

A note on time-uniform concentration inequality for matrix products

Tuan Pham

Department of Statistics and Data Science, University of Texas, Austin

tuan.pham@utexas.edu

Alessandro Rinaldo

Department of Statistics and Data Science, University of Texas, Austin

alessandro.rinaldo@austin.utexas.edu

Abstract

This short note contains a simple argument that allows us to go from fixed-time to any-time bound for the concentration of matrix products. The result presented here is motivated by the analysis of the Oja's algorithms.

1 Introduction

Let X_1, X_2, \dots, X_n be a sequence of i.i.d. positive-semidefinite (PSD) random matrices with $\mathbb{E}X_1 = \Sigma$. From now on, unless stated otherwise, $\|\cdot\|$ denotes the operator norm of matrices. The goal of this short note is to prove a time-uniform concentration inequality for the matrix product

$$\mathbf{Z}_n := \prod_{i=n}^1 (\mathbf{I}_d + \eta_i X_i).$$

In other words, we want a bound of the form

$$\mathbb{P}(\|\mathbf{Z}_n - \mathbb{E}\mathbf{Z}_n\| \geq t(\delta, n)) \leq \delta \tag{1}$$

for some threshold $t(\delta, n)$ depending on n and δ .

The fixed-time concentration analog of (1) has been studied extensively in the literature. The first result establishing a concentration inequality for \mathbf{Z}_n appears in [1, 3], where the authors derive such an inequality under the assumption that $\|X_i - \Sigma\|$ is bounded. Their argument is based on expanding the matrix product and partitioning it into blocks of independent terms, followed by a union bound. Subsequent improvements were developed in [2] (see also the references therein), where the authors exploit the uniform smoothness of the Schatten p -norms.

The goal of this note is to provide an any-time bound for \mathbf{Z}_n using a simple argument based on submartingale techniques, albeit at the cost of an additional sub-optimal term of order $\mathbb{E}\mathbf{Z}_n$.

2 Settings and results

Define

$$\begin{aligned}\mathbf{E}_n &:= \mathbb{E}\mathbf{Z}_n; \\ m_i &:= \|\mathbf{I}_d + \eta_i \boldsymbol{\Sigma}\|; \\ M_k &:= \prod_{i=1}^k m_i.\end{aligned}$$

and

$$\begin{aligned}\sigma_i &:= \eta_i \cdot \|\mathbf{X}_i - \boldsymbol{\Sigma}\|; \\ V_k &:= \sum_{i=1}^k \sigma_i^2.\end{aligned}$$

Suppose the step sizes η_i 's are chosen such that

$$M_n \sqrt{2V_n \log(d/\delta)} \leq 1. \quad (2)$$

for some $\delta > 0$. Our main result can be stated as follows.

Theorem 1. *For all $\delta > 0, \eta > 1$ and all functions $h : \mathbb{N} \rightarrow \mathbb{R}_+$ such that*

$$\sum_{k=0}^{\infty} \frac{1}{h(k)} \leq 1,$$

we have

$$\mathbb{P}\left(\exists n: \|\mathbf{Z}_n - \mathbf{E}_n\| \geq t\left(\frac{\delta}{h(k_n)}, \lfloor \eta^{k_n+1} \rfloor\right)\right) \leq \delta$$

whenever (2) holds, where

$$k_n := \min\{k = 0, 1, \dots, : \lceil \eta^k \rceil \leq n \leq \lfloor \eta^{k+1} \rfloor\}; \quad (3)$$

$$t(\delta, n) := e\|\mathbf{E}_n\| M_n \sqrt{2V_n \log(d/\delta)}. \quad (4)$$

Proof of Theorem 1. Unlike the existing results that exploit supermartingale arguments, we will use a submartingale argument instead. The argument presented below is very simple, but is sub-optimal by a factor of order $\|\mathbf{E}_n\|$. Let us split the proof into a few steps.

Step 1: Constructing the submartingale. Observe that

$$\mathbf{Y}_n = \mathbf{E}_n^{-1} \mathbf{Z}_n - \mathbf{I}_d$$

is a martingale with respect to the natural filtration.

By using the convexity of the map $X \rightarrow \|X\|$, we deduce that the process

$$\{\|\mathbf{Y}_k\|; k \geq 1\}$$

is a submartingale.

Moreover, we have the two-sided bound

$$\|\mathbf{Y}_k\| \leq \|\mathbf{Z}_k - \mathbf{E}_k\| \leq \|\mathbf{E}_k\| \times \|\mathbf{Y}_k\|.$$

for all $k \geq 1$.

The first inequality in the above is true since

$$\|\mathbf{Y}_k\| \leq \|\mathbf{E}_k^{-1}\| \times \|\mathbf{Z}_k - \mathbf{E}_k\| \leq \underbrace{\prod_{i=1}^k \|(\mathbf{I}_d + \eta_i \boldsymbol{\Sigma})^{-1}\|}_{\leq 1} \times \|\mathbf{Z}_k - \mathbf{E}_k\|.$$

The second inequality follows from the simple observation that

$$\|\mathbf{Z}_k - \mathbf{E}_k\| = \|\mathbf{E}_k \mathbf{Y}_k\| \leq \|\mathbf{E}_k\| \times \|\mathbf{Y}_k\|.$$

Thus,

$$\mathbb{P}(\forall n \geq 1 : \|\mathbf{Z}_n - \mathbf{E}_n\| \geq r_n) \leq \mathbb{P}\left(\forall n \geq 1 : \|\mathbf{Y}_n\| \geq \frac{r_n}{\|\mathbf{E}_n\|}\right). \quad (5)$$

Step 2: Bounding the submartingale. To bound the submartingale in the RHS of (5), let us first divide \mathbb{N} into intervals of the form

$$[\lfloor \eta^k \rfloor, \lfloor \eta^{k+1} \rfloor]$$

for $\eta > 1$ as specified in the statement of Theorem 1.

With k_n and $t(\delta, n)$ as in (3) and (4), respectively, we have

$$\begin{aligned} \mathbb{P}\left(\exists n : \|\mathbf{Y}_n\| \geq t\left(\frac{\delta}{h(k_n)}, \lfloor \eta^{k_n+1} \rfloor\right)\right) &\leq \mathbb{P}\left(\exists k, n : \lceil \eta^k \rceil \leq n \leq \lfloor \eta^{k+1} \rfloor, \|\mathbf{Y}_n\| \geq t\left(\frac{\delta}{h(k_n)}, \lfloor \eta^{k_n+1} \rfloor\right)\right) \\ &= \mathbb{P}\left(\exists k, n : \lceil \eta^k \rceil \leq n \leq \lfloor \eta^{k+1} \rfloor, \|\mathbf{Y}_n\| \geq t\left(\frac{\delta}{h(k)}, \lfloor \eta^{k+1} \rfloor\right)\right) \\ &\leq \sum_{k=0}^{\infty} \mathbb{P}\left(\max_{n \leq \lfloor \eta^{k+1} \rfloor} \|\mathbf{Y}_n\| \geq t\left(\frac{\delta}{h(k)}, \lfloor \eta^{k+1} \rfloor\right)\right). \end{aligned}$$

For any $n \geq 1$ and $t > 0$, by using the Doob's maximal inequality, we have

$$\mathbb{P}\left(\max_{k \leq n} \|\mathbf{Y}_k\| \geq u\right) \leq \min_{p \geq 1} \frac{\mathbb{E}[\max_{k \leq n} \|\mathbf{Y}_k\|^p]}{u^p} \leq \min_{p \geq 1} \frac{\mathbb{E}[\|\mathbf{Y}_n\|^p]}{u^p} \leq \min_{p \geq 1} \frac{\mathbb{E}[\|\mathbf{Z}_n - \mathbf{E}_n\|^p]}{u^p}.$$

To bound the last term, one can follow the proof of Corollary 5.6 in [2] to deduce that

$$\mathbb{P}\left(\max_{k \leq n} \|\mathbf{Y}_k\| \geq u M_n\right) \leq \max\{d, e\} \times \exp\left(-\frac{u^2}{2e^2 V_n}\right), \quad \text{for } u \in [0, e].$$

Thus, by choosing t as in (4), we have

$$\mathbb{P}\left(\max_{n \leq \lfloor \eta^{k+1} \rfloor} \|\mathbf{Y}_n\| \geq t\left(\frac{\delta}{h(k)}, \lfloor \eta^{k+1} \rfloor\right)\right) \leq \frac{\delta}{h(k)}.$$

The proof is completed by employing the fact that $\sum_{k \geq 1} 1/h(k) \leq 1$.

Remark 1. The boundary chosen in Theorem 1 is piece-wise constant over geometrically increasing epochs. One can easily turn it into a smooth boundary by choosing

$$f(n) = e \|\mathbf{E}_n\|_{\text{op}} M_n \sqrt{V_n (\log(d) + \log(\zeta(\alpha)/\delta) + \alpha \log(\log_\eta(n) + 1))}$$

where $\min\{\eta, \alpha\} > 1$ and $h(k) = (k+1)^\alpha \zeta(\alpha)$, with $\zeta(\cdot)$ the Riemann zeta function.

Under the assumption (2), the thresholds $\{f(n); n \geq 1\}$ satisfies

$$\mathbb{P}(\exists n \geq 1 : \|\mathbf{Z}_n - \mathbf{E}_n\| \geq f(n)) \leq \delta.$$

Remark 2. Regarding the tightness Theorem 1, compared to the point-wise (in n) concentration bound of Huang et al. (2021), which is of order $eM_n \sqrt{2V_n \log(d/\delta)}$, we have picked up an additional term, namely $\|E_n\|_{\text{op}}$. For our problem, this can be replaced by M_n .

To exemplify, let's consider the bound at fixed time n . Set $\lambda_{\max}(\Sigma) = \mu$ and assume bounded data within an Euclidean ball of radius L :

$$\|X_i - \Sigma\| \leq L.$$

Also, set $\eta_i = 1/n$ for all $i \leq n$. Then, the bound of Huang et al. (2021) is of order

$$Le^\mu \sqrt{\frac{\log(d/\delta)}{n}}$$

while our uniform bound, in addition to the extra iterated log terms, will be of order

$$Le^{2\mu} \sqrt{\frac{\log(d/\delta)}{n}}.$$

3 References

- [1] Amelia Henriksen and Rachel Ward. Concentration inequalities for random matrix products. *Linear Algebra and its Applications*, 594:81–94, 2020.
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- [3] Tarun Kathuria, Satyaki Mukherjee, and Nikhil Srivastava. On concentration inequalities for random matrix products. *arXiv preprint arXiv:2003.06319*, 2020.