

EXTENDABILITY OF GROUP ACTIONS ON K3 OR ENRIQUES SURFACES

TIANCHEN ZHAO

ABSTRACT. Let X be a K3 or Enriques surface with good reduction. Let G be a finite group acting (not necessarily linearly) on X . We give a criterion for this group action to extend to a smooth model of X in terms of the action of G on the second ℓ -adic cohomology groups. In particular, we generalize the result on the extendability of Galois actions on K3 surfaces by Chiarellotto, Lazda, and Liedtke. As an application, we prove that a symplectic linear group action is extendable if the residue characteristic does not divide its order. Lastly, we relate the good reduction of Enriques surfaces with that of their K3 double covers.

CONTENTS

1. Introduction	1
2. Preliminaries	6
3. Extendability and non-abelian group cohomology	8
4. Extendability and ℓ -adic cohomology	17
5. Applications	22
References	25

1. INTRODUCTION

Throughout the whole article, we fix the following notations.

K	a complete and discretely valued field
\mathcal{O}_K	the ring of integers of K
k	the residue field of K , which we assume to be perfect of characteristic p
K^s	a fixed separable closure of K
\bar{k}	the residue field of K^s , an algebraic closure of k
$G_{E/F}$	the Galois group of E/F , a Galois extension
$I_{E/F}$	the inertia group of E/F , a Galois extension of complete and discretely valued fields
G_F	an absolute Galois group of a field F
I_F	an absolute inertia group of a complete and discretely valued field F

Definitions 1.1. Let X be a smooth proper variety over K .

- A model of X is an proper flat algebraic space over \mathcal{O}_K whose generic fibre is isomorphic to X .
- We say X has good reduction if X has a smooth model. Also, we say X has potential good reduction if X_L has good reduction for some finite extension L/K .

In this article, we study when a group action on a smooth proper variety extends to a smooth model. In particular, the group actions are not assumed to be linear over the ground field unless explicitly stated otherwise.

Definition 1.2. Let X be a smooth proper variety over K with good reduction. Let G be a group acting on X . We say the G -action on X is extendable if the G -action extends to some smooth model \mathcal{X} of X .

Note that we only require the action to extend to some smooth model rather than a particular smooth model.

Example 1.3. Let A be an abelian variety over K with good reduction. Its smooth model coincides with the Néron model by [Art86, Corollary 1.4]. Then every group action on X is extendable by the Néron mapping property.

1.1. Good reduction and the extendability problem. The extendability of group actions plays an important role in the good reduction problem.

In [LM17], Liedtke and Matsumoto prove that if a K3 surface X over K has good reduction after a finite and separable extension L/K and $H_{\text{ét}}^2(X_{K^s}, \mathbb{Q}_\ell)$ is unramified, then X has good reduction after a finite and unramified Galois extension. The idea of the proof is as follows. By passing to the Galois closure, we assume L/K is Galois. We find a smooth model \mathcal{X} of X_L , such that the action of $I_{L/K}$ on X_L extends to \mathcal{X} . Then $\mathcal{X}/I_{L/K}$ gives a smooth model of X over the maximal unramified sub-extension of L/K .

Another example is in [CLL19]. Let X be a K3 surface over K with good reduction after a finite and unramified extension L/K . Again, we may assume L/K is Galois. Chiarellotto, Lazda, and Liedtke derive a criterion for the extendability of the action of $G_{L/K}$ on X_L . If the action does extend to \mathcal{X} , then $\mathcal{X}/G_{L/K}$ gives a smooth model of X .

In addition to the extendability of Galois actions, the extendability of linear group actions also has applications to the good reduction problem. See Corollary 1.12 for an example.

1.2. Main results. We focus on the case where X is a K3 or Enriques surface. In particular, for any smooth model \mathcal{X} of X , its special fibre \mathcal{X}_k is determined up to

isomorphism by X by [LM17, Proposition 4.7]. We call it the canonical reduction of X , denoted by X° . This is also a K3 or Enriques surface depending on X . Moreover, there is a canonical homomorphism

$$\text{sp} : \text{Aut}(X) \rightarrow \text{Aut}(X^\circ) \quad g \mapsto g_k,$$

called the specialization homomorphism. Any group action on X induces a group action on X° via specialization. See Section 2.1 for more details.

Let G be a finite group acting on X on the right. This induces a G -action on K on the left whose fixed field is $K_0 = K^G$ (we call this the base field of the G -action as in Definition 3.11). We restrict to the case where K/K_0 is finite and unramified.

Remark 1.4. We will see why we want G to act on the right in Lemma 3.16. This assumption is merely a formality. Indeed, suppose G acts on X on the left via $\rho : G \rightarrow \text{Aut}(X)$. We can convert this to a right action by taking the anti-homomorphism $\rho' : G \rightarrow \text{Aut}(X)$, $g \mapsto \rho(g^{-1})$.

Let $\ell \neq p$ be a prime number. We want to give a criterion for the extendability of group actions in terms of the ℓ -adic cohomology of X and X° . Since X has good reduction, I_K acts trivially on $H_{\text{ét}}^2(X_{K^s}, \mathbb{Q}_\ell(1))$. Then $G_k \cong G_K/I_K$ acts on $H_{\text{ét}}^2(X_{K^s}, \mathbb{Q}_\ell(1))$. Denote by $H_{\text{ét}}^2(X_{K^s}, \mathbb{Q}_\ell(1))^{G_k}$ and $H_{\text{ét}}^2(X_{\bar{k}}^\circ, \mathbb{Q}_\ell(1))^{G_k}$ respectively the G_k -invariant subspaces of $H_{\text{ét}}^2(X_{K^s}, \mathbb{Q}_\ell(1))$ and $H_{\text{ét}}^2(X_{\bar{k}}^\circ, \mathbb{Q}_\ell(1))$. As in Section 4.1, we can define a G -action on $H_{\text{ét}}^2(X_{K^s}, \mathbb{Q}_\ell(1))^{G_k}$ and similarly on $H_{\text{ét}}^2(X_{\bar{k}}^\circ, \mathbb{Q}_\ell(1))^{G_k}$ via specialization.

By adapting the method in [CLL19], we prove the following.

Theorem 1.5. *Let X be a K3 or Enriques surface over K with good reduction. Let G be a finite group acting on X with base field K_0 . Assume K/K_0 is finite and unramified. Then the G -action on X is extendable if and only if there is a G -isomorphism*

$$H_{\text{ét}}^2(X_{K^s}, \mathbb{Q}_\ell(1))^{G_k} \cong H_{\text{ét}}^2(X_{\bar{k}}^\circ, \mathbb{Q}_\ell(1))^{G_k}.$$

A further criterion is given in Theorem 4.7. In Section 4.1, we define a group \overline{G}_k that acts on X_{K^s} and $X_{\bar{k}}^\circ$, which combines the action of G on X and the Galois action on K^s and \bar{k} . Then we show that the condition in Theorem 1.5 is further equivalent to the existence of a \overline{G}_k -isomorphism $H_{\text{ét}}^2(X_{K^s}, \mathbb{Q}_\ell(1)) \cong H_{\text{ét}}^2(X_{\bar{k}}^\circ, \mathbb{Q}_\ell(1))$.

Theorem 1.5 recovers the criterion for the extendability of Galois action in [CLL19]. Let L/K be a finite and unramified extension. Let k_L be the residue field of L . Let X be a K3 surface over K such that X_L has good reduction. By passing to the Galois closure, we may assume L/K is Galois.

Corollary 1.6. *[CLL19, Theorem 1.6] The following are equivalent.*

- (1) X has good reduction.

- (2) The action of $G_{L/K}$ on X_L is extendable.
- (3) There is a G_K -isomorphism $H_{\text{ét}}^2(X_{K^s}, \mathbb{Q}_\ell) \cong H_{\text{ét}}^2(X_{\bar{k}}^\circ, \mathbb{Q}_\ell)$.
- (4) There is a $G_{L/K}$ -isomorphism $H_{\text{ét}}^2(X_{K^s}, \mathbb{Q}_\ell(1))^{G_{k_L}} \cong H_{\text{ét}}^2(X_{\bar{k}}^\circ, \mathbb{Q}_\ell(1))^{G_{k_L}}$.

Proof. Suppose the $G_{L/K}$ -action on X_L extends to a smooth model \mathcal{X} . Then $\mathcal{X}/G_{L/K}$ is a smooth model of X by [CLL19, Proposition 7.1]. This proves (2) \implies (1).

Next, suppose X has a smooth model \mathcal{X} . Then the comparison isomorphism $H_{\text{ét}}^2(X_{K^s}, \mathbb{Q}_\ell) \cong H_{\text{ét}}^2(X_{\bar{k}}^\circ, \mathbb{Q}_\ell)$ is G_K -equivariant. This proves (1) \implies (3).

For (3) \implies (4), we simply restrict the G_K -isomorphism to the G_{k_L} -invariant subspace. Finally, (4) \implies (2) follows by Theorem 1.5. \square

In case of a K -linear action, G acts on $H_{\text{ét}}^2(X_{K^s}, \mathbb{Q}_\ell)$ and on $H_{\text{ét}}^2(X_{\bar{k}}^\circ, \mathbb{Q}_\ell)$ via specialization. We have the following criterion for extendability.

Theorem 1.7. *Let X be a K3 or Enriques surface over K with good reduction. Let G be a finite group acting K -linearly on X . Then the G -action on X is extendable if and only if there is a G -isomorphism*

$$H_{\text{ét}}^2(X_{K^s}, \mathbb{Q}_\ell) \cong H_{\text{ét}}^2(X_{\bar{k}}^\circ, \mathbb{Q}_\ell).$$

In particular, this implies that the extendability of K -linear group actions is a geometric property rather than an arithmetic property.

1.3. Applications of the main results. We present some applications of our main results. We will prove them in Section 5. The first application is a criterion for the extendability of a symplectic group action in terms of the order of the group and the residue characteristic.

Definition 1.8. Let X be a K3 surface over a field F . An F -linear automorphism of X is symplectic if the induced automorphism of $H^0(X, \omega_X)$ is the identity. An F -linear group action of G on X is symplectic if for every $g \in G$, the automorphism $g : X \rightarrow X$ is symplectic.

Corollary 1.9. *Let G be a finite group acting linearly and symplectically on X . If p does not divide the order of G , then the G -action is extendable.*

Remark 1.10. Matsumoto proves this in [Mat16, Theorem 1.1] with a weaker notion of extendability. In [Mat16], an automorphism g of a K3 surface X with good reduction is defined to be extendable if there exists a finite extension L/K such that g extends to a smooth model of X_L . This corresponds to potential extendability in our terminology

Remark 1.11. Our proof of Corollary 1.9 is different from that of [Mat16, Theorem 1.1]. Despite this, we need [Mat16, Lemma 2.13] and [Mat16, Proposition 6.1].

Neither of them depend on the main result of [Mat16]. Indeed, [Mat16, Lemma 2.13] is a general fact about K3 surfaces and [Mat16, Proposition 6.1] is based on a mild generalization of the flops considered in [LM17, Section 4].

Assume $\text{char } K = 0$. This corollary can be applied to show that singular K3 surfaces have potential good reduction modulo $p \neq 2$ with a scheme model. In [Mat15, Theorem 0.1], Matsumoto shows that if $p \neq 2, 3$, every K3 surface with a Shioda-Inose structure of product type has good reduction after a field extension of ramification index 1, 2, 3, 4, or 6. In particular, this holds for singular K3 surfaces.

This corollary generalizes the argument to $p = 3$. The argument in [Mat15, Theorem 0.1] fails for $p = 3$: in step (4), Matsumoto shows that a symplectic involution on a K3 surface X extends to a smooth model \mathcal{X} using a relative elliptic fibration, that is, a fibration $\mathcal{X} \rightarrow \mathbb{P}_{\mathcal{O}_K}^1$ whose generic and special fibres are both elliptic fibrations. If $p = 3$, the special fibre might be quasi-elliptic and the argument fails.

If we apply [Mat16, Theorem 1.1], we see that the symplectic involution is potentially extendable, so we see that X has potential good reduction. However, we lose control of the ramification index. With Corollary 1.9, we see that X has good reduction after a field extension of ramification index 1, 2, 3, 4, 6, 8, or 12.

We can also relate the good reduction of Enriques surfaces with that of their K3 double covers. Recall that for every Enriques surface Y over a field F of characteristic different from 2, there exists a K3 surface X with a fixed-point-free involution ι such that $Y = X/\iota$. We say this X is a K3 double cover of Y and ι is the associated involution.

Corollary 1.12. *Assume $p \neq 2$. Let Y be an Enriques surface over K . Then the following are equivalent.*

- (1) *Y has good reduction.*
- (2) *There exists a K3 double cover X of Y , with associated involution ι , such that X has good reduction and ι_k has no fixed points.*
- (3) *There exists a K3 double cover X of Y , with associated involution ι , such that X has good reduction and $\dim H_{\text{ét}}^2(X_k^\circ, \mathbb{Q}_\ell)^{\iota_k = \text{id}} = 10$.*

Acknowledgements. I would like to express my sincere gratitude to my supervisor, Christopher Lazda, for his invaluable support in many aspects. I also want to thank Yuya Matsumoto for his helpful comments on his proof of the good reduction of K3 surfaces with a Shioda-Inose structure. This work would not have been possible without their help. Last but not least, I gratefully acknowledge the financial support from the University of Exeter.

2. PRELIMINARIES

In this section, we review the preliminaries of this article, including the specialization homomorphism, RDP models, Weyl groups, and group cohomology with coefficients in a non-abelian group.

2.1. Specialization homomorphisms. Let X be a K3 or Enriques surface over K with good reduction. Let X° be the canonical reduction of X .

Lemma 2.1. *Let $g : X \rightarrow X$ be an automorphism. For every smooth model \mathcal{X} of X , g extends uniquely to a birational self-map $\mathcal{X} \dashrightarrow \mathcal{X}$ defined away from finitely many curves on the special fibre.*

Proof. The proof is the same as [Mat16, Proposition 2.2(1)]. \square

By this lemma, every automorphism g on X extends to $U \subset \mathcal{X}$, the complement of finitely many curves on the special fibre. Then it restricts to a birational self-map of X° . Since X° is a K3 or Enriques surface, the self-map extends uniquely to an automorphism of X° . This defines a group homomorphism $\text{sp}_{\mathcal{X}} : \text{Aut}(X) \rightarrow \text{Aut}(X^\circ)$ called the specialization homomorphism.

Lemma 2.2. *The specialization map $\text{sp}_{\mathcal{X}} : \text{Aut}(X) \rightarrow \text{Aut}(X^\circ)$ is independent of the choice of \mathcal{X} .*

Proof. The proof is the same as [Mat16, Proposition 2.2(2)]. \square

Given this lemma, we may write

$$\text{sp} : \text{Aut}(X) \rightarrow \text{Aut}(X^\circ) \quad g \mapsto g_k.$$

Then any group acting on X also acts on X° via specialization. In particular, let L/K be a finite Galois extension. Let k_L be the residue field of K . The $G_{L/K}$ -action on X_L specializes to a $G_{L/K}$ -action on $X_{k_L}^\circ$. This factors through the $G_{k_L/k}$ -action on $X_{k_L}^\circ$ via the surjection $G_{L/K} \rightarrow G_{k_L/k}$.

2.2. RDP models and Weyl groups. Suppose X is a K3 or Enriques surface over K with good reduction. Although the canonical reduction of X is unique up to isomorphism, there is in general no canonical choice of smooth models of X . However, once a polarization \mathcal{L} of X is fixed, there is a canonical model of X , called the RDP model $P(X, \mathcal{L})$. It may admit mild singularities on the special fibre but we can still recover the canonical reduction from its special fibre.

Theorem 2.3. *Let X be a K3 or Enriques surface with good reduction. Let \mathcal{L} be a polarization of X . Then there exists a projective model $P(X, \mathcal{L})$ of X such that*

- (1) $P(X, \mathcal{L})$ only depends on the pair (X, \mathcal{L}) up to unique isomorphism.

- (2) *The special fibre $P(X, \mathcal{L})_k$ has at worst rational double points, and its minimal desingularization coincides with X° .*

Proof. (2) is by [LM17, Proposition 4.6] and (1) follows from the proof of [LM17, Proposition 4.7]. \square

Let X be a K3 or Enriques surface over K with good reduction. Fix a polarization \mathcal{L} of X . Then $P(X, \mathcal{L})_k$ is a surface over k with at worst rational double points and $X^\circ \rightarrow P(X, \mathcal{L})_k$ is its minimal desingularization. We denote the exceptional locus as $E_{X, \mathcal{L}} \subset X^\circ$.

Let $\Lambda_{X, \mathcal{L}}$ be the subgroup of $\text{Pic}(X^\circ)$ generated by the classes of all the integral components of $E_{X, \mathcal{L}} \subset X^\circ$. For every such component E_i , denote $n_i = h^0(E_i, \mathcal{O}_{E_i})$. Let s_{E_i} be the automorphism of either $\Lambda_{X, \mathcal{L}}$ or $\text{Pic}(X^\circ)$ defined by

$$s_{E_i} : D \mapsto D + \frac{1}{n_i}(D \cdot E_i)E_i$$

Definition 2.4. Let $\text{Pic}(X^\circ)_\mathbb{Q} = \text{Pic}(X^\circ) \otimes \mathbb{Q}$. The Weyl group $\mathcal{W}_{X, \mathcal{L}}$ is the subgroup of $\text{Aut}(\text{Pic}(X^\circ)_\mathbb{Q})$ generated by s_{E_i} for all integral components E_i of $E_{X, \mathcal{L}}$.

Remark 2.5. In [CLL19, Section 3], the Weyl group of a K3 surface is defined as the subgroup of $\text{Aut}(\text{Pic}(X^\circ))$ generated by all s_{E_i} . To generalize this definition to Enriques surfaces, we need to replace $\text{Pic}(X^\circ)$ by $\text{Pic}(X^\circ)_\mathbb{Q}$. In particular, if X is a K3 surface, this agrees with the Weyl group defined in [CLL19, Section 3], since $\text{Pic}(X^\circ)$ embeds into $\text{Pic}(X^\circ)_\mathbb{Q}$.

2.3. Group cohomology with coefficients in a non-abelian group. We recall the construction of group cohomology with coefficients in a non-abelian group in [Ser97, Section 5.1, Chapter I].

Definition 2.6. Suppose G is a group acting on a possibly non-abelian group A on the left.

- (1) An A -valued 1-cocycle for G is a function $\phi : G \rightarrow A$ such that for every $g, h \in G$,

$$\phi(gh) = \phi(g)g(\phi(h)).$$

- (2) Given two 1-cocycles ϕ and ϕ' , we say $\phi \sim \phi'$ if there exists $a \in A$ such that for all $g \in G$,

$$a\phi(g) = \phi'(g)g(a).$$

- (3) Define $H^1(G, A) = \{A\text{-valued 1-cocycles for } G\} / \sim$. This is a pointed set, with base point represented by the trivial cocycle $G \rightarrow A$, which maps every $g \in G$ to the identity in A .

Next, suppose M is a left G -module, so G acts on the possibly non-abelian group $\text{Aut}(M)$ on the left by conjugation. Denote the action of G on M by $\rho : G \rightarrow \text{Aut}(M)$.

Definition 2.7. Let $\phi : G \rightarrow \text{Aut}(M)$ be an $\text{Aut}(M)$ -valued 1-cocycle of G . Define M^ϕ , the twist of M by ϕ , as a G -module:

- (1) The underlying group is M .
- (2) The action $\rho^\phi : G \rightarrow \text{Aut}(M^\phi)$ is defined for every $g \in G$, $m \in M$,

$$\rho^\phi(g)(m) = \phi(g)(\rho(g)(m)).$$

Lemma 2.8. Let $\phi : G \rightarrow \text{Aut}(M)$ be an $\text{Aut}(M)$ -valued 1-cocycle of G . Then the cohomology class represented by ϕ is trivial if and only if there exists a G -isomorphism

$$M \cong M^\phi.$$

Proof. By definition, the cohomology class represented by ϕ is trivial if and only if there exists $f \in \text{Aut}(M)$ such that for every $g \in G$,

$$\phi(g) = f^{-1} \cdot g(f) = f^{-1} \cdot \rho(g) \cdot f \cdot \rho(g)^{-1}.$$

Next, a G -isomorphism $h : M \rightarrow M^\phi$ is a group automorphism of M such that for every $m \in M$,

$$\begin{aligned} \rho^\phi(g)(h(m)) &= h(\rho(g)(m)) \\ \iff (\phi(g) \cdot \rho(g) \cdot h)(m) &= (h \cdot \rho(g))(m) \end{aligned}$$

Therefore, $\phi(g) = h \cdot \rho(g) \cdot h^{-1} \cdot \rho(g)^{-1}$. Then the two conditions are equivalent upon taking $f = h^{-1}$. \square

3. EXTENDABILITY AND NON-ABELIAN GROUP COHOMOLOGY

To prove Theorem 1.5 and Theorem 1.7, we define a cohomology class of G that controls the extendability of the G -action.

3.1. Extendability of birational maps. The constructions in this sub-section are based on [CLL19, Section 7.1]. Let X be a K3 or Enriques surface over K with good reduction. Fix a smooth model \mathcal{X} with special fibre X° .

Given a birational map $f : \mathcal{X} \dashrightarrow \mathcal{X}^+$ between two smooth models of X , we construct an endomorphism of $\text{Pic}(X^\circ)_\mathbb{Q}$. Denote by $\Gamma_f \subset \mathcal{X} \times \mathcal{X}^+$ the graph of f and by $\Gamma_{f,k} \subset X^\circ \times \mathcal{X}_k^+$ its special fibre. Let p_1 and p_2 be the projections from $X^\circ \times \mathcal{X}_k^+$ to X° and \mathcal{X}_k^+ respectively. Then we define a homomorphism

$$\text{Pic}(\mathcal{X}_k^+) \rightarrow \text{Pic}(X^\circ) \quad D \mapsto p_{1*}(\Gamma_{f,k} \cap p_2^* D),$$

which extends by linearity to $\tilde{s}_f : \text{Pic}(\mathcal{X}_k^+)_\mathbb{Q} \rightarrow \text{Pic}(X^\circ)_\mathbb{Q}$.

Also, by the uniqueness of the canonical reduction, we have an isomorphism $f_k : X^\circ \xrightarrow{\sim} \mathcal{X}_k^+$. We precompose \tilde{s}_f with the pullback via f_k^{-1} and get an endomorphism s_f of $\text{Pic}(X^\circ)_\mathbb{Q}$,

$$s_f : \text{Pic}(X^\circ)_\mathbb{Q} \xrightarrow{(f_k^*)^{-1}} \text{Pic}(\mathcal{X}_k^+)_\mathbb{Q} \xrightarrow{\tilde{s}_f} \text{Pic}(X^\circ)_\mathbb{Q}.$$

Let $\ell \neq p$ be a prime number. The cycle class map defines an injection $[-] : \text{Pic}(X^\circ)_\mathbb{Q} \rightarrow H_{\text{ét}}^2(X_k^\circ, \mathbb{Q}_\ell(1))$. We extend s_f to $H_{\text{ét}}^2(X_k^\circ, \mathbb{Q}_\ell(1))$ as follows. First, we define a homomorphism

$$\tilde{s}_{f,\ell} : H_{\text{ét}}^2(\mathcal{X}_k^+, \mathbb{Q}_\ell(1)) \rightarrow H_{\text{ét}}^2(X_k^\circ, \mathbb{Q}_\ell(1)) \quad \alpha \mapsto p_{1*}([\Gamma_{f,k}] \cup p_2^* \alpha).$$

This is well defined since $[\Gamma_{f,k}] \in H_{\text{ét}}^4(X_k^\circ \times \mathcal{X}_k^+, \mathbb{Q}_\ell(2))$, $p_2^* \alpha \in H_{\text{ét}}^2(X_k^\circ \times \mathcal{X}_k^+, \mathbb{Q}_\ell(1))$, $[\Gamma_{f,k}] \cup p_2^* \alpha \in H_{\text{ét}}^6(X_k^\circ \times \mathcal{X}_k^+, \mathbb{Q}_\ell(3))$, and $p_{1*}([\Gamma_{f,k}] \cup p_2^* \alpha) \in H_{\text{ét}}^2(X_k^\circ, \mathbb{Q}_\ell(1))$.

Similarly, we precompose $\tilde{s}_{f,\ell}$ with the pullback via f_k^{-1} to get an endomorphism $s_{f,\ell}$ of $H_{\text{ét}}^2(X_k^\circ, \mathbb{Q}_\ell(1))$,

$$s_{f,\ell} : H_{\text{ét}}^2(X_k^\circ, \mathbb{Q}_\ell(1)) \xrightarrow{(f_k^*)^{-1}} H_{\text{ét}}^2(\mathcal{X}_k^+, \mathbb{Q}_\ell(1)) \xrightarrow{\tilde{s}_{f,\ell}} H_{\text{ét}}^2(X_k^\circ, \mathbb{Q}_\ell(1)).$$

By [Mil80, Corollary 10.7], the cycle class map sends intersection products to cup products and is compatible with pullback. By [Mil80, Theorem 11.1], the cycle class map is compatible with the trace map, therefore compatible with pushforward. Hence, the following diagram commutes.

$$(3.1) \quad \begin{array}{ccccc} \text{Pic}(X_k^\circ)_\mathbb{Q} & \xrightarrow{(f_k^*)^{-1}} & \text{Pic}(\mathcal{X}_k^+)_\mathbb{Q} & \xrightarrow{\tilde{s}_f} & \text{Pic}(X_k^\circ)_\mathbb{Q} \\ \downarrow [-] & & \downarrow [-] & & \downarrow [-] \\ H_{\text{ét}}^2(X_k^\circ, \mathbb{Q}_\ell(1)) & \xrightarrow{(f_k^*)^{-1}} & H_{\text{ét}}^2(\mathcal{X}_k^+, \mathbb{Q}_\ell(1)) & \xrightarrow{\tilde{s}_{f,\ell}} & H_{\text{ét}}^2(X_k^\circ, \mathbb{Q}_\ell(1)) \end{array}$$

Also, we have the following characterization of $\tilde{s}_{f,\ell}$.

Lemma 3.2. *[LM17, Lemma 5.6] Let ϕ be the comparison isomorphism provided by the smooth base change theorem. Then the following diagram commutes.*

$$\begin{array}{ccc} H_{\text{ét}}^2(\mathcal{X}_k^+, \mathbb{Q}_\ell(1)) & \xrightarrow{\phi} & H_{\text{ét}}^2(\mathcal{X}_{K^s}^+, \mathbb{Q}_\ell(1)) \\ \tilde{s}_{f,\ell} \downarrow & & \downarrow f_K^* \\ H_{\text{ét}}^2(X_k^\circ, \mathbb{Q}_\ell(1)) & \xrightarrow{\phi} & H_{\text{ét}}^2(X_{K^s}, \mathbb{Q}_\ell(1)) \end{array}$$

Therefore, $s_{f,\ell}$ decomposes as follows

$$\begin{aligned} s_{f,\ell} : H_{\text{ét}}^2(X_k^\circ, \mathbb{Q}_\ell(1)) & \xrightarrow{(f_k^*)^{-1}} H_{\text{ét}}^2(\mathcal{X}_k^+, \mathbb{Q}_\ell(1)) \xrightarrow{\phi} H_{\text{ét}}^2(\mathcal{X}_{K^s}^+, \mathbb{Q}_\ell(1)) \\ & \xrightarrow{f_K^*} H_{\text{ét}}^2(X_{K^s}, \mathbb{Q}_\ell(1)) \xrightarrow{\phi^{-1}} H_{\text{ét}}^2(X_k^\circ, \mathbb{Q}_\ell(1)) \end{aligned}$$

Proposition 3.3. [c.f. CLL19, Proposition 7.4]

(1) For every $\alpha_1, \alpha_2 \in H_{\text{ét}}^2(X_k^\circ, \mathbb{Q}_\ell(1))$,

$$s_{f,\ell}(\alpha_1) \cup s_{f,\ell}(\alpha_2) = \alpha_1 \cup \alpha_2.$$

In particular, for every $D_1, D_2 \in \text{Pic}(X^\circ)_\mathbb{Q}$,

$$s_f(D_1) \cdot s_f(D_2) = D_1 \cdot D_2.$$

(2) For every two composable birational maps f and g ,

$$s_{g \cdot f, \ell} = s_{f, \ell} \cdot f_k^* \cdot s_{g, \ell} \cdot (f_k^*)^{-1},$$

$$s_{g \cdot f} = s_f \cdot f_k^* \cdot s_g \cdot (f_k^*)^{-1}.$$

(3) For every birational map f , $s_{f,\ell}$ and s_f are invertible with

$$(s_{f,\ell})^{-1} = f_k^* \cdot s_{f^{-1}, \ell} \cdot (f_k^*)^{-1},$$

$$(s_f)^{-1} = f_k^* \cdot s_{f^{-1}} \cdot (f_k^*)^{-1}.$$

Proof. This proposition generalizes [CLL19, Proposition 7.4] to Enriques surfaces essentially with the same proof if we replace $\text{Pic}(X^\circ)$ in [CLL19, Proposition 7.4] by $\text{Pic}(X^\circ)_\mathbb{Q}$.

In the commutative diagram 3.1, the vertical maps are injective. Therefore, it suffices to prove these for $s_{f,\ell}$. Also, (2) \implies (3), so we are left with (1) and (2). By Lemma 3.2, we only need to prove that pullback, pushforward, and the comparison isomorphism are compatible with cup products and composition. This is clear. \square

Let \mathcal{L} be a polarization of X . Let $f : \mathcal{X} \dashrightarrow \mathcal{X}^+$ be a birational map between two smooth models of X . We are interested in when s_f lies in the Weyl group $\mathcal{W}_{X,\mathcal{L}}$. This leads to the following definition.

Definition 3.4. A smooth model \mathcal{X} of X is \mathcal{L} -terminal if the specialization \mathcal{L}_k on \mathcal{X}_k is big and nef. Equivalently, we say $(\mathcal{X}, \mathcal{L})$ is terminal.

Remark 3.5. Let \mathcal{X} be an \mathcal{L} -terminal model of X . Recall that $E_{X,\mathcal{L}}$ is the exceptional divisors of the minimal desingularization $\mathcal{X}_k \rightarrow P(X, \mathcal{L})_k$ as in Section 2.2. We can identify $E_{X,\mathcal{L}}$ with the curves $E \subset \mathcal{X}_k$ with $\mathcal{L}_k \cdot E = 0$ by [LM17, Proposition 4.6(2)].

Remark 3.6. If X has good reduction, X always admits an \mathcal{L} -terminal model by [LM17, Proposition 4.5].

Definition 3.7. Suppose \mathcal{X} is \mathcal{L} -terminal and $f : \mathcal{X} \dashrightarrow \mathcal{X}^+$ is a birational map between smooth models. Denote the pushforward of \mathcal{L} to \mathcal{X}_K^+ by \mathcal{L}^+ . Then we say f is \mathcal{L} -terminal if \mathcal{X}^+ is \mathcal{L}^+ -terminal.

Proposition 3.8. *[c.f. CLL19, Proposition 7.5] Let \mathcal{X} be an \mathcal{L} -terminal model of X . Assume all irreducible components of $E_{X,\mathcal{L}}$ are geometrically irreducible. Then for any element $w \in \mathcal{W}_{X,\mathcal{L}}$, there exists an \mathcal{L} -terminal birational map $f : \mathcal{X} \dashrightarrow \mathcal{X}^+$ such that $s_f = w$.*

Proof. This proposition generalizes [CLL19, Proposition 7.5] to Enriques surfaces, essentially with the same proof. The only difference is that we need to replace $\text{Pic}(X^\circ)$ by $\text{Pic}(X^\circ)_\mathbb{Q}$. We give a sketch of the proof.

By Proposition 3.3(2), it suffices to prove this for $w = s_E$, for an irreducible component $E \subset E_{X,\mathcal{L}}$. Since E is a (-2) -curve, there exists a blow down $\mathcal{X} \rightarrow \mathcal{X}'$ of E , which contracts E to $y \in \mathcal{X}$ by [Bad01, Theorem 3.9]. Let $\widehat{\mathcal{X}}'_y = \text{Spf}(R)$ be the formal completion of \mathcal{X}' at y and let $\widehat{\mathcal{X}} \rightarrow \widehat{\mathcal{X}}'_y$ be the formal fibre. Following the proof of [LM17, Proposition 4.2], we find an automorphism $t : \widehat{\mathcal{X}}'_y \rightarrow \widehat{\mathcal{X}}'_y$ which induces -1 on the class group $\text{Cl}(R)$. By [Art70, Theorem 3.2], there exists a dilatation of algebraic spaces $f^+ : \mathcal{X}^+ \rightarrow \mathcal{X}'$, such that the formal completion of f^+ along its exceptional locus coincides with $\widehat{\mathcal{X}} \rightarrow \widehat{\mathcal{X}}'_y \xrightarrow{t} \widehat{\mathcal{X}}'_y$. This defines a birational map

$$f : \mathcal{X} \rightarrow \mathcal{X}' \xrightarrow{t^{-1}} \mathcal{X}^+.$$

We claim that $s_f = s_E$.

Let $\Gamma_{f,k}$ be the special fibre of the graph of f and let Γ_{f_k} be the graph of f_k . Since f is an isomorphism away from E , we see that

$$\Gamma_{f,k} = \Gamma_{f_k} + b(E \times f_k(E))$$

for some integer $b \geq 0$. Then

$$s_f(D) = D + b(E \cdot D)E.$$

Also, by Proposition 3.3(1), s_f preserves the intersection pairing, thus

$$\begin{aligned} E^2 &= s_f(E)^2 \\ \implies E^2 &= E^2 + 2b(E^2)^2 + b^2(E^2)^3 \\ \implies 2b(E^2)^2 + b^2(E^2)^3 &= 0. \end{aligned}$$

Setting $E^2 = -2$ yields $b = 0$ or 1 . If $b = 0$, f extends to an isomorphism. If $b = 1$, $s_f = s_E$. Then we prove $s_f = s_E$ by arguing that f does not extend to an isomorphism as in [CLL19, Proposition 7.5].

Lastly, we need to prove that f is \mathcal{L} -terminal. Since \mathcal{X} is \mathcal{L} -terminal, $\mathcal{L}_k \cdot E = 0$. We may choose a divisor corresponding to \mathcal{L} that does not intersect E . Since f is an isomorphism away from E , it follows that \mathcal{L}_k^+ is also big and nef. \square

Conversely, we have the following.

Proposition 3.9. [c.f. CLL19, Theorem 7.6] *Let $f : \mathcal{X} \dashrightarrow \mathcal{X}^+$ be an \mathcal{L} -terminal birational map. Then*

- (1) $s_f \in \mathcal{W}_{X,\mathcal{L}}$.
- (2) f is an isomorphism if and only if $s_f = \text{id}$.

Proof. Again, this proposition generalizes [CLL19, Theorem 7.6] to Enriques surfaces with the same proof. We give a sketch of the proof.

For a finite and unramified Galois extension L/K , we have $\mathcal{W}_{X,\mathcal{L}} = \mathcal{W}_{X_L,\mathcal{L}_L}^{G_{L/K}}$ by [CLL19, Corollary 3.3]. Also, whether f extends to an isomorphism can be detected over L . Therefore, we extend K so that the irreducible components of $E_i \subset E_{X,\mathcal{L}}$ are geometrically irreducible and the singularities of $P(X,\mathcal{L})_k$ are of type ADE.

Let X^+ be the generic fibre of \mathcal{X}^+ . Then f induces an isomorphism $f_K : X \rightarrow X^+$, therefore an isomorphism $P(X,\mathcal{L}) \rightarrow P(X^+,\mathcal{L}^+)$. Next, X° and \mathcal{X}_k^+ are respectively the minimal desingularizations of $P(X,\mathcal{L})_k$ and $P(X^+,\mathcal{L}^+)_k$ with exceptional divisors $E_{X,\mathcal{L}}$ and E_{X^+,\mathcal{L}^+} . Thus, f induces an isomorphism $\mathcal{X} \setminus E_{X,\mathcal{L}} \cong \mathcal{X}^+ \setminus E_{X^+,\mathcal{L}^+}$. Let p_1 and p_2 respectively be the projections from $X^\circ \times X^\circ$ to the two factors. Then we have $s_f : D \mapsto p_{1*}(\Gamma \cap p_2^*D)$ with

$$\Gamma = \Delta_{X^\circ} + \sum_{i,j} b_{ij}(E_i \times E_j)$$

for integers $b_{ij} \geq 0$. Therefore,

$$s_f(D) = D + \sum_{i,j} b_{ij}(D \cdot E_j)E_i.$$

In particular, $s_f \in \text{Aut}(\text{Pic}(X^\circ)_{\mathbb{Q}})$ preserves the integral lattice $\Lambda_{X,\mathcal{L}} \subset \text{Pic}(X^\circ)_{\mathbb{Q}}$. Also, the isomorphism $X^\circ \rightarrow \mathcal{X}_k^+$ maps the connected components of $E_{X,\mathcal{L}}$ to those of E_{X^+,\mathcal{L}^+} , so $b_{ij} = 0$ if E_i and E_j are in different connected components. Then $s_f|_{\Lambda_{X,\mathcal{L}}}$ preserves every sub-lattice $\Lambda_l \subset E_{X,\mathcal{L}}$ corresponding to each connected component of $E_{X,\mathcal{L}}$. By Proposition 3.3(3), the same thing holds for $s_{f^{-1}}$.

Moreover, by Proposition 3.3(1), s_f preserves intersection pairing, so we write

$$s_f|_{\Lambda_{X,\mathcal{L}}} = \prod_l \tilde{w}_l$$

for $\tilde{w}_l \in O(\Lambda_l)$ in the orthogonal group of every sub-lattice Λ_l . By [KM83, Theorem 1], for every l , $\tilde{w}_l = w_l \alpha_l$ where w_l is in the Weyl group of Λ_l and α_l induced by an automorphism of the Dynkin diagram of Λ_l . By Proposition 3.8, we may absorb $\prod_l w_l$ into s_f and assume

$$s_f|_{\Lambda_{X,\mathcal{L}}} = \prod_l \alpha_l.$$

To prove (1), it suffices to prove that $s_f = \text{id}$. Let A be the intersection matrix A of $E_{X,\mathcal{L}}$ and let $B = \{b_{ij}\}$. By [CLL19, Lemma 3.5], $s_f|_{\Lambda_{X,\mathcal{L}}} = \text{id}$. Therefore, $BA = 0$. Since A is negative definite, $B = 0$ and $s_f = \text{id}$.

The proof of (2) is similar. If f is an isomorphism, $s_f = \text{id}$. Conversely, $s_f = \text{id}$ implies that $BA = 0$. Since A is negative definite, $A = 0$ and f is an isomorphism. \square

Given Proposition 3.9, we can remove the assumption in Proposition 3.8 that all irreducible components of $E_{X,\mathcal{L}}$ are geometrically irreducible.

Corollary 3.10. *Let \mathcal{X} be an \mathcal{L} -terminal model of X . For any $w \in \mathcal{W}_{X,\mathcal{L}}$, there exists an \mathcal{L} -terminal birational map $f : \mathcal{X} \dashrightarrow \mathcal{X}^+$ such that $s_f = w$.*

Proof. This is an application of Proposition 3.8 and Proposition 3.9. The proof is the same as [CLL19, Corollary 7.7]. \square

3.2. Extendability of group actions. The constructions in this sub-section are based on [CLL19, Section 7.2]. We want to apply the results from the previous sub-section to deduce a criterion for the extendability of group actions. We begin with some terminology.

Let G be a group acting on the right on a proper variety X (finite type, geometrically integral, and separated) over a field F .

For every $g \in G$, g induces an automorphism of F by pulling back the global sections. This defines a homomorphism $\sigma : G \rightarrow \text{Aut}(F)$. Denote by σ_g the automorphism induced by g on $\text{Spec } F$.

Definition 3.11. The base field of the G -action is the fixed field of $\sigma(G) \subset \text{Aut}(F)$.

Let $g \in G$. Then g is an automorphism of X semi-linear with respect to σ_g . Equivalently, denote by X^{σ_g} the base change of X by σ_g . Then the induced isomorphism $g^\# : X \rightarrow X^{\sigma_g}$ is F -linear. In short, we have the following commutative diagram, where the square is Cartesian.

$$\begin{array}{ccccc} X & \xrightarrow{g^\#} & X^{\sigma_g} & \longrightarrow & \text{Spec } F \\ & \searrow g & \downarrow & & \downarrow \sigma_g \\ & & X & \longrightarrow & \text{Spec } F \end{array}$$

Now we return to the case where X is a K3 or Enriques surface over K with good reduction. Let \mathcal{X} be a smooth model of X . Let G be a finite group acting on X on the right with base field K_0 .

Assumption 3.12. We assume that K/K_0 is finite and unramified.

Let $g \in G$. As above, we have a factorization $g : X \xrightarrow{g^\#} X^{\sigma_g} \rightarrow X$. The second isomorphism is induced by σ_g . This obviously extends to an isomorphism $\mathcal{X}^{\sigma_g} \rightarrow \mathcal{X}$, where we abuse notation and write σ_g also for the induced automorphism of $\text{Spec } \mathcal{O}_K$. Therefore, the birational map $g_{\mathcal{X}} : \mathcal{X} \dashrightarrow \mathcal{X}$ induced by g extends to an automorphism if and only if the birational map $g_{\mathcal{X}}^\# : \mathcal{X} \dashrightarrow \mathcal{X}^{\sigma_g}$ induced by $g^\#$ extends to an isomorphism.

To apply results from Section 3.1, we want to find some polarization \mathcal{L} of X and an \mathcal{L} -terminal model \mathcal{X} , such that $s_{g_{\mathcal{X}}^\#} \in \mathcal{W}_{X, \mathcal{L}}$ for every $g \in G$.

Lemma 3.13. *Suppose \mathcal{L} is a polarization of X fixed by the G -action. Let \mathcal{X} be an \mathcal{L} -terminal model. Then for every $g \in G$, $s_{g_{\mathcal{X}}^\#} \in \mathcal{W}_{X, \mathcal{L}}$.*

Proof. Note that $g^\# : \mathcal{X} \dashrightarrow \mathcal{X}^{\sigma_g}$ factors as $\mathcal{X} \xrightarrow{g} \mathcal{X} \rightarrow \mathcal{X}^{\sigma_g}$, where the second map is induced by σ_g^{-1} . The second map is just base change, so it is \mathcal{L} -terminal for any terminal pair $(\mathcal{X}, \mathcal{L})$. Also, by assumption, \mathcal{L} is fixed by the G -action, so the first map is \mathcal{L} -terminal. This proves that $g_{\mathcal{X}}^\#$ is \mathcal{L} -terminal. Then we are done by Proposition 3.9. \square

Remark 3.14. Since G is finite, the G -action fixes some polarization. Indeed, given any polarization \mathcal{L} , the tensor product $\bigotimes_{g \in G} g^* \mathcal{L}$ is G -invariant and hence defines a polarization fixed by G .

Recall that, via the specialization homomorphism, G also acts on X° on the right. Therefore, G acts on $\text{Pic}(X^\circ)$ on the left via pullback, and on $\text{Aut}(\text{Pic}(X^\circ))$ on the left via conjugation.

Equivalently, the G -action on $\text{Aut}(\text{Pic}(X^\circ))$ can be described as follows. We abuse notation and write σ_g both for the automorphisms of $\text{Spec } K$ and $\text{Spec } k$ induced respectively by $g \in \text{Aut}(X)$ and $g_k \in \text{Aut}(X^\circ)$. Given $s \in \text{Aut}(\text{Pic}(X^\circ))_{\mathbb{Q}}$, let $s^{\sigma_g} \in \text{Aut}(\text{Pic}((X^\circ)^{\sigma_g})_{\mathbb{Q}})$ be the base change of s by $\sigma_g \in \text{Aut}(\text{Spec } k)$, i.e. s^{σ_g} is the unique automorphism of $\text{Pic}((X^\circ)^{\sigma_g})_{\mathbb{Q}}$ that makes the following diagram commute.

$$\begin{array}{ccc} \text{Pic}(X^\circ)_{\mathbb{Q}} & \xrightarrow{s} & \text{Pic}(X^\circ)_{\mathbb{Q}} \\ \downarrow \sigma_g^* & & \downarrow \sigma_g^* \\ \text{Pic}((X^\circ)^{\sigma_g})_{\mathbb{Q}} & \xrightarrow{s^{\sigma_g}} & \text{Pic}((X^\circ)^{\sigma_g})_{\mathbb{Q}} \end{array}$$

We claim that g acts on $\text{Aut}(\text{Pic}(X^\circ)_{\mathbb{Q}})$ equivalently via

$$s \mapsto (g_k^\#)^* \circ s^{\sigma_g} \circ ((g_k^\#)^*)^{-1}.$$

To see this, we need to prove

$$(3.15) \quad g_k^* \circ s \circ (g_k^*)^{-1} = (g_k^\#)^* \circ s^{\sigma_g} \circ ((g_k^\#)^*)^{-1}.$$

Since $g_k = \sigma_g \circ g^\#$, we have $g_k^* = (g^\#)^* \circ \sigma_g^*$. Then the LHS and RHS of 3.15 fit in the following diagram.

$$\begin{array}{ccccc}
\mathrm{Pic}(X^\circ)_\mathbb{Q} & \xrightarrow{((g^\#)^*)^{-1}} & \mathrm{Pic}((X^\circ)^{\sigma_g})_\mathbb{Q} & \xrightarrow{(\sigma_g^*)^{-1}} & \mathrm{Pic}(X^\circ)_\mathbb{Q} \\
& & \downarrow s^{\sigma_g} & & \downarrow s \\
\mathrm{Pic}(X^\circ)_\mathbb{Q} & \xleftarrow{(g^\#)^*} & \mathrm{Pic}((X^\circ)^{\sigma_g})_\mathbb{Q} & \xleftarrow{\sigma_g^*} & \mathrm{Pic}(X^\circ)_\mathbb{Q}
\end{array}$$

To prove 3.15, it amounts to showing the square in the diagram commutes, which is immediate by the construction of s^{σ_g} .

Let \mathcal{L} be a polarization of X fixed by the G -action and let $g \in G$. Denote by \mathcal{L}^{σ_g} the base change of \mathcal{L} by σ_g . Then $g^\#$ induces an isomorphism $P(X, \mathcal{L}) \rightarrow P(X^{\sigma_g}, \mathcal{L}^{\sigma_g})$, therefore an isomorphism of their desingularizations $X^\circ \rightarrow (X^\circ)^{\sigma_g}$. In particular, $g^\#$ maps the exceptional divisors of $P(X, \mathcal{L})$ to those of $P(X^{\sigma_g}, \mathcal{L}^{\sigma_g})$. This implies that the G -action on $\mathrm{Aut}(\mathrm{Pic}(X^\circ)_\mathbb{Q})$ restricts to a G -action on $\mathcal{W}_{X, \mathcal{L}}$.

Lemma 3.16. *For any \mathcal{L} -terminal model, the function*

$$\alpha_{\mathcal{L}, \mathcal{X}} : G \rightarrow \mathcal{W}_{X, \mathcal{L}} \quad g \mapsto s_{g_{\mathcal{X}}}^\#$$

defines a 1-cocycle of G with values in $\mathcal{W}_{X, \mathcal{L}}$.

Proof. Let $g, h \in G$. We abuse notation and write σ_g also for the induced base change isomorphism $X^{\sigma_g} \rightarrow X$. Let $(h^\#)^{\sigma_g} = \sigma_g^{-1} \circ h^\# \circ \sigma_g$ be the base change of $h^\#$ by σ_g , i.e. the following diagram commutes.

$$\begin{array}{ccc}
X^{\sigma_g} & \xrightarrow{(h^\#)^{\sigma_g}} & (X^{\sigma_h})^{\sigma_g} \\
\downarrow \sigma_g & & \downarrow \sigma_g \\
X & \xrightarrow{h^\#} & X^{\sigma_h}
\end{array}$$

We use the subscript \mathcal{X} to denote the birational map induced on \mathcal{X} . The construction of $s_{h_{\mathcal{X}}}^\#$ is compatible with base change, so we have $s_{(h_{\mathcal{X}})^{\sigma_g}} = s_{(h_{\mathcal{X}})}^{\sigma_g}$.

Also, by construction, we have $\sigma_{h \circ g} = \sigma_h \circ \sigma_g$. Therefore,

$$\begin{aligned}
(h^\#)^{\sigma_g} \circ g^\# &= \sigma_g^{-1} \circ h^\# \circ \sigma_g \circ g^\# \\
&= \sigma_g^{-1} \circ (\sigma_h^{-1} \circ h) \circ \sigma_g \circ (\sigma_g^{-1} \circ g) \\
&= \sigma_{h \circ g}^{-1} \circ (h \circ g) = (h \circ g)^\#.
\end{aligned}$$

In particular, we have $(h_{\mathcal{X}}^\#)^{\sigma_g} \circ g_{\mathcal{X}}^\# = (h \circ g)_{\mathcal{X}}^\#$.

Then we compute

$$\begin{aligned}
\alpha_{\mathcal{L}, \mathcal{X}}(gh) &= s_{(hg)_{\mathcal{X}}}^\# = s_{(h_{\mathcal{X}}^\#)^{\sigma_g} \circ g_{\mathcal{X}}^\#} \\
&= s_{g_{\mathcal{X}}}^\# \circ (g_{\mathcal{X}}^\#)^* \circ s_{(h_{\mathcal{X}}^\#)^{\sigma_g}} \circ ((g_{\mathcal{X}}^\#)^*)^{-1}
\end{aligned}$$

$$\begin{aligned}
&= s_{g_{\mathcal{X}}^{\#}} \circ (g_k^{\#})^* \circ s_{h_{\mathcal{X}}^{\#}}^{\sigma_g} \circ ((g_k^{\#})^*)^{-1} \\
&= \alpha_{\mathcal{L}, \mathcal{X}}(g) \circ g(\alpha_{\mathcal{L}, \mathcal{X}}(h)).
\end{aligned}$$

This proves that $\alpha_{\mathcal{L}, \mathcal{X}}$ is a 1-cocycle. \square

Corollary 3.17. *The G -action on X extends to \mathcal{X} if and only if the 1-cocycle $\alpha_{\mathcal{L}, \mathcal{X}}$ is trivial.*

Proof. This follows by Lemma 3.16 and Proposition 3.9. \square

Another natural question is how $\alpha_{\mathcal{L}, \mathcal{X}}$ changes as we choose different \mathcal{X} .

Lemma 3.18. *Let $\mathcal{X}, \mathcal{X}^+$ be two \mathcal{L} -terminal models and $f : \mathcal{X} \dashrightarrow \mathcal{X}^+$ be the birational map induced by the identity map on X . Then for every $g \in G$,*

$$\alpha_{\mathcal{L}, \mathcal{X}^+}(g) = s_f^{-1} \circ \alpha_{\mathcal{L}, \mathcal{X}}(g) \circ g(s_f).$$

In particular, the cohomology class represented by $\alpha_{\mathcal{L}, \mathcal{X}}$ is independent of the choice of \mathcal{L} -terminal model \mathcal{X} . We denote this class by $\alpha_{\mathcal{L}}$.

Proof. View $\alpha_{\mathcal{L}, \mathcal{X}^+}$ as a function

$$\alpha_{\mathcal{L}, \mathcal{X}^+} : G \rightarrow \text{Aut}(\text{Pic}(\mathcal{X}_k^+)),$$

i.e. we don't identify $X^\circ = \mathcal{X}_k$ with \mathcal{X}_k^+ . Then we want to prove

$$f_k^* \circ \alpha_{\mathcal{L}, \mathcal{X}^+}(g) \circ (f_k^*)^{-1} = s_f^{-1} \circ \alpha_{\mathcal{L}, \mathcal{X}}(g) \circ g(s_f).$$

Denote by $g_{\mathcal{X}^+}$ the birational self-map on \mathcal{X}^+ induced by g and by f^{σ_g} the base change of f by σ_g . Then the following diagram commutes.

$$\begin{array}{ccc}
\mathcal{X} & \xrightarrow{g_{\mathcal{X}}^{\#}} & \mathcal{X}^{\sigma_g} \\
\downarrow f & & \downarrow f^{\sigma_g} \\
\mathcal{X}^+ & \xrightarrow{g_{\mathcal{X}^+}^{\#}} & (\mathcal{X}^+)^{\sigma_g}
\end{array}$$

In particular, $s_{f^{\sigma_g} \circ g_{\mathcal{X}}^{\#}} = s_{g_{\mathcal{X}^+}^{\#} \circ f}$. We compute

$$\begin{aligned}
\alpha_{\mathcal{L}, \mathcal{X}}(g) \circ g(s_f) &= s_{g_{\mathcal{X}}^{\#}} \circ (g_k^{\#})^* \circ s_{f^{\sigma_g}} \circ ((g_k^{\#})^*)^{-1} \\
&= s_{f^{\sigma_g} \circ g_{\mathcal{X}}^{\#}} = s_{g_{\mathcal{X}^+}^{\#} \circ f} \\
&= s_f \circ f_k^* \circ \alpha_{\mathcal{L}, \mathcal{X}^+}(g) \circ (f_k^*)^{-1}
\end{aligned}$$

This finishes the proof. \square

This gives a criterion for the extendability of group actions.

Theorem 3.19. *Let X be a K3 or Enriques surface over K with good reduction. Let G be a finite group acting on X with base field K_0 , such that K/K_0 is finite and unramified. Then the following are equivalent.*

- (1) *The action of G is extendable.*
- (2) *The cohomology class $\alpha_{\mathcal{L}}$ is trivial for some polarization \mathcal{L} fixed by G .*
- (3) *The cohomology class $\alpha_{\mathcal{L}}$ is trivial for every polarization \mathcal{L} fixed by G .*

Proof. Since (3) \implies (2), it suffices to prove that (1) \implies (3) and (2) \implies (1). Assume the action of G is extendable and let \mathcal{L} be a polarization of X fixed by G . Then the action extends to some \mathcal{L} -terminal model by [Mat16, Proposition 4.6], which holds for group actions that are not necessarily linear without any change to the proof. This shows (1) \implies (3).

Next, we prove (2) \implies (1). Suppose $\alpha_{\mathcal{L}}$ is trivial for some \mathcal{L} fixed by G . Let \mathcal{X} be an \mathcal{L} -terminal model of X . Then there exists some $w \in \mathcal{W}_{X,\mathcal{L}}$ such that

$$\alpha_{\mathcal{L},\mathcal{X}}(g) = w^{-1}g(w).$$

By Corollary 3.10, let $f : \mathcal{X} \dashrightarrow \mathcal{X}^+$ be an \mathcal{L} -terminal birational map such that $w^{-1} = s_f$. Then for every $g \in G$,

$$\alpha_{\mathcal{L},\mathcal{X}^+}(g) = s_f^{-1} \cdot \alpha_{\mathcal{L},\mathcal{X}}(g) \cdot g(s_f) = \text{id}.$$

This proves that the action of G extends to \mathcal{X}^+ . \square

4. EXTENDABILITY AND ℓ -ADIC COHOMOLOGY

This section is based on [CLL19, Section 8 and 9]. In this section, we define the ℓ -adic realization of $\alpha_{\mathcal{L}}$ and prove Theorem 1.5 and Theorem 1.7. Let X be a K3 or Enriques surface over K with good reduction. Let G be a finite group acting on X with base field K_0 , such that K/K_0 is finite and unramified. Denote by $\sigma : G \rightarrow G_{K/K_0}$ the action of G on K . Let \mathcal{L} be a polarization on X fixed by the G -action. Recall that we denote the subgroup of $\text{Pic}(X^\circ)$ generated by the exceptional divisors of the minimal desingularization $X^\circ \rightarrow P(X, \mathcal{L})_k$ by $\Lambda_{X,\mathcal{L}}$. Let $\Lambda_{X,\mathcal{L},\ell} = \Lambda_{X,\mathcal{L}} \otimes \mathbb{Q}_\ell$.

4.1. G -action and ℓ -adic cohomology. Note that since G does not necessarily act K -linearly on X , there is no natural G -action on X_{K^s} . Instead, we consider the fibre product $\overline{G} = G \times_{G_{K/K_0}} G_{K_0}$ of $\sigma : G \rightarrow G_{K/K_0}$ and the restriction homomorphism $G_{K_0} \rightarrow G_{K/K_0}$. We define a right \overline{G} -action on X_{K^s} by letting G act on X and G_{K_0} act on $\text{Spec } K^s$. Therefore, \overline{G} acts on $H_{\text{ét}}^2(X_{K^s}, \mathbb{Q}_\ell(1))$ on the left via pullback, and on $\text{Aut}(H_{\text{ét}}^2(X_{K^s}, \mathbb{Q}_\ell(1)))$ on the left via conjugation.

Via specialization, G also acts on X° . Let k_0 be the residue field of K_0 . We form the fibre product $\overline{G}_k = G \times_{G_{k/k_0}} G_{k_0}$. Similarly, \overline{G}_k acts on X_k° , $H_{\text{ét}}^2(X_k^\circ, \mathbb{Q}_\ell(1))$ and $\text{Aut}(H_{\text{ét}}^2(X_k^\circ, \mathbb{Q}_\ell(1)))$.

Moreover, the natural surjection $G_{K_0} \rightarrow G_{k_0}$ induces a surjection $\overline{G} \rightarrow \overline{G}_k$ with kernel I_K . Since X has good reduction, I_K acts trivially on $H_{\text{ét}}^2(X_{K^s}, \mathbb{Q}_\ell(1))$. Therefore, $\overline{G}_k \cong \overline{G}/I_K$ acts on $H_{\text{ét}}^2(X_{K^s}, \mathbb{Q}_\ell(1))$.

Note that we have the following short exact sequence

$$(4.1) \quad 1 \rightarrow G_k \rightarrow \overline{G}_k \rightarrow G \rightarrow 1$$

where the first homomorphism maps σ to (id, σ) and the second homomorphism is projection. Then G acts on $\text{Aut}_{G_k}(H_{\text{ét}}^2(X_{K^s}, \mathbb{Q}_\ell(1)))$, $\text{Aut}_{G_k}(H_{\text{ét}}^2(X_k^\circ, \mathbb{Q}_\ell(1)))$, $\text{Aut}(H_{\text{ét}}^2(X_{K^s}, \mathbb{Q}_\ell(1))^{G_k})$, and $\text{Aut}(H_{\text{ét}}^2(X_k^\circ, \mathbb{Q}_\ell(1))^{G_k})$,

4.2. ℓ -adic realization of $\alpha_{\mathcal{L}}$.

Lemma 4.2. *There exists a G -equivariant group homomorphism*

$$i_\ell : \mathcal{W}_{X, \mathcal{L}} \rightarrow \text{Aut}_{G_k}(H_{\text{ét}}^2(X_k^\circ, \mathbb{Q}_\ell(1)))$$

such that for every irreducible component $E \subset E_{X, \mathcal{L}}$, s_E maps to the automorphism

$$H_{\text{ét}}^2(X_k^\circ, \mathbb{Q}_\ell(1)) \rightarrow H_{\text{ét}}^2(X_k^\circ, \mathbb{Q}_\ell(1)) \quad \alpha \mapsto \alpha + ([E] \cup \alpha)[E].$$

Proof. We construct such i_ℓ as follows. There is a natural action of $\mathcal{W}_{X, \mathcal{L}}$ on $\Lambda_{X, \mathcal{L}, \mathbb{Q}_\ell}$, which embeds into $H_{\text{ét}}^2(X_k^\circ, \mathbb{Q}_\ell(1))$. Let $\Lambda_{X, \mathcal{L}, \mathbb{Q}_\ell}^\perp$ be the orthogonal complement of $\Lambda_{X, \mathcal{L}, \mathbb{Q}_\ell}$ with respect to the cup product. Since the intersection pairing matrix of $\Lambda_{X, \mathcal{L}, \mathbb{Q}_\ell}$ is negative definite, $H_{\text{ét}}^2(X_k^\circ, \mathbb{Q}_\ell(1)) \cong \Lambda_{X, \mathcal{L}, \mathbb{Q}_\ell} \oplus \Lambda_{X, \mathcal{L}, \mathbb{Q}_\ell}^\perp$.

We extend the action of $\mathcal{W}_{X, \mathcal{L}}$ on $\Lambda_{X, \mathcal{L}, \mathbb{Q}_\ell}$ to $H_{\text{ét}}^2(X_k^\circ, \mathbb{Q}_\ell(1))$ by letting $\mathcal{W}_{X, \mathcal{L}}$ act trivially on $\Lambda_{X, \mathcal{L}, \mathbb{Q}_\ell}^\perp$. This defines $i_\ell : \mathcal{W}_{X, \mathcal{L}} \rightarrow \text{Aut}_{G_k}(H_{\text{ét}}^2(X_k^\circ, \mathbb{Q}_\ell(1)))$. We need to check that it has the desired property.

Let $\alpha \in H_{\text{ét}}^2(X_k^\circ, \mathbb{Q}_\ell(1))$ which decomposes uniquely as $[E'] + \alpha'$ where $E' \in \Lambda_{X, \mathcal{L}, \mathbb{Q}_\ell}$ and $\alpha' \in \Lambda_{X, \mathcal{L}, \mathbb{Q}_\ell}^\perp$. Then for every irreducible component $E \subset E_{X, \mathcal{L}}$,

$$s_E(\alpha) = [E'] + (E.E')[E] + \alpha' = \alpha + (E.E')[E] = \alpha + ([E] \cup \alpha)[E].$$

This finishes the proof. \square

Proposition 4.3. *[c.f. CLL19, Theorem 8.4] Let X be a K3 or Enriques surface over K with good reduction. Let \mathcal{L} be a polarization of X and $f : \mathcal{X} \dashrightarrow \mathcal{X}^+$ an \mathcal{L} -terminal birational map between smooth models of X . Then*

$$i_\ell(s_f) = s_{f, \ell}.$$

Proof. We first prove that $i_\ell(s_f)$ and $s_{f, \ell}$ agree on the subspace $\Lambda_{X, \mathcal{L}, \mathbb{Q}_\ell}$. By the commutative diagram 3.1, for any $D \in \Lambda_{X, \mathcal{L}, \mathbb{Q}_\ell}$, $s_{f, \ell}([D]) = [s_f(D)]$. We need to

show that

$$[s_f(D)] = i_\ell(s_f)([D]).$$

Since $s_f \in \mathcal{W}_{X,\mathcal{L}}$, it suffices to check this for $s_f = s_E$. This is obvious by the construction of i_ℓ .

It remains to prove that $i_\ell(s_f)$ and $s_{f,\ell}$ agree on $\Lambda_{X,\mathcal{L},\mathbb{Q}_\ell}^\perp$. Indeed, both of them are the identity on $\Lambda_{X,\mathcal{L},\mathbb{Q}_\ell}^\perp$ by the constructions of i_ℓ and $s_{f,\ell}$. \square

4.3. Proof of Theorem 1.5. Define j_ℓ as the composition of i_ℓ with the restriction $\text{Aut}_{G_k}(H_{\text{ét}}^2(X_k^\circ, \mathbb{Q}_\ell(1))) \rightarrow \text{Aut}(H_{\text{ét}}^2(X_k^\circ, \mathbb{Q}_\ell(1))^{G_k})$,

$$j_\ell : \mathcal{W}_{X,\mathcal{L}} \xrightarrow{i_\ell} \text{Aut}_{G_k}(H_{\text{ét}}^2(X_k^\circ, \mathbb{Q}_\ell(1))) \rightarrow \text{Aut}(H_{\text{ét}}^2(X_k^\circ, \mathbb{Q}_\ell(1))^{G_k}).$$

Then j_ℓ induces a pointed map

$$j_{\ell*} : H^1(G, \mathcal{W}_{X,\mathcal{L}}) \rightarrow H^1(G, \text{Aut}(H_{\text{ét}}^2(X_k^\circ, \mathbb{Q}_\ell(1))^{G_k})).$$

Let $\beta_{\mathcal{L},\ell} = j_{\ell*}(\alpha_{\mathcal{L},\ell})$.

Lemma 4.4. *The pointed map $j_{\ell*}$ has trivial kernel.*

Proof. Denote the action of $\mathcal{W}_{X,\mathcal{L}}$ on $\Lambda_{X,\mathcal{L},\mathbb{Q}_\ell}$ by $j'_\ell : \mathcal{W}_{X,\mathcal{L}} \rightarrow \text{Aut}(\Lambda_{X,\mathcal{L},\mathbb{Q}_\ell})$, so we have a pointed map

$$j'_{\ell*} : H^1(G, \mathcal{W}_{X,\mathcal{L}}) \rightarrow H^1(G, \text{Aut}(\Lambda_{X,\mathcal{L},\mathbb{Q}_\ell})).$$

By [CLL19, Theorem 4.1], $j'_{\ell*}$ has trivial kernel, so it suffices to prove that $j_{\ell*}$ and $j'_{\ell*}$ have the same kernel.

The cycle class map gives a G -injection $\Lambda_{X,\mathcal{L},\mathbb{Q}_\ell} \rightarrow H_{\text{ét}}^2(X_k^\circ, \mathbb{Q}_\ell(1))^{G_k}$. Denote by T_ℓ the quotient and consider the short exact sequence of G -representations

$$0 \rightarrow \Lambda_{X,\mathcal{L},\mathbb{Q}_\ell} \rightarrow H_{\text{ét}}^2(X_k^\circ, \mathbb{Q}_\ell(1))^{G_k} \rightarrow T_\ell \rightarrow 0.$$

Suppose $\alpha \in H^1(G, \mathcal{W}_{X,\mathcal{L}})$. By Proposition 2.8,

$$j'_{\ell*}(\alpha) \text{ is trivial} \iff \Lambda_{X,\mathcal{L},\mathbb{Q}_\ell} \cong \Lambda_{X,\mathcal{L},\mathbb{Q}_\ell}^\alpha,$$

$$j_{\ell*}(\alpha) \text{ is trivial} \iff H_{\text{ét}}^2(X_k^\circ, \mathbb{Q}_\ell(1))^{G_k} \cong H_{\text{ét}}^2(X_k^\circ, \mathbb{Q}_\ell(1))^{G_k,\alpha}.$$

Next, by the construction of the action of $\mathcal{W}_{X,\mathcal{L}}$ on $H_{\text{ét}}^2(X_k^\circ, \mathbb{Q}_\ell(1))$, $\mathcal{W}_{X,\mathcal{L}}$ acts trivially on T_ℓ , so $T_\ell \cong T_\ell^\alpha$. Finally, by the semi-simplicity of the representations of the finite group G , we have

$$\Lambda_{X,\mathcal{L},\mathbb{Q}_\ell} \cong \Lambda_{X,\mathcal{L},\mathbb{Q}_\ell}^\alpha \iff H_{\text{ét}}^2(X_k^\circ, \mathbb{Q}_\ell(1))^{G_k} \cong H_{\text{ét}}^2(X_k^\circ, \mathbb{Q}_\ell(1))^{G_k,\alpha}.$$

This finishes the proof. \square

In particular, $\beta_{\mathcal{L},\ell}$ controls the extendability of the G -action by Theorem 3.19.

Proposition 4.5. *[c.f. CLL19, Theorem 8.6] There is a G -isomorphism*

$$H_{\text{ét}}^2(X_{K^s}, \mathbb{Q}_\ell(1))^{G_k} \cong H_{\text{ét}}^2(X_{\bar{k}}^\circ, \mathbb{Q}_\ell(1))^{G_k, \beta_{\mathcal{L}, \ell}}.$$

Proof. Since X has good reduction over K , we have a G_k -equivariant isomorphism

$$\phi : H_{\text{ét}}^2(X_{\bar{k}}^\circ, \mathbb{Q}_\ell(1)) \rightarrow H_{\text{ét}}^2(X_{K^s}, \mathbb{Q}_\ell(1)),$$

which restricts to an isomorphism $H_{\text{ét}}^2(X_{\bar{k}}^\circ, \mathbb{Q}_\ell(1))^{G_k} \rightarrow H_{\text{ét}}^2(X_{K^s}, \mathbb{Q}_\ell(1))^{G_k}$. We claim that this gives a G -isomorphism if we twist $H_{\text{ét}}^2(X_{\bar{k}}^\circ, \mathbb{Q}_\ell(1))^{G_k}$ by $\beta_{\mathcal{L}, \ell}$. Let $g \in G$ and let \bar{g} be any lift of g to \bar{G}_k . We need to show that

$$\phi \circ \beta_{\mathcal{L}, \ell}(g) \circ \bar{g}_k^* = \bar{g}^* \circ \phi.$$

By Proposition 4.3 and Lemma 3.2,

$$\beta_{\mathcal{L}, \ell}(g) = (i_\ell \circ \alpha_{\mathcal{L}})(g) = i_\ell(s_{g_{\mathcal{X}}^\#}) = s_{g_{\mathcal{X}}^\#, \ell} = \phi^{-1} \circ (g^\#)^* \circ \phi \circ ((g_k^\#)^*)^{-1}.$$

It suffices to prove that

$$(4.6) \quad \bar{g}^* \circ \phi \circ (\bar{g}_k^*)^{-1} = (g^\#)^* \circ \phi \circ ((g_k^\#)^*)^{-1}.$$

Let $\bar{\sigma}_g$ be any lift of $\sigma_g \in G_{k/k_0}$ to G_{k_0} , so $(g, \bar{\sigma}_g)$ lifts g to \bar{G}_k . Therefore, $\bar{g}^* = (g^\#)^* \circ \bar{\sigma}_g^*$ and $\bar{g}_k^* = (g_k^\#)^* \circ \bar{\sigma}_g^*$. Then the LHS and the RHS of 4.6 fit in the following diagram

$$\begin{array}{ccccc} H_{\text{ét}}^2(X_{\bar{k}}^\circ, \mathbb{Q}_\ell(1)) & \xrightarrow{((g^\#)^*)^{-1}} & H_{\text{ét}}^2((X_{\bar{k}}^\circ)^{\bar{\sigma}_g}, \mathbb{Q}_\ell(1)) & \xrightarrow{(\bar{\sigma}_g^*)^{-1}} & H_{\text{ét}}^2(X_{\bar{k}}^\circ, \mathbb{Q}_\ell(1)) \\ & & \downarrow \phi & & \downarrow \phi \\ H_{\text{ét}}^2(X_{\bar{K}}, \mathbb{Q}_\ell(1)) & \xleftarrow{(g^\#)^*} & H_{\text{ét}}^2(X_{\bar{K}}^{\bar{\sigma}_g}, \mathbb{Q}_\ell(1)) & \xleftarrow{\bar{\sigma}_g^*} & H_{\text{ét}}^2(X_{\bar{K}}, \mathbb{Q}_\ell(1)) \end{array}$$

To prove 4.6, it suffices to show that the square commutes, which is obvious. \square

Now we obtain the following theorem.

Theorem 4.7. *Let X be a K3 or Enriques surface over K with good reduction. Let G be a finite group acting on X with base field K_0 . Assume K/K_0 is finite and unramified. Then the following are equivalent.*

- (1) *The G -action on X is extendable.*
- (2) *There is a \bar{G}_k -isomorphism*

$$H_{\text{ét}}^2(X_{K^s}, \mathbb{Q}_\ell(1)) \cong H_{\text{ét}}^2(X_{\bar{k}}^\circ, \mathbb{Q}_\ell(1)).$$

- (3) *There is a G -isomorphism*

$$H_{\text{ét}}^2(X_{K^s}, \mathbb{Q}_\ell(1))^{G_k} \cong H_{\text{ét}}^2(X_{\bar{k}}^\circ, \mathbb{Q}_\ell(1))^{G_k}.$$

Proof. Suppose the G -action on X extends to a smooth model \mathcal{X} of X . The comparison isomorphism $H_{\text{ét}}^2(X_{K^s}, \mathbb{Q}_\ell(1)) \cong H_{\text{ét}}^2(X_{\bar{k}}^\circ, \mathbb{Q}_\ell(1))$ given by \mathcal{X} is equivariant with respect to the action of \overline{G}_k . Therefore, (1) \implies (2).

Also, the restriction of a \overline{G}_k -isomorphism $H_{\text{ét}}^2(X_{K^s}, \mathbb{Q}_\ell(1)) \cong H_{\text{ét}}^2(X_{\bar{k}}^\circ, \mathbb{Q}_\ell(1))$ to the G_k -invariant subspace is exactly a G -isomorphism $H_{\text{ét}}^2(X_{K^s}, \mathbb{Q}_\ell(1))^{G_k} \cong H_{\text{ét}}^2(X_{\bar{k}}^\circ, \mathbb{Q}_\ell(1))^{G_k}$. Therefore, (2) \implies (3).

Lastly, let \mathcal{L} be a polarization on X fixed by G . We have

$$\begin{aligned} H_{\text{ét}}^2(X_{\bar{k}}^\circ, \mathbb{Q}_\ell(1))^{G_k} &\cong H_{\text{ét}}^2(X_{K^s}, \mathbb{Q}_\ell(1))^{G_k} \text{ as } G\text{-representations} \\ \iff \beta_{\mathcal{L}, \ell} \text{ is trivial} &\iff \alpha_{\mathcal{L}, \ell} \text{ is trivial} \iff \text{the } G\text{-action is extendable} \end{aligned}$$

by Proposition 4.5, Lemma 4.4, and Theorem 3.19. Thus, (1) \iff (3). This finishes the proof. \square

In particular, this proves Theorem 1.5.

4.4. Proof of Theorem 1.7. The proof of Theorem 1.7 is completely analogous to that of Theorem 4.7. Suppose the G -action on X is K -linear. Then G naturally acts on $X_{\bar{k}}^\circ$ and $\text{Aut}_{\mathbb{Q}_\ell}(H_{\text{ét}}^2(X_{\bar{k}}^\circ, \mathbb{Q}_\ell(1)))$.

Define q_ℓ as the composition of i_ℓ with the inclusion $\text{Aut}_{G_k}(H_{\text{ét}}^2(X_{\bar{k}}^\circ, \mathbb{Q}_\ell(1))) \rightarrow \text{Aut}_{\mathbb{Q}_\ell}(H_{\text{ét}}^2(X_{\bar{k}}^\circ, \mathbb{Q}_\ell(1)))$,

$$q_\ell : \mathcal{W}_{X, \mathcal{L}} \xrightarrow{i_\ell} \text{Aut}_{G_k}(H_{\text{ét}}^2(X_{\bar{k}}^\circ, \mathbb{Q}_\ell(1))) \rightarrow \text{Aut}_{\mathbb{Q}_\ell}(H_{\text{ét}}^2(X_{\bar{k}}^\circ, \mathbb{Q}_\ell(1))).$$

Then q_ℓ induces a pointed map

$$q_{\ell*} : H^1(G, \mathcal{W}_{X, \mathcal{L}}) \rightarrow H^1(G, \text{Aut}_{\mathbb{Q}_\ell}(H_{\text{ét}}^2(X_{\bar{k}}^\circ, \mathbb{Q}_\ell(1)))).$$

Let $\gamma_{\mathcal{L}, \ell} = q_{\ell*}(\alpha_{\mathcal{L}, \ell})$.

Lemma 4.8. *The pointed map $q_{\ell*}$ has trivial kernel.*

Proof. Denote the action of $\mathcal{W}_{X, \mathcal{L}}$ on $\Lambda_{X, \mathcal{L}, \mathbb{Q}_\ell}$ by $j'_\ell : \mathcal{W}_{X, \mathcal{L}} \rightarrow \text{Aut}(\Lambda_{X, \mathcal{L}, \mathbb{Q}_\ell})$. The kernel of $j'_{\ell*}$ is trivial by [CLL19, Theorem 4.1]. It suffices to prove that $q_{\ell*}$ and $j'_{\ell*}$ have the same kernel.

The cycle class map gives a G -injection $\Lambda_{X, \mathcal{L}, \mathbb{Q}_\ell} \rightarrow H_{\text{ét}}^2(X_{\bar{k}}^\circ, \mathbb{Q}_\ell(1))$. Denote by S_ℓ the quotient and consider the short exact sequence of G -representations

$$0 \rightarrow \Lambda_{X, \mathcal{L}, \mathbb{Q}_\ell} \rightarrow H_{\text{ét}}^2(X_{\bar{k}}^\circ, \mathbb{Q}_\ell(1)) \rightarrow S_\ell \rightarrow 0.$$

Suppose $\alpha \in H^1(G, \mathcal{W}_{X, \mathcal{L}})$. By Proposition 2.8,

$$j'_{\ell*}(\alpha) \text{ is trivial} \iff \Lambda_{X, \mathcal{L}, \mathbb{Q}_\ell} \cong \Lambda_{X, \mathcal{L}, \mathbb{Q}_\ell}^\alpha,$$

$$q_{\ell*}(\alpha) \text{ is trivial} \iff H_{\text{ét}}^2(X_{\bar{k}}^\circ, \mathbb{Q}_\ell(1))^{G_k} \cong H_{\text{ét}}^2(X_{\bar{k}}^\circ, \mathbb{Q}_\ell(1))^{G_k, \alpha}.$$

Next, $\mathcal{W}_{X,\mathcal{L}}$ acts trivially on S_ℓ by construction, so $S_\ell \cong S_\ell^\alpha$. Then we are done by the semi-simplicity of the representations of the finite group G . \square

Proposition 4.9. *There is a G -isomorphism*

$$H_{\text{ét}}^2(X_{K^s}, \mathbb{Q}_\ell(1)) \cong H_{\text{ét}}^2(X_{\bar{k}}^\circ, \mathbb{Q}_\ell(1))^{\gamma_{\mathcal{L},\ell}}.$$

Proof. Since X has good reduction over K , we have an isomorphism

$$\phi : H_{\text{ét}}^2(X_{\bar{k}}^\circ, \mathbb{Q}_\ell(1)) \rightarrow H_{\text{ét}}^2(X_{K^s}, \mathbb{Q}_\ell(1)).$$

We claim that this gives a G -isomorphism if we twist $H_{\text{ét}}^2(X_{\bar{k}}^\circ, \mathbb{Q}_\ell(1))^{G_k}$ by $\gamma_{\mathcal{L},\ell}$. This amounts to showing that

$$\phi \circ \gamma_{\mathcal{L},\ell}(g) \circ g_k^* = g^* \circ \phi.$$

This is much simpler than the proof of Proposition 4.5 since $g = g^\#$. By Proposition 4.3 and Lemma 3.2,

$$\gamma_{\mathcal{L},\ell}(g) = (i_\ell \circ \alpha_{\mathcal{L}})(g) = i_\ell(s_{g_X}) = s_{g_X,\ell} = \phi^{-1} \circ g^* \circ \phi \circ ((g_k)^*)^{-1}.$$

This finishes the proof. \square

Proof of Theorem 1.7. Let \mathcal{L} be a polarization on X fixed by G . We have

$$\begin{aligned} H_{\text{ét}}^2(X_{\bar{k}}^\circ, \mathbb{Q}_\ell) &\cong H_{\text{ét}}^2(X_{K^s}, \mathbb{Q}_\ell) \text{ as } G\text{-representations} \\ \iff H_{\text{ét}}^2(X_{\bar{k}}^\circ, \mathbb{Q}_\ell(1)) &\cong H_{\text{ét}}^2(X_{K^s}, \mathbb{Q}_\ell(1)) \text{ as } G\text{-representations} \\ \iff \gamma_{\mathcal{L},\ell} \text{ is trivial} &\iff \alpha_{\mathcal{L},\ell} \text{ is trivial} \iff \text{the } G\text{-action is extendable} \end{aligned}$$

by Proposition 4.9, Lemma 4.8, and Theorem 3.19. \square

5. APPLICATIONS

In this section, we prove Corollary 1.9 and Corollary 1.12.

5.1. Extendability of symplectic group actions. We prove Corollary 1.9 in this sub-section. Suppose X is a K3 surface over K with good reduction and G is a finite group acting K -linearly on X .

Lemma 5.1. *If $p \nmid |G|$, for every $g \in G$, g and g_k have the same order.*

Proof. Suppose g has order n and g_k has order m . The order of g_k divides the order of g so $n = mk$ for some integer k . By [Mat16, Proposition 6.1], k is a power of p . Since $p \nmid |G|$, we have $k = 1$ so $m = n$ as we want. \square

Lemma 5.2. *If g is symplectic on X , so is g_k on X° .*

Proof. By [LM17, Proposition 4.7], there exists a smooth model \mathcal{X} such that g is defined over an open dense subset of $U \subset \mathcal{X}$ whose complement consists of finitely

many curves on the special fibres. Denote the intersection of U with X° as V , an open dense subset of X° . Denote the relative canonical divisor of U and V respectively as ω_U and ω_V .

We make the following claims.

- (1) The restriction map $H^0(X^\circ, \omega_{X^\circ}) \rightarrow H^0(V, \omega_V)$ is injective.
- (2) The restriction map $H^0(U, \omega_U) \rightarrow H^0(X, \omega_X)$ is injective.
- (3) The image of the restriction map $H^0(U, \omega_U) \rightarrow H^0(V, \omega_V)$ equals the image of $H^0(X^\circ, \omega_{X^\circ}) \rightarrow H^0(V, \omega_V)$.

Indeed, since X° is integral and ω_{X° is trivial, (1) holds. Also, U is isomorphic to \mathcal{X} away from codimension 2, so $H^0(U, \omega_U) = H^0(\mathcal{X}, \omega_{\mathcal{X}}) = \mathcal{O}_K$, which is torsion-free. Since the restriction map in (2) is the base change to from \mathcal{O}_K to K , it is injective. Finally, we have a commutative diagram

$$\begin{array}{ccc} H^0(\mathcal{X}, \omega_{\mathcal{X}}) & \longrightarrow & H^0(X^\circ, \omega_{X^\circ}) \\ \downarrow & & \downarrow \\ H^0(U, \omega_U) & \longrightarrow & H^0(V, \omega_V) \end{array}$$

The vertical map on the left is an isomorphism. The horizontal map on the top corresponds to the quotient homomorphism $\mathcal{O}_K \rightarrow k$, therefore surjective. Then (3) holds by diagram chasing.

Suppose g acts trivially on $H^0(X, \omega_X)$. By (2), g also acts trivially on $H^0(U, \omega_U)$ and thus on the image of $H^0(U, \omega_U) \rightarrow H^0(V, \omega_V)$. Then by (3), g_k acts trivially on the image of $H^0(X^\circ, \omega_{X^\circ}) \rightarrow H^0(V, \omega_V)$. Finally, by (1), g_k acts trivially on $H^0(X^\circ, \omega_{X^\circ})$. Therefore, g_k is symplectic on X° . \square

Proof of Corollary 1.9. By Theorem 1.7, we need to prove that $H_{\text{ét}}^2(X_{K^s}, \mathbb{Q}_\ell)$ and $H_{\text{ét}}^2(X_{\bar{k}}^\circ, \mathbb{Q}_\ell)$ are isomorphic as G -representations. Since \mathbb{Q}_ℓ has characteristic 0, it suffices to compare their characters. Equivalently, we prove that for all $g \in G$, g and g_k have the same characteristic polynomial on $H_{\text{ét}}^2(X_{K^s}, \mathbb{Q}_\ell)$ and $H_{\text{ét}}^2(X_{\bar{k}}^\circ, \mathbb{Q}_\ell)$ respectively.

First, g and therefore g_k are symplectic by Lemma 5.2. Then by [Mat16, Lemma 2.13], the characteristic polynomials of g and g_k only depend on their orders. Finally, by Lemma 5.1, g and g_k have the same order. This finishes the proof. \square

5.2. Good reduction of Enriques surfaces. We prove Corollary 1.12 in this sub-section. First, we recall some facts about Enriques surfaces.

Suppose the ground field F has characteristic different from 2. Then every K3 surface over F quotient by an involution without fixed points is an Enriques surface by [Bad01, Proposition 10.15].

Conversely, every Enriques surface over F is the quotient of a K3 double cover by a fixed-point-free involution. The construction is as follows. Suppose Y is an Enriques surface. We choose an isomorphism $\omega_Y^{\otimes 2} \cong \mathcal{O}_Y$ and form the \mathcal{O}_Y -algebra $\mathcal{O}_Y \oplus \omega_Y$. We take the K3 double cover to be its relative spectrum

$$X = \underline{\mathrm{Spec}}_Y(\mathcal{O}_Y \oplus \omega_Y)$$

and the associated involution ι is the unique involution that acts as the identity on \mathcal{O}_Y and the (-1) -map on ω_Y . This is a K3 surface by the proof of [Bad01, Proposition 10.14].

In fact, this construction can be done in families.

Lemma 5.3. *Assume $p \neq 2$. Let Y be an Enriques surface over K with good reduction. Let \mathcal{Y} be a smooth model of Y over \mathcal{O}_K . Then there exists a smooth proper algebraic space \mathcal{X} over \mathcal{O}_K with an involution ι such that \mathcal{X}_K and \mathcal{X}_k are respectively K3 double covers of Y and \mathcal{Y}_k with associated involutions ι_K and ι_k .*

Proof. We mimic the construction of a K3 double cover of an Enriques surface. First, since Y is an Enriques surface, ω_Y is nontrivial but $\omega_Y^{\otimes 2} \cong \mathcal{O}_Y$. The pullback induces an isomorphism $\mathrm{Pic}(\mathcal{Y}) \rightarrow \mathrm{Pic}(Y)$ that maps $\omega_{\mathcal{Y}}$ to ω_Y . Therefore, $\omega_{\mathcal{Y}}$ is nontrivial but $\omega_{\mathcal{Y}}^{\otimes 2} \cong \mathcal{O}_{\mathcal{Y}}$. We choose such an isomorphism and construct the $\mathcal{O}_{\mathcal{Y}}$ -algebra $\mathcal{O}_{\mathcal{Y}} \oplus \omega_{\mathcal{Y}}$. Let

$$\mathcal{X} = \underline{\mathrm{Spec}}_{\mathcal{Y}}(\mathcal{O}_{\mathcal{Y}} \oplus \omega_{\mathcal{Y}})$$

and ι be the unique involution that acts as the identity on $\mathcal{O}_{\mathcal{Y}}$ and the (-1) -map on $\omega_{\mathcal{Y}}$.

Since taking the relative spectrum commutes with base change, \mathcal{X}_K and \mathcal{X}_k are respectively the K3 double covers of Y and \mathcal{Y}_k determined by the base change of the isomorphism $\omega_{\mathcal{Y}}^{\otimes 2} \cong \mathcal{O}_{\mathcal{Y}}$ to K and k . Also, $Y = \mathcal{X}_K/\iota_K$ and $\mathcal{Y}_k = \mathcal{X}_k/\iota_k$. By construction, \mathcal{X} is proper and flat. It remains to show that \mathcal{X} is smooth. Indeed, since \mathcal{X}_K and \mathcal{X}_k are smooth, \mathcal{X} is smooth by flatness. \square

Now we prove Corollary 1.12.

Proof of Corollary 1.12. Suppose Y has good reduction with a smooth model \mathcal{Y} . Apply Lemma 5.3 to Y and \mathcal{Y} . Let \mathcal{X} and ι be as in Lemma 5.3. Then \mathcal{X}_K is a K3 double cover of Y with associated involution ι_K . Its specialization ι_k has no fixed points since $\mathcal{Y}_k = \mathcal{X}_k/\iota_k$ is an Enriques surface. This proves (1) \implies (2).

Also, suppose X is a K3 double cover of Y , with associated involution ι , such that X has good reduction and ι_k has no fixed points. Then X°/ι_k is an Enriques surface. By [Bad01, Section 10.10],

$$\dim H_{\mathrm{et}}^2(X_K^\circ, \mathbb{Q}_\ell)^{\iota_k = \mathrm{id}} = \dim H_{\mathrm{et}}^2((X^\circ/\iota_k)_{\bar{k}}, \mathbb{Q}_\ell) = 10.$$

This proves that (2) \implies (3).

It remains to prove that (3) \implies (1). Suppose X is a K3 double cover of Y , with associated involution ι , such that X has good reduction and $\dim H_{\text{ét}}^2(X_k^\circ, \mathbb{Q}_\ell)^{\iota_k} = 10$. We claim that ι is extendable. Indeed, let $G = \langle \iota \rangle \cong \mathbb{Z}/2$. By Theorem 1.7, it suffices to prove that there is a G -isomorphism

$$H_{\text{ét}}^2(X_{K^s}, \mathbb{Q}_\ell) \cong H_{\text{ét}}^2(X_k^\circ, \mathbb{Q}_\ell).$$

Every finite dimensional representation of G decomposes as the direct sum of the subspace fixed by ι and the subspace where ι acts as the (-1) -map. Since $H_{\text{ét}}^2(X_{K^s}, \mathbb{Q}_\ell)$ and $H_{\text{ét}}^2(X_k^\circ, \mathbb{Q}_\ell)$ both have dimension 22, it suffices to show that

$$(5.4) \quad \dim H_{\text{ét}}^2(X_{K^s}, \mathbb{Q}_\ell)^{\iota=\text{id}} = \dim H_{\text{ét}}^2(X_k^\circ, \mathbb{Q}_\ell)^{\iota=\text{id}}.$$

Since Y is an Enriques surface, by [Bad01, Section 10.10], the LHS of 5.4 is

$$\dim H_{\text{ét}}^2(X_{K^s}, \mathbb{Q}_\ell)^{\iota=\text{id}} = \dim H_{\text{ét}}^2(Y_{K^s}, \mathbb{Q}_\ell) = 10.$$

On the other hand, the RHS of 5.4 is 10 by assumption. This shows that ι is extendable.

Let \mathcal{X} be a smooth model of X to which ι extends. We want to show that $\mathcal{Y} = \mathcal{X}/\iota$ is a smooth model of Y . By construction, \mathcal{Y} is a model of Y . We need to prove that \mathcal{Y} is smooth. By flatness, it suffices to prove that \mathcal{Y}_k is smooth.

Since $p \neq 2$, we have $\mathcal{Y}_k \cong X^\circ/\iota_k$. We claim that ι_k is non-symplectic. Suppose ι_k is symplectic. By [Huy16, Proposition 3.11, Chapter 15], \mathcal{Y}_k is a surface with at worst RDPs whose minimal desingularization is a K3 surface. In particular, the Euler characteristic of \mathcal{Y}_k is 2. However, the Euler characteristic of \mathcal{Y}_k should agree with that of Y , which is 1. This is a contradiction, so ι_k is non-symplectic.

Then by the classification in [Nik83, Remark 4.5.4], \mathcal{Y}_k is either a smooth rational surface or an Enriques surface (in fact, it must be an Enriques surface but we don't need this). In particular, \mathcal{Y}_k is smooth as we want. This shows that (5) \implies (1) and finishes the proof. \square

REFERENCES

- [Art70] M. Artin. “Algebraization of Formal Moduli: II. Existence of Modifications”. In: *Annals of Mathematics* 91.1 (1970), pp. 88–135. ISSN: 0003486X, 19398980. URL: <http://www.jstor.org/stable/1970602> (visited on 10/21/2025).
- [Art86] M. Artin. “Néron Models”. In: *Arithmetic Geometry*. Ed. by Gary Cornell and Joseph H. Silverman. New York, NY: Springer New York, 1986, pp. 213–230. ISBN: 978-1-4613-8655-1. DOI: 10.1007/978-1-4613-8655-1_8. URL: https://doi.org/10.1007/978-1-4613-8655-1_8.

- [Bad01] Lucian Badescu. *Algebraic Surfaces*. Jan. 2001. ISBN: 978-1-4419-3149-8. DOI: 10.1007/978-1-4757-3512-3.
- [CLL19] Bruno Chiarellotto, Christopher Lazda, and Christian Liedtke. “A Néron–Ogg–Shafarevich criterion for K3 surfaces”. In: *Proceedings of the London Mathematical Society* 119.2 (Feb. 2019), pp. 469–514. ISSN: 1460-244X. DOI: 10.1112/plms.12237. URL: <http://dx.doi.org/10.1112/plms.12237>.
- [Huy16] Daniel Huybrechts. *Lectures on K3 Surfaces*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2016.
- [KM83] Zenji Kobayashi and Jun Morita. “AUTOMORPHISMS OF CERTAIN ROOT LATTICES”. In: *Tsukuba Journal of Mathematics* 7.2 (1983), pp. 323–336. ISSN: 03874982. URL: <http://www.jstor.org/stable/43685410> (visited on 03/11/2025).
- [LM17] Christian Liedtke and Yuya Matsumoto. “Good reduction of K3 surfaces”. In: *Compositio Mathematica* 154.1 (Sept. 2017), pp. 1–35. ISSN: 1570-5846. DOI: 10.1112/s0010437x17007400. URL: <http://dx.doi.org/10.1112/S0010437X17007400>.
- [Mat15] Yuya Matsumoto. “On good reduction of some K3 surfaces related to abelian surfaces”. In: *Tohoku Mathematical Journal* 67.1 (Jan. 2015). ISSN: 0040-8735. DOI: 10.2748/tmj/1429549580. URL: <http://dx.doi.org/10.2748/tmj/1429549580>.
- [Mat16] Yuya Matsumoto. “Extendability of automorphisms of K3 surfaces”. In: *Mathematical Research Letters* (2016). URL: <https://api.semanticscholar.org/CorpusID:50351076>.
- [Mil80] J. S. Milne. *Etale Cohomology (PMS-33)*. Princeton University Press, 1980. URL: <http://www.jstor.org/stable/j.ctt1bpmbk1> (visited on 11/13/2025).
- [Nik83] V. V. Nikulin. “Factor groups of groups of automorphisms of hyperbolic forms with respect to subgroups generated by 2-reflections. Algebrogeometric applications”. In: *Journal of Mathematical Sciences* 22 (1983), pp. 1401–1475. DOI: 10.1007/BF01094757. URL: <https://doi.org/10.1007/BF01094757>.
- [Ser97] Jean-Pierre Serre. *Galois Cohomology*. Springer Monographs in Mathematics. Springer-Verlag, 1997.