

# Estimates for convolution operators on Hardy spaces associated with ball quasi-Banach function spaces

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## Abstract

Let  $0 \leq \alpha < n$ ,  $N \in \mathbb{N}$ , and let  $X$  and  $Y$  be ball quasi-Banach function spaces on  $\mathbb{R}^n$ . We consider operators  $T_\alpha$  defined by convolution with kernels of type  $(\alpha, N)$ . Assuming that the powered Hardy–Littlewood maximal operator satisfies some Fefferman–Stein vector-valued maximal inequality on  $X$  and is bounded on the associated space, we prove that  $T_0$ ,  $\alpha = 0$ , extends to a bounded operator  $H_X(\mathbb{R}^n) \rightarrow X$  and  $H_X(\mathbb{R}^n) \rightarrow H_X(\mathbb{R}^n)$ ; and, under certain additional assumptions on  $X$  and  $Y$ ,  $T_\alpha$ ,  $0 < \alpha < n$ , extends to a bounded operator  $H_X(\mathbb{R}^n) \rightarrow Y$  and  $H_X(\mathbb{R}^n) \rightarrow H_Y(\mathbb{R}^n)$ . In particular, from these results, it follows that singular integrals and the Riesz potential satisfy such estimates, respectively. We also provide an off-diagonal Fefferman–Stein vector-valued inequality for the fractional maximal operator on the  $p$ -convexification of ball quasi-Banach function spaces.

## 1 Introduction

The concept of ball quasi-Banach function space was introduced by Y. Sawano, K.-P. Ho, D. Yang and S. Yang in [27] to unify, under a broader common framework than the one given by (quasi-)Banach function spaces, many variants of classical Hardy type spaces such as weighted Hardy spaces, Hardy-Lorentz spaces, Hardy-Orlicz spaces, Hardy-Herz spaces, Hardy-Morrey spaces, Musielak-Orlicz-Hardy spaces, and variable Hardy spaces, among others (see [27] and the references therein). For instance, if we consider the space  $L^p(\mathbb{R}^n)$ ,  $0 < p < \infty$ , with the usual quasi-norm given by  $\|f\|_p^p = \int_{\mathbb{R}^n} |f(x)|^p dx$ , then  $X := (L^p(\mathbb{R}^n), \|\cdot\|_p)$  is a ball quasi-Banach function space and its Hardy type space associated  $H_X(\mathbb{R}^n)$  coincides with  $H^p(\mathbb{R}^n)$ , where  $H_X(\mathbb{R}^n)$  is defined by (6) below, and  $H^p(\mathbb{R}^n)$  is the classical Hardy space defined in [29].

One of the principal results obtained in [27] is the atomic and molecular characterization of the Hardy space  $H_X(\mathbb{R}^n)$  associated with a ball quasi-Banach function space  $X$ .

These characterizations rely on the Fefferman-Stein vector-valued maximal inequality of the powered Hardy-Littlewood maximal operator on  $X$  and its boundedness on the associate space  $X'$  (see (2) and (3) below). As is well known, in the classic context, such decompositions are very useful when it studies the behavior of certain operators, as singular and fractional integrals, on a given Hardy type space (see for instance [13], [21], [24], [29], [30], [32]).

In [34], F. Wang et al. established a characterization, via Littlewood-Paley functions, for  $H_X(\mathbb{R}^n)$  and obtained the boundedness of Calderón-Zygmund operators on  $H_X(\mathbb{R}^n)$ . They also studied local Hardy spaces  $h_X(\mathbb{R}^n)$  associated with a ball Banach function space  $X$ . After that, D.-C. Chang et al. in [4], under some weak assumptions on the Littlewood-Paley functions, improved the existing results of the Littlewood-Paley function characterizations of  $H_X(\mathbb{R}^n)$ . Their results have applications in Morrey spaces, mixed-norm Lebesgue spaces, variable Lebesgue spaces, weighted Lebesgue spaces and Orlicz-slice spaces.

K.-P. Ho in [15], by means of extrapolation techniques, obtained mapping properties on Hardy spaces associated with ball quasi-Banach function spaces for strongly singular Calderón-Zygmund operators with applications to Hardy Orlicz-slice spaces, variable Hardy local Morrey spaces and variable Herz-Hardy spaces.

A finite atomic characterization of  $H_X(\mathbb{R}^n)$  was given by X. Yan, D. Yang and W. Yuan in [35]. As an application, they prove that the dual space of  $H_X(\mathbb{R}^n)$  is the Campanato space associated with  $X$ . Recently, Y. Tan in [33], by means of this finite atomic characterization, proved the boundedness of multilinear fractional integral operators from products of Hardy spaces associated with ball quasi-Banach function spaces into other ball quasi-Banach function space, which are related to each other. In [26], the present author pointed out that this kind of operators are not bounded from a product of Hardy spaces into a Hardy space.

Z. Nieraeth in [23], assuming the boundedness of Hardy-Littlewood maximal operator  $M$  on  $[(X')^{\frac{1}{1-\frac{\alpha}{n}}}]'$  and  $(X')^{\frac{1}{1-\frac{\alpha}{n}}}$ , proved the  $X \rightarrow ([X']^{\frac{1}{1-\frac{\alpha}{n}}})^{1-\frac{\alpha}{n}}$  boundedness for the Riesz potential  $I_\alpha$ ,  $0 < \alpha < n$ , where  $X$  is a  $\frac{n}{\alpha}$ -concave Banach function space over  $\mathbb{R}^n$ . Later, Y. Chen, H. Jia and D. Yang in [6], assuming (2) and (3) below, prove that the Riesz potential  $I_\alpha$  can be extended to a bounded operator from  $H_X(\mathbb{R}^n)$  to  $H_{X^\beta}(\mathbb{R}^n)$ ,  $\beta > 1$ , if and only in if for any ball  $B \subset \mathbb{R}^n$ ,  $|B|^{\frac{\alpha}{n}} \lesssim \|\chi_B\|_X^{(\beta-1)/\beta}$ , where  $X$  is a ball quasi-Banach function space and  $X^\beta$  denotes the  $\beta$ -convexification of  $X$ . Moreover, using extrapolation techniques, the authors also proved the  $H_X(\mathbb{R}^n) \rightarrow H_Y(\mathbb{R}^n)$  boundedness of  $I_\alpha$ , for certain ball quasi-Banach function spaces  $X$  and  $Y$ .

Let  $0 \leq \alpha < n$  and  $N \in \mathbb{N}$ . For  $0 < \alpha < n$ , a function  $K_\alpha \in C^N(\mathbb{R}^n \setminus \{0\})$  is said to be a kernel of type  $(\alpha, N)$  on  $\mathbb{R}^n$  if

$$\left| (\partial^\beta K_\alpha)(x) \right| \lesssim |x|^{\alpha-n-|\beta|} \text{ for all } |\beta| \leq N \text{ and all } x \neq 0, \quad (1)$$

where  $\partial^\beta$  is the higher order partial derivative associated to the multiindex  $\beta = (\beta_1, \dots, \beta_n)$ ,

and  $|\beta| = \beta_1 + \dots + \beta_n$ . A distribution  $K_0$  is said to be a kernel of type  $(0, N)$  on  $\mathbb{R}^n$  if it is of class  $C^N$  on  $\mathbb{R}^n \setminus \{0\}$ , satisfies (1) with  $\alpha = 0$ , and  $\|K_0 * f\|_2 \leq \|f\|_2$  for all  $f \in \mathcal{S}(\mathbb{R}^n)$ . These kernels type were studied by G. Folland and E. Stein in [12] on the homogeneous groups setting.

For  $0 \leq \alpha < n$ , let  $T_\alpha$  be the convolution operator defined, say on  $\mathcal{S}(\mathbb{R}^n)$ , by  $T_\alpha f = K_\alpha * f$ , where  $K_\alpha$  is a kernel of type  $(\alpha, N)$ . These operators include the classical singular and fractional integrals (see Sections 4.3 and 4.4 below). Our main result is contained in Theorem 27, Section 4 below, this states that if  $X$  and  $Y$  are ball quasi-Banach function spaces satisfying certain hypotheses and  $N$  is conveniently chosen, then, for  $\alpha = 0$ , the operator  $T_0$  can be extended to a bounded operator  $H_X(\mathbb{R}^n) \rightarrow X$  and  $H_X(\mathbb{R}^n) \rightarrow H_X(\mathbb{R}^n)$ ; and, for  $0 < \alpha < n$ , the operator  $T_\alpha$  can be extended to a bounded operator  $H_X(\mathbb{R}^n) \rightarrow Y$  and  $H_X(\mathbb{R}^n) \rightarrow H_Y(\mathbb{R}^n)$ .

To achieve our goals, under certain additional assumptions on  $X$  (see Section 2.2), we will use the infinite atomic decomposition and the maximal characterization of  $H_X(\mathbb{R}^n)$  established in [27] together with a density argument and some vector-valued inequalities of Section 3. Among them, we provide an off-diagonal Fefferman-Stein vector-valued inequality for the fractional maximal operator on the  $p$ -convexification of ball quasi-Banach function spaces, which is crucial to get our main result of Section 4.

The paper is organized as follows. Section 2 starts with the basics of the Hardy spaces theory associated with ball quasi-Banach function spaces. Some vector-valued maximal inequalities are established in Section 3. In Section 4, we state and prove our main results. Finally, Section 5 presents four concrete examples to illustrate our results.

**Notation:** The symbol  $A \lesssim B$  stands for the inequality  $A \leq cB$  for some constant  $c$ . We denote by  $Q(x_0, r)$  the cube centered at  $x_0 \in \mathbb{R}^n$  with side length  $r$ . Given  $\gamma > 0$  and a cube  $Q = Q(x_0, r)$ , we set  $\gamma Q = Q(x_0, \gamma r)$ . Denote by  $\mathcal{Q}$  the set of all cubes having their edges parallel to the coordinate axes. For a measurable subset  $E \subset \mathbb{R}^n$  we denote  $|E|$  and  $\chi_E$  the Lebesgue measure of  $E$  and the characteristic function of  $E$  respectively. Given a real number  $s \geq 0$ , we write  $\lfloor s \rfloor$  for the integer part of  $s$ . As usual we denote with  $\mathcal{S}(\mathbb{R}^n)$  the space of smooth and rapidly decreasing functions, with  $\mathcal{S}'(\mathbb{R}^n)$  the dual space. If  $\beta$  is the multiindex  $\beta = (\beta_1, \dots, \beta_n)$ , then  $|\beta| = \beta_1 + \dots + \beta_n$ . Given a function  $g$  on  $\mathbb{R}^n$  and  $t > 0$ , we write  $g_t(x) = t^{-n}g(t^{-1}x)$ . Let  $d$  be a non negative integer and  $\mathcal{P}_d$  the subspace of  $L^1_{loc}(\mathbb{R}^n)$  formed by all the polynomials of degree at most  $d$ . Given a measurable function  $h$ , the expression  $h \perp \mathcal{P}_d$  stands for  $\int h(x)P(x)dx = 0$  for all  $P \in \mathcal{P}_d$ .

Throughout this paper,  $C$  will denote a positive constant, not necessarily the same at each occurrence.

## 2 Preliminaries

### 2.1 Ball quasi-Banach function spaces

In the sequel,  $\mathfrak{M} = \mathfrak{M}(\mathbb{R}^n)$  is the set of all measurable functions on  $\mathbb{R}^n$  and  $\mathfrak{M}_+ = \mathfrak{M}_+(\mathbb{R}^n)$  is the cone of all non-negative measurable functions on  $\mathbb{R}^n$ .

For every  $x \in \mathbb{R}^n$  and  $r > 0$  fixed, let  $B(x, r) := \{y \in \mathbb{R}^n : |x - y| < r\}$ . Now, we define

$$\mathbb{B} := \{B(x, r) : x \in \mathbb{R}^n \text{ and } r > 0\}.$$

**Definition 1.** A mapping  $\rho : \mathfrak{M}_+ \rightarrow [0, \infty]$  is called a ball quasi-Banach function norm if it satisfy the following properties:

- (P1)  $\rho(f) = 0$  implies that  $f = 0$  a.e.;
- (P2)  $\rho(\alpha f) = |\alpha| \rho(f)$  for all  $\alpha \in \mathbb{C}$  and all  $f \in \mathfrak{M}_+$ ;
- (P3) there exists  $C \geq 1$  such that  $\rho(f + g) \leq C(\rho(f) + \rho(g))$  for all  $f, g \in \mathfrak{M}_+$ ;
- (P4) if some  $f, g \in \mathfrak{M}_+$  satisfy  $f \leq g$  a.e., then  $\rho(f) \leq \rho(g)$ ;
- (P5) if some  $f_n, f \in \mathfrak{M}_+$  satisfy  $f_n \uparrow f$  a.e., then  $\rho(f_n) \uparrow \rho(f)$ ;
- (P6) if  $B \in \mathbb{B}$ , then  $\rho(\chi_B) < \infty$ .

**Definition 2.** Let  $\rho$  be a ball quasi-Banach function norm. We then define the corresponding ball quasi-Banach function space  $X = X(\rho)$  as the set

$$X = \{f \in \mathfrak{M} : \rho(|f|) < \infty\}.$$

For each  $f \in X$ , define

$$\|f\|_X = \rho(|f|).$$

**Remark 3.** Let  $(X, \|\cdot\|_X)$  be a ball quasi-Banach function space, then it is easy to check that

- (i)  $\|f\|_X = 0$  implies that  $f = 0$  a.e.;
- (ii)  $\|\alpha f\|_X = |\alpha| \|f\|_X$  for all  $\alpha \in \mathbb{C}$  and all  $f \in X$ ;
- (iii) there exists  $C \geq 1$  such that  $\|f + g\|_X \leq C(\|f\|_X + \|g\|_X)$  for all  $f, g \in X$ ;
- (iv) if  $f \in \mathfrak{M}$ ,  $g \in X$  are such that  $|f| \leq |g|$  a.e., then  $f \in X$  and  $\|f\|_X \leq \|g\|_X$ ;
- (v) if  $0 \leq f_n \uparrow f$  a.e., then either  $f \notin X$  and  $\|f_n\|_X \uparrow \infty$ , or  $f \in X$  and  $\|f_n\|_X \uparrow \|f\|_X$ ;
- (vi) if  $B \in \mathbb{B}$ , then  $\chi_B \in X$ .

Moreover, from (P4) and (P6) follow that  $\chi_Q \in X$ , for each cube  $Q \subset \mathbb{R}^n$ .

A ball quasi-Banach function space  $X$  is called a *ball Banach function space* if the constant  $C$ , appearing in (iii) of Remark 3, is equal to 1, and for any ball  $B \in \mathbb{B}$  there exists a positive constant  $C_{(B)}$ , depending on  $B$ , such that

$$\int_B |f(x)| dx \leq C_{(B)} \|f\|_X,$$

for all  $f \in X$ . Thus, every Banach function space is a ball Banach function space (see [3, Definition 1.3]).

An interesting discussion about the well-definition of (quasi-)Banach function spaces was given by E. Lorist and Z. Nieraeth in [18] (see also [22]).

**Definition 4.** (See [3]) For any ball Banach function space  $X$ , the associate space (also called the Köthe dual)  $X'$  is defined by setting

$$X' := \left\{ f \in \mathfrak{M} : \|f\|_{X'} := \sup\{\|fg\|_{L^1(\mathbb{R}^n)} : g \in X, \|g\|_X = 1\} < \infty \right\}$$

where  $\|\cdot\|_{X'}$  is called the associate norm of  $\|\cdot\|_X$ .

**Lemma 5.** ([27, Proposition 2.3] and [36, Lemma 2.6]) Let  $X$  be a ball Banach function space. Then  $X'$  is also a ball Banach function space and  $X$  coincides with its second associate space  $X''$ . In other words, a function  $f$  belongs to  $X$  if and only if it belongs to  $X''$  and, in that case,

$$\|f\|_X = \|f\|_{X''}.$$

**Definition 6.** A ball quasi-Banach function space  $X$  is said to have an absolutely continuous quasi-norm if  $\|\chi_{E_j}\|_X \downarrow 0$  whenever  $\{E_j\}_{j=1}^\infty$  is a sequence of measurable sets that satisfies  $E_j \supset E_{j+1}$  for all  $j \in \mathbb{N}$  and  $\bigcap_{j=1}^\infty E_j = \emptyset$ .

**Definition 7.** Let  $X$  be a ball quasi-Banach function space and  $p \in (0, \infty)$ . The  $p$ -convexification  $X^p$  of  $X$  is defined by setting  $X^p := \{f \in \mathfrak{M} : |f|^p \in X\}$  equipped with the quasi-norm  $\|f\|_{X^p} := \||f|^p\|_X^{1/p}$ .

**Definition 8.** Let  $X$  be a ball quasi-Banach function space and  $p \in (0, \infty)$ . The space  $X$  is said to be  $p$ -convex if there exists a positive constant  $C$  such that, for any  $\{f_j\}_{j=1}^\infty \subset X^{1/p}$ ,

$$\left\| \sum_{j=1}^\infty |f_j| \right\|_{X^{1/p}} \leq C \sum_{j=1}^\infty \|f_j\|_{X^{1/p}}.$$

In particular, when  $C = 1$ ,  $X$  is said to be strictly  $p$ -convex.

## 2.2 Maximal functions and additional assumptions

For  $0 \leq \alpha < n$ , we define the fractional maximal operator  $M_\alpha$  by

$$(M_\alpha f)(x) = \sup_{B \ni x} |B|^{\frac{\alpha}{n}-1} \int_B |f(y)| dy,$$

where  $f$  is a locally integrable function on  $\mathbb{R}^n$  and the supremum is taken over all balls  $B$  containing  $x$ . For  $\alpha = 0$ , we have that  $M_0 = M$ , where  $M$  is the Hardy-Littlewood maximal operator on  $\mathbb{R}^n$ .

For  $\theta \in (0, \infty)$ , the *powered Hardy–Littlewood maximal operator*  $M^{(\theta)}$  is defined by

$$(M^{(\theta)}f)(x) = \left[ M(|f|^\theta)(x) \right]^{1/\theta}.$$

In what follows, we will assume the following two additional hypotheses:

A1) Let  $X$  be a ball quasi-Banach function space. Assume that, for some  $\theta, s \in (0, 1]$  with  $\theta < s$ , there exists a positive constant  $C$  such that for any sequence of functions  $\{f_j\}_{j=1}^\infty \subset L_{loc}^1(\mathbb{R}^n)$

$$\left\| \left\{ \sum_{j=1}^\infty (M^{(\theta)}f_j)^s \right\}^{1/s} \right\|_X \leq C \left\| \left\{ \sum_{j=1}^\infty |f_j|^s \right\}^{1/s} \right\|_X. \quad (2)$$

A2) Let  $X$  be a ball quasi-Banach function space. Assume that there exist  $r \in (0, 1]$ ,  $q \in (1, \infty)$  and a positive constant  $C$  such that  $X^{1/r}$  is a ball Banach function space and for any  $f \in (X^{1/r})'$

$$\left\| M^{((q/r)')} f \right\|_{(X^{1/r})'} \leq C \|f\|_{(X^{1/r})'}. \quad (3)$$

**Remark 9.** We observe that (3) is equivalent to that the Hardy–Littlewood maximal operator  $M$  be bounded on  $[(X^{1/r})']^{1/(q/r)'}$ .

Given  $L \in \mathbb{Z}_+$ , let

$$\mathcal{F}_L = \left\{ \varphi \in \mathcal{S}(\mathbb{R}^n) : \sum_{|\beta| \leq L} \sup_{x \in \mathbb{R}^n} (1 + |x|)^L \left| \partial^\beta \varphi(x) \right| := \|\varphi\|_{\mathcal{S}(\mathbb{R}^n), L} \leq 1 \right\}.$$

Let  $f \in \mathcal{S}'(\mathbb{R}^n)$ ,  $b \in (0, \infty)$ ,  $\Phi \in \mathbb{R}^n$  and  $\mathbb{R}_+^{n+1} := \mathbb{R}^n \times (0, \infty)$ . We define the following two maximal functions for  $f$ ,

$$\mathcal{M}_L^0 f(x) := \sup \{ |(\phi_t * f)(x)| : t > 0, \phi \in \mathcal{F}_L \}, \quad (4)$$

and

$$M_b^{**}(f, \Phi)(x) := \sup_{(y, t) \in \mathbb{R}_+^{n+1}} \frac{|(\Phi_t * f)(x - y)|}{(1 + t^{-1}|y|)^b}. \quad (5)$$

## 2.3 Hardy spaces associated with ball quasi-Banach function spaces

Let  $(X, \|\cdot\|_X)$  be a ball quasi-Banach function space. Now, following to [27], we introduce the Hardy type space associated with  $X$ , which is denoted by  $H_X(\mathbb{R}^n)$ , and present the atomic decomposition for elements of  $H_X(\mathbb{R}^n)$  also established in [27].

**Definition 10.** Let  $X$  be a ball quasi-Banach function space. Then the Hardy space  $H_X(\mathbb{R}^n)$  associated with  $X$  is defined as

$$H_X(\mathbb{R}^n) := \{f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{H_X(\mathbb{R}^n)} := \|M_b^{**}(f, \Phi)\|_X < \infty\}, \quad (6)$$

where  $M_b^{**}$  is the maximal operator given by (5) with  $b$  sufficiently large.

**Theorem 11.** ([27, Theorem 3.1 - (ii)]) Let  $X$  be a ball quasi-Banach function space such that the Hardy-Littlewood maximal operator  $M$  is bounded on  $X^{1/r}$  for some  $r \in (0, \infty)$ , and let  $\mathcal{M}_L^0 f$  be the grand maximal of  $f$  given by (4). Assume that  $b \in (n/r, \infty)$ . Then, when  $L \geq \lfloor b+2 \rfloor$ , if one of the quantities

$$\|M_b^{**}(f, \Phi)\|_X \text{ or } \|\mathcal{M}_L^0 f\|_X$$

is finite, then the other is also finite and mutually equivalent with the implicit positive constants independent of  $f$ .

**Remark 12.** If  $X$  and  $r \in (0, \infty)$  are as in Theorem 11 and  $b = n/r + 1$ , then for any  $L \geq \lfloor n/r + 3 \rfloor$ , we have

$$H_X(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) : \|\mathcal{M}_L^0 f\|_X < \infty\}.$$

Fixed  $L \geq \lfloor n/r + 3 \rfloor$ , we consider  $\|f\|_{H_X(\mathbb{R}^n)} = \|\mathcal{M}_L^0 f\|_X$ .

Before establishing the atomic decomposition for elements of  $H_X(\mathbb{R}^n)$ , we recall the definition of atoms.

**Definition 13.** Let  $X$  be a ball quasi-Banach function space satisfying (2) and let  $p \in [1, \infty]$ . Assume that  $d \in \mathbb{Z}_+$  satisfies  $d \geq d_X$ , where  $d_X := \lfloor n(1/\theta - 1) \rfloor$  and  $\theta \in (0, 1]$  is the constant in (2). Then the function  $a(\cdot)$  is called an  $(X, p, d)$ -atom if there exists a cube  $Q \in \mathcal{Q}$  such that  $\text{supp}(a) \subset Q$ ,

$$\|a\|_{L^p(\mathbb{R}^n)} \leq \frac{|Q|^{1/p}}{\|\chi_Q\|_X},$$

and  $a(\cdot) \perp \mathcal{P}_d$ .

**Remark 14.** For  $p \geq 1$  fixed, every  $(X, \infty, d)$ -atom is an  $(X, p, d)$ -atom.

To get our main results we need the following atomic decomposition for elements of  $H_X(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ .

**Theorem 15.** Let  $X$  be a ball quasi-Banach function space satisfying (2) for some  $\theta, s \in (0, 1]$ ,  $d \geq d_X$  be a fixed integer, and  $f \in H_X(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ , with  $p \in (1, \infty)$  fixed. Then

there exist a sequence  $\{a_j\}_{j=1}^\infty$  of  $(X, \infty, d)$ -atoms, supported on the cubes  $\{Q_j\}_{j=1}^\infty \subset \mathcal{Q}$ , and a sequence  $\{\lambda_j\}_{j=1}^\infty \subset (0, \infty)$  such that

$$f = \sum_{j=1}^{\infty} \lambda_j a_j \quad \text{in } L^p(\mathbb{R}^n), \quad (7)$$

and

$$\left\| \left\{ \sum_{j=1}^{\infty} \left( \frac{\lambda_j}{\|\chi_{Q_j}\|_X} \right)^s \chi_{Q_j} \right\}^{1/s} \right\|_X \lesssim_s \|f\|_{H_X(\mathbb{R}^n)}, \quad (8)$$

where the implicit positive constant is independent of  $f$ , but depends on  $s$ .

*Proof.* The existence of a such atomic decomposition as well as the validity of inequality (8) are guaranteed by [27, Theorem 3.7]. In principle, the convergence in (7) is in  $\mathcal{S}'(\mathbb{R}^n)$ . To see the convergence of the atomic series to  $f$  in  $L^p(\mathbb{R}^n)$ , we point out that the construction of a such atomic decomposition (see [27, Proposition 4.3]) is analogous to the one given for classical Hardy spaces (see [29, Chapter III]). Since  $f \in L^p(\mathbb{R}^n)$  we have that  $\mathcal{M}_L f \in L^p(\mathbb{R}^n)$ . So, following the proof of [25, Theorem 3.1], we obtain (7).  $\square$

From [27, Corollary 3.11] and since [27, Remark 3.12] also holds true for any  $p \in (1, \infty)$ , we have the following result.

**Proposition 16.** *Assume that  $X$  is ball quasi-Banach function space satisfying (2), (3), which is strictly  $s$ -convex, where  $s \in (0, 1]$  is as in (2), and  $X$  has an absolutely continuous quasi-norm. Then, for any  $p \in (1, \infty]$ ,  $H_X(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$  is dense in  $H_X(\mathbb{R}^n)$ .*

### 3 Auxiliary results

Our first two results equate the size of certain cubes in terms of the quasi-norm  $\|\cdot\|_X$ .

**Lemma 17.** ([35, Lemma 2.3]) *Assume that  $X$  is a ball quasi-Banach function space satisfying (2) for some  $0 < \theta < s \leq 1$ . If  $\gamma \in [1/\theta, \infty)$ , then for any cubes  $Q_1 \subset Q_2$  it holds true that*

$$\|\chi_{Q_2}\|_X \leq \left( \frac{|Q_2|}{|Q_1|} \right)^\gamma \|\chi_{Q_1}\|_X.$$

From this Lemma and Remark 3, it follows the following corollary.

**Corollary 18.** *If  $X$  is a ball quasi-Banach function space satisfying (2) for some  $0 < \theta < s \leq 1$ , then for  $\gamma \in [1/\theta, \infty)$ ,  $\delta \geq 1$ , and any cube  $Q$*

$$\|\chi_Q\|_X \leq \|\chi_{\delta Q}\|_X \leq \delta^m \|\chi_Q\|_X.$$



The following two lemmas generalize the vector-valued inequalities established in Lemmas 4.9 and 4.11 of [10] to the ball quasi-Banach functions spaces setting.

**Lemma 19.** ([5, Lemma 2.8 and Remark 2.9]) *Let  $X$  be a ball quasi-Banach function space. Let  $r \in (0, 1]$  such that  $X^{1/r}$  is a ball Banach function space and  $X$  satisfies (3) for some  $q \geq 1$ , then for any countable collection of cubes  $\{Q_j\}$  and non-negative functions  $g_j$  such that  $\text{supp}(g_j) \subset Q_j$  and  $\sum_{j=1}^{\infty} \left( \frac{1}{|Q_j|} \int_{Q_j} g_j^q \right)^{1/q} \chi_{Q_j} \in X$ ,*

$$\left\| \sum_{j=1}^{\infty} g_j \right\|_X \lesssim \left\| \sum_{j=1}^{\infty} \left( \frac{1}{|Q_j|} \int_{Q_j} g_j^q \right)^{1/q} \chi_{Q_j} \right\|_X. \quad (9)$$

**Lemma 20.** ([33, Lemma 2.4]) *Let  $0 < \alpha < n$  and  $0 < p_0 < q_0 \leq 1$  be such that  $\frac{1}{p_0} - \frac{1}{q_0} = \frac{\alpha}{n}$ , and let  $X$  and  $Y$  be ball quasi-Banach function spaces such that  $X^{1/p_0}$  and  $Y^{1/q_0}$  are ball Banach function spaces and  $(Y^{1/q_0})' = ((X^{1/p_0})')^{p_0/q_0}$ . If the Hardy-Littlewood maximal operator  $M$  is bounded on  $(Y^{1/q_0})'$ , then for any countable collection of cubes  $\{Q_j\}$ , with  $\sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \in X$ , and  $\lambda_j > 0$ ,*

$$\left\| \sum_{j=1}^{\infty} \lambda_j |Q_j|^{\frac{\alpha}{n}} \chi_{2Q_j} \right\|_Y \lesssim \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_X. \quad (10)$$

We conclude this section with two Fefferman-Stein vector-valued inequality on the  $s$ -convexification of ball quasi-Banach functions spaces. The first is for the fractional maximal operator and the second one is for the Hardy-Littlewood maximal operator.

**Proposition 21.** *Let  $0 < \alpha < n$ ,  $1 < u < \infty$  and  $0 < p_0 < q_0 \leq 1$  be such that  $\frac{1}{p_0} - \frac{1}{q_0} = \frac{\alpha}{n}$ , and let  $X$  and  $Y$  be ball quasi-Banach function spaces such that  $X^{1/p_0}$  and  $Y^{1/q_0}$  are ball Banach function spaces and  $(Y^{1/q_0})' = ((X^{1/p_0})')^{p_0/q_0}$ . If the Hardy-Littlewood maximal operator  $M$  is bounded on  $(Y^{1/q_0})'$ , then for any  $\sigma > \frac{1}{p_0}$  and any sequence of measurable functions  $\{f_j\}_{j=1}^{\infty}$  with  $\left\{ \sum_{j=1}^{\infty} |f_j|^u \right\}^{1/u} \in X^{\sigma}$ ,*

$$\left\| \left\{ \sum_{j=1}^{\infty} (M_{\frac{\alpha}{\sigma}} f_j)^u \right\}^{1/u} \right\|_{Y^{\sigma}} \lesssim \left\| \left\{ \sum_{j=1}^{\infty} |f_j|^u \right\}^{1/u} \right\|_{X^{\sigma}}. \quad (11)$$

*Proof.* Given  $0 < \alpha < n$  and  $\sigma > \frac{1}{p_0} = \frac{1}{q_0} + \frac{\alpha}{n}$ , we define

$$\mathcal{F}_{\alpha} = \left\{ \left( \left\{ \sum_{j=1}^N (M_{\frac{\alpha}{\sigma}} f_j)^u \right\}^{1/u}, \left\{ \sum_{j=1}^N |f_j|^u \right\}^{1/u} \right) : N \in \mathbb{N}, \{f_j\}_{j=1}^N \subset L_{comp}^{\infty} \right\},$$

where  $L_{comp}^\infty$  denotes the set of all the bounded measurable functions on  $\mathbb{R}^n$  with compact support. We observe that  $L_{comp}^\infty \subset X^\sigma$ , and for any  $\{f_j\}_{j=1}^N \subset L_{comp}^\infty$  it is easy to check that  $\left\{\sum_{j=1}^N |f_j|^u\right\}^{1/u} \in X^\sigma$ .

It is clear that  $1 < \sigma p_0 < \frac{\sigma n}{\alpha}$ , now for any  $v \in \mathcal{A}_1$  one has that  $v^{1/\sigma q_0} \in \mathcal{A}_{\sigma p_0, \sigma q_0}$  (for the definitions of the classes  $\mathcal{A}_1$  and  $\mathcal{A}_{p,q}$  see [14, Chapter 7] and [20, inequality (1.1)], respectively), from [20, Theorem 3] and [9, Theorem 3.23] follow that there exists an universal constant  $C > 0$  such that for any  $(F, G) \in \mathcal{F}_\alpha$  and any  $v \in \mathcal{A}_1$

$$\int [F(x)]^{\sigma q_0} v(x) dx \leq C \left( \int [G(x)]^{\sigma p_0} [v(x)]^{p_0/q_0} dx \right)^{q_0/p_0}. \quad (12)$$

On the other hand, by Lemma 5, we have

$$\|F\|_{Y^\sigma}^{\sigma q_0} = \|F^{\sigma q_0}\|_{Y^{1/q_0}} \leq C \sup \left\{ \int_{\mathbb{R}^n} |[F(x)]^{\sigma q_0} g(x)| dx : \|g\|_{(Y^{1/q_0})'} \leq 1 \right\} \quad (13)$$

for some constant  $C > 0$ .

Let  $\mathcal{R}$  be the operator defined on  $(Y^{1/q_0})'$  by

$$\mathcal{R}g(x) = \sum_{k=0}^{\infty} \frac{M^k g(x)}{2^k \|M\|_{(Y^{1/q_0})'}^k},$$

where, for  $k \geq 1$ ,  $M^k$  denotes  $k$  iterations of the Hardy-Littlewood maximal operator  $M$ ,  $M^0 = M$ , and  $\|M\|_{(Y^{1/q_0})'}$  is the operator norm of the maximal operator  $M$  on  $(Y^{1/q_0})'$ . It follows immediately from this definition that:

- (i) if  $g$  is non-negative,  $g(x) \leq \mathcal{R}g(x)$  a.e.  $x \in \mathbb{R}^n$ ;
- (ii)  $\|\mathcal{R}g\|_{(Y^{1/q_0})'} \leq 2\|g\|_{(Y^{1/q_0})'}$ ;
- (iii)  $\mathcal{R}g \in \mathcal{A}_1$  with  $[\mathcal{R}g]_{\mathcal{A}_1} \leq 2\|M\|_{(Y^{1/q_0})'}$ .

Since  $F$  is non-negative, we can take the supremum in (13) over those non-negative  $g$  only. For any fixed non-negative  $g \in (Y^{1/q_0})'$ , by (i) above we have that

$$\int [F(x)]^{\sigma q_0} g(x) dx \leq \int [F(x)]^{\sigma q_0} (\mathcal{R}g)(x) dx. \quad (14)$$

Then (iii) and (12), and Hölder's inequality yield

$$\begin{aligned} \int [F(x)]^{\sigma q_0} (\mathcal{R}g)(x) dx &\leq C \left( \int [G(x)]^{\sigma p_0} [(\mathcal{R}g)(x)]^{p_0/q_0} dx \right)^{q_0/p_0} \\ &\leq C \|G^{\sigma p_0}\|_{X^{1/p_0}}^{q_0/p_0} \|(\mathcal{R}g)^{p_0/q_0}\|_{(X^{1/p_0})'}^{q_0/p_0} \\ &= C \|G\|_{X^\sigma}^{\sigma q_0} \|\mathcal{R}g\|_{((X^{1/p_0})')^{p_0/q_0}} \end{aligned} \quad (15)$$

by hypothesis we have that  $((X^{1/p_0})')^{p_0/q_0} = (Y^{1/q_0})'$ , so

$$= C \|G\|_{X^\sigma}^{\sigma q_0} \|\mathcal{R}g\|_{(Y^{1/q_0})'}$$

now, (ii) gives

$$\leq C \|G\|_{X^\sigma}^{\sigma q_0} \|g\|_{(Y^{1/q_0})'}.$$

Thus, (14) and (15) lead to

$$\int [F(x)]^{\sigma q_0} g(x) dx \leq C \|G\|_{X^\sigma}^{\sigma q_0}, \quad (16)$$

for all non-negative  $g$  such that  $\|g\|_{(Y^{1/q_0})'} \leq 1$ . Then, (13) and (16) give (11) for all finite sequences  $\{f_j\}_{j=1}^N \subset L_{comp}^\infty \cap X^\sigma$ . By passing to the limit, we obtain (11) for all infinite sequences  $\{f_j\}_{j=1}^\infty \subset L_{comp}^\infty$  with  $\left\{\sum_{j=1}^\infty |f_j|^u\right\}^{1/u} \in X^\sigma$ . For the general case, consider  $f_{j,N} = f_j \chi_{\{x: |x| < N, |f(x)| < N\}}$ , since  $\left\{\sum_{j=1}^\infty (M_{\frac{\alpha}{\sigma}} f_{j,N})^u\right\}^{1/u} \uparrow \left\{\sum_{j=1}^\infty (M_{\frac{\alpha}{\sigma}} f_j)^u\right\}^{1/u}$  and  $\left\{\sum_{j=1}^\infty |f_{j,N}|^u\right\}^{1/u} \uparrow \left\{\sum_{j=1}^\infty |f_j|^u\right\}^{1/u}$  as  $N \rightarrow \infty$ , then (11) holds for any sequence of measurable functions  $\{f_j\}_{j=1}^\infty$  with  $\left\{\sum_{j=1}^\infty |f_j|^u\right\}^{1/u} \in X^\sigma$ .  $\square$

Proceeding as in the proof of Proposition 21, but considering now the  $\mathcal{A}_p$  class instead of the  $\mathcal{A}_{p,q}$  class and [19, Theorem 9] and [9, Corollary 3.12] instead of [20, Theorem 3] and [9, Theorem 3.23] respectively, we obtain the following result.

**Proposition 22.** *Let  $1 < u < \infty$ ,  $0 < p_0 \leq 1$  and let  $X$  be ball quasi-Banach function spaces such that  $X^{1/p_0}$  is a ball Banach function spaces. If the Hardy-Littlewood maximal operator  $M$  is bounded on  $(X^{1/p_0})'$ , then for any  $\sigma > \frac{1}{p_0}$  and any sequence of measurable functions  $\{f_j\}_{j=1}^\infty$  with  $\left\{\sum_{j=1}^\infty |f_j|^u\right\}^{1/u} \in X^\sigma$ ,*

$$\left\| \left\{ \sum_{j=1}^\infty (M f_j)^u \right\}^{1/u} \right\|_{X^\sigma} \lesssim \left\| \left\{ \sum_{j=1}^\infty |f_j|^u \right\}^{1/u} \right\|_{X^\sigma}. \quad (17)$$

**Remark 23.** *Other versions of Proposition 22 can be found in [27, Section 2.5 on p. 18].*

**Corollary 24.** *Let  $0 < p_0 \leq 1$  and let  $X$  be ball quasi-Banach function spaces such that  $X^{1/p_0}$  is a ball Banach function spaces. If the Hardy-Littlewood maximal operator  $M$  is bounded on  $(X^{1/p_0})'$ , then  $M$  is bounded on  $X^\sigma$  for any  $\sigma > \frac{1}{p_0}$ .*

*Proof.* Apply Proposition 22 with  $f_1 \in X^\sigma \setminus \{0\}$  and  $f_j = 0$  for all  $j \geq 2$ .  $\square$

## 4 Main results

### 4.1 Kernels of type $(\alpha, N)$

We recall the definition of a kernel of type  $(\alpha, N)$  on  $\mathbb{R}^n$ . Suppose  $0 \leq \alpha < n$  and  $N \in \mathbb{N}$ . For  $0 < \alpha < n$ , a kernel of type  $(\alpha, N)$  is a function  $K_\alpha$  of class  $C^N$  on  $\mathbb{R}^n \setminus \{0\}$ , which satisfies

$$\left| (\partial^\beta K_\alpha)(x) \right| \lesssim |x|^{\alpha-n-|\beta|} \text{ for all } |\beta| \leq N \text{ and all } x \neq 0, \quad (18)$$

A kernel of type  $(0, N)$  is a distribution  $K_0$  on  $\mathbb{R}^n$  which is of class  $C^N$  on  $\mathbb{R}^n \setminus \{0\}$ , satisfies (18) with  $\alpha = 0$ , and

$$\|K_0 * f\|_{L^2(\mathbb{R}^n)} \leq C \|f\|_{L^2(\mathbb{R}^n)}, \text{ for all } f \in \mathcal{S}(\mathbb{R}^n). \quad (19)$$

**Remark 25.** If  $0 < \alpha < n$  and  $K_\alpha$  is a kernel of type  $(\alpha, N)$ , from [12, Proposition 6.2], it follows that the operator  $T_\alpha : f \rightarrow K_\alpha * f$  is a bounded operator  $L^p(\mathbb{R}^n) \rightarrow L^q(\mathbb{R}^n)$  for  $1 < p < \frac{n}{\alpha}$  and  $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ .

**Remark 26.** If  $K_0$  is a kernel of type  $(0, N)$ , by (19) it follows that the operator  $U_0 : f \rightarrow K_0 * f$ ,  $f \in \mathcal{S}(\mathbb{R}^n)$ , can be extended to a bounded operator on  $L^2(\mathbb{R}^n)$ , a such extension is unique. We denote this extension by  $T_0$ . Now, it is easy to check that if  $a(\cdot) \in L^2(\mathbb{R}^n)$  and their support is contained in the cube  $Q(x_0, r)$ , then  $(T_0 a)(x) = (K_0 * a)(x)$  a.e.  $x \notin Q(x_0, 2r)$ . Moreover, from [12, Theorem 6.10], it follows that  $T_0$  is bounded on  $L^p(\mathbb{R}^n)$  for every  $1 < p < \infty$ .

### 4.2 Estimates for the operator $T_\alpha$

Given a kernel  $K_\alpha$  of type  $(\alpha, N)$  with  $0 \leq \alpha < n$ , we consider the operator  $T_\alpha$  defined by

$$T_\alpha = \begin{cases} \text{extension of the operator } U_0 \text{ on } L^2(\mathbb{R}^n), & \text{if } \alpha = 0 \\ \text{convolution operator by } K_\alpha, & \text{if } 0 < \alpha < n \end{cases}. \quad (20)$$

Our main result is contained in the following theorem.

**Theorem 27.** Given  $0 \leq \alpha < n$  and  $N \in \mathbb{N}$ , let  $T_\alpha$  be the operator by convolution with a kernel  $K_\alpha$  of type  $(\alpha, N)$  defined by (20) and let  $X$  and  $Y$  be ball quasi-Banach function spaces such that the quasi-norm of  $X$  is absolutely continuous satisfying (2), (3) and  $X$  is strictly  $s$ -convex, where  $s \in (0, 1]$  is as in (2), and  $Y$  satisfies (3) with  $q > \frac{n}{n-\alpha}$ .

(i) If for some  $0 < p_0 \leq 1$ ,  $X^{1/p_0}$  is a ball Banach function space, the Hardy-Littlewood operator  $M$  is bounded on  $(X^{1/p_0})'$  and  $N > \max\{d_X + 1, n(1/p_0 - 1)\}$ , then the operator  $T_0$  extends to a bounded operator  $H_X(\mathbb{R}^n) \rightarrow X$  and  $H_X(\mathbb{R}^n) \rightarrow H_X(\mathbb{R}^n)$ .

(ii) If for some  $0 < p_0 < q_0 \leq 1$  such that  $\frac{1}{p_0} - \frac{1}{q_0} = \frac{\alpha}{n}$  with  $0 < \alpha < n$ ,  $X^{1/p_0}$  and  $Y^{1/q_0}$  are ball Banach function spaces such that  $(Y^{1/q_0})' = ((X^{1/p_0})')^{p_0/q_0}$ , the Hardy-Littlewood operator  $M$  is bounded on  $(Y^{1/q_0})'$  and  $N > \max\{d_X + 1, n(1/p_0 - 1)\}$ , then the operator  $T_\alpha$  extends to a bounded operator  $H_X(\mathbb{R}^n) \rightarrow Y$  and  $H_X(\mathbb{R}^n) \rightarrow H_Y(\mathbb{R}^n)$ .

*Proof.* We will first prove, by means of the atomic decomposition of  $H_X(\mathbb{R}^n)$ , that the operator  $T_\alpha$ ,  $0 < \alpha < n$ , extends to a bounded operator  $H_X(\mathbb{R}^n) \rightarrow Y$  and that  $T_0$ ,  $\alpha = 0$ , extends to a bounded operator  $H_X(\mathbb{R}^n) \rightarrow X$ . For them, given  $0 \leq \alpha < n$ , let  $K_\alpha$  be a kernel of the type  $(\alpha, N)$  with  $N > \max\{d_X + 1, n(1/p_0 - 1)\}$ , where  $\frac{1}{p_0} = \frac{1}{q_0} + \frac{\alpha}{n}$  is as in (i) or (ii) according to the case. Then, given  $f \in H_X(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$  (with  $p > 1$ ), by Theorem 15 with  $d = N - 1$ , there exist a sequence of nonnegative numbers  $\{\lambda_j\}_{j=1}^\infty$ , a sequence of cubes  $\{Q_j\}_{j=1}^\infty$  and  $(X, \infty, N - 1)$ -atoms  $a_j$  supported on  $Q_j$  such that  $f = \sum_{j=1}^\infty \lambda_j a_j$  converges in  $L^p(\mathbb{R}^n)$  and

$$\left\| \left\{ \sum_{j=1}^\infty \left( \frac{\lambda_j}{\|\chi_{Q_j}\|_X} \right)^s \chi_{Q_j} \right\}^{1/s} \right\|_X \lesssim_s \|f\|_{H_X(\mathbb{R}^n)}. \quad (21)$$

For  $0 < \alpha < n$ , we have that  $Y$  satisfies (3), with  $q > \frac{n}{n-\alpha}$ . For a such  $q$ , we put  $\frac{1}{p} := \frac{1}{q} + \frac{\alpha}{n}$ . Then, by Remark 25, we have that the operator  $T_\alpha$  is bounded from  $L^p(\mathbb{H}^n)$  to  $L^q(\mathbb{H}^n)$ . For the case  $\alpha = 0$ , by Remark 26, the operator  $T_0$  is bounded on  $L^p(\mathbb{R}^n)$  for every  $1 < p < \infty$ . In this case, we consider  $p = q$  where  $q$  is as in (3). Since  $f = \sum_{j=1}^\infty \lambda_j a_j$  converges in  $L^p(\mathbb{R}^n)$ , according to the case  $0 < \alpha < n$  or  $\alpha = 0$ , we have that  $T_\alpha f = \sum_j \lambda_j T_\alpha a_j$  converges in  $L^q(\mathbb{R}^n)$ , and so

$$|T_\alpha f(x)| \leq \sum_j \lambda_j |T_\alpha a_j(x)|, \quad a.e. x \in \mathbb{R}^n.$$

Then, for  $0 < \alpha < n$ ,

$$\|T_\alpha f\|_Y \lesssim \left\| \sum_j \lambda_j \chi_{2Q_j} |T_\alpha a_j| \right\|_Y + \left\| \sum_j \lambda_j \chi_{\mathbb{R}^n \setminus 2Q_j} |T_\alpha a_j| \right\|_Y =: I_1 + I_2, \quad (22)$$

and, for  $\alpha = 0$ , we have

$$\|T_0 f\|_X \lesssim \left\| \sum_j \lambda_j \chi_{2Q_j} |T_\alpha a_j| \right\|_X + \left\| \sum_j \lambda_j \chi_{\mathbb{R}^n \setminus 2Q_j} |T_\alpha a_j| \right\|_X =: \tilde{I}_1 + \tilde{I}_2, \quad (23)$$

where  $2Q_j = Q(z_j, 2r_j)$ . To estimate  $I_1$ , we first apply Remark 25 to the expression  $\chi_{2Q_j} \cdot T_\alpha a_j$  followed by Remark 14 and Corollary 18 to obtain

$$\|T_\alpha a_j\|_{L^q(2Q_j)} \lesssim \|a_j\|_{L^p} \lesssim \frac{|Q_j|^{\frac{1}{p}}}{\|\chi_{Q_j}\|_X} \lesssim \frac{|2Q_j|^{\frac{1}{q} + \frac{\alpha}{n}}}{\|\chi_{2Q_j}\|_X},$$

so

$$\left( \frac{1}{|2Q_j|} \int_{2Q_j} |T_\alpha a_j|^q \right)^{1/q} \lesssim \frac{|Q_j|^{\frac{\alpha}{n}}}{\|\chi_{Q_j}\|_X}. \quad (24)$$

Now, successively applying Lemma 19 (considering there  $Y$  instead of  $X$ ), (24), Lemma 20, Corollary 18, the  $s$ -inequality for  $s \in (0, 1]$ , and (21), we have

$$\begin{aligned} I_1 &= \left\| \sum_j \lambda_j \chi_{2Q_j} \cdot T_\alpha a_j \right\|_Y \lesssim \left\| \sum_j \lambda_j \left( \frac{1}{|2Q_j|} \int_{2Q_j} |T_\alpha a_j|^q \right)^{1/q} \chi_{2Q_j} \right\|_Y \\ &\lesssim \left\| \sum_j \lambda_j \frac{|Q_j|^{\frac{\alpha}{n}}}{\|\chi_{Q_j}\|_X} \chi_{2Q_j} \right\|_Y \lesssim \left\| \sum_j \frac{\lambda_j}{\|\chi_{Q_j}\|_X} \chi_{Q_j} \right\|_X \\ &\lesssim \left\| \left\{ \sum_j \left( \frac{\lambda_j}{\|\chi_{Q_j}\|_X} \right)^s \chi_{Q_j} \right\}^{1/s} \right\|_X \lesssim_s \|f\|_{H_X(\mathbb{R}^n)}. \end{aligned} \quad (25)$$

Since (24) holds with  $\alpha = 0$  and  $p = q$ , proceeding as in the estimate of (25), we obtain

$$\tilde{I}_1 = \left\| \sum_j \lambda_j \chi_{2Q_j} |T_\alpha a_j| \right\|_X \lesssim_s \|f\|_{H_X(\mathbb{R}^n)}. \quad (26)$$

Now, we estimate the terms  $I_2$  and  $\tilde{I}_2$ . Being every  $a_j(\cdot)$  an  $(X, \infty, N-1)$ -atom with  $N > \max\{d_X + 1, n(1/p_0 - 1)\}$  and support on the cube  $Q_j = Q(z_j, r_j)$ , by the moment condition for  $a_j(\cdot)$  (i.e.  $a_j \perp \mathcal{P}_{N-1}$ ), and Remark 25 (for  $0 < \alpha < 0$ ) or Remark 26 (for  $\alpha = 0$ ), we obtain

$$T_\alpha a_j(x) = \int_{Q_j} (K_\alpha(x-y) - q_{x,N}(y)) a_j(y) dy, \quad \text{for all } x \notin 2Q_j,$$

where  $q_{x,N}$  is the degree  $N-1$  Taylor polynomial of the function  $y \rightarrow K_\alpha(x-y)$  expanded around  $z_j$ . By the standard estimate of the remainder term of the Taylor expansion and (18), for any  $y \in Q_j$  and any  $x \notin 2Q_j$ , we get

$$|K_\alpha(x-y) - q_{x,N}(y)| \leq Cr_j^N |x - z_j|^{-n+\alpha-N},$$

this inequality together with  $\|a_j\|_{L^\infty} \leq \|\chi_{Q_j}\|_X^{-1}$  allow us to conclude that

$$|T_\alpha a_j(x)| \lesssim \frac{r_j^{n+N}}{\|\chi_{Q_j}\|_X} |x - z_j|^{-n+\alpha-N} \lesssim \frac{\left[ M_{\frac{\alpha n}{n+N}}(\chi_{Q_j})(x) \right]^{\frac{n+N}{n}}}{\|\chi_{Q_j}\|_X}, \quad (27)$$

for all  $x \notin 2Q_j$ . Putting  $\sigma = \frac{n+N}{n}$ , (27) leads to

$$I_2 \lesssim \left\| \left\{ \sum_j \frac{\lambda_j}{\|\chi_{Q_j}\|_X} \left[ M_{\frac{\alpha}{\sigma}}(\chi_{Q_j}) \right]^\sigma \right\}^{1/\sigma} \right\|_{Y^\sigma}^\sigma.$$

Since  $\sigma = \frac{n+N}{n} > \frac{1}{p_0}$ , to apply Lemma 21 with  $u = \sigma$ , we obtain

$$I_2 \lesssim \left\| \left\{ \sum_j \frac{\lambda_j}{\|\chi_{Q_j}\|_X} \chi_{Q_j} \right\}^{1/\sigma} \right\|_{X^\sigma}^\sigma = \left\| \sum_j \frac{\lambda_j}{\|\chi_{Q_j}\|_X} \chi_{Q_j} \right\|_X \lesssim_s \|f\|_{H_X(\mathbb{R}^n)}. \quad (28)$$

Observing that (27) holds with  $\alpha = 0$  and to proceed as in the estimate of  $I_2$ , but now applying Lemma 22, we obtain

$$\tilde{I}_2 = \left\| \sum_j \lambda_j \chi_{\mathbb{R}^n \setminus 2Q_j} |T_\alpha a_j| \right\|_X \lesssim_s \|f\|_{H_X(\mathbb{R}^n)}. \quad (29)$$

So, by (22), (25), (28) and Proposition 16, the operator  $T_\alpha$ ,  $0 < \alpha < n$ , extends to a bounded operator  $H_X(\mathbb{R}^n) \rightarrow Y$ . Now, by (23), (26), (29) and Proposition 16, we have that  $T_0$  extends to a bounded operator  $H_X(\mathbb{R}^n) \rightarrow X$ .

Now, by using the atomic decomposition of  $H_X(\mathbb{R}^n)$  and the maximal characterization of  $H_Y(\mathbb{R}^n)$  and  $H_X(\mathbb{R}^n)$ , we will prove that the operator  $T_\alpha$ ,  $0 < \alpha < n$ , extends to a bounded operator  $H_X(\mathbb{R}^n) \rightarrow H_Y(\mathbb{R}^n)$  and that  $T_0$  extends to a bounded operator  $H_X(\mathbb{R}^n) \rightarrow H_X(\mathbb{R}^n)$ . Let  $0 \leq \alpha < n$  and let  $\frac{1}{p_0} = \frac{1}{q_0} + \frac{\alpha}{n}$  be as in (i) or (ii) according to the case. Given  $f \in H_X(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ , we consider its atomic decomposition  $f = \sum_j \lambda_j a_j$ , and  $N > \max\{d_X + 1, n(1/p_0 - 1)\}$ . Now, by Corollary 24, we have that, for  $r = \frac{p_0}{p_0+1}$  ( $\frac{1}{p_0} = \frac{1}{q_0} + \frac{\alpha}{n}$ ), the Hardy-Littlewood maximal operator  $M$  is bounded on  $X^{1/r}$  or on  $Y^{1/r}$ . Then, by Remark 12, we have that, for  $L \geq \left\lfloor \frac{n(p_0+1)}{p_0} + 3 \right\rfloor$ ,  $\|f\|_{H_X(\mathbb{R}^n)} = \|\mathcal{M}_L^0 f\|_X$  for  $f \in H_X(\mathbb{R}^n)$  and  $\|g\|_{H_Y(\mathbb{R}^n)} = \|\mathcal{M}_L^0 g\|_Y$  for  $g \in H_Y(\mathbb{R}^n)$ .

On the other hand, for  $0 \leq \alpha < n$ , we put  $\frac{1}{p} := \frac{1}{q} + \frac{\alpha}{n}$ , where for  $0 < \alpha < n$ ,  $q$  is as in (3) but with  $Y$  instead of  $X$  and  $q > \frac{n}{n-\alpha}$ ; for  $\alpha = 0$ ,  $q$  is as in (3). Since  $T_\alpha$  is bounded from  $L^p(\mathbb{R}^n)$  into  $L^q(\mathbb{R}^n)$ ,  $H^q(\mathbb{R}^n) \equiv L^q(\mathbb{R}^n)$  with equivalent norms, and  $T_\alpha f = \sum_j \lambda_j T_\alpha a_j$  converges in  $L^q(\mathbb{R}^n)$ , it follows, for  $0 \leq \alpha < n$ , that

$$\mathcal{M}_L^0(T_\alpha f)(x) \leq \sum_{j=1}^{\infty} \lambda_j \mathcal{M}_L^0(T_\alpha a_j)(x), \text{ a.e. } x \in \mathbb{R}^n.$$

Then, for  $0 < \alpha < n$ ,

$$\|T_\alpha f\|_{H_Y} = \|\mathcal{M}_L^0(T_\alpha f)\|_Y \lesssim \left\| \sum_j \lambda_j \chi_{2Q_j} \mathcal{M}_L^0(T_\alpha a_j) \right\|_Y$$

$$+ \left\| \sum_j \lambda_j \chi_{\mathbb{R}^n \setminus 2Q_j} \mathcal{M}_L^0(T_\alpha a_j) \right\|_Y =: J_1 + J_2.$$

For,  $\alpha = 0$  and  $L$  as above, we have

$$\begin{aligned} \|T_0 f\|_{H_X} &= \|\mathcal{M}_L^0(T_0 f)\|_X \lesssim \left\| \sum_j \lambda_j \chi_{2Q_j} \mathcal{M}_L^0(T_0 a_j) \right\|_X \\ &+ \left\| \sum_j \lambda_j \chi_{\mathbb{R}^n \setminus 2Q_j} \mathcal{M}_L^0(T_0 a_j) \right\|_X =: \tilde{J}_1 + \tilde{J}_2. \end{aligned}$$

To estimate  $J_1$  or  $\tilde{J}_1$ , we observe that for  $0 \leq \alpha < n$  and  $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ , with  $q > \frac{n}{n-\alpha}$  satisfying (3) for  $X$  or  $Y$  according the case,

$$\|\mathcal{M}_L^0(T_\alpha a_j)\|_{L^q(2Q_j)} \lesssim \|T_\alpha a_j\|_{L^q} \lesssim \|a_j\|_{L^p} \lesssim \frac{|Q_j|^{\frac{1}{p}}}{\|\chi_{Q_j}\|_X} \lesssim \frac{|2Q_j|^{\frac{1}{q} + \frac{\alpha}{n}}}{\|\chi_{2Q_j}\|_X}.$$

Then, by proceeding as above in the estimate of  $I_1$  or  $\tilde{I}_1$ , according to the case, we get

$$J_1, \tilde{J}_1 \lesssim \left\| \left\{ \sum_j \left( \frac{\lambda_j}{\|\chi_{Q_j}\|_X} \right)^s \chi_{Q_j} \right\}^{1/s} \right\|_X \lesssim_s \|f\|_{H_X}.$$

Now, we estimate  $J_2$  and  $\tilde{J}_2$ . Then, for  $x \notin 2Q_j$  and every  $t > 0$ , by the  $(N-1)$ -moment condition of the atoms, we have

$$\begin{aligned} ((T_\alpha a_j) * \phi_t)(x) &= \int_{Q_j} a_j(y) (K_\alpha * \phi_t)(x-y) dy \\ &= \int_{Q_j} a_j(y) [(K_\alpha * \phi_t)(x-y) - q_{x,t,N}(y)] dy, \end{aligned}$$

where  $y \rightarrow q_{x,t,N}(y)$  is the Taylor polynomial of the function  $y \rightarrow (K_\alpha * \phi_t)(x-y)$  at  $z_j$  of degree  $N-1$ . Let  $\phi \in \mathcal{S}(\mathbb{R}^n)$  with  $\|\phi\|_{\mathcal{S}(\mathbb{R}^n), L} \leq 1$  (where  $L \geq \left\lfloor \frac{n(p_0+1)}{p_0} + 3 \right\rfloor$  and sufficiently large), by [12, Lemma 6.9] applied with  $G = \mathbb{R}^n$ ,  $r = N$  and  $0 \leq \alpha < n$ , we have

$$|\partial^\beta (K_\alpha * \phi_t)(u)| = |((\partial^\beta K_\alpha) * \phi_t)(u)| \lesssim \|\phi\|_{\mathcal{S}(\mathbb{R}^n), L} |u|^{\alpha-n-|\beta|} \lesssim |u|^{\alpha-n-|\beta|},$$

for all  $u \neq 0$ ,  $t > 0$  and  $|\beta| \leq N$ . Then, by the standard estimate of the remainder term of the Taylor expansion, for any  $y \in Q_j$  and any  $x \notin 2Q_j$ , we obtain

$$|(K_\alpha * \phi_t)(x-y) - q_{x,t,N}(y)| \lesssim r_j^N |x-z_j|^{-n+\alpha-N},$$



and so, for  $0 \leq \alpha < n$  and  $x \notin 2Q_j$ ,

$$|(T_\alpha a_j) * \phi_t)(x)| \lesssim \frac{\left[ M_{\frac{\alpha n}{n+N}}(\chi_{Q_j})(x) \right]^{\frac{n+N}{n}}}{\|\chi_{Q_j}\|_X}.$$

This estimate does not depend on  $t > 0$  and  $\phi$  with  $\|\phi\|_{\mathcal{S}(\mathbb{R}^n), L} \leq 1$ . Then, applying the ideas to estimate  $I_2$  and  $\tilde{I}_2$  above to the present case and taking the supremum on  $t > 0$  and  $\phi \in \mathcal{F}_L$ , we obtain

$$J_2, \tilde{J}_2 \lesssim \left\| \left\{ \sum_j \left( \frac{\lambda_j}{\|\chi_{Q_j}\|_X} \right)^s \chi_{Q_j} \right\}^{1/s} \right\|_X \lesssim_s \|f\|_{H_X}.$$

Finally, by the estimates of  $J_1, J_2, \tilde{J}_1, \tilde{J}_2$  and Proposition 16, we have that  $T_\alpha, 0 < \alpha < n$ , extends to a bounded operator  $H_X(\mathbb{R}^n) \rightarrow H_Y(\mathbb{R}^n)$  and  $T_0$  extends to a bounded operator  $H_X(\mathbb{R}^n) \rightarrow H_X(\mathbb{R}^n)$ . This concludes the proof.  $\square$

### 4.3 Singular integrals

Let  $\Omega \in C^\infty(S^{n-1})$  with  $\int_{S^{n-1}} \Omega(u) d\sigma(u) = 0$ . We define the operator  $T$  by

$$Tf(x) = \lim_{\varepsilon \rightarrow 0^+} \int_{|y| > \varepsilon} \frac{\Omega(y/|y|)}{|y|^n} f(x-y) dy, \quad x \in \mathbb{R}^n. \quad (30)$$

It is well known that  $\widehat{Tf}(\xi) = m(\xi) \widehat{f}(\xi)$ , where the multiplier  $m$  is homogeneous of degree 0 and is indefinitely differentiable on  $\mathbb{R}^n \setminus \{0\}$ . Moreover, if  $k(y) = \frac{\Omega(y/|y|)}{|y|^n}$  we have

$$|\partial^\beta k(y)| \leq C|y|^{-n-|\beta|}, \quad \text{for all } y \neq 0 \text{ and all multi-index } \beta. \quad (31)$$

The operator  $T$  results bounded on  $L^p(\mathbb{R}^n)$  for every  $1 < p < +\infty$  (see [28]).

**Theorem 28.** *Let  $X$  be ball quasi-Banach function space as in Theorem 27. Then the singular operator  $T$  given by (30) extends to a bounded operator  $H_X(\mathbb{R}^n) \rightarrow X$  and  $H_X(\mathbb{R}^n) \rightarrow H_X(\mathbb{R}^n)$ .*

*Proof.* Given  $\varepsilon > 0$ , we put  $k_\varepsilon(y) = \chi_{\{|y| > \varepsilon\}}(y) \frac{\Omega(y/|y|)}{|y|^n}$  and  $T_\varepsilon f = k_\varepsilon * f$ . It is clear that  $k_\varepsilon$  satisfies (31) and  $\|k_\varepsilon * f\|_2 \leq C\|f\|_2$  for all  $f \in \mathcal{S}(\mathbb{R}^n)$ , where the constant  $C$  does not depend on  $\varepsilon$ . Then, applying the Theorem 27 - (i) with  $K_{0,\varepsilon} = k_\varepsilon$  and letting  $\varepsilon \rightarrow 0$ , the theorem follows.  $\square$

## 4.4 Fractional integrals

Let  $0 < \alpha < n$ , the Riesz potential  $I_\alpha$  on  $\mathbb{R}^n$  is defined by

$$(I_\alpha f)(x) = \int_{\mathbb{R}^n} |x-y|^{\alpha-n} f(y) dy. \quad (32)$$

It is clear that  $|\cdot|^{\alpha-n} \in C^\infty(\mathbb{R}^n \setminus \{0\}) = \bigcap_{N \in \mathbb{N}} C^N(\mathbb{R}^n \setminus \{0\})$  and satisfies the condition (18)

for every  $N \in \mathbb{N}$ . Then, to apply the Theorem 27 - (ii) with  $K_\alpha(\cdot) = |\cdot|^{\alpha-n}$ , we obtain the following result.

**Theorem 29.** *Let  $0 < \alpha < n$ , and let  $X$  and  $Y$  be ball quasi-Banach function spaces as in Theorem 27. Then the Riesz potential  $I_\alpha$  given by (32) extends to a bounded operator  $H_X(\mathbb{R}^n) \rightarrow Y$  and  $H_X(\mathbb{R}^n) \rightarrow H_Y(\mathbb{R}^n)$ .*

## 5 Examples

In this section we illustrate our results with four concrete examples of Hardy type spaces associated with ball quasi-Banach function spaces  $X$  and  $Y$  satisfying the hypotheses of Theorem 27.

**Weighted Hardy spaces.** Given  $0 < p < \infty$  and a weight  $w \in \mathcal{A}_\infty$  (see [14], [29]), the weighted Lebesgue space  $L_w^p(\mathbb{R}^n)$  is defined as the set of all the measurable functions  $f$  such that

$$\|f\|_{L_w^p} := \left( \int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{1/p} < \infty.$$

By [27, Section 7.1], the couple  $(L_w^p(\mathbb{R}^n), \|\cdot\|_{L_w^p})$  is a ball quasi-Banach function space, being the quasi-norm  $\|\cdot\|_{L_w^p}$  *absolutely continuous*. If  $p > 1$ , then  $L_w^p(\mathbb{R}^n)$  is a ball Banach function space with  $(L_w^p)' = L_{w^{1-p'}}^{p'}$ , where  $\frac{1}{p} + \frac{1}{p'} = 1$ . In [27, Section 7.1], it was pointed out that a weighted Lebesgue space may not be a Banach function space. By Fatou's Lemma, it follows that for any  $s \in (0, p)$ , the space  $L_w^p(\mathbb{R}^n)$  is *strictly  $s$ -convex*, where  $(L_w^p(\mathbb{R}^n))^{1/s} = L_w^{p/s}(\mathbb{R}^n)$ .

Given  $0 \leq \alpha < n$ , let  $X := L_w^p(\mathbb{R}^n)$ , with  $p \in (0, \frac{n}{\alpha})$  and  $w \in \mathcal{A}_\infty$ . Then, for any  $s \in (0, p)$  and  $w \in \mathcal{A}_{p/s}$ , the Hardy-Littlewood maximal operator  $M$  is bounded on  $L_w^{p/s}$  and so also on  $(L_w^{p/s})' = L_{w^{1-(p/s)'}}^{(p/s)'}$  since  $w^{1-(p/s)'} \in \mathcal{A}_{(p/s)'}$  (see [19, Theorem 9]). For  $0 < p < \frac{n}{\alpha}$ , we define  $\frac{1}{q} := \frac{1}{p} - \frac{\alpha}{n}$  and  $Y := L_{w^{q/p}}^q(\mathbb{R}^n)$ . For any  $p_0 \in (0, \min\{\frac{n}{n+\alpha}, p\})$  fixed, we put  $\frac{1}{q_0} := \frac{1}{p_0} - \frac{\alpha}{n}$ , then  $0 < p_0 \leq q_0 \leq 1$ ,  $q_0 \in (0, q)$ ,  $X^{1/p_0}$  and  $Y^{1/q_0}$  are ball Banach function spaces and  $M$  is bounded on  $(Y^{1/q_0})'$ . Since  $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n} = \frac{1}{p_0} - \frac{1}{q_0}$ , it

follows that  $(Y^{1/q_0})' = ((X^{1/p_0})')^{p_0/q_0}$ . For  $X := L_w^p(\mathbb{R}^n)$ , the assumption (2) holds true for any  $\theta, s \in (0, 1]$ ,  $\theta < s$ ,  $p \in (\theta, \frac{n}{\alpha})$  and  $w \in \mathcal{A}_{p/\theta}$  (it follows from [17, Theorem 3.1 (b)] applied on  $L_w^{p/\theta}$ , with  $r = s/\theta$  and  $|f_j|^\theta$  instead of  $f_k$ ); the assumption (3), by duality, [19, Theorem 9] and Remark 9, holds true for any  $r \in (0, \min\{1, p\})$ ,  $w \in \mathcal{A}_{p/r}$ , and  $\tilde{q} \in (\max\{1, p\}, \infty)$  sufficiently large such that  $w^{1-(p/r)'} \in \mathcal{A}_{(p/r)' / (\tilde{q}/r)'}$ . Similarly, the assumption (3) holds true for  $Y = L_{w^{q/p}}^q(\mathbb{R}^n)$ . Finally,  $H_X(\mathbb{R}^n) = H_w^p(\mathbb{R}^n)$  and  $H_Y(\mathbb{R}^n) = H_{w^{q/p}}^q(\mathbb{R}^n)$  are the weighted Hardy spaces defined in [31].

Thus, Theorem 27 applies on such  $X$  and  $Y$ .

**Variable Hardy spaces.** Let  $p(\cdot) : \mathbb{R}^n \rightarrow (0, \infty)$  be a measurable function such that

$$0 < p_- := \operatorname{ess\,inf}_{x \in \mathbb{R}^n} p(x) \leq \operatorname{ess\,sup}_{x \in \mathbb{R}^n} p(x) := p_+ < \infty.$$

Then, the variable Lebesgue space  $L^{p(\cdot)}(\mathbb{R}^n)$  consists of all the measurable functions  $f$  such that

$$\|f\|_{p(\cdot)} := \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} (\lambda^{-1} |f(x)|)^{p(x)} dx \leq 1 \right\} < \infty.$$

By [27, Section 7.8], we have that the couple  $(L^{p(\cdot)}(\mathbb{R}^n), \|\cdot\|_{p(\cdot)})$  is a ball quasi-Banach function space. Moreover, the quasi-norm  $\|\cdot\|_{p(\cdot)}$  is *absolutely continuous*. If  $p_- > 1$ , then  $L^{p(\cdot)}(\mathbb{R}^n)$  is a ball Banach function space with  $(L^{p(\cdot)}(\mathbb{R}^n))' = L^{p'(\cdot)}(\mathbb{R}^n)$ , where  $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$  for all  $x$ . Then, by [7, Proposition 2.18 and Theorem 2.61], for any  $s \in (0, p_-)$ ,  $(L^{p(\cdot)})^{1/s} = L^{p(\cdot)/s}$  and the space  $L^{p(\cdot)}$  is *strictly  $s$ -convex*.

An exponent  $p(\cdot) : \mathbb{R}^n \rightarrow (0, \infty)$  is said to be globally log-Hölder continuous if there exist positive constants  $C$  and  $p_\infty$  such that

$$|p(x) - p(y)| \leq \frac{-C}{\log(|x - y|)}, \text{ for } |x - y| \leq 1/2$$

and

$$|p(x) - p_\infty| \leq \frac{C}{\log(e + |x|)}, \text{ for all } x \in \mathbb{R}^n.$$

Let  $L^{p(\cdot)}(\mathbb{R}^n)$ , where  $p(\cdot)$  is globally log-Hölder continuous. Then, for any  $s \in (0, p_-)$ , the Hardy-Littlewood maximal operator  $M$  is bounded on  $L^{p(\cdot)/s}$  (see [8, Theorem 1.5]), and so also on  $(L^{p(\cdot)/s})' = L^{(p(\cdot)/s)'}'$ , since the exponent  $(p(\cdot)/s)'$  results globally log-Hölder continuous with  $((p(\cdot)/s)')_- > 1$ .

Given  $0 \leq \alpha < n$ , let  $p(\cdot)$  be an exponent such that  $0 < p_- \leq p_+ < \frac{n}{\alpha}$  and is globally log-Hölder continuous. Then, we define  $\frac{1}{q(\cdot)} := \frac{1}{p(\cdot)} - \frac{\alpha}{n}$ , such  $q(\cdot)$  results globally log-Hölder continuous. Let  $X := L^{p(\cdot)}(\mathbb{R}^n)$  and  $Y := L^{q(\cdot)}(\mathbb{R}^n)$ . For any  $p_0 \in (0, \min\{\frac{n}{n+\alpha}, p_-\})$  fixed, we put  $\frac{1}{q_0} := \frac{1}{p_0} - \frac{\alpha}{n}$ , then  $0 < p_0 \leq q_0 \leq 1$ ,  $q_0 \in (0, q_-)$ ,

$X^{1/p_0}$  and  $Y^{1/q_0}$  are ball Banach function spaces and  $M$  is bounded on  $(Y^{1/q_0})'$ . Since  $\frac{1}{p(\cdot)} - \frac{1}{q(\cdot)} = \frac{\alpha}{n} = \frac{1}{p_0} - \frac{1}{q_0}$ , it follows that  $(Y^{1/q_0})' = ((X^{1/p_0})')^{p_0/q_0}$ . For  $X := L^{p(\cdot)}$ , the assumption (2) holds true for any  $s \in (0, 1]$  and  $\theta \in (0, \min\{s, p_-\})$  (indeed, we apply [21, Lemma 2.4] on  $L^{p(\cdot)/\theta}$ , with  $u = s/\theta$  and  $|f_j|^\theta$  instead of  $f_j$ ); the assumption (3) holds true for any  $r \in (0, \min\{1, p_-\})$  and  $\tilde{q} \in (\max\{1, p_+\}, \infty)$  (this follows by duality, [8, Theorem 1.5] and Remark 9). Similarly, the assumption (3) holds true for  $Y = L^{q(\cdot)}$ . Finally, one has that  $H_X(\mathbb{R}^n) = H^{p(\cdot)}(\mathbb{R}^n)$  and  $H_Y(\mathbb{R}^n) = H^{q(\cdot)}(\mathbb{R}^n)$  are the variable Hardy spaces with exponents  $p(\cdot)$  and  $q(\cdot)$  defined in [21].

Then, Theorem 27 applies on such  $X$  and  $Y$ .

**Mixed-norm Hardy spaces.** Fix  $\vec{p} := (p_1, \dots, p_n) \in (0, \infty)^n$ , then the mixed-norm Lebesgue space  $L^{\vec{p}}(\mathbb{R}^n)$  is defined as the set of all the measurable functions  $f$  such that

$$\|f\|_{\vec{p}} := \left( \int_{\mathbb{R}} \cdots \left[ \int_{\mathbb{R}} |f(x_1, \dots, x_n)|^{p_1} dx_1 \right]^{p_2/p_1} \cdots dx_n \right)^{1/p_n} < \infty.$$

These spaces were introduced and studied by A. Benedek and R. Panzone in [2]. It is easy to check that the couple  $(L^{\vec{p}}(\mathbb{R}^n), \|\cdot\|_{\vec{p}})$  is a ball quasi-Banach function space, being its quasi-norm  $\|\cdot\|_{\vec{p}}$  *absolutely continuous*. For  $\vec{p} \in (1, \infty)^n$ , the space  $L^{\vec{p}}(\mathbb{R}^n)$  is a ball Banach function space with  $(L^{\vec{p}}(\mathbb{R}^n))' = L^{\vec{p}'}(\mathbb{R}^n)$ , where  $\vec{p}'$  denotes the  $n$ -tuple whose components are the conjugate values of the components of  $\vec{p}$ , in this case we write  $\frac{1}{\vec{p}} + \frac{1}{\vec{p}'} = \vec{1}$  (see [2, Theorem 1.a]). In [36, Remark 7.21], it was pointed out that these spaces may not be a Banach function spaces.

Given  $0 \leq \alpha < n$ , and  $\vec{p} \in (0, \infty)^n$  such that  $\sum_{i=1}^n \frac{1}{p_i} \in (\alpha, \infty)$ , we define

$$p_- := \min\{p_1, \dots, p_n\}, \quad p_+ := \max\{p_1, \dots, p_n\}, \quad \vec{q} := \left( \frac{np_1}{n - \alpha p_1}, \dots, \frac{np_n}{n - \alpha p_n} \right),$$

$X := L^{\vec{p}}(\mathbb{R}^n)$  and  $Y := L^{\vec{q}}(\mathbb{R}^n)$ . By [2, p. 302, lines 14-19], for any  $s \in (0, p_-)$ , the space  $L^{\vec{p}}(\mathbb{R}^n)$  is *strictly  $s$ -convex*, with  $(L^{\vec{p}})^{1/s} = L^{\vec{p}/s}$ . Now, for any  $p_0 \in (0, \min\{\frac{n}{n+\alpha}, p_-\})$ , we put  $\frac{1}{q_0} := \frac{1}{p_0} - \frac{\alpha}{n}$ , then  $0 < p_0 \leq q_0 \leq 1$ ,  $q_0 \in (0, q_-)$ ,  $X^{1/p_0}$  and  $Y^{1/q_0}$  are ball Banach function spaces such that  $(Y^{1/q_0})' = ((X^{1/p_0})')^{p_0/q_0}$ , and  $M$  is bounded on  $(Y^{1/q_0})'$  (see [16, Lemma 3.5]). For  $X := L^{\vec{p}}(\mathbb{R}^n)$ , the assumption (2) holds true for any  $s \in (0, 1]$  and  $\theta \in (0, \min\{s, p_-\})$  (apply [16, Lemma 3.7] on  $L^{\vec{p}/\theta}$ , with  $u = s/\theta$  and  $|f_j|^\theta$  instead of  $f_j$ ). Now, by duality, [16, Lemma 3.5], and Remark 9, the assumption (3) holds true for  $r \in (0, \min\{1, p_-\})$  and  $\tilde{q} \in (\max\{1, p_+\}, \infty)$ . Similarly, the assumption (3) holds true for  $Y = L^{\vec{q}}(\mathbb{R}^n)$ . Finally,  $H_X(\mathbb{R}^n) = H^{\vec{p}}(\mathbb{R}^n)$  and  $H_Y(\mathbb{R}^n) = H^{\vec{q}}(\mathbb{R}^n)$  are the mixed-norm Hardy spaces defined in [16], considering there  $\vec{a} = \vec{1}$ .

So, Theorem 27 applies on such  $X$  and  $Y$ .

**Hardy-Lorentz spaces.** Given  $0 < p, q < \infty$ , the Lorentz space  $L^{p,q}(\mathbb{R}^n)$  is defined as the collection of all the measurable function  $f$  such that

$$\|f\|_{L^{p,q}} := \left( \int_0^\infty (t^{1/p} f^*(t))^q \frac{dt}{t} \right)^{1/q} < \infty,$$

where  $f^*$ , the decreasing rearrangement function of  $f$ , is defined by setting, for any  $t \in [0, \infty)$ ,  $f^*(t) := \inf\{s > 0 : |\{x : |f(x)| > s\}| \leq t\}$ . By [27, Section 7.3], the couple  $(L^{p,q}(\mathbb{R}^n), \|\cdot\|_{L^{p,q}})$  is a ball quasi-Banach function space, whose quasi-norm  $\|\cdot\|_{L^{p,q}}$  is *absolutely continuous* and satisfies  $\| |g|^r \|_{L^{p,q}} = \|g\|_{L^{pr,qr}}^r$  for all  $0 < p, q, r < \infty$ . For any  $s \in (0, \min\{p, q\})$ , by [3, Theorem 4.6], the space  $(L^{p,q})^{1/s} = L^{p/s, q/s}$  is a Banach function space and so  $L^{p,q}$  is *strictly s-convex* (see [3, Theorem 1.6]). When  $1 < p, q < \infty$ ,  $(L^{p,q})' = L^{p',q'}$ , where  $\frac{1}{p} + \frac{1}{p'} = 1$  and  $\frac{1}{q} + \frac{1}{q'} = 1$  (see [3, Theorem 4.7]).

Given  $0 \leq \alpha < n$  and  $0 < p, q < \frac{n}{\alpha}$ , we put  $\frac{1}{u} := \frac{1}{p} - \frac{\alpha}{n}$  and  $\frac{1}{v} := \frac{1}{q} - \frac{\alpha}{n}$ . Now, we consider  $X := L^{p,q}(\mathbb{R}^n)$  and  $Y := L^{u,v}(\mathbb{R}^n)$ . For any  $p_0 \in (0, \min\{\frac{n}{n+\alpha}, p, q\})$ , we put  $\frac{1}{q_0} := \frac{1}{p_0} - \frac{\alpha}{n}$ , then  $0 < p_0 \leq q_0 \leq 1$ ,  $q_0 \in (0, \min\{u, v\})$ ,  $X^{1/p_0}$  and  $Y^{1/q_0}$  are ball Banach function spaces such that  $M$  is bounded on  $(Y^{1/q_0})'$  (see ). Since  $\frac{1}{p} - \frac{1}{u} = \frac{1}{p_0} - \frac{1}{q_0} = \frac{1}{q} - \frac{1}{v}$ , we have  $(Y^{1/q_0})' = ((X^{1/p_0})')^{p_0/q_0}$ . For  $X := L^{p,q}(\mathbb{R}^n)$ , the assumption (2) holds true for any  $s \in (0, 1]$  and  $\theta \in (0, \min\{s, p, q\})$  (apply conveniently [11, Theorem 2.3 (iii)]). Now, by duality, [11, Theorem 2.3 (iii)] and Remark 9, the assumption (3) holds true for any  $r \in (0, \min\{1, p, q\})$  and  $\tilde{q} \in (\max\{1, p, q\}, \infty)$ . Similarly, the assumption (3) holds true for  $Y = L^{u,v}(\mathbb{R}^n)$ . Finally,  $H_X(\mathbb{R}^n) = H^{p,q}(\mathbb{R}^n)$  and  $H_Y(\mathbb{R}^n) = H^{u,v}(\mathbb{R}^n)$  are the Hardy-Lorentz spaces defined in [1].

Then, Theorem 27 applies on such  $X$  and  $Y$ .

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