Differential Privacy with Dependent Data

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November 2025

Abstract

Dependent data underlies many statistical studies in the social and health sciences, which often involve sensitive or private information. Differential privacy (DP) and in particular *user-level* DP provide a natural formalization of privacy requirements for processing dependent data where each individual provides multiple observations to the dataset. However, dependence introduced, e.g., through repeated measurements challenges the existing statistical theory under DP-constraints. In i.i.d. settings, noisy Winsorized mean estimators have been shown to be minimax optimal for standard (*item-level*) and *user-level* DP estimation of a mean $\mu \in \mathbb{R}^d$. Yet, their behavior on potentially dependent observations has not previously been studied. We fill this gap and show that Winsorized mean estimators can also be used under dependence for bounded and unbounded data, and can lead to asymptotic and finite sample guarantees that resemble their i.i.d. counterparts under a weak notion of dependence. For this, we formalize dependence via log-Sobolev inequalities on the joint distribution of observations. This enables us to adapt the stable histogram by Karwa and Vadhan (2018) to a non-i.i.d. setting, which we then use to estimate the private projection intervals of the Winsorized estimator. The resulting guarantees for our item-level mean estimator extend to *user-level* mean estimation and transfer to the local model via a randomized response histogram. Using the mean estimators as building blocks, we provide extensions to random effects models, longitudinal linear regression and nonparametric regression. Therefore, our work constitutes a first step towards a systematic study of DP for dependent data.

1 Introduction

Differential privacy (DP) tools have been deployed at scale in numerous applications in industry and government agencies (Erlingsson et al., 2014; Ding et al., 2017; Tang et al., 2017; Garfinkel et al., 2019). The standard DP framework seeks to release statistics while protecting single data points as it assumes every individual contributes a single data point to a dataset of size n. We will follow the literature and call this standard approach *item-level* DP. In this framework, privacy is achieved by releasing randomized outputs that are calibrated in a way which guarantees that the effect that any single data point can have in the computation of this output is hidden by the randomization (Dwork et al., 2006; Dwork and Roth, 2014).

A recent line of work called *user-level* differential privacy (uDP) has studied the setting where each person/user contributes multiple observations to a dataset (Liu et al., 2020; Levy et al., 2021; Narayanan et al., 2022; Acharya et al., 2023; Bassily and Sun, 2023; Ghazi et al., 2023; Asi and Liu, 2024). For simplicity, we assume that every user contributes T data points. Naturally, the goal now is to protect all T points that a user contributes to the dataset. The challenge is that standard DP

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techniques protect only individual data points and either yield privacy guarantees that degrade as the users contribute more points or get the right privacy guarantee by adding too much noise if the algorithms are built using the group property of DP (Dwork and Roth, 2014, Theorem 2.2).

In this work, we introduce DP tools for dependent data and are particularly interested in longitudinal data, also commonly called panel data, consisting of multiple dependent observations collected from the same individuals over some time frame. Longitudinal data are immensely important in the social sciences and in medical applications (Diggle, 2002; Baltagi, 2008; Fitzmaurice et al., 2012; Hsiao, 2022). While user-level DP seems to be the natural definition of privacy for longitudinal data, this connection has not been exploited in the literature. In fact, there is relatively little work on modeling dependent data under differential privacy constraints.

We were mainly inspired by Karwa and Vadhan (2018), as their algorithm serves as building block for all our procedures, as well as recent work in the emerging field of uDP, in particular Levy et al. (2021) and Kent et al. (2024).

Our main contributions can be summarized in the following key points:

(a) DP for dependent and unbounded data: our work seems to be the first to study multiple DP algorithms on dependent observations. In particular, we introduce *log-Sobolev dependence*, a notion of dependence using log-Sobolev inequalities on the joint distribution of observations that replaces typical i.i.d. assumptions. The building block of our methodology is a Winsorized mean estimation algorithm inspired by the one of Karwa and Vadhan (2018), developed for Gaussian i.i.d. data. While additionally handling dependence, our algorithm inherits two appealing features from their work: it enable estimation with unbounded observations and does not require any knowledge about the unknown mean.

(b) DP for longitudinal data: we introduce various algorithms for uDP estimation for dependent and in particular longitudinal data, allowing for dependence across both users and time. This is a significant departure from the overwhelming majority of the theoretical analysis of DP algorithms which are designed and analyzed leveraging i.i.d. assumptions, even in uDP settings (Levy et al., 2021; Kent et al., 2024; Agarwal et al., 2025). Our algorithms are shown to lead to optimal finite-sample error rates under log-Sobolev dependence. Thereby, the condition proves to be general enough to cover a wide range of interesting statistical models that had not been studied previously in the literature, including nonparametric regression, simple random effects models and linear regression for longitudinal data with dependent errors.

(c) Histogram learning: we extend the analysis of the histogram estimator by Vadhan (2017) to log-Sobolev dependent data using a Dvoretzky-Kiefer-Wolfowitz-type inequality by Bobkov and Götze (2010). This is a critical intermediate result in the construction of our mean estimation, because the private histogram is called as a first step when crudely estimating the midpoint of a private projection interval of length $O(\log n)$ that is used by our Winsorized mean estimator. This slight adjustment significantly extends the scope of the techniques introduced in Vadhan (2017) beyond i.i.d. Gaussians and enables dependence even in the item-level setting.

(d) Item-level and user-level DP: a conceptually interesting contribution is to highlight how all the existing uDP algorithms are built on a known item-level DP algorithm. This connection allows us to give more perspective on the existing in-expectation minimax DP and uDP lower bounds. To do so, we provide an in-expectation analysis for projection estimators à la Karwa-Vadhan that seems to be novel even for i.i.d. Gaussian data. The bound suggests that an impossibility result in uDP, saying that learning is impossible if only the number of observations per users $T \to \infty$ (Levy et al., 2021, Theorem 8), is connected to the impossibility of learning in the item-level setting when the variance tends to 0 too quickly. We thereby identify a disconnect

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between existing item- and user-level lower bounds (Cai et al., 2021; Levy et al., 2021).

(e) Local DP: while the focus of our presentation is on the central model of DP that assumes the existence of a trusted data curator, we extend all our results to the local model of DP where the central server is not trusted and the privacy mechanism is enforced in the data collection phase (Kasiviswanathan et al., 2011; Duchi et al., 2018). In particular, we show how replacing the histogram in our central DP algorithms with a local analog leads to near-optimal estimators under the local model while keeping all the desirable properties that we highlighted in the previous points. This means, we can again allow for unbounded observations, dependence between users and among their observations in a user-level local DP (uLDP) framework.

1.1 Related Work

Private estimation of location parameters like mean and median is a core statistical question frequently investigated in the DP literature. The first DP mean estimator can be traced back to an application of noisy sums in Dwork et al. (2006) and asymptotic guarantees for private trimmed mean and median estimators were obtained in Dwork and Lei (2009). Smith (2011) seem to have been the first work to propose and study the asymptotic properties of a noisy two-step Winsorized mean estimator similar to the one we consider, i.e., one that first crudely estimates the mean, projects the data onto an interval around this estimate and then adds noise for privacy calibrated to the resulting finite sensitivity. Lower-bounds for mean estimation were derived in Bun et al. (2013); Steinke and Ullman (2017); Foygel Barber and Duchi (2014); Steinke and Ullman (2017); Bun and Steinke (2019); Cai et al. (2021). In particular, Cai et al. (2021) provided in-expectation minimax upper and lower bounds matching the rate due to Steinke and Ullman (2017), for a Winsorized mean estimator lacking a data-driven projection interval. In contrast, Karwa and Vadhan (2018) started a line of work on in-probability guarantees for Winsorized mean estimators that neither impose boundedness assumptions on observations nor on the mean. Their algorithm is based on a (ε, δ)-DP "stable" histogram estimator that is used to find a private confidence interval for the mean of univariate Gaussians. Subsequently, Kamath et al. (2019) leveraged this approach to multivariate Gaussians with unknown covariance matrix and Kamath et al. (2020) extended this approach to heavy-tailed distributions.

We note that there are also numerous other works studying the problem of private mean and median estimation, in particular aiming to avoid assuming a bounded sample space, and in many cases making connections to robustness in the statistics community (Avella-Medina and Brunel, 2020; Avella-Medina, 2020, 2021; Avella-Medina et al., 2023; Li et al., 2023; Yu et al., 2024; Ramsay et al., 2022) as well as in computer science venues (Tzamos et al., 2020; Liu et al., 2021, 2022; Hopkins et al., 2023; Alabi et al., 2023; Chhor and Sentenac, 2023). These constructions are not as closely related to our work.

User-level DP was first introduced by McMahan et al. (2017) to ensure privacy in the context of federated training of language models. Subsequently, the definition spread to other work in this context (Wang et al. (2019); Augenstein et al. (2020)) and user-level private data-aggregation in SQL-databases (Wilson et al. (2020)). Statistical results investigating the interplay between the number of users n, the number of observations per user T and the magnitude of the noise necessary for user-level privacy in the context of empirical risk minimization were obtained in Amin et al. (2019); Epasto et al. (2020); Levy et al. (2021); Narayanan et al. (2022); Kent et al. (2024). Recently, Agarwal et al. (2025) and Zhao et al. (2024) derived the first user-level mean estimators for independent users with independent unbounded observations in the central model.

The problem of estimation under local differential privacy has been explored in a wide range of

statistical problems including mean estimation, density estimation, nonparametric regression and hypothesis testing, just to cite a few Wasserman and Zhou (2010); Duchi et al. (2018); Gaboardi and Rogers (2018); Butucea et al. (2020); Berrett and Butucea (2020); Berrett et al. (2021); Sart (2023); Pensia et al. (2024). In particular user-level local differential privacy has recently been studied in Girgis et al. (2022); Acharya et al. (2023); Kent et al. (2024).

Finally, we note that there is some incipient literature on differential privacy with special temporal structure. This includes data problems such as change point detection (Zhang et al., 2021; Berrett and Yu, 2021; Li et al., 2022) and multi-armed bandits (Mishra and Thakurta, 2015; Sajed and Sheffet, 2019; Hu and Hegde, 2022; Ou et al., 2024) which are typically studied under item-level DP and independent observation assumptions. There is also some initial work on time series with DP that again has tackled the problem from an item-level DP perspective. For example, Zhang et al. (2022) considered a parametric approach with sequential AR(1) data, Amorino et al. (2025) studied diffusion processes and Kroll (2024); Butucea et al. (2025) considered the problem of nonparametric spectral density estimation.

2   Preliminaries

We first present notation that will be used throughout this work and give basic background on differential privacy and log-Sobolev inequalities. At the end of this section we introduce the main dependence assumption that we use to model dependent and especially longitudinal data.

2.1   Notation

We define $[\mu \pm b] := [\mu - b, \mu + b]$, $[\mu \pm b) := [\mu - b, \mu + b)$ and $(\mu \pm b] := (\mu - b, \mu + b]$. Moreover, $[n] = \{1, 2, \ldots, n\}$, $\mathbb{R}_{\geq} = [0, \infty)$ and $\mathbf{1}\{A\}$ is the indicator function for a set A. For the minimum and maximum of $a, b \in \mathbb{R}$ we write $a \wedge b$ and $a \vee b$. When a sequence of random variables $X_1, X_1, X_2, \ldots$ converges in probability to a random variable X_*, we write $X_n \xrightarrow{p} X_*$. When $\xi_1, \ldots, \xi_d \sim \mathrm{Lap}(l, s)$ i.i.d., we say $\Xi := (\xi_1, \ldots, \xi_d) \sim \mathrm{Lap}(l\mathbf{1}_d, sI_d)$. Here, let $\mathbf{1}_d$ be the all ones vector in $\mathbb{R}^d$. For matrices $A, B \in \mathbb{R}^{d \times d}$ we denote their positive definite and positive semi-definite order as $A \prec B$ and $A \preceq B$ and refer to them as Loewner orderings. Further, $\|A\|_{op}$ and $\|A\|_F$ are the operator and Frobenius norm of A. For a matrix $M = [M_1^\top, \ldots, M_n^\top]^\top \in \mathbb{R}^{n \times d}$ we denote its column vectors by $M_{.j}$ for $j \in [d]$. We write the L_2-projection of a vector $x \in \mathbb{R}^d$ onto a convex set C as $\Pi_C(x) := \arg\min_{y \in R} \|x - y\|_2$. For two sequences $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ we use the following order symbols: $b_n = O(a_n)$ or $a_n \lesssim b_n$ if $\exists C > 0$ and $n_0 \in \mathbb{N}$ such that $\forall n \geq n_0, |\frac{b_n}{a_n}| \leq C$; if this holds up to log-factors we write $\tilde{O}$ or $a_n \lesssim b_n$; $b_n = o(a_n)$ if $\lim_{n \to \infty} |\frac{b_n}{a_n}| = 0$. We write $a_n \asymp b_n$ if $a_n \lesssim b_n$ and $b_n \lesssim a_n$. For $x, y \in \mathcal{X}^n$, their Hamming distance is $d_H(x, y) := \sum_{i=1}^n \mathbf{1}\{x_i \neq y_i\}$, i.e., the number of elements which differ. Throughout the text we let $\kappa, c > 0$ denote absolute constants.

2.2   Differential Privacy

There are several competing notions of DP (See Mironov (2017); Dong et al. (2022); Dwork and Rothblum (2016); Bun and Steinke (2016); Dwork et al. (2006)), but we will work with the most common notion (ε, δ)-DP introduced in Dwork et al. (2006). In its definition below, we denote the Borel σ-algebra on $\mathbb{R}^p$ by $\mathcal{B}(\mathbb{R}^p)$.

Definition 2.1. (ε, δ)-*Differential Privacy (Central Model). Let $\varepsilon, \delta > 0$ and let $\mathcal{A} : \mathcal{X}^n \to \mathbb{R}^d$ be a randomized algorithm, mapping $x \in \mathcal{X}^n$ to a $\mathcal{B}(\mathbb{R}^d)$-measurable random variable $\mathcal{A}(x)$. Then, $\mathcal{A}$ is*

(ε, δ)-DP if for all $x, y \in \mathcal{X}^n$ s.t. $d_H(x, y) \leq 1$ we have

$$\mathbb{P}\left(\mathcal{A}(x) \in S\right) \leq e^\varepsilon \cdot \mathbb{P}\left(\mathcal{A}(y) \in S\right) + \delta, \quad \text{for all } S \in \mathcal{B}(\mathbb{R}^p).$$

Furthermore, we say that

1. $\mathcal{A}$ is item-level-DP if $x_i, y_i \in \mathbb{R}^{1 \times d}$ for all $i \in [n]$,
2. $\mathcal{A}$ is user-level-DP if $x_i, y_i \in \mathbb{R}^{T \times d}$ for all $i \in [n]$.

Remark 2.2. *The parameters (ε, δ) control the noise in the randomized algorithm and are commonly referred to as privacy budget parameters. If an estimator is (ε, δ)-user-level DP, we write (ε, δ)-uDP.*

The so-called Laplace mechanism described in Algorithm 1 is one of the most common ways to randomize an estimator to make it $(\varepsilon, 0)$-DP.

Theorem 2.3. *(Theorem 3.6, Dwork and Roth (2014)). Algorithm 1 is $(\varepsilon, 0)$-DP.*

Algorithm 1 LaplaceMechanism(f, x, ε):

Require: $f : \mathcal{X}^n \to \mathbb{R}^d$, $x \in \mathcal{X}^n \subset \mathbb{R}^n$, $\varepsilon > 0$
 1: $D \leftarrow \{(x, y) : x, y \in \mathcal{X}^n, d_H(x, y) \leq 1\}$
 2: $\Delta \leftarrow \max_{(x,y) \in D} \|f(x) - f(y)\|_1$
 3: **return** $f(x) + \Xi$ where $\Xi \sim \text{Lap}(0, \frac{\Delta}{\varepsilon} I_d)$

The Laplace mechanism above and further mechanisms presented in Dwork and Roth (2014) are the foundation of DP. They can be combined to form more complex DP algorithms by composing the output of simple DP algorithms. The effective privacy budget of an algorithm that outputs a composition of various DP algorithms can be quantified by the composition theorems below.

Theorem 2.4. *(Corollary 3.21, Corollary B.2, Dwork and Roth (2014)). Let $\mathcal{A}$ be a privacy mechanism that composes d mechanisms that are each $(\varepsilon', \delta/d)$-DP.*

(Basic Composition) If $\varepsilon, \delta > 0$ and $\varepsilon' = \varepsilon/d$, then $\mathcal{A}$ is (ε, δ)-DP.
(Advanced Composition) If $\varepsilon, \delta, \varrho \in (0, 1)$ and $\varepsilon' = \varepsilon/\sqrt{8d \log(1/\varrho)}$, then $\mathcal{A}$ is $(\varepsilon, \delta + \varrho)$-DP.

2.3 Log-Sobolev Inequalities

Our work uses log-Sobolev inequalities to control dependence along the columns of the data matrix $X^n \in \mathbb{R}^{n \times d}$. Log-Sobolev inequalities are functional inequalities extensively studied in the fields of optimal transport, Markov diffusions and concentration of measure (cf. Villani (2009); Bakry et al. (2014); Ledoux (2001); Boucheron et al. (2013)). Log-Sobolev inequalities impose sub-Gaussian tail-decay and thus imply concentration as stated in Theorem 2.6. Moreover, a log-Sobolev inequality controls the variance of any test function $f : \mathbb{R}^d \to \mathbb{R}$, because it implies a Poincaré inequality (Proposition 5.1.3, Bakry et al. (2014)). To be more precise in the treatment of log-Sobolev inequalities to follow, we define them below.

Definition 2.5. *(Log-Sobolev Inequality). Let ν be a probability measure on $\mathbb{R}^d$. For an integrable function $f : \mathbb{R}^d \to \mathbb{R}_{\geq}$ s.t. $\int_{\mathbb{R}^d} f |\log(f)| d\nu < \infty$ define the entropy of f as the following functional:*

$$\text{Ent}_\nu(f) := \int_{\mathbb{R}^d} f \log(f) d\nu - \left(\int_{\mathbb{R}^d} f d\nu\right) \log\left(\int_{\mathbb{R}^d} f d\nu\right).$$

Then, the probability measure ν fulfills a log-Sobolev inequality with constant $\rho > 0$ and we write $\nu \in \text{LSI}(\rho)$ if for all $f : \mathbb{R}^d \to \mathbb{R}$ s.t. $\text{Ent}_\nu(f^2) < \infty$ and $\nabla f \in L^2(\nu)$ we have

$$\text{Ent}_\nu(f^2) \leq 2\rho \cdot \mathbb{E}[\|\nabla f(X)\|_2^2].$$

A generic approach to establish a log-Sobolev inequality is through its relation to the curvature of the density of a probability measure with respect to the Lebesgue measure. Specifically, strong log-concavity immediately implies a log-Sobolev inequality through the Bakry-Emery Criterion that can be found in Appendix A; see Theorem A.1.

The following Lipschitz concentration inequality will be the main probabilistic workhorse for the derivation of our algorithms under dependence.

Theorem 2.6. *(Log-Sobolev Lipschitz Concentration; Theorem 5.3, Ledoux (2001)). Let the probability measure ν on $\mathbb{R}^d$ have log-Sobolev constant ρ. Further, let $f : \mathbb{R}^n \to \mathbb{R}$ be an L-Lipschitz function. Then, when $X \sim \nu$ and $t \geq 0$ we have:*

$$\mathbb{P}\left(|f(X) - \mathbb{E}[f(X)]| \geq t\right) \leq 2 \exp\left(-\frac{t^2}{2\rho L^2}\right).$$

2.4 Log-Sobolev Dependence

To the best of our knowledge, all the existing theoretical analysis of item-level differentially private estimators has relied on simplifying i.i.d. assumptions similar or stronger to the one we state next.

Assumption 2.7. *The matrix $X^n = (X_1^\top, \ldots, X_n^\top)^\top \in \mathbb{R}^{n \times d}$ has i.i.d rows.*

Observe that Assumption 2.7 does not impose a known bounded domain for the data as is frequently the case in the differential privacy literature. We will see that our estimators will allow for this relaxed i.i.d. assumption and show that slight modifications of existing methods can be shown to work well for unbounded data domains. This is an interesting side product of our analysis. However, our main contribution will be to explicitly allow for dependent data.

We introduce a *log-Sobolev dependence* condition on the data matrix X^n that imposes log-Sobolev inequality constraints to model the dependence of the entries of X^n. To capture the dependence between n observations across d dimensions we could require that $X^n \sim \mathcal{D}$ for some joint probability measure $\mathcal{D}$ that fulfills a log-Sobolev inequality. However, this would be an unnecessarily rigid condition for our purposes as we only need to control the variance and concentration of the column vectors $X_{\cdot 1}^n, \ldots, X_{\cdot d}^n$ for the design of our algorithms for histogram learning and mean estimation. This motivates our formulation of Assumption 2.8.

Assumption 2.8. *(Log-Sobolev dependence). A matrix $X^n \in \mathbb{R}^{n \times d}$ fulfills this assumption with constants (ρ, M) if for every column $X_{\cdot j}^n$:*

(a) $X_{\cdot j}^n \sim \mathcal{D}_j$ *on $\mathbb{R}^n$ where $\mathcal{D}_j$ fulfills* LSI(ρ),
(b) *The average marginal cdf $\bar{F}_j(x) := \frac{1}{n} \sum_{i=1}^n \mathbb{P}(X_{ij} \leq x)$ is M-Lipschitz,*

Remark 2.9. *Lipschitzness of $\bar{F}_j$ is implied by X_{ij} having marginal densities bounded by M. When we use constants (ρ, ∞), we specify $M = \infty$ and mean that Condition (b) does not have to hold.*

Assumption 2.8 allows us to use a Dvoretzky-Kiefer-Wolfowitz-type inequality in the analysis of the histogram estimator of Section 3. This is a key subroutine for our main mean estimators in Section 4. Interested readers may find this inequality due to Bobkov and Götze (2010) in Theorem C.2 in the appendix. Assumption 2.8(b) is a technicality required in the proof of Bobkov and Götze (2010)'s result. Besides its role in Theorem C.2, we use Assumption 2.8(a) as a model of dependence within the columns $X_{\cdot j}^n \in \mathbb{R}^n$. Corollary 2.10 shows that imposing it for all columns $X_{\cdot j}^n$ is enough to control each column's covariance and the variance of their mean. The proof can be found in Appendix B.

Corollary 2.10. *For $X^n \in \mathbb{R}^{n \times d}$ fulfilling Assumption 2.8 with constants (ρ, ∞),*

$$\mathrm{Cov}(X_{\cdot j}^n) \preceq \rho I_n, \quad and \quad \mathbb{E}[\|\bar{X}_n - \mathbb{E}[\bar{X}_n]\|_2^2] \leq \frac{d\rho}{n}, \quad where \ \bar{X}_n := \frac{1}{n} X^\top \mathbf{1}_n.$$

When $\rho \asymp 1$, the variance of $\bar{X}_n$ qualitatively behaves as if the $X_1, \ldots, X_n$ were independent and each had variance bounded by ρ. Since $\bar{X}_n$ is unbiased, its MSE vanishes as $n \to \infty$ as long as $\rho = o(n)$. The next lemma shows that Assumption 2.8(a) implies sub-Gaussian concentration of the mean $\bar{X}_n \in \mathbb{R}^d$ over the n observations. The result is well know but we give a proof for completeness in Appendix B.

Lemma 2.11. *Let $X^n \in \mathbb{R}^{n \times d}$ fulfill Assumption 2.8 with constants (ρ, ∞) and $\mathbb{E}[X_i] = \mu$. Then,*

$$\mathbb{P}\left(\left\| \bar{X}_n - \mu \right\|_2 \leq \sqrt{\frac{2d\rho \log(2d/\alpha)}{n}} \right) \geq \mathbb{P}\left(\left\| \bar{X}_n - \mu \right\|_\infty \leq \sqrt{\frac{2\rho \log(2d/\alpha)}{n}} \right) \geq 1 - \alpha.$$

We will repeatedly exploit the L_∞-norm concentration of Lemma 2.11 to derive our multivariate mean estimators. This lemma shows that, for the empirical mean, Assumption 2.8 trades off dependence and concentration via the log-Sobolev constant ρ. If $\rho \asymp 1$, the concentration in Lemma 2.11 resembles that of i.i.d. sub-Gaussian random variables. In turn, if $\rho \asymp n$ there is no effect of averaging and the mean $\bar{X}_n$ behaves like each single X_i.

3 Histogram Estimator

We now turn to histogram learning which is one of the building blocks of our mean estimator. The main feature of our histogram estimator is that it allows for dependent observations with an unbounded domain. The estimator in Algorithm 2 is a stable histogram that estimates the probability mass in possibly infinitely many non-overlapping bins $(B_k)_{k \in \mathbb{Z}}$. It was introduced by Korolova et al. (2009) and first fully described by Bun et al. (2016). To fix ideas, in later sections we will work with bins of the form $B_k = (2hk \pm h]$ for $k \in \mathbb{Z}$ and some given bin width parameter h. In the current section we allow for general non-overlapping bins[1].

Algorithm 2 StableHistogram($X^n, (B_k)_{k \in \mathbb{Z}}, \varepsilon, \delta$):

Require: $X^n \in \mathbb{R}^n$, $\varepsilon, \delta > 0$

1: **for all** $k \in \mathbb{Z}$ **do**
2: $\quad \hat{p}_k \leftarrow \frac{1}{n} \sum_{i=1}^n \mathbf{1}\{X_i \in B_k\}$
3: $\quad \xi_k \leftarrow \xi$ where $\xi \sim \mathrm{Lap}(0, \frac{2}{\varepsilon n})$
4: $\quad \tilde{p}_k \leftarrow \begin{cases} 0 & \text{if } \hat{p}_k = 0 \\ 0 & \text{if } \hat{p}_k + \xi_k < \frac{2}{\varepsilon n} \log(\frac{2}{\delta}) + \frac{1}{n} \\ \hat{p}_k + \xi_k, & \text{otherwise} \end{cases}$
5: **return** $\tilde{\boldsymbol{p}} := (\ldots, \tilde{p}_{-1}, \tilde{p}_0, \tilde{p}_1, \tilde{p}_2, \ldots)$

3.1 Privacy and Utility Guarantees

We note that while the privacy of Algorithm 2 is known in the literature, we provide a proof of this result in the continuous setting for completeness; see Lemma C.1. Indeed, this case is not formally covered in Theorem 7.3.5 Vadhan (2017).

Karwa and Vadhan (2018) seem to have been the first authors to use stable histograms for DP mean estimation. More specifically they use this idea to privately estimate the mean and variance of i.i.d. Gaussian random variables. Their results rely on the following histogram utility guarantee.

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[1]Note that Aliakbarpour et al. (2024, Lemma 2.4) considered an alternative to stable histograms which relies on truncated Laplace noise. We do not explore this approach in this work.

Lemma 3.1. *(Adjusted from Lemma 2.3, Karwa and Vadhan (2018)). Let $\eta, \varepsilon, \delta > 0$. Further, let $X^n \in \mathbb{R}^n$ have i.i.d. entries, a $(B_k)_{k \in \mathbb{Z}}$ be a collection of disjoint bins covering $\mathbb{R}$, and $p_k := \mathbb{P}(X_1 \in B_k)$. Then, the output $\tilde{\boldsymbol{p}}$ of Algorithm 2 satisfies*

$$\mathbb{P}\left(\max_{k \in \mathbb{Z}} |\tilde{p}_k - p_k| \leq \eta \right) \geq 1 - \left(1 + \frac{e^\varepsilon}{\delta} \right) n \exp\left(-\frac{\varepsilon n \eta}{4} \right) - \exp\left(-\frac{n\eta^2}{2} \right).$$

Our analysis of the stability-based histogram estimator generalizes the one by Karwa and Vadhan (2018) to dependent observations. Concretely, we extend their Lemma 2.3 for observations $X^n \in \mathbb{R}^n$ whose joint distribution fulfills a log-Sobolev inequality following essentially their proof strategy. This leads to Lemma 3.2 below that is proven in Appendix C.

Lemma 3.2. *Let $\eta, \varepsilon, \delta > 0$ and the random matrix $X^n \in \mathbb{R}^n$ fulfill Assumption 2.8 with constants (ρ, M). Let $(B_k)_{k \in \mathbb{Z}}$ be a collection of disjoint bins covering $\mathbb{R}$ and $p_k := \frac{1}{n} \sum_{i=1}^n \mathbb{P}(X_i \in B_k)$. Then, the output $\tilde{\boldsymbol{p}}$ of Algorithm 2 satisfies*

$$\mathbb{P}\left(\max_{k \in \mathbb{Z}} |\tilde{p}_k - p_k| \leq \eta \right) \geq 1 - \left(1 + \frac{e^\varepsilon}{\delta} \right) n \exp\left(-\frac{\varepsilon n \eta}{4} \right) - \frac{16}{\eta} \exp\left(-\frac{1}{864} \frac{n\eta^3}{\rho M^2} \right).$$

The critical difference between the proof of Lemma 3.1 and 3.2 is that we replace the Dvoretsky-Kiefer-Wolfowitz (DKW) inequality used by Karwa and Vadhan (2018) with the DKW-type inequality obtained by Bobkov and Götze (2010) which allows for dependent observations. We restate the DKW-type inequality of Bobkov and Götze (2010) in Theorem C.2. It ensures ensures that $\hat{p}_k$ approximates well p_k uniformly over all bins. When computing the private estimates $\tilde{p}_k$ from $\hat{p}_k$, the stability mechanism accumulates at most n errors of the form $|\tilde{p}_k - \hat{p}_k|$ that have fast exponential tail-decay. We note that the deviation in Lemma 3.2 suggests a slower estimation error rate of $O(n^{-1/3})$ which can be sub-optimal in some dependent settings. However, this slower error will suffice for our purposes in the mean estimation algorithm described in the next section.

3.2 Finding Private Data-Driven Projection Intervals

Our main application of the histogram learning algorithm is to find data-driven projection intervals that will be used by our main mean estimator. The idea is to find a slowly diverging interval that contains all the observations with high-probability. This follows the blueprint of Karwa and Vadhan (2018). The projection interval will be the bin containing most of the mass and its two neighboring bins. This simple algorithm fits within a broader family of private midpoint algorithms used in the central and local item- and user-level DP mean estimation literature (Smith, 2011; Kamath et al., 2020; Levy et al., 2021; Kent et al., 2024; Agarwal et al., 2025).

Algorithm 3 ProjectionInterval$(X^n, \tau, \varepsilon, \delta)$:

Require: $X^n \in \mathbb{R}^n$, $\tau, \varepsilon, \delta > 0$

1: $B_k \leftarrow (2\tau k \pm \tau]$ $\hfill \forall k \in \mathbb{Z}$

2: $\tilde{\boldsymbol{p}} \leftarrow \text{StableHistogram}(X^n, (B_k)_{k \in \mathbb{Z}}, \varepsilon, \delta)$ $\hfill \tilde{\boldsymbol{p}} = (\ldots, \tilde{p}_{-1}, \tilde{p}_0, \tilde{p}_1, \tilde{p}_2, \ldots)$

3: $\hat{k} \leftarrow \arg\max_{k \in \mathbb{Z}} \tilde{p}_k$

4: $\hat{m} \leftarrow \begin{cases} 2\tau\hat{k} & \text{if } \exists k \in \mathbb{Z} : \tilde{p}_k > 0 \\ 0 & \text{otherwise} \end{cases}$

5: **return** $\hat{I} := [\hat{m} \pm 3\tau]$

Our theoretical analysis will work with a notion of (τ, γ)-concentration that is similar to the one introduced by Levy et al. (2021) in the context of user-level privacy. While this is not really needed, it makes it easier to compare our results to previous work. It will also help highlighting how more observations per user can improve the estimation error even under the more stringent user-level DP setting. This will become very clear in the applications of Section 5.

Assumption 3.3. $((\tau, \gamma)_p$-*Concentration, Adjusted from Definition 2, Levy et al. (2021)). A matrix $X^n \in \mathbb{R}^{n \times d}$ is $(\tau, \gamma)_p^{x_0}$-concentrated or $(\tau, \gamma)_p$-concentrated around $x_0 \in \mathbb{R}^d$ if for all $i \in [n]$ with probability at least $1 - \gamma/n$ we have $\|X_i^\top - x_0\|_p \leq \tau$. We call τ the concentration radius.*

Algorithm 3 returns an interval whose midpoint is the center of the bin of a stable histogram with the biggest estimated mass (cf. Karwa and Vadhan (2018), Algorithm 1). Our innovation is in the analysis of the algorithm which leads to Lemma 3.4 below, proven in Appendix C.2. The lemma generalizes Theorem 3.1 by Karwa and Vadhan (2018) using $(\tau, \gamma)_\infty$-concentration and Lemma 3.2 enabled by the DKW-type inequality due to Bobkov and Götze (2010). Most importantly, this allows for dependence between observations within the model of log-Sobolev dependence, but also enables non-identically distributed observations. Moreover, using the original DKW inequality, our proof strategy also generalizes the analysis of Karwa and Vadhan (2018) for i.i.d. Gaussians to general i.i.d. and $(\tau, \gamma)_\infty$-concentrated observations.

Lemma 3.4. *Algorithm 3 is (ε, δ)-DP for $\varepsilon, \delta > 0$. Let $\gamma \in (0, 1 \wedge \frac{n}{4})$, let $X^n \in \mathbb{R}^n$ be $(\tau, \gamma)_\infty$-concentrated around $x_0 \in \mathbb{R}$ and fulfill Assumption 2.7 or 2.8 with constants (ρ, M) s.t. $\rho M^2 \lesssim 1$. Let $\hat{m}$ denote the midpoint of the interval returned by Algorithm 3. Then, with probability at least $1 - O(\frac{n}{\delta} \cdot e^{-\kappa n \varepsilon} \vee e^{-\kappa' n})$ for absolute constants $\kappa, \kappa' > 0$,*

$$\hat{m} \in \left[x_0 \pm 2\tau \right].$$

Remark 3.5. *We state Lemma 3.4 in terms of $(\tau, \gamma)_\infty^{x_0}$-concentration of $X^n \in \mathbb{R}^n$, because the analysis to follow heavily relies on this notion of concentration. However, note that the proof permits relaxing to $(\tau', \gamma)_\infty^{x_0}$-concentration of each single X_i, which can have considerably smaller τ'. This is exploited in our numerical experiments in Section 7.1.2 to sharpen rates.*

Since $(\tau, \gamma)_\infty$-concentration is an abstract notion of concentration, we instantiate Lemma 3.4 for n i.i.d. σ^2-sub-Gaussian random variables with mean $\mu \in \mathbb{R}$ to give an example. These are $(\tau, \gamma)_\infty^\mu$-concentrated with $\tau = \sqrt{2\sigma^2 \log(2n/\gamma)}$, yielding Corollary 3.6 that is proven in Appendix C.2.

Corollary 3.6. *Let $\varepsilon, \delta > 0$ and $X^n \in \mathbb{R}^n$ have i.i.d. and sub-Gaussian entries with variance proxy σ^2 and mean $\mu \in \mathbb{R}$. Let $\hat{m}$ denote the midpoint of the interval returned by Algorithm 3. Then, for $\gamma \in (0, 1 \wedge \frac{n}{4})$ and with absolute constants $\kappa, \kappa' > 0$ we have*

$$\mathbb{P}\left(\hat{m} \in \left[\mu \pm 2\sqrt{2\sigma^2 \log(2n/\gamma)} \right] \right) \geq 1 - O\left(\frac{n}{\delta} \cdot e^{-\kappa n \varepsilon} \vee e^{-\kappa' n} \right).$$

The corollary exemplifies that we can estimate the true mean $\mu \in \mathbb{R}$ using the private midpoint $\hat{m}$ up to an error of size $\asymp \sigma$ with high probability, ignoring log-factors in n, γ.

4 Item-Level DP Estimation with Dependent Data

We are ready to present our main algorithm. Our proposal follows a popular idea for private mean estimation: compute a Winsorized mean estimator and make it private with the Laplace mechanism. This approach is common both in the item- and user-level mean estimation literature (See, e.g., Smith (2011); Karwa and Vadhan (2018); Levy et al. (2021); Kent et al. (2024); Agarwal et al. (2025)).

Estimators within this framework restrict their global sensitivity by projecting observations onto an interval of fixed length. Given observations $X_1, \ldots, X_n \in \mathbb{R}$, we follow the one-dimensional item-level mean estimator construction of Karwa and Vadhan (2018) given in three steps:

1. Compute a private crude mean estimator $\hat{m}$ and range $\hat{I} := [\hat{m} \pm \varsigma]$,
2. Project observations X_i onto $\hat{I}$ and average them,
3. Privatize the projected average by adding Laplace noise with variance $\asymp \varsigma/(\varepsilon n)$.

In this construction, the choice of $\hat{m}$ and the parameter ς are critical for the theoretical analysis and empirical performance of the estimator. We will find both simultaneously using the private histogram-based projection estimator from the previous section. Item-level mean estimators in $\mathbb{R}^d$ can be obtained through coordinate-wise application of the one-dimensional estimator. Conceptually this is equivalent to first constructing an L_∞-ball around a rough mean estimator $\hat{m} \in \mathbb{R}^d$, projecting observations $X_1, \ldots, X_n \in \mathbb{R}^d$ onto this hypercube, averaging them and adding isotropic Laplace noise in $\mathbb{R}^d$. The theory worked out in this section will be presented for dependent data in the item-level setting, but it will also be the cornerstone for all the user-level estimators in Section 5.

Our main theoretical results are the finite-sample error and in-expectation MSE bounds in Theorems 4.2 and 4.3. These main results as well as all the guarantees derived for intermediate algorithms are obtained under log-Sobolev dependence. An interesting side product of our analysis is that we generalize some known results even in the i.i.d. setting by allowing for both unbounded data domains and unbounded parameter spaces.

4.1 Mean Estimators

Algorithm 4 presents our main mean estimation routine. It is a one-dimensional noisy Winsorized mean estimator that is shown to work well under log-Sobolev dependence. The estimator relies on projecting the data to the private range $\hat{I} := [\hat{m} \pm 3\tau]$ provided by Algorithm 3.

Algorithm 4 WinsorizedMean1D$(X, \tau, \varepsilon, \delta, \ldots)$

Require: $X \in \mathbb{R}^n$, $\tau, \varepsilon, \delta > 0$
  1: $\hat{I} \leftarrow$ ProjectionInterval$(X, \tau, \frac{\varepsilon}{2}, \delta, \ldots)$ where $\hat{I} = [\hat{m} \pm 3\tau]$ and $|\hat{I}| = 6\tau$
  2: **return** $\frac{1}{n}\sum_{i=1}^{n} \Pi_{\hat{I}}(X_i) + \xi$ with $\xi \sim \mathrm{Lap}(0, \frac{12\tau}{n\varepsilon})$

By construction, all observations $X_1, \ldots, X_n \in \mathbb{R}$ lie in the private projection interval $\hat{I}$ with high probability. This guarantees that with high probability the noisy Winsorized mean estimator behaves like a noisy empirical mean. Lemma 4.1 formalizes this behavior and is proven in Appendix D.

Lemma 4.1. *Algorithm 4 denoted by $\mathcal{A}$ is (ε, δ)-DP for $\varepsilon, \delta > 0$. Assume $X^n \in \mathbb{R}^n$ is $(\tau, \gamma)_\infty^{x_0}$-concentrated and make Assumption 2.7 or 2.8 s.t. $\rho M^2 \lesssim 1$. Then, for absolute constants $\kappa, \kappa' > 0$ the event $\mathcal{E} := \{\forall i \in [n] : \Pi_{\hat{I}}(X_i) = X_i\}$ has probability at least $1 - \gamma - O(n/\delta \cdot e^{-\kappa n\varepsilon} \vee e^{-\kappa' n})$ and*

$$\mathcal{A}(X^n) \cdot \mathbf{1}\{\mathcal{E}\} = (\bar{X}_n + \xi) \cdot \mathbf{1}\{\mathcal{E}\}, \quad \text{where} \quad \xi \sim \mathrm{Lap}\left(0, \frac{12\tau}{n\varepsilon}\right).$$

While Lemma 4.1 does not itself provide a statistical error guarantee, it is helpful for deriving the finite-sample result of Subsection 4.2.1 as it lets us analyze Algorithm 4 via the proxy $\bar{X}_n + \xi$.

A multivariate version of Lemma 4.1 is presented in Lemma D.1 in Appendix D. It is analogous to Theorem 2 by Levy et al. (2021) and obtained through our multivariate mean estimator given in Algorithm 5. The estimator is a simple coordinate-wise application of Algorithm 4. Therefore, the assumptions of $(\tau, \gamma)_\infty$-concentrated data $X^n \in \mathbb{R}^{n \times d}$ around a point $x_0 \in \mathbb{R}^d$ in combination

with log-Sobolev dependence or i.i.d. data are again natural in this context. Note that $(\tau_p, \gamma)_p$-concentration for all $p \in \mathbb{N}$ are equivalent in $\mathbb{R}$ but not in $\mathbb{R}^d$ since for a fixed γ the order of τ_p as a function of d might differ significantly (cf. Levy et al. (2021); Kent et al. (2024)).

Algorithm 5 WinsorizedMeanHD$(X^n, \tau, \varepsilon, \delta, \varrho, \dots)$

Require: $X^n \in \mathbb{R}^{n \times d}$, $\tau > 0$, $\delta, \varrho, \varepsilon \in (0,1)$
 1: $\varepsilon' \leftarrow \varepsilon / \sqrt{8d \log(1/\varrho)}$
 2: $\delta' \leftarrow \delta/d$
 3: $\bar{X}_j \leftarrow$ WinsorizedMean1D$(X_{\cdot j}, \tau, \varepsilon', \delta', \dots)$ $\forall j \in [d]$
 4: **return** $(\bar{X}_1, \dots, \bar{X}_d)^\top$

4.2 Theoretical Guarantees

4.2.1 High-Probability Bound

We translate the utility guarantee of Theorem D.1 into a finite-sample upper bound on the MSE of the estimator in Algorithm 5. The estimation error in Theorem 4.2 below is quantified by the sum of two terms: the statistical error and the cost of privacy. The theorem is proven in Appendix D.1.1.

Theorem 4.2. *Algorithm 5 denoted by $\mathcal{A}$ is $(\varepsilon, \delta + \varrho)$-DP for $\delta, \varrho, \varepsilon \in (0,1)$. Furthermore, let $\gamma \in (0, 1 \wedge \frac{n}{4})$ and $X^n \in \mathbb{R}^{n \times d}$ be $(\tau, \gamma)_\infty^\mu$-concentrated and fulfill Assumption 2.7 or 2.8 s.t. $\rho M^2 \lesssim 1$. Then, for absolute constants $\kappa, \kappa' > 0$, with probability at least $1 - 2d\gamma - O(d^2 n/\delta \cdot e^{-\kappa n \varepsilon'})$,*

$$\|\mathcal{A}(X^n) - \mu\|_2 \lesssim \|\bar{X}_n - \mu\|_2 + \tau \sqrt{\frac{d^2 \log(1/\varrho) \log(3/(d\gamma))^2}{n^2 \varepsilon^2}}.$$

As shown in Lemma 2.11, under log-Sobolev dependence with constant ρ, the statistical rate $\|\bar{X}_n - \mu\|_2$ is upper bounded by $\sqrt{2d\rho \log(2d/\alpha)/n}$. Thus, up to constants and logarithmic terms Theorem 4.2 yields an MSE bound for mean estimation of order $d/n + d^2/(n\varepsilon)^2$ in probability. Expressing the theorem in terms of sample complexity bounds recovers Kamath et al. (2019, Theorem 4.1) in the i.i.d. Gaussian setting when $X^n \sim \mathcal{N}(0, \sigma^2 I_n)$. It is tempting to apply the integral identity of the expectation to translate an in-probability bound like Theorem 4.2 into an in-expectation bound. However, this will not work becuase of the term $O(d^2 n/\delta \cdot e^{-\kappa n \varepsilon'})$ in the probability. Alternatively, splitting the expectation on the event in Theorem 4.2 and its complement is equally troublesome, since we cannot easily control the MSE on the low probability event.

4.2.2 In-Expectation Analysis

While the main focus of this work is on non-asymptotic bounds like the finite-sample guarantee of Theorem 4.2 and instantiations thereof in Section 5, we also give an in-expectation MSE upper bound in Theorem 4.3. We do so to provide an analysis of our estimator that is comparable to the existing upper and in particular also lower bounds in the user-level DP literature like Corollary 5 in Levy et al. (2021), and Theorem 3.1 or Theorem 3.2 of Cai et al. (2021) in the item-level DP literature. This allows us to make direct comparisons to known results in the i.i.d. setting. In particular, we can match the i.i.d. rates under weak dependence as shown in Theorem 4.3.

The procedure used for the in-expectation and consistency result in Theorem 4.3 is again Algorithm 5. However, we employ a sample splitting technique like in the proof of Theorem 6 by Kent et al. (2024). For simplicity, we assume that we have $2n$ samples instead of just n that are split into two independent halves $Z^n := [Z_1, \dots, Z_n]^\top$ and $X^n := [X_1, \dots, X_n]^\top$. We estimate the private midpoint

$\hat{m} := \hat{m}(Z^n)$ with Algorithm 3 using only the first half and project the second half onto the hypercube $\hat{I}^d := \hat{I}_1 \times \cdots \times \hat{I}_d$ where $\hat{I}_j = [\hat{m}_j \pm 3\tau]$ to then estimate the mean using $\Pi_{\hat{I}^d}(X_1), \ldots, \Pi_{\hat{I}^d}(X_n)$. This lets us prove the statement conditional on $\hat{I}^d$. Through this conditioning, $\hat{I}^d$ is deterministic which makes $\Pi_{\hat{I}^d}(X_i)$ a contraction and significantly simplifies controlling the estimator's variance. More concretely, we propose the estimator:

$$\mathcal{A}_{\hat{m}(Z^n)}(X^n) = \frac{1}{n}\sum_{i=1}^n \Pi_{\hat{I}^d}(X_i) + \Xi, \quad \text{where} \quad \Xi \sim \text{Lap}\left(0, \frac{12\tau}{\varepsilon n}I_d\right). \tag{1}$$

Theorem 4.3. *The estimator* (1) *is* $(\varepsilon, \delta + \varrho)$-*DP for* $\varepsilon, \delta, \varrho \in (0,1)$. *Further, let* $X^n, Z^n \in \mathbb{R}^{n \times d}$ *have rows with mean* $\mu \in \mathbb{R}^d$. *Make Assumption 2.7 with* ρ-*sub-Gaussian entries or alternatively Assumption 2.8 s.t.* $\rho M^2 \lesssim 1$. *Then, choosing* $\tau^2 \asymp \rho \log(4n)$ *we have*

$$\mathbb{E}[\|\mathcal{A}_{\hat{m}(Z^n)}(X^n) - \mu\|_2^2] \lesssim d\rho \log(4n)\left(\frac{1}{n} + \frac{d\log(1/\varrho)}{n^2\varepsilon^2} + \frac{dn}{\delta e^{\kappa n \varepsilon'/2}}\right) + \frac{d\|\mu\|_2^2}{\delta e^{\kappa n \varepsilon'}}.$$

Theorem 4.3 is proven in Appendix D.1.2. In the i.i.d. setting where the rows have ρ-sub-Gaussian entries, the upper bound in Theorem 4.3 recovers the error rates obtained in prior work such as Theorem 3.2 by Cai et al. (2021) or Corollary 5 by Levy et al. (2021). However, note that the construction of our estimator does not need to assume a known bound on $\|\mu\|_2^2$. Unlike other results in the literature, $\|\mu\|_2^2$ only shows up in the analysis of our estimator.

4.3 Minimax Optimality

To assess the optimality of our mean estimators, we compare our upper bounds to the statistical minimax lower bound given in Cai et al. (2021). Their results were obtained for i.i.d. observations $X_1, \ldots, X_n \in \mathbb{R}^d$ each with i.i.d. ρ-sub-Gaussian entries in the item-level DP setting. For sake of exposition, we restate their theorem below. We will see that this bound implies that our mean estimator is minimax optimal in the i.i.d. setting of item-level DP.

Theorem 4.4. *(Theorem 3.1, Cai et al. (2021)). Let* $X^n \in \mathbb{R}^{n \times d}$ *fulfill Assumption 2.7, have* ρ-*sub-Gaussian entries and rows with mean* $\|\mu\|_\infty^2 < \rho$. *Call the class of such distributions* $\mathcal{P}$. *Let* $\varepsilon \in (0,1)$ *and* $\delta \in ((ne^{\varepsilon n})^{-1}, n^{-(1+\omega)})$ *for* $\omega > 0$. *Assume* $\log(\delta)/\log(n)$ *is non-increasing in* n, *and* $d \gtrsim \log(1/\delta)$. *Then, if* $n\varepsilon \gtrsim \sqrt{d\log(1/\delta)}$ *it holds that*

$$\inf_{\mathcal{A} \in \mathcal{A}_{(\varepsilon,\delta)}^{\text{item}}} \sup_{\nu \in \mathcal{P}} \mathbb{E}[\|\mathcal{A}(X^n) - \mu\|_2^2] \gtrsim d\rho\left(\frac{1}{n} + \frac{d\log(1/\delta)}{n^2\varepsilon^2}\right),$$

where $\mathcal{A}_{(\varepsilon,\delta)}^{\text{item}}$ *is the family of all* (ε, δ)-*DP estimators.*

Under Assumption 2.7 which subsumes Assumption 2.8 with i.i.d. rows, the distributional conditions of Theorem 4.3 are more general than those of Theorem 4.4, because the latter restricts the population mean to satisfy $\|\mu\|_\infty^2 < \rho$. To see this, note that if we assume $\|\mu\|_\infty^2 \lesssim \rho n e^{\kappa n \varepsilon'/2}$ and $\delta \in (n\varepsilon'/e^{\kappa n \varepsilon'}, n^{-1}]$, the upper bound of Theorem 4.3 will match the lower bound of Theorem 4.4 up to logarithmic terms. This is slightly more restrictive than the condition on δ required in the lower bound, but given that our analysis does not need a known bound on the norm of μ and even allows the norm to grow exponentially in n, we consider this a minor inconvenience. For a formal statement of this discussion see Corollary D.7 in Appendix D.1.2.

Remark 4.5. *Interestingly, the rate in Theorem 4.3 also matches the lower bound up to a* $O(\log n)$-*factor under log-Sobolev dependence of Assumption 2.8 as long as* $\rho \asymp 1$. *Therefore, in this weak*

dependence scenario, the rate of differentially private mean estimation in $n, d, \varepsilon, \delta$ is unchanged relative to i.i.d. sub-Gaussian observations. However, as soon as $\rho \gtrsim n^\omega$ for $\omega > 0$, dependence decreases the rate in the upper bound and Theorem 4.3 is not tight anymore. We leave deriving lower bounds under dependence incorporating the log-Sobolev constant ρ as an open problem.

Remark 4.6. *The lower bound in Theorem 4.4 requires the condition $\|\mu\|_\infty^2 < \rho$ which connects the mean and sub-Gaussian variance proxy ρ. In particular, the lower bound is not very interesting when $\rho \to 0$. In contrast, our upper bound suggests that the MSE does not vanish as $\rho \to 0$ but n stays constant when we do not connect the norm of μ to ρ. Indeed, the fourth term in the upper bound of Theorem 4.3 is exponentially small in n but does not vanish when $\rho \to 0$. While this limit might not seem practically relevant in the item-level setting, it has important implications for user-level estimation related to the impossibility of learning if only the number of observations T per user goes to infinity. See also Remark 5.7.*

4.4 Extension to Nonparametric Regression

Nonparametric regression aims to estimate a function $f : [0, 1] \to \mathbb{R}$ from n noisy observations of the form $Y_i = f(x_i) + \epsilon_i$ with $\epsilon^n = (\epsilon_1, \ldots, \epsilon_n) \sim \mathcal{N}(0, \Sigma^n)$. Commonly, the function f is assumed to be part of a family of functions $\mathcal{F}$ such as a Hilbert, Hölder, Sobolev or Besov space. While the x_i may be random, we make a fixed design assumption and view x_i as fixed points on a grid on $[0, 1]$.

We note that the problem of private nonparametric regression has been considered for i.i.d. data in Awan et al. (2020); Golowich (2021); Cai et al. (2023) in the central DP model and in Berrett et al. (2021); Györfi and Kroll (2025) in the local DP model. We permit correlated observations and want to focus on their effect on estimation. For simplicity, we restrict ourselves to $\mathcal{F}$ being the space of bounded Lipschitz functions. This setting is formalized in Definition 4.7.

Definition 4.7. *(Fixed Design Nonparametric Regression). Suppose $f : [0, 1] \to \mathbb{R}$ is L_f-Lipschitz and $\|f\|_\infty < \infty$ and define the equidistant design points $x_i = i/n$ for $0 \le i \le n-1$. Let $\epsilon^n \sim \mathcal{N}(0, \Sigma^n)$ be noise with $\sigma_{\min}^2 I_n \preceq \Sigma^n \preceq \sigma_{\max}^2 I_n$ and $0 < \sigma_{\min}^2 \le \sigma_{\max}^2$. We define the nonparametric fixed design regression model through the observations $Y_i = f(x_i) + \epsilon_i$.*

Given observations, the underlying function f is commonly estimated using classical kernel-based estimators like the nearest neighbor, Priestley-Chao, Nadaraya-Watson, Gasser-Müller or local polynomial estimators. Alternatively, the task is translated into mean estimation using Fourier or wavelet transforms, or linear regression via smoothing splines (See Tsybakov (2008)). Nonparametric estimation under DP mainly relies on such reformulations as can be seen, e.g., in Subsection 5.2.2 of Duchi et al. (2018), Cai et al. (2025) or Section 6 of Cai et al. (2023). Here we show that, in our fixed design setting, we can construct optimal pointwise DP estimators of f building on the classical nonparametric Priestley-Chao regression estimator in Definition 4.8 that was introduced by Priestley and Chao (1972).

Definition 4.8. *Let $x^n, Y^n \in \mathbb{R}^n$ contain design points and responses. Let $K : \mathbb{R} \to \mathbb{R}_\ge$ be a kernel and let $\mu_1(K) := \int |s| K(s) ds$. Suppose K is L_K-Lipschitz, $\int_\mathbb{R} K(u) du = 1$ and $\mu_1(K), \|K\|_\infty < \infty$. Let $b > 0$ be the bandwidth. Then, we define the Priestley-Chao estimator as*

$$\hat{f}_n(\cdot) := \sum_{i=0}^{n-1} \frac{x_{i+1} - x_i}{b} K\left(\frac{\cdot - x_i}{b}\right) Y_i.$$

Since the Priestley-Chao regression estimator with equidistant x_i s.t. $x_{i+1} - x_i = 1/n$ is an average of $\hat{f}_i^n(\cdot) := Y_i \frac{1}{b} K(\frac{\cdot - x_i}{x})$, we can easily privatize it using our Algorithm 4. In order to apply Theorem 4.2 to obtain a finite-sample generalization bound for the privatized estimator, we first need to

establish $(\tau, \gamma)_\infty$-concentration of the non-private $\hat{f}_i^n(x)$ around $f(x)$ for a fixed x. We state this intermediate result formally in the following lemma. It is proven in Appendix D.2.

Lemma 4.9. *Let $\hat{f}_n$ be the estimator of Definition 4.8 with a kernel that is a 1-sub-Gaussian density with $L_K, \|K\|_\infty < \infty$. Assume the model of Definition 4.7 with constants L_f and $\sigma_{\max}^2$. Then, for all $n \in \mathbb{N}$, bandwidths $b > 0$ and $x \in [\zeta, 1 - \zeta]$ where $\zeta = b\sqrt{2\log(2/b)}$ with probability at least $1 - \alpha$,*

$$\left| \hat{f}_n(x) - f(x) \right| \leq b\left(L_f \mu_1(K) + \|f\|_\infty\right) + \frac{L_f \|K\|_\infty + \|f\|_\infty L_K}{nb} + \sqrt{\frac{2\sigma_{\max}^2 \|K\|_\infty \log(2/\alpha)}{nb}}.$$

Further, $\hat{f}^n(x) \in \mathbb{R}^n$ with $\hat{f}_i^n(x) := Y_i \frac{1}{b} K\left(\frac{x_i - x}{b}\right)$ is $(\tau, \gamma)_\infty$-concentrated around $f(x)$ where

$$\tau = b\left(L_f \mu_1(K) + \|f\|_\infty\right) + \frac{1}{b}\left(L_f \|K\|_\infty + \|f\|_\infty L_K + \sqrt{2\sigma_{\max}^2 \|K\|_\infty \log(2n/\gamma)}\right).$$

The proof idea behind Lemma 4.9 is to decompose the mean absolute error into bias and mean absolute deviation. The bias is bounded deterministically leveraging the Lipschitzness of f, while the mean absolute deviation is controlled using the L_2-Lipschitzness of $\hat{f}_n$ in the errors $\epsilon^n \in \mathbb{R}^n$ and Gaussian-Lipschitz concentration. While such an approach is common when showing in-expectation bounds for kernel estimators, the literature contains only few finite-sample bounds for kernel estimators. The authors are not aware of another one for a Priestley-Chao regression estimator, in particular for correlated Gaussian noise. The closest results we found in the literature are the following: a finite-sample bound for a distance-based kernel regressor for estimating a functions first derivative Kpotufe and Boularias (2012, Theorem 1) and generalization bound for a Nadaraya-Watson estimator assuming a uniformly bounded function as we do, but also bounded noise (Srivastava et al., 2021; Wang et al., 2024).

The following lemma shows that besides their concentration, the summands defining $\hat{f}^n(x)$ are log-Sobolev dependent. A proof is given in Appendix D.2.

Lemma 4.10. *Let $\hat{f}_n$ be the estimator of Definition 4.8 with kernel K on the model of Definition 4.7. Then, $\hat{f}^n(x) \in \mathbb{R}^n$ with $\hat{f}_i^n(x) = Y_i \frac{1}{b} K\left(\frac{x_i - x}{b}\right)$ fulfills Assumption 2.8 with $\rho M^2 \lesssim \sigma_{\max}^2 / \sigma_{\min}^2 \cdot \|K\|_\infty$.*

Now, by Theorem 4.2 under Assumption 2.8 we get the finite-sample generalization bound in Corollary 4.11 in whose display we suppress the constants $L_f, L_K, \|f\|_\infty, \|K\|_\infty$ and logarithmic terms for clarity. Corollary 4.11 is proven in Appendix D.2.

Corollary 4.11. *Let $\hat{f}^n(x) \in \mathbb{R}^n$ with $\hat{f}_i^n(x) = Y_i \frac{1}{b} K\left(\frac{x_i - x}{b}\right)$, and denote by $\mathcal{A}$ Algorithm 5 taking as input $\hat{f}^n(x)$ for $x \in [0, 1]$. Then, $\mathcal{A}$ is (ε, δ)-DP for $\varepsilon, \delta \in (0, 1)$. Furthermore, if $\gamma \in (0, 1 \wedge \frac{n}{4})$, $\sigma_{\max}^2 \gtrsim 1$ and $x \in [\zeta, 1 - \zeta]$ where $\zeta = b\sqrt{2\log(2/b)}$ and $b > 0$. Then, if Y^n is as in Definition 4.7, with probability at least $1 - 3\gamma - O(n/\delta \cdot e^{-\kappa n \varepsilon})$ where $\kappa > 0$ is an absolute constant,*

$$\left| \mathcal{A}(\hat{f}^n(x)) - f(x) \right| \lessapprox b + \sigma_{\max}\left(\sqrt{\frac{1}{nb}} + \sqrt{\frac{1}{n^2 b^2 \varepsilon^2}}\right).$$

For an optimally chosen $b \asymp (\sigma_{\max}^2 / n)^{1/3} \vee \sqrt{\sigma_{\max}/n\varepsilon}$ this becomes

$$\left| \mathcal{A}(\hat{f}^n(x)) - f(x) \right| \lessapprox \left(\frac{\sigma_{\max}^2}{n}\right)^{1/3} \vee \sqrt{\frac{\sigma_{\max}}{n\varepsilon}} \lesssim \left(\frac{\sigma_{\max}^2}{n}\right)^{1/3} + \sqrt{\frac{\sigma_{\max}}{n\varepsilon}}.$$

Albeit the result is given in probability instead of in expectation, the error rate in the second statement of Corollary 4.11 resembles that of integrated and pointwise in-expectation lower bounds such as Corollary 1, Corollary 2 in Cai et al. (2024) or Theorem 6.1 of Cai et al. (2023) in the i.i.d. setting, i.e., when $\Sigma^n \asymp I_n$.

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5 User-Level DP Estimators for Dependent Data

Remember that in the user-level DP setting each of the n users contribute T observations that shall be protected as a whole. While all the theoretical work in user-level DP that we are aware of assumes independent bounded data across both users and time, our results simultaneously cover unbounded observations and enable dependence both across the n users and their T observations.

The user-level DP estimators that we introduce in this section will showcase the versatility of the theoretical analysis of our item-level DP Winsorized mean estimator of the previous section. The construction of the user-level estimators will follow from the same 3 steps. Namely, given a data matrix $X \in \mathbb{R}^{n \times d}$ we will need to show that:

1. X is $(\tau, \gamma)_\infty^\mu$-concentrated,
2. X fulfills Assumption 2.7 or 2.8,
3. The sample average $\bar{X}_n$, fulfills $\|\bar{X}_n - \mu\|_2 \le R_{stat}^\mu(n, \alpha)$ with probability at least $1 - \alpha$.

In the longitudinal problems that we will consider below, the data matrix X^n will be a collection of n statistics computed from the T observations of each individual user. In particular, by constructing specific data matrices X^n, we will reduce user-level mean estimation, a simple random effects model and longitudinal linear regression to item-level mean estimation. This reduction allows us to obtain finite-sample statistical guarantees for all these models from Theorem 4.2.

5.1 User-Level Mean Estimation

Similar to the item-level case, we control dependence by Assumption 2.8 but applied to a data matrix $X^n \in \mathbb{R}^{nT \times d}$ consisting of nT observations as defined next.

Definition 5.1. *(User-Level Data). Collect the observations $X_u \in \mathbb{R}^{T \times d}$ of users $u \in [n]$ over time points $t \in [T]$ in the data matrix $X^n := [X_1^\top, \dots X_n^\top]^\top \in \mathbb{R}^{nT \times d}$. Define a user-average as $\hat{\mu}_u := \frac{1}{T} X_u^\top \mathbf{1}_T$ for $u \in [n]$ and collect them in $\hat{\mu}^n := [\hat{\mu}_1, \dots, \hat{\mu}_n]^\top \in \mathbb{R}^{n \times d}$. We call X^n a user-level data matrix and $\hat{\mu}^n$ the user-level averages.*

Our user-level mean estimator feeds the matrix $\hat{\mu}^n \in \mathbb{R}^{n \times d}$ to Algorithm 5, which then computes a private counterpart to the following empirical mean over the nT observations:

$$\bar{X}_{nT} := \frac{1}{nT} \sum_{i=1}^{nT} X_{ij} = \frac{1}{n} \sum_{u=1}^n \hat{\mu}_u = \frac{1}{n} (\hat{\mu}^n)^\top \mathbf{1}_n.$$

Feeding the user-level averages $\hat{\mu}^n$ to Algorithm 5 ensures privacy for each single $\hat{\mu}_u$ and thus all observations of a user at once, which is exactly what user-level DP requires. Utility also follows by that of Algorithm 5 with the main difference being that the T observations per user essentially make $\hat{\mu}_u$ concentrate with a shrinking radius of order $O(1/\sqrt{T})$. The main ingredient needed to make these claims rigorous is Theorem 4.2. For this we need to show that $\hat{\mu}^n$ is $(\tau, \gamma)_\infty$-concentrated around the mean μ. This is established in Lemma 5.2 which additionally gives a non-asymptotic deviation error rate for the non-private $\bar{X}_{nT}$. The proof of is given in Appendix E.1.

Lemma 5.2. *Let $X^n \in \mathbb{R}^{nT \times d}$ be a user-level data matrix. Assume $\mathbb{E}[X^n] = \mathbf{1}_{nT} \mu^\top$, with mean $\mu \in \mathbb{R}^d$ and let X^n fulfill Assumption 2.8 s.t. $\rho M^2 \lesssim 1$. Then, $\hat{\mu}^n$ is $(\tau, \gamma)_\infty^\mu$-concentrated with $\tau = \sqrt{2\rho \log(2dn/\gamma)/T}$ and with probability at least $1 - \alpha$,*

$$\left\| \bar{X}_{nT} - \mu \right\|_2 \le \sqrt{\frac{2d\rho \log(2d/\alpha)}{nT}}.$$

Remark 5.3. *The concentration radius in Lemma 5.2 is actually $\tau = \sqrt{2\rho_\star \log(2dn/\gamma)/T}$ where $\rho_\star = \max_{u\in[n],j\in[d]} \rho_{uj}$ and $\rho_{uj} \leq \rho$ are the marginal log-Sobolev constants of $(X_u)_{\cdot j}$. If $\rho_\star$ is known, this τ is constant in n up to logarithms if $\rho_{uj} \lesssim 1$ even under strong dependence, i.e., $\rho \asymp n^\omega, \omega \geq 0$.*

Now, Lemma 5.2 directly implies an error bound for our Winsorized mean estimator through Theorem 4.2. We state this guarantee next and prove it in Appendix E.1.

Corollary 5.4. *Let $\mathcal{A}$ denote Algorithm 5 taking input $\hat{\mu}^n$ as in Definition 5.1. Then, $\mathcal{A}$ is $(\varepsilon, \delta+\varrho)$-uDP for $\delta, \varrho, \varepsilon \in (0,1)$. Further, let $\gamma \in (0, 1 \wedge \frac{n}{4})$ and assume the user-level data matrix $X^n \in \mathbb{R}^{nT\times d}$ is s.t. $\mathbb{E}[X^n] = \mathbf{1}_{nT}\mu^\top$ and fulfills Assumption 2.8 s.t. $\rho M^2 \lesssim 1$. Then, with probability at least $1 - 3d\gamma - O(d^2n/\delta \cdot e^{-\kappa n\varepsilon'})$ for an absolute constant $\kappa > 0$,*

$$\|\mathcal{A}(\hat{\mu}^n) - \mu\|_2 \lesssim \sqrt{\frac{d\rho \log(2/\gamma)}{nT}} + \sqrt{\frac{d^2\rho \log(2dn/\gamma)\log(1/\varrho)\log(3/(d\gamma))^2}{Tn^2\varepsilon^2}}.$$

The rate in Corollary 5.4 is additive in the statistical rate of order $d/(nT)$ and a cost of privacy term of order $d^2/(Tn^2\varepsilon^2)$. Both also arise in the user-level in-expectation bound in Levy et al. (2021, Corollary 1) which was derived for bounded data. Moreover, when limiting our finite-sample result to the i.i.d. setting, our rates match those present in the upper and lower bounds of Theorem 1.2 and Theorem 1.4 by Agarwal et al. (2025) in the limit when $k \to \infty$ directional moments are bounded.

Remark 5.5. *While the error in Corollary 5.4 goes to 0 as $T \to \infty$, the probability of this event cannot be made arbitrarily small unless $n \to \infty$. This is analogous to the impossibility result Levy et al. (2021, Theorem 8) which shows that learning under user-level DP is impossible in general if only $T \to \infty$ but the number of users n stays constant.*

5.1.1 In-Expectation Analysis

Theorem 4.3 also immediately implies an MSE upper bound in the user-level setting. We obtain it by running the item-level mean estimator (1) on observations $\hat{\mu}^n = [\hat{\mu}_1^\top, \ldots, \hat{\mu}_n^\top]^\top \in \mathbb{R}^{n\times d}$ like in Definition 5.1. This construction assumes an additional independent copy $\hat{\mu}_Z^n$ of $\hat{\mu}^n$ which is used to estimate a crude private midpoint $\hat{m} := \hat{m}(\hat{\mu}_Z^n)$ with Algorithm 3. We then project the second half $\hat{\mu}^n$ onto the hypercube $\hat{I}^d := \hat{I}_1 \times \cdots \times \hat{I}_d$ where $\hat{I}_j = [\hat{m}_j \pm 3\tau]$ to then estimate the mean using $\Pi_{\hat{I}^d}(\hat{\mu}_1), \ldots, \Pi_{\hat{I}^d}(\hat{\mu}_n)$.

Corollary 5.6. *Construct $\hat{\mu}^n$ and $\hat{\mu}_Z^n$ from user-level data matrices X^n and Z^n as in Definition 5.1. Let $\mathcal{A}_{\hat{m}(\hat{\mu}_Z^n)}$ denote the estimator (1) computed on $\hat{\mu}^n$, with midpoint $\hat{m}(\hat{\mu}_Z^n)$. Then, $\mathcal{A}_{\hat{m}(\hat{\mu}_Z^n)}$ is $(\varepsilon, \delta + \varrho)$-uDP for $\varepsilon, \delta, \varrho \in (0,1)$. Further, let $X^n = [X_1^\top, \cdots, X_n^\top]^\top \in \mathbb{R}^{nT\times d}$ have rows with mean $\mu \in \mathbb{R}^d$ and satisfy Assumption 2.7 with ρ-sub-Gaussian entries or Assumption 2.8 s.t. $\rho M^2 \lesssim 1$. Let $\hat{\mu}_Z^n$ be an independent copy of $\hat{\mu}^n$ and choose $\tau^2 \asymp \rho\log(4n)$ in Algorithm 3. Then,*

$$\mathbb{E}[\|\mathcal{A}_{\hat{m}(\hat{\mu}_Z^n)}(\hat{\mu}^n) - \mu\|_2^2] \lesssim d\rho\log(4n)\left(\frac{1}{nT} + \frac{d\log(1/\varrho)}{Tn^2\varepsilon^2} + \frac{dn}{\delta Te^{\kappa n\varepsilon'/2}}\right) + \frac{d\|\mu\|_2^2}{\delta e^{\kappa n\varepsilon'}}.$$

Corollary 5.6 is proven in Appendix E.1. For reasonable choices of δ and ε the estimator is consistent as $n \to \infty$ and if $\|\mu\|_2^2 = o(\delta e^{\kappa n\varepsilon'})$. As the $\hat{\mu}_u$ themselves are averages, their variances are of order ρ/T and thus by analogy with $\bar{X}_{nT}$ we expect an MSE that is smaller by order $1/T$ compared to the item-level case. However, the fourth term in Corollary 5.6 shows that this is not the case.

Remark 5.7. *We see from Corollary 5.6 that for $\delta \in (n\varepsilon'/e^{\kappa n\varepsilon'}, n^{-1}]$ the rate in n and ε is the same as in the item-level setting of Theorem 4.3. Furthermore, the fourth term does not vanish for a fixed n as $T \to \infty$, so our bound exhibits the same impossibility of learning in the $T \to \infty$ limit*

that was established in Levy et al. (2021) even though a closer look at Levy et al. (2021, Theorem 8) reveals that their lower bound is realized for $\delta < (2ne^{n\varepsilon})^{-1}$ which is excluded by Corollary 5.6. Still, our rates are comparable to the best existing estimators under i.i.d. assumptions. Note also that Theorem 4.4 is not applicable because the local mean estimators $\hat{\mu}_u$ are $\frac{\rho}{T}$-sub-Gaussian which makes the condition $\|\mu\|_\infty^2 < \frac{\rho}{T}$ that would be required by Theorem 4.4 excessively stringent.

5.2 Random Effects Location Model

Random effects models are commonly used for the analysis of clustered or correlated data (Pinheiro and Bates, 2000; Fahrmeir and Tutz, 2001; Demidenko, 2013). We restrict our presentation here to a simple one-dimensional user-level random effects location model presented in Definition 5.8. We do so to fully focus on the dependence between users introduced by random effects and investigate their impact on the convergence rate of our Winsorized mean estimator. Note, however, that our theoretical guarantees of Section 4 are powerful enough to also handle random effects in linear regression settings like the one in Subsection 5.3.

Definition 5.8. *(Random Effects Location Model). Let $\mu \in \mathbb{R}$ and $\sigma_U^2, \sigma^2 < \infty$. Assume independent noise $U_g \sim \mathcal{N}(0, \sigma_U^2)$ and $\epsilon_u \sim \mathcal{N}(0, \Sigma_u)$ with $\Sigma_u \in \mathbb{R}^{T \times T}$ and $\Sigma_u \preceq \sigma^2$. For all users $u \in [n]$, groups $g \in [G]$ and times $t \in [T]$, define a random effects model through observations $Y_{gut} := \mu + U_g + \epsilon_{ut}$ collected in $Y^n \in \mathbb{R}^{nT}$.*

In addition to the random effects that introduce dependence between users of the same group g, the model in Definition 5.8 also contains noise that is independent between users but correlated across a user's observations over time. Hence, it contains both, inter- and intra-user dependence. Nevertheless, we will see that the analysis of the model in Definition 5.8 is simple because it can be viewed as an instance of the user-level mean estimation problem covered in Subsection 5.1. Hence, we only need to show that the resulting user-level data matrix fulfills log-Sobolev dependence. We do that in the following lemma.

Lemma 5.9. *Let $Y^n \in \mathbb{R}^{nT}$ be the design of the random effects model in Definition 5.8. Writing n_g for the group sizes, define $\rho := \max_{g \in [G]} \left(\sigma_U^2 n_g T + \sigma^2 \right)$ and $M^2 := (2\pi(\sigma_U^2 + \sigma^2))^{-1}$. Then, Y^n is a user-level log-Sobolev data matrix and Assumption 2.8 holds with constants (ρ, M).*

Our proof of Lemma 5.9 can be found in Appendix E.2. It exploits the joint Gaussian distribution of observations Y_{gut} in Definition 5.8. The next result is a direct consequence of Corollary 5.4. It gives an upper bound on the estimation error of privately estimating the location parameter μ.

Corollary 5.10. *Construct $\hat{\mu}^n \in \mathbb{R}^n$ from $Y^n \in \mathbb{R}^{nT}$ and denote by $\mathcal{A}$ the output of Algorithm 5 with input $\hat{\mu}^n$. Then $\mathcal{A}$ is (ε, δ)-uDP for $\delta, \varepsilon \in (0, 1)$. Furthermore, let $\gamma \in (0, 1 \wedge \frac{n}{4})$ and $Y^n \in \mathbb{R}^{nT}$ be the design of Definition 5.8 with mean $\mu \in \mathbb{R}$ and maximum group size $n_{g^\star}$. Then, if $\rho M^2 \lesssim 1$, with probability at least $1 - 3\gamma - O(n/\delta \cdot e^{-\kappa n \varepsilon})$ for an absolute constant $\kappa > 0$,*

$$\|\mathcal{A}(\hat{\mu}^n) - \mu\|_2 \lessapprox \sqrt{\frac{\sigma_U^2 \cdot n_{g^\star} T + \sigma^2}{nT}} + \sqrt{\frac{\sigma_U^2 \cdot n_{g^\star} T + \sigma^2}{Tn^2\varepsilon^2}}.$$

Remark 5.11. *The rate in the error bound only matches the baseline of independent users with uncorrelated noise if $\sigma_U^2 n_{g^\star} T + \sigma^2 \asymp 1$. Note that $\sigma^2 \asymp 1$ can be realized, e.g., through sparse correlations, a Toeplitz covariance structure with fast decay or equicorrelated noise with variance $\sigma^2 \asymp 1/T$. However, $\sigma_U^2 \asymp 1/(n_{g^\star} T)$ is stringent and will hold for instance when the maximum group size $n_{g^\star} \asymp 1$ and $\sigma_U^2 \asymp 1/T$. Otherwise, the latter example results in slower rates relative to the i.i.d. case, but the rates are sharp in this case as shown in the next example.*

Example 5.12. (Slow Rates for the Empirical Mean). The behavior of the error rate in Corollary

5.10 as a function of n_{g^*} is not due to a loose bound when deriving the log-Sobolev dependence of Y^n. To see this, assume $\epsilon_u = 0$. Then, in the derivation of Lemma 5.9 the operator norm of the covariance matrix $\Sigma^n \in \mathbb{R}^{nT \times nT}$ of the joint is exactly

$$\|\Sigma^n\|_{op} = \max_{g \in [G]} \|\Sigma_g^U\|_{op} = \max_{g \in [G]} \|\sigma_U^2 \mathbf{1}_{n_g T} \mathbf{1}_{n_g T}^\top\|_{op} = \sigma_U^2 \cdot T \cdot \max_{g \in [G]} n_g.$$

This suggests that already one group with $n_g \asymp n$ kills the MSE-rate for user-level mean estimation.

Note that this phenomenon is due to the nature of this statistical model and not the user-level setting. To show this, let $X^n \in \mathbb{R}^n$ with $X^n \sim \mathcal{N}(\mu, \Sigma^n)$ where $\mu \in \mathbb{R}^d$ and

$$\Sigma^n := \begin{bmatrix} \mathbf{1}_{n_g} \mathbf{1}_{n_g}^\top & 0 \\ 0 & 0 \end{bmatrix} + I_n.$$

As $\bar{X}_n$ is unbiased for $g(\mu) = \frac{1}{n} \mathbf{1}_n^\top \mu$, the MSE reduces to the variance that we may compute exactly:

$$\text{MSE}(\bar{X}_n) = \text{Var}(\bar{X}_n) = \frac{1}{n^2} \text{Var}\left(\sum_{i=1}^n X_i\right) = \frac{1}{n^2} \sum_{i,j}^n \text{Cov}(X_i, X_j) = \frac{1}{n^2}\left(\sum_{i,j}^{n_g} 1 + n\right) = \frac{n_g^2}{n^2} + \frac{1}{n}.$$

Here, the asymptotic rate of $\bar{X}_n$ is also constant if $n_g \asymp n$. This is optimal for unbiased estimators by the Cramér-Rao Lower Bound. For any unbiased estimator T of $g(\mu)$,

$$\text{Var}(T) \geq \nabla_\mu g(\mu)^\top I(\mu)^{-1} \nabla_\mu g(\mu) = \left(\frac{1}{n}\mathbf{1}_n\right)^\top \Sigma^n \left(\frac{1}{n}\mathbf{1}_n\right) = \frac{n_g^2}{n} + \frac{1}{n}.$$

Note also that assuming $\mu = \tilde{\mu}\mathbf{1}_n$ with $\tilde{\mu} \in \mathbb{R}$ as in Corollary 5.10 there is potential for improvement through additional information. For instance, an unbiased estimator can do better than $\bar{X}_n$ if it knows which observations belong to the equi-correlated group. Whether and how such information can be obtained privately to construct better estimators is unclear to the authors at this point.

5.3 User-level DP Linear Regression for Longitudinal Data

While the body of literature on item-level differentially private linear regression is growing Wang (2018); Alabi et al. (2020); Liu et al. (2023); Avella-Medina et al. (2023); Brown et al. (2024); Bombari et al. (2025), little work addresses linear regression in user-level DP and nothing has been worked out for dependent longitudinal data. Here, we apply our Winsorized mean estimator to show that it can be used to estimate the regression coefficients $\beta \in \mathbb{R}^p$ in a user-level linear regression model as formalized in Definition 5.13.

Definition 5.13. *(User-Level Linear Model). Let $u \in [n]$ be users and $t \in [T]$ time points. Let $\beta \in \mathbb{R}^p$, $X^n = [X_1^\top, \ldots, X_n^\top]^\top \in \mathbb{R}^{nT \times p}$ and $\epsilon^n \in \mathbb{R}^{nT}$ where $\epsilon^n \sim \mathcal{N}(0, \Sigma^n)$. Assume Σ^n is block-diagonal with blocks $\Sigma_u \in \mathbb{R}^{T \times T}$. Define the user-level linear regression model through observations $Y^n := [Y_1^\top, \ldots, Y_n^\top]^\top = X^n\beta + \epsilon^n$.*

When $T = 1$ this user-level linear model reduces to the standard linear model with i.i.d. Gaussian errors. When $T > 1$ each user provides several observations that can be correlated due to the block-diagonal structure of the covariance matrix Σ^n. When T is reasonably large, a simple idea for privately estimating β is to estimate β locally for each user and combine these estimates by privately averaging the local non-private estimates over the n users. This strategy amounts to a divide and conquer algorithm that reduces the problem of user-level linear regression under DP to item-level

differentially private mean estimation. Since the non-private counterpart of the uDP estimator we have in mind is no longer a standard least squares estimator, it is natural to wonder how efficient this estimator is relative to standard and generalized least squares. The next lemma clarifies this point. Its proof can be found in Appendix E.3.

Lemma 5.14. *Let $X^n = [X_1^\top, \ldots, X_n^\top]^\top, Y^n = [Y_1^\top, \ldots, Y_n^\top]^\top, \Sigma^n = \mathrm{diag}(\Sigma_1, \ldots, \Sigma_n)$ be the design, observations and covariance matrix in Definition 5.13. Define the following estimators of β:*

1. *$\hat{\beta} := \frac{1}{n} \sum_{u=1}^n \hat{\beta}_u$ with $\hat{\beta}_u = (X_u^\top X_u)^{-1} X_u^\top Y_u$,*
2. *$\hat{\beta}_w := \frac{1}{n} \sum_{u=1}^n \hat{\beta}_{wu}$ with $\hat{\beta}_{wu} := (X_u^\top \Sigma_u^{-1} X_u)^{-1} X_u^\top \Sigma_u^{-1} Y_u$,*
3. *$\hat{\beta}_{OLS} := (\sum_{u=1}^n X_u^\top X_u)^{-1} \sum_{u=1}^n X_u^\top Y_u$,*
4. *$\hat{\beta}_{wOLS} := (\sum_{u=1}^n X_u^\top \Sigma_u^{-1} X_u)^{-1} \sum_{u=1}^n X_u^\top \Sigma_u^{-1} Y_u$.*

Then, $\hat{\beta}, \hat{\beta}_u, \hat{\beta}_w, \hat{\beta}_{wu}$ are unbiased for β and have the following variance relations:

(i) *$\mathrm{Cov}(\hat{\beta}_{OLS}) \not\preceq \mathrm{Cov}(\hat{\beta})$,*
(ii) *$\mathrm{Cov}(\hat{\beta}_{OLS}) \preceq \mathrm{Cov}(\hat{\beta})$ if $\Sigma_u = \sigma^2 I_T$,* $\qquad\qquad\qquad\qquad \forall u \in [n]$
(iii) *$\mathrm{Cov}(\hat{\beta}_{OLS}) = \mathrm{Cov}(\hat{\beta})$ if $X_u^\top X_u = D$ for some constant $D \succeq 0$,* $\qquad \forall u \in [n]$
(iv) *$\mathrm{Cov}(\hat{\beta}_{wOLS}) \preceq \mathrm{Cov}(\hat{\beta}_w) \preceq \mathrm{Cov}(\hat{\beta})$.*

Recall that by the Gauss-Markov Theorem $\hat{\beta}_{wOLS}$ is the most efficient unbiased linear estimator; more precisely for all unbiased linear estimator $\hat{\theta}$ we have $\mathrm{Cov}(\hat{\beta}_{wOLS}) \preceq \mathrm{Cov}(\hat{\theta})$. However, $\hat{\beta}_{wOLS}$ and its unweighted version $\hat{\beta}_{OLS}$ are not distributed estimators and cannot be privatized as easily as $\hat{\beta}$ and $\hat{\beta}_w$. Lemma 5.14 shows that having access to Σ_u, we should use $\hat{\beta}_w$ as it is a better divide-and-conquer estimator than $\hat{\beta}$. Yet, if we only have access to an upper bound on Σ_u and therefore cannot reweigh, using $\hat{\beta}$ is in some cases equally efficient as $\hat{\beta}_{OLS}$ and generally neither one dominates the other in the general case when $\Sigma_u \neq \sigma^2 I_T$. For a simple counterexample with $\Sigma_u \asymp I_T$ but with heterogeneous variances see the proof of Lemma 5.14 in Appendix E.3. Hence, even in some non-private settings $\hat{\beta}$ is a reasonable distributed estimator. We work with a private counterpart of this estimator that is straightforward to compute and analyze using our private mean estimator. The following lemma establishes the concentration result of the local per user estimators $\hat{\beta}_1, \ldots, \hat{\beta}_n$ that we need to get a finite sample guarantee for our private estimator. The proof of Lemma 5.15 is given in Appendix E.3.

Lemma 5.15. *Let $X^n = [X_1^\top, \ldots, X_n^\top]^\top$ and $Y^n = [Y_1^\top, \ldots, Y_n^\top]^\top$ be as in Definition 5.13. Define $\hat{M}_u := \frac{1}{T} X_u^\top X_u$ and assume $\theta I_p \preceq \hat{M}_u \preceq I_p \vartheta$ and $\Sigma_u \preceq \sigma^2 I_T$. Then, for the estimator $\hat{\beta} = \frac{1}{n} \sum_{u=1}^n \hat{\beta}_u$ with $\hat{\beta}_u = (X_u^\top X_u)^{-1} X_u^\top Y_u$, with probability at least $1 - \alpha$,*

$$\left\| \hat{\beta} - \beta \right\|_2 \leq \sqrt{\frac{p \vartheta \sigma^2}{n T \theta^2}} + \sqrt{\frac{2 \vartheta \sigma^2 \log(1/\alpha)}{n T \theta^2}}.$$

Further, $\hat{\beta}^n := [\hat{\beta}_1, \ldots, \hat{\beta}_n]^\top \in \mathbb{R}^{n \times p}$ is $(\tau, \gamma)_\infty^\beta$-concentrated with $\tau = \sqrt{2(\vartheta/\theta^2)\sigma^2 \log(2dn/\gamma)/T}$.

Lemma 5.15 allows us to instantiate Theorem 4.2, which immediately yields the following result.

Corollary 5.16. *Algorithm 5 is $(\varepsilon, \delta + \varrho)$-uDP on input $\hat{\beta}^n$ for $\delta, \varrho, \varepsilon \in (0, 1)$. Let $\gamma \in (0, 1 \wedge \frac{n}{4})$, and let $X^n \in \mathbb{R}^{nT \times p}$ and $Y^n \in \mathbb{R}^{nT}$ be as in Definition 5.13. Construct $\hat{\beta}^n \in \mathbb{R}^n$ as in Lemma 5.15 from X^n, Y^n. Then, with probability at least $1 - 3p\gamma - O(p^2 n/\delta \cdot e^{-\kappa \varepsilon' n})$ where $\kappa > 0$,*

$$\left\| \mathcal{A}(\hat{\beta}^n) - \beta \right\|_2 \lessapprox \sqrt{\frac{p \vartheta \sigma^2}{n T \theta^2}} + \sqrt{\frac{p^2 \vartheta \sigma^2}{\theta^2 T n^2 \varepsilon^2}}.$$

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6 Extensions to Local Differential Privacy

The central model of DP considered up to now requires that items/users trust a central server to aggregate and privatize their data. Whenever such trust is not given, items or users themselves have to privatize their data instead. Such a setting is called the *local* model of DP (Kasiviswanathan et al., 2011; Duchi et al., 2018).

Definition 6.1. *((ε, δ)-Differential Privacy, Local Model). Let $\varepsilon, \delta > 0$ and let $\mathcal{A} : \mathcal{X} \to \mathbb{R}^d$ be a randomized algorithm, mapping $x \in \mathcal{X}$ to a $\mathcal{B}(\mathbb{R}^d)$-measurable random variable $\mathcal{A}(x)$. Then, $\mathcal{A}$ is (ε, δ)-LDP in the local model if for all $x, y \in \mathcal{X}^n$ we have*

$$\mathbb{P}\left(\mathcal{A}(x_i) \in S\right) \leq e^{\varepsilon} \cdot \mathbb{P}\left(\mathcal{A}(y_i) \in S\right) + \delta, \quad \text{for all } S \in \mathcal{B}(\mathbb{R}^p) \text{ and } i \in [n].$$

Furthermore, we say that

1. *$\mathcal{A}$ is item-level-LDP if $x_i, y_i \in \mathbb{R}^{1 \times d}$ for all $i \in [n]$,*
2. *$\mathcal{A}$ is user-level-LDP if $x_i, y_i \in \mathbb{R}^{T \times d}$ for all $i \in [n]$.*

The randomized response mechanism in Algorithm 6 was the first randomized algorithm used to ensure this local notion of data privacy. It was already introduced by Warner (1965) in the context of survey sampling long before local DP was formalized. The randomized response mechanism is well-known to be ($\varepsilon, 0$)-LDP; see Dwork and Roth (2014, Section 3.2).

Theorem 6.2. *Algorithm 6 is ($\varepsilon, 0$)-LDP.*

Algorithm 6 RandomizedResponse(x, ε):

Require: $x \in \{0, 1\}$, $\varepsilon > 0$
 1: $u \leftarrow U \sim \text{Unif}[0, 1]$
 2: $\pi \leftarrow e^{\epsilon}/(1 + e^{\epsilon})$
 3: $\tilde{x} \leftarrow x \cdot \mathbf{1}\{U \leq \pi\} + (1 - x) \cdot \mathbf{1}\{U > \pi\}$
 4: **return** $\tilde{x}, \pi$

In the rest of this section we use the randomized response mechanism in conjunction with the Laplace mechanism of Algorithm 1 to lift the theory on Winsorized mean estimation under dependence developed so far to the local model – albeit under slightly stronger assumptions. In addition to $(\tau, \gamma)_\infty$-concentration of the data and log-Sobolev dependence or i.i.d. rows of the data matrix, we need to assume that $\|\mu\|_\infty \leq \mathsf{B}$. This is because we replace the stable histogram underlying our private midpoints by a histogram estimator based on randomized responses that cannot handle an infinite number of bins. Besides this replacement our approach remains unchanged and therefore proves to be substantially more general than existing techniques as we allow for unbounded dependent data. In particular, in the case of bounded i.i.d. observations we recover the upper bound in Theorem 6 by Kent et al. (2024). Modulo histogram estimation, the proofs of our results in the central and local model all build on general statements that we instantiate for both cases.

6.1 Histogram Estimator

Our local histogram estimator again allows for dependent observations. The estimator in Algorithm 7 is a histogram estimator that relies on the randomized response mechanism for privacy also in the local model. The histogram's construction is similar to that used by Kent et al. (2024). However, we do not require merging multiple neighboring bins and include a debiasing step that ensures that it is consistent. Unlike Algorithm 2 intended for the central model, this histogram can only handle

finitely many disjoint bins but it is $(\varepsilon, 0)$-LDP. Remember that the stable histogram is (ε, δ)-DP with $\delta > 0$.

Algorithm 7 RandomizedHistogram$(X, (B_k)_{k \in [N_{\mathsf{bins}}]}, \varepsilon, \delta)$:

Require: $X \in \mathbb{R}^n$, $\varepsilon, \delta = 0$
 1: **for all** $k \in [N_{\mathsf{bins}}]$ **do**
 2:     $c_{ik} \leftarrow \mathbf{1}\{X_i \in B_k\}$ $\forall i \in [n]$
 3:     $\tilde{c}_{ik}, \pi \leftarrow \text{RandomizedResponse}(c_{ik}, \varepsilon/2)$ $\forall i \in [n]$
 4:     $\tilde{p}_k \leftarrow \frac{1}{n} \sum_{i=1}^n \tilde{c}_{ik}$
 5:     $\tilde{p}_k^{\mathrm{debiased}} \leftarrow \frac{\tilde{p}_k - (1-\pi)}{2\pi - 1}$
 6: **return** $\tilde{\boldsymbol{p}} := (\tilde{p}_1^{\mathrm{debiased}}, \tilde{p}_2^{\mathrm{debiased}}, \ldots, p_{N_{\mathsf{bins}}}^{\mathrm{debiased}})$

In Lemma 6.3 we provide a privacy and utility guarantee for Algorithm 7 when estimating the true probability mass $p_1, p_2, \ldots, p_{N_{\mathsf{bins}}}$ of bins $(B_k)_{k \in [N_{\mathsf{bins}}]}$ given log-Sobolev dependent or i.i.d. data. The proof can be found in Appendix F.2

Lemma 6.3. *Algorithm 7 is $(\epsilon, 0)$-LDP for $\varepsilon > 0$. Let $\eta > 0$, and $X^n \in \mathbb{R}^n$ fulfill Assumption 2.7 (A1) or Assumption 2.8 (A2) with constants (ρ, M). Further, let $(B_k)_{k=1}^{N_{\mathsf{bins}}}$ be a finite collection of disjoint bins covering $[-\mathsf{B}, \mathsf{B}] \subset \mathbb{R}$ and define $p_k := \mathbb{P}(X_1 \in B_k)$. Then, for the output $\tilde{\boldsymbol{p}}$ we have*

$$\mathbb{P}\left(\max_{k \in N_{\mathsf{bins}}} |\tilde{p}_k - p_k| \geq \eta\right) \leq 2N_{\mathsf{bins}} \cdot \exp\left(-\frac{n\eta^2}{4} \tanh^2\left(\frac{\varepsilon}{4}\right)\right) + \begin{cases} 2\exp\left(-\frac{n\eta^2}{8}\right) & (A1), \\ \frac{16}{\eta}\exp\left(-\frac{2}{27}\frac{n\eta^3}{4^3 \rho M^2}\right) & (A2). \end{cases}$$

Lemma 6.3 is a local DP counterpart of Lemma 3.2 in Section 3. The lemma mainly differs in that it assumes a finite number of bins, and provides a $(\varepsilon, 0)$-LDP guarantee instead of the (ε, δ)-DP guarantee of the stable histogram. Since both lemmas are shown by decomposing the error $|\tilde{p}_k - p_k|$ into the privacy loss $|\tilde{p}_k - \hat{p}_k|$ and the estimation error $|\hat{p}_k - p_k|$, the second term in the right hand side of both statements is identical. Their first terms, obtained by bounding the privacy loss with high probability differs in that the $1/\delta$ factor in front of the exponential is replaced by the number of bins N_{bins} and the privacy budget parameter ε shows up as $\tanh^2(\varepsilon)$ instead of just ε. When $\varepsilon \lesssim 1$, we bound this instead by ε^2 which gives minimax optimal rates for the local Winsorized mean estimator as we will see below.

6.2 Mean Estimators

The construction of our local mean estimators are analogous to their central model counterparts in Section 4. Here, the only difference is that we use the randomized histogram when estimating the private midpoint of the projection interval and every item or user privatize their projected observation by adding Laplace noise before sending it to the central server. Algorithm 8 formalizes this protocol. Remember that the randomized histogram can only handle a finite number of bins and thus the algorithm requires $\mathsf{B} \in (0, \infty)$ s.t. $\|\mu\|_\infty \leq \mathsf{B}$ as input.

Algorithm 8 WinsorizedMean1D$(X^n, \tau, \varepsilon, \delta, \mathsf{B})$

Require: $X^n \in \mathbb{R}^n$, $\tau, \varepsilon, \delta > 0$
 1: $\hat{I} \leftarrow \text{ProjectionInterval}(X, \tau, \frac{\varepsilon}{2}, \delta, B)$ where $\hat{I} = [\hat{m} \pm 3\tau]$ Algorithm 9
 2: **return** $\frac{1}{n} \sum_{i=1}^n \Pi_{\hat{I}}(X_i) + \xi_i$ with i.i.d. $\xi_i \sim \mathrm{Lap}(0, \frac{12\tau}{\varepsilon})$

In Lemma F.5 we show that $\mathcal{A}(X^n) = \bar{X}_n + \bar{\bar{\Xi}}_n$ with high probability, where $\bar{\bar{\Xi}}_n = \frac{1}{n}\sum_{i=1}^{n} \Xi_i$ with $\Xi_i \sim \text{Lap}\left(0, \frac{12\tau}{\varepsilon'}I_d\right)$ and $\varepsilon' = \varepsilon/\sqrt{8d\log(1/\varrho)}$. We exploit this equivalence to provide the finite-sample error bound in Theorem 6.4. The proof is given in Appendix F.5.

Theorem 6.4. *Let $\mathcal{A}$ denote Algorithm 5 calling Algorithm 8 in line 3. $\mathcal{A}$ is (ε, ϱ)-LDP for $\varepsilon, \varrho \in (0, 1)$. Let $\kappa > 0$, $\gamma \in (0, 1 \wedge \frac{n}{4})$ and let $X^n \in \mathbb{R}^{n \times d}$ be $(\tau, \gamma)_\infty^\mu$-concentrated with $\|\mu\|_\infty \leq \mathsf{B}$ and make Assumption 2.7 or 2.8 s.t. $\rho M^2 \lesssim 1$. Then, with probability at least $1 - 2d\gamma - O(\mathsf{B}/\tau + 1) \cdot de^{-\kappa n(\varepsilon')^2}$,*

$$\|\mathcal{A}(X^n) - \mu\|_2 \lesssim \|\bar{X}_n - \mu\|_2 + \tau\sqrt{\frac{d^2\log(1/\varrho)\log(3/(d\gamma))^2}{n\varepsilon^2}}.$$

As is typical also for local DP, the error rate in Theorem 6.4 is additive in the statistical rate and a cost of privacy term. Here, classical statistical rates of order $\sqrt{d\rho/n}$ are dominated by the cost of privacy even for fixed $\varepsilon > 0$ due to the dependence in d and $\log(1/\varrho)$. Further, the dependence on ε differs significantly from our result in the central model, because any dependence of the form $\varepsilon \asymp 1/n^\omega$ for $\omega > 0$ immediately affects the error rate in n.

In the local model, the high probability event $\mathcal{A}(X^n) = \bar{X}_n + \bar{\bar{\Xi}}_n$ easily translates into a MSE in-expectation bound. This is, because we can control the error in the complement event using that $\|\hat{m}\|_\infty \leq \mathsf{B} + \tau$. This inequality holds by design of the local private midpoint estimation in Algorithm 9. The next result is proven in Appendix F.5

Theorem 6.5. *Let $\mathcal{A}$ denote Algorithm 5 calling Algorithm 8 in Line 3. $\mathcal{A}$ is (ε, ϱ)-LDP for $\varepsilon, \varrho \in (0, 1)$. Further, assume $X^n \in \mathbb{R}^{n \times d}$ is $(\tau, \gamma)_\infty$-concentrated around $\mu \in \mathbb{R}^d$ with $\|\mu\|_\infty \leq \mathsf{B}$. Make Assumption 2.7 or Assumption 2.8 s.t. $\rho M^2 \lesssim 1$. Then,*

$$\mathbb{E}[\|\mathcal{A}(X^n) - \mu\|_2^2] \lesssim \mathbb{E}\left[\|\bar{X}_n - \mu\|_2^2\right] + \frac{d^2\tau^2\log(1/\varrho)}{n\varepsilon^2} + \begin{cases} d^2\mathsf{B}^2 \cdot \left(\gamma + \frac{\mathsf{B}/\tau}{e^{\kappa n(\varepsilon')^2}}\right) & \text{if } \mathsf{B} \geq \tau \\ d^2\tau^2 \cdot \left(\gamma + \frac{1}{e^{\kappa n(\varepsilon')^2}}\right) & \text{if } \mathsf{B} \leq \tau. \end{cases}$$

The interpretation of this result is the same as for the in-probability bound. Statistical rates of order $d\rho/n$ are dominated by the cost of privacy term if we choose $\gamma \asymp 1/(n\varepsilon^2)$. To give a clean interpretation we instantiate the theorem in the user-level setting that subsumes the item-level one. The resulting statement is given in Corollary 6.6 below.

Corollary 6.6. *Let $\mathcal{A}$ denote Algorithm 5 calling Algorithm 8 in Line 3. Then, $\mathcal{A}$ is (ε, ϱ)-uLDP for $\varepsilon, \varrho \in (0, 1)$. Further, let $X^n \in \mathbb{R}^{nT \times d}$ and $\hat{\mu}^n$ be as in Definition 5.1. Let X^n have rows with mean $\mu \in \mathbb{R}^d$ s.t. $\|\mu\|_\infty \leq \mathsf{B}$ and fulfill Assumption 2.7 with ρ-sub-Gaussian entries or Assumption 2.8 s.t. $\rho M^2 \lesssim 1$. Then, choosing $\tau^2 \asymp \rho\log(2dTn^2\varepsilon^2)/T$,*

$$\mathbb{E}[\|\mathcal{A}(X^n) - \mu\|_2^2] \lesssim \frac{d^2\rho\log(2dTn^2\varepsilon^2)\log(1/\varrho)}{Tn\varepsilon^2} + \begin{cases} d^2\mathsf{B}^2 \cdot \left(\frac{1}{Tn\varepsilon^2} + \frac{\mathsf{B}}{\sqrt{\rho}}\frac{\sqrt{T}}{e^{\kappa n(\varepsilon')^2}}\right) & \text{if } \mathsf{B} \geq \tau \\ 0 & \text{if } \mathsf{B} \leq \tau. \end{cases}$$

Above, we again distinguish the cases of whether or not our apriori bound $\|\mu\|_\infty \leq \mathsf{B}$ localizes the mean more than the concentration radius τ or not. Whether we are in the item- or user-level setting plays an interesting role in this distinction. In the item-level setting where $T = 1$ the concentration radius τ grows in n and at some point will surpass B. Thus, $\mathsf{B} \leq \tau$ is the more relevant case there and when $d = 1$ our bound attains the $1/(n\varepsilon^2)$-rate that Corollary 1 Duchi et al. (2018) predicts for bounded random variables. More interestingly, Corollary 6.6 generalizes this result to ρ-sub-Gaussian random variables with potentially unbounded domain and bounded first moment. For small T, the user-level setting behaves the same as the item-level one. However, as T grows beyond

$\rho \log(2dTn^2\varepsilon^2)/\mathsf{B}^2$ we pay the second term in the upper bound which corresponds to the prize of estimating the projection interval. If we specialize to observations that are bounded in L_∞-norm, for which $\rho = \mathsf{B}^2$ and we set $\mathsf{B} = 1$, our result recovers Theorem 6 by Kent et al. (2024) up to logarithmic factors.

6.2.1 Extensions beyond Item-Level Mean Estimation

Instantiating Theorem 6.4 immediately yields finite-sample risk bounds for item-level nonparametric regression, user-level mean estimation, random effects location estimation and linear regression with longitudinal data. The following gives a brief overview on these results, starting with nonparametric regression in Corollary 6.7. This result is a local DP analog of Corollary 4.11.

Corollary 6.7. *Let $\mathcal{A}$ denote Algorithm 5 taking $\hat{f}^n(x)$ as input with $x \in [0,1]$. $\mathcal{A}$ is $(\varepsilon,0)$-DP for $\varepsilon \in (0,1)$. Let $\gamma \in (0, 1 \wedge \frac{n}{4})$ and assume Y^n is as in Definition 4.7 and $\sigma_{\max}^2 \gtrsim 1$. Then, for all $x \in [\zeta, 1-\zeta]$ where $\zeta = b\sqrt{2\log(2/b)}$ and $b > 0$, with probability at least $1 - 3\gamma - O(\|f\|_\infty/(L_f b) \cdot e^{-\kappa n \varepsilon^2})$,*

$$\left| \mathcal{A}(\hat{f}^n(x)) - f(x) \right| \lesssim b + \sigma_{\max} \left(\frac{1}{nb} + \sqrt{\frac{1}{nb}} + \sqrt{\frac{1}{nb^2\varepsilon^2}} \right).$$

For an optimally chosen $b \asymp (\sigma_{\max}^2/n)^{1/3} \vee (\sigma_{\max}/\sqrt{n}\varepsilon)^{1/2}$ this becomes

$$\left| \mathcal{A}(\hat{f}^n(x)) - f(x) \right| \lesssim \left(\frac{\sigma_{\max}^2}{n} \right)^{1/3} \vee \sqrt{\frac{\sigma_{\max}}{\sqrt{n}\varepsilon}} \lesssim \left(\frac{\sigma_{\max}^2}{n} \right)^{1/3} + \sqrt{\frac{\sigma_{\max}}{\sqrt{n}\varepsilon}}.$$

The pointwise error rate in Corollary 6.7 shown in Appendix F.6 resembles the almost optimal integrated in-expectation rate in Theorem 3.1 and Theorem 4.1 by Györfi and Kroll (2025) obtained with basis transformation estimators. Our result privatizes the Priestley-Chao regression estimator of Definition 4.8 and thus complements their approach (See also Subsection 2.2.3, Kent et al. (2024)).

The next three results concern user-level estimation for mean estimation, location random effects models and longitudinal regression. We first give a general user-level mean estimation result that is the local counterpart of Corollary 5.4.

Corollary 6.8. *Let $\mathcal{A}$ denote Algorithm 5 calling Algorithm 8 in Line 3 with input $\hat{\mu}^n \in \mathbb{R}^{n \times d}$ in Definition 5.1. Then $\mathcal{A}$ is (ε, ϱ)-uLDP for $\varepsilon, \varrho \in (0,1)$. Further, let $\gamma \in (0, 1 \wedge \frac{n}{4})$ and let $X^n \in \mathbb{R}^{nT \times d}$ be a user-level data matrix with mean s.t. $\|\mu\|_\infty \leq \mathsf{B}$ that fulfills Assumption 2.8 s.t. $\rho M^2 \lesssim 1$. Then, with probability at least $1 - 3d\gamma - \tilde{O}(\sqrt{T/\rho} \cdot \mathsf{B} \cdot d e^{-\kappa n (\varepsilon')^2})$,*

$$\|\mathcal{A}(\hat{\mu}^n) - \mu\|_2 \lesssim \sqrt{\frac{d\rho \log(4d/\gamma)}{nT}} + \sqrt{\frac{d^2\rho \log(2dn/\gamma)\log(1/\varrho)\log(3/(d\gamma))^2}{Tn\varepsilon^2}}.$$

Note that the cost of privacy here always dominates the statistical rate of order $\sqrt{d/(nT)}$. While we have an exponentially quickly decaying exponential term in the probability like in the central model, the probability now grows in T. Parallel to the user-level mean estimator in Kent et al. (2024), it is therefore sensible to only consider $T \leq e^{\kappa n (\varepsilon')^2}$ many observations per user to prevent the probability from blowing up as only $T \to \infty$ while n stays fixed. The same holds for the direct corollary below that applies Corollary 6.8 in the setting of a user-level random effects location model.

Corollary 6.9. *Let $\mathcal{A}$ denote Algorithm 5 calling Algorithm 8 in Line 3 on input $\hat{\mu}^n$. $\mathcal{A}$ is $(\varepsilon, 0)$-uDP for $\varepsilon \in (0,1)$. Let $\gamma \in (0, 1 \wedge \frac{n}{4})$, $n_{g^\star}$ be the maximum group size and let $Y^n \in \mathbb{R}^{nT}$ and $\hat{\mu}^n$ be as*

in Definition 5.8. Assuming that $\|\mu\|_\infty \leq \mathsf{B}$, with probability at least $1 - 3\gamma - \tilde{O}(\sqrt{T/\rho} \cdot \mathsf{B} \cdot e^{-\kappa n \varepsilon^2})$,

$$\|\mathcal{A}(\hat{\mu}^n) - \mu\|_2 \lesssim\approx \sqrt{\frac{\sigma_U^2 \cdot n_{g^\star} T + \sigma^2}{nT}} + \sqrt{\frac{\sigma_U^2 \cdot n_{g^\star} T + \sigma^2}{Tn\varepsilon^2}}.$$

Beware that we suppress log-factors and the cost of privacy again dominates, even when $d = 1$ and $\varepsilon \asymp 1$. Clearly, any privacy budget $\varepsilon \asymp n^{-\omega}$ with $\omega > 0$ decreases the rate below $1/n$. The effect of dependence on estimation is the same as discussed in Remark 5.11 covering the central model.

Lastly, we revisit user-level linear regression in Corollary 6.10.

Corollary 6.10. *Let $\mathcal{A}$ denote Algorithm 5 calling Algorithm 8 in Line 3 on input $\hat{\beta}^n$ constructed as in Lemma 5.15 from X^n, Y^n. Then, $\mathcal{A}$ is (ε, ϱ)-uLDP for $\varepsilon, \varrho \in (0, 1)$. Further, let $\gamma \in (0, 1 \wedge \frac{n}{4})$. Assume that X^n, Y^n follow the model in Definition 5.13 with true regression parameter β s.t. $\|\beta\|_\infty \leq \mathsf{B}$. Then, with probability at least $1 - 3p\gamma - \tilde{O}(\sqrt{T\rho} \cdot \mathsf{B} \cdot pe^{-\kappa n(\varepsilon')^2})$,*

$$\left\|\mathcal{A}(\hat{\beta}^n) - \beta\right\|_2 \lesssim\approx \sqrt{\frac{p\vartheta\sigma^2}{nT\theta^2}} + \sqrt{\frac{p^2\vartheta\sigma^2}{\theta^2 Tn\varepsilon^2}}.$$

Note that to keep the probability of the error bound small in Corollary 6.10 we should use only $T \leq e^{\kappa n(\varepsilon')^2}$ many observations per user. This is again similar to the results in Kent et al. (2024).

7 Simulations

We examine empirically the finite sample guarantees of our noisy Winsorized mean estimator. While our simulations focus on item-level mean estimation in the central model, we also briefly discuss user-level mean estimation and mean estimation under the model of local DP.

7.1 Item-Level Mean Estimation

For our item-level simulations we consider a random sample of dependent univariate Gaussian random variables representing univariate observations of n dependent individuals. More precisely, we have observations of the form $X^n \sim \mathcal{N}(\mu\mathbf{1}_n, \Sigma^n)$ with $\mu \in \mathbb{R}$ and $\Sigma^n \in \mathbb{R}^{n \times n}$. In this case, X^n is log-Sobolev dependent with constant $\rho = \|\Sigma^n\|_{op}$. Hence, we can easily consider both i.i.d. observations where $\Sigma^n \asymp I_n$, and weak and strongly dependent X_i where $\|\Sigma^n\|_{op} \asymp 1$ and $\|\Sigma^n\|_{op} \asymp n^\omega$ for $\omega > 0$. As in our theoretical analysis, we mostly assume the log-Sobolev constant $\rho = \|\Sigma^n\|_{op}$ to be known. On top, without loss of generality, we assume that $\Sigma_{ii}^n = 1$ for all $i \in [n]$ to fix a scale.

7.1.1 Cost of Privacy in Small Samples

It is well known that for a privacy parameter $\varepsilon \gtrsim 1/\sqrt{n}$ the statistical rate dominates the cost of privacy in mean estimation in the central model. Hence, for $\varepsilon \gtrsim 1/\sqrt{n}$ and in particular for constant ε we expect the MSE rate of our Winsorized mean estimator to be of order $1/n$. This suggests that privacy can be obtained almost for free in this regime. However, when considering real world applications constants enter the picture and especially on small samples the concrete choice of ε expresses itself in substantially different MSEs for fixed n. To showcase this, Figure 1 compares the non-private empirical mean estimator $\bar{X}_n$ with our private estimator $\mathcal{A}$ of Algorithm 4 parametrized with privacy budget parameters $\varepsilon \in \{1/\sqrt{n}, 0.1, 0.5, 1\}$ and $\delta = 1/n^2$. The two subfigures on the left depict functionals of the squared errors over 2000 replications of the empirical measure at increasing sample sizes n. Concretely, the upper part depicts the MSE, the median squared error and the
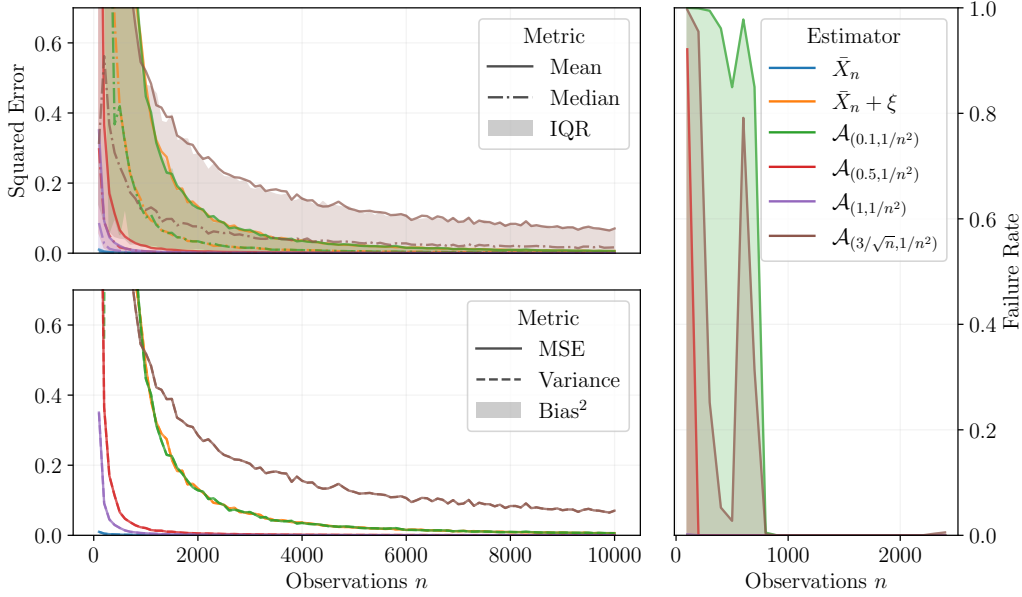
Figure 1: Comparison of the empirical mean $\bar{X}_n$ with Algorithm 4, denoted $\mathcal{A}_{(\varepsilon,\delta)}$.

interquartile range (IQR) of squared errors. The lower part shows a bias-variance decomposition where the bias squared is given as the pointwise distance between the solid and dashed line for fixed n. As expected, increasing ε lets the private estimators approach the non-private $\bar{X}_n$. As theory predicts, the estimator with $\varepsilon \asymp 1/\sqrt{n}$ has a $1/n$ rate throughout. In contrast, when $\varepsilon \in \{0.1, 0.5, 1\}$ The cost of privacy is of order $1/n^2$ stemming from the added Laplace noise, but for small sample sizes the privacy error can dominate the statistical error of order $1/n$. This interpretation is supported by the non-private $\bar{X}_n + \xi$ that does not experience truncation but still behaves like $\mathcal{A}_{(0.1,1/n^2)}$ and the bias-variance decomposition that shows that the MSE of $\mathcal{A}$ is almost entirely driven by variance. In fact, the bias is not even visible for $\mathcal{A}$. The right half of Figure 1 presents the empirical failure rate of the stable histogram underlying $\mathcal{A}$, i.e., the percentage of times Algorithm 3 outputs $\mathbf{0}$. We observe that this rate rapidly goes to 0 and few non-zero values occur for $n \geq 1000$.

7.1.2 Improving Constants Theoretically and Empirically

We revisit our projection interval construction in order to further reduce the cost of privacy. In Remark 3.5 we noted that $(\tau, \gamma)_\infty$-concentration of $X^n \in \mathbb{R}^n$ is not necessary for Lemma 3.4 to hold. Instead, for the private midpoint algorithm to output a crude mean estimate not further than $2\tau'$ from μ, it suffices that each X_i is $(\tau', \gamma)_\infty$-concentrated with $\gamma \in (0, \frac{1}{4})$. More concretely, if X^n is log-Sobolev dependent with constant ρ, $(\tau')^2 \asymp \rho \log(2/\gamma)$ is $\rho \log(n)$ smaller than $\tau^2 \asymp \rho \log(2n/\gamma)$. Such a decrease in the size of the projection interval immediately yields privacy noise with smaller variance and therefore smaller constants in the MSE upper bounds and also simulations as can be seen in Figure 2. This is obtained through bins of width $2\tau'$ in the histogram. Carrying this change through the analysis leads to $\hat{I} = [\hat{m} \pm \tau + 2\tau']$ instead of $\hat{I} = [\hat{m} \pm 3\tau]$ in Algorithm 4 to ensure that no X_i is affected by the projection with high probability. On top, we also exploit Remark 5.3 and use the marginal log-Sobolev constants.
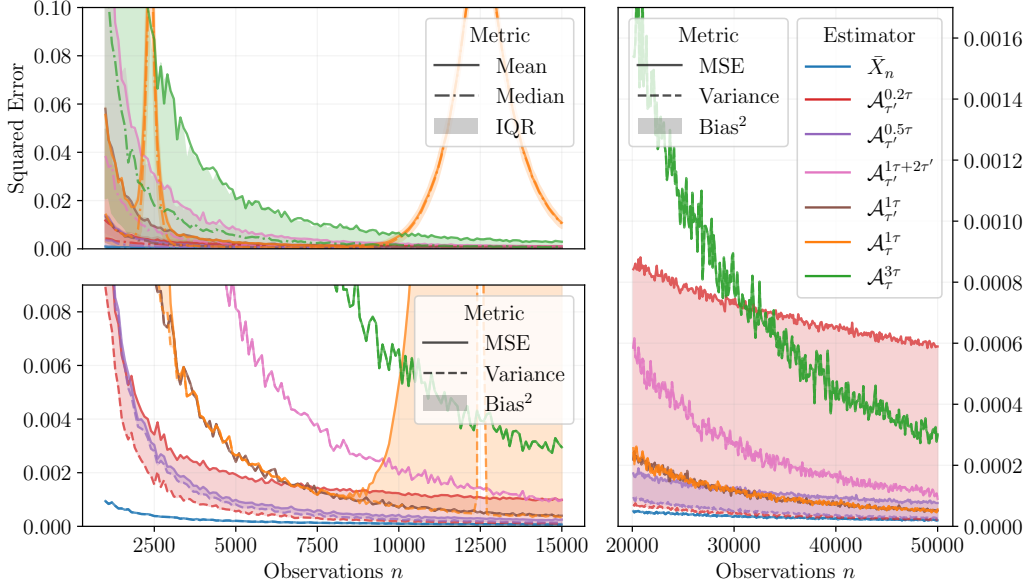
Figure 2: Comparison of $(0.1, 1/n^2)$-DP estimators $\mathcal{A}_h^\kappa$ where κ is the radius of the projection interval and h the length of the histogram bins. Note that here $\tau' \asymp \sqrt{\log(2/\gamma)}$ and $\tau \asymp \sqrt{\log(2n/\gamma)}$.

We further optimize constants empirically. For this, we fix privacy budget parameters $(\varepsilon, \delta) = (0.1, 1/n^2)$ and run Algorithm 4 with different projection intervals $\hat{I}$ and potentially adjusted binwidths for the underlying histogram estimator. Firstly, this lets us empirically assess the impact of choosing the projection interval $\hat{I} = [\hat{m} \pm \tau + 2\tau']$ instead of $\hat{I} = [\hat{m} \pm 3\tau]$. Secondly, we experiment with different combinations of binwidths for the stable histogram and even smaller projection intervals. In Figure 2 we compare Winsorized mean estimators $\mathcal{A}_h^\kappa$ where $\kappa > 0$ indicates the radius of the projection interval $\hat{I}$ and h the binwidth used by the histogram. Note that the resulting algorithms are not necessarily grounded in theory, but $\mathcal{A}_\tau^{3\tau}$ and $\mathcal{A}_{\tau'}^{\tau+2\tau'}$ are. Comparing these two, we observe that our theoretical approach to reducing constants already considerably decreases the MSE. Our experiments go beyond our theory with the private Winsorized estimators $\mathcal{A}_{\tau'}^\kappa$ for $\kappa \in \{1\tau, 0.5\tau, 0.2\tau\}$. The subfigure on the lower left of Figure 2 suggests that an interval length of 1τ still works empirically and results in even smaller MSE. For an interval length of 0.5τ, the same subfigure suggests that introducing reasonable bias through a more aggressive projection decreases the MSE for the $n \leq 50,000$ we consider, but for 0.2τ the bias already dominates for small n and cannot be offset by the decreased variance. The right hand side of the figure suggests that introducing bias by shrinking the length of $\hat{I}$ is particularly beneficial for small sample sizes. Yet, for bigger n even bias that is at first reasonable ends up pushing the MSE above that of estimators with small or no bias.

Our empirical results indicate that our bounds are sharp in the sense that shrinking the bin width aggressively beyond our theory eventually worsens the rate. Indeed, Algorithm $\mathcal{A}_\tau^{1\tau}$ shows that a less principled approach where we simply shrink the length of the projection interval without adjusting the bin width fails as it introduces excess bias and variance. As the edge between some bins B_k and B_{k+1} approaches the true mean μ from the left as n grows, the midpoint of B_{k+1} that initially contains μ moves away from μ and for symmetric distributions like our Gaussian, up to 50% of the data are eventually projected. This explains that the bias first increases and then decreases again

when B_k has more mass and its midpoint approaches μ from the left. When the edge is almost exactly at μ, the estimator's variance explodes because single observations let the midpoint $\hat{m}$ jump between the midpoints of B_k and B_{k+1}.

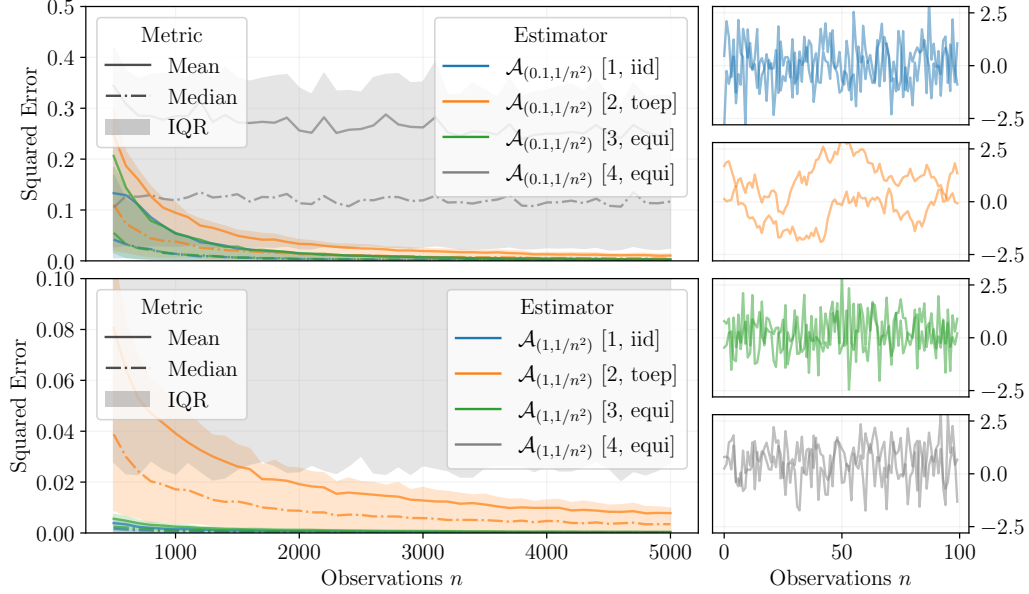### 7.1.3 Cost of Dependence in Small Samples

Figure 3: (ε, δ)-DP estimators $\mathcal{A}_{(\varepsilon, \delta)}$ on $X^n \in \mathbb{R}^n$ with different covariances $\Sigma^n \in \mathbb{R}^{n \times n}$.

Our theory predicts that the main driving factor of the MSE rate under log-Sobolev dependence is the log-Sobolev constant $\rho = \|\Sigma^n\|_{op}$. Thus, whenever $\|\Sigma^n\|_{op}$ is constant, our estimators should have the same $1/n$ or $1/(n^2 \varepsilon^2)$ but MSEs differing in constants. At the same time, when $\|\Sigma^n\|_{op} \asymp n$ we expect the MSE to converge to a non-zero value. To verify this, Figure 3 depicts the MSE, median squared error and IQR of squared errors for (ε, δ)-DP estimators $\mathcal{A}_{(\varepsilon, \delta)}$ on observations X^n with $\Sigma^n = I_n$, Σ^n a Toeplitz covariance matrix with decay 0.95, i.e., $\Sigma^n = (0.95^{|j-i|})_{i,j \in [n]}$ and two equi-covariance matrices Σ^n with variance and covariance pairs $(1, 1/(n-1))$ and $(1, 1/4)$, respectively. We depict two samples from each of these models in the four subfigures on the right of Figure 3. On the left, the upper subplot depicts a high privacy regime where $(\varepsilon, \delta) = (0.1, 1/n^2)$. Here, we see that the three weakly dependent settings where $\|\Sigma^n\|_{op} \asymp 1$ behave similarly and have comparable squared errors. In turn, in the lower subplot where $(\varepsilon, \delta) = (1, 1/n^2)$, the three weakly dependent settings have considerably different squared errors and the Toeplitz model far exceeds the others. To understand this note that Remark 5.3 implies that the cost of privacy term only depends on the marginal log-Sobolev constants, which here are the $\Sigma_{ii}^n = 1$. The statistical rate, in turn, always depends on $\|\Sigma^n\|_{op}$ which, e.g., for the Toeplitz model is ≈ 38. When $\varepsilon = 0.1$, the privacy term seems to dominate whereas for $\varepsilon = 1$, it is small enough so that the different constants in the statistical rate are visible. In both subplots, the strongly dependent observations in the second equi-covariance model have MSE converging to $1/4$ with rate $1/n$.
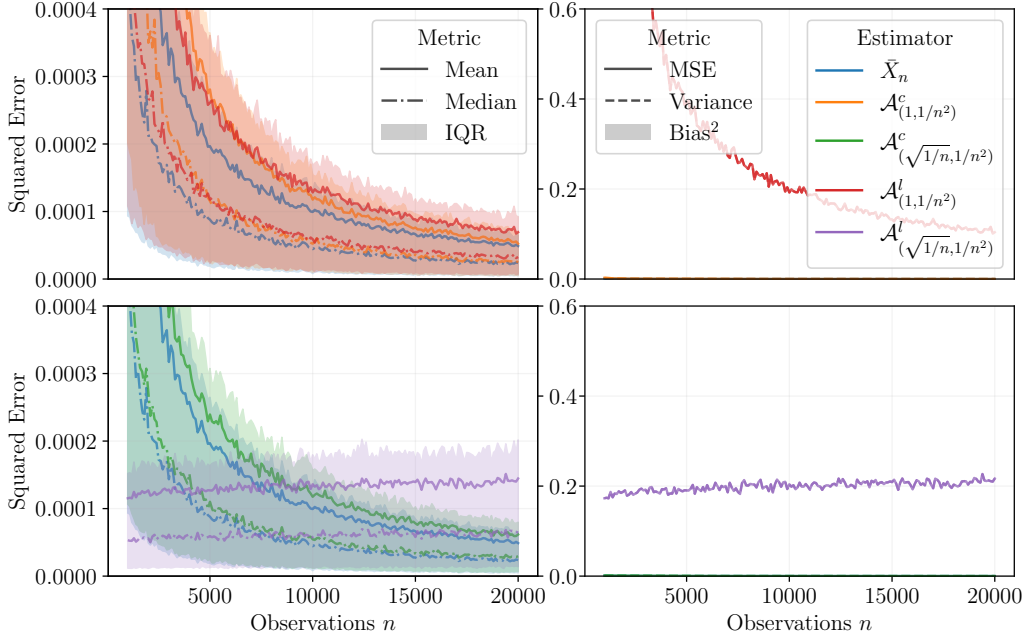
7.1.4 Central versus Local Model

Figure 4: Comparison of (ε, δ)-DP and LDP estimators $\mathcal{A}^c_{(\varepsilon,\delta)}$ and $\mathcal{A}^l_{(\varepsilon,\delta)}$.

In the central and the local model, the privacy budget parameter ε directly impacts the MSE-rate. Ignoring constants and logarithmic terms our theory predicts that the cost of privacy is $1/(n^2\varepsilon^2)$ and $1/(n\varepsilon^2)$, respectively. Thus, when ε is constant in n, both models of privacy have the same $1/n$-estimation rate. This changes as $\varepsilon \asymp 1/n^\omega$ for $\omega > 0$. Then, our theory predicts that in the central model for $0 \leq \omega \leq 1/2$ the statistical rate dominates the cost of privacy. In the local model, however, for any $\omega > 0$ the cost of privacy should immediately dominate the statistical rate. We investigate this difference in the simulations in Figure 4. The left hand side of the plot uses two different axes where $\bar{X}_n$ and the central estimators $\mathcal{A}^c$ use the left axis and the local estimators $\mathcal{A}^l$ are plotted on the middle axis that is shared with the right hand side of the figure. There, we plot the estimators on their true scale. This layout allows us to compare the rate and the magnitude of the MSEs. We use $\varepsilon = 1$ in the two subfigures on top. While we observe a rate of $1/n$ in both models of DP, the right hand side shows that constants are substantially larger when privatizing locally. In the bottom, $\varepsilon \asymp \sqrt{1/n}$ and we still observe an MSE-rate of $1/n$ in the central model whereas the MSE in the local model is constant or even slightly increasing due to logarithmic terms.

7.1.5 Plug-in Variance Estimation

So far we assumed that $\rho = \|\Sigma^n\|_{op}$ or more precisely the Σ_{ii} are known a priori. To make our estimator more applicable, we now present a way to estimate the concentration radius of the data τ in a private and data-driven way. For this, we rely on Remark 5.3 which tells us that to compute τ it is enough to know the marginal sub-Gaussian variance proxies or log-Sobolev constants ρ_i corresponding to the distribution of $X_i \in \mathbb{R}$ for all $i \in [n]$. For Gaussian observations that we
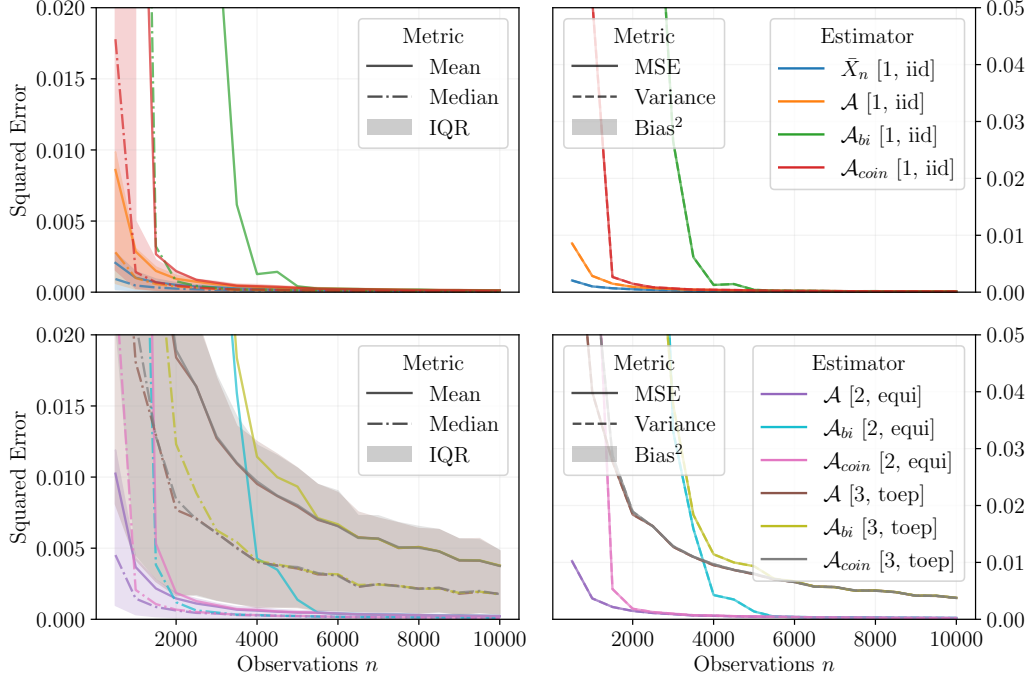
Figure 5: Comparison of the empirical mean $\bar{X}_n$ with $(1, 1/n^2)$-DP estimator $\mathcal{A}$ using known variance and $(2, 1/n^2)$-DP estimators $\mathcal{A}_{bi}$ and $\mathcal{A}_{coin}$ using plug-in variance estimators.

investigate here, these constants are the marginal variances. We therefore introduce two private variance estimators that we use as plug-ins for the $\rho_i = \Sigma_{ii}$. Both get a very rough estimate of the first moment, and upper and lower bounds of the variance as input and iteratively and alternatingly refine them. For the variance step, Algorithm 10 in Appendix G implements a bisection that starts with the upper bound on the variance and halves it unless the lower bound is reached or a confidence interval around the first moment has too low coverage. The variance step of Algorithm 13 in Appendix G is that of the CoinPress Algorithm due to Biswas et al. (2020). We call the resulting Winsorized mean estimators with these plug-in variance estimators $\mathcal{A}_{bi}$ and $\mathcal{A}_{coin}$, respectively. Figure 5 showcases their performance with privacy budget parameters $(\varepsilon, \delta) = (1, 1/n^2)$ each for the mean and the variance estimation. For the true mean and variance we use $\mu = 100$ and $\sigma^2 = 1$ whereas the provided rough first moment estimate is 300 and the upper and lower bounds on the variance are $(0.1, 10000)$. We use three dependence models once with an i.i.d., equi-covariance and Toeplitz covariance matrix: $\Sigma^n = I_n, \Sigma^n = 3/4 \cdot I_n + 1/4 \cdot \mathbf{1}_n \mathbf{1}_n^\top$ and $\Sigma^n = (0.95^{|j-i|})_{i,j \in [n]}$. The small difference between $\bar{X}_n$ and the $(1, 1/n^2)$-DP estimator $\mathcal{A}$ using the known variance suggests that we are in a regime with low cost of privacy for mean estimation. Keeping in mind that we allow for an additional privacy budget for the variance estimation, the adjusted CoinPress clearly outperforms the bisection for smaller $1000 \leq n \leq 6000$. For large enough $n \geq 6000$, the performance of the plug-in estimators is almost indistinguishable from that of $\mathcal{A}$. This suggests that there the cost of privacy paid for variance estimation is low.

7.2 User-Level Mean Estimation

In our user-level simulations we assume that n users each contribute T observations collected in $X_u \in \mathbb{R}^T$. For the resulting user-level data matrix $X^n := [X_1^\top, \ldots, X_n^\top]^\top \in \mathbb{R}^{nT}$ we want to assume log-Sobolev dependence. For our simulations we therefore suppose that $X^n \sim \mathcal{N}(\mu \mathbf{1}_{nT}, \Sigma^n)$ where $\Sigma^n \in \mathbb{R}^{nT \times nT}$. Throughout, we assume that $\Sigma^n = \mathrm{diag}(\Sigma_1, \cdots, \Sigma_n)$ where Σ_u are Toeplitz covariance matrices with decay 0.95. This means that we model users to be independent but their repeated measurements to have decaying covariances over time. The results are depicted in Figure 6. The plot's left hand side shows the MSEs over $k = 1000$ simulations of the $(0.1, 1/n^2)$-DP estimator $\mathcal{A}_{(0.1, 1/n^2)}$ as a function of n for three different numbers of timepoints $T \in \{1, 10, 100\}$ per user. The right hand side, in turn, fixes the number of users as $n = 1000$ and plots the MSE as a function of $100 \leq T \leq 1000$. Here, we also consider the estimator $\mathcal{A}_{(\varepsilon, 1/n^2)}$ for three different privacy budgets $\varepsilon \in \{0.1, 0.5, 1\}$. As expected, the left hand side in this weakly dependent setting shows that increasing T by one order of magnitude has a similar effect on reducing the MSE as decreasing the variance in an item-level setting by one order of magnitude. In the right, we observe the $1/T$ rate of our Winsorized estimator we expect when n is held constant.
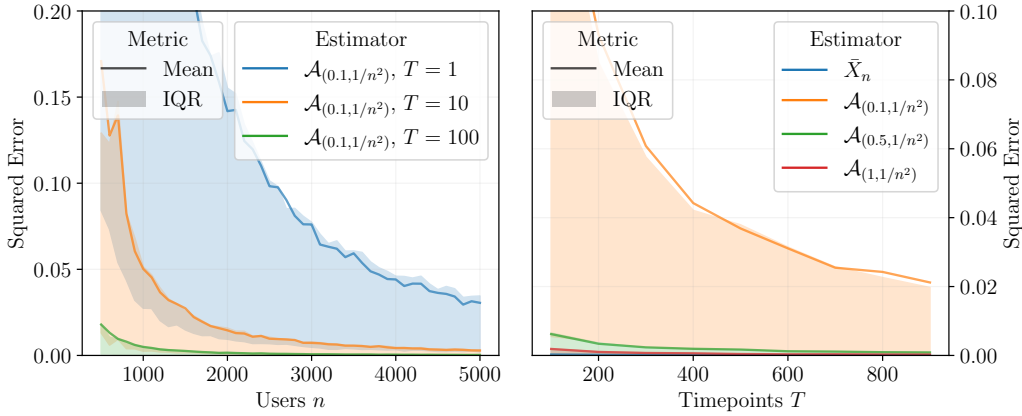

Figure 6: MSE of user-level estimators as a function of users and timepoints.

8 Final remarks

We have provided some first differentially private estimation tools for dependent data. The core procedure is a mean estimation algorithm based on noisy Winsorized means that had been introduced by Karwa and Vadhan (2018) in the context of item-level DP for univariate i.i.d. Gaussian data. Our work shows that simple variants of this algorithm cannot only handle dependent data, but can also lead to optimal estimation rates in the context of user-level DP and local DP for various estimation problems such as nonparametric regression, random effects models and longitudinal linear regression. In this sense, our work also provides a simple unifying estimation framework for a wide range of problems in DP that tend to be studied separately.

From a technical perspective, the key tool allowing for log-Sobolev dependent observations is a DKW-type inequality due to Bobkov and Götze (2010). It provides us a way to control the performance of our intermediate histogram estimators which are essential for adaptively finding appropriate projection intervals for our Winsorized mean estimators. Further relaxing the log-Sobolev dependence

assumption would require DKW-inequalities under weaker conditions.

Our work opens numerous natural future research directions. Perhaps the most pressing one in practice is to develop inference tools that work well with moderate sample sizes under dependence. For this it will be important to develop good private variance estimators and we believe that ideas based on the CoinPress procedure of Biswas et al. (2020) are promising. Finally, some of the many important statistical questions that naturally involve dependent data include multi-armed bandits, online learning, federated learning and in general noisy DP optimization procedures. We hope to address some of these in future work.

A   Auxiliary Results

Theorem A.1. *(Bakry-Emery, Theorem 21.2, Villani (2009), Corollary 5.7.2, Bakry et al. (2014)). Let the probability measure ν on $\mathbb{R}^n$ have density $p(x) \propto \exp(-V(x))$ for all $x \in \mathbb{R}^n$. Suppose there is $\alpha > 0$ s.t. $\nabla^2 V(x) \succeq \alpha I_n$ for all $x \in \mathbb{R}^n$. Then, the following log-Sobolev inequality holds:*

$$\mathrm{Ent}_\nu(f^2) \leq \frac{2}{\alpha} \cdot \mathbb{E}[\|\nabla f(X)\|_2^2].$$

Theorem A.1 states a sufficient condition for a log-Sobolev inequality. It is not a necessary condition as illustrated, e.g., by the Holley-Stroock Perturbation Theorem (Chewi (2025, Proposition 2.3.1)).

Example A.2. *(Gaussian Log-Sobolev Inequality via Bakry-Emery). Let $\mu \in \mathbb{R}^d$ and $\Sigma \in \mathbb{R}^{d \times d}$ s.t. $\Sigma \preceq \beta I_d$. A multivariate Gaussian measure with mean μ and covariance matrix Σ has the following density p with respect to Lebesgue measure*

$$p(x) \propto \exp\left(-\frac{1}{2}(x - \mu)^\top \Sigma^{-1}(x - \mu)\right).$$

Computing the negative log-Hessian now yields $\nabla_x^2 V(x) = -\nabla_x^2 \log(p(x)) = \Sigma^{-1} \succeq 1/\beta I_d$ due to the covariance bound $\Sigma \preceq \beta I_d$. Hence, by Theorem A.1, the measure $\mathcal{N}(\mu, \Sigma)$ fulfills LSI(β).

The following stability result for log-Sobolev inequalities is useful for our purposes as it implies that marginals of distributions fulfilling a log-Sobolev inequality also fulfill one with the same constant.

Lemma A.3. *(Lipschitz Contractions for Log-Sobolev Inequality, Colombo et al. (2017, p. 1492)). Let ν on $\mathbb{R}^d$ fulfill LSI(ρ) and $T : \mathbb{R}^d \to \mathbb{R}^n$ be L-Lipschitz. Then, for the push-forward measure $\nu = T_\# \mu$ on $\mathbb{R}^n$ and $\varphi : \mathbb{R}^n \to \mathbb{R}$ s.t. $\mathrm{Ent}_\nu(\varphi^2) < \infty$ and $\nabla \varphi \in L^2(\nu)$ we have*

$$\mathrm{Ent}_\nu(\varphi) \leq 2\rho L^2 \cdot \mathbb{E}_\nu[\|\nabla_y \varphi(Y)\|_2^2].$$

Remark A.4. *Let $T : \mathbb{R}^d \to \mathbb{R}^k$ be the map corresponding to marginalization, i.e., dropping $d - k$ coordinates. Let the set of indices of the remaining coordinates be $\mathcal{M}$. Then, T is 1-Lipschitz:*

$$\|T(x) - T(y)\|_2 = \sqrt{\sum_{i \in \mathcal{M}} (x_i - y_i)^2} \leq \sqrt{\sum_{i \in [d]} (x_i - y_i)^2} = \|x - y\|_2, \quad \forall x, y \in \mathbb{R}^d.$$

Hence, any marginal of a measure fulfilling a log-Sobolev inequality has the same ρ.

The next two results follow from Gaussian and Laplace Lipschitz concentration respectively.

Lemma A.5. *Let $X_1, \ldots, X_n \in \mathbb{R}^d$ and $X_i \sim \mathcal{N}(\mu, \Sigma_i)$ independently. Then, with $\bar{\Sigma}_n := \frac{1}{n}\sum_{i=1}^n \Sigma_i$,*

$$\mathbb{P}\left(\left\|\frac{1}{n}\sum_{i=1}^n X_i - \mu\right\|_2 \geq \sqrt{\frac{\mathrm{tr}(\bar{\Sigma}_n)}{n}} + \sqrt{\frac{2\|\bar{\Sigma}_n\|_{op}\log(1/\alpha)}{n}}\right) \leq \alpha.$$

Proof. This well-known result immediately follows from the affine transformation property of Gaussians combined with Gaussian Lipschitz concentration (Boucheron et al., 2013, Theorem 5.6) applied to the L_2-norm. □

Lemma A.6. *Let $\Xi_1, \ldots, \Xi_n \sim \mathrm{Lap}(0, bI_d)$ independently. Then, it holds that*

$$\mathbb{P}\left(\left\|\frac{1}{n}\sum_{i=1}^n \Xi_i\right\|_2 \geq b\left(\sqrt{\frac{2d}{n}} + \sqrt{\frac{4\log(3/\alpha)^2}{n}}\right)\right) \leq \alpha.$$

Proof. This proof collects $\Xi := [\Xi_1, \ldots, X_h]^\top \sim \mathrm{Lap}(0, bI_{dh})$ and expresses the average in terms of Ξ. This allows us to apply Poincaré Lipschitz concentration (Chewi, 2025, Theorem 2.4.3).

Concretely, we write $\bar{\bar{\Xi}}_n = A\Xi$ where $A = \frac{1}{n}[I_d, \ldots, I_d] \in \mathbb{R}^{d\times dn}$ and $\|A\|_{op} = 1/\sqrt{n}$. We now show that $\|\bar{\bar{\Xi}}_n\|_2$ is Lipschitz. For $\Xi, \Xi' \in \mathbb{R}^{dn}$ and by the reverse triangle inequality,

$$\left|\|\bar{\bar{\Xi}}_n\|_2 - \|\bar{\bar{\Xi}}'_n\|_2\right| = \left|\|A\Xi\|_2 - \|A\Xi'\|_2\right| \leq \|A(\Xi - \Xi')\|_2 \leq \|A\|_{op}\|\Xi - \Xi'\|_2 \leq \frac{1}{\sqrt{n}}\|\Xi - \Xi'\|_2.$$

Further, by concave Jensen's $\mathbb{E}[\|\bar{\bar{\Xi}}_n\|_2 \leq \sqrt{\mathbb{E}[\|\bar{\bar{\Xi}}_n\|_2^2]} \leq b\sqrt{2d/n}$ where the last inequality holds since

$$\mathbb{E}\left[\|\bar{\bar{\Xi}}_n\|_2^2\right] = \mathbb{E}\left[\left\|\frac{1}{n}\sum_{i=1}^n \Xi_i\right\|_2^2\right] = \mathbb{E}\left[\sum_{j=1}^d\left(\frac{1}{n}\sum_{i=1}^n \Xi_{ij}\right)^2\right] = \frac{d}{n^2}\cdot\mathbb{E}\left[\left(\sum_{i=1}^n \Xi_{i1}\right)^2\right]$$

$$= \frac{d}{n^2}\cdot\sum_{i=1}^n\sum_{k=1}^n\mathbb{E}\left[\Xi_{i1}\Xi_{k1}\right] = \frac{d}{n^2}\cdot\sum_{i=1}^n\mathbb{E}\left[\Xi_{i1}^2\right] = \frac{d}{n}\cdot\mathrm{Var}(\Xi_{i1}) = \frac{d}{n}\cdot 2b^2.$$

Note that the distribution of Ξ is a product distribution with Poincaré constant $4b^2$ (Boucheron et al., 2013, Exercise 3.22 + Theorem 3.1). Hence, for $t \geq 0$ Poincaré Lipschitz concentration yields

$$\mathbb{P}\left(\|\bar{\bar{\Xi}}_h\|_2 \geq b\sqrt{\frac{2d}{h}} + t\right) \leq \mathbb{P}\left(\|\bar{\bar{\Xi}}_h\|_2 \geq \mathbb{E}[\|\bar{\bar{\Xi}}_h\|_2] + t\right) \leq 3\exp\left(-\frac{\sqrt{h}t}{2b}\right).$$

A choice of $t = \frac{2b}{\sqrt{n}}\log(3/\alpha)$ then recovers the statement. □

B Proofs of Section 2

Proof of Corollary 2.10.

The LSI(ρ) condition on $X_{\cdot j}^n$'s distribution implies a Poincaré constant ρ. Thus, for $\varphi : \mathbb{R}^n \to \mathbb{R}$,

$$\mathrm{Var}(\varphi(X_{\cdot j}^n)) \leq \rho\cdot\mathbb{E}[\|\nabla\varphi(X_{\cdot j}^n)\|_2^2].$$

For the first statement we use the test function $\phi(X_{\cdot j}^n) = \theta^\top X_{\cdot j}^n$ with $\theta \in \mathbb{R}^n$. Hence, for all $\theta \in \mathbb{R}^n$:

$$\theta^\top\mathrm{Cov}(X_{\cdot j}^n)\theta \leq \rho\cdot\mathbb{E}[\theta^\top\theta] = \theta^\top(\rho I_n)\theta.$$

The Loewner ordering follows by definition. To show the second statement we evaluate with another test function. For each $j \in [d]$ define $f(X_{\cdot j}^n) := (\bar{X}_n)_j = \frac{1}{n}\sum_{i=1}^n X_{ij}$. For $x, y \in \mathbb{R}^n$,

$$|f(x) - f(y)| \leq \frac{1}{n}\left|\sum_{i=1}^n x_i - y_i\right| \leq \frac{1}{n}\sum_{i=1}^n|x_i - y_i| \leq \frac{1}{\sqrt{n}}\|x - y\|_2.$$

Hence, f is $1/\sqrt{n}$-Lipschitz. By definition of the variance of a random vector, we have

$$\mathbb{E}[\|\bar{X}_n - \mathbb{E}[\bar{X}_n]\|_2^2] = \sum_{j=1}^{d} \mathbb{E}[|(\bar{X}_n)_j - \mathbb{E}[(\bar{X}_n)_j]|^2] = \sum_{j=1}^{d} \mathrm{Var}((\bar{X}_n)_j).$$

Evaluating the Poincaré inequality with the test function f yields

$$\mathrm{Var}((\bar{X}_n)_j) = \mathrm{Var}(f(X_{\cdot j}^n)) \leq \rho \cdot \mathbb{E}[\|\nabla f(X_{\cdot j}^n)\|_2^2] \leq \frac{\rho}{n}.$$

$\square$

Proof of Lemma 2.11.

The proof relies on log-Sobolev Lipschitz concentration applied to a dimension-wise estimator $(\bar{X}_n)_l = \frac{1}{n}\sum_{j=1}^{n} X_{jl} = \frac{1}{n}\mathbf{1}^\top X_{\cdot l}^n =: f(X_{\cdot l}^n)$ of μ and union bounding over dimensions.

To apply Lipschitz concentration, we first compute the mean of f that is $\mathbb{E}[f(X_{\cdot l}^n)] = \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}[X_{il}] = \mu_l$. Second, f is $1/\sqrt{n}$-Lipschitz as shown in the proof of Corollary 2.10.

By Assumption 2.8 for all $l \in [d]$, the vector $X_{\cdot l}^n$ is distributed according to a distribution that is LSI(ρ). By log-Sobolev Lipschitz concentration of Theorem 2.6 we have:

$$\mathbb{P}\left(|(\bar{X}_n)_l - \mu_l| \geq t\right) = \mathbb{P}\left(|f(X_{\cdot l}) - \mathbb{E}[f(X_{\cdot l})]| \geq t\right) \leq 2\exp\left(-\frac{nt^2}{2\rho}\right).$$

Through the choice $t := \sqrt{2\rho \log(2d/\alpha)/n}$ and a union bound over dimensions we obtain:

$$\mathbb{P}\left(\|\bar{X}_n - \mu\|_\infty \geq \sqrt{\frac{2\rho \log(2d/\alpha)}{n}}\right) = \mathbb{P}\left(\exists l \in [d] : |(\bar{X}_n)_l - \mu_l| \geq \sqrt{\frac{2\rho \log(2d/\alpha)}{m}}\right) \leq \alpha.$$

Using that $\frac{1}{\sqrt{d}}\|x\|_2 \leq \|x\|_\infty$ for all $x \in \mathbb{R}^d$ on the complement event yields the statement. $\square$

C   Proofs of Section 3

C.1   Proofs for Stable Histogram

We briefly describe the main idea behind the proof of the privacy guarantee stated in Lemma C.1. The stability mechanism of the histogram estimator releases all $\tilde{\boldsymbol{p}} = (\ldots, \tilde{p}_{-1}, \tilde{p}_0, \tilde{p}_1, \tilde{p}_2, \ldots)$ which by construction contains at most n non-zero elements. This is not trivial since if we only add noise to the bins B_k with $\hat{p}_k > 0$, even though we protect the counts, we release information about where these bins are. We cannot fix this by adding unbounded noise to all bins $(B_k)_{k \in \mathbb{Z}}$, as this blows up the estimation error. Therefore, the algorithm zeroes out the unstable bins whose noisy count is comparable to one plus the $(1 - \delta/2)$-quantile of a standard Laplace. This stability mechanism prevents leaking the location of non-zero bins at the cost of a $\delta > 0$ in the privacy guarantee.

Lemma C.1. *(Adjusted from Theorem 7.3.5, Vadhan (2017)). Algorithm 2 is (ε, δ)-DP for $\varepsilon, \delta > 0$.*

Proof. The proof exploits that for two neighboring datasets only two empirical frequencies $\hat{p}_k$ can be affected. If the corresponding bins both contain at least one data point, a Laplace mechanism is used contributing the ε and otherwise thresholding yields the δ.

Let $\mathcal{A}$ denote the histogram learner of Algorithm 2. For two non-random $x, x' \in \mathbb{R}^n$ with $d_H(x, x') \leq 1$ we want to show that $\mathcal{A}$ is (ε, δ)-DP, i.e., for all measurable S,

$$\mathbb{P}\left(\mathcal{A}(x) \in S\right) \leq e^{\varepsilon} \mathbb{P}\left(\mathcal{A}(x') \in S\right) + \delta. \tag{2}$$

Remember $\hat{p}_k = \hat{p}_k(x) = \frac{1}{n} \sum_{i=1}^n \mathbf{1}\{x_i \in B_k\}$ and $\tilde{p}_k$ is the corresponding noisy estimate for all $k \in \mathbb{Z}$. We write also $\hat{p}'_k := \hat{p}_k(x')$ and $\tilde{p}'_k$ for the noisy estimate. Let $M := \{k \in \mathbb{Z} : \hat{p}_k \neq \hat{p}'_k\}$ be the set of bins affected by the change of the single entry between x and x'. As only one pair (x_j, x'_j) differs, we have $|M| \leq 2$. For $k, l \in M$ three cases occur:

Case 1: x_j, x'_j are in the same bin B_k.
Case 2: x_j, x'_j are in different bins B_k, B_l where $\hat{p}_k = \frac{1}{n}, \hat{p}'_k = 0$ and $\hat{p}_l = 0, \hat{p}'_l = \frac{1}{n}$.
Case 3: x_j, x'_j are in different bins B_k, B_l where $\hat{p}_k, \hat{p}'_k, \hat{p}_l, \hat{p}'_l > 0$.

For all other $k \in M^c$, $\tilde{p}_k$ and $\tilde{p}'_k$ are equal in distribution implying constants $(0, 0)$ in Equation (2) for this part of the output of $\mathcal{A}$. It hence suffices to show DP for the remaining k in the three cases.

For **Case 1** the output of $\mathcal{A}$ is unchanged and we have $\mathbb{P}\left(\mathcal{A}(x) \in S\right) = \mathbb{P}\left(\mathcal{A}(x') \in S\right)$ for all measurable sets S. This yields Equation (2) with constants $(0, 0)$.

For **Case 2** when $\hat{p}_l = 0$, by definition of $\mathcal{A}$, $\tilde{p}_l = 0$. With $t = \frac{2}{\varepsilon n} \log(\frac{2}{\delta}) + \frac{1}{n}$ we then have

$$\mathbb{P}\left(\tilde{p}_l \neq \tilde{p}'_l\right) = \mathbb{P}\left(\tilde{p}'_l \neq 0\right) = \mathbb{P}\left(\{\tilde{p}'_l \neq 0\} \cap \{\hat{p}'_l + \xi_k < t\}\right) + \mathbb{P}\left(\{\tilde{p}'_l \neq 0\} \cap \{\hat{p}'_l + \xi_k \geq t\}\right)$$

$$= 0 + \mathbb{P}\left(\hat{p}'_l + \xi_k \geq \frac{2}{\varepsilon n} \log\left(\frac{2}{\delta}\right) + \frac{1}{n}\right)$$

$$= \mathbb{P}\left(\xi_k \geq \frac{2}{\varepsilon n} \log\left(\frac{2}{\delta}\right)\right) \leq \frac{\delta}{2}.$$

By a symmetry argument we also have $\mathbb{P}\left(\tilde{p}_k \neq \tilde{p}'_k\right) \leq \frac{\delta}{2}$. A union bound yields $\mathbb{P}(\mathcal{A}(x) \neq \mathcal{A}(x')) = \mathbb{P}(\{\tilde{p}_l \neq \tilde{p}'_l\} \cup \{\tilde{p}_k \neq \tilde{p}'_k\}) \leq \delta$. Now, we can show Equation (2) with constants $(0, \delta)$ as follows:

$$\mathbb{P}\left(\mathcal{A}(x) \in S\right) = \mathbb{P}\left(\{\mathcal{A}(x) \in S\} \cap \{\mathcal{A}(x) = \mathcal{A}(x')\}\right) + \mathbb{P}\left(\{\mathcal{A}(x) \in S\} \cap \{\mathcal{A}(x) \neq \mathcal{A}(x')\}\right)$$

$$\leq \mathbb{P}\left(\mathcal{A}(x') \in S\right) + \delta.$$

For **Case 3** we first note that the sensitivity of $\hat{p}_k(\cdot)$ is $1/n$. Indeed, for all $k \in \mathbb{Z}$ and $d_H(x, x') \leq 1$:

$$\Delta = \sup_{x, x'} |\hat{p}_k(x) - \hat{p}_k(x')| = \sup_{x, x'} \left| \frac{1}{n} \left(\mathbf{1}\{x_j \in B_k\} - \mathbf{1}\{x'_j \in B_k\}\right) \right| \leq \frac{1}{n}.$$

The noisy statistic $\check{p}_k = \hat{p}_k + \xi_k$ ensures that Equation (2) holds for $\check{p}_k$ with constants $(\varepsilon, 0)$ by the Laplace mechanism, because $\xi_k \sim \mathrm{Lap}(0, b)$ and $b = \frac{2\Delta}{\varepsilon} = \frac{2}{\varepsilon n}$. Therefore,

$$\tilde{p}_k = \check{p}_k \cdot \mathbf{1}\{\check{p}_k \geq \frac{2}{\varepsilon n} \log(\frac{2}{\delta}) + \frac{1}{n}\}$$

also fulfills Equation (2) with constants $(\varepsilon, 0)$ by post-processing. $\qquad\square$

Proof of Lemma 3.2.
Algorithm 2 is (ε, δ)-DP by Lemma C.1.

For utility, we adapt the proof of Lemma 2.3 by Karwa and Vadhan (2018). The triangle inequality and the law of total probability yield the upper bound

$$\mathbb{P}\left(\max_{k\in\mathbb{Z}}|\tilde{p}_k - p_k| > \eta\right) \leq \mathbb{P}\left(\max_{k\in\mathbb{Z}}|\tilde{p}_k - \hat{p}_k| > \frac{\eta}{2}\right) + \mathbb{P}\left(\max_{k\in\mathbb{Z}}|\hat{p}_k - p_k| > \frac{\eta}{2}\right)$$

$$= \mathbb{E}\left[\mathbb{P}\left(\max_{k\in\mathbb{Z}}|\tilde{p}_k - \hat{p}_k| > \frac{\eta}{2}\Big|X^n\right)\right] + \mathbb{P}\left(\max_{k\in\mathbb{Z}}|\hat{p}_k - p_k| > \frac{\eta}{2}\right). \quad (3)$$

By conditioning on X^n in the first term, we isolate the estimation error due to privacy of the stability-based histogram given a specific realization. Hence, by Lemma C.4 we have

$$\mathbb{E}\left[\mathbb{P}\left(\max_{k\in\mathbb{Z}}|\tilde{p}_k - \hat{p}_k| \geq \frac{\eta}{2}\Big|X^n\right)\right] \leq n\left(1 + \frac{e^{\varepsilon}}{\delta}\right)\exp\left(-\frac{\varepsilon n\eta}{4}\right).$$

For the estimation error of $\hat{p}_k$ we then invoke Lemma C.3 under Assumption 2.8. This yields

$$\mathbb{P}\left(\max_{k\in\mathbb{Z}}|\hat{p}_k - p_k| \geq \frac{\eta}{2}\right) \leq \frac{16}{\eta}\exp\left(-\frac{2}{27}\frac{n\eta^3}{4^3\rho M^2}\right).$$

Plugging both bounds into the inequality (3) shows the desired result. □

Proof of Lemma 3.1.

Analogous to the proof of Lemma 3.2. Uses Lemma C.3 under the i.i.d. assumption on the entries of $X^n \in \mathbb{R}^n$. Thus, it uses the Dvoretzky-Kiefer-Wolfowitz inequality of Corollary 1 in Massart (1990) instead of Theorem 1.2 by Bobkov and Götze (2010). □

Auxiliary Results for Stable Histogram

Theorem C.2. *(Theorem 1.2, Bobkov and Götze (2010)) Let $X \sim \nu$ on $\mathbb{R}^n$ where ν is a probability measure fulfilling* LSI(ρ). *Assume that the average marginal cumulative distribution function $\bar{F}(x) = \frac{1}{n}\sum_{i=1}^{n}\mathbb{P}(X_i \leq x)$ is M-Lipschitz. Then, for any $r > 0$*

$$\mathbb{P}\left(\sup_{x\in\mathbb{R}}|\bar{F}(x) - \hat{F}_n(x)| \geq r\right) \leq \frac{4}{r}\exp\left(-\frac{2}{27}\frac{nr^3}{\rho M^2}\right).$$

Lemma C.3. *Let $\eta > 0$ and $(B_k)_{k\in\mathbb{Z}}$ be disjoint bins covering $[-\mathsf{B},\mathsf{B}] \subseteq \mathbb{R}$. Assume $X \in \mathbb{R}^n$ has i.i.d. entries or fulfills Assumption 2.8 with constants (ρ, M). Let $\hat{p}_k := \frac{1}{n}\sum_{i=1}^{n}\mathbf{1}\{X_i \in B_k\}$ be the empirical mass of bin B_k. Then, when estimating the $p_k = \mathbb{E}[\hat{p}_k]$ uniformly over $k \in \mathbb{Z}$, the estimation error is*

$$\mathbb{P}\left(\max_{k\in\mathbb{Z}}|\hat{p}_k - p_k| \geq \eta\right) \leq \begin{cases} 2\exp\left(-\frac{n\eta^2}{2}\right) & \text{if } X_1,\ldots,X_n \text{ are i.i.d.,} \\ \frac{8}{\eta}\exp\left(-\frac{2}{27}\frac{n\eta^3}{2^3\rho M^2}\right) & \text{if } X \text{ fulfills Assumption 2.8.} \end{cases}$$

Proof. Let $\hat{F}_n(x) = \frac{1}{n}\sum_{i=1}^{n}\mathbf{1}\{X_i \leq x\}$ be the empirical cdf and $\bar{F}(x) = \mathbb{E}[\hat{F}_n(x)]$ the average marginal cdf. As X^n fulfills Assumption 2.8, Theorem C.2 yields:

$$\mathbb{P}\left(\sup_{x\in\mathbb{R}}|\bar{F}(x) - \hat{F}_n(x)| \geq r\right) \leq \frac{4}{r}\exp\left(-\frac{2}{27}\frac{nr^3}{\rho M^2}\right).$$

Let $-\infty = x_0 < x_1 < \cdots < x_\infty = \infty$ be the endpoints of the bins $(B_k)_{k \in \mathbb{Z}}$. Then,

$$p_k = \bar{F}(x_k) - \bar{F}(x_{k-1}) \quad \text{and} \quad \hat{p}_k = \hat{F}_n(x_k) - \hat{F}_n(x_{k-1}).$$

Using these relations and a triangle inequality, for all $k \in \mathbb{Z}$:

$$
\begin{aligned}
|\hat{p}_k - p_k| &= |\bar{F}(x_k) - \bar{F}(x_{k-1}) - (\hat{F}_n(x_k) - \hat{F}_n(x_{k-1}))| \\
&\leq |\bar{F}(x_k) - \hat{F}_n(x_k)| + |\bar{F}(x_{k-1}) - \hat{F}_n(x_{k-1}))|.
\end{aligned}
$$

Therefore, Theorem C.2 shows that

$$
\begin{aligned}
\mathbb{P}\left(\max_{k \in \mathbb{Z}} |\hat{p}_k - p_k| \geq \eta\right) &\leq \mathbb{P}\left(\exists k \in \mathbb{Z} : |\bar{F}(x_k) - \hat{F}_n(x_k)| + |\bar{F}(x_{k-1}) - \hat{F}_n(x_{k-1}))| \geq \eta\right) \\
&\leq \mathbb{P}\left(\sup_{x \in \mathbb{R}} |\bar{F}(x) - \hat{F}_n(x)| \geq \frac{\eta}{2}\right) \leq \frac{8}{\eta} \exp\left(-\frac{2}{27} \frac{n\eta^3}{2^3 \rho M^2}\right).
\end{aligned}
$$

The proof for $X^n \in \mathbb{R}^n$ with i.i.d. entries is analogous but uses the Dvoretzky-Kiefer-Wolfowitz inequality of Corollary 1 Massart (1990) instead of Theorem C.2 by Bobkov and Götze (2010). $\quad\square$

Lemma C.4. *Let $\eta, \varepsilon > 0$ and let $(B_k)_{k \in \mathbb{Z}}$ be a disjoint set of bins covering $\mathbb{R}$. Then, for input vector $x^n \in \mathbb{R}^n$, Algorithm 2 is (ϵ, δ)-DP and the histogram output $\tilde{\boldsymbol{p}}$ satisfies:*

$$\mathbb{P}\left(\max_{k \in \mathbb{Z}} |\tilde{p}_k - \hat{p}_k| \geq \eta\right) \leq n\left(1 + \frac{e^{\varepsilon/2}}{\delta}\right) \exp\left(-\frac{\varepsilon n \eta}{2}\right).$$

Proof. Algorithm 2 is (ε, δ)-DP by Lemma C.1.

For utility, we follow the proof of Lemma 2.3 in Karwa and Vadhan (2018) to bound the privacy loss of the stability-based histogram. We perform a case distinction on whether or not $\hat{p}_k = 0$.

Case 1: We have $\tilde{p}_k = 0$ for all $k \in \mathbb{A} := \{k \in \mathbb{Z} : \hat{p}_k = 0\}$ by design of Algorithm 2. This yields the trivial bound $\mathbb{P}\left(\max_{k \in \mathbb{A}} |\tilde{p}_k - \hat{p}_k| \geq \eta\right) = 0$.

Case 2: For the indices $k \in \mathbb{A}^c$ where $\hat{p}_k > 0$ we have:

$$
\begin{aligned}
&\mathbb{P}\left(|\tilde{p}_k - \hat{p}_k| \geq \eta\right) \\
&= \mathbb{P}\left(\{|\tilde{p}_k - \hat{p}_k| \geq \eta\} \cap \{\hat{p}_k + \xi_k \geq t\}\right) + \mathbb{P}\left(\{|\tilde{p}_k - \hat{p}_k| \geq \eta\} \cap \{\hat{p}_k + \xi_k < t\}\right) \\
&= \mathbb{P}\left(\{|\xi_k| \geq \eta\} \cap \{\hat{p}_k + \xi_k \geq t\}\right) + \mathbb{P}\left(\{|\hat{p}_k| \geq \eta\} \cap \{\hat{p}_k + \xi_k < t\}\right) \\
&\leq \mathbb{P}\left(|\xi_k| \geq \eta\right) + \mathbb{P}\left(\xi_k < t - \eta\right) \\
&= \mathbb{P}\left(|\xi_k| \geq \eta\right) + \mathbb{P}\left(\xi_k > \eta - t\right).
\end{aligned}
$$

Above, the first equality holds as we split the event into two disjoint sets. By definition of Algorithm 2 we have that $\tilde{p}_k = \hat{p}_k + \xi_k$ and $\tilde{p}_k = 0$ if $\tilde{p}_k < t$, which justifies the second equality. The inequality holds by inclusion. The last equality holds as the distribution of ξ_k is symmetric around zero. Using that $\xi_k \sim \text{Lap}(0, \frac{2}{\varepsilon n})$ we see that

$$\mathbb{P}\left(|\xi_k| \geq l\right) = \exp\left(-\frac{\varepsilon n l}{2}\right),$$

Furthermore, choosing $t = \frac{2}{\varepsilon n} \log\left(\frac{2}{\delta}\right) + \frac{1}{n}$ we see that

$$\mathbb{P}\left(\xi_k > l - t\right) = \frac{1}{2}\exp\left(-\frac{\varepsilon n(l-t)}{2}\right)$$

$$= \frac{1}{2}\exp\left(-\frac{\varepsilon n l}{2}\right)\exp\left(\frac{\varepsilon n}{2}\left(\frac{2}{\varepsilon n}\log\left(\frac{2}{\delta}\right) + \frac{1}{n}\right)\right)$$

$$= \frac{e^{\varepsilon/2}}{\delta}\exp\left(-\frac{\varepsilon n l}{2}\right)$$

Plugging both into the bound, we have

$$\mathbb{P}\left(|\tilde{p}_k - \hat{p}_k| \geq \eta\right) \leq \left(1 + \frac{e^{\varepsilon/2}}{\delta}\right)\exp\left(-\frac{\varepsilon n \eta}{2}\right).$$

By a union bound over all $k \in \mathbb{Z}$ and using that only n of the probabilities can be non-zero,

$$\mathbb{P}\left(\max_{k\in\mathbb{Z}}|\tilde{p}_k - \hat{p}_k| > \eta\right) \leq \sum_{k\in\mathbb{Z}}\mathbb{P}\left(|\tilde{p}_k - \hat{p}_k| > \eta\right) \leq n\left(1 + \frac{e^{\varepsilon/2}}{\delta}\right)\exp\left(-\frac{\varepsilon n \eta}{2}\right).$$

$\square$

C.2 Proofs for Private Projection Interval

Proof of Lemma 3.4.

The privacy of Algorithm 3 follows from that of Algorithm 2, shown in Lemma C.1.

For utility, by Lemma 3.2 with $\eta = 1/16$ we have that

$$\mathbb{P}\left(\max_{k\in\mathbb{Z}}|\tilde{p}_k - p_k| \leq \frac{1}{16}\right) \geq 1 - \left(1 + \frac{e^{\varepsilon}}{\delta}\right)n\exp\left(-\frac{\varepsilon n}{4\cdot 16}\right) - \frac{16}{\eta}\exp\left(-\frac{1}{864}\frac{n}{16^3\rho M^2}\right)$$

$$= 1 - O\left(\frac{n}{\delta}\cdot e^{-\kappa n\varepsilon} \vee e^{-\kappa' n}\right).$$

Here, we suppress the constant ρM^2 in the big-O notation, use that $\delta \leq e^{\varepsilon}$ for all $\varepsilon > 0$. Combining the guarantee above with the $(\tau, \gamma)_\infty$-concentration of X^n allows us to call Lemma C.5 to get

$$\mathbb{P}\left(\hat{m} \in \left[x_0 \pm 2\tau\right]\right) \geq 1 - O\left(\frac{n}{\delta}\cdot e^{-\kappa n\varepsilon} \vee e^{-\kappa' n}\right).$$

The proof under Assumption 2.7 is analogous but uses Lemma 3.1 instead of Lemma 3.2. $\square$

Proof of Corollary 3.6.

As X_i are sub-Gaussian with variance proxy σ^2

$$\mathbb{P}\left(|X_i - \mu| > t\right) \leq 2\exp\left(-\frac{t^2}{2\sigma^2}\right) =: \frac{\gamma}{n}.$$

Hence, X^n is $(\tau, \gamma)^\mu_\infty$-concentrated with $\tau = \sqrt{2\sigma^2\log(2n/\gamma)}$. Under Assumption 2.7 by Lemma 3.4

$$\mathbb{P}\left(\hat{m} \in \left[\mu \pm 2\sqrt{2\sigma^2\log(2n/\gamma)}\right]\right) \geq 1 - O\left(\frac{n}{\delta}\cdot e^{-\kappa n\varepsilon} \vee e^{-\kappa' n}\right).$$

This holds under the condition that $\gamma \in (0, 1 \wedge \frac{n}{4})$. $\square$

Auxiliary Results for Private Projection Intervals

Lemma C.5. *Let $\gamma \in (0, 1 \wedge \frac{n}{4})$ and let $X^n \in \mathbb{R}^n$ be $(\tau, \gamma)_\infty^{x_0}$-concentrated. Assume there is a histogram estimator $\mathcal{A}$ that outputs $(\dots, \tilde{p}_{-2}, \tilde{p}_{-1}, \tilde{p}_0, \tilde{p}_1, \tilde{p}_2, \dots)$ s.t. w.p. at least $1 - \alpha$ we have $\max_{k \in \mathbb{Z}} |\tilde{p}_k - p_k| \leq 1/16$. Then, with probability at least $1 - \alpha$ the midpoint $\hat{m}$ in Algorithm 3 fulfills*

$$\hat{m} \in \left[x_0 \pm 2\tau \right].$$

Proof. By $(\tau, \gamma)_\infty$-concentration, for all $i \in [n]$ with probability at least $1 - \gamma/n =: 1 - \alpha$ we have $X_i \in I := [x_0 \pm \tau]$. We construct a set S of bin-indices $k \in \mathbb{Z}$ whose bins cover I:

$$S := \{k \in \mathbb{Z} : I \cap B_k \neq \emptyset\}.$$

Note that given the binwidth 2τ we have $|S| \leq 2$. Since bins in S cover I, by a union bound

$$\sum_{k \in S} p_k = \frac{1}{n} \sum_{i=1}^n \sum_{k \in S} \mathbb{P}(X_i \in B_k) \geq \frac{1}{n} \sum_{i=1}^n \mathbb{P}(\bigcup_{k \in S} \{X_i \in B_k\})$$

$$= \frac{1}{n} \sum_{i=1}^n \mathbb{P}(X_i \in \bigcup_{k \in S} B_k) \geq \frac{1}{n} \sum_{i=1}^n \mathbb{P}(X_i \in I) = \frac{1}{n} \sum_{i=1}^n \mathbb{P}(|X_i - x_0| \leq \tau) \geq 1 - \alpha. \quad (4)$$

By the pigeon hole principle, i.e., a lower bound by the uniform distribution over the $|S|$ bins,

$$\exists k^\star \in S \quad \text{such that} \quad p_{k^\star} \geq \frac{1 - \alpha}{|S|} \geq \frac{1 - \alpha}{2}. \quad (5)$$

At the same time for all $k \in S^c$

$$p_k = \frac{1}{n} \sum_{i=1}^n \mathbb{P}(X_i \in B_k) \leq \frac{1}{n} \sum_{i=1}^n \mathbb{P}(X_i \in \mathbb{R} \setminus I) \leq \frac{1}{n} \sum_{i=1}^n \mathbb{P}(|X_i - x_0| \geq \tau) \leq \alpha. \quad (6)$$

We now translate the bounds on p_k above into bounds on $\tilde{p}_k$. If there is an η such that

$$\mathbb{P}\left(\forall k \in \mathbb{Z} : \tilde{p}_k \in [p_k \pm \eta]\right) = \mathbb{P}\left(\max_{k \in \mathbb{Z}} |\tilde{p}_k - p_k| \leq \eta\right) \geq 1 - \alpha,$$

then using (5) and (6) we have $\tilde{p}_{k^\star} \geq p_{k^\star} - \eta \geq \frac{1-\alpha}{2} - \eta$ for $k^* \in S$ and $\tilde{p}_k \leq p_k + \eta \leq \alpha + \eta$ for all $k \in S^c$. To ensure that any such $k^\star$ is selected in the arg max, we need $\tilde{p}_{k^\star} > \tilde{p}_k$ for all $k \in S^c$. This holds if:

$$\frac{1 - \alpha}{2} - \eta > \alpha + \eta \quad \Leftrightarrow \quad \alpha < \frac{1 - 4\eta}{3}.$$

Since by assumption $\eta = 1/16$, the last condition becomes $\alpha < 1/4$ and it holds as $\alpha = \gamma/n$ and we assume that $\gamma < n/4$. Hence, with probability at least $1 - \alpha$ we select $\hat{k} \in S$. The center of B_k for any $k \in S$ can at most be τ away from I. The distance from I's endpoints to x_0 is also at most τ. A triangle inequality recovers $|\hat{m} - x_0| \leq 2\tau$. $\square$

D Proofs of Section 4

Proof of Lemma 4.1.

The private midpoint estimator of Algorithm 3 is $(\frac{\varepsilon}{2}, \delta)$-DP by Lemma 3.4 as we call it with $\frac{\varepsilon}{2}$ in Algorithm 4. Further, the Laplace mechanism therein is $(\frac{\varepsilon}{2}, 0)$-DP by Theorem 2.3. Therefore, algorithm $\mathcal{A}$ is (ε, δ)-DP through basic composition in Theorem 2.4.

For utility, by Lemma 3.4 under Assumption 2.7 or 2.8 we have

$$\mathbb{P}\left(\hat{m} \in [x_0 \pm 2\tau]\right) \geq 1 - O(n/\delta \cdot e^{-\kappa n\varepsilon} \vee e^{-\kappa' n}).$$

Hence, by an application of Lemma D.2 with $C = 2$, the event $\mathcal{E} := \{\forall i \in [n] : \Pi_{[\hat{m} \pm 3\tau]}(X_i) = X_i\}$ has probability at least $1 - \gamma - O(n/\delta \cdot e^{-\kappa n\varepsilon} \vee e^{-\kappa' n})$ and

$$\mathcal{A}(X) \cdot \mathbf{1}\{\mathcal{E}\} = \left(\bar{X}_n + \xi\right) \cdot \mathbf{1}\{\mathcal{E}\}, \quad \text{almost surely.}$$

$\square$

Lemma D.1. *Algorithm 5 denoted by $\mathcal{A}$ is $(\varepsilon, \delta + \varrho)$-DP for $\delta, \varrho, \varepsilon \in (0, 1)$. Let $\gamma \in (0, 1 \wedge \frac{n}{4})$ and $X^n \in \mathbb{R}^{n \times d}$ be $(\tau, \gamma)_\infty^{x_0}$-concentrated. Make Assumption 2.7 or 2.8 s.t. $\varrho M^2 \lesssim 1$. Then, the event $\mathcal{E} := \{\forall i \in [n], j \in [d] : \Pi_{R_j}(X_{ij}) = X_{ij}\}$ has probability at least $1 - d\gamma - O(d^2 n/\delta \cdot e^{-\kappa n\varepsilon'})$ and a.s.*

$$\mathcal{A}(X) \cdot \mathbf{1}\{\mathcal{E}\} = \left(\bar{X}_n + \Xi\right) \cdot \mathbf{1}\{\mathcal{E}\}, \quad \text{where} \quad \Xi \sim \text{Lap}\left(0, \frac{12\tau}{n\varepsilon'} I_d\right) \quad \text{and} \quad \varepsilon' = \varepsilon/\sqrt{8d \log(1/\varrho)}.$$

Proof. Since the d one-dimensional mean estimators fulfill (ε, δ)-DP, the coordinate-wise multi-dimensional estimator is private by advanced composition in Theorem 2.4. Thus, we use the privacy parameters $\varepsilon', \delta', \varrho$ defined in Algorithm D.1. Note that this is where ϱ occurs.

X^n is $(\tau, \gamma)_\infty^{x_0}$-concentrated. Applying Algorithm 4 per dimension with $(X_{\cdot j}^n, \tau, \varepsilon', \delta')$, by Lemma 4.1 $\mathcal{E}_j := \{\forall i \in [n] : \Pi_{R_j}(X_{ij}) = X_{ij}\}$ has probability at least $1 - \gamma - O(n/\delta \cdot e^{-\kappa n\varepsilon'} \vee e^{-\kappa' n})$ and

$$\mathcal{A}_1(X_{\cdot j}^n) \cdot \mathbf{1}\{\mathcal{E}_j\} = \left(\frac{1}{n} \sum_{i=1}^{n} X_{ij} + \xi_j\right) \cdot \mathbf{1}\{\mathcal{E}_j\}, \quad \text{almost surely,}$$

where we denote the output of Algorithm 4 by $\mathcal{A}_1$. By Lemma D.3 with $h = 1$ and $b = \frac{12\tau}{n\varepsilon'}$, $\mathcal{E} := \{\forall j \in [d], i \in [n] : \Pi_{R_j}(X_{ij}) = X_{ij}\}$ has probability at least $1 - d\gamma - O(dn/\delta' \cdot e^{-\kappa n\varepsilon'})$ and a.s.

$$\mathcal{A}(X^n) \cdot \mathbf{1}\{\mathcal{E}\} = \left(\frac{1}{n} \sum_{i=1}^{n} X_{i\cdot}^\top + \Xi\right) \cdot \mathbf{1}\{\mathcal{E}\}, \quad \text{with} \quad \Xi := (\xi_1, ..., \xi_d)^\top.$$

Because the ξ_j are i.i.d. Laplace random variables, $\Xi \sim \text{Lap}(0, \frac{12\tau}{n\varepsilon} I_d)$. We now recover the statement using that $\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_{i\cdot}^\top$ and $\delta' = \delta/d$. $\square$

Auxiliary Results for Mean Estimators

The statements of the following Lemma D.2 and Lemma D.3 are general enough to incorporate both the central and local model, with $h = 1$ and $h = n$, respectively. We provide these general versions to highlight that both models of DP can be treated in a unified way.

Lemma D.2. *Let $X^n \in \mathbb{R}^n$ be $(\tau, \gamma)_\infty$-concentrated around $x_0 \in \mathbb{R}$. Let algorithm $\tilde{\mathcal{A}}$ output a midpoint $\hat{m}$ s.t. $\hat{m} \in [x_0 \pm C\tau']$ with probability at least $1 - \alpha$. Denote Algorithm 4 calling $\tilde{\mathcal{A}}$ in Line 1 of $\mathcal{A}$. Then, the event $\mathcal{E} := \{\forall i \in [n] : \Pi_{[\hat{m} \pm \tau + C\tau']}(X_i) = X_i\}$ has probability at least $1 - \gamma - \alpha$ and*

$$\mathcal{A}(X^n) \cdot \mathbf{1}\{\mathcal{E}\} = \left(\bar{X}_n + \bar{\xi}_h\right) \cdot \mathbf{1}\{\mathcal{E}\}, \quad \text{almost surely.}$$

Proof. To prove the statement, we note that X_i is only projected if it is more than $\tau + C\tau'$ away from the midpoint. The triangle inequality gives

$$|X_i - \hat{m}| = |X_i - x_0 + x_0 - \hat{m}| \leq |X_i - x_0| + |x_0 - \hat{m}|.$$

Note that $(\tau, \gamma)_\infty^{x_0}$-concentration of X^n and a union bound imply that with probability at least $1 - \gamma$ we have that $|X_i - x_0| \leq \tau$ for all $i \in [n]$. Furthermore, by our assumption $\hat{m} \in [x_0 \pm C\tau']$ with probability at least $1 - \alpha$. Therefore, a union bound shows that with probability at least $1 - \gamma - \alpha$,

$$|X_i - \hat{m}| \leq |X_i - x_0| + |x_0 - \hat{m}| \leq \tau + C\tau', \quad \forall i \in [n].$$

In other words, $\mathbb{P}(\mathcal{E}) \geq 1 - \gamma - \alpha$. The almost sure equality on $\mathcal{E}$ holds since there

$$\mathcal{A}(X^n) = \frac{1}{n} \sum_{i=1}^n \Pi_{\hat{I}}(X_i) + \bar{\xi}_h = \bar{X}_n + \bar{\xi}_h.$$

$\square$

Lemma D.3. *Let $\mathcal{A}_1 : \mathbb{R}^n \to \mathbb{R}$ be an estimator of $x_0 \in \mathbb{R}$ with arguments $(X_{\cdot j}^n, \tau, \varepsilon, \delta)$ for any $j \in [d]$ where $\tau, \varepsilon, \delta > 0$ and $X^n \in \mathbb{R}^{n \times d}$. Assume that for $h \in \mathbb{N}$, on an event $\mathcal{E}_j$ that has probability at least $1 - \alpha$, we almost surely have*

$$\mathcal{A}_1(X_{\cdot j}^n) \cdot \mathbf{1}\{\mathcal{E}_j\} = \left(\frac{1}{n} \sum_{i=1}^n X_{ij} + \frac{1}{h} \sum_{l=1}^h \xi_{lj}\right) \cdot \mathbf{1}\{\mathcal{E}_j\}, \quad \text{with i.i.d. } \xi_{lj} \sim \text{Lap}(0, b).$$

Let $\mathcal{A}$ be Algorithm 5 calling $\mathcal{A}_1$ in Line 3. Then, $\mathcal{E} = \bigcap_{j=1}^d \mathcal{E}_j$ has probability at least $1 - d\alpha$ and

$$\mathcal{A}(X^n) \cdot \mathbf{1}\{\mathcal{E}\} = \left(\frac{1}{n} \sum_{i=1}^n X_{i \cdot}^\top + \frac{1}{h} \sum_{l=1}^h \Xi_l\right) \cdot \mathbf{1}\{\mathcal{E}\}, \quad \text{with} \quad \Xi_l = (\xi_{l1}, \ldots \xi_{ld})^\top.$$

Proof. By assumption the one-dimensional mean estimator applied to $X_{\cdot j}^n$ is a noised version of the empirical mean on $\mathcal{E}_j$. We apply the mean estimator $\mathcal{A}_1$ per dimension with arguments $(X_{\cdot j}^n, \tau, \varepsilon, \delta)$ to form the multivariate estimator $\mathcal{A}$. We then combine the guarantees per coordinate through a union bound over $j \in [d]$. Concretely, on the event $\mathcal{E}_j$ with $\mathbb{P}(\mathcal{E}_j) \geq 1 - \alpha$:

$$\mathcal{A}(X_{\cdot j}^n) = \frac{1}{n} \sum_{i=1}^n X_{ij} + \frac{1}{h} \sum_{l=1}^h \xi_{lj}.$$

Letting $\mathcal{E} = \bigcap_{j=1}^n \mathcal{E}_j$ we see that an union bound over $\mathcal{E}_1^c, \ldots, \mathcal{E}_d^c$ gives $\mathbb{P}(\mathcal{E}) \geq 1 - d\alpha$. Due to the coordinate-wise construction of $\mathcal{A}$ this recovers the statement. $\square$

Remark D.4. *Lemma D.3 is essentially analogous to Levy et al. (2021, Theorem 2) by Lemma D.5 which implies that $\mathcal{A}(X^n) \sim_\beta \bar{X}_n + \Xi$, where $X \sim_\beta Y$ means that the total variation distance between the distributions of X and Y is smaller or equal to β.*

Lemma D.5. *Let X, Y be two random variables on (Ω, Σ, μ) and (Ω, Σ, ν), respectively. Let there be an event E on which $X = Y$ and $\mathbb{P}(E) \geq 1 - \beta$. Then, $X \sim_\beta Y$, i.e., X and Y are β-close.*

Proof. To show that $X \sim_\beta Y$ we need to show the inequality below:

$$d_{TV}(\mu, \nu) = \sup_{A \in \Sigma} |\mu(A) - \nu(A)| = \sup_{A \in \Sigma} |\mathbb{P}(X \in A) - \mathbb{P}(Y \in A)| \leq \beta.$$

We know that $X(\omega) = Y(\omega)$ for all $\omega \in E \subset \Omega$. Hence, for all $A \in \Sigma$ we have $\mathbb{P}(\{X \in A\} \cap E) = \mathbb{P}(\{Y \in A\} \cap E)$. To use this equality, we decompose the probabilities in the TV-distance as follows:

$$\mathbb{P}(X \in A) = \mathbb{P}(\{X \in A\} \cap E) + \mathbb{P}(\{X \in A\} \cap E^c),$$
$$\mathbb{P}(Y \in A) = \mathbb{P}(\{Y \in A\} \cap E) + \mathbb{P}(\{Y \in A\} \cap E^c).$$

Plugging the decompositions into $d_{TV}(\mu, \nu)$ and we get

$$\begin{aligned}
d_{TV}(\mu, \nu) &= \sup_{A \in \Sigma} |\mathbb{P}(\{X \in A\} \cap E) + \mathbb{P}(\{X \in A\} \cap E^c) - \mathbb{P}(\{Y \in A\} \cap E) - \mathbb{P}(\{Y \in A\} \cap E^c)| \\
&= \sup_{A \in \Sigma} |\mathbb{P}(\{X \in A\} \cap E^c) - \mathbb{P}(\{Y \in A\} \cap E^c)| \\
&\leq \sup_{A \in \Sigma} \max \{\mathbb{P}(\{X \in A\} \cap E^c), \mathbb{P}(\{Y \in A\} \cap E^c)\} \\
&\leq \mathbb{P}(E^c) \leq \beta.
\end{aligned}$$

$\square$

D.1 Proofs of Theoretical Guarantees

D.1.1 Proof of Finite Sample Bound

Proof of Theorem 4.2.

The privacy guarantee follows from Lemma D.1. This proof directly follows from Lemma D.6. To apply it, we use that by Lemma D.1 there is an event $\mathcal{E}$ that has probability at least $1 - d\gamma - O(d^2 n/\delta \cdot e^{-\kappa n \varepsilon'} \vee e^{-\kappa' n})$ s.t. almost surely,

$$\mathcal{A}(X^n) \cdot \mathbf{1}\{\mathcal{E}\} = (\bar{X}_n + \Xi) \cdot \mathbf{1}\{\mathcal{E}\}, \quad \text{where } \Xi \sim \text{Lap}\left(0, \frac{12\tau}{n\varepsilon'} I_d\right).$$

Applying Lemma D.6 with $h = 1$, with probability at least $1 - \alpha - d\gamma - O(d^2 n/\delta \cdot e^{-\kappa n \varepsilon'} \vee e^{-\kappa' n})$,

$$\begin{aligned}
\|\mathcal{A}(X^n) - \mu\|_2 &\leq \|\bar{X}_n - x_0\|_2 + \frac{12\tau}{n\varepsilon'}\left(\sqrt{2d} + \sqrt{4\log(3/\alpha)^2}\right) \\
&= \|\bar{X}_n - x_0\|_2 + 12\tau\left(\sqrt{\frac{16d^2 \log(1/\varrho)}{n^2 \varepsilon^2}} + \sqrt{\frac{32d \log(1/\varrho) \log(3/\alpha)^2}{n^2 \varepsilon^2}}\right) \\
&\lesssim \|\bar{X}_n - x_0\|_2 + \tau\sqrt{\frac{d^2 \log(1/\varrho) \log(3/\alpha)^2}{n^2 \varepsilon^2}}.
\end{aligned}$$

The statement is obtained using that $e^{-\kappa' n} \leq e^{-\kappa' n \varepsilon'}$ as $\varepsilon < 1$ and setting $\alpha = d\gamma$. $\square$

Lemma D.6 below is general enough to handle both the central and local model of DP with $h = 1$ and $h = n$, respectively.

Lemma D.6. *Let $\mathcal{A} : \mathbb{R}^{n \times d} \to \mathbb{R}^d$ be an estimator of $x_0 \in \mathbb{R}^d$, $X^n := [X_1^\top, \dots, X_n^\top]^\top \in \mathbb{R}^{n \times d}$ and i.i.d. $\Xi_j \sim \text{Lap}(0, bI_d)$ for all $j \in [h]$. Assume that on an event $\mathcal{E}$ having probability at least $1 - \upsilon$,*

$$\mathcal{A}(X^n) \cdot \mathbf{1}\{\mathcal{E}\} = (\bar{X}_n + \bar{\Xi}_h) \cdot \mathbf{1}\{\mathcal{E}\}, \quad \textit{almost surely.}$$

Then, with probability at least $1 - \alpha - \upsilon$

$$\|\mathcal{A}(X^n) - x_0\|_2 \leq \|\bar{X}_n - x_0\|_2 + b\left(\sqrt{\frac{2d}{h}} + \sqrt{\frac{4 \log(3/\alpha)^2}{h}}\right).$$

Proof. With $t = b(\sqrt{2d/h} + \sqrt{4 \log(1/\alpha)^2/h})$, define the events

$$A := \{\|\mathcal{A}(X^n) - x_0\|_2 \leq \|\bar{X}_n - x_0\|_2 + t\}) \quad \text{and} \quad L := \{\|\bar{\Xi}_h\|_2 \leq t\}.$$

Then, on $\mathcal{E} \cap L$, by a triangle inequality

$$\|\mathcal{A}(X^n) - x_0\|_2 = \|\bar{X}_n + \bar{\Xi}_h - x_0\|_2 \leq \|\bar{X}_n - x_0\|_2 + \|\bar{\Xi}_h\|_2 \leq \|\bar{X}_n - x_0\|_2 + t.$$

Thus, $\mathcal{E} \cap L \subseteq A$ and by a union bound and Lemma A.6 it holds that

$$\mathbb{P}(A) \geq \mathbb{P}(\mathcal{E} \cap L) = 1 - \mathbb{P}(\mathcal{E}^c \cup L^c) \geq 1 - \mathbb{P}(\mathcal{E}^c) - \mathbb{P}(L^c) \geq 1 - \upsilon - \alpha.$$

$\square$

D.1.2 Proof of In-Expectation Bound

As $\mathcal{A}_{\hat{m}(Z^n)}$ is constructed per coordinate, the proof reduces to the one-dimensional case of Lemma D.8 where $X^n, Z^n \in \mathbb{R}^n$. Therein, by iterated expectation and a bias-variance decomposition conditional on $\hat{I}$ that collects Z^n and the randomness of Algorithm 3:

$$\mathbb{E}[|\mathcal{A}_{\hat{m}(Z^n)}(X^n) - \mu|^2] \leq \mathbb{E}[\text{Var}(A_{\hat{m}(Z^n)}(X^n)|\hat{I}) + \text{Bias}(A_{\hat{m}(Z^n)}(X^n)|\hat{I})^2].$$

The conditional variance is bounded easily, because $\hat{I}$, X^n and Ξ are independent and the projection is a contraction. Our main technical innovation is a bound on the conditional bias. This is straightforward for $X_i \in [-B, B]$, since then similar to Appendix B.1 in Duchi et al. (2018),

$$\mathbb{E}[\text{Bias}(A_{\hat{m}(Z^n)}(X^n)|\hat{I})^2] \lesssim \mathbb{E}[\mathbb{E}[|X_i - \hat{m}|^2|\hat{I}] \cdot \mathbb{P}(X_i \notin \hat{I}|\hat{I})] \leq B^2 \cdot \mathbb{P}(X_i \notin \hat{I}).$$

Our contribution is to avoid the boundedness assumption in the argument above. In particular, this involves handling the failure event of the private midpoint estimator in Algorithm 3 caused by the stable histogram outputting $\tilde{p} = \mathbf{0}$. On this event with exponentially small probability in $n\varepsilon$ as shown in Lemma D.10 in Appendix D.1.2, the MSE is bounded by $\|\mu\|_2^2$. When the private midpoint algorithm does not fail, we exploit the design of the stable histogram estimator in Algorithm 2 to bound the bias. Concretely, we use that the midpoint $\hat{m}(Z^n)$ can at most be of order τ away from one of the X_i and can thus be controlled using maximal inequalities.

Proof of Theorem 4.3.

The estimator $\mathcal{A}_{\hat{m}(Z^n)}$ used in this theorem differs from the one in Algorithm 5 only by computing the midpoint $\hat{m} := \hat{m}(Z^n) \in \mathbb{R}^d$ using an additional independent dataset Z^n instead of X^n.

The estimator $\mathcal{A}_{\hat{m}(Z^n)}$ is private by reasoning analogous to Theorem D.1.

By definition of the MSE of an estimator, we have the first line in the following:

$$\mathrm{MSE}(\mathcal{A}_{\hat{m}(Z^n)}) := \mathbb{E}[\|\mathcal{A}_{\hat{m}(Z^n)}(X^n) - \mu\|_2^2] = \sum_{j=1}^{d} \mathbb{E}[|\mathcal{A}_{1,\hat{m}(Z^n_{\cdot j})}(X^n_{\cdot j}) - \mu_j|^2]$$

$$\lesssim \sum_{j=1}^{d} \left(\rho \log(4n) \left(\frac{1}{n} + \frac{1}{n^2 (\varepsilon')^2} + \frac{n}{\delta' e^{cn\varepsilon'/(2 \cdot 4^3)}} \right) + \frac{\mu_j^2}{\delta' e^{cn\varepsilon'}} \right)$$

$$\lesssim d\rho \log(4n) \left(\frac{1}{n} + \frac{d \log(1/\varrho)}{n^2 \varepsilon^2} + \frac{dn}{\delta e^{cn\varepsilon'/(2 \cdot 4^3)}} \right) + \frac{d\|\mu\|_2^2}{\delta e^{cn\varepsilon'}}$$

$$\lesssim d\rho \log(4n) \left(\frac{1}{n} + \frac{d \log(1/\varrho)}{n^2 \varepsilon^2} + \frac{dn}{\delta e^{cn\varepsilon'/(2 \cdot 4^3)}} \right) + \frac{d\|\mu\|_2^2}{\delta e^{cn\varepsilon'/4^3}}.$$

The first inequality holds by Lemma D.8 that is applicable as $A_{\hat{m}(Z^n)}$ is constructed from coordinate-wise estimators $\mathcal{A}_{1,\hat{m}(Z^n_{\cdot j})}$ specified in Algorithm 4 with the midpoints $\hat{m}_j$ computed on $Z^n_{\cdot j}$ and the Winsorized means computed on $X^n_{\cdot j}$. The privacy parameters

$$\varepsilon' = \varepsilon/\sqrt{8d \log(1/\varrho)} < 1 \quad \text{and} \quad \delta' = \delta/d$$

are adjusted according to advanced composition in Theorem 2.4, which yields the second to last inequality. Finally, the last inequality holds, because $c \geq c/4^3$. In the statement $\kappa := c/4^3$. $\qquad\square$

Corollary D.7. *Let $\mathcal{A}_{\hat{m}(Z^n)}$ denote Algorithm 5 with midpoint $\hat{m}(Z^n)$. $\mathcal{A}_{\hat{m}(Z^n)}$ is (ε, δ)-DP for $\varepsilon, \delta \in (0,1)$. Let $X^n, Z^n \in \mathbb{R}^{n \times d}$ have rows with mean $\mu \in \mathbb{R}^d$ s.t. $\|\mu\|_\infty^2 \lesssim \rho n e^{\kappa n\varepsilon'/2}$. Make Assumption 2.7 with ρ-sub-Gaussian entries or Assumption 2.8 s.t. $\rho M^2 \lesssim 1$. For $\delta \in (n\varepsilon'/e^{\kappa n\varepsilon'}, 1/n]$,*

$$\mathbb{E}[\|\mathcal{A}_{\hat{m}(Z^n)}(X^n) - \mu\|_2^2] \lesssim d\rho \log(4n) \left(\frac{1}{n} + \frac{d \log(2/\delta)}{n^2 \varepsilon^2} \right).$$

Proof. By Theorem 4.3 with $\varrho = \delta$ and using that $\|\mu\|_2^2 \leq d\|\mu\|_\infty^2 \lesssim d\rho n e^{\kappa n\varepsilon'/2}$ we have

$$\mathbb{E}[\|\mathcal{A}_{\hat{m}(Z^n)}(X^n) - \mu\|_2^2] \lesssim d\rho \log(4n) \left(\frac{1}{n} + \frac{d \log(1/\varrho)}{n^2 \varepsilon^2} + \frac{dn}{\delta e^{\kappa n\varepsilon'/2}} \right) + \frac{d\|\mu\|_2^2}{\delta e^{\kappa n\varepsilon'}}$$

$$\lesssim d\rho \log(4n) \left(\frac{1}{n} + \frac{d \log(1/\varrho)}{n^2 \varepsilon^2} + \frac{dn}{\delta e^{\kappa n\varepsilon'/2}} \right) + \frac{d^2 \rho n}{\delta e^{\kappa n\varepsilon'/2}}$$

$$\lesssim d\rho \log(4n) \left(\frac{1}{n} + \frac{d \log(1/\varrho)}{n^2 \varepsilon^2} + \frac{d}{\delta^2 e^{\kappa n\varepsilon'/2}} \right).$$

Here, for the last inequality we use that $\delta \leq 1/n$ and $1 \leq \log(4n)$. We now choose δ s.t. the second dominates the third term. Note that $\varepsilon' \asymp \varepsilon/\sqrt{d \log(1/\varrho)}$ and thus we require

$$\frac{d}{\delta^2 e^{\kappa n\varepsilon'/2}} \lesssim \frac{d}{(n\varepsilon')^2} \quad \Leftrightarrow \quad \delta \gtrsim \frac{n\varepsilon'}{e^{\kappa n\varepsilon'/4}}.$$

The statement then follows by setting $\varrho = \delta$ and δ is replaced by $\delta/2$ to obtain (ε, δ)-DP. $\qquad\square$

Lemma D.8. *Let $\mathcal{A}_{\hat{m}(Z^n)}$ denote Algorithm 4 with midpoint $\hat{m}$ computed on $Z^n \in \mathbb{R}^n$. $\mathcal{A}_{\hat{m}(Z)}$ is (ε, δ)-DP for $\varepsilon, \delta \in (0,1)$. Let $X^n \in \mathbb{R}^n$ fulfill Assumption 2.7 and have ρ-sub-Gaussian entries or Assumption 2.8 s.t. $\rho M^2 \lesssim 1$. If Z^n is an independent copy of X^n and choosing $\tau^2 \asymp \rho \log(4n)$,*

$$\mathbb{E}[|\mathcal{A}_{\hat{m}(Z^n)}(X^n) - \mu|^2] \lesssim \rho \log(4n) \left(\frac{1}{n} + \frac{1}{n^2 \varepsilon^2} + \frac{n}{\delta e^{cn\varepsilon/(2 \cdot 4^3)}} \right) + \frac{\mu^2}{\delta e^{cn\varepsilon}}.$$

Proof. Algorithm 5 is private by reasoning analogous to Lemma 4.1.

Let the the output of Algorithm 3 on Z^n be the projection interval $\hat{I} := \hat{I}(Z^n)$. For brevity, define the estimator T and its noise-free version T_0 as

$$T := \mathcal{A}_{\hat{m}(Z^n)}(X^n) = \frac{1}{n} \sum_{i=1}^n \Pi_{\hat{I}}(X_i) + \xi \quad \text{and} \quad T_0 := \frac{1}{n} \sum_{i=1}^n \Pi_{\hat{I}}(X_i).$$

This construction of T ensures that X^n and $\hat{I}$ are independent and simplifies the analysis. By definition of $\mathrm{MSE}(T)$, iterated expectation and a bias-variance decomposition applied to the conditional mean squared error $\mathbb{E}[|T - \mu|^2 | \hat{I}]$ we obtain

$$\mathrm{MSE}(T) := \mathbb{E}[|T - \mu|^2] = \mathbb{E}[\mathbb{E}[|T - \mu|^2 | \hat{I}]] = \mathbb{E}[\mathbb{E}[|T - \mathbb{E}[T|\hat{I}]|^2 | \hat{I}] + |\mathbb{E}[T|\hat{I}] - \mu|^2]$$
$$= \mathbb{E}[\mathrm{Var}(T|\hat{I}) + \mathrm{Bias}(T|\hat{I})^2].$$

We now bound the conditional variance and bias of T separately. For the conditional variance, by the mutual independence of ξ, X^n and $\hat{I}$ and the variance of a Laplace random variable

$$\mathrm{Var}(T|\hat{I}) = \mathrm{Var}(T_0 + \xi | \hat{I}) = \mathrm{Var}(T_0 | \hat{I}) + \mathrm{Var}(\xi)$$
$$= \mathrm{Var}(T_0 | \hat{I}) + 2b^2 \leq \frac{\rho}{n} + 2b^2 \lesssim \frac{\rho}{n} + \frac{\tau^2}{n^2 \varepsilon^2}.$$

Above, the inequality holds by Corollary D.12 or D.11 which are both applicable, because we condition on $\hat{I}$ that is independent of X^n. Further, we use that $b \asymp \tau/(n\varepsilon)$.

It remains to upper bound the conditional bias of T. For the following we use that $\mathbb{E}[\xi] = 0$, $\xi, \hat{I}$ are independent and Jensen's inequality:

$$\mathrm{Bias}(T|\hat{I})^2 := |\mathbb{E}[T|\hat{I}] - \mu|^2 = |\mathbb{E}[T_0|\hat{I}] + \mathbb{E}[\Xi] - \mu|^2$$
$$= |\mathbb{E}[T_0 - \bar{X}_n | \hat{I}]|^2 = \left| \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n (\Pi_{\hat{I}}(X_i) - X_i) | \hat{I} \right] \right|^2 \leq \frac{1}{n} \sum_{i=1}^n \mathbb{E}[|\Pi_{\hat{I}}(X_i) - X_i| | \hat{I}]^2.$$

Note that the distance between X_i and $\Pi_{\hat{I}}(X_i)$ is upper bounded as

$$|\Pi_{\hat{I}}(X_i) - X_i| \leq \begin{cases} 0 & \text{if } X_i \in \hat{I} \\ X_i - (\hat{m} + 3\tau) & \text{if } X_i \notin \hat{I} \text{ and } X_i > \hat{m} \\ \hat{m} - 3\tau - X_i & \text{if } X_i \notin \hat{I} \text{ and } X_i < \hat{m}. \end{cases}$$

These three cases can be combined into the following bound

$$|\Pi_{\hat{I}}(X_i) - X_i| \leq (|X_i - \hat{m}| - 3\tau) \cdot \mathbf{1}\{X_i \notin \hat{I}\} \leq |X_i - \hat{m}| \cdot \mathbf{1}\{X_i \notin \hat{I}\}.$$

We introduce the event $\mathcal{E}_{zero} := \{\forall k \in \mathbb{Z} : \tilde{p}_k = 0\}$ on which the histogram fails. On $\mathcal{E}_{zero}$ we have that $\hat{I} := [\pm 3\tau]$ and otherwise $\hat{I} := [\hat{m} \pm 3\tau]$. Distinguishing both cases, for each index $i \in [n]$,

$$
\begin{aligned}
\mathbb{E}[|\Pi_{\hat{I}}(X_i) - X_i||\hat{I}]^2 &\le \mathbb{E}[|X_i - \hat{m}| \cdot \mathbf{1}\{X_i \notin \hat{I}\}|\hat{I}]^2 \\
&= \mathbb{E}[|X_i| \cdot \mathbf{1}\{X_i \notin \hat{I}\} \cdot \mathbf{1}\{\mathcal{E}_{zero}\} + |X_i - \hat{m}| \cdot \mathbf{1}\{X_i \notin \hat{I}\} \cdot \mathbf{1}\{\mathcal{E}_{zero}^c\}|\hat{I}]^2 \\
&\le \mathbb{E}[|X_i| \cdot \mathbf{1}\{\mathcal{E}_{zero}\} + |X_i - \hat{m}| \cdot \mathbf{1}\{\mathcal{E}_{zero}^c\} \cdot \mathbf{1}\{X_i \notin \hat{I}\}|\hat{I}]^2 \\
&\lesssim \underbrace{\mathbb{E}[|X_i| \cdot \mathbf{1}\{\mathcal{E}_{zero}\}|\hat{I}]^2}_{T_1} + \underbrace{\mathbb{E}[|X_i - \hat{m}| \cdot \mathbf{1}\{\mathcal{E}_{zero}^c\} \cdot \mathbf{1}\{X_i \notin \hat{I}\}|\hat{I}]^2}_{T_2} .
\end{aligned}
$$

Using the Cauchy-Schwarz inequality, the independence of $X_i, \hat{I}$ and $\mathrm{Var}(X_i) \le \rho$ which holds for ρ-sub-Gaussian random variables (Vershynin, 2025, Exercise 2.40), we obtain the bound

$$
T_1 = \mathbb{E}[|X_i| \cdot \mathbf{1}\{\mathcal{E}_{zero}\}|\hat{I}]^2 \le \mathbb{E}[|X_i|^2] \cdot \mathbb{E}[\mathbf{1}\{\mathcal{E}_{zero}\}^2|\hat{I}] \le (\rho + \mu^2) \cdot \mathbb{P}(\mathcal{E}_{zero}|\hat{I}).
$$

Taking expectation, using Lemma D.10 and that $\delta, \varepsilon < 1$ with $c > 0$ that remains the same throughout the whole proof, we have

$$
\mathbb{E}[T_1] \le (\rho + \mu^2) \cdot \mathbb{E}[\mathbb{P}(\mathcal{E}_{zero}|\hat{I})] = (\rho + \mu^2) \cdot \mathbb{P}(\mathcal{E}_{zero}) \lesssim (\rho + \mu^2) \cdot 1/\delta \cdot e^{-cn\varepsilon}.
$$

Similarly, for T_2 the Cauchy-Schwarz inequality yields

$$
T_2 = \mathbb{E}[|X_i - \hat{m}| \cdot \mathbf{1}\{\mathcal{E}_{zero}^c\} \cdot \mathbf{1}\{X_i \notin \hat{I}\}|\hat{I}]^2 \le \underbrace{\mathbb{E}[(|X_i - \hat{m}| \cdot \mathbf{1}\{\mathcal{E}_{zero}^c\})^2|\hat{I}]}_{T_3} \cdot \underbrace{\mathbb{E}[\mathbf{1}\{X_i \notin \hat{I}\}^2|\hat{I}]}_{T_4} .
$$

We bound T_3 via a triangle inequality. By independence of $X_i, \hat{I}$ and because $\rho \lesssim \tau^2$ we have $\mathrm{Var}(X_i|\hat{I}) = \mathrm{Var}(X_i) \le \rho \le 2\tau^2$. Therefore,

$$
\begin{aligned}
T_3 := \mathbb{E}[|X_i - \hat{m}|^2 \cdot \mathbf{1}\{\mathcal{E}_{zero}^c\}|\hat{I}] &\le \mathbb{E}[(|X_i - \mu| + |\mu - \hat{m}|)^2|\hat{I}] \cdot \mathbf{1}\{\mathcal{E}_{zero}^c\} \\
&\lesssim (\mathbb{E}[|X_i - \mu|^2|\hat{I}] + \mathbb{E}[|\mu - \hat{m}|^2|\hat{I}]) \cdot \mathbf{1}\{\mathcal{E}_{zero}^c\} \\
&\le \mathrm{Var}(X_i|\hat{I}) + \mathbb{E}[|\mu - \hat{m}|^2|\hat{I}] \cdot \mathbf{1}\{\mathcal{E}_{zero}^c\} \\
&\le \rho + |\mu - \hat{m}|^2 \cdot \mathbf{1}\{\mathcal{E}_{zero}^c\} \\
&\lesssim \tau^2 + \max_{i \in [n]} |\mu - Z_i|^2 .
\end{aligned}
$$

Above, the last inequality holds for the following reason. By design of Algorithm 3 and Algorithm 2 that compute $\hat{m}$, on $\mathcal{E}_{zero}^c$ any bin B_k with $\tilde{p}_k > 0$ must have at least one Z_i inside. Bins have width 2τ. Hence, the midpoint m_k must be less than τ away from any $Z_i \in B_k$. The same holds for $\hat{m}$, the center of B_k with the biggest $\tilde{p}_k$. This reasoning deterministically yields the last inequality:

$$
\begin{aligned}
\min_{i \in [n]} Z_i - \tau \le \hat{m} \le \max_{i \in [n]} Z_i + \tau, \quad &\Rightarrow \quad |\mu - \hat{m}| \le \max_{i \in [n]} |\mu - Z_i| + \tau, \\
&\Rightarrow \quad |\mu - \hat{m}|^2 \le 2 \max_{i \in [n]} |\mu - Z_i|^2 + 2\tau^2 .
\end{aligned}
$$

To bound T_4 we again use that X_i and $\hat{I}$ are independent. Observe that under Assumption 2.8, by Theorem 2.6 applied with $f(x) := x_i$ that is 1-Lipschitz, the X_i are sub-Gaussian. Hence, we have

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$\mathbb{P}(|X_i - \mu| \geq \tau) \leq \gamma/n$ for $\tau^2 \asymp \rho \log(2n/\gamma)$. Then, by a triangle inequality

$$
\begin{aligned}
T_4 = \mathbb{E}[\mathbf{1}\{X_i \notin \hat{I}\}|\hat{I}] = \mathbb{P}\left(|X_i - \hat{m}| > 3\tau|\hat{I}\right) \\
\leq \mathbb{P}\left(|X_i - \mu| \geq \tau|\hat{I}\right) + \mathbb{P}\left(|\mu - \hat{m}| > 2\tau|\hat{I}\right) \\
\leq \mathbb{P}\left(|X_i - \mu| \geq \tau\right) + \mathbf{1}\{\hat{m} \notin [\mu \pm 2\tau]\} \\
\leq \gamma/n + \mathbf{1}\{\hat{m} \notin [\mu \pm 2\tau]\}.
\end{aligned}
$$

For convenience, define $\mathcal{E}_{bad} := \{\hat{m} \notin [\mu \pm 2\tau]\}$ and $M_n^p := \max_{i \in [n]} |Z_i - \mu|^p$ for $p \geq 1$. As the maximum is monotone, $(M_n^1)^p = M_n^p$ for such p. To control $\mathbb{E}[\text{Bias}(T|\hat{I})^2]$ we now bound $T_3 T_4$ in expectation using the Cauchy-Schwarz inequality and the fact that $\mathbf{1}\{\cdot\} = \mathbf{1}\{\cdot\}^2$:

$$
\begin{aligned}
\mathbb{E}[T_3 T_4] &\lesssim \mathbb{E}\left[(\tau^2 + M_n^2) \cdot (\gamma/n + \mathbf{1}\{\mathcal{E}_{bad}\})\right] \\
&= \gamma/n \cdot \left(\tau^2 + \mathbb{E}[M_n^2]\right) + \tau^2 \cdot \mathbb{P}\left(\mathcal{E}_{bad}\right) + \mathbb{E}[M_n^2 \cdot \mathbf{1}\{\mathcal{E}_{bad}\}] \\
&\leq \tau^2 \cdot (\gamma/n + \mathbb{P}\left(\mathcal{E}_{bad}\right)) + \gamma/n \cdot \mathbb{E}[M_n^2] + \sqrt{\mathbb{E}[M_n^4] \cdot \mathbb{P}(\mathcal{E}_{bad})}.
\end{aligned}
$$

Again, using that the X_i are sub-Gaussian, by Corollary D.13, Lemma 3.4 and since $\varepsilon < 1$ we have

$$
\mathbb{E}[M_n^2] \lesssim \rho \log(2n), \quad \mathbb{E}[M_n^4] \lesssim \rho^2 \log(2n)^2 \quad \text{and} \quad \mathbb{P}(\mathcal{E}_{bad}) \lesssim \frac{n}{\delta e^{cn\varepsilon/4^3}}.
$$

Plugging the above bounds into $\mathbb{E}[T_3 T_4]$ and using that $1 \leq n/\delta$ and $e^{-\kappa n \varepsilon} \leq e^{-\kappa n \varepsilon/2}$ yields

$$
\begin{aligned}
\mathbb{E}[T_3 T_4] &\lesssim \tau^2 \cdot \left(\frac{\gamma}{n} + \frac{n}{\delta e^{cn\varepsilon/4^3}}\right) + \frac{\gamma}{n} \cdot \rho \log(2n) + \rho \log(2n) \cdot \sqrt{\frac{n}{\delta e^{cn\varepsilon/4^3}}} \\
&\lesssim \tau^2 \cdot \left(\frac{\gamma}{n} + \frac{n}{\delta e^{cn\varepsilon/(2 \cdot 4^3)}}\right).
\end{aligned}
$$

Collecting terms we have thus obtained the conditional bias bound to follow:

$$
\begin{aligned}
\mathbb{E}[\text{Bias}(T|\hat{I})^2] &\leq \frac{1}{n} \sum_{i=1}^n \mathbb{E}[\mathbb{E}[|\Pi_{\hat{I}}(X_i) - X_i||\hat{I}]^2] \leq \frac{1}{n} \sum_{i=1}^n \mathbb{E}[T_1] + \mathbb{E}[T_3 T_4] \\
&\lesssim \frac{\mu^2}{\delta e^{cn\varepsilon}} + \tau^2 \left(\frac{\gamma}{n} + \frac{n}{\delta e^{cn\varepsilon/(2 \cdot 4^3)}}\right).
\end{aligned}
$$

The bounds on the conditional variance and bias then imply that

$$
\begin{aligned}
\text{MSE}(T) = \mathbb{E}[\text{Var}(T|\hat{I}) + \text{Bias}(T|\hat{I})^2] \\
\lesssim \left(\frac{\rho}{n} + \frac{\tau^2}{n^2 \varepsilon^2}\right) + \tau^2 \left(\frac{\gamma}{n} + \frac{n}{\delta e^{cn\varepsilon/(2 \cdot 4^3)}}\right) + \frac{\mu^2}{\delta e^{cn\varepsilon}} \\
\lesssim \tau^2 \left(\frac{\gamma + 1}{n} + \frac{1}{n^2 \varepsilon^2} + \frac{n}{\delta e^{cn\varepsilon/(2 \cdot 4^3)}}\right) + \frac{\mu^2}{\delta e^{cn\varepsilon}}.
\end{aligned}
$$

Plugging in $\tau^2 \asymp \rho \log(2n/\gamma)$ and choosing $\gamma = 1/2$ recovers the statement. $\qquad \square$

Proofs for Failure of Stable Histogram

Lemma D.9. *Fix observations $x \in \mathbb{R}^n$ and bins $(B_k)_{k \in \mathbb{Z}}$. Define $\hat{p}_k := \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}\{x_i \in B_k\}$ and assume that $\hat{p}_k \geq \alpha$ with $\alpha > 0$. Then, the output $\tilde{p}_k$ of Algorithm 2 fulfills*

$$\mathbb{P}\left(\{\tilde{p}_k = 0\} \cap \{\hat{p}_k \geq \alpha\}\right) \leq \frac{e^{\varepsilon/2}}{\delta} \cdot \exp\left(-\frac{\alpha n \varepsilon}{2}\right).$$

Proof. By design of Algorithm 2, when $\tilde{p}_k = 0$, straightforward calculations yield

$$
\begin{aligned}
\mathbb{P}\left(\{\tilde{p}_k = 0\} \cap \{\hat{p}_k \geq \alpha\}\right) &= \mathbb{P}\left(\left\{\xi_k < \frac{2}{\varepsilon n} \log\left(\frac{1}{\delta}\right) + \frac{1}{n} - \hat{p}_k\right\} \cap \{\hat{p}_k \geq \alpha\}\right) \\
&\leq \mathbb{P}\left(\xi_k < \frac{2}{\varepsilon n} \log\left(\frac{1}{\delta}\right) + \frac{1}{n} - \alpha\right) \\
&= \mathbb{P}\left(\xi_k > \alpha - \frac{1}{n} - \frac{2}{\varepsilon n} \log\left(\frac{1}{\delta}\right)\right) \\
&\leq \exp\left(-\frac{\varepsilon n}{2}\left\{\alpha - \frac{1}{n} - \frac{2}{\varepsilon n} \log\left(\frac{1}{\delta}\right)\right\}\right) \\
&= \frac{e^{\varepsilon/2}}{\delta} \cdot \exp\left(-\frac{\alpha n \varepsilon}{2}\right).
\end{aligned}
$$

Here, we use that $-\xi_k = \xi_k$ in distribution and we apply a standard Laplace tail bound. $\qquad\square$

Lemma D.10. *Let $\gamma \in (0, 1 \wedge \frac{n}{4})$ and let $X^n \in \mathbb{R}^n$ be $(\tau, \gamma)_\infty^{x_0}$-concentrated. Then, for Algorithm 2*

$$\mathbb{P}\left(\forall k \in \mathbb{Z} : \tilde{p}_k = 0\right) \leq \frac{e^{\varepsilon/2}}{\delta} \cdot \exp\left(-\frac{n\varepsilon}{8}\right) + \begin{cases} 2\exp\left(-\frac{n}{128}\right) & \text{if } X^n \text{ fulfills Assumption 2.7,} \\ 64 \exp\left(-\frac{2n}{48^3 \rho M^2}\right) & \text{if } X^n \text{ fulfills Assumption 2.8.} \end{cases}$$

Proof. We start by reducing the problem of bounding the probability that all bins are zero to bounding the probability that a bin with high empirical mass is thresholded.

Define $\mathcal{E}_{zero} := \{\tilde{p}_k = 0, \forall k \in \mathbb{Z}\}$ and $\mathcal{E}_{zero}^k := \{\tilde{p}_k = 0\}$. Note that

$$\mathbb{P}\left(\mathcal{E}_{zero}\right) = \mathbb{P}\left(\forall k \in \mathbb{Z} : \tilde{p}_k = 0\right) \leq \mathbb{P}\left(\tilde{p}_l = 0\right) = \mathbb{P}\left(\mathcal{E}_{zero}^l\right), \quad \text{for all } l \in \mathbb{Z}.$$

As we want to apply Lemma D.9, we want to use this for a bin with high mass and accordingly also high empirical mass. By $(\tau, \gamma)_\infty$-concentration, for all $i \in [n]$ with probability at least $1 - \gamma/n =: 1 - \alpha$ we have $X_i \in I := [x_0 \pm \tau]$. We construct a set S of bin-indices $k \in \mathbb{Z}$ that cover I:

$$S := \{k \in \mathbb{Z} : I \cap B_k \neq \emptyset\}.$$

Note that $|S| \leq 2$, (4) and (5) hold, and take $k^\star$ as in (5).

We will now bound $\mathbb{P}(\mathcal{E}_{zero}^{k^\star})$ assuming that X^n fulfills the i.i.d. Assumption (2.7). We split it on the event where $\hat{p}_{k^\star}$ estimates $p_{k^\star}$ well and where it does not:

$$
\begin{aligned}
\mathbb{P}\left(\mathcal{E}_{zero}^{k^\star}\right) &= \mathbb{P}\left(\mathcal{E}_{zero}^{k^\star} \cap \left\{\max_{k \in \mathbb{Z}} |\hat{p}_k - p_k| < \eta\right\}\right) + \mathbb{P}\left(\mathcal{E}_{zero}^{k^\star} \cap \left\{\max_{k \in \mathbb{Z}} |\hat{p}_k - p_k| \geq \eta\right\}\right) \\
&\leq \mathbb{P}\left(\mathcal{E}_{zero}^{k^\star} \cap \left\{\max_{k \in \mathbb{Z}} |\hat{p}_k - p_k| < \eta\right\}\right) + \mathbb{P}\left(\max_{k \in \mathbb{Z}} |\hat{p}_k - p_k| \geq \eta\right) \\
&\leq \mathbb{P}\left(\mathcal{E}_{zero}^{k^\star} \cap \{|\hat{p}_{k^\star} - p_{k^\star}| < \eta\}\right) + 2\exp\left(-\frac{n\eta^2}{2}\right).
\end{aligned}
$$

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Here, the last inequality holds by Lemma C.3. On the event $\mathcal{E}_{zero}^{k^\star} \cap \{|\hat{p}_{k^\star} - p_{k^\star}| < \eta\}$ we know that there is $k^\star \in \mathbb{Z}$ s.t. $p_{k^\star} \geq \frac{1-\alpha}{2}$ and therefore $\hat{p}_{k^\star} > p_{k^\star} - \eta \geq \frac{1-\alpha-2\eta}{2}$. When $\frac{1-\alpha-2\eta}{2} > 0$ we know that $\hat{p}_{k^\star} > 0$ and hence $\tilde{p}_{k^\star}$ can only be 0 through thresholding. Formally, by Lemma D.9

$$\mathbb{P}\left(\mathcal{E}_{zero}^{k^\star} \cap \{|\hat{p}_{k^\star} - p_{k^\star}| \leq \eta\}\right) \leq \mathbb{E}\left[\mathbb{P}\left(\{\tilde{p}_{k^\star} = 0\} \cap \left\{\hat{p}_{k^\star} \geq \frac{1-\alpha-2\eta}{2}\right\} \Big| X^n\right)\right]$$

$$\leq \frac{e^{\varepsilon/2}}{\delta} \cdot \exp\left(-\frac{(1-\alpha-2\eta)n\varepsilon}{4}\right)$$

$$\leq \frac{e^{\varepsilon/2}}{\delta} \cdot \exp\left(-\frac{(3/4-2\eta)n\varepsilon}{4}\right).$$

For the last inequality we use that $\alpha < 1/4$ by assumption. We set $\eta = 1/8$ and collect terms:

$$\mathbb{P}\left(\mathcal{E}_{zero}\right) \leq \mathbb{P}\left(\mathcal{E}_{zero}^{k^\star}\right) \leq \frac{e^{\varepsilon/2}}{\delta} \cdot \exp\left(-\frac{n\varepsilon}{4\cdot 2}\right) + 2\exp\left(-\frac{n}{2\cdot 8^2}\right).$$

The proof under Assumption 2.8 is analogous. □

Auxiliary Results for Asymptotic Analysis

Corollary D.11. *Let $X^n \in \mathbb{R}^n$ fulfill Assumption 2.8 with constants (ρ, ∞). For a fixed closed convex set $C \subseteq \mathbb{R}$ and the L_2-projection $\Pi_C(x) := \arg\min_{y\in C} |x-y|^2$ we have*

$$\mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{n} \Pi_C(X_i)\right) \leq \frac{\rho}{n}.$$

Proof. Since X^n fulfills Assumption 2.8 with constants (ρ, ∞), the distribution of X^n fulfills LSI(ρ). The non-expansiveness of the L_2-projection shows that for all $x, y \in \mathbb{R}^n$ we have

$$\left|\frac{1}{n}\sum_{i=1}^{n} \Pi_C(x_i) - \frac{1}{n}\sum_{i=1}^{n} \Pi_C(y_i)\right| \leq \frac{1}{n}\sum_{i=1}^{n} |\Pi_C(x_i) - \Pi_C(y_i)| \leq \frac{1}{n}\sum_{i=1}^{n} |x_i - y_i| \leq \frac{1}{\sqrt{n}}\|x-y\|_2.$$

Thus, $f(x) := \frac{1}{n}\sum_{i=1}^{n} \Pi_C(x_i)$ is $1/\sqrt{n}$-Lipschitz. By a Poincaré inequality

$$\mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{n} \Pi_C(X_i)\right) = \mathrm{Var}(f(X)) \leq \rho \cdot \mathbb{E}[\|\nabla f(X)\|_2^2] \leq \frac{\rho}{n}.$$

□

Corollary D.12. *Let $X^n \in \mathbb{R}^n$ fulfill Assumption 2.7 and have σ^2-sub-Gaussian entries. For a fixed closed convex set $C \subseteq \mathbb{R}$ and the L_2-projection $\Pi_C(x) := \arg\min_{y\in C} |x-y|^2$ we have*

$$\mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{n} \Pi_C(X_i)\right) \leq \frac{\sigma^2}{n}.$$

Proof. By independence of $X_1, \ldots, X_n$ we have the equality in the following:

$$\mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{n} \Pi_C(X_i)\right) = \frac{1}{n^2}\sum_{i=1}^{n} \mathrm{Var}(\Pi_C(X_i)) \leq \frac{1}{n^2}\sum_{i=1}^{n} \mathrm{Var}(X_i) \leq \frac{\sigma^2}{n}.$$

The last inequality holds as $\mathrm{Var}(X_i) \leq \sigma^2$ for sub-Gaussian X_i (Vershynin, 2025, Exercise 2.40). □

Corollary D.13. *Let $Z_1, \ldots, Z_n$ have mean μ and be σ^2-sub-Gaussian. Then,*

$$\mathbb{E}[\max_{i \in [n]} |Z_i - \mu|^4] \lesssim \sigma^4 \log(2n)^2 \quad and \quad \mathbb{E}[\max_{i \in [n]} |Z_i - \mu|^2] \lesssim \sigma^2 \log(2n).$$

Proof. First note that a union bound leads to the following tail-bound:

$$\mathbb{P}\left(\max_{i \in [n]} |Z_i - \mu| \geq t\right) = \mathbb{P}\left(\bigcup_{i \in [n]} \{|Z_i - \mu| \geq t\}\right) \leq \sum_{i \in [n]} \mathbb{P}\left(|Z_i - \mu| \geq t\right) \leq 2n \exp\left(-\frac{t^2}{2\sigma^2}\right).$$

Taking the 4-th power, the change of variables $a = t^4 \Leftrightarrow t^2 = \sqrt{a}$ immediately give us

$$\mathbb{P}\left(\max_{i \in [n]} |Z_i - \mu_j|^4 \geq a\right) \leq 2n \exp\left(-\frac{\sqrt{a}}{2\sigma^2}\right).$$

The desired in-expectation results follows from the integral identity and the tails bounds we have established. Indeed,

$$\mathbb{E}[\max_{i \in [n]} |Z_i - \mu|^4] = \int_0^\infty \mathbb{P}\left(\max_{i \in [n]} |Z_i - \mu|^4 \geq a\right) da \leq 2n \int_0^\infty \exp\left(-\frac{\sqrt{a}}{2\sigma^2}\right) da = 16n\sigma^4$$

The order of this bound can be improved as follows:

$$\begin{aligned}
\mathbb{E}[\max_{i \in [n]} |Z_i - \mu|^4] &= \int_0^\infty \mathbb{P}\left(\max_{i \in [n]} |Z_i - \mu|^4 \geq a\right) da \\
&\leq \int_0^b 1 \, da + \int_b^\infty 2n \exp\left(-\frac{\sqrt{a}}{2\sigma^2}\right) da \\
&\leq b + 2n \left(8\sigma^4 + 4\sigma^2 \sqrt{b}\right) e^{-\frac{\sqrt{b}}{2\sigma^2}}
\end{aligned}$$

Minimizing over b then yields $b = 4\sigma^4 \log(2n)^2$ and the maximal inequality becomes

$$\mathbb{E}[\max_{i \in [n]} |Z_i - \mu|^4] \leq b + 2n \left(8\sigma^4 + 4\sigma^2 \sqrt{b}\right) e^{-\frac{\sqrt{b}}{2\sigma^2}} = 4\sigma^4 \log(2n)^2 + 8\sigma^4 + \frac{8\sigma^4 \log(2n)}{2n}.$$

The first statement follows as the above is bounded in order by $\sigma^4 \log(2n)^2$. The second statement follows because higher moment bounds imply lower moment bounds by Jensen's inequality. $\qquad \square$

D.2 Proofs for Nonparametric Regression

Proof of Lemma 4.9.

Throughout, we extend f by 0 outside of its support $[0, 1]$. The proof starts with a triangle inequality that lets us decompose the mean absolute error of $f, \hat{f}_n$ into bias and mean absolute deviation,

$$|f(x) - \hat{f}_n(x)| \leq \underbrace{|f(x) - \mathbb{E}[\hat{f}_n(x)]|}_{\text{Bias}(\hat{f}_n)} + \underbrace{|\mathbb{E}[\hat{f}_n(x)] - \hat{f}_n(x)|}_{\text{Mad}(\hat{f}_n)}.$$

Introducing the integral $I := \int_{\mathbb{R}} f(t) \frac{1}{b} K(\frac{x-t}{b}) dt$, the bias can be further decomposed by adding and subtracting I and using a triangle inequality:

$$\text{Bias}(\hat{f}_n) = |f(x) - \mathbb{E}[\hat{f}_n(x)]| \leq |f(x) - I| + |I - \mathbb{E}[\hat{f}_n(x)]|$$

$$\overset{(\star)}{\leq} b\left(L_f \mu_1(K) + \|f\|_\infty\right) + \frac{L_f \|K\|_\infty + \|f\|_\infty L_K}{nb}.$$

Here, it remains to establish $(\star)$. We handle both terms separately, starting as follows:

$$|f(x) - I| = \left| \int_{\mathbb{R}} (f(x) - f(t)) \frac{1}{b} K\left(\frac{t-x}{b}\right) dt \right|$$

$$\leq \int_0^1 |f(x) - f(t)| \frac{1}{b} K\left(\frac{t-x}{b}\right) dt + \int_{\mathbb{R} \setminus [0,1]} |f(x)| \frac{1}{b} K\left(\frac{t-x}{b}\right) dt =: T + T_{tail}.$$

By Lipschitzness of f, we obtain the following bound on T:

$$T = \int_0^1 |f(x) - f(t)| \frac{1}{b} K\left(\frac{t-x}{b}\right) dt \leq L_f \int_0^1 |x - t| \frac{1}{b} K\left(\frac{t-x}{b}\right) dt \leq b L_f \mu_1(K).$$

Given the assumption that $x \in [\zeta, 1 - \zeta]$ and the 1-sub-Gaussian tails,

$$T_{tail} \leq \|f\|_\infty \int_{\mathbb{R} \setminus [-\frac{x}{b}, \frac{1-x}{b}]} K(u) du \leq \|f\|_\infty \cdot \mathbb{P}(|X| \geq \zeta/b) \leq 2e^{\zeta^2/b^2} \|f\|_\infty \leq b \|f\|_\infty.$$

Above, the last inequality holds assuming $\zeta = b\sqrt{2\log(2/b)}$. For the second term we need to bound to obtain $(\star)$, our choice of the Kernel and the zero mean of ϵ_i yield

$$T_2 = \left| I - \mathbb{E}[\hat{f}_n(x)] \right| = \left| \frac{1}{b} \int_0^1 f(t) K\left(\frac{t-x}{b}\right) dt - \frac{1}{nb} \sum_{i=1}^n \mathbb{E}[Y_i] K\left(\frac{x_i - x}{b}\right) \right|$$

$$= \frac{1}{b} \left| \int_0^1 f(t) K\left(\frac{t-x}{b}\right) dt - \frac{1}{n} \sum_{i=1}^n f(x_i) K\left(\frac{x_i - x}{b}\right) \right| \leq \frac{L_f \|K\|_\infty}{nb} + \frac{\|f\|_\infty L_K}{nb}.$$

Here, the last inequality is by Lemma D.14. Unlike the deterministic bias bound, bounding the mean absolute deviation requires a probabilistic argument:

$$\text{Mad}(\hat{f}_n) = |\mathbb{E}[\hat{f}_n(x)] - \hat{f}_n(x)| = \left| \frac{1}{n} \sum_{i=1}^n (f(x_i) - Y_i) \frac{1}{b} K\left(\frac{x_i - x}{b}\right) \right| = \frac{1}{n} \left| \sum_{i=1}^n \epsilon_i \frac{1}{b} K\left(\frac{x_i - x}{b}\right) \right|.$$

We now show that the $\text{Mad}(\hat{f}_n)$ is an ℓ-Lipschitz function of $\epsilon^n := (\epsilon_1, \ldots, \epsilon_n)$ with $\ell = O((nb)^{-1/2})$. By the triangle and Cauchy-Schwarz inequality, for all $\epsilon, \epsilon' \in \mathbb{R}^n$ we have

$$\frac{1}{n} \left| \left| \sum_{i=1}^n \epsilon_i' \frac{1}{b} K\left(\frac{x_i - x}{b}\right) \right| - \left| \sum_{i=1}^n \epsilon_i \frac{1}{b} K\left(\frac{x_i - x}{b}\right) \right| \right| \leq \frac{1}{n} \left| \sum_{i=1}^n (\epsilon_i' - \epsilon_i) \frac{1}{b} K\left(\frac{x_i - x}{b}\right) \right|$$

$$\leq \frac{1}{n} \left(\sum_{i=1}^n |\epsilon_i' - \epsilon_i|^2 \right)^{1/2} \left(\sum_{i=1}^n \frac{1}{b} K\left(\frac{x_i - x}{b}\right) \right)^{1/2}$$

$$\leq \frac{1}{n} \|\epsilon' - \epsilon\|_2 \frac{\sqrt{n \|K\|_\infty}}{\sqrt{b}} = \sqrt{\frac{\|K\|_\infty}{nb}} \|\epsilon' - \epsilon\|_2.$$

By log-Sobolev Lipschitz concentration of 2.6 with constants $\ell = \sqrt{\|K\|_\infty/nb}$ and $\sigma_{\max}^2$,

$$\mathbb{P}\left(\mathrm{Mad}(\hat{f}_n) \geq \sqrt{\frac{2\sigma_{\max}^2\|K\|_\infty \log(2/\alpha)}{nb}}\right) = \mathbb{P}\left(\mathrm{Mad}(\hat{f}_n) \geq t\right) = \exp\left(-\frac{t^2}{2\ell^2\sigma_{\max}^2}\right) = \alpha.$$

On the complement and therefore with probability $1 - \alpha$, combining this with our deterministic bound on the bias, we obtain the following bound

$$|f(x) - \hat{f}_n(x)| \leq b\left(L_f\mu_1(K) + \|f\|_\infty\right) + \frac{L_f\|K\|_\infty + \|f\|_\infty L_K}{nb} + \sqrt{\frac{2\sigma_{\max}^2\|K\|_\infty \log(2/\alpha)}{nb}}.$$

We now obtain $(\tau, \gamma)_\infty$-concentration from the general statement. As it holds for $n = 1$ and since $\hat{f}_i(x) = Y_i\frac{1}{b}K\left(\frac{x_i-x}{b}\right)$, for any single (x_i, Y_i) with probability at least $1 - \alpha$

$$|f(x) - \hat{f}_1(x)| \leq b\left(L_f\mu_1(K) + \|f\|_\infty\right) + \frac{L_f\|K\|_\infty + \|f\|_\infty L_K}{b} + \sqrt{\frac{2\sigma_{\max}^2\|K\|_\infty \log(2/\alpha)}{b}}.$$

Since $b \leq 1 \Leftrightarrow 1 \leq 1/b$, the statement above yields $(\tau, \gamma)_\infty$-concentration of $\hat{f}^n(x)$. Thereby,

$$\tau = b\left(L_f\mu_1(K) + \|f\|_\infty\right) + \frac{1}{b}\left(L_f\|K\|_\infty + \|f\|_\infty L_K + \sqrt{2\sigma_{\max}^2\|K\|_\infty \log(2n/\gamma)}\right).$$

Finally, we note that since K is a 1-sub-Gaussian density we have $\mu_1(K) \leq 1$. $\qquad\square$

Proof of Lemma 4.10.
Define $f^n := (f(x_1), \ldots, f(x_n))^\top$ and $D := \frac{1}{b}\mathrm{diag}(K_b(x_1), \ldots, K_b(x_n))$, with $K_b(t) := K(\frac{x-t}{b})$. Note that $D_{ii} = \frac{1}{b}K_b(x_i) \leq \frac{\|K\|_\infty}{b}$. By assumption $\epsilon^n \sim \mathcal{N}(0, \Sigma^n)$ in $\mathrm{LSI}(\sigma_{\max}^2)$. As the x_i are fixed,

$$Z_\epsilon := DY = D(f^n + \epsilon^n) = Df^n + D\epsilon^n, \quad \mathbb{E}[Z_\epsilon] = Df^n \quad \text{and} \quad \mathrm{Cov}(Z_\epsilon) = D\Sigma^n D^\top.$$

By Gaussian invariance under affine transformations Z_ϵ is Gaussian and more concretely $Z_\epsilon \sim \mathcal{N}(Df^n, D\Sigma^n D^\top)$. Now, by Example A.2 the log-Sobolev constant of this distribution is given by the operator norm of the covariance matrix which we bound as follows

$$\|\mathrm{Cov}(Z_\epsilon)\|_{op} \leq \|D\Sigma^n D\|_{op} \leq \|D\|_{op}^2\|\Sigma^n\|_{op} \leq \frac{\sigma_{\max}^2\|K\|_\infty}{b^2} =: \rho.$$

Any marginal of Z_ϵ is Gaussian with variance $\sigma_i^2 := \mathrm{Var}((Z_\epsilon)_i) = \Sigma_{ii}/b^2 \geq \sigma_{\min}^2/b^2$. Thus,

$$d\mathcal{N}(x; \mu_i, \sigma_i^2) \leq (2\pi\sigma_i^2)^{-1/2} \leq (2\pi\sigma_{\min}^2/b^2)^{-1/2} =: M.$$

Accordingly, $\rho M^2 \lesssim \sigma_{\max}^2/\sigma_{\min}^2 \cdot \|K\|_\infty$. $\qquad\square$

Proof of Corollary 4.11.
In Lemma 4.9 we derived $(\tau, \gamma)_\infty$-concentration of $\hat{f}^n(x)$ around $f(x)$ with

$$\tau \lesssim b + \frac{1}{b}\left(1 + \sqrt{\sigma_{\max}^2\|K\|_\infty \log(2n/\gamma)}\right) \lesssim \frac{1}{b}\sqrt{\sigma_{\max}^2\|K\|_\infty \log(2n/\gamma)} =: \frac{C}{b}.$$

There, we also derived a finite sample error bound showing that with probability at least $1 - \alpha$,

$$\left|\hat{f}_n(x) - f(x)\right| \lesssim b + \frac{1}{nb} + \sqrt{\frac{\sigma_{\max}^2 \|K\|_\infty \log(2/\alpha)}{nb}}.$$

Moreover, we showed in Lemma 4.10 that $\hat{f}^n(x)$ fulfills Assumption 2.8. This makes Theorem 4.2 with $d = 1$ applicable under Assumption 2.8. Thus, via a union bound and setting $\alpha = \gamma$, with probability at least $1 - 3\gamma - O(n/\delta \cdot e^{-\kappa n\varepsilon})$,

$$\begin{aligned}
\left|\mathcal{A}(\hat{f}^n(x)) - f(x)\right| &\lesssim \left|\hat{f}_n(x) - f(x)\right| + \tau \sqrt{\frac{\log(1/\varrho)\log(3/\gamma)^2}{n^2\varepsilon^2}} \\
&\lesssim \left|\hat{f}_n(x) - f(x)\right| + \sqrt{\frac{C^2 \log(1/\varrho)\log(3/\gamma)^2}{b^2 n^2 \varepsilon^2}} \\
&\lesssim b + \frac{1}{nb} + \sqrt{\frac{\sigma_{\max}^2 \|K\|_\infty \log(2/\gamma)}{nb}} + \sqrt{\frac{C^2 \log(1/\varrho)\log(3/\gamma)^2}{n^2 b^2 \varepsilon^2}}.
\end{aligned}$$

Modulo logarithms, assuming $\sigma_{\max}^2 \gtrsim 1$ and as $1 \leq 1/\varepsilon$ matching the second with the fourth term,

$$\left|\mathcal{A}(\hat{f}^n(x)) - f(x)\right| \lessapprox b + \frac{1}{nb} + \sigma_{\max}\left(\sqrt{\frac{1}{nb}} + \sqrt{\frac{1}{n^2 b^2 \varepsilon^2}}\right) \lesssim b + \sigma_{\max}\left(\sqrt{\frac{1}{nb}} + \sqrt{\frac{1}{n^2 b^2 \varepsilon^2}}\right).$$

We call the three terms in the right hand side T_1, T_2 and T_3. To choose the bandwidth b, we balance T_1, T_2 and T_1, T_3 separately. For this, we perform a case distinction on whether T_2 or T_3 dominate:

Case 1: If $T_2 \gtrsim T_3$, i.e., when $\varepsilon \gtrsim 1/\sqrt{nb}$ we balance T_1 and T_2 as follows:

$$T_1 \asymp T_2 \quad \Leftrightarrow \quad b \asymp \frac{\sigma_{\max}}{\sqrt{nb}} \quad \Leftrightarrow \quad b \asymp \left(\frac{\sigma_{\max}^2}{n}\right)^{1/3} \quad \Rightarrow \quad \varepsilon \gtrsim \left(\frac{1}{n\sigma_{\max}}\right)^{1/3}.$$

Case 2: If $T_3 \gtrsim T_2$, i.e., when $\varepsilon \lesssim 1/\sqrt{nb}$ we balance T_1 and T_3 as follows:

$$T_1 \asymp T_3 \quad \Leftrightarrow \quad b \asymp \frac{\sigma_{\max}}{\sqrt{n^2 b^2 \varepsilon^2}} \quad \Leftrightarrow \quad b \asymp \sqrt{\frac{\sigma_{\max}}{n\varepsilon}} \quad \Rightarrow \quad \varepsilon \lesssim \left(\frac{1}{n\sigma_{\max}}\right)^{1/3}.$$

Hence, T_2 dominates when $\varepsilon \lesssim (n\sigma_{\max})^{-1/3}$ and otherwise T_3. To incorporate both cases, we choose $b \asymp (\sigma_{\max}^2/n)^{1/3} \vee (\sigma_{\max}/n\varepsilon)^{1/2}$ and plug this into the bound on the rMSE. Doing the case distinction, we obtain the statement by bounding the maximum by the sum of both terms. $\qquad \square$

Auxiliary Results for Nonparametric Regression

Lemma D.14. *(Adjusted from Lemma A.1, Gugushvili and Klaassen (2012)). Let $f : [0,1] \to \mathbb{R}$ be L_f-Lipschitz. Further, assume that the kernel $K : \mathbb{R} \to \mathbb{R}_\geq$ is L_K-Lipschitz. Then, with the mesh $x_i := i/n$ for $0 \leq i \leq n$, for all $x \in [0,1]$,*

$$\frac{1}{b}\left|\int_0^1 f(t)K\left(\frac{x-t}{b}\right)dt - \frac{1}{n}\sum_{i=0}^{n-1} f(x_i)K\left(\frac{x-x_i}{b}\right)\right| \leq \frac{L_f\|K\|_\infty}{nb} + \frac{L_K\|f\|_\infty}{nb}.$$

If instead of Lipschitzness $K(u) := \frac{1}{2}\mathbf{1}\{|u| \leq 1\}$ then the same holds with $L_K = 1$.

Proof. For brevity of notation denote the left hand side of the statement by I and $K_b(u) := K\left(\frac{x-u}{b}\right)$. To bound I we use that the x_i are a grid on $[0,1]$ and $|x_{i+1} - x_i| = 1/n$. Concretely, we first split the integral into intervals $[x_i, x_{i+1}]$. Second, we add and subtract $f(t)K(x_i)$ to use a triangle inequality:

$$I = \frac{1}{b}\left|\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}}f(t)K_b(t)dt - \frac{1}{n}\sum_{i=0}^{n-1}f(x_i)K_b(x_i)\right| \le \frac{1}{b}\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}}|f(t)K_b(t) - f(x_i)K_b(x_i)|\,dt$$

$$\le \frac{1}{b}\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}}|f(t)K_b(t) - f(t)K_b(x_i)|\,dt + \frac{1}{b}\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}}|f(t)K_b(x_i) - f(x_i)K_b(x_i)|\,dt$$

$$\le \frac{1}{b}\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}}|f(t)|\,|K_b(t) - K_b(x_i)|\,dt + \frac{1}{b}\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}}|K_b(x_i)|\,|f(t) - f(x_i)|\,dt$$

$$\le \frac{L_K\|f\|_\infty}{b}\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}}|t - x_i|\,dt + \frac{L_f\|K\|_\infty}{b}\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}}|t - x_i|\,dt$$

$$\le \frac{L_K\|f\|_\infty}{nb} + \frac{L_f\|K\|_\infty}{nb}.$$

Here, the last inequality holds because $|t - x_i| \le 1/n$ and $|x_{i+1} - x_i| = 1/n$, and thus

$$\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}}|t - x_i|dt \le \frac{1}{n}\sum_{i=0}^{n-1}[t]_{x_i}^{x_{i+1}} = \frac{1}{n}\sum_{i=0}^{n-1}x_{i+1} - x_i \le \frac{1}{n}\sum_{i=0}^{n-1}|x_{i+1} - x_i| = \frac{1}{n}.$$

For the special case when $K(u) = \frac{1}{2}\mathbf{1}\{|u| \le 1\}$ we have

$$I \le \frac{1}{b}\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}}|f(t)|\,|K_b(t) - K_b(x_i)|\,dt + \frac{1}{b}\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}}|K_b(x_i)|\,|f(t) - f(x_i)|\,dt$$

$$\le \frac{\|f\|_\infty}{b}\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}}|K_b(t) - K_b(x_i)|\,dt + \frac{L_f\|K\|_\infty}{nb}.$$

Since K is neither Lipschitz nor continuous we have to make a more direct argument. Note that

$$\mathbf{1}(t, x_i) := |K_b(t) - K_b(x_i)| = \frac{1}{2}\left|\mathbf{1}\{x - b \le t \le x + b\} - \mathbf{1}\{x - b \le x_i \le x + b\}\right|$$

$$= \frac{1}{2}\mathbf{1}\{\{x - b \le t \le x + b\}\text{ or }\{x - b \le x_i \le x + b\}\}.$$

This can only evaluate to 1 at the boundary $x - b$ or $x + b$ and hence $\mathbf{1}(t, x_i) \ne 0$ only for two x_l and x_u such that $x - b \in [x_l, x_{l+1}]$ and $x + b \in [x_u, x_{u+1}]$. Thus,

$$\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}}|K_b(t) - K_b(x_i)|\,dt \le \left(\int_{x_l}^{x_{l+1}}K(t, x_l)dt + \int_{x_u}^{x_{u+1}}\mathbf{1}(t, x_u)dt\right)$$

$$\le \frac{1}{2}\left(\int_{x_l}^{x_{l+1}}dt + \int_{x_u}^{x_{u+1}}dt\right) = \frac{1}{n}.$$

Plugging into our bound on I we recover the statement $I \le \frac{\|f\|_\infty}{nb} + \frac{L_f\|K\|_\infty}{nb}$. $\qquad\square$

E Proofs of Section 5

E.1 Proofs for User-Level Mean Estimation

Proof of Lemma 5.2.

First, we show that $\hat{\mu}^n$ is $(\tau, \gamma)_\infty$-concentrated. To justify Remark 5.3, this proof is more elaborate than necessary, involving marginal log-Sobolev constants. Second, we show the bound on $\|\bar{X}_{nT} - \mu\|_2$.

By Assumption 2.8 the distribution of $X_{\cdot j}^n$ is LSI(ρ) for all $j \in [d]$. For all $u \in [n]$ and $j \in [d]$, the $(X_u)_{\cdot j}$ are sub-matrices of $X_{\cdot j}^n$ and thus have distributions $\mathcal{D}_{uj}$ that are marginals of the distributions of $X_{\cdot j}^n$. By Lemma A.3 $\mathcal{D}_{uj}$ fulfills a log-Sobolev inequality with constant ρ too.

However, because log-Sobolev constants are defined as the smallest constant s.t. a log-Sobolev inequality holds, the log-Sobolev constants ρ_{uj} of $\mathcal{D}_{uj}$ can be smaller than ρ. Using these and noting that $(\hat{\mu}_u)_j = \frac{1}{T}(X_u)_{\cdot j}^\top \mathbf{1}_T$, by Lemma 2.11 we have

$$\mathbb{P}\left(|(\hat{\mu}_u)_j - \mu_j| \le \sqrt{\frac{2\rho_{uj} \log(2/\alpha)}{T}}\right) \ge 1 - \alpha.$$

Union bounding over $j \in [d]$ and setting $\alpha = \gamma/dn$ yields $(\tau, \gamma)_\infty$-concentration with

$$\tau = \max_{u \in [n]} \max_{j \in [d]} \sqrt{\frac{2\rho_{uj} \log(2dn/\gamma)}{T}} \le \sqrt{\frac{2\rho \log(2dn/\gamma)}{T}}.$$

The characterization $\bar{X}_{nT} = \frac{1}{n}(\hat{\mu}^n)^\top \mathbf{1}_n$ allows us to invoke the L_2-norm result of Lemma 2.11:

$$\mathbb{P}\left(\|\bar{X}_{nT} - \mu\|_2 \le \sqrt{\frac{2\rho \log(2d/\alpha)}{nT}}\right) \ge 1 - \alpha.$$

$\square$

Proof of Corollary 5.4.

This result is a direct consequence of Theorem 4.2 under Assumption 2.8. By Lemma 5.2 $\hat{\mu}^n$ is $(\tau, \gamma)_\infty^\mu$-concentrated with $\tau = \sqrt{2\rho \log(2dn/\gamma)/T}$ and with probability at least $1 - \alpha$

$$\|\bar{X}_{nT} - \mu\|_2 \le \sqrt{\frac{2d\rho \log(2d/\alpha)}{nT}}.$$

Setting $\alpha = d\gamma$ and combing this with Theorem 4.2 via a union bound, with probability at least $1 - 3d\gamma - O(d^2 n/\delta \cdot e^{-\kappa n \varepsilon'})$ we have

$$
\begin{aligned}
\|\mathcal{A}(\hat{\mu}^n) - \mu\|_2 &\lesssim \|\bar{X}_{nT} - \mu\|_2 + \tau \sqrt{\frac{d^2 \log(1/\varrho) \log(3/(d\gamma))^2}{n^2 \varepsilon^2}} \\
&\lesssim \sqrt{\frac{2d\rho \log(2/\gamma)}{nT}} + \sqrt{\frac{2\rho \log(2dn/\gamma)}{T}} \sqrt{\frac{d^2 \log(1/\varrho) \log(3/(d\gamma))^2}{n^2 \varepsilon^2}} \\
&\lesssim \sqrt{\frac{d\rho \log(2/\gamma)}{nT}} + \sqrt{\frac{d^2 \rho \log(2dn/\gamma) \log(1/\varrho) \log(3/(d\gamma))^2}{Tn^2 \varepsilon^2}}.
\end{aligned}
$$

$\square$

Proof of Corollary 5.6.

By definition of the per-user means collected in $\hat{\mu}^n$ we have

$$(\hat{\mu}_u)_j = \frac{1}{T} \left(X_u^\top \mathbf{1}_T \right)_j = \frac{1}{T} (X_u)_{\cdot j}^\top \mathbf{1}_T = \frac{1}{T} \sum_{i=1}^{T} (X_u)_{ij}.$$

Under Assumption 2.7 the entries $(X_u)_{ij}$ are ρ-sub-Gaussian and independent. Hence, the mean $(\hat{\mu}_u)_j$ is ρ/T-sub-Gaussian. Under Assumption 2.8 we can use that $(\hat{\mu}_u)_j$ is $1/\sqrt{T}$-Lipschitz in $(X_u)_{\cdot j}$ and thus by Lemma A.3 its distribution has log-Sobolev constant ρ/T. The same holds for the entries of $\hat{\mu}_Z^n$. The result then follows immediately from Theorem 4.3 with ρ replaced by ρ/T. $\qquad\square$

E.2 Proofs for Random Effects Location Model

Proof of Lemma 5.9.

As the noise terms U_g and ϵ_{ut} are independent and have zero-mean,

$$\mathbb{E}[Y_{gut}] = \mathbb{E}[\mu + U_g + \epsilon_{ut}] = \mu,$$
$$\mathrm{Var}(Y_{gut}) = \mathrm{Var}(U_g + \epsilon_{ut}) = \mathrm{Var}(U_g) + \mathrm{Var}(\epsilon_{ut}) = \sigma_U^2 + (\Sigma_u)_{tt}.$$

To show that $\mathcal{D}$ fulfills a Log-Sobolev inequality, we show that it is Gaussian and derive a bound on the operator norm of its covariance matrix Σ^n. Entry-wise, Y^n is a sum of Gaussian random variables. Since the random variables U_g, ϵ_u are mutually independent for all $u \in [n], g \in [G]$, the joint is Gaussian. It has mean $\mu \mathbf{1}_{nT}$ and its covariance matrix Σ^n has the following entries:

$$\begin{aligned}
\mathrm{Cov}(Y_{gut}, Y_{ijk}) &= \mathrm{Cov}(U_g + \epsilon_{ut}, U_i + \epsilon_{jk}) \\
&= \mathrm{Cov}(U_g, U_i) + \mathrm{Cov}(U_g, \epsilon_{jk}) + \mathrm{Cov}(\epsilon_{ut}, U_i) + \mathrm{Cov}(\epsilon_{ut}, \epsilon_{jk}) \\
&= \mathrm{Cov}(U_g, U_i) + \mathrm{Cov}(\epsilon_{ut}, \epsilon_{jk})
\end{aligned}$$

Here, we index with tuples (gut) and (ijk). Depending on the indices there are then four cases:

Case 1: If $(gut) \neq (ijk)$ we have $\mathrm{Cov}(Y_{gut}, Y_{ijk}) = 0$.
Case 2: If $(gut) = (ijk)$ we have $\mathrm{Cov}(Y_{gut}, Y_{ijk}) = \sigma_U^2 + (\Sigma_u)_{tt}$.
Case 3: If $g = i, u \neq j$ we have $\mathrm{Cov}(Y_{gut}, Y_{ijk}) = \mathrm{Cov}(U_g, U_g) = \sigma_U^2$.
Case 4: If $g = i, u = j, t \neq k$ we have

$$\mathrm{Cov}(Y_{gut}, Y_{ijk}) = \mathrm{Cov}(U_g, U_g) + \mathrm{Cov}(\varepsilon_{ut}, \varepsilon_{uk}) = \sigma_U^2 + (\Sigma_u)_{tk}.$$

Hence, the covariance matrix Σ^n and the submatrix Σ_g^U for a group g of size n_g have the form

$$\Sigma^n = \begin{pmatrix} \Sigma_1^U & 0 & \cdots & 0 \\ 0 & \Sigma_2^U & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & \Sigma_G^U \end{pmatrix}, \quad \Sigma_g^U = \sigma_U^2 \mathbf{1}_{n_g T} \mathbf{1}_{n_g T}^\top + \begin{pmatrix} \Sigma_1 & 0 & \cdots & 0 \\ 0 & \Sigma_2 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & \Sigma_{n_g} \end{pmatrix}.$$

Together with the mean $\mu \mathbf{1}_{nT}$ this explicit form of Σ^n specifies the Gaussian distribution $\mathcal{D} := \mathcal{N}(\mu \mathbf{1}_{nT}, \Sigma^n)$. By Example A.2 the log-Sobolev constant of $\mathcal{D}$ is therefore $\|\Sigma^n\|_{op}$ or an upper bound thereof. As the operator norm of a block-diagonal matrix is the maximum norm of its blocks,

$$\|\Sigma^n\|_{op} = \max_{g \in [G]} \|\Sigma_g^U\|_{op} \leq \max_{g \in [G]} \left(\|\sigma_U^2 \mathbf{1}_{n_g T} \mathbf{1}_{n_g T}^\top\|_{op} + \max_{u \in \mathcal{G}_g} \|\Sigma_u\|_{op} \right) \leq \max_{g \in [G]} \left(\sigma_U^2 \cdot T n_g + \sigma^2 \right).$$

Here, $\mathcal{G}_g$ is the collection of users in group g. Now, we show that the marginal densities are bounded. Above we derived $\mathbb{E}[Y_{gut}] = \mu$ and $\mathrm{Var}(Y_{gut}) = \sigma_U^2 + (\Sigma_u)_{tt} \leq \sigma_U^2 + \sigma^2 =: \sigma_Y^2$. We further know that any marginal of a Gaussian is also Gaussian. Therefore, for all $x \in \mathbb{R}$ the density of any Y_{gut} is

$$\frac{1}{\sqrt{2\pi\sigma_Y^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma_Y^2}\right) \leq \frac{1}{\sqrt{2\pi\sigma_Y^2}} =: M.$$

Hence, $\frac{d}{dx}\bar{F}(x) = \frac{1}{nT}\sum_{u=1}^n \sum_{t=1}^T \frac{d}{dx}\mathbb{P}(Y_{gut} \leq x) \leq M$ for all $x \in \mathbb{R}$ and Assumption 2.8 holds. As a consequence the matrix $Y^n \in \mathbb{R}^{nT}$ fulfills the user-level log-Sobolev design of Definition 5.1 because it has the required block structure and its entries Y_{gut} have identical mean μ. $\qquad\square$

Proof of Corollary 5.10.

The result is a direct consequence of Corollary 5.4 enabled by Lemma 5.9. $\qquad\square$

E.3   Proofs for Longitudinal Linear Regression

Proof of Lemma 5.14.

First, we derive the bias and covariance matrix for a general weighted OLS estimator. Based on this, we show unbiasedness of $\hat{\beta}, \hat{\beta}_u$, and $\hat{\beta}_w, \hat{\beta}_{wu}$. Second, we prove the claims for the variances.

We begin reminding the reader of some properties of generalized least squares estimators. Note that for observations $\Upsilon = Z\theta + \epsilon$ with noise $\epsilon \in \mathbb{R}^l$ where $\epsilon \sim \mathcal{N}(0, \Lambda)$, the generalized least squares estimator $\hat{\theta} := (Z^\top W Z)^{-1} Z^\top W \Upsilon$ with weight matrix $W \in \mathbb{R}^{p \times p}$ and design $Z \in \mathbb{R}^{l \times p}$ is unbiased for θ. Indeed, for fixed W and Z we have that

$$\mathbb{E}[\hat{\theta}] = (Z^\top W Z)^{-1} Z^\top W \mathbb{E}[Z\theta + \epsilon] = (Z^\top W Z)^{-1} Z^\top W Z\theta = \theta. \tag{7}$$

Its covariance matrix can be computed as

$$\mathrm{Cov}(\hat{\theta}) = \mathrm{Cov}((Z^\top W Z)^{-1} Z^\top W \Upsilon) = (Z^\top W Z)^{-1} Z^\top W \Lambda W Z (Z^\top W Z)^{-1}. \tag{8}$$

We note that all the estimators we are interested in can be written in this form, so they are all unbiased. It remains to compute their variance.

Write $\epsilon^n = [\epsilon_1, \ldots, \epsilon_n]^\top \in \mathbb{R}^{nT}$. Since $\hat{\beta}_u$ is an OLS estimator with weight $W = I_T$, design X_u, observations Y_u and noise $\epsilon_u \sim \mathcal{N}(0, \Sigma_u)$ we have

$$\mathbb{E}[\hat{\beta}_u] = \beta, \quad \text{and} \quad \mathbb{E}[\hat{\beta}] = \mathbb{E}\left[\frac{1}{n}\sum_{u=1}^n \hat{\beta}_u\right] = \frac{1}{n}\sum_{u=1}^n \mathbb{E}[\hat{\beta}_u] = \frac{1}{n}\sum_{u=1}^n \beta = \beta.$$

Similarly, $\hat{\beta}_{wu}$ is a weighted OLS estimator with weight $W = \Sigma_u$ and therefore

$$\mathbb{E}[\hat{\beta}_{wu}] = \beta \quad \text{and} \quad \mathbb{E}[\hat{\beta}] = \mathbb{E}\left[\frac{1}{n}\sum_{u=1}^n \hat{\beta}_{wu}\right] = \frac{1}{n}\sum_{u=1}^n \mathbb{E}[\hat{\beta}_{wu}] = \frac{1}{n}\sum_{u=1}^n \beta = \beta.$$

We can again apply the general result to obtain the variance of $\hat{\beta}_u$ and $\hat{\beta}_{wu}$,

$$\mathrm{Cov}(\hat{\beta}_u) = (X_u^\top X_u)^{-1} X_u^\top \Sigma_u X_u (X_u^\top X_u)^{-1},$$
$$\mathrm{Cov}(\hat{\beta}_{wu}) = (X_u^\top \Sigma_u^{-1} X_u)^{-1} X_u^\top \Sigma_u^{-1} \Sigma_u \Sigma_u^{-1} X_u (X_u^\top \Sigma_u^{-1} X_u)^{-1} = (X_u^\top \Sigma_u^{-1} X_u)^{-1}.$$

For $u, v \in [n]$, by the independence of ϵ_u, ϵ_v due to the block-diagonal structure of Σ^n we have $\mathrm{Cov}(\hat\beta_u, \hat\beta_v) = 0$. Hence, we can compute the variance of the estimator $\hat\beta$ as

$$\mathrm{Cov}(\hat\beta) = \mathrm{Cov}\left(\frac{1}{n}\sum_{u=1}^{n}\hat\beta_u\right) = \frac{1}{n^2}\sum_{u=1}^{n}\mathrm{Cov}(\hat\beta_u) = \frac{1}{n^2}\sum_{u=1}^{n}(X_u^\top X_u)^{-1}X_u^\top \Sigma_u X_u (X_u^\top X_u)^{-1}.$$

A similar variance calcualtion for $\hat\beta_w$ leads to

$$\mathrm{Cov}(\hat\beta_w) = \mathrm{Cov}\left(\frac{1}{n}\sum_{u=1}^{n}\hat\beta_{wu}\right) = \frac{1}{n^2}\sum_{u=1}^{n}\mathrm{Cov}\left(\hat\beta_{wu}\right) = \frac{1}{n^2}\sum_{u=1}^{n}\left(X_u^\top \Sigma_u^{-1} X_u\right)^{-1}.$$

Invoking the general result for the variance of $\hat\beta_{OLS}$

$$\mathrm{Cov}(\hat\beta_{OLS}) = \left(\sum_{u=1}^{n}X_u^\top X_u\right)^{-1}\left(\sum_{u=1}^{n}X_u^\top \Sigma_u X_u\right)\left(\sum_{u=1}^{n}X_u^\top X_u\right)^{-1}.$$

With weight $W = (\Sigma^n)^{-1}$ the same result yields

$$\mathrm{Cov}(\hat\beta_{wOLS}) = \left(\sum_{u=1}^{n}X_u^\top \Sigma_u^{-1} X_u\right)^{-1}.$$

We are now ready to find the desired variance relations.

(i) Generally, for $\Sigma_u \neq \sigma^2 I_d$ the Loewner ordering $\mathrm{Cov}(\hat\beta_{OLS}) \preceq \mathrm{Cov}(\hat\beta)$ does not hold, i.e.,

$$\left(\sum_{u=1}^{n}X_u^\top X_u\right)^{-1}\left(\sum_{u=1}^{n}X_u^\top \Sigma_u X_u\right)\left(\sum_{u=1}^{n}X_u^\top X_u\right)^{-1} \npreceq \frac{1}{n^2}\sum_{u=1}^{n}\left(X_u^\top X_u\right)^{-1} X_u^\top \Sigma_u X_u \left(X_u^\top X_u\right)^{-1}.$$

Already when $n = 2, T = 3$ and $p = 2$, the inequality above fails for the following example:

$$X_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}, \quad B_1 = I_3, \quad B_2 = 4 \cdot I_3.$$

We have that $X_1^\top X_1 = I_2$, $X_2^\top X_2 = 2 \cdot I_2$, $X_1^\top B_1 X_1 = I_2$ and $X_2^\top B_2 X_2 = 8 \cdot I_2$. Thus,

$$\begin{aligned}
\mathrm{Cov}(\hat\beta_{OLS}) &= (X_1^\top X_1 + X_2^\top X_2)^{-1}(X_1^\top B_1 X_1 + X_2^\top B_2 X_2)(X_1^\top X_1 + X_2^\top X_2)^{-1} \\
&= (3 \cdot I_2)^{-1}(I_2 + 8 \cdot I_2)(3 \cdot I_2)^{-1} \\
&= (1/3 \cdot I_2)(9 \cdot I_2)(1/3 \cdot I_2) = I_2.
\end{aligned}$$

Moreover, we can explicitly compute

$$\begin{aligned}
\mathrm{Cov}(\hat\beta) &= 1/2^2 \cdot ((X_1^\top X_1)^{-1}X_1^\top B_1 X_1 (X_1^\top X_1)^{-1} + (X_2^\top X_2)^{-1}X_2^\top B_2 X_2 (X_2^\top X_2)^{-1}) \\
&= 1/4 \cdot (I_2 I_2 I_2 + (1/2 \cdot I_2)(8 \cdot I_2)(1/2 \cdot I_2)) \\
&= 1/4 \cdot (I_2 + 2 \cdot I_2) = 3/4 \cdot I_2.
\end{aligned}$$

Overall, we have $I_2 = \mathrm{Cov}(\hat\beta_{OLS}) \npreceq \mathrm{Cov}(\hat\beta) = 3/4 \cdot I_2$.

(ii) When restricting to the case $\Sigma_u = \sigma^2 I_d$, Jensen's inequality shows that

$$\text{Cov}(\hat{\beta}_{OLS}) = \left(\sum_{u=1}^{n} X_u^\top X_u\right)^{-1} \left(\sum_{u=1}^{n} X_u^\top \Sigma_u X_u\right) \left(\sum_{u=1}^{n} X_u^\top X_u\right)^{-1} = \sigma^2 \left(\sum_{u=1}^{n} X_u^\top X_u\right)^{-1}$$

$$\preceq \frac{1}{n^2} \sum_{i=1}^{n} \sigma^2 \left(X_u^\top X_u\right)^{-1} = \sum_{u=1}^{n} \left(X_u^\top X_u\right)^{-1} X_u^\top \Sigma_u X_u \left(X_u^\top X_u\right)^{-1} = \text{Cov}(\hat{\beta}).$$

(iii) When $X_u^\top X_u = D \succeq 0$ for all users, we see that

$$\text{Cov}(\hat{\beta}_{OLS}) = (nD)^{-1} \left(\sum_{u=1}^{n} X_u^\top \Sigma_u X_u\right) (nD)^{-1} = \frac{1}{n^2} \sum_{u=1}^{n} D^{-1} X_u^\top \Sigma_u X_u D^{-1} = \text{Cov}(\hat{\beta}).$$

(iv) Applying the Gauss-Markov Theorem to $\hat{\beta}_{wu}$ as the weighted OLS and $\hat{\beta}_u$ as the other linear unbiased estimator, we have $\text{Cov}(\hat{\beta}_{wu}) \preceq \text{Cov}(\hat{\beta}_u)$. Thus,

$$\text{Cov}(\hat{\beta}_w) = \frac{1}{n^2} \sum_{u=1}^{n} \text{Cov}(\hat{\beta}_{wu}) \preceq \frac{1}{n^2} \sum_{u=1}^{n} \text{Cov}(\hat{\beta}_u) = \text{Cov}(\hat{\beta}).$$

Moreover, Jensen's inequality shows that

$$\text{Cov}(\hat{\beta}_{wOLS}) = \left(\sum_{u=1}^{n} X_u^\top \Sigma_u^{-1} X_u\right)^{-1} \preceq \frac{1}{n^2} \sum_{u=1}^{n} \left(X_u^\top \Sigma_u^{-1} X_u\right)^{-1} = \text{Cov}(\hat{\beta}_w).$$

$\square$

Proof of Lemma 5.15.
We first prove that $\hat{\beta}_u$ is $(\tau, \gamma)_\infty^\beta$-concentrated. Second, we derive the concentration of $\hat{\beta}$.
Since $\hat{\beta}_u$ is a linear map of $Y_u \sim \mathcal{N}(X_u \beta, \Sigma_u)$, it follows from (7) and (8) that $\hat{\beta}_u \sim \mathcal{N}(\beta, \Sigma_{\hat{\beta}_u})$ where $\Sigma_{\hat{\beta}_u} = (X_u^\top X_u)^{-1} X_u^\top \Sigma_u X_u (X_u^\top X_u)^{-1}$. We now bound the operator norm of $\Sigma_{\hat{\beta}_u}$.

$$\|\Sigma_{\hat{\beta}_u}\|_{op} = \|(X_u^\top X_u)^{-1} X_u^\top \Sigma_u X_u (X_u^\top X_u)^{-1}\|_{op} \leq \|(X_u^\top X_u)^{-1}\|_{op}^2 \cdot \|X_u^\top \Sigma_u X_u\|_{op}$$

$$\leq \|(X_u^\top X_u)^{-1}\|_{op}^2 \cdot \|X_u^\top X_u\|_{op} \cdot \|\Sigma_u\|_{op} \leq \lambda_{\max}(X_u^\top X_u)/\lambda_{\min}^2(X_u^\top X_u) \cdot \lambda_{\max}(\Sigma_u).$$

Writing this in terms of the empirical second moment $\hat{M}_u = \frac{1}{T} X_u^\top X_u$ and using the bounds $\theta \preceq \hat{M}_u \preceq \vartheta$ and $\Sigma_u \preceq \sigma^2$ we obtain an upper bound on the operator norm

$$\|\Sigma_{\hat{\beta}_u}\|_{op} \leq \frac{1}{T} \frac{\lambda_{\max}(\hat{M}_u)\lambda_{\max}(\Sigma_u)}{\lambda_{\min}^2(\hat{M}_u)} \leq \frac{\vartheta \sigma^2}{T\theta^2} =: \rho.$$

To show $(\tau, \gamma)_\infty^\beta$-concentration we use that $\hat{\beta}_u$ is Gaussian. Example A.2 then implies that its distribution is LSI(ρ). Hence, using the upper bound on the operator norm, $\hat{\beta}_u^\top \in \mathbb{R}^{1 \times d}$ fulfills Assumption 2.8 with constants (ρ, ∞) and by Lemma 2.11 we have $(\tau, \gamma)_\infty^\beta$-concentration as

$$\mathbb{P}\left(\left\|\hat{\beta}_u - \beta\right\|_\infty \leq \sqrt{\frac{2\vartheta \sigma^2 \log(2dn/\gamma)}{m\theta^2}}\right) \leq \gamma/n.$$

The desired result follows from Lemma A.5 applied to $\hat{\beta}_1, \ldots, \hat{\beta}_n$ combined with $\|\Sigma_{\hat{\beta}_u}\|_{op} \leq \rho$. Concretely, we use that $\frac{1}{n}\sum_{u=1}^n \Sigma_{\hat{\beta}_u} \preceq \rho$. On top, we exploit that $\mathrm{tr}(\Sigma_{\hat{\beta}_u}) \leq p\|\Sigma_{\hat{\beta}_u}\|_{op}$ to obtain

$$\mathbb{P}\left(\left\|\hat{\beta} - \beta\right\|_2 \geq \sqrt{\frac{p\vartheta\sigma^2}{nT\theta^2}} + \sqrt{\frac{2\vartheta\sigma^2\log(1/\alpha)}{nT\theta^2}}\right) \leq \mathbb{P}\left(\left\|\hat{\beta} - \beta\right\|_2 \geq \sqrt{\frac{\mathrm{tr}(\Sigma_{\hat{\beta}_u})}{n}} + \sqrt{\frac{2\rho\log(1/\alpha)}{n}}\right) \leq \alpha.$$

$\square$

Proof of Corollary 5.16.
By Lemma 5.15 $\hat{\beta}^n$ is the matrix $(\tau, \gamma)_\infty^\beta$-concentrated with $\tau = \sqrt{2\vartheta\sigma^2\log(2pn/\gamma)/(T\theta^2)}$ and with probability at least $1 - \alpha$ it holds that

$$\left\|\hat{\beta} - \beta\right\|_2 \leq \sqrt{\frac{p\vartheta\sigma^2}{nT\theta^2}} + \sqrt{\frac{2\vartheta\sigma^2\log(1/\alpha)}{nT\theta^2}}.$$

Furthermore, assumption 2.7 holds for $\hat{\beta}^n$ as the rows $\hat{\beta}_u$ are independent of each other due to the block structure of Σ^n. Thus by the concentration derived above with $\alpha = p\gamma$ combined with Theorem 4.2 via a union bound, with probability at least $1 - 3p\gamma - O(p^2 n/\delta \cdot e^{-\kappa n\varepsilon'})$,

$$\left\|\mathcal{A}(\hat{\beta}^n) - \beta\right\|_2 \lesssim \left\|\hat{\beta} - \beta\right\|_2 + \tau\sqrt{\frac{p^2\log(1/\varrho)\log(3/(p\gamma))^2}{n^2\varepsilon^2}}$$
$$\lesssim \sqrt{\frac{p\vartheta\sigma^2}{nT\theta^2}} + \sqrt{\frac{\vartheta\sigma^2\log(1/(p\gamma))}{nT\theta^2}} + \sqrt{\frac{p^2\vartheta\sigma^2\log(2dn/\gamma)\log(1/\varrho)\log(3/(p\gamma))^2}{\theta^2 Tn^2\varepsilon^2}}.$$

$\square$

F Proofs of Section 6

F.1 Finding Private Data-Driven Projection Intervals (Local)

Algorithm 9 ProjectionInterval($X^n, \tau, \varepsilon, \delta, \mathsf{B}$):
___
Require: $X^n \in \mathbb{R}^n,\ \tau, \varepsilon > 0, \delta = 0$
  1: $S \leftarrow \{k \in \mathbb{Z} : |2\tau k| \leq \mathsf{B}\}$
  2: $B_k^+ \leftarrow (2\tau k \pm \tau]$ $\forall k \in S$
  3: $\tilde{\boldsymbol{p}} \leftarrow \text{RandomizedHistogram}(X, (B_k^+)_{k \in S}, \varepsilon/2, 0)$
  4: $\hat{k} \leftarrow \arg\max_{k \in S} \tilde{p}_k$
  5: $\hat{m} \leftarrow 2\tau\hat{k}$
  6: **return** $[\hat{m} \pm 3\tau]$

Our main application of the randomized histogram is to compute private data-driven projection intervals that we use in our local Winsorized mean estimator. Our approach hereby resembles the one in the central model. In Algorithm 9 we again choose the projection interval as the bin of the histogram that has the highest estimated mass and its two neighboring bins. Because the randomized histogram requires finitely many bins as input, we cannot search for the bin with highest mass over the entirety of $\mathbb{R}$ anymore. Instead we restrict ourselves to the interval $[-\mathsf{B}, \mathsf{B}] \subset \mathbb{R}$. For mean estimation, this will impose the assumption that the mean $\mu \in \mathbb{R}^d$ to be estimated fulfills $\|\mu\|_\infty \leq \mathsf{B}$.

This boundedness assumption relaxes boundedness assumptions on the single observations X_i as they are made, e.g., by Kent et al. (2024). Yet, it is more restrictive than the boundedness we assume for our in-expectation bound in the central model, because the constant B actually has to be known for the construction of the algorithm and not just during its analysis in expectation. Hence, this peculiarity of the private projection interval estimator shows up in the finite sample MSE-risk bounds for our local estimators. These build on top of the estimation guarantee for the center of the projection interval that is given in Lemma F.1 below whose proof we defer for now.

Lemma F.1. *Let $\varepsilon > 0$, $\gamma \in (0, 1 \wedge \frac{n}{4})$. Suppose $X^n \in \mathbb{R}^n$ is $(\tau, \gamma)_\infty$-concentrated around $x_0 \in [-\mathsf{B}, \mathsf{B}] \subset \mathbb{R}$ and fulfills Assumption 2.7 or Assumption 2.8 with $\rho M^2 \lesssim 1$. Then, Algorithm 9 is $(\varepsilon, 0)$-DP and with probability at least $1 - O(\mathsf{B}/\tau + 1) \cdot e^{-\kappa n \tanh^2(\varepsilon/4)} \vee e^{-\kappa' n}$ with $\kappa, \kappa' > 0$:*

$$\hat{m} \in \left[x_0 \pm 2\tau \right].$$

F.2 Proofs for Randomized Histogram

Proof of Lemma 6.3.

The randomized histogram is private by the reasoning in Lemma F.2.

For the proof of utility, define $\hat{p}_k := \frac{1}{n} \sum_{i=1}^n \mathbf{1}\{X_i \in B_k\}$. An application of the triangle inequality $|\tilde{p}_k - p_k| \leq |\tilde{p}_k - \hat{p}_k| + |\hat{p}_k - p_k|$ and the total law of probability yield:

$$\mathbb{P}\left(\max_{k \in \mathbb{Z}} |\tilde{p}_k - p_k| \geq \eta \right) \leq \mathbb{E}\left[\mathbb{P}\left(\max_{k \in \mathbb{Z}} |\tilde{p}_k - \hat{p}_k| \geq \frac{\eta}{2} \Big| X^n \right) \right] + \mathbb{P}\left(\max_{k \in \mathbb{Z}} |\hat{p}_k - p_k| \geq \frac{\eta}{2} \right).$$

The above isolates the privacy loss of the randomized histogram due to conditioning on X^n. This makes Lemma F.2 applicable and we obtain

$$\mathbb{E}\left[\mathbb{P}\left(\max_{k \in \mathbb{Z}} |\tilde{p}_k - \hat{p}_k| \geq \frac{\eta}{2} \Big| X^n \right) \right] \leq 2 N_{\mathsf{bins}} \cdot \exp\left(-\frac{n\eta^2}{4} \tanh^2\left(\frac{\varepsilon}{4} \right) \right).$$

The estimation error can now be bounded using Lemma C.3, which gets us

$$\mathbb{P}\left(\max_{k \in \mathbb{Z}} |\hat{p}_k - p_k| \geq \frac{\eta}{2} \right) \leq \begin{cases} 2 \exp\left(-\frac{n\eta^2}{8} \right) & \text{if } X_1, \ldots, X_n \text{ are i.i.d.,} \\ \frac{16}{\eta} \exp\left(-\frac{2}{27} \frac{n\eta^3}{4^3 \rho M^2} \right) & \text{if } X \text{ fulfills Assumption 2.8.} \end{cases}$$

The statement is then recovered by collecting terms. $\qquad\square$

Lemma F.2. *Let $\eta, \varepsilon > 0$. Moreover, let $(B_k)_{k \in [N_{\mathsf{bins}}]}$ be disjoint bins covering $[-\mathsf{B}, \mathsf{B}] \subset \mathbb{R}$. Then, on input of a fixed vector $x \in \mathbb{R}^n$, Algorithm 7 is $(\varepsilon, 0)$-DP and its output $\tilde{\boldsymbol{p}}$ fulfills:*

$$\mathbb{P}\left(\max_{k \in \mathbb{Z}} |\tilde{p}_k - \hat{p}_k| \geq \eta \right) \leq 2 N_{\mathsf{bins}} \cdot \exp\left(-n\eta^2 \tanh^2\left(\frac{\varepsilon}{4} \right) \right).$$

Proof. The privacy follows from that of the randomized response mechanism. Concretely, for $x, x' \in \mathbb{R}^n$ s.t. $d(x, x') \leq 1$ at most two bins can be affected. Basic composition and the $(\varepsilon, 0)$-DP guarantee of Algorithm 6 in Theorem 6.2 yield the claimed privacy. The debiasing step does not affect the privacy guarantee by post-processing.

For utility, we first show that the $\tilde{p}_k^{\text{debiased}}$ are indeed unbiased for $\hat{p}_k$. For this, let π be the probability that the randomization keeps the data unchanged in Algorithm 6. Then,

$$\mathbb{E}\left[\tilde{p}_k^{\text{debiased}}\right] = \mathbb{E}\left[\frac{\tilde{p}_k - (1-\pi)}{2\pi - 1}\right] = \frac{\mathbb{E}[\tilde{p}_k] - (1-\pi)}{2\pi - 1} \stackrel{(a)}{=} \frac{(2\pi-1)\hat{p}_k + (1-\pi) - (1-\pi))}{2\pi - 1} = \hat{p}_k.$$

Above, equality (a) holds by linearity of the expectation since

$$\mathbb{E}\left[\tilde{p}_k\right] = \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n}\tilde{c}_{ik}\right] = \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} c_{ik} \cdot \mathbf{1}\{U_i \le \pi\} + (1-c_{ik}) \cdot \mathbf{1}\{U_i > \pi\}\right]$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left[c_{ik} \cdot \mathbb{P}\left(U_i \le \pi\right) + (1-c_{ik}) \cdot \mathbb{P}\left(U_i > \pi\right)\right]$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left[c_{ik} \cdot \pi + (1-c_{ik}) \cdot (1-\pi)\right]$$

$$= \frac{1}{n}\sum_{i=1}^{n} c_{ik}(2\pi - 1) + (1-\pi)$$

$$= (2\pi - 1)\hat{p}_k + (1-\pi).$$

Note also that

$$\tilde{p}_k^{\text{debiased}} - \hat{p}_k = \tilde{p}_k^{\text{debiased}} - \mathbb{E}\left[\tilde{p}_k^{\text{debiased}}\right] = \frac{\tilde{p}_k - (1-\pi))}{2\pi - 1} - \frac{\mathbb{E}\left[\tilde{p}_k\right] - (1-\pi))}{2\pi - 1} = \frac{\tilde{p}_k - \mathbb{E}[\tilde{p}_k]}{2\pi - 1}.$$

Here, the numerator is a sum of independent terms $\tilde{c}_{ik} \in [0,1]$ minus their expectation since

$$\tilde{p}_k - \mathbb{E}[\tilde{p}_k] = \frac{1}{n}\sum_{i=1}^{n}\tilde{c}_{ik} - \mathbb{E}\left[\tilde{c}_{ik}\right] = \frac{1}{n}\sum_{i=1}^{n} c_{ik} \cdot \mathbf{1}\{U_i \le \pi\} + (1-c_{ik}) \cdot \mathbf{1}\{U_i > \pi\} - \mathbb{E}[\tilde{c}_{ik}].$$

As the U_i and the $\tilde{c}_{ik}$ are i.i.d., an application of Hoeffding's inequality yields

$$\mathbb{P}\left(\left|\tilde{p}_k^{\text{debiased}} - \hat{p}_k\right| \ge \eta\right) = \mathbb{P}\left(\left|\frac{\tilde{p}_k - \mathbb{E}[\tilde{p}_k]}{2\pi - 1}\right| \ge \eta\right) = \mathbb{P}\left(\left|\frac{\sum_{i=1}^{n}\tilde{c}_{ik} - \mathbb{E}[\tilde{c}_{ik}]}{2\pi - 1}\right| \ge n\eta\right) \le 2e^{-n\eta^2(2\pi-1)^2}.$$

Using that $(2\pi-1)^2 = \tanh^2(\varepsilon/4)$ and union bounding over the N_{bins} bins recovers the statement. $\quad\square$

F.3 Proofs for Private Projection Interval (Local)

Proof of Lemma F.1.

The privacy of Algorithm 9 follows directly from that of Algorithm 7 shown in Lemma F.2.

Algorithm 7 uses bins $B_k := (2\tau k \pm \tau)$ of width 2τ to cover the interval $[-\mathsf{B}, \mathsf{B}]$. Hence, the collection of bins $(B_k)_{k \in S}$ used is of size $N_{\mathsf{bins}} \le \lceil \mathsf{B}/\tau \rceil + 1 \le \mathsf{B}/\tau + 2$. With this, under Assumption 2.8 Lemma 6.3 yields the following guarantee:

$$\mathbb{P}\left(\max_{k \in \mathbb{Z}} |\tilde{p}_k - p_k| \ge \frac{1}{16}\right) \le \left(\frac{\mathsf{B}}{\tau} + 2\right) 2\exp\left(-\frac{n}{4 \cdot 16}\tanh^2\left(\frac{\varepsilon}{4}\right)\right) - 16^2 \exp\left(-\frac{1}{864}\frac{n}{16^3 \rho M^2}\right)$$

$$\lesssim \left(\frac{\mathsf{B}}{\tau} + 1\right) e^{-\kappa n \tanh^2\left(\frac{\varepsilon}{4}\right)} \vee e^{-\kappa' n}.$$

Above, we suppress the constant $\rho M^2 \lesssim 1$. Given that X^n is $(\tau, \gamma)^{x_0}_\infty$-concentrated, Lemma C.5 translates this guarantee into one for the midpoint $\hat{m}$:

$$\mathbb{P}\left(\hat{m} \in \left[x_0 \pm 2\tau\right]\right) \geq 1 - O\left(\left(\frac{\mathsf{B}}{\tau} + 1\right) e^{-\kappa n \tanh^2\left(\frac{\varepsilon}{4}\right)} \vee e^{-\kappa' n}\right).$$

The proof is analogous and instead uses Lemma 6.3 under Assumption 2.7. □

F.4   Proofs for Mean Estimators (Local)

Lemma F.3. *Algorithm 8 denoted by $\mathcal{A}$ is $(\varepsilon, 0)$-DP with $\varepsilon > 0$. Assume $X^n \in \mathbb{R}^n$ is $(\tau, \gamma)^{x_0}_\infty$-concentrated with $\|x_0\|_\infty \leq \mathsf{B}$ and fulfills Assumption 2.7 or 2.8 s.t. $\rho M^2 \lesssim 1$. Then, the event $\mathcal{E} := \{\forall i \in [n] : \Pi_{\hat{I}}(X_i) = X_i\}$ has probability at least $1 - \gamma - O(\mathsf{B}/\tau + 1) \cdot e^{-\kappa n \tanh^2(\varepsilon/4)} \vee e^{-\kappa' n}$ and*

$$\mathcal{A}(X^n) \cdot \mathbf{1}\{\mathcal{E}\} = \left(\bar{X}_n + \bar{\xi}_n\right) \cdot \mathbf{1}\{\mathcal{E}\}, \quad \textit{almost surely.}$$

Proof. The private midpoint estimator of Algorithm 3 is $(\frac{\varepsilon}{2}, 0)$-DP by Lemma F.1 as we call it with $\frac{\varepsilon}{2}$ in Algorithm 8. Further, the Laplace mechanism therein is $(\frac{\varepsilon}{2}, 0)$-DP by Theorem 2.3. Therefore, algorithm $\mathcal{A}$ is $(\varepsilon, 0)$-DP through basic composition in Theorem 2.4.

For utility, by Lemma F.1 under Assumption 2.7 or 2.8 we have

$$\mathbb{P}\left(\hat{m} \in [x_0 \pm 2\tau]\right) \geq 1 - O\left(\mathsf{B}/\tau + 1\right) \cdot e^{-\kappa n \tanh^2(\varepsilon/4)} \vee e^{-\kappa' n}.$$

Hence, by an application of Lemma D.2 with $C = 2$, the event $\mathcal{E} := \{\forall i \in [n] : \Pi_{[\hat{m} \pm 3\tau]}(X_i) = X_i\}$ has probability at least $1 - \gamma - O(\mathsf{B}/\tau + 1) \cdot e^{-\kappa n \tanh^2(\varepsilon/4)} \vee e^{-\kappa' n}$ and

$$\mathcal{A}(X^n) \cdot \mathbf{1}\{\mathcal{E}\} = \left(\bar{X}_n + \bar{\xi}_n\right) \cdot \mathbf{1}\{\mathcal{E}\}, \quad \text{almost surely.}$$

□

Remark F.4. *If we assume $\varepsilon \in (0, 1)$ we can make the bound above exponential in the quadratic of ε instead of the hyperbolic tangent. This holds as $\tanh(1/2)x \leq \tanh(x/2)$ for $x \in [0, 1]$ and thus,*

$$\tanh^2\left(\frac{\varepsilon}{4}\right) \leq -\tanh^2\left(\frac{1}{4}\right)\varepsilon^2 =: -\kappa''\varepsilon^2.$$

As a result, for $\varepsilon \in (0, 1)$ and $\mathcal{E}$ as above, with probability $1 - \gamma - O(\mathsf{B}/\tau \cdot e^{-\kappa''' n \varepsilon^2})$ for $\kappa''' > 0$,

$$\mathcal{A}(X^n) \cdot \mathbf{1}\{\mathcal{E}\} = \left(\bar{X}_n + \bar{\xi}_n\right) \cdot \mathbf{1}\{\mathcal{E}\}, \quad \textit{almost surely.}$$

Lemma F.5. *Let $\varepsilon, \varrho \in (0, 1)$ and $\gamma \in (0, 1 \wedge \frac{n}{4})$. Algorithm 5 denoted by $\mathcal{A}$ calling Algorithm 8 in Line 3 is (ε, ϱ)-LDP. Suppose $X^n \in \mathbb{R}^{n \times d}$ is $(\tau, \gamma)^{x_0}_\infty$-concentrated with $\|x_0\|_\infty \leq \mathsf{B}$. Further, let X^n fulfill Assumption 2.7 or 2.8 s.t. $\rho M^2 \lesssim 1$. Then, the event $\mathcal{E} := \{\forall i \in [n], j \in [d] : \Pi_{R_j}(X_{ij}) = X_{ij}\}$ has probability at least $1 - d\gamma - O(\mathsf{B}/\tau + 1) \cdot de^{-\kappa n (\varepsilon')^2}$ with $\varepsilon' = \varepsilon/\sqrt{8d \log(1/\varrho)}$. Moreover, a.s.*

$$\mathcal{A}(X^n) \cdot \mathbf{1}\{\mathcal{E}\} = \left(\bar{X}_n + \bar{\Xi}_n\right) \cdot \mathbf{1}\{\mathcal{E}\}, \quad \textit{where} \quad \Xi_i \sim \mathrm{Lap}\left(0, \frac{12\tau}{\varepsilon'} I_d\right).$$

Proof. Since the d one-dimensional mean estimators fulfill $(\varepsilon, 0)$-DP, the coordinate-wise multi-dimensional estimator is private by composition. We use the privacy parameters $\varepsilon', \delta' = 0, \varrho$ defined in Algorithm D.1 governed by advanced composition in Theorem 2.4. This introduces ϱ.

X^n is $(\tau, \gamma)_\infty^{x_0}$-concentrated. Applying Algorithm 8 denoted by $\mathcal{A}_1$ with $(X_{\cdot j}^n, \tau, \varepsilon', \delta')$ per dimension, by Lemma F.3, $\mathcal{E}_j := \{\forall i \in [n] : \Pi_{\hat{I}}(X_i) = X_i\}$ has $\mathbb{P}(\mathcal{E}_j) \geq 1 - \gamma - O(\mathsf{B}/\tau + 1) \cdot e^{-\kappa n \tanh^2(\varepsilon)}$ and

$$\mathcal{A}_1(X_{\cdot j}) \cdot \mathbf{1}\{\mathcal{E}\} = \left(\frac{1}{n} \sum_{i=1}^n X_{ij} + \xi_{ij}\right) \cdot \mathbf{1}\{\mathcal{E}\}, \quad \text{almost surely.}$$

Using the remark above and by Lemma D.3 with $h = n$ and $b = \frac{12\tau}{\varepsilon'}$, on the event $\mathcal{E} = \bigcap_{j \in [d]} \mathcal{E}_j$ that has probability at least $1 - d\gamma - O(\mathsf{B}/\tau + 1) \cdot de^{-\kappa n(\varepsilon')^2}$ it holds that, almost surely

$$\mathcal{A}(X) \cdot \mathbf{1}\{\mathcal{E}\} = \left(\frac{1}{n} \sum_{i=1}^n X_{i\cdot}^\top + \Xi_i\right) \cdot \mathbf{1}\{\mathcal{E}\}, \quad \text{with } \Xi_i = (\xi_{i1}, ..., \xi_{id})^\top.$$

Note that by definition it holds that $\bar{X}_n + \bar{\Xi}_n = \frac{1}{n} \sum_{i=1}^n X_{i\cdot}^\top + \Xi_i$. The statement is then recovered as the the Ξ_{ij} are i.i.d. and therefore $\Xi_i \sim \mathrm{Lap}(0, \frac{12\tau}{\varepsilon'})$. $\square$

F.5 Proofs of Theoretical Guarantees (Local)

Proof of Theorem 6.4.
This proof directly follows from Lemma D.6. By Lemma F.5 there is an event $\mathcal{E}$ that has probability at least $1 - \gamma - O(\mathsf{B}/\tau + 1) \cdot de^{-\kappa n(\varepsilon')^2}$ s.t. almost surely,

$$\mathcal{A}(X^n) \cdot \mathbf{1}\{\mathcal{E}\} = \left(\bar{X}_n + \bar{\Xi}_n\right) \cdot \mathbf{1}\{\mathcal{E}\}, \quad \text{where } \Xi \sim \mathrm{Lap}\left(0, \frac{12\tau}{\varepsilon'} I_d\right).$$

Applying Lemma D.6 with $h = n$, with probability at least $1 - \alpha - d\gamma - O(\mathsf{B}/\tau + 1) \cdot de^{-\kappa n(\varepsilon')^2}$,

$$\|\mathcal{A}(X) - \mu\|_2 < \|\bar{X}_n - \mu\|_2 + \frac{12\tau}{\varepsilon'} \left(\sqrt{\frac{2d}{n}} + \sqrt{\frac{4\log(3/\alpha)^2}{n}}\right)$$

$$\leq \|\bar{X}_n - \mu\|_2 + 12\tau \left(\sqrt{\frac{16d^2 \log(1/\varrho)}{n\varepsilon^2}} + \sqrt{\frac{32d \log(1/\varrho) \log(3/\alpha)^2}{n\varepsilon^2}}\right).$$

The statement follows by omitting absolute constants in $\lesssim$-notation and bounding

$$\sqrt{16d^2 \log(1/\varrho)} + \sqrt{32d \log(1/\varrho) \log(3/\alpha)^2} \lesssim \sqrt{d^2 \log(1/\varrho) \log(3/\alpha)^2}.$$

Finally, the statement is obtained by setting $\alpha = d\gamma$. $\square$

Proof of Theorem 6.5.
Define the noiseless estimator $T_0 := \frac{1}{n} \sum_{i=1}^n \Pi_{\hat{I}^d}(X_i)$ where $\Pi_{\hat{I}^d}(X_i) := [\Pi_{\hat{I}_1}(X_{i1}), \ldots, \Pi_{\hat{I}_1}(X_{ij})]^\top$ stems from the coordinate-wise application of Algorithm 8. Then, by independence of $\bar{\Xi}_n$ and T_0

$$\mathbb{E}\left[\|\mathcal{A}(X^n) - \mu\|_2^2\right] = \mathbb{E}\left[\|T_0 + \bar{\Xi}_n - \mu\|_2^2\right] = \mathbb{E}\left[\|\bar{\Xi}_n\|_2^2\right] + \mathbb{E}\left[\|T_0 - \mu\|_2^2\right].$$

Here, $\mathbb{E}[\|\bar{\Xi}_n\|_2^2] \le 2db^2/n$ as $\bar{\Xi}_n$ is an average of $\Xi_1, \ldots, \Xi_n \overset{iid}{\sim} \mathrm{Lap}(0, bI_d)$. We now bound the second term. For this, define the event on which the projections in Algorithm 5 don't have an effect:

$$\mathcal{E} := \{\forall i \in [n], j \in [d] : \Pi_{\hat{I}_j}(X_{ij}) = X_{ij}\}.$$

By Lemma F.5 we have then have $\mathbb{P}(T_0 = \bar{X}_n) \ge \mathbb{P}(\mathcal{E}) \ge 1 - d\gamma - O(\mathsf{B}/\tau + 1) \cdot de^{-\kappa n(\varepsilon')^2}$. Thus, on $\mathcal{E}$ it holds that $T_0 = \bar{X}_n$ and we may split the expectation as follows

$$\mathbb{E}\left[\|T_0 - \mu\|_2^2\right] = \mathbb{E}\left[\|T_0 - \mu\|_2^2 \cdot \mathbf{1}\{\mathcal{E}\}\right] + \mathbb{E}\left[\|T_0 - \mu\|_2^2 \cdot \mathbf{1}\{\mathcal{E}^c\}\right]$$
$$\le \mathbb{E}\left[\|\bar{X}_n - \mu\|_2^2\right] + 2\mathbb{E}\left[(\|T_0\|_2^2 + \|\mu\|_2^2) \cdot \mathbf{1}\{\mathcal{E}^c\}\right].$$

By Assumption we have $\|\mu\|_2^2 \le d\|\mu\|_\infty^2 \le d\mathsf{B}^2$. Further, by Jensen's inequality

$$\|T_0\|_2^2 \le \sum_{j=1}^d \frac{1}{n} \sum_{i=1}^n \Pi_{R_j}(X_{ij})^2 \le \sum_{j=1}^d \frac{1}{n} \sum_{i=1}^n (\hat{m}_j + \tau)^2 \le 2d(\mathsf{B}^2 + \tau^2).$$

Combining both now yields the following upper bound

$$\mathbb{E}\left[(\|T_0\|_2^2 + \|\mu\|_2^2) \cdot \mathbf{1}\{\mathcal{E}^c\}\right] \lesssim d(\mathsf{B}^2 + \tau^2) \cdot \mathbb{P}(\mathcal{E}^c)$$
$$\lesssim d(\mathsf{B}^2 + \tau^2) \cdot \left(d\gamma + \left(\frac{\mathsf{B}}{\tau} + 1\right) \cdot de^{-\kappa n(\varepsilon')^2}\right)$$
$$\lesssim d^2\gamma(\mathsf{B}^2 + \tau^2) + \frac{d^2}{e^{\kappa n(\varepsilon')^2}}\left(\frac{\mathsf{B}^3}{\tau} + \mathsf{B}^2 + \mathsf{B}\tau + \tau^2\right).$$

Collecting terms and plugging in $b \asymp \tau/(\varepsilon')$ we then obtain

$$\mathbb{E}\left[\|\mathcal{A}(X^n) - \mu\|_2^2\right] \le \mathbb{E}\left[\|\bar{\Xi}_n\|_2^2\right] + \mathbb{E}\left[\|T_0 - \mu\|_2^2\right]$$
$$\lesssim \mathbb{E}\left[\|\bar{X}_n - \mu\|_2^2\right] + \frac{d\tau^2}{n(\varepsilon')^2} + d^2(\mathsf{B}^2 + \tau^2) \cdot \left(\gamma + \frac{\mathsf{B}/\tau + 1}{e^{\kappa n(\varepsilon')^2}}\right).$$

The statement follows as $\varepsilon' \asymp \varepsilon/\sqrt{d\log(1/\varrho)}$ and considering the cases $\mathsf{B} \ge \tau$ and $\mathsf{B} \le \tau$. $\qquad\square$

Proof of Corollary 6.6.

Given that Assumption 2.8 holds, by Corollary 2.10 we have $\mathbb{E}[\|\bar{X}_{nT} - \mu\|_2^2] \le d\rho/(nT)$. Combining this with Theorem 6.5 we thus obtain

$$\mathbb{E}[\|\mathcal{A}(X^n) - \mu\|_2^2] \lesssim \mathbb{E}\left[\|\bar{X}_n - \mu\|_2^2\right] + d^2\left(\frac{\tau^2\log(1/\varrho)}{n\varepsilon^2} + (\mathsf{B}^2 + \tau^2) \cdot \left(\gamma + \frac{\mathsf{B}/\tau + 1}{e^{\kappa n(\varepsilon')^2}}\right)\right)$$
$$\le \frac{d\rho}{nT} + \frac{d^2\tau^2\log(1/\varrho)}{n\varepsilon^2} + d^2(\mathsf{B}^2 + \tau^2) \cdot \left(\gamma + \frac{\mathsf{B}/\tau + 1}{e^{\kappa n(\varepsilon')^2}}\right) =: T_1 + T_2 + T_3.$$

Again using Assumption 2.8, X^n is $(\tau, \gamma)_\infty^\mu$-concentrated with $\tau = \sqrt{2\rho\log(2dn/\gamma)/T}$ by Lemma 2.11. We specify $\gamma \asymp (Tn\varepsilon^2)^{-1}$ which yields $\tau^2 \asymp \rho\log(2dTn^2\varepsilon^2)/T$. The term T_2 then becomes

$$T_2 = \frac{d^2\tau^2\log(1/\varrho)}{n\varepsilon^2} = \frac{d^2\rho\log(2dTn^2\varepsilon^2)\log(1/\varrho)}{Tn\varepsilon^2}.$$

Note that T_1 is always dominated by T_2 since $\varepsilon \lesssim 1$ by assumption. For the third term T_3 we get

$$T_3 = d^2(\mathsf{B}^2 + \tau^2) \cdot \left(\gamma + \frac{\mathsf{B}/\tau + 1}{e^{\kappa n(\varepsilon')^2}}\right)$$

$$= d^2\left(\mathsf{B}^2 + \frac{\rho \log(2dTn^2\varepsilon^2)}{T}\right) \cdot \left(\frac{1}{Tn\varepsilon^2} + \left(\sqrt{\frac{\mathsf{B}^2 T}{\rho \log(2dTn^2\varepsilon^2)}} + 1\right)\frac{1}{e^{\kappa n(\varepsilon')^2}}\right).$$

In the setting when $\mathsf{B} \geq \tau$ and hence $\mathsf{B}/\tau \geq 1$, collecting terms yields

$$\mathbb{E}[\|\mathcal{A}(X^n) - \mu\|_2^2] \lesssim \frac{d^2 \rho \log(2dTn^2\varepsilon^2)\log(1/\varrho)}{Tn\varepsilon^2} + d^2\mathsf{B}^2 \cdot \left(\frac{1}{Tn\varepsilon^2} + \sqrt{\frac{\mathsf{B}^2 T}{\rho \log(2dTn^2\varepsilon^2)}}\frac{1}{e^{\kappa n(\varepsilon')^2}}\right)$$

$$\lesssim \frac{d^2 \rho \log(2dTn^2\varepsilon^2)\log(1/\varrho)}{Tn\varepsilon^2} + d^2\mathsf{B}^2 \cdot \left(\frac{1}{Tn\varepsilon^2} + \frac{\mathsf{B}}{\sqrt{\rho}}\frac{\sqrt{T}}{e^{\kappa n(\varepsilon')^2}}\right).$$

Instead if $\mathsf{B} \leq \tau$ it holds that $\mathsf{B}/\tau \leq 1$ and therefore

$$\mathbb{E}[\|\mathcal{A}(X^n) - \mu\|_2^2] \lesssim \frac{d^2 \rho \log(2dTn^2\varepsilon^2)\log(1/\varrho)}{Tn\varepsilon^2} + \frac{d^2 \rho \log(2dTn^2\varepsilon^2)}{T} \cdot \left(\frac{1}{Tn\varepsilon^2} + \frac{1}{e^{\kappa n(\varepsilon')^2}}\right)$$

$$\lesssim \frac{d^2 \rho \log(2dTn^2\varepsilon^2)}{T} \cdot \left(\frac{\log(1/\varrho)}{n\varepsilon^2} + \frac{1}{e^{\kappa n(\varepsilon')^2}}\right)$$

$$\lesssim \frac{d^2 \rho \log(2dTn^2\varepsilon^2)\log(1/\varrho)}{Tn\varepsilon^2}.$$

Under Assumption 2.7 we exploit that the rows of X^n are i.i.d. with ρ-sub-Gaussian entries. This yields $\mathbb{E}[\|\bar{X}_n - \mu\|_2^2] \leq d\rho/n$ and $(\tau,\gamma)_\infty^\mu$-concentration with the same τ. The rest is analogous. $\quad\square$

F.6 Proofs for Nonparametric Regression (Local)

Proof of Corollary 6.7.

This proof is analogous to the proof of Corollary 4.11 covering the central model.

We want to combine the concentration results of Lemma 4.9 with Theorem 6.4 for $d = 1$. Note $\hat{f}^n(x)$ is concentrated around $f(x)$ and $\sup_{x \in [0,1]} |f(x)| \leq \|f\|_\infty$ and thus the theorem's boundedness condition holds. Further, by Lemma 4.10 we may call the theorem under Assumption 2.8.

In Lemma 4.9 we derived $(\tau,\gamma)_\infty$-concentration of $\hat{f}^n(x)$ around $f(x)$ with

$$\tau \asymp b + \frac{1}{b}\left(1 + \sqrt{\sigma_{\max}^2\|K\|_\infty \log(2n/\gamma)}\right) \lesssim \frac{1}{b}\sqrt{\sigma_{\max}^2\|K\|_\infty \log(2n/\gamma)} =: \frac{C}{b}.$$

There, we also derived a statistical rate for the rMSE, meaning that with probability at least $1 - \alpha$

$$\left|\hat{f}_n(x) - f(x)\right| \lesssim b + \frac{1}{nb} + \sqrt{\frac{\sigma_{\max}^2\|K\|_\infty \log(2/\alpha)}{nb}}.$$

Setting $\alpha = \gamma$ and combining this with Theorem 6.4 via a union bound, with probability at least

$1 - 3\gamma - O(M/\tau \cdot e^{-\kappa n\varepsilon^2})$ we have

$$\left|\mathcal{A}(\hat{f}^n(x)) - f(x)\right| \lesssim \left|\hat{f}_n(x) - f(x)\right| + \tau\sqrt{\frac{\log(1/\varrho)\log(3/\gamma)^2}{n^2\varepsilon^2}}$$

$$\lesssim \left|\hat{f}_n(x) - f(x)\right| + \sqrt{\frac{C^2\log(1/\varrho)\log(3/\gamma)^2}{b^2n\varepsilon^2}}$$

$$\lesssim b + \frac{1}{nb} + \sqrt{\frac{\sigma_{\max}^2\|K\|_\infty \log(2/\gamma)}{nb}} + \sqrt{\frac{C^2\log(1/\varrho)\log(3/\gamma)^2}{nb^2\varepsilon^2}}.$$

Note that by the expression for τ above we have $\tau \gtrsim b$, which lets us bound the big-O in the probability. Modulo logarithms, and using $1 \leq 1/b$, $1 \lesssim 1/\varepsilon$, $1/n \leq 1$ and $1 \lesssim \sigma_{\max}^2$ to merge the second and fourth term yields

$$\left|\mathcal{A}(\hat{f}^n(x)) - f(x)\right| \lessgtr b + \frac{1}{nb} + \sigma_{\max}\left(\sqrt{\frac{1}{nb}} + \sqrt{\frac{1}{n^2b^2\varepsilon^2}}\right) \lesssim b + \sigma_{\max}\left(\sqrt{\frac{1}{nb}} + \sqrt{\frac{1}{nb^2\varepsilon^2}}\right).$$

We call the three terms on the right hand side above T_1, T_2 and T_3. Finally, to balance T_1, T_3 and T_2, T_3, we perform the following case distinction:

Case 1: If $T_2 \gtrsim T_3$, i.e., when $\varepsilon \gtrsim 1/\sqrt{b}$ we balance T_1 and T_2 as follows:

$$T_1 \asymp T_2 \quad \Leftrightarrow \quad b \asymp \frac{\sigma_{\max}}{\sqrt{nb}} \quad \Leftrightarrow \quad b \asymp \left(\frac{\sigma_{\max}^2}{n}\right)^{1/3} \quad \Rightarrow \quad \varepsilon \gtrsim \left(\frac{n}{\sigma_{\max}^2}\right)^{1/6}.$$

Case 2: If $T_3 \gtrsim T_2$, i.e., when $\varepsilon \lesssim 1/\sqrt{b}$ we balance T_1 and T_3 as follows:

$$T_1 \asymp T_3 \quad \Leftrightarrow \quad b \asymp \frac{\sigma_{\max}}{\sqrt{nb^2\varepsilon^2}} \quad \Leftrightarrow \quad b \asymp \sqrt{\frac{\sigma_{\max}}{\sqrt{n}\varepsilon}} \quad \Rightarrow \quad \varepsilon \lesssim \left(\frac{n}{\sigma_{\max}^2}\right)^{1/6}.$$

We therefore know that T_2 dominates when $\varepsilon \lesssim \varepsilon_c := (n/\sigma_{\max}^2)^{1/6}$ and otherwise T_3 does. Choosing $b \asymp (\sigma_{\max}^2/n)^{1/3} \vee (\sigma_{\max}/(\sqrt{n}\varepsilon))^{1/2}$ incorporates both cases. We then recover the statement by plugging in b and bounding the resulting maximum by a sum of both terms. $\qquad\square$

Proof of Corollary 6.8.
The proof is analogous to that of Corollary 6.8. As in the central model, Assumption 2.8 holds and we also assume $\|\mu\|_\infty \leq \mathsf{B}$. We then combine Lemma 5.2 with Theorem 6.4 instead of Theorem 4.2. The statement follows by plugging $\tau \asymp \sqrt{\rho\log(2dn/\gamma)/T}$ into the bound on the probability. $\qquad\square$

Proof of Corollary 6.9.
The result is a direct consequence of Corollary 6.8 enabled by Lemma 5.9. $\qquad\square$

Proof of Corollary 6.10.
The proof is analogous to that of Corollary 5.16. Because of the block structure of Σ^n and since $\|\beta\|_\infty \leq \mathsf{B}$ by assumption, we use Theorem 6.4 under Assumption 2.7 instead of Theorem 4.2. $\qquad\square$

G Variance Estimation Algorithms

Lastly, we present the plug-in variance estimators used in Subsection 7.1.5. They both take $\sigma_{\min}^2, \sigma_{\max}^2$ and $\hat{m}$ as input, i.e., a lower and upper bounds on the variance as well as a potentially very rough estimate for the first moment. The algorithms iteratively update the upper bound and the first moment estimate, and eventually output an estimate of the variance.

G.1 Variance Estimation by Bisection

Our first plug-in variance estimator repeatedly halves the provided upper bound on the variance as long as the coverage of the resulting confidence interval has high enough empirical coverage.

Algorithm 10 VarianceBisection$(X^n, \varepsilon, \delta, \varrho, \sigma_{\min}^2, \sigma_{\max}^2, \hat{m}, \gamma)$:

Require: $X^n \in \mathbb{R}$, $\varepsilon, \delta, \varrho > 0$, $\sigma_{\min}^2 \sigma_{\max}^2 \in (0, \infty)$, $\hat{m} \in \mathbb{R}$, $\gamma > 0$
1: $N_{\text{iter}} \leftarrow \lceil \log_2(\sigma_{\max}^2 / \sigma_{\min}^2) \rceil$
2: $\varepsilon' \leftarrow \varepsilon / \sqrt{16 N_{\text{iter}} \log(1/\varrho)}$
3: **while** $\text{Coverage}(X^n, \varepsilon', \sigma_{\max}, \hat{m}) \geq 1 - \gamma$ **do**
4: $\quad \hat{m} \leftarrow \text{RefineMidpoint}(X^n, \varepsilon', \sigma_{\max}, \hat{m})$
5: $\quad \sigma_{\max}^2 \leftarrow \sigma_{\max}^2 / 2$
6: **return** $\sigma_{\max}^2$

The intervals coverage is computed privately in Algorithm 11 using a Laplace mechanism and will by design tend to give a conservative large variance estimate. Algorithm 12 is used to refine the mean estimator using the current variance upper bounds and privacy is ensured by the Laplace mechanism.

Algorithm 11 Coverage$(X^n, \varepsilon, \sigma, \hat{m})$:

Require: $X^n \in \mathbb{R}$, $\varepsilon > 0$, $\sigma \in (0, \infty)$
1: $\hat{\tau} \leftarrow \sqrt{2}\sigma$
2: $\hat{I} \leftarrow [\hat{m} \pm \hat{\tau}]$
3: $\hat{p} \leftarrow \frac{1}{n} \sum_{i=1}^n \mathbf{1}\{X_i \in \hat{I}\}$
4: **return** $\hat{p} + \xi$ with $\xi \sim \text{Lap}(0, \frac{2}{n\varepsilon})$

Algorithm 12 RefineMidpoint$(X^n, \varepsilon, \sigma, \hat{m})$:

Require: $X^n \in \mathbb{R}$, $\varepsilon > 0$, $\sigma \in (0, \infty)$, $\hat{m} \in \mathbb{R}$
1: $\hat{\tau} \leftarrow \sqrt{2}\sigma$
2: $\hat{I} \leftarrow [\hat{m} \pm \hat{\tau}]$
3: **return** $\frac{1}{n} \sum_{i=1}^n \Pi_{\hat{I}}(X_i) + \xi$ with $\xi \sim \text{Lap}(0, \frac{2\tau}{n\varepsilon})$

G.2 Variance Estimation via CoinPress

The second plug-in variance estimator we employ is an adjusted version of Algorithms 4 and 8 (CoinPress) by Biswas et al. (2020). Their original implementation assumes a zero mean and argues that under i.i.d. samples non-zero means this is not restrictive since one can always work with an i.i.d. sample of size $n/2$ symmetrized observations $Y_i := X_{i+1} - X_i$ and just double the variance.

Under dependence, however, differencing observations biases the variance estimate. Algorithm 13 avoids this issue by centering observations using the current estimate of the first moment $\hat{m}$.

Algorithm 13 AdaptiveCoinPress$(X^n, \varepsilon, \delta, \varrho, \sigma_{\min}^2, \sigma_{\max}^2, \hat{m}, \gamma)$:

Require: $X^n \in \mathbb{R}$, $\varepsilon, \delta, \varrho > 0$, $\sigma_{\min}^2 \sigma_{\max}^2 \in (0, \infty)$, $\hat{m} \in \mathbb{R}$, $\gamma > 0$
1: $N_{\text{iter}} \leftarrow \lceil \log_2(\sigma_{\max}^2 / \sigma_{\min}^2) \rceil$
2: $\varepsilon' \leftarrow \varepsilon / \sqrt{16 N_{\text{iter}} \log(1/\varrho)}$
3: $\sigma \leftarrow \sigma_{\max}$
4: **for** $t \in [N_{iter}]$ **do**
5: $\quad \hat{m} \leftarrow$ RefineMidpoint$(X^n, \varepsilon', \sigma_{\max}, \hat{m})$
6: $\quad (\sigma, \sigma_{\max}^2) \leftarrow$ RefineVariance$(X^n, \varepsilon', \sigma, \hat{m})$
7: **return** $\sigma_{\max}^2$

The variance estimation à la CoinPress is then performed on centered observations in Algorithm 14. The algorithm implements a one-dimensional version of Biswas et al. (2020, Algorithm 3).

Algorithm 14 RefineVariance$(X^n, \varepsilon, \sigma, \hat{m}, \gamma)$:

Require: $X^n \in \mathbb{R}$, $\varepsilon > 0$, $\sigma \in (0, \infty)$, $\hat{m} \in \mathbb{R}$, $\gamma > 0$
1: $\beta \leftarrow (1 + 2\sqrt{\log(1/\gamma)} + 2\log(1/\gamma))^{1/2}$
2: $w \leftarrow \Pi_{[\pm\beta]^n}((X^n - \hat{m} \cdot \mathbf{1}_n)/\sigma)$
3: $z \leftarrow 0 \vee (\frac{1}{n}\|w\|_2^2 + \xi)$ with $\xi \sim \text{Lap}(0, \frac{2\beta^2}{n\varepsilon})$
4: $\sigma \leftarrow \sigma \cdot (z + \sqrt{1/n} + 1/(2n))^{1/2}$
5: **return** $(\sigma, z \cdot \sigma^2)$

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