

# Scaled relative graphs for pairs of operators beyond classical monotonicity

Jan Quan

Alexander Bodard

Konstantinos Oikonomidis

Panagiotis Patrinos

**Abstract**—We introduce a generalization of the scaled relative graph (SRG) to pairs of operators, enabling the visualization of their relative incremental properties. This novel SRG framework provides the geometric counterpart for the study of nonlinear resolvents based on paired monotonicity conditions. We demonstrate that these conditions apply to linear operators composed with monotone mappings, a class that notably includes NPN transistors, allowing us to compute the response of multivalued, nonsmooth and highly nonmonotone electrical circuits.

## I. INTRODUCTION

In the study of dynamical systems and control algorithms, a central problem is understanding the input-output behavior of systems and their underlying operators. Classical monotone operator theory provides a unifying mathematical framework for modeling feedback interconnections, optimization dynamics, and equilibrium systems. Especially for ensuring stability and convergence, monotonicity is fundamental in many settings, from proximal algorithms to feedback systems governed by maximal monotone mappings. Looking beyond these classical assumptions is necessary, however, since many relevant control and learning systems are described by nonmonotone operators and thus require a different approach.

A new concept, *pair monotonicity*, has been introduced in order to better analyze the differential inclusions of sweeping processes in [1]. Notably, this notion was further utilized in [2] to characterize the firm-nonexpansiveness of generalized resolvent operators. Departing from standard monotonicity that pairs the inputs and outputs of an operator, pair monotonicity describes the incremental properties of the output of one operator compared to the output of another, thus making it useful for characterizing highly nonmonotone systems. Pair monotonicity provides a powerful way to incorporate prior knowledge by choosing the second operator judiciously. As this approach is purely algebraic, this choice is quite difficult in practice.

Rather than relying on algebra, a graphical tool is invaluable for building intuition. To this end, the scaled relative graph (SRG) has emerged as a powerful framework, by mapping the action of operators onto the (extended) complex plane. Yielding a nonlinear generalization of the classical Nyquist diagram, this tool has since seen applications in various systems and control contexts, such as graphical system analysis [3], [4], [5] and reset control systems [6].

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KU Leuven, Department of Electrical Engineering ESAT-STADIUS – Kasteelpark Arenberg 10, box 2446, 3001 Leuven, Belgium  
jan.quan@kuleuven.be

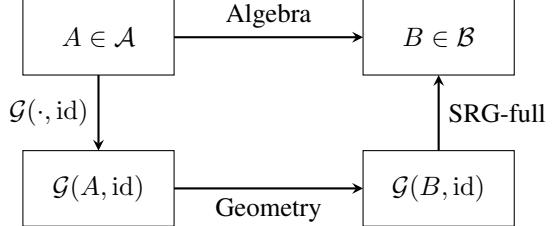


Fig. 1: Visualization standard SRG approach. Instead of traditional algebraic inequalities to characterize the properties of  $B$ , geometric manipulations transform the SRG of  $A$  to find the SRG of  $B$  (or a superset thereof). Then, SRG-fullness is used to conclude the properties of  $B$ .

Nevertheless, the SRG was originally used for the analysis of operator properties, where the high-level approach is shown in Fig. 1. Of particular interest is showing that operators are firmly nonexpansive [7, Def. 4.1] or contractive, upon which convergence of the associated fixed-point iteration can be shown using the Krasnosel'skii–Mann [7, Cor. 5.17] and Banach fixed theorem [7, Thm. 1.50] respectively. With the emergence of the new pair monotonicity, it is natural to ask how the SRG can be extended to handle these novel properties. This is the main subject of study in this paper.

Concretely, our contribution is threefold.

- We introduce a scaled relative graph for pairs of operators and establish important calculus rules. Furthermore, we extend the notions of SRG-fullness and semimonotonicity to this setting, which naturally covers pairs of monotone operators, extending classical monotonicity.
- We apply this novel tool to provide purely geometric proofs for the core properties of two nonlinear resolvents that have been used to solve inclusion problems with monotone pairs.
- We show the practical utility of this paired monotonicity condition by analyzing operators that can be written as a linear operator composed with a monotone mapping, which covers nonlinear transistors. Leveraging this property, we are able to compute the response of a nonmonotone common-emitter amplifier circuit.

## A. Notation

In the following,  $\mathcal{H}$  denotes a real Hilbert space with inner product  $\langle \cdot, \cdot \rangle$  and induced norm  $\|\cdot\|$ . We denote the set of complex and extended-complex numbers by  $\mathbb{C}$  and  $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$  respectively. For a complex number  $z \in \mathbb{C}$ , we adopt the convention  $z + \infty = \infty$ ,  $z/\infty = 0$ ,  $z/0 = \infty$ , and  $z \cdot \infty = \infty$ , while we avoid indeterminate forms  $\infty + \infty$ ,  $0/0$ ,  $\infty/\infty$ ,  $0 \cdot \infty$ . For  $z \in \mathbb{C}$ ,  $z^*$  denotes its complex conjugate while

for a bounded, linear operator  $M$ ,  $M^*$  denotes its adjoint. For a set-valued mapping  $A : \mathcal{H} \rightrightarrows \mathcal{H}$ , we define its domain  $\text{dom } A := \{x \in \mathcal{H} \mid A(x) \neq \emptyset\}$ , its range  $\text{rge } A := \{u \in \mathcal{H} \mid u \in A(x) \text{ for some } x \in \mathcal{H}\}$  and its graph  $\text{gph } A := \{(x, u) \in \mathcal{H} \times \mathcal{H} \mid u \in A(x)\}$ . The inverse mapping  $A^{-1} : \mathcal{H} \rightrightarrows \mathcal{H}$ , which always exists, is defined by  $A^{-1}(u) := \{x \in \mathcal{H} \mid u \in A(x)\}$ . The set of zeros is denoted by  $\text{Zer}(A) := A^{-1}(0) = \{x \in \mathcal{H} \mid 0 \in A(x)\}$  and the set of fixed points by  $\text{Fix}(A) := \{x \in \mathcal{H} \mid A(x) = x\}$ .  $A$  is firmly nonexpansive if  $\langle x - \bar{x}, u - \bar{u} \rangle \geq \|u - \bar{u}\|^2, \forall (x, u), (\bar{x}, \bar{u}) \in \text{gph } A$ , and contractive if it is  $L$ -Lipschitz with  $L < 1$ . We denote the identity operator by  $\text{id}$ . The closed disk with center  $c \in \mathbb{C}$  and radius  $r > 0$  is defined as  $D(c, r) := \{z \in \mathbb{C} : |z - c| \leq r\}$ . We also denote the half-plane  $\mathbb{C}_{\geq \alpha} := \{z \in \mathbb{C} : \text{Re}(z) \geq \alpha\} \cup \{\infty\}$  with  $\alpha \in \mathbb{R}$ . Finally, we adopt the standard set operations in vector spaces [8, p. 6].

## II. SCALED RELATIVE GRAPHS OF OPERATOR PAIRS

First, we introduce the notion of scaled relative graphs for pairs of operators  $(A, B)$ , which map the incremental properties of an operator  $A : \mathcal{H} \rightrightarrows \mathcal{H}$  with respect to another operator  $B : \mathcal{H} \rightrightarrows \mathcal{H}$ . This reduces to the classical relative graph when  $B = \text{id}$ . To this end, let  $(x, u), (\bar{x}, \bar{u}) \in \mathcal{H} \times \mathcal{H}$  and define the corresponding complex conjugate pair

$$z_{\pm}(u - \bar{u}, x - \bar{x}) := \frac{\|u - \bar{u}\|}{\|x - \bar{x}\|} \exp(\pm i \angle(x - \bar{x}, u - \bar{u})),$$

where the angle  $\angle(x - \bar{x}, u - \bar{u}) \in [0, \pi]$  is defined as  $\arccos(\frac{\langle x - \bar{x}, u - \bar{u} \rangle}{\|x - \bar{x}\| \|u - \bar{u}\|})$  if  $x \neq \bar{x}$  and  $u \neq \bar{u}$ , and 0 otherwise. The SRG of a pair of operators then consists of the union of these pairs, where  $A$  and  $B$  are evaluated at the same inputs.

**Definition II.1** (SRG of a pair of operators). *Let  $A, B : \mathcal{H} \rightrightarrows \mathcal{H}$ , the scaled relative graph of a pair of operators  $(A, B)$  is*

$$\mathcal{G}(A, B) := \left\{ z_{\pm}(u_A - \bar{u}_A, u_B - \bar{u}_B) \middle| \begin{array}{l} u_A \in A(x), \bar{u}_A \in A(\bar{x}) \\ u_B \in B(x), \bar{u}_B \in B(\bar{x}) \end{array} \right\}$$

for all  $x, \bar{x} \in \text{dom } A \cap \text{dom } B$  such that  $u_B \neq \bar{u}_B$ . Additionally,  $\mathcal{G}(A, B)$  includes  $\infty$  if there exist  $(x, u_A), (\bar{x}, \bar{u}_A) \in \text{gph } A$  and  $(x, u_B), (\bar{x}, \bar{u}_B) \in \text{gph } B$  such that  $u_B = \bar{u}_B$  (including when  $x = \bar{x}$ ) and  $u_A \neq \bar{u}_A$ . Further, we define the SRG of a class of operator pairs  $\mathcal{P}$  as  $\mathcal{G}(\mathcal{P}) := \bigcup_{(A, B) \in \mathcal{P}} \mathcal{G}(A, B)$ .

We now derive some basic identities that can be shown based on the definition and similar proof techniques as in [9]. The main difference arises from  $B$  not necessarily being bijective, such that handling the  $\infty$  case requires extra care. Due to the technicality of the proofs in this section, they are relegated to the appendix.

**Proposition II.1** (Basic calculus). *Let  $A, B, C : \mathcal{H} \rightrightarrows \mathcal{H}$ , and  $\alpha, \beta \in \mathbb{R} \setminus \{0\}$ . Then,*

- (i)  $\mathcal{G}(\alpha A, \beta B) = (\alpha/\beta) \mathcal{G}(A, B)$ .
- (ii)  $\mathcal{G}(B, A) = (\mathcal{G}(A, B))^{-1} := \{(z^{-1})^* \in \overline{\mathbb{C}} \mid z \in \mathcal{G}(A, B)\}$ .
- (iii)  $\mathcal{G}(\text{id}, A) = (\mathcal{G}(A, \text{id}))^{-1} = \mathcal{G}(A^{-1}, \text{id})$ .

- (iv) If  $\mathcal{G}(A, C) \neq \emptyset \neq \mathcal{G}(B, C)$ , and either  $\mathcal{G}(A, C)$  or  $\mathcal{G}(B, C)$  satisfies the chord property (see [9, Fig. 6]), then  $\mathcal{G}(A + B, C) \subseteq \mathcal{G}(A, C) + \mathcal{G}(B, C)$ .
- (v) If  $\mathcal{G}(B, A) \neq \emptyset \neq \mathcal{G}(C, A)$ , and either  $\mathcal{G}(B, A)$  or  $\mathcal{G}(C, A)$  satisfies the chord property (see [9, Fig. 6]), then  $\mathcal{G}(A, B + C) \subseteq (\mathcal{G}(A, B)^{-1} + \mathcal{G}(A, C)^{-1})^{-1}$ .

If one of the operators is single-valued or satisfies some mild additional properties, even more calculus rules can be established.

**Proposition II.2** (Additional calculus). *Let  $A, B : \mathcal{H} \rightrightarrows \mathcal{H}$  and suppose  $F : \mathcal{H} \rightarrow \mathcal{H}$  is single-valued. Then,*

- (i) If  $F$  is not constant, then  $\mathcal{G}(F, F) = \{1\}$ .
- (ii)  $\mathcal{G}(A + F, F) = 1 + \mathcal{G}(A, F)$ .
- (iii)  $\mathcal{G}(F \circ (A + F)^{-1}, \text{id}) = \mathcal{G}(F, A + F)$ .
- (iv)  $\mathcal{G}(A \circ F, B \circ F) \subseteq \mathcal{G}(A, B)$  with equality if  $F$  is surjective.
- (v) If  $\mathcal{G}(F, \text{id}) \subseteq D(0, L)$  and  $\mathcal{G}(A, B) \subseteq D(0, l)$  for some  $L, l > 0$ , then  $\mathcal{G}(F \circ A, B) \subseteq D(0, Ll)$ .
- (vi) If  $\mathcal{G}(A, F) \subseteq D(0, L)$  and  $\mathcal{G}(F, \text{id}) \subseteq D(0, l)$  for some  $L, l > 0$ , then  $\mathcal{G}(A, \text{id}) \subseteq D(0, Ll)$ .
- (vii) If  $M$  is a bounded, invertible linear operator on  $\mathcal{H}$  and  $\mathcal{G}(A, B) \subseteq \mathbb{C}_{\geq 0}$ , then  $\mathcal{G}((M^{-1})^* \circ A, M \circ B) \subseteq \mathbb{C}_{\geq 0}$ .

A crucial property in the context of the original SRG is the concept of SRG-fullness of operator classes [9, Sec. 3.3], since these allow for membership checking based on geometric containment of the SRG, which forms the final step of the standard SRG approach, as shown in Fig. 1. A generalization to our framework is straightforward.

**Definition II.2** (SRG-full operator classes of pairs). *Let  $\mathcal{P}$  be a class of operator pairs, then  $\mathcal{P}$  is SRG-full if*

$$(A, B) \in \mathcal{P} \iff \mathcal{G}(A, B) \subseteq \mathcal{G}(\mathcal{P}).$$

Similar to [9, Thm. 2], classes defined through some nonnegative homogeneous function  $h : \mathbb{R}^3 \rightarrow \mathbb{R}$ , i.e.,  $h(\kappa a, \kappa b, \kappa c) = \kappa h(a, b, c)$  for all  $\kappa \geq 0$ , satisfy this desirable property.

**Proposition II.3.** *Let  $\mathcal{P}$  be a class of operator pairs, then  $\mathcal{P}$  is SRG-full if there exists some nonnegative homogeneous function  $h : \mathbb{R}^3 \rightarrow \mathbb{R}$  such that  $(A, B) \in \mathcal{P}$  if and only if  $\forall (x, u_A), (\bar{x}, \bar{u}_A) \in \text{gph } A, (x, u_B), (\bar{x}, \bar{u}_B) \in \text{gph } B$*

$$h(\|u_A - \bar{u}_A\|^2, \|u_B - \bar{u}_B\|^2, \langle u_A - \bar{u}_A, u_B - \bar{u}_B \rangle) \leq 0.$$

In the context of the original SRG, one particularly interesting SRG-full class is the one defined by  $h : (a, b, c) \mapsto pa + \mu b - c$ , which covers  $(\mu, \rho)$ -semimonotone operators, first introduced in [10, Def. 4.1], as was shown in [11, Prop. 3.2]. In the following, we generalize this class to the setting of operator pairs, inspired by the operator pair monotonicity introduced in [1, Def. 3], [2, Eq. (5)]. Note that classical  $(\mu, \rho)$ -semimonotonicity is recovered by taking  $B = \text{id}$ .

**Definition II.3** (Semimonotone operator pairs). *Let  $A, B : \mathcal{H} \rightrightarrows \mathcal{H}$ , and  $\mu, \rho \in \mathbb{R}$ . Then  $(A, B)$  is  $(\mu, \rho)$ -semimonotone*

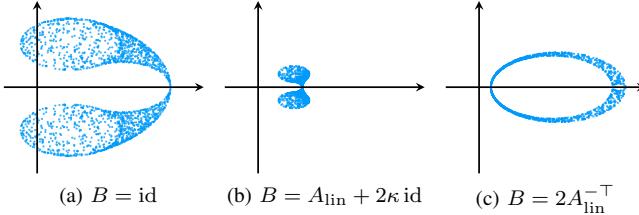


Fig. 2: Numerical SRG of  $(A_{\text{lin}}, B)$ .

if  $\forall (x, u_A), (\bar{x}, \bar{u}_A) \in \text{gph } A, (x, u_B), (\bar{x}, \bar{u}_B) \in \text{gph } B :$   
 $\langle u_A - \bar{u}_A, u_B - \bar{u}_B \rangle \geq \mu \|u_B - \bar{u}_B\|^2 + \rho \|u_A - \bar{u}_A\|^2.$

The class of all  $(\mu, \rho)$ -semimonotone operator pairs is denoted by  $\mathcal{S}_{\mu, \rho}$ .

We will also denote  $\mathcal{M}_\mu := \mathcal{S}_{\mu, 0}$  for the  $\mu$ -monotone pairs of operators. When  $B = \text{id}$  and  $\mu > 0$  we retrieve the class of  $\mu$ -strongly monotone operators, while we recover the class of  $|\mu|$ -hypomonotone operators when  $\mu < 0$ , similarly to [10, Rem. 4.2]. The SRG of the class of  $(\mu, \rho)$ -semimonotone operator pairs is now shown to be exactly the same as in [11, Prop. 3.4], for which we moreover derive an alternative representation.

**Proposition II.4.** Let  $\mu, \rho \in \mathbb{R}$ . Then,

$$\mathcal{G}(\mathcal{S}_{\mu, \rho}) = \{z \in \mathbb{C} \mid \text{Re}(z) \geq \mu + \rho|z|^2\} (\cup \{\infty\} \text{ if } \rho \leq 0).$$

In particular,

$$\mathcal{G}(\mathcal{M}_\mu) = \{z \in \mathbb{C} \mid \text{Re}(z) \geq \mu\} \cup \{\infty\}. \quad (1)$$

*Proof sketch.* The forward inclusion of Proposition II.4 is shown using the properties of semimonotone operators, while the reverse inclusion follows by the fact that classical semimonotone operators (i.e., with  $B = \text{id}$ ) are contained in  $\mathcal{G}(\mathcal{S}_{\mu, \rho})$ , such that [11, Prop. 3.4] can be invoked.  $\square$

**Example II.1.** Consider the linear operator

$$A_{\text{lin}} = \begin{bmatrix} \frac{1}{2} & 2 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad (2)$$

from [9, Fig. 5]. Fig. 2 visualizes, for different operators  $B$ , the numerical SRG of the pair  $(A_{\text{lin}}, B)$ . For  $B = \text{id}$ , this reduces to the standard SRG of  $A_{\text{lin}}$ . Inspired by [2, Lem. 5.1], we also consider the choice  $B = A_{\text{lin}} + 2\kappa \text{id}$  where  $\kappa = 0.45|\alpha|$  and  $|\alpha|$  is the smallest eigenvalue of  $A_{\text{lin}}$  in absolute value. Finally, we also consider  $B = 2A_{\text{lin}}^{-\top}$ .

It is clear that  $(A_{\text{lin}}, \text{id})$  is not monotone by (1) and Definition II.2. Further, the second and third SRGs suggest that  $(A_{\text{lin}}, A_{\text{lin}} + 2\kappa \text{id}), (A_{\text{lin}}, 2A_{\text{lin}}^{-\top}) \in \mathcal{M}_0$ , though this cannot be concluded by sampling alone. That they indeed have this property follows by respectively [2, Lem. 5.1] and Proposition IV.1(i).

**Example II.2.** Let  $A_{\text{NPN}}$  be the NPN transistor modeled with Ebers–Moll, see [11, Sec. 4.2],

$$A_{\text{NPN}} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} := \left\{ R \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \mid \begin{array}{l} u_1 \in A_{\text{D}}(v_1) \\ u_2 \in A_{\text{D}}(v_2) \end{array} \right\}$$

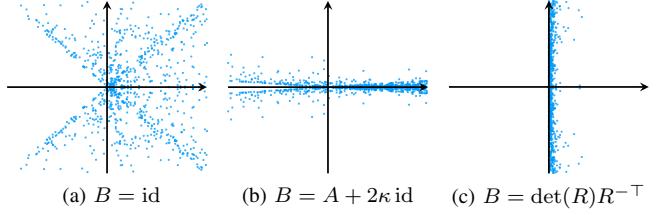


Fig. 3: Numerical SRG of  $(A_{\text{NPN}}, B)$ .

with  $R := \begin{bmatrix} 1 & -\alpha_R \\ -\alpha_F & 1 \end{bmatrix}$ ,  $0 \leq \alpha_R, \alpha_F < 1$  and  $(A_{\text{D}}, \text{id}) \in \mathcal{M}_0$  a diode model. Fig. 3 shows, for different operators  $B$ , the numerical SRG of pairs  $(A_{\text{NPN}}, B)$ . For  $B = \text{id}$ , we observe that the NPN transistor is angle-bounded [11, Def. 3.6], as proven in [11, Prop. 4.4], and that it is not monotone by (1) and Definition II.2. Unlike in Example II.1, the choice  $B = A_{\text{NPN}} + 2\kappa \text{id}$ , where  $\kappa = 0.45|\alpha|$  and  $|\alpha|$  is the smallest eigenvalue of  $R$  in absolute value [2, Lem. 5.1], does not yield a monotone pair of operators. Finally, for  $B = \det(R)R^{-\top}$  the sampled SRG suggests monotonicity of the pair  $(A_{\text{NPN}}, \det(R)R^{-\top})$ , which Corollary IV.1 later establishes formally.

### III. PROPERTIES OF NONLINEAR RESOLVENTS

Let  $A : \mathcal{H} \rightrightarrows \mathcal{H}$  and consider the zero inclusion problem

$$0 \in A(x)$$

that arises ubiquitously in optimization and systems theory. A classical way to solve this problem is to pose it as finding a fixed point of a related operator. The most well-known example is the *resolvent*  $J_{\gamma A} = (\gamma A + \text{id})^{-1}$ , which leads to the celebrated proximal point algorithm. Often, this fixed point operator is shown to be firmly nonexpansive or contractive, from which convergence readily follows by the Krasnosel'skiĭ–Mann or Banach fixed point theorem.

We now proceed with introducing two nonlinear resolvents called the warped resolvent [12] and transformed resolvent [2].

**Definition III.1.** Let  $A : \mathcal{H} \rightrightarrows \mathcal{H}$  and suppose  $F : \mathcal{H} \rightarrow \mathcal{H}$  is single-valued. The transformed resolvent of  $A$  with respect to  $F$  is defined as  $T_A^F := F \circ (A + F)^{-1}$ , and the warped resolvent of  $A$  with respect to  $F$  is defined as  $J_A^F := (A + F)^{-1} \circ F$ , provided that  $\text{rge } F \subseteq \text{rge}(A + F)$ .

These resolvents are of interest since  $\text{Fix } T_{\gamma A}^F = F(\text{Zer } A)$  and  $\text{Fix } J_{\gamma A}^F = \text{Zer } A$  in light of [2, Prop. 3.2]. Therefore, if we can show firm-nonexpansiveness or contraction, then standard theory shows convergence, as described above. These properties have indeed been shown under the pair of monotonicity framework in [2].

We now provide the geometric picture of these derivations by giving purely SRG-based proofs. This approach yields concise proofs that capture the core insights and also allow us to see that the established results are tight.

First, we show that if a pair of operators is  $\alpha$ -monotone for some  $\alpha \geq 0$ , then the transformed resolvent is firmly nonexpansive or even contractive. Then, a similar result

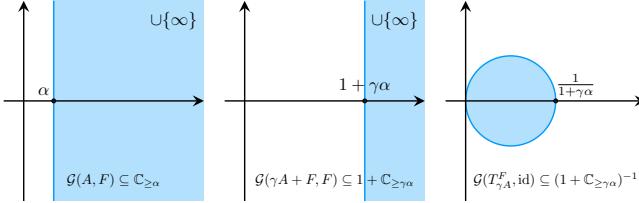


Fig. 4: Geometry of proof [Proposition III.1](#).

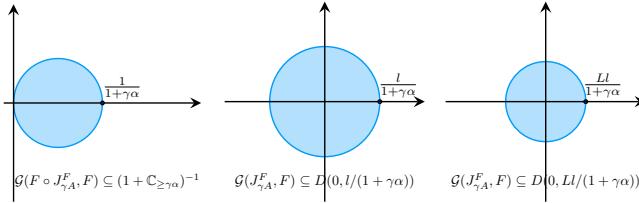


Fig. 5: Geometry of proof [Proposition III.2](#).

is shown for the warped resolvent, under some additional assumptions on  $F$ .

**Proposition III.1.** *Let  $A : \mathcal{H} \rightrightarrows \mathcal{H}$ ,  $\gamma > 0$ , and suppose  $F : \mathcal{H} \rightarrow \mathcal{H}$  is single-valued. If  $(A, F) \in \mathcal{M}_\alpha$  with  $\alpha \geq 0$ , then the transformed resolvent  $T_{\gamma A}^F$  has  $\mathcal{G}(T_{\gamma A}^F, \text{id}) \subseteq D(1/(2+2\gamma\alpha), 1/(2+2\gamma\alpha))$ . In particular, if  $\alpha = 0$ , then the transformed resolvent is firmly nonexpansive. If  $\alpha > 0$ , then the transformed resolvent is Lipschitz continuous with factor  $1/(1+\alpha\gamma) < 1$ , i.e., contractive.*

*Proof.* The geometry of this proof is shown in [Fig. 4](#).

Since  $(A, F) \in \mathcal{M}_\alpha$ , we find from [\(1\)](#) that  $\mathcal{G}(A, F) \subseteq \mathbb{C}_{\geq\alpha}$ . Then, by [Proposition II.1\(i\)](#), we obtain  $\mathcal{G}(\gamma A, F) \subseteq \mathbb{C}_{\geq\gamma\alpha}$ . Further, from the single-valuedness of  $F$ , [Proposition II.2\(ii\)](#) ensures that  $\mathcal{G}(\gamma A + F, F) \subseteq 1 + \mathbb{C}_{\geq\gamma\alpha}$ . Lastly, the property [Proposition II.2\(iii\)](#) and the inversion rule [Proposition II.1\(ii\)](#) yield that  $\mathcal{G}(T_{\gamma A}^F, \text{id}) = \mathcal{G}(F, \gamma A + F) \subseteq (1 + \mathbb{C}_{\geq\gamma\alpha})^{-1} = D(1/(2+2\gamma\alpha), 1/(2+2\gamma\alpha))$ .

Since  $\mathcal{G}(T_{\gamma A}^F, \text{id})$  is the standard SRG of  $T_{\gamma A}^F$ , it follows from [\[9, Prop. 1 and Thm. 2\]](#) that  $T_{\gamma A}^F$  is firmly nonexpansive if  $\alpha = 0$  and Lipschitz continuous with factor  $1/(1+\alpha\gamma) < 1$  if  $\alpha > 0$ .  $\square$

**Proposition III.2.** *Let  $A : \mathcal{H} \rightrightarrows \mathcal{H}$ ,  $\gamma > 0$ , and suppose  $F : \mathcal{H} \rightarrow \mathcal{H}$  is single-valued. If  $(A, F) \in \mathcal{M}_\alpha$  with  $\alpha \geq 0$ , then the warped resolvent satisfies  $\mathcal{G}(F \circ J_{\gamma A}^F, F) \subseteq D(1/(2+2\gamma\alpha), 1/(2+2\gamma\alpha))$ . If moreover  $F^{-1}$  is well-defined and  $l$ -Lipschitz, and  $F$  is  $L$ -Lipschitz, then  $\mathcal{G}(J_{\gamma A}^F, \text{id}) \subseteq D(0, ll/(1+\gamma\alpha))$ .*

*Proof.* The geometry of the proof is shown in [Fig. 5](#).

Since  $F \circ J_{\gamma A}^F = T_{\gamma A}^F \circ F$ , [Proposition III.1](#) and the precomposition rule [Proposition II.2\(iv\)](#) ensure that  $\mathcal{G}(F \circ J_{\gamma A}^F, F) \subseteq D(1/(2+2\gamma\alpha), 1/(2+2\gamma\alpha))$ . If moreover  $F^{-1}$  is well-defined and  $l$ -Lipschitz, then  $\mathcal{G}(F^{-1}, \text{id}) \subseteq D(0, l)$  by [\[9, Prop. 1\]](#). From [Proposition II.2\(v\)](#) and the fact that  $D(1/(2+2\gamma\alpha), 1/(2+2\gamma\alpha)) \subseteq D(0, 1/(1+\gamma\alpha))$ , it then follows that  $\mathcal{G}(F^{-1} \circ F \circ J_{\gamma A}^F, F) = \mathcal{G}(J_{\gamma A}^F, F) \subseteq lD(0, 1/(1+\gamma\alpha)) = D(0, l/(1+\gamma\alpha))$ . Lastly, by [Proposition II.2\(vi\)](#) and the fact

that  $\mathcal{G}(F, \text{id}) \subseteq D(0, L)$  from [\[9, Prop. 1\]](#), we conclude that  $\mathcal{G}(J_{\gamma A}^F, \text{id}) \subseteq D(0, lL/(1+\gamma\alpha))$ .  $\square$

**Remark III.1.** In [\[2\]](#), strong monotonicity of pairs of operators is defined differently from our  $\mathcal{M}_\alpha$ . Nevertheless, if  $(A, B)$  satisfies [\[2, Eq. \(6\)\]](#), i.e.,  $\forall(x, u_1), (\bar{x}, \bar{u}_1) \in \text{gph } A, (x, u_2), (\bar{x}, \bar{u}_2) \in \text{gph } B$ ,

$$\langle u_1 - \bar{u}_1, u_2 - \bar{u}_2 \rangle \geq \alpha \|x - \bar{x}\|^2$$

and  $B$  is  $L$ -Lipschitz continuous, then  $\|x - \bar{x}\|^2 \geq L^{-2} \|u_2 - \bar{u}_2\|^2$  and it follows that  $(A, B) \in \mathcal{M}_{\alpha L^{-2}}$  per [Definition II.3](#), so [Proposition III.1](#) exactly recovers [\[2, Prop. 3.4\]](#).

**Example III.1.** Note that the semimonotone operator pair definition allows for great flexibility in the choice of  $(A, F)$ . For  $\mu = \rho = 0$  and  $F = \text{id} - A$ , [Definition II.3](#) recovers the class of firmly nonexpansive operators:

$$\begin{aligned} \langle A(x) - A(\bar{x}), x - A(x) - (\bar{x} - A(\bar{x})) \rangle &\geq 0 \\ \iff \langle A(x) - A(\bar{x}), x - \bar{x} \rangle &\geq \|A(x) - A(\bar{x})\|^2, \end{aligned}$$

for all  $x, \bar{x} \in \text{dom } A$ . In that case, the transformed resolvent becomes  $T_A^F = F = \text{id} - A$ , i.e., the standard forward step.

**Example III.2.** We can recover more relaxed conditions by using the so-called nonlinear preconditioning technique [\[13\], \[14\], \[15\], \[16\]](#): choosing  $F = \text{id} - \nabla\psi \circ A$  with  $\nabla\psi$  the gradient of a Legendre function [\[14, p. 6\]](#) and  $\text{rge } A \subseteq \text{dom } \nabla\psi$ . In that case, the semimonotonicity inequality with  $\mu = \rho = 0$  takes the form

$$\begin{aligned} \langle A(x) - A(\bar{x}), x - \bar{x} \rangle \\ \geq \langle \nabla\psi(A(x)) - \nabla\psi(A(\bar{x})), A(x) - A(\bar{x}) \rangle. \end{aligned}$$

Clearly, this implies that  $A$  is a monotone operator while if  $\psi = \frac{1}{2} \|\cdot\|^2$ , we recover [Example III.1](#). We remark that by choosing a suitable  $\psi$ , we can make the inequality above less restrictive than the one in [Example III.1](#) as shown in [\[16\]](#). The corresponding transformed resolvent becomes  $T_A^F = (\text{id} - \nabla\psi \circ A)((\text{id} - \nabla\psi) \circ A + \text{id})^{-1}$ .

**Example III.3.** As in [Example III.1](#), let  $F = \text{id} - A$ . Suppose that  $A = \gamma \nabla\psi \circ \nabla f$  where  $f : \mathbb{R}^2 \rightarrow \mathbb{R} : x \mapsto \frac{1}{4} \sum_{i=1}^2 x_i^4$  is a quartic two-dimensional function, and  $\gamma = 0.1$  is a fixed step size. Then, the transformed resolvent reduces to a nonlinearly preconditioned forward step  $T_A^F = \text{id} - \gamma \nabla\psi \circ \nabla f$  [\[16\]](#). Typically,  $\nabla\psi$  satisfies  $\nabla\psi(y) = 0$  if and only if  $y = 0$ , in which case the zeros of  $A$  correspond to the stationary points of  $f$ .

[Fig. 6](#) visualizes the numerical SRG of the pair  $(A, F)$  for different *separable* nonlinear preconditioners of the form  $\nabla\psi = (h'(y_1), h'(y_2))$ . Without preconditioning, i.e., for  $h'(y) = y$ , the pair is clearly not monotone. However, in the case of hard clipping, i.e.,  $h'(y) = \Pi_{[-1,1]}(y)$ , and when preconditioning with  $h'(y) = \text{arcsinh}(y)$  the numerical SRG suggests monotonicity of the pair  $(A, F)$ .

Lastly, inspired by [\[2, Ex. 2.3 and 2.4\]](#), we provide two more examples to showcase the strength of the SRG approach.

**Example III.4.** Let  $A : \mathcal{H} \rightrightarrows \mathcal{H}$ . Suppose we have the decomposition  $A = B + F$  where  $A : \mathcal{H} \rightrightarrows \mathcal{H}$ , and

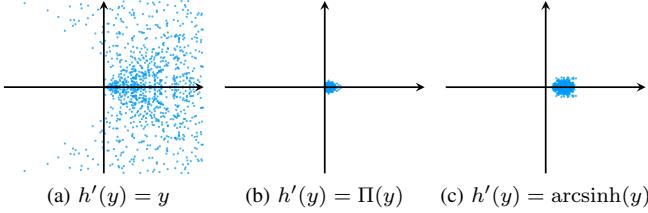


Fig. 6: Numerical SRG of  $(\gamma \nabla \psi \circ \nabla f, \text{id} - \gamma \nabla \psi \circ \nabla f)$  for  $f(x) = \frac{1}{4} \sum_{i=1}^2 x_i^4$  and  $\nabla \psi = (h'(y_1), h'(y_2))$ . We denote by  $\Pi$  the Euclidean projection onto the interval  $[-1, 1]$ .

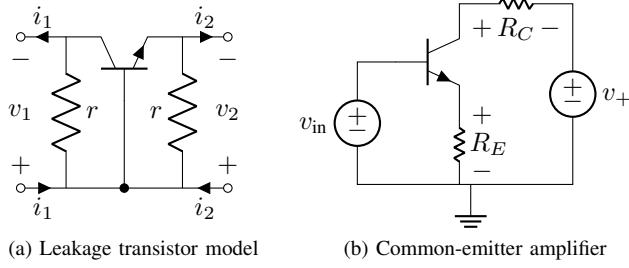


Fig. 7: Examples of nonmonotone electrical circuits.

$F : \mathcal{H} \rightarrow \mathcal{H}$  is single-valued. If  $(F, B) \in \mathcal{M}_0$ , then  $(A, F) \in \mathcal{M}_1$ .

*Proof.* Since  $(F, B) \in \mathcal{M}_0$ , we have that  $\mathcal{G}(F, B) \subseteq \mathbb{C}_{\geq 0}$ . Then, from the inversion rule [Proposition II.1\(ii\)](#),  $\mathcal{G}(B, F) \subseteq (\mathbb{C}_{\geq 0})^{-1} = \mathbb{C}_{\geq 0}$ . Further, by [Proposition II.1\(iii\)](#),  $\mathcal{G}(B + F, F) \subseteq 1 + \mathbb{C}_{\geq 0} = \mathbb{C}_{\geq 1}$  and we conclude that  $(A, F) = (B + F, F) \in \mathcal{M}_1$  by [Definition II.2](#) and [Proposition II.4](#).  $\square$

*Example III.5.* Let  $A, B : \mathcal{H} \rightrightarrows \mathcal{H}$  and suppose  $F : \mathcal{H} \rightarrow \mathcal{H}$  is single-valued. If  $(A, \text{id}) \in \mathcal{M}_0$  and  $(B, F) \in \mathcal{M}_\alpha$  for some  $\alpha \geq 0$ , then  $(A \circ F + B, F) \in \mathcal{M}_\alpha$ .

*Proof.* Since  $(A, \text{id}) \in \mathcal{M}_0$  and  $(B, F) \in \mathcal{M}_\alpha$ , we have  $\mathcal{G}(A, \text{id}) \subseteq \mathbb{C}_{\geq 0}$  and  $\mathcal{G}(B, F) \subseteq \mathbb{C}_{\geq \alpha}$ . Then, applying precomposition, [Proposition II.2\(iv\)](#),  $\mathcal{G}(A \circ F, F) \subseteq \mathbb{C}_{\geq 0}$ , and from [Proposition II.1\(iv\)](#) (passing to chord completions if necessary, see [\[17, Def. 4\]](#)), we obtain that  $\mathcal{G}(A \circ F + B, F) \subseteq \mathbb{C}_{\geq 0} + \mathbb{C}_{\geq \alpha} = \mathbb{C}_{\geq \alpha}$ , which readily leads to the desired result by SRG-fullness.  $\square$

Lastly, we provide a result that recovers part of [\[7, Cor. 25.6\]](#) in the case that  $F = \text{id}$ , and is useful for deriving (linearly) preconditioned algorithms, as we will do in [Example IV.2](#).

**Proposition III.3.** *Let  $A : \mathcal{H} \rightrightarrows \mathcal{H}$  and suppose  $F : \mathcal{H} \rightarrow \mathcal{H}$  is single-valued. Let  $M$  be a bounded, invertible linear operator on  $\mathcal{H}$ . Suppose  $(A, F) \in \mathcal{M}_0$ , then  $((M^{-1})^* A M^{-1}, M F M^{-1}) \in \mathcal{M}_0$ .*

*Proof.* Since  $(A, F) \in \mathcal{M}_0$ , we have  $\mathcal{G}(A, F) \subseteq \mathbb{C}_{\geq 0}$ . Then, from [Proposition II.2\(vii\)](#), we obtain  $\mathcal{G}((M^{-1})^* A, M F) \subseteq \mathbb{C}_{\geq 0}$ . Then, by precomposition, [Proposition II.2\(iv\)](#), we conclude that  $\mathcal{G}((M^{-1})^* A M^{-1}, M F M^{-1}) \subseteq \mathbb{C}_{\geq 0}$ , so  $((M^{-1})^* A M^{-1}, M F M^{-1}) \in \mathcal{M}_0$ .  $\square$

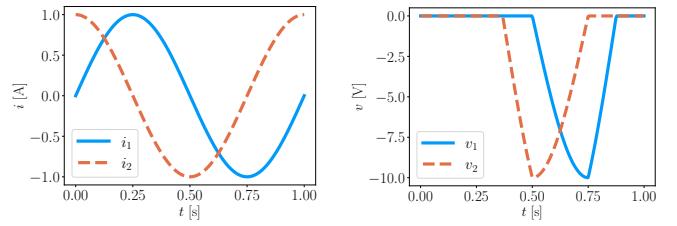


Fig. 8: Solution to the inclusion problem (3) for the same configuration as in [\[11, Fig. 6\]](#): leakage resistance  $r = 10 \Omega$  and a desired sinusoidal current  $i$  are given.

#### IV. APPLICATION TO CIRCUIT THEORY

In this section, we apply the paired monotonicity framework to solve two inclusions involving nonsmooth, multi-valued and highly nonmonotone operators in the context of circuit theory.

To this end, we first show that linear mappings composed with monotone mappings fit naturally into this framework, of which the NPN transistor is a part of. While the proof of this result is entirely algebraic, we would like to emphasize that it is our generalized SRG that has provided the necessary intuition and easy visualization to see which classes can be covered.

**Proposition IV.1.** *Let  $M \in \mathbb{R}^{n \times n}$  and  $B : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ . Let  $A = M \circ B$ . Suppose that  $(B, \text{id}) \in \mathcal{M}_0$ . Then,*

- (i) *If  $M$  is nonsingular, then  $(A, c M^{-\top}) \in \mathcal{M}_0$  for any  $c > 0$ .*
- (ii) *If  $\text{rank}(M) = n - 1$ , then  $(A, y x^\top) \in \mathcal{M}_0$  for some  $x, y \in \mathbb{R}^n \setminus \{0\}$ .*

*Proof.* “[IV.1\(i\)](#)”: Let  $(x, u), (\bar{x}, \bar{u}) \in \text{gph}(c M^{-\top})$  and  $(x, u_A), (\bar{x}, \bar{u}_A) \in \text{gph } A$ . Then,

$$\begin{aligned} \langle u - \bar{u}, u_A - \bar{u}_A \rangle &= \langle c M^{-\top}(u - \bar{u}), M(u_B - \bar{u}_B) \rangle \\ &= c \langle x - \bar{x}, u_B - \bar{u}_B \rangle \end{aligned}$$

for  $u_B \in B(x)$ ,  $\bar{u}_B \in B(\bar{x})$ . From the monotonicity of  $B$ , we find that  $\langle x - \bar{x}, u_B - \bar{u}_B \rangle \geq 0$ , so  $(A, c M^{-\top}) \in \mathcal{M}_0$ .

“[IV.1\(ii\)](#)”: By a similar proof as in [Proposition IV.1\(i\)](#),  $(A, \text{adj } M^\top) \in \mathcal{M}_0$  provided that  $(\det M) \geq 0$ , where  $\text{adj } M$  denotes the adjugate of  $M$ , i.e., the transpose of the cofactor matrix. To derive this, the defining property of the adjugate  $(\text{adj } M)M = (\det M)I$  is used. Further, if  $\text{rank}(M) = n - 1$ , then  $\det(M) = 0$  and  $\text{rank}(\text{adj } M) = 1$ , so  $\text{adj } M$  admits the full-rank factorization  $\text{adj } M = xy^\top$  for some  $x, y \in \mathbb{R}^n \setminus \{0\}$  and the result is proven. See [\[18, Sec. 0.8.2\]](#) for a more detailed discussion on the used properties of adjugates.  $\square$

**Corollary IV.1.** *The NPN transistor  $A_{\text{NPN}}$  in [Example II.2](#) satisfies  $(A_{\text{NPN}}, B) \in \mathcal{M}_0$ , where  $B = (\det R)R^{-\top} = \begin{bmatrix} 1 & \alpha_F \\ \alpha_R & 1 \end{bmatrix}$ .*

In the following example<sup>1</sup>, we consider the same experiment as in [\[11, Prop. 5.1\]](#) and provide a transformed proximal point iteration that converges without any stepsize restriction.

<sup>1</sup>The code for reproducing the experiments can be found at [https://github.com/alexanderbodard/SRGs\\_for\\_pairs](https://github.com/alexanderbodard/SRGs_for_pairs).

*Example IV.1.* We first consider the leaky transistor shown in Fig. 7a. The associated inclusion problem is

$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} \in A_{\text{NPN}} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \frac{1}{r} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} =: A_{\text{NPN},r} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad (3)$$

where  $r > 0$  and  $A_D$  is the ideal diode defined by  $A_D(v) = \{0\}$  if  $v < 0$ ;  $A_D(0) = [0, +\infty)$ ; and  $A_D(v) = \emptyset$  otherwise. Define  $\tilde{A}_{\text{NPN},r} := A_{\text{NPN},r} - i$ , with  $i \in \mathbb{R}^2$ . Then, any sequence  $(v^k)_{k \in \mathbb{N}}$  satisfying the update rule

$$v^{k+1} = T_{\gamma \tilde{A}_{\text{NPN},r}}^B$$

with  $B$  as in Corollary IV.1 and stepsize  $\gamma > 0$  converges weakly to  $Bv^*$  where  $v^*$  is a solution of (3).

Crucially, our iteration converges with any stepsize  $\gamma > 0$  whereas that of [11, Prop. 5.1] only works for  $\gamma > r(\sqrt{2}-1)$ , which is quite a severe restriction when  $r \gg 1$ , i.e., the case that usually occurs in practice. Fig. 8 shows the experimental results for the same configuration as in [11, Fig. 6].

*Proof.* By Corollary IV.1, we find that  $(A_{\text{NPN}}, B) \in \mathcal{M}_0$ , while per Definition II.3 with  $\mu = \rho = 0$ , we also obtain that  $((1/r) \text{id}, B) \in \mathcal{M}_0$ . Similar to Example III.5, we can use Proposition II.1(iv) to find  $(A_{\text{NPN}} + (1/r) \text{id}, B) = (A_{\text{NPN},r}, B) \in \mathcal{M}_0$ . Since adding a constant does not change incremental properties, we have  $(\tilde{A}_{\text{NPN},r}, B) \in \mathcal{M}_0$ . From Proposition III.1, we find that  $T_{\gamma \tilde{A}_{\text{NPN},r}}^B$  is firmly non-expansive. It follows by [7, Cor. 5.17] that  $(v^k)_{k \in \mathbb{N}}$  converges weakly to  $\text{Fix } T_{\gamma \tilde{A}_{\text{NPN},r}}^B = B(\text{Zer } \tilde{A}_{\text{NPN},r})$ .  $\square$

We next show how to leverage Proposition III.3 to derive a linearly preconditioned proximal point algorithm based on the transformed resolvent, before applying this to a (nonmonotone) common-emitter amplifier circuit.

Let  $A : \mathcal{H} \rightrightarrows \mathcal{H}$  and suppose  $F : \mathcal{H} \rightarrow \mathcal{H}$  is single-valued. Let  $M$  be a bounded, invertible linear operator on  $\mathcal{H}$ . We can then invoke Proposition III.1 and Krasnosel'skiĭ–Mann [7, Cor. 5.17] to show that the iteration

$$x^{k+1} = MFM^{-1} \circ (\gamma(M^{-1})^* AM^{-1} + MFM^{-1})^{-1} x^k$$

with stepsize  $\gamma > 0$  converges weakly to an element of  $MFM^{-1}(\text{Zer } M^{-\top} AM^{-1})$ , provided it exists. Now, define  $M\bar{x}^k = x^k$ . With some algebra, the iteration is then equivalent to

$$\bar{x}^{k+1} \in F \circ (A + M^* MF)^{-1} M^* M\bar{x}^k$$

Conversely, starting from a positive definite  $P$ , by letting  $M = P^{1/2}$ , we find that the iteration

$$\bar{x}^{k+1} \in F \circ (A + PF)^{-1} P\bar{x}^k \quad (4)$$

converges weakly to a point in  $F(\text{Zer } A)$  if one exists. Further, instead of Féjer monotonicity in the Euclidean norm [7, Cor. 5.17(i)], one now obtains Féjer monotonicity in the matrix  $P$ -norm [18, Eq. (5.2.6)].

We now apply this iteration to a common-emitter amplifier circuit shown in Fig. 7b. Note that the circuit considered here corresponds to that in [11, Fig. 5(b)] with  $r \rightarrow \infty$ , i.e., a

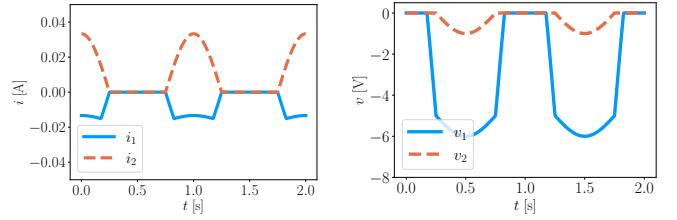


Fig. 9: Solution to the inclusion problem (5) for the common-emitter amplifier circuit in Fig. 7b with linear resistors  $R_E = 30 \Omega$ ,  $R_C = 300 \Omega$ , and circuit parameters  $v_{\text{in}} = \cos(2\pi t)V$ ,  $v_+ = 5V$ . These plots were obtained through the iteration (6) with parameters  $\gamma = 10^{-3}$ ,  $\tau = 100$ .

setting which existing theory for Chambolle-Pock does not cover [11, Prop. 5.2].

*Example IV.2.* From [11, Eq. (9)], the behavior of a common-emitter amplifier circuit can be retrieved by solving an inclusion problem of the form

$$0 \in \begin{bmatrix} A(i) \\ B(v) \end{bmatrix} + \begin{bmatrix} 0 & I_2^\top \\ -I_2 & 0 \end{bmatrix} \begin{bmatrix} i \\ v \end{bmatrix} + \begin{bmatrix} s_v \\ s_i \end{bmatrix} \quad (5)$$

where

$$A := \begin{bmatrix} R_C & 0 \\ 0 & R_E \end{bmatrix}, B := A_{\text{NPN}}, s_v := \begin{bmatrix} v_+ - v_{\text{in}} \\ -v_{\text{in}} \end{bmatrix}, s_i := 0$$

with  $v_{\text{in}} \in \mathbb{R}$ ,  $v_+, R_C, R_E > 0$  and  $I_2$  denoting the identity matrix. Such a structure for solving circuits was studied in [19] in the monotone setting and in [11] in the semi-monotone setting. We now consider the pair of monotonicity setting, that considers parameters that can not be covered by semimonotonicity.

Let  $R$  be associated with  $A_{\text{NPN}}$  as in Example II.2 and suppose that  $A^\top R^{-1} + R^{-\top} A \succeq 0$ ,  $\tau, \gamma > 0$  and  $\gamma\tau\|R\|^2 < 1$ , where  $\|R\|$  denotes the spectral norm of  $R$ . Consider the iteration

$$\begin{aligned} \bar{i}^k &= (A + \gamma^{-1} R^{-1})^{-1} (\gamma^{-1} i^k - R^\top v^k - s_v) \\ \bar{v}^k &= (B + \tau^{-1} R^{-\top})^{-1} (-Ri^k + \tau^{-1} v^k + 2\bar{i}^k - s_i) \\ i^{k+1} &= R^{-1} \bar{i}^k, \quad v^{k+1} = R^{-\top} \bar{v}^k. \end{aligned} \quad (6)$$

Then,  $(Ri^k, R^\top v^k)_{k \in \mathbb{N}}$  converges weakly to a solution of (5), provided a solution exists.

Fig. 9 shows the result of applying iteration (6) to the common-emitter amplifier circuit in Fig. 7b. For these numerical values, the required conditions hold.

*Proof.* First, note that  $(B, R^{-\top}), (A, R^{-1}) \in \mathcal{M}_0$  where the first follows by Corollary IV.1 and the second by the assumption and Definition II.3. Similarly, the skew-symmetric term in (5) can also be shown to be monotone with respect to  $R^{-1} \oplus R^{-\top}$ , where  $\oplus$  denotes the direct sum.

By using the sum rule, Proposition II.1(iv) and the fact that constant terms do not affect the incremental properties, it follows that the complete operator in (5) is monotone with respect to  $F := (R^{-1} \oplus R^{-\top})$ . Therefore, we can apply (4) with a positive definite  $P$ . Similar to how Chambolle–Pock [20] arises from a specific choice of  $P$  in the classical preconditioned proximal point algorithm to decouple the

equations [21, Eq.(1.1)], we propose a similar preconditioner  $P$ , noting that our setting is different due to the appearance of  $F$ .

Let  $P = \begin{bmatrix} \gamma^{-1}I_2 & -R^\top \\ -R & \tau^{-1}I_2 \end{bmatrix}$  which is positive definite by the Schur complement [18, Thm. 7.7.7] and our assumptions on  $\tau, \gamma$ . The iteration (4) applied to (5) then becomes (6) after some algebraic manipulations. The convergence then follows from the discussion above.  $\square$

## V. CONCLUSION

In this paper, we introduced a novel scaled relative graph for pairs of operators. This framework naturally provides the geometric counterpart for recently introduced assumptions of paired monotonicity of operators. We have shown the practical relevance of these properties by calculating the response of two highly nonmonotone circuits, thus extending known theoretical guarantees.

We believe that the proposed scaled relative graph for pairs of operators may prove valuable for stability analyses of feedback systems, and will greatly simplify the further analysis of other classes of nonmonotone circuits that can be handled by tailored splitting methods.

## APPENDIX

For the proofs we will require the following equivalent formulation of the complex conjugate pair  $z_\pm$ :

$$z_\pm(u - \bar{u}, x - \bar{x}) = \frac{\langle u - \bar{u}, x - \bar{x} \rangle}{\|x - \bar{x}\|^2} \pm i \frac{\|\Pi_{\{x-\bar{x}\}^\perp}(u - \bar{u})\|}{\|x - \bar{x}\|}. \quad (7)$$

where  $\Pi_{\{x-\bar{x}\}^\perp}$  is the projection onto the subspace orthogonal to  $x - \bar{x}$ .

### Proof of Proposition II.1

*Proof.* “II.1(i)”: Follows by Definition II.1 along with the properties of the inner product and the norm.

“II.1(ii)”: The equality follows easily for points not 0 or  $\infty$  by definition. If  $\infty \in \mathcal{G}(A, B)$ , there exist  $(x, u_A), (\bar{x}, \bar{u}_A) \in \text{gph } A$  and  $(x, u_B), (\bar{x}, \bar{u}_B) \in \text{gph } B$  such that  $u_B = \bar{u}_B$  and  $u_A \neq \bar{u}_A$ . This immediately implies that  $0 \in \mathcal{G}(B, A)$ . The zero case follows similarly.

“II.1(iii)”: The first equality follows by Proposition II.1(ii) and the second by [9, Thm.5] since  $\mathcal{G}(A, \text{id})$  is the standard SRG.

“II.1(iv)”: The inclusion for the finite points follows similarly to [9, Thm.6]. If  $\infty \in \mathcal{G}(A + B, C)$ , then there exist  $(x, u_C), (\bar{x}, \bar{u}_C) \in \text{gph } C$  and  $(x, u_A), (\bar{x}, \bar{u}_A) \in \text{gph } A$ ,  $(x, u_B), (\bar{x}, \bar{u}_B) \in \text{gph } B$  such that  $u_A + u_B \neq \bar{u}_A + \bar{u}_B$  and  $u_C = \bar{u}_C$ . Clearly then at least one of  $u_A \neq \bar{u}_A$  or  $u_B \neq \bar{u}_B$  holds, and thus  $\infty \in \mathcal{G}(A, C) + \mathcal{G}(B, C)$ , where we also used that both sets are assumed nonempty.

“II.1(v)”: Combine Proposition II.1(iv) with Proposition II.1(ii), and use that the inverse is defined elementwise, so subset inclusions are preserved.  $\square$

### Proof of Proposition II.2

*Proof.* “II.2(i)”: Take  $(x, F(x)), (\bar{x}, F(\bar{x})) \in \text{gph } F$ . Clearly,  $\infty \notin \mathcal{G}(F, F)$  since  $F$  is a function and we would have

$F(x) \neq F(\bar{x})$  while  $x \neq \bar{x}$ . Since  $F$  is not constant, for all points with  $F(x) \neq F(\bar{x})$ , which must exist, we have that  $z_\pm = \frac{\|F(x) - F(\bar{x})\|}{\|F(x) - F(\bar{x})\|} \exp(\pm i0) = \{1\}$ . Thus  $\mathcal{G}(F, F) = \{1\}$ .

“II.2(ii)”: Take  $z \in \mathcal{G}(A + F, F) \setminus \{\infty\}$ . Then, there exist  $x, \bar{x}$  and  $(x, u_A), (\bar{x}, \bar{u}_A) \in \text{gph } A$  such that  $z \in z_\pm(F(x) + u_A - (F(\bar{x}) + \bar{u}_A), F(x) - F(\bar{x}))$ . Since  $z \neq \infty$ ,  $F(x) \neq F(\bar{x})$  and  $z_\pm(F(x) + u_A - (F(\bar{x}) + \bar{u}_A), F(x) - F(\bar{x})) = 1 + \frac{\langle u_A - \bar{u}_A, F(x) - F(\bar{x}) \rangle}{\|F(x) - F(\bar{x})\|} \pm i \frac{\|\Pi_{\{F(x)-F(\bar{x})\}^\perp}(u_A - \bar{u}_A)\|}{\|F(x) - F(\bar{x})\|} \subseteq 1 + \mathcal{G}(A, F)$ . Now,  $\infty \in \mathcal{G}(A + F, F)$  implies  $F(x) = F(\bar{x})$  and  $u_A + F(x) \neq \bar{u}_A + F(\bar{x})$ , i.e.  $u_A \neq \bar{u}_A$  which then means that  $\infty \in \mathcal{G}(A, F)$ . The opposite inclusions follow similarly.

“II.2(iii)”: Take  $z \in \mathcal{G}(F \circ (A + F)^{-1}) \setminus \{\infty\}$ . Then, there exist  $(x, u), (\bar{x}, \bar{u}) \in \text{gph}(F \circ (A + F)^{-1})$  such that  $z \in z_\pm(u - \bar{u}, x - \bar{x})$ . This moreover implies the existence of  $(x, y), (\bar{x}, \bar{y}) \in \text{gph}(A + F)^{-1}$  with  $u = F(y)$  and  $\bar{u} = F(\bar{y})$ . Equivalently,  $(y, x), (\bar{y}, \bar{x}) \in \text{gph}(A + F)$ . But then, by definition,  $z_\pm(u - \bar{u}, x - \bar{x}) \in \mathcal{G}(F, A + F)$ . Now, if  $z = \infty$  there exists  $x \in \mathcal{H}$  such that  $(x, u), (x, \bar{u}) \in \text{gph}(F \circ (A + F)^{-1})$  and since  $F$  is single-valued,  $(x, y), (x, \bar{y}) \in \text{gph}(A + F)^{-1}$  such that  $y \neq \bar{y}$ . This implies that  $(A + F)(y) \cap (A + F)(\bar{y}) \neq \emptyset$  and thus that  $\infty \in \mathcal{G}(F, A + F)$ . The opposite direction follows similarly.

“II.2(iv)”: Take  $z \in \mathcal{G}(A \circ F, B \circ F) \setminus \{\infty\}$ . Then, there exist  $(x, u), (\bar{x}, \bar{u}) \in \text{gph}(A \circ F)$  and  $(x, v), (\bar{x}, \bar{v}) \in \text{gph}(B \circ F)$  such that  $z \in z_\pm(u - \bar{u}, v - \bar{v})$  and  $v \neq \bar{v}$ . This implies  $(F(x), u), (F(\bar{x}), u) \in \text{gph } A$  and  $(F(x), v), (F(\bar{x}), \bar{v}) \in \text{gph } B$  with  $v \neq \bar{v}$ . Then, by definition,  $z \in \mathcal{G}(A, B)$ . Now take  $z = \infty$ . Then, for some pair as before,  $v = \bar{v}$  and  $u \neq \bar{u}$  meaning also that  $\infty \in \mathcal{G}(A, B)$ .

Now assume moreover that  $F$  is surjective. Take  $z \in \mathcal{G}(A, B) \setminus \{\infty\}$ . Then, there exist  $(x, u_A), (\bar{x}, \bar{u}_A) \in \text{gph } A$  and  $(x, u_B), (\bar{x}, \bar{u}_B) \in \text{gph } B$  such that  $z \in z_\pm(u_A - \bar{u}_A, u_B - \bar{u}_B)$  and  $u_B \neq \bar{u}_B$ . Clearly, since  $F$  is surjective, there exist  $y, \bar{y} \in \mathcal{H}$  such that  $F(y) = x$  and  $F(\bar{y}) = \bar{x}$ . Then,  $(y, u_A), (\bar{y}, \bar{u}_A) \in \text{gph } A$  and  $(y, u_B), (\bar{y}, \bar{u}_B) \in \text{gph } B$  with  $u_B \neq \bar{u}_B$  and thus  $z \in z_\pm(u_A - \bar{u}_A, u_B - \bar{u}_B) \subset \mathcal{G}(A \circ F, B \circ F)$ . The case  $z = \infty$  follows similarly.

“II.2(v)”: Take  $z \in \mathcal{G}(F \circ A, B)$ . If  $z = \infty$ , then there exist  $(x, u), (\bar{x}, \bar{u}) \in \text{gph } F \circ A$  and  $(x, v), (\bar{x}, \bar{v}) \in \text{gph } B$  such that  $v = \bar{v}$  and  $u \neq \bar{u}$ . Therefore, there also exist  $(x, y), (\bar{x}, \bar{y}) \in \text{gph } A$  such that  $u = F(y), \bar{u} = F(\bar{y})$  and since  $F$  is single-valued and  $u \neq \bar{u}$  we have that  $y \neq \bar{y}$ . This implies  $\infty \in \mathcal{G}(A, B)$  which would contradict the assumption that  $\mathcal{G}(A, B) \subseteq D(0, l)$ , and therefore,  $\infty \notin \mathcal{G}(F \circ A, B)$ . Now assume that  $z \neq \infty$ . This means that with pairs as before such that  $v \neq \bar{v}$  we have  $|z| = \frac{\|u - \bar{u}\|}{\|v - \bar{v}\|} = \frac{\|F(y) - F(\bar{y})\|}{\|v - \bar{v}\|} \leq L \frac{\|y - \bar{y}\|}{\|v - \bar{v}\|}$ , since  $F$  is a  $L$ -Lipschitz operator in light of [9, Prop.1 and Thm.2]. This implies that  $|z| \leq L|\bar{z}|$  for  $\bar{z} \in \mathcal{G}(A, B)$  and thus that  $|z| \leq L \sup\{|\bar{z}| : \bar{z} \in \mathcal{G}(A, B)\}$ . Lastly, from the hypothesis,  $\sup\{|\bar{z}| : \bar{z} \in \mathcal{G}(A, B)\} \leq l$ , so we obtain the claimed result  $\mathcal{G}(A, \text{id}) \subseteq D(0, Ll)$ .

“II.2(vi)”: Take  $z \in \mathcal{G}(A, \text{id})$ . If  $z = \infty$ , then there exist  $(x, u), (\bar{x}, \bar{u}) \in \text{gph } A$  such that  $u \neq \bar{u}$  and  $x = \bar{x}$ . Since  $F$  is single-valued, this implies that there exist  $(x, u), (\bar{x}, \bar{u}) \in \text{gph } A$  such that  $u \neq \bar{u}$  and  $F(x) = F(\bar{x})$ ,

i.e.,  $\infty \in \mathcal{G}(A, F)$ , which contradicts the assumption that  $\mathcal{G}(A, F) \subseteq D(0, L)$ , so  $\infty \notin \mathcal{G}(A, \text{id})$ . In particular, we have that  $A$  is single-valued.

Further, for  $z \neq \infty$ , as in the proof of [Proposition II.2\(v\)](#), we know that  $F$  is an  $l$ -Lipschitz operator. Similarly, from  $\mathcal{G}(A, F) \subseteq D(0, L)$  we have that  $z \in \mathcal{G}(A, F)$  implies  $|z| \leq L$  and thus that  $\|A(x) - \bar{A}(\bar{x})\| \leq L\|F(x) - F(\bar{x})\|$  for all  $x, \bar{x} \in \text{dom } A$ . Then, if  $z \in \mathcal{G}(A, \text{id})$  we have that  $|z| = \frac{\|A(x) - \bar{A}(\bar{x})\|}{\|x - \bar{x}\|}$  for some  $x, \bar{x} \in \text{dom } A$ . Using the previous inequality along with the Lipschitz continuity of  $F$  we obtain  $|z| \leq lL$ , which is the claimed result.

**“II.2(vii)”:** To begin with, note that since  $M$  is bounded and invertible, we have that  $(M^*)^{-1} = (M^{-1})^*$ . Now for any  $z \in \mathcal{G}(A, B)$ ,  $z \in \mathbb{C}_{\geq 0}$ . Thus, for all  $(x, u_A), (\bar{x}, \bar{u}_A) \in \text{gph } A$  and  $(x, u_B), (\bar{x}, \bar{u}_B) \in \text{gph } B$  such that  $u_B \neq \bar{u}_B$  we have that  $\langle u_A - \bar{u}_A, u_B - \bar{u}_B \rangle \geq 0$  or that  $\langle (M^*)^{-1}(u_A - \bar{u}_A), M(u_B - \bar{u}_B) \rangle \geq 0$  by definition of the adjoint [\[8, Thm. 4.10\]](#). Since  $M$  is invertible, it holds for any  $(x, u), (\bar{x}, \bar{u}) \in \text{gph}((M^*)^{-1} \circ A, M \circ B)$  and  $(x, v), (\bar{x}, \bar{v}) \in \text{gph}(M \circ B)$  with  $v \neq \bar{v}$  that  $\langle u - \bar{u}, v - \bar{v} \rangle \geq 0$ . The  $\infty$  case follows similarly and we conclude that  $\mathcal{G}((M^{-1})^* \circ A, M \circ B) \subseteq \mathbb{C}_{\geq 0}$ .  $\square$

### Proof of Proposition II.3

*Proof.* Notice that  $(A, B) \in \mathcal{P} \implies \mathcal{G}(A, B) \subseteq \mathcal{G}(\mathcal{P})$  by construction, and it remains to show that  $\mathcal{G}(A, B) \subseteq \mathcal{G}(\mathcal{P}) \implies (A, B) \in \mathcal{P}$ . For all points  $(x, u_A), (\bar{x}, \bar{u}_A) \in \text{gph } A$  and  $(x, u_B), (\bar{x}, \bar{u}_B) \in \text{gph } B$ , such that  $u_B \neq \bar{u}_B$ , the same technique from [\[9, Thm. 2\]](#) yields that  $\mathcal{G}(A, B) \subseteq \mathcal{G}(\mathcal{P}) \implies h(\|u_A - \bar{u}_A\|^2, \|u_B - \bar{u}_B\|^2, \langle u_A - \bar{u}_A, u_B - \bar{u}_B \rangle) \leq 0$ .

Now suppose  $u_B = \bar{u}_B$ , while  $u_A \neq \bar{u}_A$ , then  $\infty \in \mathcal{G}(A, B) \subseteq \mathcal{G}(\mathcal{P})$ . Thus, there is some  $(C, D) \in \mathcal{P}$  such that  $(x', u'_C), (\bar{x}', \bar{u}'_C) \in \text{gph } C$ ,  $(x', u'_D), (\bar{x}', \bar{u}'_D) \in \text{gph } D$ , and  $u'_D = \bar{u}'_D$ ,  $u'_C \neq \bar{u}'_C$ . Since  $(C, D) \in \mathcal{P}$ , we have that  $h(\|u'_C - \bar{u}'_C\|^2, 0, 0) \leq 0$ . Multiplying by  $\|u_A - \bar{u}_A\|^2 / \|u'_C - \bar{u}'_C\|^2$  and using homogeneity, we find  $h(\|u_A - \bar{u}_A\|^2, 0, 0) \leq 0$ . Lastly, if  $u_B = \bar{u}_B$  and  $u_A = \bar{u}_A$ , then  $h$  is evaluated at all zeros, and its output must therefore be zero by homogeneity. In all three cases, we have  $h(\|u_A - \bar{u}_A\|^2, \|u_B - \bar{u}_B\|^2, \langle u_A - \bar{u}_A, u_B - \bar{u}_B \rangle) \leq 0$  and since the points were arbitrary evaluations of  $A, B$ , we have that  $(A, B) \in \mathcal{P}$  by the hypothesis.  $\square$

### Proof of Proposition II.4

*Proof.* Take  $(A, B) \in \mathcal{S}_{\mu, \rho}$  and  $(x, u_A), (\bar{x}, \bar{u}_A) \in \text{gph } A$ ,  $(x, u_B), (\bar{x}, \bar{u}_B) \in \text{gph } B$  be arbitrary such that  $u_B \neq \bar{u}_B$ . It follows from [Definition II.3](#) that  $\langle u_A - \bar{u}_A, u_B - \bar{u}_B \rangle \geq \mu\|u_B - \bar{u}_B\|^2 + \rho\|u_A - \bar{u}_A\|^2$ , and dividing by  $\|u_B - \bar{u}_B\|^2$  yields  $\frac{\langle u_A - \bar{u}_A, u_B - \bar{u}_B \rangle}{\|u_B - \bar{u}_B\|^2} \geq \mu + \rho \frac{\|u_A - \bar{u}_A\|^2}{\|u_B - \bar{u}_B\|^2}$  which is equivalent to  $\text{Re}(z_{\pm}(u_A - \bar{u}_A, u_B - \bar{u}_B)) \geq \mu + \rho|z_{\pm}(u_A - \bar{u}_A, u_B - \bar{u}_B)|$ , see [\(7\)](#). Now suppose  $\rho > 0$ . Clearly, if  $u_B = \bar{u}_B$  then also  $u_A = \bar{u}_A$  from [Definition II.3](#) and thus  $\infty \notin \mathcal{G}(\mathcal{S}_{\mu, \rho})$ .

For the opposite inclusion. Suppose  $\rho = 0$ , then the set on the right-hand side is exactly  $\mathbb{C}_{\geq \mu}$ , and from [\[9, Prop. 1\]](#), we find  $\mathbb{C}_{\geq \mu} = \cup_{(A, \text{id}) \in \mathcal{M}_0} \mathcal{G}(A, \text{id}) \subseteq \mathcal{G}(\mathcal{M}_0) = \mathcal{G}(\mathcal{S}_{\mu, 0})$ .

Otherwise, if  $\rho > 0$  let  $z = x + iy$  satisfy  $\text{Re}(z) \geq \mu + \rho|c|^2$ , then  $\rho x^2 - x + \rho y^2 + \mu \leq 0$ . Further, dividing by  $\rho$  and completing the square, we obtain

$$\left( x - \frac{1}{2\rho} \right)^2 + y^2 \leq \left( \frac{1}{2\rho} \right)^2 - \frac{\mu}{\rho} = \frac{1 - 4\mu\rho}{4\rho^2}.$$

We remark that this defines the same disk as in [\[1, Prop. 3.4\]](#). Therefore, the set on the right-hand side is  $\cup_{(A, \text{id}) \in \mathcal{S}_{\mu, \rho}} \mathcal{G}(A, \text{id}) \subseteq \mathcal{S}_{\mu, \rho}$ . The case  $\rho < 0$  follows similarly.  $\square$

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