

# An Elementary Proof of a Minimax Theorem

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## Abstract

Here, we give a self-contained and elementary proof of a minimax theorem due to Fan [1] in a simplified setting that can be taught in an advanced undergraduate course. Our proof follows [2] with some simplifications.

## 1 Introduction

Let  $X \subseteq \mathbb{R}^d$ ,  $Y \subseteq \mathbb{R}^k$ , and  $f : X \times Y \rightarrow \mathbb{R}$ . We always assume  $X$  and  $Y$  are nonempty. We are concerned here with *minimax theorems*, which give conditions under which it holds that

$$(1) \quad \sup_{\mathbf{x} \in X} \inf_{\mathbf{y} \in Y} f(\mathbf{x}, \mathbf{y}) = \inf_{\mathbf{y} \in Y} \sup_{\mathbf{x} \in X} f(\mathbf{x}, \mathbf{y}).$$

The purpose of this note is to provide an elementary proof of a minimax theorem that is accessible to undergraduate students. Minimax theorems date back to von Neumann [4] in 1928, who established the first minimax theorem in the special case that  $f(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T A \mathbf{y}$  and  $X$  and  $Y$  are the spaces of probability vectors in  $\mathbb{R}^d$  and  $\mathbb{R}^k$ , respectively. This initial setting arose in two player zero sum games. Since then, many extensions to more general functions  $f(\mathbf{x}, \mathbf{y})$  have been given (see, e.g., [3] and references therein), and minimax theorems have found powerful applications in primal dual optimization.

One way to think about the difference between the two sides of (1) is in terms of two player games, where the first player is choosing  $\mathbf{x} \in X$  to maximize  $f(\mathbf{x}, \mathbf{y})$ , while the second player is choosing  $\mathbf{y} \in Y$  to minimize  $f(\mathbf{x}, \mathbf{y})$ . When the supremum is on the outside, like on the left hand side of (1), this amounts to the second player getting to observe the first player's choice  $\mathbf{x}$  before making their choice  $\mathbf{y}$ . This confers an advantage to the second player. Vice versa, when the infimum is on the outside, like on the right hand side of (1), the first player gets to observe the second player's choice  $\mathbf{y}$  before making their choice of  $\mathbf{x}$ , which confers an advantage to the first player. Thus, we expect to always see an inequality, which is verified in the following result.

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**Lemma 1.** Let  $X \subseteq \mathbb{R}^d$ ,  $Y \subseteq \mathbb{R}^k$ , and  $f : X \times Y \rightarrow \mathbb{R}$ . Then

$$(2) \quad \sup_{\mathbf{x} \in X} \inf_{\mathbf{y} \in Y} f(\mathbf{x}, \mathbf{y}) \leq \inf_{\mathbf{y} \in Y} \sup_{\mathbf{x} \in X} f(\mathbf{x}, \mathbf{y}).$$

*Proof.* For any  $\mathbf{x}_0 \in X$  we have

$$\inf_{\mathbf{y} \in Y} f(\mathbf{x}_0, \mathbf{y}) \leq \inf_{\mathbf{y} \in Y} \sup_{\mathbf{x} \in X} f(\mathbf{x}, \mathbf{y}).$$

Taking the supremum over  $\mathbf{x}_0 \in X$  gives (2).  $\square$

By Lemma 1, to prove a minimax theorem, it is sufficient to show the opposite inequality, namely that

$$(3) \quad \inf_{\mathbf{y} \in Y} \sup_{\mathbf{x} \in X} f(\mathbf{x}, \mathbf{y}) \leq \sup_{\mathbf{x} \in X} \inf_{\mathbf{y} \in Y} f(\mathbf{x}, \mathbf{y}).$$

However, this direction is much more difficult to prove, and does not always hold.

**Example 1.** Let  $X = \{-1, 1\} \subseteq \mathbb{R}$ ,  $Y = [-1, 1] \subseteq \mathbb{R}$ , and  $f(x, y) = xy$ . Then

$$\max_{x \in X} \min_{y \in Y} f(x, y) = \max_{x \in X} (-1) = -1,$$

while

$$\min_{y \in Y} \max_{x \in X} f(x, y) = \min_{y \in Y} |y| = 0.$$

As we shall see below, convexity of the sets  $X$  and  $Y$  play a key role in minimax theorems. Note that  $X = \{-1, 1\}$  is not a convex set.

## 2 Minimax and Saddle Points

We now introduce some notation. We say that  $\mathbf{x}^* \in X$  is an *outer max* of  $f$  if the supremum on the left hand side of (1) is attained at  $\mathbf{x}^*$ . Likewise, we say  $\mathbf{y}^* \in Y$  is an *outer min* of  $f$  if the infimum on the right hand side of (1) is attained at  $\mathbf{y}^*$ . We also define the notion of a *saddle point* of  $f$ .

**Definition 2.** A pair  $(\mathbf{x}^*, \mathbf{y}^*) \in X \times Y$  is called a *saddle point* of  $f$  if

$$(4) \quad f(\mathbf{x}, \mathbf{y}^*) \leq f(\mathbf{x}^*, \mathbf{y}^*) \leq f(\mathbf{x}^*, \mathbf{y}) \quad \text{for all } (\mathbf{x}, \mathbf{y}) \in X \times Y.$$

A saddle point  $(\mathbf{x}^*, \mathbf{y}^*) \in X \times Y$  of  $f$  clearly satisfies

$$(5) \quad \max_{\mathbf{x} \in X} f(\mathbf{x}, \mathbf{y}^*) = f(\mathbf{x}^*, \mathbf{y}^*) = \min_{\mathbf{y} \in Y} f(\mathbf{x}^*, \mathbf{y}).$$

In fact, (4) is equivalent to (5). In other words,  $(\mathbf{x}^*, \mathbf{y}^*)$  is a saddle point if  $\mathbf{x} \mapsto f(\mathbf{x}, \mathbf{y}^*)$  is maximized at  $\mathbf{x}^*$ , and  $\mathbf{y} \mapsto f(\mathbf{x}^*, \mathbf{y})$  is minimized at  $\mathbf{y}^*$ .

When the supremum and infimum are attained, minimax theorems are equivalent to the problem of finding a *saddle point*.

**Proposition 3.** *Let  $f : X \times Y \rightarrow \mathbb{R}$ . If  $f$  has a saddle point  $(\mathbf{x}^*, \mathbf{y}^*) \in X \times Y$ , then (1) holds and  $\mathbf{x}^*$  is an outer max of  $f$ , while  $\mathbf{y}^*$  is an outer min of  $f$ . Conversely, if (1) holds,  $\mathbf{x}^* \in X$  is an outer max of  $f$ , and  $\mathbf{y}^* \in Y$  is an outer min of  $f$ , then  $(\mathbf{x}^*, \mathbf{y}^*)$  is a saddle point of  $f$ .*

*Proof.* If  $(\mathbf{x}^*, \mathbf{y}^*) \in X \times Y$  is a saddle point, so (4) holds, then we have

$$\inf_{\mathbf{y} \in Y} \sup_{\mathbf{x} \in X} f(\mathbf{x}, \mathbf{y}) \leq \max_{\mathbf{x} \in X} f(\mathbf{x}, \mathbf{y}^*) = f(\mathbf{x}^*, \mathbf{y}^*),$$

and

$$\sup_{\mathbf{x} \in X} \inf_{\mathbf{y} \in Y} f(\mathbf{x}, \mathbf{y}) \geq \min_{\mathbf{y} \in Y} f(\mathbf{x}^*, \mathbf{y}) = f(\mathbf{x}^*, \mathbf{y}^*)$$

which yields (3). By Lemma 1 we have that (1) holds, and so

$$\inf_{\mathbf{y} \in Y} \sup_{\mathbf{x} \in X} f(\mathbf{x}, \mathbf{y}) = f(\mathbf{x}^*, \mathbf{y}^*) = \sup_{\mathbf{x} \in X} \inf_{\mathbf{y} \in Y} f(\mathbf{x}, \mathbf{y}).$$

It follows that  $\mathbf{x}^*$  is an outer max of  $f$ , while  $\mathbf{y}^*$  is an outer min.

Conversely, suppose that (1) holds, and let  $\mathbf{x}^*$  be an outer max of  $f$  and  $\mathbf{y}^*$  an outer min. This means that

$$\sup_{\mathbf{x} \in X} \inf_{\mathbf{y} \in Y} f(\mathbf{x}, \mathbf{y}) = \inf_{\mathbf{y} \in Y} f(\mathbf{x}^*, \mathbf{y}) \leq f(\mathbf{x}^*, \mathbf{y}^*),$$

and

$$\inf_{\mathbf{y} \in Y} \sup_{\mathbf{x} \in X} f(\mathbf{x}, \mathbf{y}) = \sup_{\mathbf{x} \in X} f(\mathbf{x}, \mathbf{y}^*) \geq f(\mathbf{x}^*, \mathbf{y}^*).$$

Since (1) holds we have

$$\sup_{\mathbf{x} \in X} f(\mathbf{x}, \mathbf{y}^*) = f(\mathbf{x}^*, \mathbf{y}^*) = \inf_{\mathbf{y} \in Y} f(\mathbf{x}^*, \mathbf{y}),$$

which implies that  $(\mathbf{x}^*, \mathbf{y}^*)$  is a saddle point of  $f$ . □

### 3 A Minimax Theorem

We now give our main result. The proof follows Nikaidô's argument [2], with some modifications.

**Theorem 4.** *Let  $X \subseteq \mathbb{R}^d$  and  $Y \subseteq \mathbb{R}^k$  be convex and compact sets with nonempty interiors. Let  $f : X \times Y \rightarrow \mathbb{R}$  be continuous, and assume  $\mathbf{x} \mapsto f(\mathbf{x}, \mathbf{y})$  is concave for all  $\mathbf{y} \in Y$  and  $\mathbf{y} \mapsto f(\mathbf{x}, \mathbf{y})$  is convex for all  $\mathbf{x} \in X$ . Then  $f$  has a saddle point  $(\mathbf{x}^*, \mathbf{y}^*) \in X \times Y$ .*

*Proof.* Let us define

$$\Phi(\mathbf{x}, \mathbf{y}) = \int_Y \int_X (f(\mathbf{u}, \mathbf{y}) - f(\mathbf{x}, \mathbf{v}))_+^2 \, d\mathbf{u} \, d\mathbf{v},$$

where  $a_+ = \max(a, 0)$ .<sup>1</sup> Since  $f$  is continuous on the compact set  $X \times Y$ , so is  $\Phi$ , and hence  $\Phi$  attains its minimum at a point  $(\mathbf{x}^*, \mathbf{y}^*) \in X \times Y$ . The proof will be completed by showing that  $\Phi(\mathbf{x}^*, \mathbf{y}^*) = 0$ , from which we obtain that  $(\mathbf{x}^*, \mathbf{y}^*)$  is a saddle point.

To see that  $\Phi(\mathbf{x}^*, \mathbf{y}^*) = 0$ , we will take a variation of  $\Phi$  that remains inside our convex domain  $X \times Y$ . For any  $0 < t < 1$  and  $(\mathbf{x}, \mathbf{y}) \in X \times Y$  we use convexity/concavity of  $f$  to obtain

$$\begin{aligned} \Phi(\mathbf{x}^*, \mathbf{y}^*) &\leq \Phi(\mathbf{x}^* + t(\mathbf{x} - \mathbf{x}^*), \mathbf{y}^* + t(\mathbf{y} - \mathbf{y}^*)) \\ &= \int_Y \int_X (f(\mathbf{u}, (1-t)\mathbf{y}^* + t\mathbf{y}) - f((1-t)\mathbf{x}^* + t\mathbf{x}, \mathbf{v}))_+^2 \, d\mathbf{u} \, d\mathbf{v} \\ &\leq \int_Y \int_X [(1-t)f(\mathbf{u}, \mathbf{y}^*) + tf(\mathbf{u}, \mathbf{y}) - (1-t)f(\mathbf{x}^*, \mathbf{v}) - tf(\mathbf{x}, \mathbf{v})]_+^2 \, d\mathbf{u} \, d\mathbf{v}. \end{aligned}$$

Writing  $g(\mathbf{u}, \mathbf{v}) = f(\mathbf{u}, \mathbf{y}^*) - f(\mathbf{x}^*, \mathbf{v})$  and  $h(\mathbf{u}, \mathbf{v}; \mathbf{x}, \mathbf{y}) = f(\mathbf{u}, \mathbf{y}) - f(\mathbf{x}, \mathbf{v})$  for notational convenience, we can rearrange the above to obtain

$$(6) \quad \int_Y \int_X [g(\mathbf{u}, \mathbf{v}) + t(h(\mathbf{u}, \mathbf{v}; \mathbf{x}, \mathbf{y}) - g(\mathbf{u}, \mathbf{v}))]_+^2 - g(\mathbf{u}, \mathbf{v})_+^2 \, d\mathbf{u} \, d\mathbf{v} \geq 0.$$

We now divide by  $t > 0$ , take the limit as  $t \rightarrow 0^+$ , and use that

$$\lim_{t \rightarrow 0^+} \frac{1}{t} [(a + tb)_+^2 - a_+^2] = 2a_+b$$

for all  $a, b \in \mathbb{R}$ . This yields

$$\int_Y \int_X g(\mathbf{u}, \mathbf{v})_+ (h(\mathbf{u}, \mathbf{v}; \mathbf{x}, \mathbf{y}) - g(\mathbf{u}, \mathbf{v})) \, d\mathbf{u} \, d\mathbf{v} \geq 0,$$

which can be rearranged to read

$$(7) \quad \Phi(\mathbf{x}^*, \mathbf{y}^*) \leq \int_Y \int_X g(\mathbf{u}, \mathbf{v})_+ h(\mathbf{u}, \mathbf{v}; \mathbf{x}, \mathbf{y}) \, d\mathbf{u} \, d\mathbf{v}.$$

Multiply by  $g(\mathbf{x}, \mathbf{y})_+$  on both sides and integrate over  $(\mathbf{x}, \mathbf{y}) \in X \times Y$  to obtain

$$\begin{aligned} \Phi(\mathbf{x}^*, \mathbf{y}^*) \int_Y \int_X g(\mathbf{x}, \mathbf{y})_+ \, d\mathbf{x} \, d\mathbf{y} \\ \leq \int_Y \int_X \int_Y \int_X g(\mathbf{u}, \mathbf{v})_+ g(\mathbf{x}, \mathbf{y})_+ h(\mathbf{u}, \mathbf{v}; \mathbf{x}, \mathbf{y}) \, d\mathbf{u} \, d\mathbf{v} \, d\mathbf{x} \, d\mathbf{y} = 0 \end{aligned}$$

since  $h$  is skew-symmetric (i.e.,  $h(\mathbf{u}, \mathbf{v}; \mathbf{x}, \mathbf{y}) = -h(\mathbf{x}, \mathbf{y}; \mathbf{u}, \mathbf{v})$ ). Therefore  $\Phi(\mathbf{x}^*, \mathbf{y}^*) = 0$ .  $\square$

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<sup>1</sup>Some care should be taken in defining the integral. Since  $X$  and  $Y$  are convex, they are measurable sets. Since they are compact and convex with *nonempty* interiors, each is the closure of an open and bounded set, which has positive Lebesgue measure. Then we can write  $\int_X f(\mathbf{x}) \, d\mathbf{x} = \int_{\mathbb{R}^d} f(\mathbf{x}) \mathbb{1}_X(\mathbf{x}) \, d\mathbf{x}$  for any continuous  $f : X \rightarrow \mathbb{R}$ , where  $\mathbb{1}_X$  is the indicator function of the set  $X$ . In particular, if  $f : X \rightarrow \mathbb{R}$  is continuous and *nonnegative* and  $\int_X f \, d\mathbf{x} = 0$ , then  $f \equiv 0$  on  $X$ .

In many applications of minimax theorems, one of the sets  $X$  or  $Y$  is unbounded, and in particular, not compact, though  $f$  has more structure. The minimax theorem of Fan [1] applies in this setting, though the proof is rather involved. Here, we take the simpler route of extending Theorem 4 to the unbounded case when  $f$  has quadratic dependence on  $\mathbf{y}$  and the mixed terms are linear. In particular, we assume now that  $f$  has the form

$$(8) \quad f(\mathbf{x}, \mathbf{y}) = \frac{1}{2} \mathbf{y}^T S \mathbf{y} - \mathbf{y}^T A \mathbf{x} - g(\mathbf{x}),$$

for appropriate conditions on  $S$ ,  $A$ , and  $g$ . In particular, we prove the following result.

**Theorem 5.** *Let  $X \subseteq \mathbb{R}^d$  be convex and compact with nonempty interior and assume  $\mathbf{0} \in X$ . Let  $f : X \times \mathbb{R}^k \rightarrow \mathbb{R}$  be given by (8), where  $g : X \rightarrow \mathbb{R}$  is continuous and convex,  $S = S^T$  is a  $k \times k$  symmetric and positive semidefinite matrix, and  $A$  is any  $k \times d$  matrix. Then it holds that*

$$(9) \quad \inf_{\mathbf{y} \in \mathbb{R}^k} \max_{\mathbf{x} \in X} f(\mathbf{x}, \mathbf{y}) = \max_{\mathbf{x} \in X} \inf_{\mathbf{y} \in \mathbb{R}^k} f(\mathbf{x}, \mathbf{y}).$$

*Proof.* For any fixed  $\mathbf{x} \in X$ ,  $\mathbf{y} \mapsto f(\mathbf{x}, \mathbf{y})$  is a quadratic function which has a global minimizer if and only if  $A\mathbf{x} \in \text{img } S$ . This holds, for example, when  $\mathbf{x} = \mathbf{0}$ . In the case  $A\mathbf{x} \in \text{img } S$ , every minimizer  $\mathbf{y}$  satisfies  $S\mathbf{y} = A\mathbf{x}$  and there is a unique minimizer  $\mathbf{y} \in \text{img } S$ , which satisfies

$$\sigma_{\min}(S) \|\mathbf{y}\| \leq \|S\mathbf{y}\| = \|A\mathbf{x}\| \leq \|A\| \|\mathbf{x}\| \leq C \|A\| \quad \text{where } C = \max_{\mathbf{x} \in X} \|\mathbf{x}\|,$$

where  $\sigma_{\min}(S) > 0$  is the smallest (positive) singular value of  $S$ . Thus, when  $A\mathbf{x} \in \text{img } S$ ,  $\mathbf{y} \mapsto f(\mathbf{x}, \mathbf{y})$  admits its minimum value over  $\mathbb{R}^k$  on the closed ball  $B_R$  of radius  $R = C\|A\|/\sigma_{\min}(S)$ . When  $A\mathbf{x} \notin \text{img } S$  we have  $\inf_{\mathbf{y} \in \mathbb{R}^k} f(\mathbf{x}, \mathbf{y}) = -\infty$ , so the maximum over  $\mathbf{x} \in X$  will avoid such choices. It follows that the outer max of  $f$  is attained and we have

$$\max_{\mathbf{x} \in X} \inf_{\mathbf{y} \in \mathbb{R}^k} f(\mathbf{x}, \mathbf{y}) = \max_{\mathbf{x} \in X} \min_{\mathbf{y} \in B_R} f(\mathbf{x}, \mathbf{y}).$$

We can now apply Theorem 4 to find that

$$\max_{\mathbf{x} \in X} \inf_{\mathbf{y} \in \mathbb{R}^k} f(\mathbf{x}, \mathbf{y}) = \min_{\mathbf{y} \in B_R} \max_{\mathbf{x} \in X} f(\mathbf{x}, \mathbf{y}) \geq \inf_{\mathbf{y} \in \mathbb{R}^k} \max_{\mathbf{x} \in X} f(\mathbf{x}, \mathbf{y}),$$

which when combined with Lemma 1 completes the proof.  $\square$

**Remark 6.** By Theorem 5,  $f$  has an outer max  $\mathbf{x}^* \in X$ , and  $A\mathbf{x}^* \in \text{img } S$ . If  $f$  has an outer min  $\mathbf{y}^* \in \mathbb{R}^k$  as well, then by Proposition 3, the pair  $(\mathbf{x}^*, \mathbf{y}^*) \in X \times \mathbb{R}^k$  is a saddle point, that is

$$(10) \quad \min_{\mathbf{y} \in \mathbb{R}^k} \max_{\mathbf{x} \in X} f(\mathbf{x}, \mathbf{y}) = \max_{\mathbf{x} \in X} f(\mathbf{x}, \mathbf{y}^*) = f(\mathbf{x}^*, \mathbf{y}^*) = \min_{\mathbf{y} \in \mathbb{R}^k} f(\mathbf{x}^*, \mathbf{y}) = \max_{\mathbf{x} \in X} \inf_{\mathbf{y} \in \mathbb{R}^k} f(\mathbf{x}, \mathbf{y}).$$

We also note that we can clearly restrict the outer max to the set

$$X_0 = \{\mathbf{x} \in X \mid A\mathbf{x} \in \text{img } S\},$$

in which case the inner infimum is a true minimum and we can write

$$(11) \quad \min_{\mathbf{y} \in \mathbb{R}^k} \max_{\mathbf{x} \in X} f(\mathbf{x}, \mathbf{y}) = \max_{\mathbf{x} \in X} f(\mathbf{x}, \mathbf{y}^*) = f(\mathbf{x}^*, \mathbf{y}^*) = \min_{\mathbf{y} \in \mathbb{R}^k} f(\mathbf{x}^*, \mathbf{y}) = \max_{\mathbf{x} \in X_0} \min_{\mathbf{y} \in \mathbb{R}^k} f(\mathbf{x}, \mathbf{y}).$$

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