geq 1$. Then if $m \gtrsim d^2 + \log^4(1/\delta)$, we require $T \gtrsim \frac{\log^2(1/\delta)}{\Delta^2} \log(dm/\delta)$ i.e., T is only required to grow logarithmically w.r.t both m and dimension d .

We also have the bound

$$\mathfrak{M} \lesssim \frac{1}{\sqrt{\Delta}} \max \left\{ d^{3/2} \log \left(\frac{dT}{\delta \Delta} \right), \log(dm/\delta) \right\} =: \mathfrak{M}_1^*,$$

hence, for λ as in (2.8),

$$\frac{1}{m} \|\hat{a} - a^*\|_2^2 \lesssim \frac{(\mathfrak{M}_1^*)^2}{mT} + \frac{\mathfrak{M}_1^*}{m\sqrt{T}} \|\tilde{D} a^*\|_1$$

holds w.p at least $1 - \delta$.

Example 4 (2D grid, $S = \emptyset$, moderate-sized T). For the 2D grid, we have $\mu, \mu' \lesssim \sqrt{\log m}$ as shown in Table 1. Also, $|\mathcal{E}| \asymp m$. Take $v \asymp \log^2(1/\delta)$ as before, then conditions 2.9-2.12 correspond to

$$T \gtrsim \max \left\{ \frac{\log^2(1/\delta)}{\Delta^2} \Phi_S^2 \log^2(dm/\delta), \frac{d^2 + \log^4(1/\delta)}{m \Delta^2}, \Phi_S d \Delta_G \sqrt{\frac{\log m}{m}} \right\},$$

where $\Phi_S = 1 + \|\tilde{D} a^*\|_1$. As before, consider for simplicity the very smooth regime where $\Phi_S, \Delta_G \leq c$ for a constant $c \geq 1$. Then if $\frac{m}{\log m} \gtrsim d^2 + \log^4(1/\delta)$, we require $T \gtrsim \frac{\log^2(1/\delta)}{\Delta^2} \log^2(dm/\delta)$.

Using the bound

$$\mathfrak{M} \lesssim \frac{1}{\sqrt{\Delta}} \max \left\{ d^{3/2} \log \left(\frac{dT}{\delta \Delta} \right), \log^{3/2}(dm/\delta) \right\} =: \mathfrak{M}_2^*.$$

we then have for λ as in (2.8), that

$$\frac{1}{m} \|\hat{a} - a^*\|_2^2 \lesssim \frac{(\mathfrak{M}_2^*)^2}{mT} + \frac{\mathfrak{M}_2^*}{m\sqrt{T}} \|\tilde{D} a^*\|_1$$

holds w.p at least $1 - \delta$.

Remark 4 (Estimating each A_l^* separately). As a sanity check, it is worth comparing the error bounds in the above examples with that obtained by simply estimating each A_l^* individually via the ordinary least-squares (OLS) estimator. Let $\widehat{A}_{OLS,l}$ be the OLS estimate at node l , then it was shown in [13] that w.p at least $1 - \delta$,

$$\left\| \widehat{A}_{OLS,l} - A_l^* \right\|_2 \lesssim \sqrt{\frac{\log(1/\delta) + d}{T}} \quad \text{if } T \gtrsim \Delta^{-1}(\log(1/\delta) + d).$$

By taking a union bound, this means that w.p at least $1 - \delta$,

$$\frac{1}{m} \sum_{l=1}^m \left\| \widehat{A}_{OLS,l} - A_l^* \right\|_F^2 \lesssim \frac{d \log(m/\delta) + d^2}{T} \quad \text{if } T \gtrsim \Delta^{-1}(\log(m/\delta) + d). \quad (2.17)$$

For sufficiently smooth A_l^* 's, we saw in Examples 3 and (4) that the dependence of T can at times be logarithmic w.r.t both m and d . Furthermore, notice that if $T = \Theta(\log(m))$, then the MSE in (2.17) is $O(1)$ w.r.t m while the corresponding bounds in Examples 3 and 4 are $o(1)$.

3 Proof of Theorem 1

Since \widehat{a} is a solution of (2.3) and a^* is feasible, we have

$$\frac{1}{2m} \|\tilde{x} - Q\widehat{a}\|_2^2 + \lambda \|\tilde{D}\widehat{a}\|_1 \leq \frac{1}{2m} \|\tilde{x} - Qa^*\|_2^2 + \lambda \|\tilde{D}a^*\|_1.$$

Using (2.2) in the above inequality, we obtain after some simple calculations

$$\frac{1}{2m} \|Q(a^* - \widehat{a})\|_2^2 \leq \frac{1}{m} \langle \widehat{a} - a^*, Q^\top \eta \rangle + \lambda \|\tilde{D}a^*\|_1 - \lambda \|\tilde{D}\widehat{a}\|_1. \quad (3.1)$$

Let $\Pi, \widetilde{\Pi}$ denote the projection matrices for $\text{null}(D)$ and $\text{null}(\tilde{D})$ respectively. Since G is connected, we have

$$\Pi = \frac{1}{m} \mathbf{1}_m \mathbf{1}_m^\top \quad \text{and} \quad \widetilde{\Pi} = (\frac{1}{m} \mathbf{1}_m \mathbf{1}_m^\top) \otimes I_{d^2}.$$

Moreover, $\tilde{D}^\dagger \tilde{D} = (D^\dagger D) \otimes I_{d^2} = I_{md^2} - \widetilde{\Pi}$ is the projection matrix for the orthogonal complement of $\text{null}(\tilde{D})$. With this in mind, we can bound

$$\begin{aligned} \langle \widehat{a} - a^*, Q^\top \eta \rangle &= \eta^\top Q \widetilde{\Pi}(\widehat{a} - a^*) + \eta^\top Q(\tilde{D}^\dagger \tilde{D})(\widehat{a} - a^*) \\ &\leq \left\| \widetilde{\Pi} Q^\top \eta \right\|_2 \|\widehat{a} - a^*\|_2 + \left\| ((D^\dagger)^\top \otimes I_{d^2}) Q^\top \eta \right\|_\infty \left\| \tilde{D}(\widehat{a} - a^*) \right\|_1. \end{aligned} \quad (3.2)$$

The following lemma's bound the terms $\left\| \widetilde{\Pi} Q^\top \eta \right\|_2$ and $\left\| ((D^\dagger)^\top \otimes I_{d^2}) Q^\top \eta \right\|_\infty$. The crux of the proof is based on tail bounds for the norm of self-normalized vector-valued martingales, using ideas in the proof of [1, Theorem 1].

Lemma 1. There exist constants $c_1 > 0$, $c_2 \in (0, 1)$ such that the following is true. For any $\delta \in (0, 1)$, denote

$$\zeta_1(m, T, \delta) := c_1 \left(\sum_{l=1}^m \sum_{t=0}^{T-1} \text{Tr}(\Gamma_t(A_l^*)) \right) \log(1/\delta).$$

Then for $\delta \in (0, c_2)$, it holds with probability at least $1 - 2\delta$ that

$$\left\| \widetilde{\Pi} Q^\top \eta \right\|_2 \leq \sqrt{2} \left(\frac{\zeta_1(m, T, \delta)}{m} + 1 \right)^{1/2} \left(\log(1/\delta) + \frac{d^2}{2} \log \left(\frac{\zeta_1(m, T, \delta)}{m} + 1 \right) \right)^{1/2} =: F_1.$$

Lemma 2. *There exist constants $c_1, c_2 > 0$ and $c_3 \in (0, 1)$ such that the following is true. For any $\delta \in (0, 1)$, denote*

$$\zeta_2(m, T, \delta) := c_1 \mu^2 \left[\max_{l \in [m], i \in [d]} e_i^\top \left(\sum_{t=1}^T \Gamma_{t-1}(A_l^*) \right) e_i \right] \log^2 \left(\frac{d^2 |\mathcal{E}|}{\delta} \right).$$

Then for any $\delta \in (0, c_3)$, it holds with probability at least $1 - 2\delta$ that

$$\left\| ((D^\dagger)^\top \otimes I_{d^2}) Q^\top \eta \right\|_\infty \leq c_2 \zeta_2^{1/2}(m, T, \delta) =: F_2.$$

The proofs are provided in Sections 3.1 and 3.2. Conditioned on the events of Lemma's 1 and 2, we obtain from (3.1) and (3.2) that

$$\frac{1}{2m} \|Q(a^* - \hat{a})\|_2^2 \leq \frac{F_1}{m} \|\hat{a} - a^*\|_2 + \frac{F_2}{m} \left\| \tilde{D}(\hat{a} - a^*) \right\|_1 + \lambda \left\| \tilde{D}a^* \right\|_1 - \lambda \left\| \tilde{D}\hat{a} \right\|_1.$$

Choosing $\lambda \geq \frac{2}{m} \max\{F_1, F_2\}$, this implies

$$\frac{1}{m} \|Q(a^* - \hat{a})\|_2^2 \leq \lambda \|\hat{a} - a^*\|_2 + \lambda \left\| \tilde{D}(\hat{a} - a^*) \right\|_1 + 2\lambda \left\| \tilde{D}a^* \right\|_1 - 2\lambda \left\| \tilde{D}\hat{a} \right\|_1. \quad (3.3)$$

Now for any $\mathcal{S} \subseteq [d^2 |\mathcal{E}|]$, it is not difficult to verify that

$$\left\| \tilde{D}(\hat{a} - a^*) \right\|_1 + \left\| \tilde{D}a^* \right\|_1 - \left\| \tilde{D}\hat{a} \right\|_1 \leq 2 \left\| (\tilde{D}(\hat{a} - a^*))_{\mathcal{S}} \right\|_1 + 2 \left\| (\tilde{D}a^*)_{\mathcal{S}^c} \right\|_1.$$

Plugging this in (3.3), we obtain the inequality

$$\frac{1}{m} \|Q(a^* - \hat{a})\|_2^2 \leq \lambda \|\hat{a} - a^*\|_2 + 3\lambda \left\| (\tilde{D}(\hat{a} - a^*))_{\mathcal{S}} \right\|_1 + 4\lambda \left\| (\tilde{D}a^*)_{\mathcal{S}^c} \right\|_1 - \lambda \left\| (\tilde{D}(\hat{a} - a^*))_{\mathcal{S}^c} \right\|_1. \quad (3.4)$$

This implies that $\hat{a} - a^*$ lies in the set $\mathcal{C}_{\mathcal{S}}$, where

$$\mathcal{C}_{\mathcal{S}} := \left\{ h : \left\| (\tilde{D}h)_{\mathcal{S}^c} \right\|_1 \leq \|h\|_2 + 3 \left\| (\tilde{D}h)_{\mathcal{S}} \right\|_1 + 4 \left\| (\tilde{D}a^*)_{\mathcal{S}^c} \right\|_1 \right\}. \quad (3.5)$$

Moreover, (3.4) also implies

$$\begin{aligned} \frac{1}{m} \|Q(a^* - \hat{a})\|_2^2 &\leq \lambda \|\hat{a} - a^*\|_2 + 3\lambda \left\| (\tilde{D}(\hat{a} - a^*))_{\mathcal{S}} \right\|_1 + 4\lambda \left\| (\tilde{D}a^*)_{\mathcal{S}^c} \right\|_1 \\ &\leq \lambda \left(1 + 3 \frac{\sqrt{|\mathcal{S}|}}{\kappa_{\mathcal{S}}} \right) \|\hat{a} - a^*\|_2 + 4\lambda \left\| (\tilde{D}a^*)_{\mathcal{S}^c} \right\|_1 \quad (\text{recall Definition 1}). \end{aligned} \quad (3.6)$$

Note that conditioned on Lemmas 1 and 2, the inequality (3.6) holds simultaneously for all $\mathcal{S} \subseteq [|\mathcal{E}| d^2]$. From this point, our goal is to establish a restricted eigenvalue (RE) condition wherein for some $\kappa > 0$ and any fixed \mathcal{S} (the choice of which is arbitrary), $\|Qh\|_2^2 \geq \kappa \|h\|_2^2$ holds simultaneously for all $h \in \mathcal{C}_{\mathcal{S}}$. Plugging this in (3.6) will yield the desired bound on $\|\hat{a} - a^*\|_2$ after some simplifications.

RE analysis. Recall from earlier the definition of $\tilde{\Pi}$, we can write

$$h = h_1 + h_2 \text{ where } h_1 = \tilde{D}^\dagger \tilde{D}h \text{ and } h_2 = \tilde{\Pi}h.$$

Also note that for any $h \in \mathcal{C}_S$

$$\|Qh\|_2^2 \geq \left(\inf_{h \in \mathcal{C}_S \cap \mathbb{S}^{md^2-1}} \|Qh\|_2^2 \right) \|h\|_2^2$$

and so our focus now is on lower bounding the term within parentheses. To this end, we begin by observing that

$$\inf_{h \in \mathcal{C}_S \cap \mathbb{S}^{md^2-1}} \|Qh\|_2^2 \geq \inf_{h \in \mathcal{C}_S \cap \mathbb{S}^{md^2-1}} \|Qh_1\|_2^2 + \inf_{h \in \mathcal{C}_S \cap \mathbb{S}^{md^2-1}} \|Qh_2\|_2^2 - 2 \sup_{h \in \mathcal{C}_S \cap \mathbb{S}^{md^2-1}} |\langle Qh_1, Qh_2 \rangle|, \quad (3.7)$$

hence the strategy now is to suitably bound the terms in the RHS above. This is stated in the following lemma.

Lemma 3 (RE for h_1 term). *There exist constants $c_1, c_2, C_1 \geq 1$ such that the following is true. For any $v \geq 1$, it holds w.p at least $1 - 2 \exp(-c_2 \sqrt{v})$ that*

$$\forall h \in \mathcal{C}_S \cap \mathbb{S}^{md^2-1} : \|Qh_1\|_2^2 \geq T \|h_1\|_2^2 - c_1 F_3 - G_3 \sqrt{v},$$

where F_3, G_3 are defined as follows.

$$\begin{aligned} F_3 &:= C_1 \beta^2 \left[(\mu')^2 \left(4 \frac{\sqrt{|\mathcal{S}|}}{\kappa_S} + 4 \left\| (\tilde{D}a^*)_{\mathcal{S}^c} \right\|_1 + 1 \right)^2 \log(d^2 |\mathcal{E}|) \right. \\ &\quad \left. + \mu' \sqrt{T} \left(4 \frac{\sqrt{|\mathcal{S}|}}{\kappa_S} + 4 \left\| (\tilde{D}a^*)_{\mathcal{S}^c} \right\|_1 + 1 \right) \sqrt{\log(d^2 |\mathcal{E}|)} + \sqrt{T} \right], \\ G_3 &:= C_1 \beta^2 \left(\mu' \left(4 \frac{\sqrt{|\mathcal{S}|}}{\kappa_S} + 4 \left\| (\tilde{D}a^*)_{\mathcal{S}^c} \right\|_1 + 1 \right) \sqrt{\log(d^2 |\mathcal{E}|)} + \sqrt{T} \right), \end{aligned}$$

where κ_S is the compatibility factor from Definition 1.

The proof, which is outlined in Section 3.3, makes use of a concentration result of [15] for controlling the suprema of second order subgaussian chaos processes involving positive semidefinite (p.s.d) matrices (recalled as Theorem 2 in Appendix C.2).

Lemma 4 (RE for h_2 term). *There exist constants $c_1, c_2, c_3 \geq 1$ such that the following is true. For any $v \geq 1$, it holds w.p at least $1 - 2 \exp(-c_3 \sqrt{v})$ that*

$$\forall h \in \mathcal{C}_S \cap \mathbb{S}^{md^2-1} : \|Qh_2\|_2^2 \geq T \|h_2\|_2^2 - \frac{c_1}{m} \beta^2 (\sqrt{mT} + d)d - \frac{c_2}{m} \beta^2 (\sqrt{mT} + d)v.$$

The proof, presented in Section 3.4, also relies on the aforementioned concentration inequality due to [15]. We now present a bound on the “cross term” involving both h_1 and h_2 .

Lemma 5 (Control of the cross term). *There exist absolute constants $c_1 \geq 1$ and $c_2 \in (0, 1)$ such that the following holds. For any $\delta \in (0, c_2)$, with probability at least $1 - \delta$, for all $h \in \mathcal{C}_S \cap \mathbb{S}^{md^2-1}$,*

$$|\langle Qh_1, Qh_2 \rangle| \leq c_1 \frac{\mu}{\sqrt{m}} \left(1 + \frac{\sqrt{|\mathcal{S}|}}{\kappa_S} + \left\| (Da^*)_{\mathcal{S}^c} \right\|_1 \right) \left[d \Delta_G + \beta^2 \sqrt{T} \log\left(\frac{|\mathcal{E}| d}{\delta}\right) \right], \quad (3.8)$$

where Δ_G as defined in (2.6)

The proof of this lemma is given in Section 3.5.

Putting everything together. Using Lemma's 3, 4 and 5 in (3.7), we get absolute constants $c_1, \dots, c_6 > 0$ such that, with probability at least $1 - 4e^{-c_1\sqrt{v}} - \delta$,

$$\begin{aligned} \inf_{h \in C_S \cap \mathbb{S}^{md^2-1}} \|Qh\|_2^2 &\geq T\|h_1\|_2^2 - c_2 F_3 - c_3 G_3 \sqrt{v} \\ &\quad + T\|h_2\|_2^2 - \frac{c_4}{m} \beta^2 (\sqrt{mT} + d) d - \frac{c_5}{m} \beta^2 (\sqrt{mT} + d) v \\ &\quad - \frac{c_6 \mu}{\sqrt{m}} \left(1 + \frac{\sqrt{|S|}}{\kappa_S} + \|(D a^*)_{S^c}\|_1 \right) \left[d \Delta_G + \beta^2 \sqrt{T} \log\left(\frac{|\mathcal{E}| d}{\delta}\right) \right]. \end{aligned}$$

Using $G_3 \leq F_3$ and $\|h_1\|_2^2 + \|h_2\|_2^2 = 1$, the bound simplifies to

$$\begin{aligned} \inf_{h \in C_S \cap \mathbb{S}^{md^2-1}} \|Qh\|_2^2 &\geq T - c \left(F_3 \sqrt{v} + \frac{\beta^2}{m} (\sqrt{mT} + d) (d + v) \right. \\ &\quad \left. + \frac{\mu}{\sqrt{m}} \left(1 + \frac{\sqrt{|S|}}{\kappa_S} + \|(\tilde{D} a^*)_{S^c}\|_1 \right) \left[d \Delta_G + \beta^2 \sqrt{T} \log\left(\frac{|\mathcal{E}| d}{\delta}\right) \right] \right). \end{aligned}$$

Therefore, if the following (sufficient) conditions hold:

- (i) $F_3 \sqrt{v} \leq \frac{T}{6c}$,
- (ii) $\frac{\beta^2}{m} (\sqrt{mT} + d) (d + v) \leq \frac{T}{6c}$,
- (iii) $\frac{\mu}{\sqrt{m}} \left(1 + \frac{\sqrt{|S|}}{\kappa_S} + \|(\tilde{D} a^*)_{S^c}\|_1 \right) \left[d \Delta_G + \beta^2 \sqrt{T} \log\left(\frac{|\mathcal{E}| d}{\delta}\right) \right] \leq \frac{T}{6c}$,

then $\inf_{h \in C_S \cap \mathbb{S}^{md^2-1}} \|Qh\|_2^2 \geq T/2$. Plugging this into (3.6) yields

$$\frac{1}{m} \|a^* - \hat{a}\|_2^2 \leq \frac{2\lambda}{T} \left(1 + 3 \frac{\sqrt{|S|}}{\kappa_S} \right) \|\hat{a} - a^*\|_2 + \frac{8\lambda}{T} \|(\tilde{D} a^*)_{S^c}\|_1,$$

which implies the stated error bound of Theorem 1 after some simple calculations.

3.1 Proof of Lemma 1

We start by observing that

$$Q^\top \eta = \begin{bmatrix} (X_1 \otimes I_d) \eta_1 \\ \vdots \\ (X_m \otimes I_d) \eta_m \end{bmatrix} \tag{3.9}$$

where we recall $\eta_l = \text{vec}(E_l)$, and the definitions of $X_l, E_l \in \mathbb{R}^{d \times T}$ from Section 2.2. Denoting $M_{l,T} := E_l X_l^\top = \sum_{s=1}^T \eta_{l,s+1} x_{l,s}^\top$, note that $(X_l \otimes I_d) \eta_l = \text{vec}(M_{l,T})$, and we have

$$\left\| \tilde{\Pi} Q^\top \eta \right\|_2^2 = \frac{1}{m} \left\| \sum_{l=1}^m \text{vec}(M_{l,T}) \right\|_2^2 = \frac{1}{m} \left\| \sum_{l=1}^m M_{l,T} \right\|_F^2. \tag{3.10}$$

For any system $l \in [m]$, denote $\mathcal{F}_{l,t} := \sigma(\eta_{l,1}, \dots, \eta_{l,t})$ to be the sigma algebra at time $t \geq 1$, leading to a filtration $(\mathcal{F}_{l,t})_{t=1}^\infty$. Since $M_{l,t} = M_{l,t-1} + \eta_{l,t+1}x_{l,t}^\top$, and $M_{l,t}$ is $\mathcal{F}_{l,t+1}$ -measurable, hence $\mathbb{E}[M_{l,t} | \mathcal{F}_{l,t}] = M_{l,t-1}$, and $(M_{l,t})_{t=1}^T$ is a martingale.

Before deriving a high-probability upper bound for $\|\tilde{\Pi}Q^\top \eta\|_2^2$, let us examine what its expectation looks like. To this end, note that

$$\mathbb{E} \left[\|M_{l,t}\|_F^2 | \mathcal{F}_{l,t} \right] = \|M_{l,t-1}\|_F^2 + d \|x_{l,t}\|_2^2.$$

Together with (3.10), and the fact that $M_{l,T}, M_{l',T}$ are independent and centered, this implies

$$\mathbb{E} \left\| \tilde{\Pi}Q^\top \eta \right\|_2^2 = \frac{1}{m} \sum_{l=1}^m \mathbb{E} \left[\|M_{l,T}\|_F^2 \right] = \frac{d}{m} \sum_{l=1}^m \sum_{s=1}^T \mathbb{E} \left[\|x_{l,s}\|_2^2 \right] = \frac{d}{m} \sum_{l=1}^m \mathbb{E} \left[\|\text{vec}(X_l)\|_2^2 \right] = \frac{d}{m} \sum_{l=1}^m \left\| \tilde{A}_l^* \right\|_F^2,$$

where we recall \tilde{A}_l^* from (2.5).

Now to find an upper bound on $\|\tilde{\Pi}Q^\top \eta\|_2^2$, we first write

$$\sum_{l=1}^m \text{vec}(E_l X_l^\top) = \sum_{l=1}^m \sum_{s=1}^T \text{vec}(\eta_{l,s+1} x_{l,s}^\top) = \sum_{s=1}^T \left(\sum_{l=1}^m x_{l,s} \otimes \eta_{l,s+1} \right) =: S_T.$$

Clearly, $(S_t)_{t=1}^T$ is a martingale and $\|\tilde{\Pi}Q^\top \eta\|_2^2 = \frac{1}{m} \|S_T\|_2^2$. Denoting $V_T := \sum_{s=1}^T \sum_{l=1}^m (x_{l,s} x_{l,s}^\top) \otimes I_d$, we obtain for any $\bar{V} \succ 0$ that

$$\frac{1}{m} \|S_T\|_2^2 = \frac{1}{m} \left\| (V_T + \bar{V})^{1/2} (V_T + \bar{V})^{-1/2} S_T \right\|_2^2 \leq \frac{1}{m} \|V_T + \bar{V}\|_2 \left\| (V_T + \bar{V})^{-1/2} S_T \right\|_2^2. \quad (3.11)$$

The term $(V_T + \bar{V})^{-1/2} S_T$ is a self-normalized vector valued martingale. In a completely analogous manner to the proof of [37, Proposition 1], which in turn follows the steps in the proof of [1, Theorem 1], it is easy to show that (see Appendix E for details)

$$\mathbb{P} \left(\left\| (V_T + \bar{V})^{-1/2} S_T \right\|_2^2 \leq 2 \log \left(\frac{\det[(V_T + \bar{V})^{1/2}] \det[\bar{V}^{-1/2}]}{\delta} \right) \right) \geq 1 - \delta. \quad (3.12)$$

We will now bound $\|V_T\|_2$, which together with (3.11), (3.12), and a suitable choice of \bar{V} will complete the proof.

To this end, note that

$$\|V_T\|_2 = \left\| \sum_{s=1}^T \sum_{l=1}^m (x_{l,s} x_{l,s}^\top) \right\|_2 \leq \sum_{s=1}^T \sum_{l=1}^m \|x_{l,s}\|_2^2 = \tilde{\eta}^\top (\tilde{A}^*)^\top \tilde{A}^* \tilde{\eta} \quad (3.13)$$

where $\tilde{\eta}$ is formed by column-stacking $\eta_{l,1}, \dots, \eta_{l,T}$; $\tilde{\eta}$ is formed by column-stacking $\tilde{\eta}_1, \dots, \tilde{\eta}_m$, and $\tilde{A}^* := \text{blkdiag}(\tilde{A}_1^*, \dots, \tilde{A}_m^*)$. Invoking the tail-bound in [11, Theorem 2.1] for random positive-semidefinite quadratic forms, and denoting $\Sigma := (\tilde{A}^*)^\top \tilde{A}^*$, we have for any $t > 0$,

$$\mathbb{P} \left(\tilde{\eta}^\top \Sigma \tilde{\eta} \geq \text{Tr}(\Sigma) + 2\sqrt{\text{Tr}(\Sigma^2)t} + 2\|\Sigma\|_2 t \right) \leq e^{-t}.$$

Since $\|\Sigma\|_2 \leq \text{Tr}(\Sigma)$ and $\text{Tr}(\Sigma^2) \leq \|\Sigma\|_2 \text{Tr}(\Sigma) \leq (\text{Tr}(\Sigma))^2$, we obtain for any $t \geq 1$ the simplified bound

$$\mathbb{P}\left(\tilde{\eta}^\top \Sigma \tilde{\eta} \geq 5t \text{Tr}(\Sigma)\right) \leq e^{-t}. \quad (3.14)$$

Using the fact

$$\text{Tr}(\Sigma) = \sum_{l=1}^m \text{Tr}((\tilde{A}_l^*)^\top \tilde{A}_l^*) = \sum_{l=1}^m \sum_{t=0}^{T-1} \text{Tr}(\Gamma_t(A_l^*))$$

in (3.14), and choosing $t = \log(1/\delta)$ for $\delta \in (0, e^{-1})$, we obtain from (3.13) that

$$V_T \preceq 5 \left(\sum_{l=1}^m \sum_{t=0}^{T-1} \text{Tr}(\Gamma_t(A_l^*)) \right) \log(1/\delta) I_{d^2}.$$

Applying this in (3.12) and choosing $\bar{V} = mI_{d^2}$, we then obtain from (3.11) (with some minor simplifications) the stated error bound on $\|\tilde{\Pi}Q^\top \eta\|_2 = \frac{1}{\sqrt{m}} \|S_T\|_2$.

3.2 Proof of Lemma 2

Recall the expression of D^\dagger from Definition 1, we then have

$$(D^\dagger)^\top \otimes I_{d^2} = \begin{bmatrix} s_1^\top \otimes I_{d^2} \\ \vdots \\ s_{|\mathcal{E}|}^\top \otimes I_{d^2} \end{bmatrix}$$

which together with (3.9) implies

$$[(D^\dagger)^\top \otimes I_{d^2}] Q^\top \eta = \begin{bmatrix} \sum_{l=1}^m (s_1)_l (X_l \otimes I_d) \eta_l \\ \vdots \\ \sum_{l=1}^m (s_{|\mathcal{E}|})_l (X_l \otimes I_d) \eta_l \end{bmatrix}.$$

Hence we obtain

$$\left\| [(D^\dagger)^\top \otimes I_{d^2}] Q^\top \eta \right\|_\infty = \max_{i \in [\mathcal{E}]} \left\| \sum_{l=1}^m (s_i)_l (X_l \otimes I_d) \eta_l \right\|_\infty.$$

For a given $w = (w_1, \dots, w_m)^\top \in \mathbb{R}^m$ with $\|w\|_2 \leq \mu$, we will now bound $\left\| \sum_{l=1}^m w_l (X_l \otimes I_{d^2}) \eta_l \right\|_\infty$, and then take a union bound over $\{s_1, \dots, s_{|\mathcal{E}|}\}$ to conclude.

To this end, denoting

$$v = \sum_{l=1}^m w_l (X_l \otimes I_{d^2}) \eta_l = \sum_{s=1}^T \sum_{l=1}^m w_l (x_{l,s} \otimes \eta_{l,s+1}) \in \mathbb{R}^{d^2},$$

we can consider v to be formed by column-stacking the vectors $v_1, v_2, \dots, v_d \in \mathbb{R}^{d^2}$ where

$$v_i = \sum_{s=1}^T \sum_{l=1}^m w_l (x_{l,s})_i \eta_{l,s+1} \quad \text{for } i = 1, \dots, d.$$

For a given $i, j \in [d]$, we will now first bound $|(v_i)_j|$ with high probability, and then taken a union bound to bound $\|v\|_\infty$. To this end, we start by writing

$$(v_i)_j = \underbrace{\sum_{s=1}^T \sum_{l=1}^m w_l(x_{l,s})_i (\eta_{l,s+1})_j}_{:=S_T} \quad \text{and} \quad (\bar{v}_i)_j := \sum_{s=1}^T \sum_{l=1}^m w_l^2(x_{l,s})_i^2 \geq 0.$$

Similar to Section 3.1, we can see that $(S_t)_{t=1}^T$ is a martingale. By writing

$$|(v_i)_j| = ((\bar{v}_i)_j + a)^{1/2} \left| ((\bar{v}_i)_j + a)^{-1/2} (v_i)_j \right| \quad (3.15)$$

for any fixed $a > 0$, and noting that $((\bar{v}_i)_j + a)^{-1/2} (v_i)_j$ is a (scalar-valued) self-normalized martingale, we can show in a completely analogous manner to the proof of [1, Theorem 1] (as explained in Section 3.1) that for any $\delta \in (0, 1)$,

$$\mathbb{P} \left(\left| ((\bar{v}_i)_j + a)^{-1/2} (v_i)_j \right|^2 \leq 2 \log \left(\frac{((\bar{v}_i)_j + a)^{1/2}}{\delta a^{1/2}} \right) \right) \geq 1 - \delta. \quad (3.16)$$

It remains to bound $(\bar{v}_i)_j$. Recall \tilde{A}_l^* from (2.5). It will be useful to denote $\tilde{A}_l^*(s)$ to be the s 'th “row-block” of matrices of \tilde{A}_l^* , i.e.,

$$\tilde{A}_l^*(s) := [(A_l^*)^{s-1} \cdots A_l^* I_d 0 \cdots 0].$$

Moreover, denote $\tilde{A}_l^*(s, i) := e_i^\top \tilde{A}_l^*(s)$ to be the i 'th row vector of $\tilde{A}_l^*(s)$, for $i \in [d]$. Then we can write $(x_{l,s})_i = \tilde{A}_l^*(s, i) \tilde{\eta}_l$; recall $\tilde{\eta}_l$ and $\tilde{\eta}$ from Section 3.1. Using the expression for $(\bar{v}_i)_j$, this implies

$$\begin{aligned} (\bar{v}_i)_j &= \sum_{s=1}^T \sum_{l=1}^m w_l^2(x_{l,s})_i^2 \\ &= \sum_{s=1}^T \sum_{l=1}^m \tilde{\eta}_l^\top (w_l^2 \tilde{A}_l^*(s, i)^\top \tilde{A}_l^*(s, i)) \tilde{\eta}_l \\ &= \tilde{\eta}^\top \left[\underbrace{\text{blkdiag} \left(w_1^2 \sum_{s=1}^T (\tilde{A}_1^*(s, i))^\top \tilde{A}_1^*(s, i), \dots, w_m^2 \sum_{s=1}^T (\tilde{A}_m^*(s, i))^\top \tilde{A}_m^*(s, i) \right)}_{=: \Sigma} \right] \tilde{\eta}. \end{aligned}$$

Since $\|w\|_2^2 \leq \mu^2$ by assumption,

$$\implies \text{Tr}(\Sigma) = \sum_{l=1}^m w_l^2 \sum_{s=1}^T \text{Tr} \left(\tilde{A}_l^*(s, i)^\top \tilde{A}_l^*(s, i) \right) \leq \mu^2 \max_{\substack{l \in [m] \\ i \in [d]}} e_i^\top \left(\sum_{s=1}^T \Gamma_{s-1}(A_l^*) \right) e_i.$$

Then invoking the tail bound in (3.14) with $t = \log(1/\delta)$ for $\delta \in (0, e^{-1})$, we obtain with probability at least $1 - \delta$,

$$(\bar{v}_i)_j \leq 5\mu^2 \left[\max_{\substack{l \in [m] \\ i \in [d]}} e_i^\top \left(\sum_{s=1}^T \Gamma_{s-1}(A_l^*) \right) e_i \right] \log(1/\delta) =: \zeta'_2(m, T, \delta). \quad (3.17)$$

Using (3.17) and (3.16), and choosing $a = \zeta'_2(m, T, \delta)$, we have thus far shown the following (via a union bound over $[d^2]$). For any given $w \in \mathbb{R}^m$ with $\|w\|_2 \leq \mu$, it holds with probability at least $1 - 2\delta$ that for all $i, j \in [d]$,

$$\left|((\bar{v}_i)_j + a)^{-1/2}(v_i)_j\right|^2 \leq 2 \log\left(\frac{\sqrt{2}d^2}{\delta}\right) \quad \text{and} \quad ((\bar{v}_i)_j + a)^{1/2} \leq \sqrt{2\zeta'_2\left(m, T, \frac{\delta}{d^2}\right)}. \quad (3.18)$$

Using (3.18) and (3.15), this implies that for any given $w \in \mathbb{R}^m$ with $\|w\|_2 \leq \mu$, it holds with probability at least $1 - 2\delta$ that

$$\left\| \sum_{l=1}^m w_l (X_l \otimes I_{d^2}) \eta_l \right\|_\infty \leq 2\zeta'^{1/2}_2\left(m, T, \frac{\delta}{d^2}\right) \log^{1/2}(\sqrt{2}d^2/\delta).$$

The statement of the lemma now follows by taking a union bound over $\{s_1, \dots, s_{|\mathcal{E}|}\}$.

3.3 Proof of Lemma 3

Denoting $\tilde{Q} = Q(D^\dagger \otimes I_{d^2}) = Q\tilde{D}^\dagger$, we have $\|Qh_1\|_2^2 = \|\tilde{Q}\tilde{D}h\|_2^2$. Recall from Definition 1 that $D^\dagger = [s_1 \ s_2 \ \dots \ s_{|\mathcal{E}|}]$ where $s_i = (s_{i,1}, \dots, s_{i,m})^\top \in \mathbb{R}^m$ for $i = 1, \dots, |\mathcal{E}|$. Let us denote

$$(D^\dagger)^\top = [s'_1 \ s'_2 \ \dots \ s'_m] \in \mathbb{R}^{|\mathcal{E}| \times m} \quad \text{where} \quad s'_l = (s'_{1,l}, \dots, s'_{|\mathcal{E}|,l})^\top.$$

Then, it is not difficult to verify that

$$\tilde{Q} = \begin{bmatrix} {s'_1}^\top \otimes (X_1^\top \otimes I_d) \\ {s'_2}^\top \otimes (X_2^\top \otimes I_d) \\ \vdots \\ {s'_m}^\top \otimes (X_m^\top \otimes I_d) \end{bmatrix}$$

where we observe that \tilde{Q} consists of independent ‘‘blocks’’ of rows, with the entries within a block being dependent.

It will be useful to denote $u(h) = \tilde{D}h$, and the matrix version of $u(h)$ by $U(h)$, where $u(h) = \text{vec}(U(h))$ and

$$U(h) = [U_1(h) \ U_2(h) \ \dots \ U_{|\mathcal{E}|}(h)] \in \mathbb{R}^{d \times (|\mathcal{E}|d)}.$$

Then we can write $\|Qh_1\|_2^2$ as

$$\begin{aligned} \|Qh_1\|_2^2 &= \|\tilde{Q}u(h)\|_2^2 = \sum_{l=1}^m \left\| \left((s'_l \otimes X_l)^\top \otimes I_d \right) u(h) \right\|_2^2 \\ &= \sum_{l=1}^m \|U(h)(s'_l \otimes X_l)\|_F^2 \\ &= \sum_{l=1}^m \left\| \left(\sum_{i=1}^{|\mathcal{E}|} s_{i,l} U_i(h) \right) X_l \right\|_F^2 \\ &= \sum_{l=1}^m \left\| \underbrace{\left[I_T \otimes \left(\sum_{i=1}^{|\mathcal{E}|} s_{i,l} U_i(h) \right) \right]}_{=: P_l(u(h))} \text{vec}(X_l) \right\|_2^2 = \sum_{l=1}^m \|P_l(u(h)) \text{vec}(X_l)\|_2^2. \end{aligned}$$

Using (2.5), the definition of \tilde{A}_l^* and (2.1), the definition of η_l and η , we have $\text{vec}(X_l) = \tilde{A}_l^* \eta_l$. This implies

$$\|Qh_1\|_2^2 = \sum_{l=1}^m \left\| P_l(u(h)) \tilde{A}_l^* \eta_l \right\|_2^2 = \|P(u(h))\eta\|_2^2 \quad (3.19)$$

where we denote

$$P(u(h)) := \text{blkdiag} \left[P_1(u(h)) \tilde{A}_1^*, \dots, P_m(u(h)) \tilde{A}_m^* \right]. \quad (3.20)$$

So our focus is now on lower bounding $\|P(u(h))\eta\|_2^2$ uniformly over $h \in \mathcal{C}_S \cap \mathbb{S}^{md^2-1}$. More specifically, denoting the set of matrices

$$\mathcal{P} = \left\{ P(u(h)) : h \in \mathcal{C}_S \cap \mathbb{S}^{md^2-1} \right\}$$

we will control the quantity

$$\sup_{h \in \mathcal{C}_S \cap \mathbb{S}^{md^2-1}} \left| \|P(u(h))\eta\|_2^2 - \mathbb{E} \|P(u(h))\eta\|_2^2 \right| \quad (3.21)$$

by invoking the concentration bound of [15], recalled as Theorem 2 in Appendix C.2. This theorem captures the complexity of the set \mathcal{P} via Talagrand's γ_2 functionals; their definition and properties are recalled in Appendix C.1. Thereafter, we will bound the γ_2 functionals of \mathcal{P} in terms of the γ_2 functionals of the set

$$\tilde{D}(\mathcal{C}_S \cap \mathbb{S}^{md^2-1}) = \left\{ \tilde{D}h : h \in \mathcal{C}_S \cap \mathbb{S}^{md^2-1} \right\}$$

using the Lipschitz property outlined in Appendix C.1. Finally, the γ_2 functionals of $\tilde{D}(\mathcal{C}_S \cap \mathbb{S}^{md^2-1})$ will be bounded in terms of the Gaussian width of this set, namely $w(\tilde{D}(\mathcal{C}_S \cap \mathbb{S}^{md^2-1}))$, which will then be controlled using arguments analogous to that in [2, Lemma F.1]. This, along with some simplifications will conclude the proof.

With the above strategy in mind, let us first establish that

$$\mathbb{E} \|P(u(h))\eta\|_2^2 \geq T \left\| \tilde{D}^\dagger \tilde{D}h \right\|_2^2 = T \|h_1\|_2^2, \quad \forall h \in \mathcal{C}_S \cap \mathbb{S}^{md^2-1}. \quad (3.22)$$

Indeed, first note that

$$\mathbb{E} \|P(u(h))\eta\|_2^2 = \|P(u(h))\|_F^2 = \sum_{l=1}^m \left\| P_l(u(h)) \tilde{A}_l^* \right\|_F^2.$$

Recall the expressions for $P_l(u(h))$ and \tilde{A}_l^* , we then see that

$$\left\| P_l(u(h)) \tilde{A}_l^* \right\|_F^2 \geq T \left\| \sum_{i=1}^{|\mathcal{E}|} s_{i,l} u_i(h) \right\|_F^2 = T \|U(h)(s'_l \otimes I_d)\|_F^2.$$

This in turn implies

$$\begin{aligned} \sum_{l=1}^m \left\| P_l(u(h)) \tilde{A}_l^* \right\|_F^2 &\geq T \sum_{l=1}^m \|U(h)(s'_l \otimes I_d)\|_F^2 \\ &= T \|U(h)[s'_1 \otimes I_d \ \dots \ s'_m \otimes I_d]\|_F^2 \\ &= T \left\| U(h)((D^\dagger)^\top \otimes I_d) \right\|_F^2 \\ &= T \left\| U(h)(D^\dagger \otimes I_d)^\top \right\|_F^2 = T \left\| (D^\dagger \otimes I_{d^2})u(h) \right\|_2^2 = T \left\| \tilde{D}^\dagger \tilde{D}h \right\|_2^2, \end{aligned} \quad (3.23)$$

thus establishing (3.22). Observe that in the penultimate equality above, we used the fact

$$\text{vec}(U(h)(D^\dagger \otimes I_d)^\top) = ((D^\dagger \otimes I_d) \otimes I_d) \text{vec}(U(h)) = (D^\dagger \otimes I_{d^2})u(h).$$

Bounding $d_2(\mathcal{P})$ and $d_F(\mathcal{P})$. To use Theorem 2, we need to bound the terms $d_2(\mathcal{P})$ and $d_F(\mathcal{P})$. To this end, we have for any $h \in \mathcal{C}_S \cap \mathbb{S}^{md^2-1}$ that

$$\|P(u(h))\|_F^2 = \sum_{l=1}^m \|P_l(u(h))\tilde{A}_l^*\|_F^2 \leq \beta^2 \sum_{l=1}^m \|P_l(u(h))\|_F^2 \leq \beta^2 T,$$

and also,

$$\begin{aligned} \|P(u(h))\|_2 &= \max_{l=1,\dots,m} \|P_l(u(h))\tilde{A}_l^*\|_2 \leq \beta \max_l \|P_l(u(h))\|_2 = \beta \max_{l \in [m]} \|U(h)(s'_l \otimes I_d)\|_2 \\ &\leq \beta \sqrt{\sum_{l=1}^m \|U(h)(s'_l \otimes I_d)\|_2^2} \\ &\leq \beta \sqrt{\sum_{l=1}^m \|U(h)(s'_l \otimes I_d)\|_F^2} \\ &= \beta \|U(h)((D^\dagger)^\top \otimes I_d)\|_F \\ &= \beta \|\tilde{D}^\dagger \tilde{D}h\|_2 \quad (\text{as seen in (3.23)}) \\ &\leq \beta. \end{aligned}$$

Hence we have shown that

$$d_2(\mathcal{P}) \leq \beta \text{ and } d_F(\mathcal{P}) \leq \beta\sqrt{T}. \quad (3.24)$$

Bounding $\gamma_2(\mathcal{P}, \|\cdot\|_2)$. For any $h, h' \in \mathcal{C}_S \cap \mathbb{S}^{md^2-1}$ we have

$$\begin{aligned} \|P(u(h)) - P(u(h'))\|_2 &\leq \beta \max_l \|P_l(u(h)) - P_l(u(h'))\|_2 \\ &\leq \beta \max_l \|(U(h) - U(h'))(s'_l \otimes I_d)\|_2 \\ &\leq \beta \mu' \|U(h) - U(h')\|_2 \\ &\leq \beta \mu' \|U(h) - U(h')\|_F \\ &\leq \beta \mu' \|u(h) - u(h')\|_2. \end{aligned}$$

Since the mapping $\tilde{D}(\mathcal{C}_S \cap \mathbb{S}^{md^2-1}) \mapsto \mathcal{P}$ via (3.20) is onto, thus invoking the Lipschitz property of γ_α functionals (see Appendix C.1), we obtain in conjunction with Talagrand's majorizing measure theorem (see (C.1)) that for some constants $c, c_1 > 0$,

$$\gamma_2(\mathcal{P}, \|\cdot\|_2) \leq c\beta\mu' \gamma_2(\tilde{D}(\mathcal{C}_S \cap \mathbb{S}^{md^2-1}), \|\cdot\|_2) \leq c_1\beta\mu' w(\tilde{D}(\mathcal{C}_S \cap \mathbb{S}^{md^2-1})). \quad (3.25)$$

Bounding the Gaussian width of $\tilde{D}(\mathcal{C}_S \cap \mathbb{S}^{md^2-1})$. We will show this using the following set inclusion. The proof is outlined in Appendix D.

Proposition 3. *For $r > 0$, and a positive integer n , recall that $\mathbb{B}_p^n(r)$ denotes the ℓ_p ball of radius r in \mathbb{R}^n . Also recall the compatibility factor κ_S from Definition 1. Then, it holds that*

$$\tilde{D}(\mathcal{C}_S \cap \mathbb{S}^{md^2-1}) \subseteq \mathbb{B}_1^{|\mathcal{E}|d^2} \left(4 \frac{\sqrt{|\mathcal{S}|}}{\kappa_S} + 4 \left\| (\tilde{D}a^*)_{\mathcal{S}^c} \right\|_1 + 1 \right).$$

It then follows from Proposition 3 along with standard properties of the Gaussian width (see e.g., [39, Proposition 7.5.2]), that

$$\begin{aligned} w\left(\tilde{D}(\mathcal{C}_S \cap \mathbb{S}^{md^2-1})\right) &\leq w\left(\mathbb{B}_1^{|\mathcal{E}|d^2} \left(4 \frac{\sqrt{|\mathcal{S}|}}{\kappa_S} + 4 \left\| (\tilde{D}a^*)_{\mathcal{S}^c} \right\|_1 + 1 \right)\right) \\ &\leq C \left(4 \frac{\sqrt{|\mathcal{S}|}}{\kappa_S} + 4 \left\| (\tilde{D}a^*)_{\mathcal{S}^c} \right\|_1 + 1 \right) \sqrt{\log(d^2 |\mathcal{E}|)} \quad (\text{see [39, Example 7.5.8]}) \end{aligned}$$

for some constant $C \geq 1$. Using this together with (3.25), we have hence shown that

$$\gamma_2(\mathcal{P}, \|\cdot\|_2) \leq c_2 \beta \mu' \left(4 \frac{\sqrt{|\mathcal{S}|}}{\kappa_S} + 4 \left\| (\tilde{D}a^*)_{\mathcal{S}^c} \right\|_1 + 1 \right) \sqrt{\log(d^2 |\mathcal{E}|)}, \quad (3.26)$$

for a constant $c_2 \geq 1$.

Putting it together. It will be useful to bound the terms F , G and H stated in Theorem 2 using (3.26) and (3.24). To this end, we obtain for a suitably large constant $C_1 \geq 1$ that

$$\begin{aligned} H &\leq \beta^2 =: \bar{H}, \\ G &\leq C_1 \beta^2 \left(\mu' \left(4 \frac{\sqrt{|\mathcal{S}|}}{\kappa_S} + 4 \left\| (\tilde{D}a^*)_{\mathcal{S}^c} \right\|_1 + 1 \right) \sqrt{\log(d^2 |\mathcal{E}|)} + \sqrt{T} \right) =: \bar{G}, \\ F &\leq C_1 \beta^2 \left((\mu')^2 \left(4 \frac{\sqrt{|\mathcal{S}|}}{\kappa_S} + 4 \left\| (\tilde{D}a^*)_{\mathcal{S}^c} \right\|_1 + 1 \right)^2 \log(d^2 |\mathcal{E}|) \right. \\ &\quad \left. + \mu' \sqrt{T} \left(4 \frac{\sqrt{|\mathcal{S}|}}{\kappa_S} + 4 \left\| (\tilde{D}a^*)_{\mathcal{S}^c} \right\|_1 + 1 \right) \sqrt{\log(d^2 |\mathcal{E}|)} + \sqrt{T} \right) =: \bar{F}. \end{aligned}$$

Note that $\bar{F} \geq \bar{G} \geq \bar{H}$. Now applying Theorem 2 to the quantity (3.21), and recalling the relation (3.19) along with the bound in (3.22), we have thus far shown that there exist constants c_1, c_2 such that for any $t > 0$,

$$\mathbb{P}\left(\forall h \in \mathcal{C}_S \cap \mathbb{S}^{md^2-1} : \|Qh_1\|_2^2 \geq T \|h_1\|_2^2 - c_1 \bar{F} - t\right) \leq 2 \exp\left(-c_2 \min\left\{\frac{t^2}{\bar{G}^2}, \frac{t}{\bar{H}}\right\}\right).$$

We now set $t = \bar{G}\sqrt{v}$ for any $v \geq 1$, $t = \bar{G}\sqrt{v}$ for any $v \geq 1$, and simply note that

$$\min\left\{\frac{t^2}{\bar{G}^2}, \frac{t}{\bar{H}}\right\} \geq \sqrt{v}$$

since $\bar{G} \geq \bar{H}$, as mentioned earlier. This leads to the statement of the lemma.

3.4 Proof of Lemma 4

We have

$$h_2 = \tilde{\Pi}h = \left[\left(\frac{1}{m} \mathbf{1}_m \mathbf{1}_m^\top \right) \otimes I_{d^2} \right] h$$

and $\|Qh_2\|_2^2 = \|Q\tilde{\Pi}h\|_2^2$. Plugging

$$Q = \text{blkdiag} \left(\underbrace{X_l^\top \otimes I_d}_{T \times d} \right)_{l=1}^m \quad \text{and} \quad \tilde{\Pi} = \frac{1}{m} (\mathbf{1}_m \otimes I_{d^2}) (\mathbf{1}_m \otimes I_{d^2})^\top$$

we compute

$$Q\tilde{\Pi}h = \left(\frac{1}{\sqrt{m}} Q (\mathbf{1}_m \otimes I_{d^2}) \right) \underbrace{\left(\frac{\mathbf{1}_m^\top}{\sqrt{m}} \otimes I_{d^2} \right)}_{=:u(h)} h.$$

Also note that

$$h_2 = \tilde{\Pi}h = \left(\frac{\mathbf{1}_m}{\sqrt{m}} \otimes I_{d^2} \right) \underbrace{u(h)}_{d^2 \times 1} = \frac{1}{\sqrt{m}} \begin{bmatrix} I_{d^2} \\ \vdots \\ I_{d^2} \end{bmatrix} u(h) = \frac{1}{\sqrt{m}} \begin{bmatrix} u(h) \\ \vdots \\ u(h) \end{bmatrix}_{md^2 \times 1}$$

which implies $\|h_2\|_2^2 = \|u(h)\|_2^2$. So, we have

$$\begin{aligned} Q\tilde{\Pi}h &= \left[\frac{1}{\sqrt{m}} Q (\mathbf{1}_m \otimes I_{d^2}) \right] u(h) = \frac{1}{\sqrt{m}} \begin{bmatrix} X_1^\top \otimes I_d \\ \vdots \\ X_m^\top \otimes I_d \end{bmatrix} u(h) \\ &\Rightarrow \|Q\tilde{\Pi}h\|_2^2 = \frac{1}{m} \sum_{l=1}^m \left\| (X_l^\top \otimes I_d) u(h) \right\|_2^2. \end{aligned}$$

Now, denote $U(h) \in \mathbb{R}^{d \times d}$ to be matrix version of $u(h) \in \mathbb{R}^{d^2 \times 1}$, so that $u(h) = \text{vec}(U(h))$. Then we have,

$$\begin{aligned} \left\| (X_l^\top \otimes I_d) u(h) \right\|_2^2 &= \left\| \underbrace{U(h)}_{d \times d} X_l \right\|_F^2 = \left\| (I_T \otimes U(h)) \underbrace{\text{vec}(X_l)}_{dT \times l} \right\|_2^2 \\ &= \left\| (I_T \otimes U(h)) \tilde{A}_l^* \eta_l \right\|_2^2. \end{aligned}$$

where we use that $\text{vec}(X_l) = \tilde{A}_l^* \eta_l$. So,

$$\|Q\tilde{\Pi}h\|_2^2 = \frac{1}{m} \sum_{l=1}^m \left\| (I_T \otimes U(h)) \tilde{A}_l^* \eta_l \right\|_2^2 = \frac{1}{m} \|P(U(h))\eta\|_2^2$$

where $P(U(h)) =: \text{blkdiag} \left((I_T \otimes U(h)) \tilde{A}_l^* \right)_{l=1}^m$. We define

$$\mathcal{P} := \left\{ P(U(h)) : h \in \mathcal{C}_S \cap \mathbb{S}^{md^2-1} \right\}.$$

We want to control

$$M := \frac{1}{m} \sup_{h \in \mathcal{C}_S \cap \mathbb{S}^{md^2-1}} \left| \|P(U(h))\eta\|_2^2 - \mathbb{E} \|P(U(h))\eta\|_2^2 \right|$$

for which we will use Theorem 2.

Controlling M . We start by lower bounding $\mathbb{E}\|P(U(h))\eta\|_2^2$. We compute

$$\begin{aligned}\mathbb{E}\|P(U(h))\eta\|_2^2 &= \|P(U(h))\|_F^2 = \sum_{l=1}^m \left\| (I_T \otimes U(h)) \tilde{A}_l^* \right\|_F^2 \\ &\geq Tm \|U(h)\|_F^2 \quad \left(\begin{array}{l} \text{similar calculation} \\ \text{as done for } h_1 \text{ term} \end{array} \right) \\ &= Tm \|u(h)\|_2^2 = Tm \|h_2\|_2^2.\end{aligned}$$

where we used $\|u(h)\|_2^2 = \|h_2\|_2^2$ as shown earlier.

2.1 Bounding $d_2(\mathcal{P})$ and $d_F(\mathcal{P})$.

For all $h \in \mathcal{C}_{\mathcal{S}} \cap \mathbb{S}^{md^2-1}$

$$\begin{aligned}\|P(U(h))\|_F^2 &= \sum_{l=1}^m \|(I_T \otimes U(h)) \tilde{A}_l^*\|_F^2 \leq \beta^2 m \|I_T \otimes U(h)\|_F^2 \\ &= \beta^2 m T \|U(h)\|_F^2 = \beta^2 m T \|h_2\|_2^2 \leq \beta^2 m T\end{aligned}$$

On the other hand,

$$\begin{aligned}\|P(U(h))\|_2 &= \max_l \left\| (I_T \otimes U(h)) \tilde{A}_l^* \right\|_2 \leq \beta \|I_T \otimes U(h)\|_2 \\ &= \beta \|U(h)\|_2 \leq \beta \|h_2\|_2 \leq \beta\end{aligned}$$

So, we get

$$d_F(\mathcal{P}) \leq \beta \sqrt{mT}, \quad d_2(\mathcal{P}) \leq \beta. \quad (3.27)$$

2.2 Bounding $\gamma_2(\mathcal{P}, \|\cdot\|_2)$.

We have that

$$\begin{aligned}\|P(U(h)) - P(U(h'))\|_2 &= \max_l \left\| ([I_T \otimes U(h)] - [I_T \otimes U(h')]) \tilde{A}_l^* \right\|_2 \\ &\leq \beta \|I_T \otimes (U(h) - U(h'))\|_2 \\ &= \beta \|U(h) - U(h')\|_2 \\ &\leq \beta \|U(h) - U(h')\|_F \\ &= \beta \|u(h) - u(h')\|_2.\end{aligned}$$

Denote as before the set

$$\mathcal{U} := \left\{ u(h) = \left(\frac{\mathbf{1}_m^\top}{\sqrt{m}} \otimes I_{d^2} \right) h : h \in \mathcal{C}_{\mathcal{S}} \cap \mathbb{S}^{md^2-1} \right\} \subset \mathbb{R}^{d^2}.$$

Since the map from \mathcal{U} to \mathcal{P} is onto, we have that

$$\gamma_2(\mathcal{P}, \|\cdot\|_2) \leq c\beta \gamma_2(\mathcal{U}, \|\cdot\|_2).$$

Now we will bound $\gamma_2(\mathcal{U}, \|\cdot\|_2)$ by showing a set-inclusion type lemma as before. The difference is that now \mathcal{U} contains elements of the form

$$\left(\frac{\mathbf{1}_m^\top}{\sqrt{m}} \otimes I_{d^2} \right) h.$$

Set inclusion. Denote

$$\mathcal{U}' := \left\{ u(h) = \left(\frac{\mathbf{1}_m^\top}{\sqrt{m}} \otimes I_{d^2} \right) h : h \in \mathcal{C}_{\mathcal{S}} \cap B_2^{md^2}(1) \right\} \subset \mathbb{R}^{d^2}.$$

Then it's easy to see that $\mathcal{U} \subset \mathcal{U}'$. Now for any $h \in \mathcal{C}_{\mathcal{S}} \cap B_2^{md^2}(1)$, we have

$$\begin{aligned} \|u(h)\|_1 &= \left\| \left[\frac{\mathbf{1}_m^\top}{\sqrt{m}} \otimes I_{d^2} \right] h \right\|_1 = \frac{1}{\sqrt{m}} \left\| \left[\underbrace{I_{d^2} \dots I_{d^2}}_{m \text{ times}} \right] h \right\|_1 \\ &\leq \frac{1}{\sqrt{m}} \|h\|_1 \leq d \|h\|_2 \leq d, \end{aligned} \tag{3.28}$$

and also

$$\|u(h)\|_2 = \left\| \left[\frac{\mathbf{1}_m^\top}{\sqrt{m}} \otimes I_{d^2} \right] h \right\|_2 \leq \underbrace{\left\| \frac{\mathbf{1}_m^\top}{\sqrt{m}} \otimes I_{d^2} \right\|_2}_{=1} \underbrace{\|h\|_2}_{\leq 1} \leq 1. \tag{3.29}$$

Together, (3.28) and (3.29) imply $\mathcal{U}' \subseteq B_1^{d^2}(d) \cap B_2^{d^2}(1)$. So, we have shown that

$$\mathcal{U} \subseteq \mathcal{U}' \subseteq B_1^{d^2}(d) \cap B_2^{d^2}(1).$$

Now using [39, Excercise 9.26] we obtain

$$w(B_1^{d^2}(d) \cap B_2^{d^2}(1)) \leq c \sqrt{d^2 \log \left(e \frac{d^2}{d^2} \right)} = cd$$

which implies that $w(\mathcal{U}) \leq cd$ and $\gamma_2(\mathcal{U}, \|\cdot\|_2) \leq c'd$ for some constant $c' > 0$. Hence, for a constant $c_1 \geq 1$, we have

$$\gamma_2(\mathcal{P}, \|\cdot\|_2) \leq c_1 \beta d. \tag{3.30}$$

Next we will bound the F, G, H terms which we recall below.

$$\begin{aligned} F &= \gamma_2(\mathcal{P}, \|\cdot\|_2) [\gamma_2(\mathcal{P}, \|\cdot\|_2) + d_F(\mathcal{P})] + d_F(\mathcal{P}) d_2(\mathcal{P}), \\ G &= d_2(\mathcal{P}) [\gamma_2(\mathcal{P}, \|\cdot\|_2) + d_F(\mathcal{P})], \\ H &= d_2^2(\mathcal{P}). \end{aligned}$$

Using (3.27) - (3.30) we obtain the following bounds:

$$\begin{aligned} H &\leq \beta^2 =: \bar{H} \\ G &\leq \beta \left[c_1 \beta d + \beta \sqrt{mT} \right] \leq c_1 \beta^2 [\sqrt{mT} + d] =: \bar{G} \\ F &\leq c_1 \beta d \left[c_1 \beta d + \beta \sqrt{mT} \right] + \beta^2 \sqrt{mT} \leq c'_1 \beta^2 d [d + \sqrt{mT}] + \beta^2 \sqrt{mT} \\ &\leq c'_2 \beta^2 d (d + \sqrt{mT}) = c_2 \bar{G} d = \bar{F} \end{aligned}$$

where $c_1, c_2 \geq 1$. So we have $\bar{F} \geq \bar{G} \geq \bar{H}$. Now, Theorem 2 implies that there exist $c_1, c_2 > 0$ s.t for any $t > 0$,

$$\begin{aligned} \mathbb{P} \left(\forall h \in \mathcal{C}_{\mathcal{S}} \cap \mathbb{S}^{md^2-1} : \|P(U(h))\eta\|_F^2 \geq \mathbb{E} \|P(U(h))\eta\|_F^2 - c_1 \bar{F} - t \right) \\ \leq 2 \exp \left(-c_2 \min \left\{ \frac{t^2}{\bar{G}^2}, \frac{t}{\bar{H}} \right\} \right). \end{aligned}$$

Now using $\|Q\tilde{\Pi}h\|_2^2 = \frac{1}{m} \|P(U(h))\eta\|_2^2$ and $\mathbb{E}\|P(U(h))\eta\|_2^2 \geq Tm \|h_2\|_2^2$ we obtain

$$\mathbb{P}\left(\forall h \in \mathcal{C}_{\mathcal{S}} \cap \mathbb{S}^{md^2-1} : \|Qh_2\|_2^2 \geq T \|h_2\|_2^2 - \frac{c_1 \bar{G}d}{m} - \frac{t}{m}\right) \leq 2 \exp\left(-c_2 \min\left\{\frac{t^2}{\bar{G}^2}, \frac{t}{\bar{H}}\right\}\right).$$

Set $t = \bar{G}v$ for $v \geq 1$, then

$$\min\left\{\frac{t^2}{\bar{G}^2}, \frac{t}{\bar{H}}\right\} \leq \sqrt{v} \quad (\text{Since } \bar{G} \geq \bar{H})$$

which implies the statement of the lemma.

3.5 Proof of Lemma 5

We start by writing

$$h_1 = \tilde{D}^\dagger \tilde{D}h = \tilde{D}^\dagger (\tilde{D}h)_{\mathcal{S}} + \tilde{D}^\dagger (\tilde{D}h)_{\mathcal{S}^c},$$

and

$$\begin{aligned} h_2 &= \tilde{\Pi}h = \left(\frac{\mathbf{1}_m \mathbf{1}_m^\top}{m} \otimes I_{d^2}\right) h = \left(\frac{\mathbf{1}_m}{\sqrt{m}} \otimes I_{d^2}\right) \underbrace{\left(\frac{\mathbf{1}_m^\top}{\sqrt{m}} \otimes I_{d^2}\right)}_{\tilde{h}} h \\ &= \left(\frac{\mathbf{1}_m}{\sqrt{m}} \otimes I_{d^2}\right) \tilde{h} = U\tilde{h} \end{aligned}$$

where $U := \frac{\mathbf{1}_m}{\sqrt{m}} \otimes I_{d^2}$ has orthonormal columns.

1. A bookkeeping inequality. Starting from

$$\langle Qh_1, Qh_2 \rangle = h_1^\top Q^\top Qh_2$$

and substituting h_1, h_2 we obtain

$$\begin{aligned} |\langle Qh_1, Qh_2 \rangle| &= \left| \left[\tilde{D}^\dagger (\tilde{D}h)_{\mathcal{S}} + \tilde{D}^\dagger (\tilde{D}h)_{\mathcal{S}^c} \right]^\top Q^\top QU \tilde{h} \right| \\ &\leq \left\| \left[(\tilde{D}h)_{\mathcal{S}}^\top (\tilde{D}^\dagger)^\top + (\tilde{D}h)_{\mathcal{S}^c}^\top (\tilde{D}^\dagger)^\top \right] Q^\top QU \right\|_2 \|h_2\|_2 \quad (\text{using } \|\tilde{h}\|_2 = \|h_2\|_2) \\ &\leq \left(\left\| (\tilde{D}h)_{\mathcal{S}}^\top (\tilde{D}^\dagger)^\top Q^\top QU \right\|_2 + \left\| (\tilde{D}h)_{\mathcal{S}^c}^\top (\tilde{D}^\dagger)^\top Q^\top QU \right\|_2 \right) \|h_2\|_2. \end{aligned}$$

The first term is bounded as follows.

$$\begin{aligned} \left\| (\tilde{D}h)_{\mathcal{S}}^\top (\tilde{D}^\dagger)^\top Q^\top QU \right\|_2 &= \left\| U^\top Q^\top Q (\tilde{D}^\dagger) (\tilde{D}h)_{\mathcal{S}} \right\|_2 \\ &\leq \left\| (\tilde{D}h)_{\mathcal{S}} \right\|_1 \left\| \underbrace{U^\top}_{d^2 \times md^2} \underbrace{Q^\top Q}_{md^2 \times md^2} \underbrace{(\tilde{D}^\dagger)}_{md^2 \times |E|d^2} \right\|_{1 \rightarrow 2} = \left\| (\tilde{D}h)_{\mathcal{S}} \right\|_1 \max_{j \in [|E|d^2]} \left\| U^\top Q^\top Q (\tilde{D}^\dagger) e_j \right\|_2 \end{aligned}$$

where we recall the fact

$$\|X\|_{1 \rightarrow 2} = \max_j \|X_{:j}\|_2.$$

For the second term we have

$$\begin{aligned} \left\| \left(\tilde{D}h \right)_{S^c}^\top \left(\tilde{D}^\dagger \right)^\top Q^\top QU \right\|_2 &\leq \left\| (\tilde{D}h)_{S^c} \right\|_1 \max_{j \in [|\mathcal{E}|d^2]} \left\| U^\top Q^\top Q \left(\tilde{D}^\dagger \right) e_j \right\|_2 \\ &\leq \left(1 + 3 \left\| (\tilde{D}h)_{S^c} \right\|_1 + 4 \left\| (\tilde{D}a^*)_{S^c} \right\|_1 \right) \max_{j \in [|\mathcal{E}|d^2]} \left\| U^\top Q^\top Q \left(\tilde{D}^\dagger \right) e_j \right\|_2 \end{aligned}$$

where we used $h \in \mathcal{C}_S \cap \mathbb{S}^{md^2-1}$. Putting the two bounds together we get

$$|\langle Qh_1, Qh_2 \rangle| \leq \left(1 + 4 \left\| (\tilde{D}h)_S \right\|_1 + 4 \left\| (\tilde{D}a^*)_{S^c} \right\|_1 \right) \max_{j \in [|\mathcal{E}|d^2]} \left\| U^\top Q^\top Q b_j \right\|_2 \|h_2\|_2, \quad (3.31)$$

where $b_j := \tilde{D}^\dagger e_j$ denotes the j -th column of \tilde{D}^\dagger .

Interpretation. Equation (3.31) shows that the entire problem boils down to a single quantity

$$M := \max_{j \in [|\mathcal{E}|d^2]} \left\| U^\top Q^\top Q b_j \right\|_2,$$

plus the harmless $\|h_2\|_2 \leq 1$.

2. Controlling M . Let u_i be the i -th column of U ($i \in [d^2]$). Then

$$M = \frac{1}{\sqrt{m}} \max_j \left(\sum_{i=1}^{d^2} (u_i^\top Q^\top Q b_j)^2 \right)^{1/2} \leq \frac{d}{\sqrt{m}} \max_{i \in [d^2], j \in [|\mathcal{E}|d^2]} |u_i^\top Q^\top Q b_j|. \quad (3.32)$$

Block/Kronecker expansion. Partition $u_i = (u_{i,1}^\top, \dots, u_{i,m}^\top)^\top$ and $b_j = (b_{j,1}^\top, \dots, b_{j,m}^\top)^\top$ with $u_{i,l}, b_{j,l} \in \mathbb{R}^{d^2}$ and let $U_{i,l}, B_{j,l} \in \mathbb{R}^{d \times d}$ be their matrix “unvec” versions, i.e., $\text{vec}(U_{i,l}) = u_{i,l}$ and $\text{vec}(B_{j,l}) = b_{j,l}$. Using $(X \otimes I)\text{vec}(B) = \text{vec}(BX^\top)$ and $\langle A, C \rangle = \text{vec}(A)^\top \text{vec}(C)$,

$$u_i^\top (Q^\top Q) b_j = \sum_{l=1}^m u_{i,l}^\top ((X_l X_l^\top) \otimes I_d) b_{j,l} = \sum_{l=1}^m \langle B_{j,l}^\top U_{i,l}, X_l X_l^\top \rangle. \quad (3.33)$$

Controllability expansion. With the block-lower-triangular matrix \tilde{A}_l^* from (2.5) and its t -th block row $\tilde{A}_l^*(t)$, one has

$$\mathbb{E}[X_l X_l^\top] = \sum_{t=1}^T \tilde{A}_l^*(t) \tilde{A}_l^*(t)^\top = \sum_{t=1}^T \sum_{s=0}^{t-1} (A_l^*)^s ((A_l^*)^s)^\top =: G_l,$$

where $G_l := \sum_{t=1}^T \Gamma_{t-1}(A_l^*)$ and $\Gamma_{t-1}(A) = \sum_{s=0}^{t-1} A^s (A^s)^\top$. Thus (3.33) becomes

$$\mathbb{E}[u_i^\top Q^\top Q b_j] = \sum_{l=1}^m \langle Z_l^\top, G_l \rangle, \quad Z_l := U_{i,l}^\top B_{j,l}. \quad (3.34)$$

Step 1 (deterministic offset). Recall $D^\dagger = [s_1 \ s_2 \ \dots \ s_{|\mathcal{E}|}]$. Notice from the structure of $U_{i,l}, B_{j,l}$ that for any $i \in [d^2], j \in [|\mathcal{E}|d^2]$, there exists a corresponding $k \in [|\mathcal{E}|]$ and $a, b \in [d]$ such that $Z_l = s_{k,l} e_a e_b^\top$, where e_s is a canonical vector of \mathbb{R}^d . Then writing $\bar{G} := \frac{1}{m} \sum_{l=1}^m G_l$,

$$\mathbb{E}[u_i^\top Q^\top Q b_j] = \sum_{l=1}^m s_{k,l} (G_l)_{b,a} = \sum_{l=1}^m s_{k,l} ((G_l)_{b,a} - \bar{G}_{b,a}) \leq \|s_k\|_2 \sqrt{\sum_{l=1}^m ((G_l)_{b,a} - \bar{G}_{b,a})^2}$$

where we used $\mathbf{1}_m^\top s_k = \sum_{l=1}^m s_{k,l} = 0$ for any $k \in [|\mathcal{E}|]$ since $s_k \in \text{span}(D^\dagger)$ and $\text{span}(D^\dagger) \perp \text{span}(\mathbf{1}_m)$. Using the definition of μ we get

$$\boxed{\mathbb{E}[u_i^\top Q^\top Q b_j] \leq \mu \sqrt{\sum_{l=1}^m ((G_l)_{b,a} - \bar{G}_{b,a})^2}} \quad (3.35)$$

Step 2 (Hanson–Wright fluctuation). Recall the definition of $\tilde{\eta}_l := (\eta_{l,1}^\top, \dots, \eta_{l,T}^\top)^\top \in \mathbb{R}^{dT}$ and let

$$W_l = \tilde{A}_l^{*\top} (I_T \otimes Z_l) \tilde{A}_l^*, \quad W = \text{blkdiag}(W_1, \dots, W_m).$$

Then $u_i^\top Q^\top Q b_j = \tilde{\eta}^\top W \tilde{\eta}$ with $\tilde{\eta} := (\tilde{\eta}_1; \dots; \tilde{\eta}_m)$, and the Hanson–Wright inequality yields, for universal $c > 0$,

$$\mathbb{P}\left(\left|\tilde{\eta}^\top W \tilde{\eta} - \mathbb{E}[\tilde{\eta}^\top W \tilde{\eta}]\right| \geq t\right) \leq 2 \exp\left(-c \min\{t^2/\|W\|_F^2, t/\|W\|_2\}\right).$$

Using $\|A^\top B A\|_F \leq \|A\|_2^2 \|B\|_F$ and $\|A^\top B A\|_2 \leq \|A\|_2^2 \|B\|_2$ with $\beta := \max_l \|\tilde{A}_l^*\|_2$ (see (2.5)), and noting $\|I_T \otimes Z_l\|_F = \sqrt{T} \|Z_l\|_F = \sqrt{T} |s_{k,l}|$ and $\|I_T \otimes Z_l\|_2 = \|Z_l\|_2 = |s_{k,l}|$, one gets

$$\|W\|_F^2 = \sum_{l=1}^m \|W_l\|_F^2 \leq \beta^4 T \sum_{l=1}^m s_{k,l}^2 \leq \beta^4 T \mu^2, \quad \|W\|_2 \leq \beta^2 \mu.$$

Hence, for $\delta \in (0, c_2)$ for a suitably small constant $c_2 < 1$, it holds with probability at least $1 - \delta$ that

$$\boxed{|u_i^\top Q^\top Q b_j - \mathbb{E}[u_i^\top Q^\top Q b_j]| \leq c_1 \sqrt{T} \beta^2 \mu \log(\frac{1}{\delta})}, \quad (3.36)$$

for a suitably large constant $c_1 > 1$.

Putting (3.35) - (3.36) into M . Apply (3.36) with a union bound over $i \in [d^2]$ and $j \in [|\mathcal{E}|d^2]$ and plug into (3.32). We then have that there exist absolute constants $c_1 > 1$ and $c_2 \in (0, 1)$ such that, for every $\delta \in (0, c_2)$,

$$\boxed{\mathbb{P}\left[M \leq \underbrace{\frac{d\mu}{\sqrt{m}} \max_{a,b \in [d]} \left(\sum_{l=1}^m ((G_l)_{b,a} - \bar{G}_{b,a})^2 \right)^{1/2}}_{=: \Delta_G} + c_1 \beta^2 \mu \sqrt{\frac{T}{m}} \log\left(\frac{|\mathcal{E}|d}{\delta}\right)\right] \geq 1 - \delta}. \quad (3.37)$$

3. Putting pieces together. Starting from the bookkeeping inequality (3.31),

$$|\langle Qh_1, Qh_2 \rangle| \leq M \left(1 + 4\|(\tilde{D}h)_S\|_1 + 4\|(\tilde{D}a^*)_{S^c}\|_1 \right) \|h_2\|_2,$$

use $\|(\tilde{D}h)_S\|_1 \leq \frac{\sqrt{|S|}}{\kappa_S} \|h\|_2 \leq \frac{\sqrt{|S|}}{\kappa_S}$ and $\|h_2\|_2 \leq 1$. Inserting the bound for M from (3.37), we obtain that, with probability at least $1 - \delta$, simultaneously for all $h \in C_S \cap \mathbb{S}^{md^2-1}$,

$$|\langle Qh_1, Qh_2 \rangle| \leq c_3 \frac{\mu}{\sqrt{m}} \left(1 + \frac{\sqrt{|S|}}{\kappa_S} + \|(\tilde{D}a^*)_{S^c}\|_1 \right) \left[d\Delta_G + \beta^2 \sqrt{T} \log\left(\frac{|\mathcal{E}|d}{\delta}\right) \right]. \quad (3.38)$$

This is exactly the claim in Lemma 5.

4 Experiments

In this section, we validate the performance of the graph total variation regularizer for estimating the LDS matrices A_l^* on a series of synthetic experiments. To this end, we consider linear dynamical systems on a graph on m nodes where each node $l \in \{1, \dots, m\}$ is associated with a time series that evolves as

$$x_{l,t+1} = A_l^* x_{l,t} + \eta_{l,t}, \quad \eta_{l,t} \stackrel{\text{i.i.d.}}{\sim} \mathcal{N}(0, I_d), \quad (4.1)$$

with (unless stated otherwise) $d = 2$. The system matrices $(A_l^*)_{l=1}^m$ vary across nodes but are structured by the underlying graph. We compare (a) *our graph-TV LDS*; (b) an *individual OLS (OLS_{ind}) estimator*, where a separate LDS is learnt per node; (c) *the pooled OLS (OLS_{pooled})*, where we learn a single common A across all nodes; (d) *a Laplacian LDS* [37], similar to our graph-TV estimator but that uses an ℓ_2^2 penalty of the form $\|\tilde{D}a\|_2^2$ on the differences across edges rather than a TV penalty; and (e) *a group-lasso estimator*, that groups coefficients together (across the graph) and imposes a lasso penalty. This lasso penalty allows us to compare graph penalties against another type of structural prior (here, group sparsity).

4.1 Experimental Protocol

Graphs and ground truths. To evaluate the importance of the graph topology in the final performance of the graph-TV regularizer, we consider various types of graph topologies: the chain graph, the 2d grid, the star graph, as well as an Erdős-Rényi random graph, parameterized by the connection probability between any pair of nodes, with varying levels of sparsity.

Each node l is assigned a matrix of the form

$$A_l^* = \begin{bmatrix} \beta_l^* & 0.1 \\ 0 & 0.6 \end{bmatrix} \in \mathbb{R}^{2 \times 2}, \quad (4.2)$$

where $(\beta_l^*)_{l=1}^m$ are chosen in one of the following ways.

(i) Piecewise constant. In this case, nodes are assigned to one of three groups, and

$$\beta_l^* \in \{s, 0, -s\}, \quad (4.3)$$

with s a scaling parameter that controls the maximal jump size: $s = \|\tilde{D}a^*\|_\infty$.

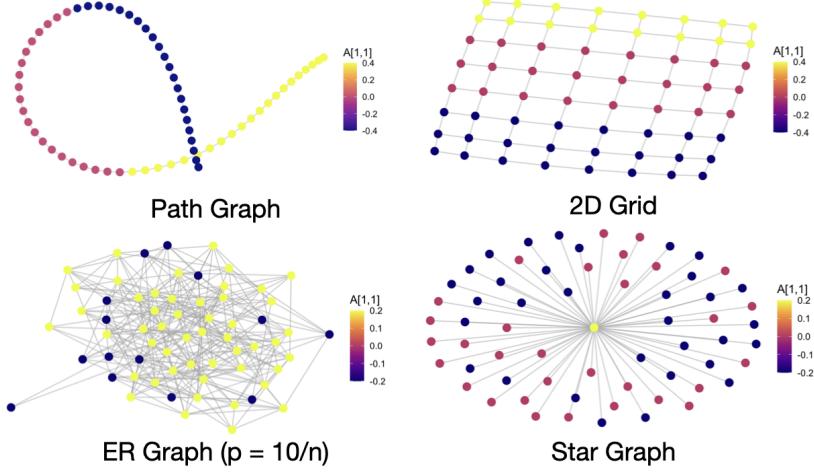


Figure 1: Examples of coefficients β_l^* generated, overlaid on the graph. Here, the coefficients $(\beta_l^*)_{l=1}^m$ are chosen to be piecewise constant across the graph, and the maximal jump size is constrained to $\|\tilde{D}a^*\|_\infty = 0.4$.

(ii) **Smoothly varying.** In this case, we compute the 2D spectral embedding for each node in the graph and define the signal as (for ω a parameter)

$$\beta_l^* = s(\cos(\omega t_1) \times \cos(\omega t_2)), \quad \text{where } t_1 = l \bmod \sqrt{m}, \quad t_2 = \lfloor l/\sqrt{m} \rfloor. \quad (4.4)$$

In each case, the signal is scaled to achieve either the same ℓ_∞ norm in the signal difference across edges (i.e., $\|\tilde{D}a^*\|_\infty = \|D\beta^*\|_\infty$, with $\beta^* \in \mathbb{R}^m$ is made similar across signals). This ensures appropriate comparison across all estimators. Figure 1 shows examples of the piecewise constant β_l^* generated on different graph topologies, while Figure 2 shows examples of its smooth counterpart.

Estimators and parameter selection. We vectorize each $A_l^* \in \mathbb{R}^{d \times d}$ and stack them in a single vector $a^* = (\text{vec}(A_1^*)^\top, \dots, \text{vec}(A_m^*)^\top)^\top \in \mathbb{R}^{md^2}$. For node l , with time series $X_l \in \mathbb{R}^{d \times T_{\text{train}}}$ and response $\tilde{X}_l \in \mathbb{R}^{d \times T_{\text{train}}}$, the local design is $Q_l = (X_l^\top \otimes I_d) \in \mathbb{R}^{(T_{\text{train}}d) \times d^2}$. We form $Q = \text{blkdiag}(Q_1, \dots, Q_m)$ and $\tilde{x} = \text{vec}(\tilde{X}_l) \in \mathbb{R}^{dT_{\text{train}}}$.

TV-joint LDS. We solve the generalized lasso (as per equation 2.3)

$$\min_{a \in \mathbb{R}^{md^2}} \frac{1}{2m} \|Qa - \tilde{x}\|_2^2 + \lambda \|\tilde{D}a\|_1, \quad (4.5)$$

using a full regularization path (up to 200 steps). The regularization parameter λ is selected by minimizing the validation MSE (defined below).

Data generation, splits, and metrics Given a graph and a collection of LDS matrices $(A_l^*)_{l=1}^m$, we simulate $T_{\text{tot}} = T_{\text{train}} + 2b + T_{\text{val}} + T_{\text{test}}$ steps per node from (4.1), with b as a buffer parameter (here, chosen to be 100) that ensures that the validation and testing sets are (approximately) independent of one another and of the training set. We use the first T_{train} steps for training, T_{val} steps for validation, and the last T_{test} for testing. Unless stated otherwise, we set $T_{\text{val}} = 4$, $T_{\text{test}} = 8$, and repeat over seeds $r = 1, \dots, n_{\text{rep}}$.

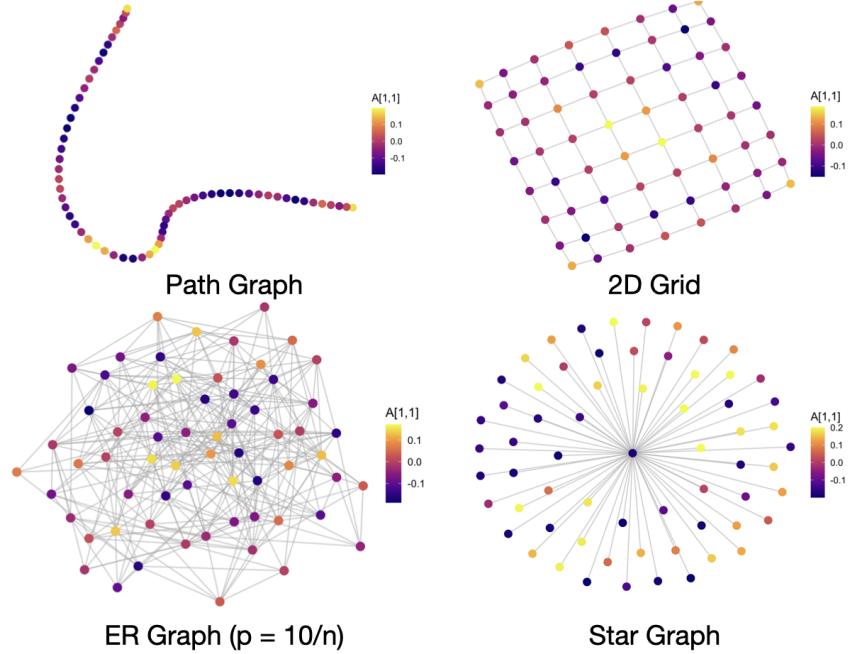


Figure 2: Examples of coefficients β_l generated, overlaid on the graph. Here, the coefficients $(\beta_l^*)_{l=1}^m$ are chosen to be smoothly varying across the graph, as per (4.4).

We report the parameter and test prediction MSEs, defined as

$$\text{(Parameter MSE)} \quad \text{MSE} = \frac{1}{m} \sum_{l=1}^m \| \widehat{A}_l - A_l^* \|_F^2, \quad (4.6)$$

$$\text{(Test prediction MSE)} \quad \text{MSE}_{\text{pred}} = \frac{1}{m T_{\text{test}}} \sum_{l=1}^m \| \widehat{A}_l X_{\text{test}}^{(l)} - \widetilde{X}_{\text{test}}^{(l)} \|_F^2, \quad (4.7)$$

where $X_{\text{test}}^{(l)}, \widetilde{X}_{\text{test}}^{(l)} \in \mathbb{R}^{d \times T_{\text{test}}}$. Curves show means and 95% normal confidence intervals (mean \pm $1.96 \times$ standard error) over seeds.

4.2 Results

(a) Error vs topology. We begin evaluating the error as a function of the graph topology. To this end, we fix the number of nodes to 64 and generate different graph topologies and corresponding piecewise constant signals. Figure 3 shows the error of the different estimators as a function of the maximal jump size $\| \widetilde{D}a^* \|_\infty$. Note that, since β_l^* is the only varying coefficient in the matrix A_l^* , $\| \widetilde{D}a^* \|_\infty = \| D\beta^* \|_\infty$. To emphasize the link with models 4.3 and 4.4, we will use the latter notation to denote the gap size. As expected, our estimator significantly outperforms the naive estimator $\hat{\beta}_{\text{OLS}_{\text{ind}}}$, and improves upon the pooled estimator $\hat{\beta}_{\text{OLS}_{\text{pool}}}$, particularly in regimes where the jump size is large. In particular, for the path graph, the TV (ℓ_1) denoiser substantially improves upon the Laplacian (ℓ_2^2) regularizer.

(b) Error vs. training length T_{train} . We also vary $T_{\text{train}} \in \{4, 8, 16, 32, 64, 128, 256\}$ (log-scale x -axis). We expect TV to clearly outperform OLS_{ind} at small T by borrowing strength across nodes, while approaching OLS_{ind} as T_{train} grows; OLS_{pooled} is expected to perform well when T_{train}

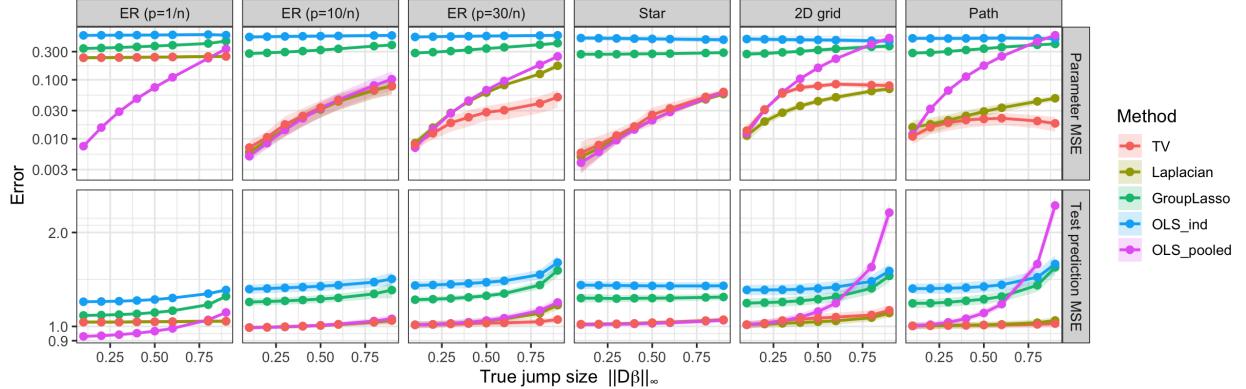


Figure 3: Performance of the TV regularizer and other competing methods as a function of the maximal jump size $\|D\beta^*\|_\infty$ on a piecewise constant signal (as per (4.3)).

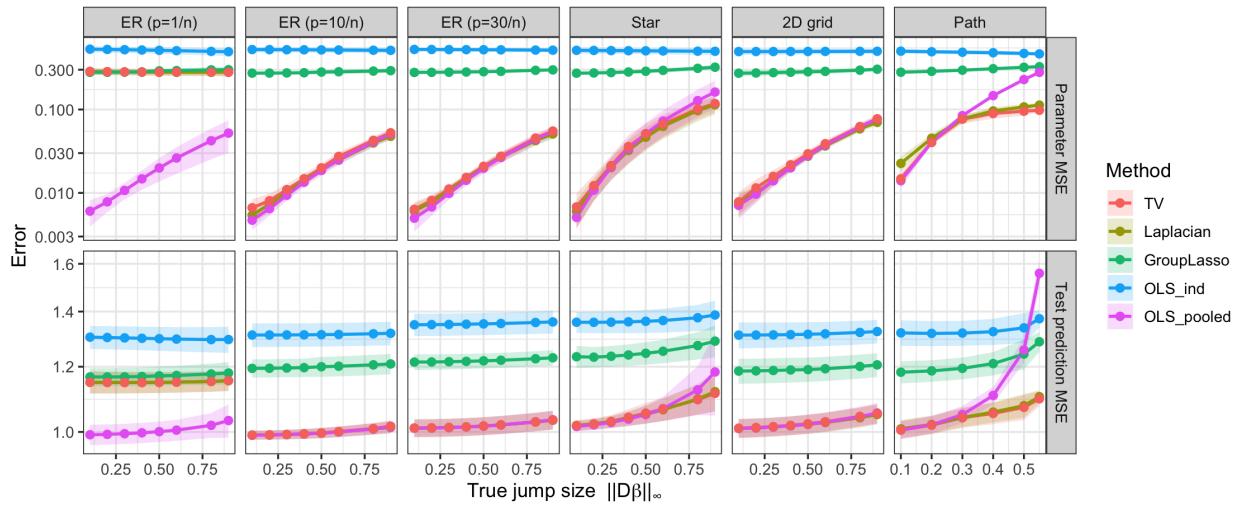
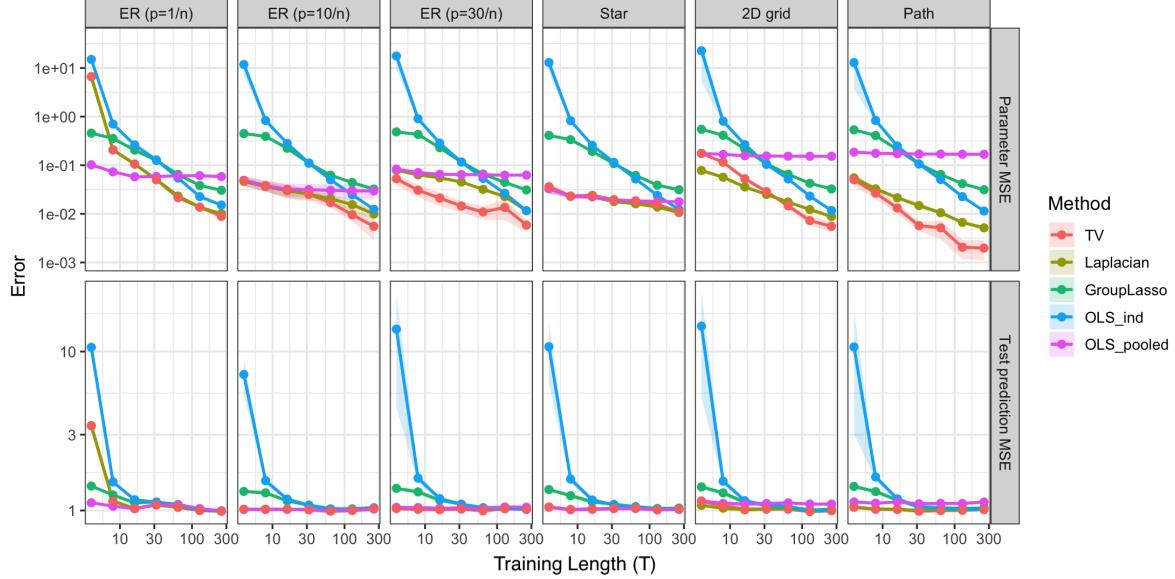
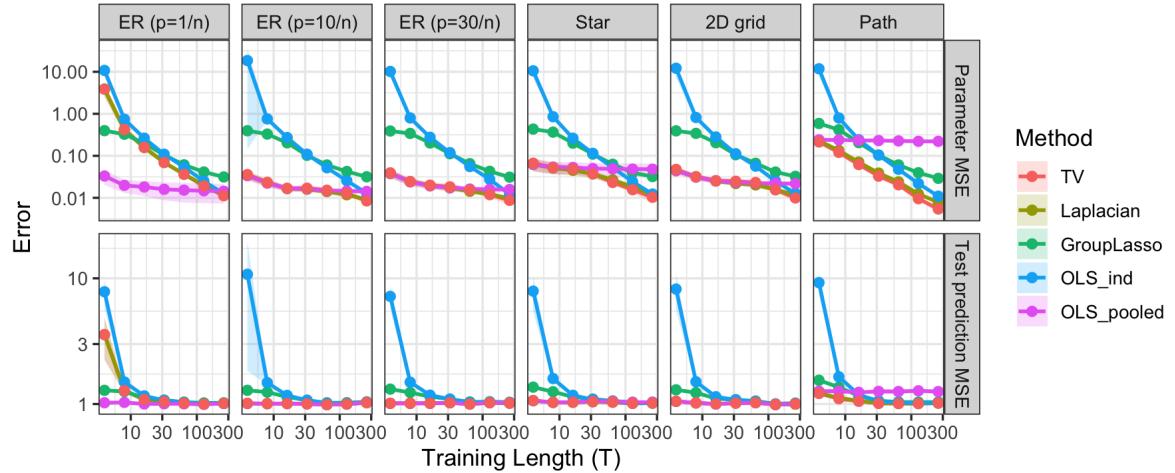


Figure 4: Performance of the TV regularizer as a function of the maximal jump size $\|D\beta^*\|_\infty$ on smooth signal (defined as per (4.4)).

is extremely small, as the benefits of the smoothing induced by fitting the estimator over the entire graph are expected to outweigh the bias incurred in this case. The results for the different topologies and signals are reported in Figures 5a and 5b, and confirm our expected findings. In particular, for the path graph and dense Erdős–Rényi graph, the TV regularizer consistently outperforms all other estimators, including the Laplacian estimator. We also note that for the piecewise-constant signal, the graph TV estimator offers significant advantages in terms of estimation MSE over the Laplacian estimator (e.g. in the path and ER cases). Even in the 2D grid, as T_{train} increases, we see that the TV regularizer improves upon the Laplacian. This is expected, as the TV regularizer's ability to preserve sharp contrasts is advantageous for signals defined by a few sparse, high-magnitude differences rather than small, distributed ones.



(a) Parameter and test prediction MSE as a function of training time length T and graph topology. The signal generated in these plots is piecewise constant (as per Equation 4.3), with $\|D\beta^*\|_\infty = 0.5$. Lines and dots indicate MSE mean averaged over 15 trials $\pm 95\%$ CI.



(b) Parameter and test prediction MSE as a function of training time length T and graph topology. The signal generated here was smooth (as per Equation 4.4), with $\|D\beta^*\|_\infty = 0.5$. Lines and dots indicate MSE mean averaged over 15 trials $\pm 95\%$ CI.

Figure 5: Parameter and test prediction MSE as a function of training time length T for different graph topologies.

(c) Error vs. number of nodes m . We hold T fixed (here, $T = 10$) and increase the graph size (m). The results are shown in Figure 6 for a piecewise constant signal. We note that our TV estimator improves with larger m . By contrast, OLS_{ind} does not benefit from added nodes, while especially in the path or grid graph, the pooled OLS (OLS_{pooled}) oversmoothes the signal, inducing a substantial parameter MSE compared to both graph-regularized penalties.

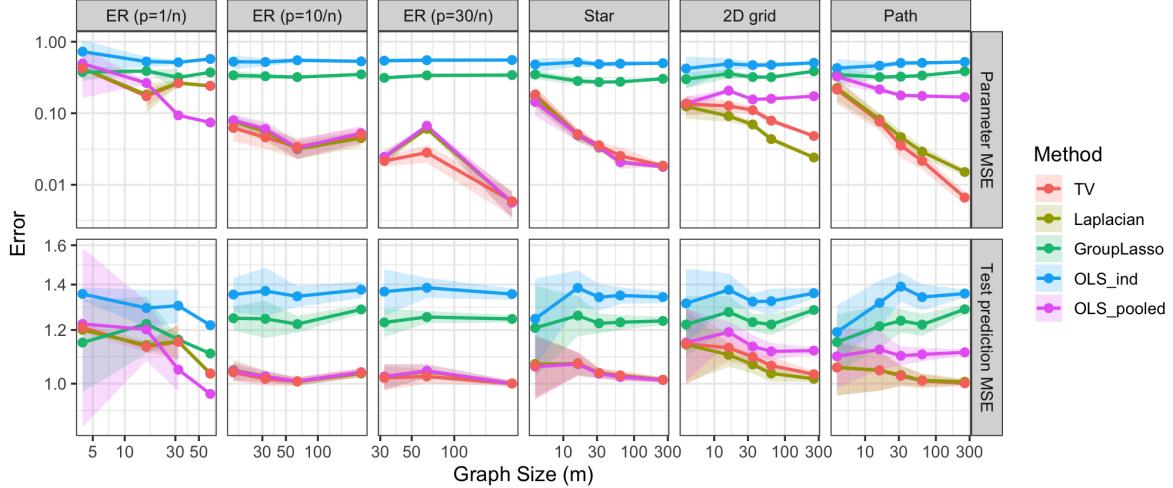


Figure 6: Parameter and test prediction MSE versus m for a piecewise constant signal, for different graph topologies. Lines and dots indicate MSE mean averaged over 15 trials $\pm 95\%$ CI.

4.3 Real Data Experiments: EPA data

We evaluate the proposed estimator on data from the U.S. Environmental Protection Agency (EPA) national air-quality monitoring network. Specifically, we analyze daily arithmetic-mean concentrations of nitric oxide (NO), particulate matter with aerodynamic diameter $\leq 10\mu\text{m}$ (PM_{10}), and daily mean temperature from 2010-2024.⁷ These measurements are obtained from 398 monitoring stations across the United States. We process the data by retaining measurements from primary instruments with at least 75% temporal coverage, removing observations flagged as special events, and applying variable-specific transformations to approximate normality (none for temperature, log for PM_{10} , and a log–log transform $\log(\log(x + 1))$ for NO). Stations are further filtered so that, within each year, fewer than 5% of observations are missing; this reduces the number of stations by roughly 45% on average (e.g., 221 stations in 2020; see Figure 7). Missing values are imputed via last observation carried forward. Using the reported geographic coordinates, we construct a spatial graph by connecting each station to its five nearest neighbors; the resulting graph is shown in Figure 7, with colors indicating NO on April 9, 2024 (day 100).

We compare our method for fitting a $d = 2$ -LDS to our 3 EPA datasets to the benchmarks described in the previous section. We also compare the results to a persistence baseline that, for each forecast horizon, predicts using the value observed at time $t = 0$. For each year, we draw a start day s uniformly from the first half of the year and define the training window as days s to $s + T_{\text{train}}$. To ensure independence across splits, we insert 20-day buffer windows between train, validation, and test sets. Validation error is computed at 10 evenly spaced evaluation points over the 100-day validation window, and test error at 10 evenly spaced points over the remaining test window. This procedure is repeated six times per year; results are first averaged within year, then aggregated across 15 years. A visualization of the coefficients obtained by the different methods is provided in Figure 10.

Results. Figure 10 illustrates the structural differences and implicit regularization of each method. With $T_{\text{train}} = 15$ (a data-limited regime), the station-wise OLS fits are highly heterogeneous, as

⁷Data are available from the EPA: <https://www.epa.gov/outdoor-air-quality-data/download-daily-data>.

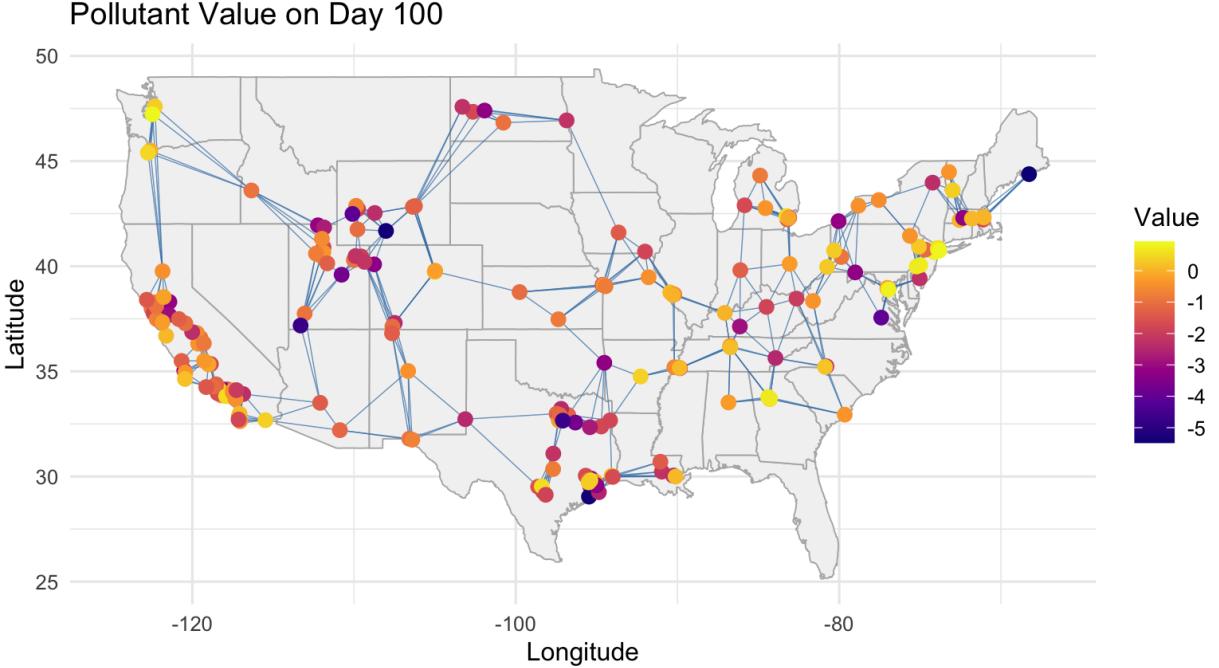
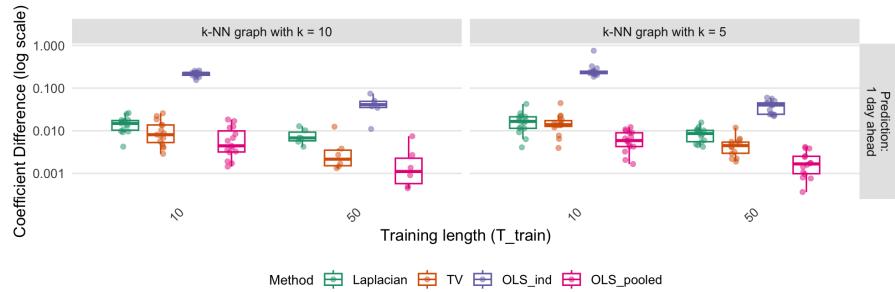


Figure 7: Air quality stations, along with the induced kNN graph ($k = 5$).

evidenced by the wide spread of coefficients. By contrast, pooled OLS yields a single global coefficient shared across stations. Both the graph total-variation (TV) denoiser and the Laplacian regularizer produce coefficients that vary smoothly over the spatial graph. As expected, TV recovers piecewise-constant (or piecewise-smooth) structure with sharp transitions aligned to graph boundaries, whereas Laplacian regularization yields a smoother, more diffuse spatial distribution.

To quantify performance, we evaluate two criteria: (a) the stability of the estimated coefficients across repeated fits; and (b) the prediction error. In the absence of ground truth parameters, coefficient stability serves as a proxy for estimation accuracy: a suitable regularizer should yield reproducible coefficients and support more reliable inferences. Figure 8 highlights the results for prediction on the next day (horizon = 1) for the NO dataset, while Figure 9 shows the same for the PM10 dataset. Table 2 shows the average test RMSE across all 3 datasets, where $T_{\text{train}} = 10$, with the same 1-day horizon. Consistent with our simulations, we note that the prediction MSE for the TV estimator matches that of the best competing method. However, we note substantial differences in coefficient stabilities across datasets. In particular, we note that the Laplacian estimator tends to produce more variable coefficients across runs. In contrast, the TV estimator is relatively more consistent across fittings. Figures 8 and 9 also highlight the performance of the method as a function of the training length and the number of neighbors k . In particular, we observe that as k and T_{train} increase, the difference in coefficient stability between the Laplacian and TV regularizer increases.

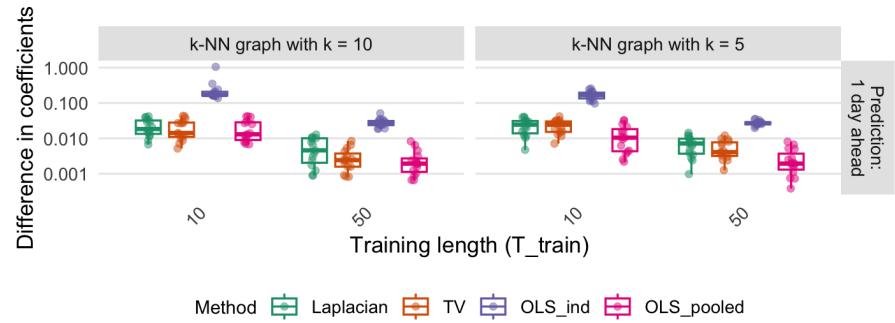


(a) Coefficient Stability: NO

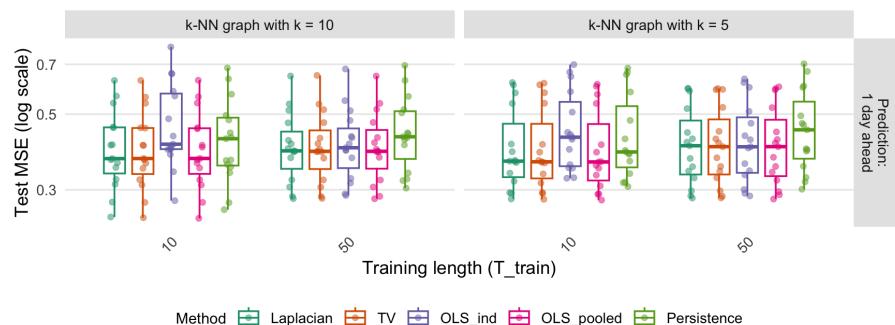


(b) Test MSE: NO

Figure 8: Coefficient stability and test RMSE for predicting NO concentration. Boxplots show the distribution of errors over 15 independent trials (years).



(a) Coefficient Stability: PM10



(b) Test MSE: PM10

Figure 9: Coefficient stability and Test RMSE for predicting PM10 concentration. Boxplots show the distribution of errors over 15 independent trials (years).

	NO		PM10		Temperature	
	Stability	Test RMSE	Stability	Test RMSE	Stability	Test RMSE
Persistence	0 ± 0	0.822 ± 0.0508	0 ± 0	0.431 ± 0.0626	0 ± 0	68.8 ± 18.4
OLS ind	0.214 ± 0.0144	0.802 ± 0.0461	0.246 ± 0.116	0.478 ± 0.0717	0.168 ± 0.0241	84.9 ± 19.2
OLS Pooled	0.00693 ± 0.00279	0.673 ± 0.0391	0.0198 ± 0.00676	0.403 ± 0.0547	0.0130 ± 0.00525	69.4 ± 17.8
Laplacian	0.0148 ± 0.00292	0.660 ± 0.0385	0.0226 ± 0.00613	0.404 ± 0.0546	0.0237 ± 0.00576	77.2 ± 17.7
TV	0.0105 ± 0.00352	0.666 ± 0.0391	0.0209 ± 0.00653	0.403 ± 0.0545	0.0233 ± 0.00491	71.1 ± 18.1

Table 2: Mean test MSE ($\pm 1.96 \times \text{sd}$) by method, for forecasting pollutant one day ahead. Both NO and PM10 are fitted using the 10 nearest-neighbor graph, while TEMP uses a 5 nearest neighbor graph.

In this setting, both the Laplacian and graph total-variation (TV) denoisers outperform the alternative estimators, with performance between the Laplacian and TV being comparably close.

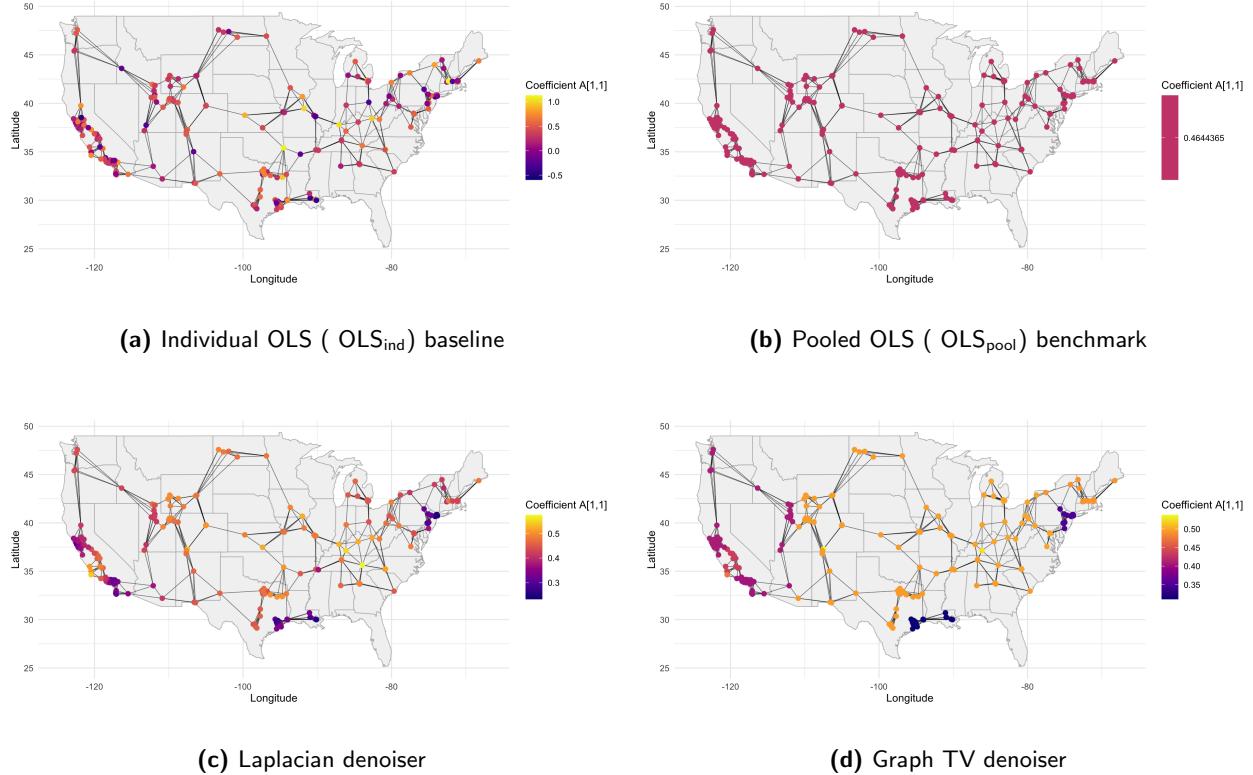


Figure 10: Monitoring stations colored by the value of the estimated LDS coefficient $\hat{A}_l[1,1]$ under four methods. Nodes are stations; (optional) gray edges indicate the k -NN graph ($k = 5$).

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A Proof of Proposition 2

With a slight abuse of notation, we will sometimes identify the set $\mathcal{T}_\mathcal{E}$ as a subset of \mathcal{E} , for convenience. This is clear since each $\{l, l'\} \in \mathcal{E}$ can be identified by a unique label $j \in |\mathcal{E}|$, based on the ordering of the rows of D .

Then for each $j \in \mathcal{T}_\mathcal{E}$, we can define the set $\mathcal{P}_j \subseteq [d^2]$ as

$$\mathcal{P}_j := \{i \mod d^2 : i \in \mathcal{T} \text{ and } (j-1)d^2 + 1 \leq i \leq jd^2\},$$

where \mathcal{P}_j will be identified⁸ by $\mathcal{P}_{\{l, l'\}}$. Now for $\theta \in \mathbb{R}^{md^2}$ formed by column-stacking $\theta_1, \dots, \theta_m \in \mathbb{R}^{d^2}$, we obtain the bound

$$\begin{aligned} \|(\tilde{D}\theta)_\mathcal{T}\|_1 &= \sum_{\{l, l'\} \in \mathcal{T}_\mathcal{E}} \sum_{i \in \mathcal{P}_{\{l, l'\}}} |(\theta_l)_i - (\theta_{l'})_i| \\ &\leq \sqrt{|\mathcal{T}|} \sqrt{\sum_{\{l, l'\} \in \mathcal{T}_\mathcal{E}} \sum_{i \in \mathcal{P}_{\{l, l'\}}} |(\theta_l)_i - (\theta_{l'})_i|^2} \\ &\leq \sqrt{|\mathcal{T}|} \sqrt{2 \sum_{\{l, l'\} \in \mathcal{T}_\mathcal{E}} \sum_{i \in \mathcal{P}_{\{l, l'\}}} ((\theta_l)_i^2 + (\theta_{l'})_i^2)} \\ &\leq \sqrt{|\mathcal{T}|} \sqrt{2 \sum_{\{l, l'\} \in \mathcal{T}_\mathcal{E}} (\|\theta_l\|_2^2 + \|\theta_{l'}\|_2^2)} \\ &\leq \sqrt{|\mathcal{T}|} \min \left\{ \sqrt{2\Delta}, 2\sqrt{|\mathcal{T}_\mathcal{E}|} \right\} \|\theta\|_2, \end{aligned}$$

which completes the proof.

B Proof of Corollary 1

We bound ζ_1 and ζ_2 (recall Lemma’s 1 and 2). Using $\|A^k\|_2 \leq \rho_{\max}^k$ and $\|X\|_F \leq \sqrt{d} \|X\|_2$, we obtain

$$\text{Tr}(\Gamma_t(A_l^*)) = \sum_{k=0}^t \|A_l^{*k}\|_F^2 \leq d \sum_{k=0}^t \rho_{\max}^{2k} \leq \frac{d}{1 - \rho_{\max}^2}.$$

For any canonical basis vector e_i ,

$$e_i^\top \Gamma_{t-1}(A_l^*) e_i = \sum_{k=0}^{t-1} \|A_l^{*k} e_i\|_2^2 \leq \sum_{k=0}^{t-1} \|A_l^{*k}\|_2^2 \leq \frac{1}{1 - \rho_{\max}^2}, \quad \forall l \in [m]. \quad (\text{B.1})$$

We now get the following upper bounds for ζ_1 and ζ_2 . Using

$$\sum_{l=1}^m \sum_{t=0}^{T-1} \text{Tr}(\Gamma_t(A_l^*)) \leq mT \frac{d}{1 - \rho_{\max}^2},$$

⁸Essentially, for each edge $j \in \mathcal{T}_\mathcal{E}$ which “appears” in \mathcal{T} , we want to identify (through \mathcal{P}_j) the set of coordinates that are “attached” to $\mathcal{T}_\mathcal{E}$. Alternatively, the set \mathcal{T} is completely specified by: (a) $\mathcal{T}_\mathcal{E}$ and, (b) the collection $(\mathcal{P}_j)_{j \in \mathcal{T}_\mathcal{E}}$

we obtain the bound

$$\zeta_1(m, T, \delta) \leq \frac{c_1 m d T}{1 - \rho_{\max}^2} \log(1/\delta) \quad (\text{B.2})$$

On the other hand, (B.1) readily implies

$$\zeta_2(m, T, \delta) \leq \frac{c_1 \mu^2 T}{1 - \rho_{\max}^2} \log^2\left(\frac{d^2 |\mathcal{E}|}{\delta}\right). \quad (\text{B.3})$$

Now we bound $\beta := \max_l \|\tilde{A}_l^*\|_2$. This is done using [13, Lemma 5], which in our notation implies

$$\|\tilde{A}_l^*\|_2 \leq \sum_{t=0}^{T-1} \|(A_l^*)^t\|_2 \leq \sum_{t=0}^{T-1} \|A_l^*\|_2^t \leq \sum_{t=0}^{\infty} \|A_l^*\|_2^t \leq \frac{1}{1 - \rho_{\max}}.$$

Hence we have $\beta \leq \frac{1}{1 - \rho_{\max}}$. Using (B.2) and (B.3), there exist constants $c, c', c'', c_1 > 1$ and $c_2 \in (0, 1)$ such that for $\delta \in (0, c_2)$, we can respectively bound F_1, F_2 as follows.

$$\begin{aligned} F_1 &\leq \sqrt{2} \left(\frac{c_1 d T}{1 - \rho_{\max}^2} \log(1/\delta) + 1 \right)^{1/2} \left(\log(1/\delta) + \frac{d^2}{2} \log \left(\frac{c_1 d T}{1 - \rho_{\max}^2} \log(1/\delta) + 1 \right) \right)^{1/2} \\ &\leq c \left(\frac{dT}{1 - \rho_{\max}^2} \log(1/\delta) \right)^{1/2} \left(d^2 \log(1/\delta) + d^2 \log \left(\frac{cdT}{1 - \rho_{\max}^2} \log(1/\delta) + 1 \right) \right)^{1/2} \\ &\leq c' \left(\frac{d^3 T}{1 - \rho_{\max}^2} \log(1/\delta) \right)^{1/2} \log^{1/2} \left(\frac{cdT}{\delta(1 - \rho_{\max}^2)} \right) \\ &\leq c'' \left(\frac{d^3 T}{1 - \rho_{\max}^2} \right)^{1/2} \log \left(\frac{dT}{\delta(1 - \rho_{\max}^2)} \right). \end{aligned}$$

Furthermore,

$$F_2 \leq c \zeta_2^{1/2}(m, T, \delta) \leq c_1 \frac{\mu \sqrt{T}}{\sqrt{1 - \rho_{\max}^2}} \log \left(\frac{d |\mathcal{E}|}{\delta} \right).$$

The statement of the corollary now follows in a straightforward manner.

C Some technical tools

C.1 Talagrand's functionals and Gaussian width

Recall Talagrand's γ_α functionals [33] which can be interpreted as a measure of the complexity of a (not necessarily convex) set.

Definition 2 ([33]). *Let (\mathcal{T}, d) be a metric space. We say that a sequence of subsets of \mathcal{T} , namely $(\mathcal{T}_r)_{r \geq 0}$ is an admissible sequence if $|\mathcal{T}_0| = 1$ and $|\mathcal{T}_r| \leq 2^{2^r}$ for every $r \geq 1$. Then for any $0 < \alpha < \infty$, the γ_α functional of (\mathcal{T}, d) is defined as*

$$\gamma_\alpha(\mathcal{T}, d) := \inf \sup_{t \in \mathcal{T}} \sum_{r=0}^{\infty} 2^{r/\alpha} d(t, \mathcal{T}_r)$$

with the infimum being taken over all admissible sequences of \mathcal{T} .

The following properties of the γ_α functional are useful to note.

1. It follows directly from Definition 2 that for any two metrics d_1, d_2 such that $d_1 \leq ad_2$ for some $a > 0$, we have that $\gamma_\alpha(\mathcal{T}, d_1) \leq a\gamma_\alpha(\mathcal{T}, d_2)$.
2. For $\mathcal{T}' \subset \mathcal{T}$, we have that $\gamma_\alpha(\mathcal{T}', d) \leq C_\alpha \gamma_\alpha(\mathcal{T}, d)$ for $C_\alpha > 0$ depending only on α .
3. If $f : (\mathcal{T}, d_1) \rightarrow (\mathcal{U}, d_2)$ is onto and for some constant L satisfies

$$d_2(f(x), f(y)) \leq Ld_1(x, y) \quad \text{for all } x, y \in \mathcal{T},$$

then $\gamma_\alpha(\mathcal{U}, d_2) \leq C_\alpha L \gamma_\alpha(\mathcal{T}, d_1)$ for $C_\alpha > 0$ depending only on α .

Properties 2 and 3 are stated in [32, Theorem 1.3.6] for the γ_α functional defined in [33, Definition 2.2.19] with $C_\alpha = 1$. However, γ_α as in Definition 2 is equivalent to that in [33, Definition 2.2.19] up to a constant depending only on α ; see [33, Section 2.3].

Remark 5. Note that [32, Theorem 1.3.6] is stated for another version of γ_α functional (see [33, Definition 2.2.19]) which is equivalent to the one in Definition 2 up to a constant C_α depending only on α (see [33, Section 2.3]).

By Talagrand's majorizing measure theorem [33, Theorem 2.4.1], the expected suprema of centered Gaussian processes $(X_s)_{s \in \mathcal{S}}$ are characterized by $\gamma_2(\mathcal{S}, d)$ as

$$c\gamma_2(\mathcal{S}, d) \leq \mathbb{E} \sup_{s \in \mathcal{S}} X_s \leq C\gamma_2(\mathcal{S}, d) \tag{C.1}$$

for universal constants $c, C > 0$, with d denoting the canonical distance $d(s, s') := (\mathbb{E}[X_s - X_{s'}]^2)^{1/2}$. For instance, if $\mathcal{S} \subset \mathbb{R}^{n \times m}$, and $X_s = \langle G, s \rangle$ for a $n \times m$ matrix G with iid standard Gaussian entries, we obtain $d(s, s') = \|s - s'\|_F^2$. Then (C.1) implies $\mathbb{E} \sup_{s \in \mathcal{S}} \langle G, s \rangle \asymp \gamma_2(\mathcal{S}, \|\cdot\|_F)$. Here, $\mathbb{E} \sup_{s \in \mathcal{S}} \langle G, s \rangle$ is known as the *Gaussian width* of the set \mathcal{S} , denoted as $w(\mathcal{S})$. For an overview of the properties of the Gaussian width, the reader is referred to [39].

C.2 Suprema of second order subgaussian chaos processes

For a set of matrices \mathcal{A} , define the terms

$$d_F(\mathcal{A}) := \sup_{A \in \mathcal{A}} \|A\|_F, \quad d_2(\mathcal{A}) := \sup_{A \in \mathcal{A}} \|A\|_2. \tag{C.2}$$

These can be interpreted as other types of complexity measures of the set \mathcal{A} . The following result from [15] is a concentration bound for the suprema of second order subgaussian chaos processes involving positive semidefinite (p.s.d) matrices.

Theorem 2 ([15]). *Let \mathcal{A} be a set of matrices and ξ be a vector whose entries are independent, zero-mean, variance 1, and are L -subgaussian random variables. Denote*

$$\begin{aligned} F &= \gamma_2(\mathcal{A}, \|\cdot\|_2)[\gamma_2(\mathcal{A}, \|\cdot\|_2) + d_F(\mathcal{A})] + d_F(\mathcal{A})d_2(\mathcal{A}) \\ G &= d_2(\mathcal{A})[\gamma_2(\mathcal{A}, \|\cdot\|_2) + d_F(\mathcal{A})], \quad \text{and} \quad H = d_2^2(\mathcal{A}) \end{aligned}$$

where d_2, d_F are as in (C.2). Then there exist constants $c_1, c_2 > 0$ depending only on L such that for any $t > 0$ it holds that

$$\mathbb{P} \left(\sup_{A \in \mathcal{A}} \left| \|A\xi\|_2^2 - \mathbb{E}[\|A\xi\|_2^2] \right| \geq c_1 F + t \right) \leq 2 \exp \left(-c_2 \min \left\{ \frac{t^2}{G^2}, \frac{t}{H} \right\} \right).$$

D Proof of Proposition 3

For any $h \in \mathcal{C}_S \cap \mathbb{S}^{md^2-1}$, we have for $u(h) = \tilde{D}h$ that

$$\begin{aligned}
\|u(h)\|_1 &= \|\tilde{D}h\|_1 = \|(\tilde{D}h)_S\|_1 + \|(\tilde{D}h)_{S^c}\|_1 \\
&\leq 4\|(\tilde{D}h)_S\|_1 + 4\|(\tilde{D}a^*)_{S^c}\|_1 + 1 && (\text{since } h \in \mathcal{C}_S \cap \mathbb{S}^{md^2-1}) \\
&\leq 4\frac{\sqrt{|S|}}{\kappa_S} \|h\|_2 + 4\|(\tilde{D}a^*)_{S^c}\|_1 + 1 && (\text{Recall Definition 1}) \\
&= 4\frac{\sqrt{|S|}}{\kappa_S} + 4\|(\tilde{D}a^*)_{S^c}\|_1 + 1. && (\text{since } h \in \mathbb{S}^{md^2-1})
\end{aligned}$$

E Additional details from Section 3.1

Denote the sigma algebra $\mathcal{F}_t := \sigma(\eta_{l,1}, \dots, \eta_{l,t})_{l=1}^m$, we then obtain a filtration $(\mathcal{F})_{t=1}^\infty$. The idea now is to show the following result, analogous to that in [1, Lemma 9].

Lemma 6. *For any $u \in \mathbb{R}^d$ and $t \geq 1$, consider*

$$M_t^u := \exp \left(\sum_{s=1}^t \sum_{l=1}^m \left[\langle u, x_{l,s} \otimes \eta_{l,s+1} \rangle - \frac{\|\mu\|_{(x_{l,s} x_{l,s}^\top) \otimes I_d}^2}{2} \right] \right)$$

Let $\bar{\tau}$ be a stopping time w.r.t $(\mathcal{F})_{t=1}^\infty$. Then $M_{\bar{\tau}}^u$ is well defined and $\mathbb{E}[M_{\bar{\tau}}^u] \leq 1$.

Proof. Denote

$$D_t^u = \exp \left(\sum_{l=1}^m \left[\langle u, x_{l,t} \otimes \eta_{l,t+1} \rangle - \frac{\|\mu\|_{(x_{l,t} x_{l,t}^\top) \otimes I_d}^2}{2} \right] \right)$$

Observe that D_t^u and M_t^u are both \mathcal{F}_{t+1} measurable. Hence it follows that

$$\begin{aligned}
\mathbb{E}[D_t^u | \mathcal{F}_t] &= \prod_{l=1}^m \mathbb{E} \left[\exp \left(\sum_{l=1}^m \left[\langle u, x_{l,t} \otimes \eta_{l,t+1} \rangle - \frac{\|\mu\|_{(x_{l,t} x_{l,t}^\top) \otimes I_d}^2}{2} \right] \right) \middle| \mathcal{F}_t \right] \\
&\leq 1 \text{ a.s}
\end{aligned}$$

since $\eta_{l,t}$ has independent 1-subgaussian entries (for each l, t). Using this bound, we obtain

$$\mathbb{E}[D_t^u | \mathcal{F}_t] \leq D_1^u D_2^u \cdots D_{t-1}^u = M_{t-1}^u$$

which implies $(M_t^u)_{t=1}^\infty$ is a super-martingale and also that $\mathbb{E}[M_t^u] \leq 1$. Now one can show using the same arguments as presented in the proof of [1, Lemma 9] to show that $M_{\bar{\tau}}^u$ is well defined and $\mathbb{E}[M_{\bar{\tau}}^u] \leq 1$; the details are omitted. \square

Equipped with Lemma 6, the bound in (3.12) now follows in an analogous manner to that in the proof of [1, Theorem 1].

F Control of the inverse scaling factor

F.1 2D grid

We introduce the following notation.

- Let $G_{N \times N} = (V, E)$ denote the $N \times N$ two-dimensional rectangular grid with vertex set $V = \{0, \dots, N-1\}^2$. Hence $|V| = m = N^2$ and let D_2 be the *incidence matrix* of the 2D grid.
- We write the pseudoinverse of D_2 as

$$S = D_2^\dagger = L^\dagger D_2^\top,$$

where L^\dagger is the pseudoinverse of L .

It was shown in [12] that

$$\max_{e \in E} \|S_{:,e}\|_2 \leq C\sqrt{\log m},$$

i.e. the *column* norms grow only logarithmically. We now bound the *row* norms.

Lemma 7. *For any vertex $v \in V$, the squared ℓ_2 norm of the v -th row of S equals the v -th diagonal entry of L^\dagger , i.e.,*

$$\|S_{v,:}\|_2^2 = (L^\dagger)_{vv}.$$

Proof. Let $e_v \in \mathbb{R}^n$ be the v -th canonical basis vector. Because $S = L^\dagger D_2^\top$,

$$S_{v,:}^\top = e_v^\top L^\dagger D_2^\top, \implies \|S_{v,:}\|_2^2 = e_v^\top L^\dagger D_2^\top D_2 L^\dagger e_v.$$

Since $D_2^\top D_2 = L$ and $L^\dagger LL^\dagger = L^\dagger$,

$$\|S_{v,:}\|_2^2 = e_v^\top L^\dagger e_v = (L^\dagger)_{vv}.$$

□

Consequently, an upper bound on $\max_{v \in V} (L^\dagger)_{vv}$ implies the desired row-norm bound.

One-dimensional path. Let L_1 be the $N \times N$ Laplacian of the path graph P_N . It possesses the discrete cosine eigenbasis⁹

$$L_1 = V_1 \Lambda_1 V_1^\top, \quad \lambda_k = 2 - 2 \cos\left(\frac{k\pi}{N}\right), \quad k = 0, \dots, N-1.$$

For $k \neq 0$, the entries of the k -th eigenvector satisfy $|(V_1)_{ik}| \leq \sqrt{\frac{2}{N}}$; for $k = 0$, $(V_1)_{i0} = \frac{1}{\sqrt{N}}$ (see, e.g. [31, Chapter 1.5]).

⁹In a slight abuse of notation, we write the eigenvalues here as $\lambda_0 \leq \lambda_1 \leq \dots \leq \lambda_{N-1}$ for convenience.

Two-dimensional grid. Using the (Kronecker) Cartesian product,

$$L = L_1 \otimes I_N + I_N \otimes L_1 = V_2 \Lambda_2 V_2^\top,$$

where $V_2 = V_1 \otimes V_1$ and $\Lambda_2 = \Lambda_1 \otimes I_N + I_N \otimes \Lambda_1$. Writing a vertex v as a pair (i, j) , the corresponding diagonal entry of L^\dagger expands to

$$(L^\dagger)_{vv} = \sum_{\substack{k, \ell=0 \\ (k, \ell) \neq (0, 0)}}^{N-1} \frac{(V_1)_{ik}^2 (V_1)_{j\ell}^2}{\lambda_k + \lambda_\ell}. \quad (\text{F.1})$$

Indeed, let $\phi_{k\ell}$ denote the eigenvector and $\mu_{k\ell}$ the eigenvalue of L , we have

$$\mu_{k\ell} := \lambda_k + \lambda_\ell, \quad \phi_{k\ell} := v_k \otimes v_\ell, \quad k, \ell \in \{0, \dots, N-1\}.$$

Because L is symmetric, its pseudoinverse is obtained by inverting the positive eigenvalues and leaving the zero mode at 0, i.e.,

$$L^\dagger = V_2 \Lambda_2^\dagger V_2^\top = \sum_{\substack{0 \leq k, \ell \leq N-1 \\ (k, \ell) \neq (0, 0)}} \frac{1}{\lambda_k + \lambda_\ell} \phi_{k\ell} \phi_{k\ell}^\top, \quad (\Lambda_2^\dagger)_{k\ell, k\ell} := \begin{cases} \frac{1}{\lambda_k + \lambda_\ell}, & (k, \ell) \neq (0, 0), \\ 0, & (k, \ell) = (0, 0), \end{cases}$$

and

$$\begin{aligned} (L^\dagger)_{vv} &= e_{(i,j)}^\top L^\dagger e_{(i,j)} = \sum_{\substack{0 \leq k, \ell \leq N-1 \\ (k, \ell) \neq (0, 0)}} \frac{(\phi_{k\ell}(i, j))^2}{\lambda_k + \lambda_\ell} \\ &= \sum_{\substack{0 \leq k, \ell \leq N-1 \\ (k, \ell) \neq (0, 0)}} \frac{(v_k(i) v_\ell(j))^2}{\lambda_k + \lambda_\ell} = \sum_{\substack{0 \leq k, \ell \leq N-1 \\ (k, \ell) \neq (0, 0)}} \frac{(V_1)_{ik}^2 (V_1)_{j\ell}^2}{\lambda_k + \lambda_\ell}. \end{aligned}$$

Note¹⁰ that $\lambda_k = 4 \sin^2(\pi k/(2N)) \asymp (\pi k/N)^2$ for $k = 0, 1, \dots, N-1$, which implies $\lambda_k \geq c$ (for some absolute $c > 0$) once $k \geq N/(2\pi)$. Hence, we now split the double sum (F.1) into ‘‘low-’’ and ‘‘high-frequency’’ blocks.

Low frequencies: $k, \ell \leq \frac{N}{2\pi}$

Using $|(V_1)_{ik}|, |(V_1)_{j\ell}| \leq \sqrt{\frac{2}{N}}$ and $\lambda_k \asymp (\pi k/N)^2$,

$$\sum_{1 \leq k, \ell \leq N/(2\pi)} \frac{(V_1)_{ik}^2 (V_1)_{j\ell}^2}{\lambda_k + \lambda_\ell} \leq \frac{4}{N^2} \sum_{1 \leq k, \ell \leq N/(2\pi)} \frac{N^2/\pi^2}{k^2 + \ell^2} = \frac{4}{\pi^2} \sum_{k, \ell=1}^{\lfloor N/(2\pi) \rfloor} \frac{1}{k^2 + \ell^2}.$$

Let $R := \lfloor N/(2\pi) \rfloor$ and define $\mathcal{A}_j := \{(k, \ell) \in \mathbb{N}^2 : 2^j < \sqrt{k^2 + \ell^2} \leq 2^{j+1}\}$, $j = 0, \dots, J := \lfloor \log_2 R \rfloor$. Then for $(k, \ell) \in \mathcal{A}_j$, $1/(k^2 + \ell^2) \leq 2^{-2j}$, so

$$\sum_{k=1}^R \sum_{\ell=1}^R \frac{1}{k^2 + \ell^2} \leq \sum_{j=0}^J \frac{|\mathcal{A}_j|}{2^{2j}}.$$

¹⁰Since $\sin x \asymp x$ for $x \in [0, \pi/2]$.

Since $|\mathcal{A}_j| \leq (2^{j+1})^2 = 4 \cdot 2^{2j}$, each shell contributes at most 4, hence

$$\sum_{k=1}^R \sum_{\ell=1}^R \frac{1}{k^2 + \ell^2} \leq 4(J+1) = O(\log R).$$

Substituting into $\frac{4}{\pi^2} \sum_{1 \leq k, \ell \leq R} \frac{1}{k^2 + \ell^2}$ gives the low-frequency bound $O(\log N)$.

High frequencies

If k or ℓ exceeds $N/(2\pi)$, then $\lambda_k + \lambda_\ell \geq c$ for some absolute $c > 0$. Together with $|(V_1)_{ik}|^2, (V_1)_{j\ell}|^2 \leq 2/N$, the contribution is $O(1)$.

Combining both regions,

$$(L^\dagger)_{vv} \leq C \log N \implies \max_{v \in V} (L^\dagger)_{vv} \leq C \log N,$$

for an absolute constant C . This proves the following proposition.

Proposition 4 (Bound on μ' for 2D grid). *Let D_2 be the incidence matrix of the $N \times N$ two-dimensional grid and $S = D_2^\dagger$ its Moore–Penrose inverse. Then*

$$\boxed{\max_{v \in V} \|S_{v,:}\|_2 \leq C \sqrt{\log m}}, \quad m = N^2,$$

where $C > 0$ is an absolute constant.

F.2 n -dimensional grid

We can extend the proof of Proposition 4 to the general case of a n -dimensional grid.

Proposition 5 (Bound on μ' for n -dimensional grid). *Let $G_{N,n}$ be the N^n grid, D_n its incidence, $L_n = D_n^\top D_n$, $S = D_n^\dagger$. Then for some $C_n > 0$ (dimension only) where $C_n = O(2^n)$,*

$$\max_v \|S_{v,:}\|_2^2 = (L_n^\dagger)_{vv} \leq \begin{cases} C_1 N, & n = 1, \\ C_2 \log N, & n = 2, \\ C_n, & n \geq 3. \end{cases}$$

Proof. Lemma 7 implies that $\|S_{v,:}\|_2^2 = (L_n^\dagger)_{vv}$ and we have $L_n = \sum_{r=1}^n I^{\otimes(r-1)} \otimes L_1 \otimes I^{\otimes(n-r)}$ with $L_1 = V_1 \Lambda_1 V_1^\top$, $\lambda_k = 2 - 2 \cos(k\pi/N)$. Then for $v = (i_1, \dots, i_n)$,

$$(L_n^\dagger)_{vv} = \sum_{\mathbf{k} \neq 0} \frac{\prod_{r=1}^n (V_1)_{i_r k_r}^2}{\sum_{r=1}^n \lambda_{k_r}} \tag{F.2}$$

where $\mathbf{k} = (k_r)$. Using $(V_1)_{ik}^2 \leq 2/N$ (and $1/N$ for $k = 0$) gives a uniform numerator bound $(2/N)^n$. Let

$$R := \left\lfloor \frac{N}{2\pi} \right\rfloor.$$

For $1 \leq k \leq R$ we have $\frac{k\pi}{N} \leq \frac{1}{2}$ and (by Taylor's inequality) $2 - 2 \cos x \geq \frac{x^2}{2}$ for $x \in [0, 1/2]$. Hence

$$\lambda_k \geq \frac{1}{2} \left(\frac{\pi k}{N} \right)^2, \quad \sum_{r=1}^n \lambda_{k_r} \geq \frac{\pi^2}{2N^2} \|\mathbf{k}\|_2^2 \quad \text{whenever } \|\mathbf{k}\|_\infty \leq R, \mathbf{k} \neq \mathbf{0}.$$

Then, we get

$$\sum_{\substack{\mathbf{k} \neq \mathbf{0} \\ \|\mathbf{k}\|_\infty \leq R}} \frac{\prod_{r=1}^n (V_1)_{i_r, k_r}^2}{\sum_{r=1}^n \lambda_{k_r}} \leq \left(\frac{2}{N}\right)^n \cdot \frac{2N^2}{\pi^2} \sum_{\substack{\mathbf{k} \neq \mathbf{0} \\ \|\mathbf{k}\|_\infty \leq R}} \frac{1}{\|\mathbf{k}\|_2^2} = \frac{C2^n}{N^{n-2}} S_n(R),$$

where

$$S_n(R) := \sum_{\substack{\mathbf{k} \in \{0, \dots, R\}^n \\ \mathbf{k} \neq \mathbf{0}}} \frac{1}{\|\mathbf{k}\|_2^2}.$$

Next, we prove the following lemma.

Lemma 8. *There exists $C_n > 0$ such that*

$$S_n(R) \leq \begin{cases} C_1, & n = 1, \\ C_2 \log R, & n = 2, \\ C_n R^{n-2}, & n \geq 3. \end{cases}$$

Proof of Lemma 8. Partition into dyadic shells

$$\mathcal{A}_j := \{\mathbf{k} \in \mathbb{Z}_{\geq 0}^n : 2^j < \|\mathbf{k}\|_2 \leq 2^{j+1}\}, \quad j = 0, 1, \dots, \lfloor \log_2 R \rfloor.$$

On \mathcal{A}_j we have $\|\mathbf{k}\|_2^{-2} \leq 2^{-2j}$, and $|\mathcal{A}_j| \leq C' 2^{(j+1)n}$ (crude cardinality bound by volume). Thus

$$S_n(R) \leq \sum_{j=0}^{\lfloor \log_2 R \rfloor} \frac{|\mathcal{A}_j|}{2^{2j}} \leq C' 2^n \sum_{j=0}^{\lfloor \log_2 R \rfloor} 2^{j(n-2)}.$$

The sum is $O(1)$ if $n = 1$, equals $O(\log R)$ if $n = 2$, and is $O(R^{n-2})$ if $n \geq 3$. \square

Lemma 8 implies

$$\sum_{\substack{\mathbf{k} \neq \mathbf{0} \\ \|\mathbf{k}\|_\infty \leq R}} \frac{\prod_{r=1}^n (V_1)_{i_r, k_r}^2}{\sum_{r=1}^n \lambda_{k_r}} \leq \begin{cases} C N, & n = 1, \\ C \log N, & n = 2, \\ C, & n \geq 3. \end{cases}$$

If $\|\mathbf{k}\|_\infty > R$, then some $k_r \geq R+1$ and $\frac{k_r \pi}{N} \geq \frac{1}{2}$, hence $\lambda_{k_r} = 2 - 2 \cos\left(\frac{k_r \pi}{N}\right) \geq c_0$ for a universal $c_0 > 0$. Thus

$$\sum_{r=1}^n \lambda_{k_r} \geq c_0, \quad \frac{\prod_{r=1}^n (V_1)_{i_r, k_r}^2}{\sum_{r=1}^n \lambda_{k_r}} \leq \frac{1}{c_0} \left(\frac{2}{N}\right)^n.$$

Summing over at most N^n indices gives the uniform bound

$$\sum_{\|\mathbf{k}\|_\infty > R} \frac{\prod_{r=1}^n (V_1)_{i_r, k_r}^2}{\sum_{r=1}^n \lambda_{k_r}} \leq \frac{1}{c_0} \left(\frac{2}{N}\right)^n N^n = \frac{2^n}{c_0} = C_n.$$

Combining the two blocks in yields the result. \square

G Controlling Δ_G

Lemma 9 (Control of Δ_G by edgewise Frobenius variation). *Let $G = ([m], \mathcal{E})$ be a connected graph and suppose each local system matrix is stable: $\|A_\ell^*\|_2 \leq \rho_{\max} < 1$ for all $\ell \in [m]$. Define the controllability aggregate*

$$G_\ell := \sum_{t=1}^T \Gamma_{t-1}(A_\ell^*) \quad \text{with} \quad \Gamma_{t-1}(A) = \sum_{s=0}^{t-1} A^s (A^s)^\top,$$

and let $\bar{G} := \frac{1}{m} \sum_{\ell=1}^m G_\ell$. Recall

$$\Delta_G := \max_{a,b \in [d]} \left\{ \sum_{\ell=1}^m \left[(G_\ell)_{b,a} - (\bar{G})_{b,a} \right]^2 \right\}^{1/2}.$$

Then

$$\Delta_G \leq \frac{L_T(\rho_{\max})}{\sqrt{\lambda_{m-1}(G)}} \left(\sum_{\{\ell,\ell'\} \in \mathcal{E}} \|A_\ell^* - A_{\ell'}^*\|_F^2 \right)^{1/2}, \quad L_T(\rho) := 2 \sum_{s=1}^{T-1} (T-s)s \rho^{2s-1} \leq \frac{2\rho T}{(1-\rho^2)^2}.$$

Proof. Step 1 (entrywise Lipschitz control of $G(\cdot)$). Fix $a, b \in [d]$ and set $g_{a,b}(A) := e_b^\top G(A) e_a = \sum_{t=1}^T \sum_{s=0}^{t-1} e_b^\top A^s (A^s)^\top e_a$. For stable A, B with $\|A\|_2, \|B\|_2 \leq \rho_{\max}$, the standard telescoping bound

$$\|A^s - B^s\|_2 \leq \sum_{r=0}^{s-1} \|A\|_2^{s-1-r} \|A - B\|_2 \|B\|_2^r \leq s \rho_{\max}^{s-1} \|A - B\|_2$$

implies

$$\|A^s (A^s)^\top - B^s (B^s)^\top\|_2 \leq \|A^s\|_2 \|A^s - B^s\|_2 + \|A^s - B^s\|_2 \|B^s\|_2 \leq 2s \rho_{\max}^{2s-1} \|A - B\|_2.$$

Hence

$$|g_{a,b}(A) - g_{a,b}(B)| \leq \sum_{t=1}^T \sum_{s=1}^{t-1} 2s \rho_{\max}^{2s-1} \|A - B\|_2 = L_T(\rho_{\max}) \|A - B\|_2 \leq L_T(\rho_{\max}) \|A - B\|_F. \quad (\text{G.1})$$

Step 2 (from Lipschitz to nodewise variance). Let $\bar{A}^* := \frac{1}{m} \sum_{\ell=1}^m A_\ell^*$. The empirical mean minimizes squared deviations, so for each fixed (a, b) ,

$$\sum_{\ell=1}^m \left[(G_\ell)_{b,a} - (\bar{G})_{b,a} \right]^2 \leq \sum_{\ell=1}^m \left[(G_\ell)_{b,a} - g_{a,b}(\bar{A}^*) \right]^2.$$

Apply Step 1 entrywise with $A = A_\ell^*$ and $B = \bar{A}^*$ to get

$$\sum_{\ell=1}^m \left[(G_\ell)_{b,a} - (\bar{G})_{b,a} \right]^2 \leq L_T^2(\rho_{\max}) \sum_{\ell=1}^m \|A_\ell^* - \bar{A}^*\|_F^2.$$

Step 3 (graph Poincaré, matrix-valued signal). For any matrix-valued signal on the nodes, the graph Poincaré inequality yields

$$\sum_{\ell=1}^m \|A_\ell^* - \bar{A}^*\|_F^2 \leq \frac{1}{\lambda_{m-1}(G)} \sum_{\{\ell,\ell'\} \in \mathcal{E}} \|A_\ell^* - A_{\ell'}^*\|_F^2.$$

Combining Step 2 with Step 3 and taking the maximum over (a, b) we obtain the stated bound. \square

References

- [1] Yasin Abbasi-yadkori, Dávid Pál, and Csaba Szepesvári. Improved algorithms for linear stochastic bandits. In *Advances in Neural Information Processing Systems*, volume 24, 2011.
- [2] Sumanta Basu and George Michailidis. Regularized estimation in sparse high-dimensional time series models. *The Annals of Statistics*, 43(4):1535–1567, 2015.
- [3] Sumanta Basu, Ali Shojaie, and George Michailidis. Network granger causality with inherent grouping structure. *Journal of Machine Learning Research*, 16(13):417–453, 2015.
- [4] John T Bosworth. Linearized aerodynamic and control law models of the x-29a airplane and comparison with flight data. *National Aeronautics and Space Administration, Office of Management . . .*, 4356, 1992.
- [5] Stéphane Boucheron, Gábor Lugosi, and Pascal Massart. *Concentration Inequalities: A Nonasymptotic Theory of Independence*. Oxford University Press, 2013.
- [6] M.C. Campi and E. Weyer. Finite sample properties of system identification methods. *IEEE Transactions on Automatic Control*, 47(8):1329–1334, 2002.
- [7] Yiting Chen, Ana M. Ospina, Fabio Pasqualetti, and Emiliano Dall’Anese. Multi-task system identification of similar linear time-invariant dynamical systems. *2023 62nd IEEE Conference on Decision and Control (CDC)*, pages 7342–7349, 2023.
- [8] Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, and Clifford Stein. *Introduction to Algorithms, Third Edition*. The MIT Press, 3rd edition, 2009.
- [9] Salar Fattah, Nikolai Matni, and Somayeh Sojoudi. Learning sparse dynamical systems from a single sample trajectory. In *2019 IEEE 58th Conference on Decision and Control (CDC)*, pages 2682–2689, 2019.
- [10] Shi Gu, Fabio Pasqualetti, Matthew Cieslak, Qawi K. Telesford, Alfred B. Yu, Ari E. Kahn, John D. Medaglia, Jean M. Vettel, Michael B. Miller, Scott T. Grafton, and Danielle S. Bassett. Controllability of structural brain networks. *Nature Communications*, 6, 2014.
- [11] Daniel Hsu, Sham Kakade, and Tong Zhang. A tail inequality for quadratic forms of subgaussian random vectors. *Electronic Communications in Probability*, 17:1–6, 2012.
- [12] Jan-Christian Hüttter and Philippe Rigollet. Optimal rates for total variation denoising. volume 49, pages 1115–1146, 2016.
- [13] Yassir Jedra and Alexandre Proutiere. Finite-time identification of stable linear systems optimality of the least-squares estimator. In *2020 59th IEEE Conference on Decision and Control (CDC)*, pages 996–1001, 2020.
- [14] Anders Bredahl Kock and Laurent Callot. Oracle inequalities for high dimensional vector autoregressions. *Journal of Econometrics*, 186(2):325–344, 2015.
- [15] Felix Krahmer, Shahar Mendelson, and Holger Rauhut. Suprema of chaos processes and the restricted isometry property. *Communications on Pure and Applied Mathematics*, 67(11):1877–1904, 2014.

- [16] T.L Lai and C.Z Wei. Asymptotic properties of general autoregressive models and strong consistency of least-squares estimates of their parameters. *Journal of Multivariate Analysis*, 13(1):1–23, 1983.
- [17] Tze Lai and Ching-Zong Wei. Extended least squares and their applications to adaptive control and prediction in linear systems. *IEEE Transactions on Automatic Control*, 31(10):898–906, 1986.
- [18] Tze Leung Lai and Ching Zong Wei. Least Squares Estimates in Stochastic Regression Models with Applications to Identification and Control of Dynamic Systems. *The Annals of Statistics*, 10(1):154 – 166, 1982.
- [19] Stephanie R Land and Jerome H Friedman. Variable fusion: A new adaptive signal regression method. *Dept. Statistics, Carnegie Mellon Univ. Pittsburgh, PA, USA, Rep*, 656, 1997.
- [20] Yuan Li, Benjamin Mark, Garvesh Raskutti, Rebecca Willett, Hyebin Song, and David Neiman. Graph-based regularization for regression problems with alignment and highly correlated designs. *SIAM Journal on Mathematics of Data Science*, 2(2):480–504, 2020.
- [21] Po-Ling Loh and Martin J. Wainwright. High-dimensional regression with noisy and missing data: Provable guarantees with nonconvexity. *The Annals of Statistics*, 40(3):1637 – 1664, 2012.
- [22] Igor Melnyk and Arindam Banerjee. Estimating structured vector autoregressive models. In *Proceedings of The 33rd International Conference on Machine Learning*, pages 830–839, 2016.
- [23] Aditya Modi, Mohamad Kazem Shirani Faradonbeh, Ambuj Tewari, and George Michailidis. Joint learning of linear time-invariant dynamical systems. *Automatica*, 164:111635, 2024.
- [24] Francesco Ortelli and Sara van de Geer. Prediction bounds for higher order total variation regularized least squares. *The Annals of Statistics*, 49(5):2755–2773, 2021.
- [25] Garvesh Raskutti, Martin J. Wainwright, and Bin Yu. Restricted eigenvalue properties for correlated gaussian designs. *J. Mach. Learn. Res.*, 11:2241–2259, 2010.
- [26] Veeranjaneyulu Sadhanala, Yu-Xiang Wang, and Ryan J Tibshirani. Total variation classes beyond 1d: Minimax rates, and the limitations of linear smoothers. *Advances in Neural Information Processing Systems*, 29:3521–3529, 2016.
- [27] Tuhin Sarkar and Alexander Rakhlin. Near optimal finite time identification of arbitrary linear dynamical systems. In *Proceedings of the 36th International Conference on Machine Learning, ICML*, volume 97, pages 5610–5618, 2019.
- [28] Mohamad Kazem Shirani Faradonbeh, Ambuj Tewari, and George Michailidis. Finite time identification in unstable linear systems. *Automatica*, 96:342–353, 2018.
- [29] Max Simchowitz, Horia Mania, Stephen Tu, Michael I. Jordan, and Benjamin Recht. Learning without mixing: Towards a sharp analysis of linear system identification. In *Proceedings of the 31st Conference On Learning Theory*, volume 75, pages 439–473, 2018.

- [30] Pragya Srivastava, Erfan Nozari, Jason Z. Kim, Harang Ju, Dale Zhou, Cassiano Becker, Fabio Pasqualetti, George J. Pappas, and Danielle S. Bassett. Models of communication and control for brain networks: distinctions, convergence, and future outlook. *Network Neuroscience*, 4(4):1122–1159, 2020.
- [31] Gilbert Strang. *Computational Science and Engineering*. Wellesley-Cambridge Press, Philadelphia, PA, 2007.
- [32] Michel Talagrand. *The Generic Chaining: Upper and Lower Bounds of Stochastic Processes*. Springer, 2005.
- [33] Michel Talagrand. *Upper and lower bounds for stochastic processes*, volume 60. Springer, 2014.
- [34] Ryan J Tibshirani. *The solution path of the generalized lasso*. Stanford University, 2011.
- [35] Huy Tran, Sansen Wei, and Claire Donnat. The generalized elastic net for least squares regression with network-aligned signal and correlated design. *IEEE Transactions on Signal and Information Processing over Networks*, 2025.
- [36] Hemant Tyagi. Joint estimation of smooth graph signals from partial linear measurements. *arXiv:2505.23240*, 2025.
- [37] Hemant Tyagi. Joint learning of linear dynamical systems under smoothness constraints. *Information and Inference: A Journal of the IMA*, 14(3):iaaf026, 2025.
- [38] Hemant Tyagi and Denis Efimov. Learning linear dynamical systems under convex constraints. *arxiv:2303.15121*, 2025.
- [39] Roman Vershynin. *High-Dimensional Probability: An Introduction with Applications in Data Science (2nd ed.)*. Cambridge University Press, 2025.
- [40] M. Vidyasagar and R.L. Karandikar. A learning theory approach to system identification and stochastic adaptive control. *Journal of Process Control*, 18(3):421–430, 2008.
- [41] Di Wang and Ruey S. Tsay. Rate-optimal robust estimation of high-dimensional vector autoregressive models. *The Annals of Statistics*, 51(2):846 – 877, 2023.
- [42] Han Wang, Leonardo Felipe Toso, and James Anderson. Fedsysid: A federated approach to sample-efficient system identification. In *Proceedings of The 5th Annual Learning for Dynamics and Control Conference*, volume 211, pages 1308–1320, 2023.
- [43] Yu-Xiang Wang, James Sharpnack, Alexander J Smola, and Ryan J Tibshirani. Trend filtering on graphs. *Journal of Machine Learning Research*, 17(105):1–41, 2016.
- [44] Lei Xin, George Chiu, and Shreyas Sundaram. Learning the dynamics of autonomous linear systems from multiple trajectories. In *2022 American Control Conference (ACC)*, pages 3955–3960, 2022.
- [45] Lei Xin, Lintao Ye, George T.-C. Chiu, and Shreyas Sundaram. Identifying the dynamics of a system by leveraging data from similar systems. *2022 American Control Conference (ACC)*, pages 818–824, 2022.
- [46] Yang Zheng and Na Li. Non-asymptotic identification of linear dynamical systems using multiple trajectories. *IEEE Control Systems Letters*, 5(5):1693–1698, 2020.