

# Lower central series of the Riordan group over the field with two elements

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## Abstract

The Riordan group  $\mathcal{R}$  over the field  $\mathbb{F}_2$  is a split extension of the Appell subgroup by the Nottingham group  $\mathcal{N}(\mathbb{F}_2)$ . Using the lower central series of the Nottingham group obtained by C. Leedham-Green and S. McKay, the lower central series of  $\mathcal{R}(\mathbb{F}_2)$  is calculated. It is also proved that the abelianization of the Riordan group over any commutative ring with identity is isomorphic to the direct product of the abelianization of the corresponding Lagrange subgroup, the additive group of the ring, and its multiplicative groups of units.

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## 1 Introduction

The Riordan group, introduced by L. Shapiro et al. [17], is a group of infinite lower triangular matrices called Riordan arrays, whose columns consist of the coefficients of certain formal power series. For the detailed introduction to the subject, the reader is referred to the books by P. Barry [3] and L. Shapiro et al. [19], and a survey article by D. Davenport et al. [7]. Here are a few basic definitions and notations, which will be used in the paper.

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Let  $\mathbb{D}[[t]]$  be the set all formal power series (f.p.s.) in indeterminate  $t$  with coefficients in an arbitrary commutative ring  $\mathbb{D}$  with identity, and denote its multiplicative group of units by  $\mathbb{D}^*$ . Take two f.p.s.  $g(t) = \sum_{k=0}^{\infty} g_k t^k$  and  $f(t) = \sum_{k=1}^{\infty} f_k t^k$ , such that  $g_0, f_1 \in \mathbb{D}^*$ , and for all  $n, k \geq 0$ , define

$$d_{n,k} := [t^n]g(t)f(t)^k,$$

where  $[t^n]h(t)$  stands for the coefficient of  $t^n$  in the expansion of the f.p.s.  $h(t)$ . Then the Riordan array  $(g, f)$  is defined to be the infinite lower triangular matrix

$$A = (d_{n,k})_{n,k \geq 0}.$$

Since each row of such a matrix has only finitely many nonzero terms, and  $g_0, f_1 \in \mathbb{D}^*$ , these arrays form a group with the usual matrix multiplication, called the *Riordan group over  $\mathbb{D}$* . We will denote this group by  $\mathcal{R}(\mathbb{D})$ , or simply by  $\mathcal{R}$  when the ground ring  $\mathbb{D}$  will be clear from the context. In terms of the pairs of f.p.s. the group operation is written as

$$(g_1(t), f_1(t)) \cdot (g_2(t), f_2(t)) = (g_1(t)g_2(f_1(t)), f_2(f_1(t))). \quad (1.1)$$

The Riordan array  $I = (1, t)$  is the group identity, and the inverse of the Riordan array  $(g(t), f(t))$  is the pair

$$(g(t), f(t))^{-1} = \left( \frac{1}{g(\bar{f}(t))}, \bar{f}(t) \right),$$

where we used the standard notation  $\bar{f}(t)$  for the compositional inverse of  $f(t)$ , i.e.  $\bar{f}(f(t)) = t$  and  $f(\bar{f}(t)) = t$ .

As we see from 1.1,  $\mathcal{R}(\mathbb{D})$  is a semidirect product of two proper subgroups: the Appell subgroup and the Lagrange (or associated) subgroup ([3], [19]). The Appell subgroup  $\mathcal{A}(\mathbb{D})$  is abelian, normal, and consists of the Riordan arrays  $(g(t), t)$ . The Lagrange subgroup  $\mathcal{N}(\mathbb{D})$  is not commutative, and consists of the Riordan arrays  $(1, f(t))$ . When  $\mathbb{D}^*$  has at least two elements,  $\mathcal{N}(\mathbb{D})$  properly contains the *substitution group of formal power series*  $\mathcal{J}(\mathbb{D})$ , which was studied in the 1950s by S. Jennings, [9] (see also review article by I. Babenko [1]). Elements of the substitution group, as Riordan arrays, are the pairs  $(1, f(t))$ , where the f.p.s.  $f(t) = \sum_{k \geq 1} f_k t^k$  has the first coefficient  $f_1 = 1$ . The fact that the Riordan group is a semidirect product  $\mathcal{R}(\mathbb{D}) \cong \mathcal{A}(\mathbb{D}) \ltimes \mathcal{N}(\mathbb{D})$  can be written as a splitting short exact sequence

$$0 \longrightarrow \mathcal{A}(\mathbb{D}) \xrightarrow{\mu} \mathcal{R}(\mathbb{D}) \xrightarrow{s} \mathcal{N}(\mathbb{D}) \longrightarrow 1, \quad (1.2)$$

where 0 indicates that  $\mathcal{A}(\mathbb{D})$  is commutative. Projection homomorphism  $\rho$  is defined by  $\rho((g(t), f(t))) := (1, f(t))$ , and the splitting homomorphism

$s$  is the inclusion  $s((1, f(t))) := (1, f(t))$ . These two maps clearly satisfy  $\rho \circ s = id|_{\mathcal{N}(\mathbb{D})}$ . The homomorphism  $\mu$  is an inclusion too.

When the ground ring  $\mathbb{D} = \mathbb{F}_2 = \{0, 1\}$ , the Lagrange subgroup  $\mathcal{N}(\mathbb{F}_2)$  is isomorphic to the Nottingham group, denoted by  $\mathcal{N}(\mathbb{F}_2)$  as well. The lower central series of  $\mathcal{N}(\mathbb{F}_2)$  was calculated by C.R. Leedham-Green and S. McKay in [13], §12.4 (see also article [5] by R. Camina). Since I will use these calculations to obtain the lower central series of  $\mathcal{R}(\mathbb{F}_2)$ , I will follow closely their notations, which seem to be standardly accepted in the literature dedicated to the Nottingham group. Notice however that the corresponding notations for the Lagrange subgroup and truncated Lagrange subgroups used in the literature dedicated to the Riordan group are usually different. When  $\mathbb{D} = \mathbb{F}_q$  and  $q$  is odd, the Lagrange subgroup is also called the ‘full’ group of f.p.s. under substitution, and contains the Nottingham group  $\mathcal{N}(\mathbb{F}_q)$  as a proper normal subgroup, [5].

It was proved by A. Luzón et al. [14], that the Riordan group  $\mathcal{R}(\mathbb{D})$  is isomorphic to the inverse limit of the inverse system of Riordan matrices of finite size. Analogous result for the groups of formal power series under substitution was proved by D.L. Johnson in [10]. I will briefly recall the construction at the beginning of next section, since such Riordan matrices play a key role here. If the ring  $\mathbb{D}$  is finite, the group  $\mathcal{R}(\mathbb{D})$  is isomorphic to the inverse limit of an inverse system of finite groups, and therefore is a *profinite group*. A group  $G$  is profinite if it is a compact Hausdorff topological group whose open subgroups form a base for the neighborhoods of the identity. Since for any finite ring  $\mathbb{D}$ ,  $\mathcal{A}(\mathbb{D})$  is a closed normal subgroup of infinite index, the Riordan group  $\mathcal{R}(\mathbb{D})$  is not just infinite. Topology on the inverse limit is induced by the product topology on the infinite Cartesian product, where each finite group comes with the discrete topology. It is the weakest topology in which the projection map from the inverse limit group onto each finite group is continuous. For an introduction to the theory of profinite groups the reader is referred to one of the monographs [8], [13], [20]. When  $\mathbb{D} = \mathbb{F}_2$ ,  $\mathcal{R}(\mathbb{D})$  is a pro-2 group, being the inverse limit of finite 2-groups

$$\mathcal{R}(\mathbb{D}) \cong \varprojlim(T\mathcal{R}_n(\mathbb{D}), P_{n-1}),$$

where for each  $n \geq 1$ , the 2-group  $T\mathcal{R}_n(\mathbb{D})$  consists of the Riordan arrays of finite size

$$(1 + \alpha_1 t + \cdots + \alpha_n t^n, t + \beta_2 t^2 + \cdots + \beta_n t^n), \quad \alpha_i, \beta_i \in \mathbb{F}_2, 1 \leq i \leq n.$$

The homomorphism  $P_{n-1} : T\mathcal{R}_n \rightarrow T\mathcal{R}_{n-1}$  deletes the last row and column from the array  $T\mathcal{R}_n$ , i.e.

$$\begin{aligned} P_{n-1} & \left( (1 + \alpha_1 t + \cdots + \alpha_{n-1} t^{n-1} + \alpha_n t^n, t + \beta_2 t^2 + \cdots + \beta_{n-1} t^{n-1} + \beta_n t^n) \right) \\ &= (1 + \alpha_1 t + \cdots + \alpha_{n-1} t^{n-1}, t + \beta_2 t^2 + \cdots + \beta_{n-1} t^{n-1}). \end{aligned}$$

The main objective of this article is the lower central series of  $\mathcal{R}(\mathbb{F}_2)$ , i.e.

$$\mathcal{R}(\mathbb{F}_2) = \gamma_1(\mathcal{R}(\mathbb{F}_2)) \geq \gamma_2(\mathcal{R}(\mathbb{F}_2)) \geq \dots,$$

where  $\gamma_i(G) = [\gamma_{i-1}(G), G]$  for all  $i \geq 2$ . Our result is stated in Theorem 7 in terms of the lower central series of the Nottingham group  $\mathcal{N}(\mathbb{F}_2)$ , which is given in [13], §12.4. In particular,  $\mathcal{N}^{ab}(\mathbb{F}_2) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$ . When  $p$  is odd, the corresponding computations for  $\mathcal{N}(\mathbb{F}_p)$  are given in Theorem 2, [5]. It is important to notice that by a subgroup of a topological group  $G$  one means a nonempty closed subset of the topological space  $G$ , which is also closed under multiplication and inversion. The commutator of two closed subgroups of a topological group is not automatically closed as the topological subset, therefore to have a well defined lower central series, one needs the notion of a *topologically generating set*. In particular, if  $X$  and  $Y$  are normal closed subgroups of  $\mathcal{R}(\mathbb{D})$ , we define  $[X, Y]$  as the smallest closed subgroup containing all the commutators  $[x, y] = x^{-1}y^{-1}xy$ , where  $x \in X$  and  $y \in Y$  (see [9], §2). That is how we will understand all commutators  $[X, Y]$  here, unless it will be specifically stated otherwise.

Using results of R. Camina [5], and C.R. Leedham-Green and S. McKay [13], on the lower central series of the Nottingham group over a finite field with characteristic  $p > 2$ , Gi-S. Cheon, N.-P. Chung, and M.-N. Phung calculated in [6] the lower central series of the corresponding Riordan group. S.I. Bogataya and S.A. Bogatyri utilizing extensive calculations in [4] found the high commutants and gave explicit abelianization homomorphism of the Jennings - Lagrange - Nottingham group  $\mathcal{N}(\mathbb{F}_2)$ . D.L. Johnson, [10] calculated the lower central series of the group of f.p.s. under substitution over the rationals, and I.K. Babenko and S.A. Bogatyri in [2] found the abelianization of the group of f.p.s. under substitution with integer coefficients. Recently, A. Luzón, M.A. Morón, and L.F. Prieto-Martínez calculated the derived series of the Riordan group over a field of characteristic 0, [15]. The present work is motivated by this research and offers further insights into the algebraic structure of the Riordan group.

The paper is organized as follows. In section 2, we will review the inverse limit description of the Riordan group, and use the short exact sequence 1.2 to derive a commutative diagram connecting the groups  $\mathcal{R}$  and  $\mathcal{N}$  with their abelianizations. From this diagram we obtain an isomorphism  $\mathcal{R}^{ab}(\mathbb{D}) \cong \mathbb{D}^* \times \mathbb{D} \times \mathcal{N}^{ab}(\mathbb{D})$ , for an arbitrary commutative ring  $\mathbb{D}$  with identity. This isomorphism, together with calculations of I. Babenko and S. Bogatyri, [2], gives the answer to a question posed by L. Shapiro in [18] about the commutator group of the subgroup of  $\mathcal{R}(\mathbb{Z})$ , that has all 1s on the main diagonal (see Corollary 4 below). In section 3 we will classify all truncated Appell subgroups over  $\mathbb{F}_2$  as finite abelian groups, and prove our main result,

Theorem 7. When  $\mathbb{D} = \mathbb{F}_2$ ,  $\gamma_n(\mathcal{R}) = \mathcal{A}_{2n-3} \ltimes \gamma_n(\mathcal{N})$ , and

$$\gamma_i(\mathcal{R})/\gamma_{i+1}(\mathcal{R}) \cong \begin{cases} (\mathbb{Z}_2)^3 \times \mathbb{Z}_4 & \text{if } i = 1 \\ (\mathbb{Z}_2)^4 & \text{if } i > 1 \text{ is even} \\ (\mathbb{Z}_2)^6 & \text{if } i > 1 \text{ is odd.} \end{cases}$$

At the end, we will show how to embed the dihedral group  $D_{2^{n+1}}$ ,  $\forall n \geq 0$  into the truncated Riordan group  $T\mathcal{R}_{2^n}(\mathbb{F}_2)$ .

## 2 Short exact sequences

Let  $\mathbb{D}$  be an arbitrary, fixed, commutative ring with identity 1. For any  $n \geq 0$ , take the general linear group of all invertible  $(n+1) \times (n+1)$  matrices over  $\mathbb{D}$ ,  $\mathrm{GL}(n+1, \mathbb{D})$ , and denote by  $T\mathcal{R}_n(\mathbb{D}) := \Pi_n(\mathcal{R}(\mathbb{D}))$  the image of a natural homomorphism (called *truncation*)

$$\Pi_n : \mathcal{R}(\mathbb{D}) \longrightarrow \mathrm{GL}(n+1, \mathbb{D}) \quad (2.1)$$

defined by

$$\Pi_n((d_{i,j})_{i,j \geq 0}) = (d_{i,j})_{0 \leq i,j \leq n}.$$

Deleting the last row and column from  $T\mathcal{R}_{n+1}(\mathbb{D})$  produces  $T\mathcal{R}_n(\mathbb{D})$ , and this way we obtain another natural homomorphism, which will be denoted by  $P_n : T\mathcal{R}_{n+1}(\mathbb{D}) \longrightarrow T\mathcal{R}_n(\mathbb{D})$ . In terms of the formal notations we have

$$P_n((d_{i,j})_{0 \leq i,j \leq n+1}) = (d_{i,j})_{0 \leq i,j \leq n},$$

and clearly  $\Pi_n = P_n \circ \Pi_{n+1}$ . The homomorphism  $P_n$  is onto, and for all  $n \geq 0$ , the homomorphisms  $\Pi_{n+1}$ ,  $\Pi_n$  and  $P_n$  make the commutative triangle

$$\begin{array}{ccc} & \mathcal{R}(\mathbb{D}) & \\ \swarrow \Pi_{n+1} & & \searrow \Pi_n \\ T\mathcal{R}_{n+1}(\mathbb{D}) & \xrightarrow{P_n} & T\mathcal{R}_n(\mathbb{D}) \end{array} \quad (2.2)$$

Finite size Riordan arrays  $T\mathcal{R}_n(\mathbb{D})$  can be constructed iteratively using the  $A$  and  $Z$  sequences, and it was shown in [14], that the Riordan group  $\mathcal{R}(\mathbb{D})$  is isomorphic to the inverse limit

$$\mathcal{R}(\mathbb{D}) \cong \varprojlim(T\mathcal{R}_{n+1}(\mathbb{D}), P_n)$$

(c.f. Proposition 1. in [10], for the group of f.p.s. under substitution).

Since  $\mathcal{R}(\mathbb{D}) \cong \mathcal{A}(\mathbb{D}) \ltimes \mathcal{N}(\mathbb{D})$ , restricting the homomorphism  $\Pi_n$  onto the subgroups  $\mathcal{A}(\mathbb{D})$  and  $\mathcal{N}(\mathbb{D})$  will produce truncated Appell and Lagrange

subgroups, and for every  $n \geq 0$ , we get  $T\mathcal{R}_n(\mathbb{D}) \cong T\mathcal{A}_n(D) \ltimes T\mathcal{N}_n(\mathbb{D})$ . Truncated group  $T\mathcal{N}_n(\mathbb{D})$  was also introduced by R. Camina in [5], as the factor group of  $\mathcal{N}(\mathbb{D})$  by the kernel of

$$\Phi_n : (f : t \mapsto \sum_{i=1}^{\infty} \alpha_i t^i) \mapsto (f_n : t \mapsto \sum_{i=1}^n \alpha_i t^i).$$

Clearly, the maps  $\Phi_n$  and  $\Pi_n|_{\mathcal{N}(\mathbb{D})}$  are the same after identifying the f.p.s.  $\sum_{i=1}^{\infty} \alpha_i t^i$  with the Riordan array  $(1, \sum_{i=1}^{\infty} \alpha_i t^i)$ . The kernel of  $\Phi_n$  is a normal subgroup of  $\mathcal{N}(\mathbb{D})$  denoted by  $\mathcal{N}_n(\mathbb{D})$ , so  $T\mathcal{N}_n(\mathbb{D}) \cong \mathcal{N}(\mathbb{D})/\mathcal{N}_n(\mathbb{D})$ . When  $\mathbb{D}$  is finite, these normal subgroups  $T\mathcal{N}_n$  form a countable basis for the neighborhoods of the identity in  $\mathcal{N}(\mathbb{D})$ . Analogously, when  $\mathbb{D}$  is finite, the kernels of  $\Pi_n(\mathcal{R}(\mathbb{D}))$ ,  $n \geq 0$  form a countable basis for the neighborhoods of the identity  $(1, t) \in \mathcal{R}(\mathbb{D})$ .

It will be convenient to use certain elements of the group  $\mathcal{N}_n(\mathbb{D})$ , so let us recall some notations from [5] and [13]. The Riordan array  $(1, t + \alpha t^{n+1})$  will be denoted by  $e_n[\alpha]$ , or simply by  $e_n$  if  $\alpha = 1$ , and  $e_0 = (1, t)$  will stand for the identity element. Sequence 1.2 naturally induces the following split short exact sequence of the truncated groups for all  $n \in \mathbb{N}$

$$0 \longrightarrow T\mathcal{A}_n(\mathbb{D}) \xrightarrow{\mu_n} T\mathcal{R}_n(\mathbb{D}) \xrightarrow{\rho_n} T\mathcal{N}_n(\mathbb{D}) \longrightarrow 1, \quad (2.3)$$

where maps  $\mu_n$ ,  $\rho_n$ , and  $s_n$  are the restrictions of their corresponding analogs from 1.2. If we have a f.p.s.  $g(t)$ , we can take  $g_n(t)$ , the  $n$ -th degree Taylor polynomial of it for any  $n \geq 0$ . Truncated Riordan arrays  $(g_n(t), t)$  are also known as Toeplitz matrices of size  $(n+1) \times (n+1)$  with zeroes above the main diagonal. Restriction of the homomorphism  $P_n$  onto the Appell subgroup  $T\mathcal{A}_{n+1}(\mathbb{D})$  gives an epimorphism

$$P_n|_{T\mathcal{A}_{n+1}(\mathbb{D})} : T\mathcal{A}_{n+1}(\mathbb{D}) \longrightarrow T\mathcal{A}_n(\mathbb{D}),$$

with  $T\mathcal{A}_0(\mathbb{D}) \cong \mathbb{D}^*$ . Since a Toeplitz matrix with zeroes above the main diagonal is completely determined by its most left column,  $\ker(P_n|_{T\mathcal{A}_{n+1}(\mathbb{D})}) \cong \mathbb{D}$  for all  $n \geq 0$ . Hence, there is the exact sequence

$$0 \longrightarrow \mathbb{D} \longrightarrow T\mathcal{A}_{n+1}(\mathbb{D}) \xrightarrow{P_n} T\mathcal{A}_n(\mathbb{D}) \longrightarrow 0, \quad (2.4)$$

where for brevity,  $P_n$  was also used to denote the restriction homomorphism  $P_n|_{T\mathcal{A}_{n+1}(\mathbb{D})}$ . Notice that  $\mathbb{D}$  here (and later in a similar context) stands for the additive group of the ring, i.e.  $\mathbb{D}^+$ . We also have  $T\mathcal{A}_1(\mathbb{D}) \cong \mathbb{D} \times \mathbb{D}^*$ , where the isomorphism is given by the map

$$\begin{pmatrix} u & 0 \\ x & u \end{pmatrix} \mapsto (xu^{-1}, u) \in \mathbb{D} \times \mathbb{D}^*.$$

In general however,  $T\mathcal{A}_n(\mathbb{D}) \not\cong \mathbb{D}^n \times \mathbb{D}^*$ . Since the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

corresponding to  $(1+t, t)$ , has order 4 in  $T\mathcal{A}_2(\mathbb{F}_2)$ , we have  $T\mathcal{A}_2(\mathbb{F}_2) \cong C_4$ , the cyclic group of order 4. We will discuss the group structure of  $T\mathcal{A}_n(\mathbb{F}_2)$  in details in the next section. Similarly, restricting  $P_n$  onto the subgroup  $T\mathcal{N}_{n+1}(\mathbb{D})$ , one obtains the short exact sequence

$$0 \longrightarrow \mathbb{D} \longrightarrow T\mathcal{N}_{n+1}(\mathbb{D}) \xrightarrow{P_n} T\mathcal{N}_n(\mathbb{D}) \longrightarrow 1, \quad (2.5)$$

with  $T\mathcal{N}_1(\mathbb{D}) \cong \mathbb{D}^*$  and

$$T\mathcal{N}_2(\mathbb{D}) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & u & 0 \\ 0 & x & u^2 \end{pmatrix} \mid u \in \mathbb{D}^*, x \in \mathbb{D} \right\} \cong \mathbb{D} \ltimes \mathbb{D}^*.$$

It is convenient to connect together three exact sequences 2.3, 2.4, and 2.5, in one commutative diagram (see Lemma 1. and diagram (2.6) in the article [12] by T.-X. He and the author).

$$\begin{array}{ccccccc} & 0 & & 0 & & 0 & \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 & \longrightarrow & \mathbb{D} & \longrightarrow & \mathbb{D} \times \mathbb{D} & \xrightarrow{\quad} & \mathbb{D} \longrightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 & \longrightarrow & T\mathcal{A}_{n+1}(\mathbb{D}) & \xrightarrow{\mu_{n+1}} & T\mathcal{R}_{n+1}(\mathbb{D}) & \xrightarrow{\rho_{n+1}} & T\mathcal{N}_{n+1}(\mathbb{D}) \longrightarrow 1 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 & \longrightarrow & T\mathcal{A}_n(\mathbb{D}) & \xrightarrow{\mu_n} & T\mathcal{R}_n(\mathbb{D}) & \xrightarrow{\rho_n} & T\mathcal{N}_n(\mathbb{D}) \longrightarrow 1 \\ & \downarrow & & \downarrow & & \downarrow & \\ & 0 & & 1 & & 1 & \end{array} \quad (2.6)$$

To simplify the writing, I will drop the symbol  $\mathbb{D}$  from the notations of all Appell, Riordan, and Lagrange groups. Also, to be in agreement with the notation  $e_n[\alpha] = (1, t + \alpha t^{n+1})$ , I will use  $a_k(u, \beta)$  for the Riordan array  $(u + \beta t^k, t)$ , where  $u \in \mathbb{D}^*$  and  $\beta \in \mathbb{D}$ . In such notations the kernel of the projection  $P_n : T\mathcal{R}_{n+1} \rightarrow T\mathcal{R}_n$  is generated by  $a_{n+1}(1, \beta)$  and  $e_n[\alpha]$ , for some elements  $\alpha, \beta$  from the ground ring.

Recall that for the groups consisting of finite size Riordan arrays, the discrete topology is used, and the commutator subgroup  $[G, G]$  is considered as the *abstract* subgroup of  $G$ . In particular, every element of  $[G, G]$  is a product of finitely many commutators. We will show next that for all  $n \geq 1$ , the abelianization of the truncated Riordan group  $T\mathcal{R}_n^{ab}$  is isomorphic to the direct product of  $\mathbb{D}^* \times \mathbb{D}$  with the abelianization of the truncated Lagrange subgroup  $T\mathcal{N}_n^{ab}$  (Theorem 3. below). Next lemma proves that for any  $\alpha \in \mathbb{D}$ , and any integer  $k \geq 2$ , the array  $a_k(1, \alpha)$  can be written as a commutator, i.e.  $(1 + \alpha t^k, t) = [x, y]$ , and hence  $a_k(1, \alpha) \in \gamma_2(T\mathcal{R}_n)$ ,  $\forall n \geq k$ .

**Lemma 1.** *For any  $\alpha \in \mathbb{D}$  and  $k \geq 2$ ,  $a_k(1, \alpha) \in \gamma_2(\mathcal{R})$ . We also have  $a_1(u, \alpha) \notin \gamma_2(T\mathcal{R}_n)$  for all  $n \geq 1$ , unless  $u = 1 \wedge \alpha = 0$ .*

*Proof.* Since  $\mathbb{D}^* \times \mathbb{D}$  is commutative, the composition of homomorphisms

$$P_2 \circ \cdots \circ P_n : T\mathcal{R}_{n+1} \rightarrow T\mathcal{R}_1 = \{a_1(u, \alpha) \mid u \in \mathbb{D}^*, \alpha \in \mathbb{D}\} \cong \mathbb{D} \times \mathbb{D}^*$$

maps any commutator to the identity, so  $a_1(u, \alpha) \notin \gamma_2(T\mathcal{R}_n)$ , unless it is the identity. To prove the first statement, take polynomials  $g(t) = 1 + t$ , and  $f(t) = t + \alpha t^k + \alpha t^{k+1}$ ,  $k \geq 2$ . Then direct computations show

$$\begin{aligned} [(g(t), t), (1, \bar{f}(t))] &= \left( \frac{1}{1+t}, f(t) \right) (1+t, \bar{f}(t)) \\ &= \left( \frac{1+t+\alpha t^k + \alpha t^{k+1}}{1+t}, t \right) = (1 + \alpha t^k, t), \end{aligned}$$

as required.  $\square$

If we define subgroups  $\mathcal{A}_n$  in a similar way to the subgroups  $\mathcal{N}_n$ , i.e. as

$$\mathcal{A}_n := \ker (\Pi_n|_{\mathcal{A}} : \mathcal{A} \longrightarrow T\mathcal{A}_n), \quad (2.7)$$

Lemma 1 immediately implies that  $\mathcal{A}_1 = \mathcal{A} \cap \gamma_2(\mathcal{R})$ . Therefore, considering  $\gamma_2(\mathcal{R})$  as the discrete subgroup of  $\mathcal{R}$ , we obtain

**Corollary 2.** *For every integer  $n \geq 1$ , there is an isomorphism*

$$T\mathcal{A}_n / (T\mathcal{A}_n \cap \gamma_2(T\mathcal{R}_n)) \cong \left\{ \begin{pmatrix} u & 0 \\ \alpha & u \end{pmatrix} \mid u \in \mathbb{D}^*, \alpha \in \mathbb{D} \right\} = T\mathcal{A}_1 \cong \mathbb{D} \times \mathbb{D}^*,$$

and also

$$\mathcal{A} / (\mathcal{A} \cap \gamma_2(\mathcal{R})) = \mathcal{A} / \mathcal{A}_1 \cong T\mathcal{A}_1 \cong \mathbb{D} \times \mathbb{D}^*.$$

If one applies the abelianization functor to the short exact sequence 2.3 with arbitrary  $n \geq 1$ , one derives the following commutative diagram, where the projections  $F_{\mathcal{R}}$  and  $F_{\mathcal{N}}$  are the abelianization homomorphisms. Recall that the abelianization functor is right exact, but it is straightforward to

check the exactness at each group of 2.8 by standard diagram chasing. Explanation of all maps is given below.

$$\begin{array}{ccccccc}
& & 1 & & 1 & & \\
& & \downarrow & & \downarrow & & \\
& & \gamma_2(T\mathcal{R}_n) & \xrightarrow{\rho_n^{(2)}} & \gamma_2(T\mathcal{N}_n) & \longrightarrow 1 & \\
& & \downarrow \iota_{\mathcal{R}} & \swarrow s_n^{(2)} & \downarrow \iota_{\mathcal{N}} & & \\
0 & \longrightarrow T\mathcal{A}_n & \xrightarrow{\mu_n} & T\mathcal{R}_n & \xrightarrow{\rho_n} & T\mathcal{N}_n & \longrightarrow 1 \\
& \downarrow F_{\mathcal{A}} & & \downarrow F_{\mathcal{R}} & \swarrow s_n & \downarrow F_{\mathcal{N}} & \\
0 & \longrightarrow \ker(\rho_n^{ab}) & \xrightarrow{\mu_n^{ab}} & T\mathcal{R}_n^{ab} & \xrightarrow{\rho_n^{ab}} & T\mathcal{N}_n^{ab} & \longrightarrow 0 \\
& & & \downarrow s_n^{ab} & & \downarrow & \\
& & & 0 & & 0 &
\end{array} \tag{2.8}$$

Take the epimorphism  $\rho_n$  and restrict it onto the commutator subgroup to get the epimorphism  $\rho_n^{(2)}$ . Then  $\rho_n^{ab}$  is the induced homomorphism between the abelianizations, and since  $F_{\mathcal{N}}$  and  $\rho_n$  are epimorphisms,  $\rho_n^{ab}$  is onto as well. In a similar way, take the splitting monomorphism  $s_n$  and restrict it onto  $\gamma_2(T\mathcal{N}_n)$  to obtain the monomorphism  $s_n^{(2)}$ , which satisfies

$$\rho_n^{(2)} \circ s_n^{(2)} = Id_{\gamma_2(T\mathcal{N}_n)}.$$

The homomorphism  $s_n^{ab} : T\mathcal{N}_n^{ab} \rightarrow T\mathcal{R}_n^{ab}$  is the induced homomorphism between the quotient groups. Commutativity  $s_n^{ab} \circ F_{\mathcal{N}} = F_{\mathcal{R}} \circ s_n$  follows directly from the definition of  $s_n^{ab}$ . The fact that  $s_n^{ab}$  satisfies  $\rho_n^{ab} \circ s_n^{ab} = Id_{T\mathcal{N}_n^{ab}}$ , follows from the commutativity  $\rho_n^{ab} \circ F_{\mathcal{R}} = F_{\mathcal{N}} \circ \rho_n$ . Indeed, take any  $\bar{x} \in T\mathcal{N}_n^{ab}$  and any  $x \in T\mathcal{N}_n$  such that  $F_{\mathcal{N}}(x) = \bar{x}$ . Then according to the definition of  $s_n^{ab}$ ,

$$\rho_n^{ab} \circ s_n^{ab}(\bar{x}) = \rho_n^{ab} \circ F_{\mathcal{R}} \circ s_n(x) = F_{\mathcal{N}} \circ \rho_n \circ s_n(x) = F_{\mathcal{N}}(x) = \bar{x}.$$

In particular, the splitting map  $s_n^{ab}$  is a monomorphism too.

Monomorphisms  $\iota_{\mathcal{R}}$  and  $\iota_{\mathcal{N}}$  are the inclusion homomorphism corresponding to the abelianization projections  $F_{\mathcal{R}}$  and  $F_{\mathcal{N}}$ , and the map  $F_{\mathcal{A}}$  is defined as the composition  $F_{\mathcal{A}} := F_{\mathcal{R}} \circ \mu_n$ . To see why  $F_{\mathcal{A}}$  is onto, one can use diagram chasing as follows. Take any  $x \in \ker(\rho_n^{ab})$ , and any  $\bar{x} \in T\mathcal{R}_n$  s.t.  $F_{\mathcal{R}}(\bar{x}) = x$ . Since  $F_{\mathcal{N}} \circ \rho_n(\bar{x}) = 0$ ,  $\exists y \in \gamma_2(T\mathcal{R}_n)$  s.t.  $\iota_{\mathcal{N}} \circ \rho_n^{(2)}(y) = \rho_n(\bar{x})$ . Then  $\rho_n(\iota_{\mathcal{R}}(y^{-1}) \cdot \bar{x}) = 1$ , i.e.  $\iota_{\mathcal{R}}(y^{-1}) \cdot \bar{x} \in T\mathcal{A}_n$ , and  $F_{\mathcal{A}}(\iota_{\mathcal{R}}(y^{-1}) \cdot \bar{x}) = F_{\mathcal{R}} \circ \mu_n(\iota_{\mathcal{R}}(y^{-1}) \cdot \bar{x}) = F_{\mathcal{R}}(\bar{x}) = x$ . Map  $\mu_n^{ab}$  is the inclusion  $\ker(\rho_n^{ab}) \hookrightarrow T\mathcal{R}_n^{ab}$ . The kernel of  $F_{\mathcal{A}}$  equals the intersection of  $T\mathcal{A}_n$  with the kernel of  $F_{\mathcal{R}}$ , i.e.

$$\ker(F_{\mathcal{A}}) = T\mathcal{A}_n \cap \gamma_2(T\mathcal{R}_n).$$

We saw in Corollary 2 that  $T\mathcal{A}_n/(T\mathcal{A}_n \cap \gamma_2(T\mathcal{R}_n)) \cong \mathbb{D}^* \times \mathbb{D}$ , so  $\ker(\rho_n^{ab}) \cong \mathbb{D}^* \times \mathbb{D}$ . It is clear that  $T\mathcal{A}_n \cap \gamma_2(T\mathcal{R}_n) \leq \ker(\rho_n^{(2)})$ . If we take any  $x \in \ker(\rho_n^{(2)}) < \gamma_2(T\mathcal{R}_n)$ , then  $\rho_n \circ \iota_{\mathcal{R}}(x) = 0$ , but  $\iota_{\mathcal{R}}$  is an inclusion, so we must have

$$x \in \ker(\rho_n) = T\mathcal{A}_n,$$

i.e.  $x \in T\mathcal{A}_n \cap \gamma_2(T\mathcal{R}_n)$ , and thus  $T\mathcal{A}_n \cap \gamma_2(T\mathcal{R}_n) = \ker(\rho_n^{(2)})$ .

If we denote this intersection  $T\mathcal{A}_n \cap \gamma_2(T\mathcal{R}_n)$  by  $\widetilde{T\mathcal{A}}_n$ , then we can put all the information from corollary 2 and the diagram 2.8 into the following commutative diagram, where  $n \geq 1$ .

$$\begin{array}{ccccccc}
& 0 & & 1 & & 1 & \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & \widetilde{T\mathcal{A}}_n & \xrightarrow{\mu_n^{(2)}} & \gamma_2(T\mathcal{R}_n) & \xrightarrow{\rho_n^{(2)}} & \gamma_2(T\mathcal{N}_n) & \longrightarrow 1 \\
& \downarrow & & \downarrow \iota_{\mathcal{R}} & \swarrow s_n^{(2)} & \downarrow \iota_{\mathcal{N}} & \\
0 \longrightarrow & T\mathcal{A}_n & \xrightarrow{\mu_n} & T\mathcal{R}_n & \xrightarrow{\rho_n} & T\mathcal{N}_n & \longrightarrow 1 \\
& \downarrow F_{\mathcal{A}} & & \downarrow F_{\mathcal{R}} & \swarrow s_n & \downarrow F_{\mathcal{N}} & \\
0 \longrightarrow & \mathbb{D}^* \times \mathbb{D} & \xrightarrow{\mu_n^{ab}} & T\mathcal{R}_n^{ab} & \xrightarrow{\rho_n^{(ab)}} & T\mathcal{N}_n^{ab} & \longrightarrow 0 \\
& \downarrow & & \downarrow & \swarrow s_n^{ab} & \downarrow & \\
& 0 & & 0 & & 0 &
\end{array} \tag{2.9}$$

When a short exact sequence of abelian groups

$$0 \longrightarrow H \longrightarrow G \xrightarrow{\quad} Q \longrightarrow 0$$

splits,  $G \cong H \times Q$ . Therefore the bottom exact sequence in 2.9 implies that for all  $n \geq 1$ , the abelianization of the truncated Riordan group is isomorphic to the direct product of the abelianization of the truncated Lagrange subgroup with  $\mathbb{D}^* \times \mathbb{D}$ , where the elements of  $\mathbb{D}^* \times \mathbb{D}$  correspond to the elements of  $T\mathcal{A}_1(\mathbb{D}) = T\mathcal{R}_1(\mathbb{D})$ .

Moreover, if we start with the exact sequence 1.2 instead of 2.3, and follow the same approach we used to derive 2.9, we obtain a completely analogous commutative diagram 2.10, where all truncated groups are replaced by the corresponding “original” groups  $\mathcal{A}(\mathbb{D})$ ,  $\mathcal{R}(\mathbb{D})$ , and  $\mathcal{N}(\mathbb{D})$  considered as abstract groups (i.e. with the discrete topology). All the maps in 2.10 are defined identically to the corresponding maps in 2.9, and by Corollary 2,  $\mathcal{A} \cap \gamma_2(\mathcal{R}) = \mathcal{A}_1$ .

$$\begin{array}{ccccccc}
& & 0 & & 1 & & 1 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{A}_1 & \xrightarrow{\mu^{(2)}} & \gamma_2(\mathcal{R}) & \xrightleftharpoons[\substack{s^{(2)} \\ \iota_{\mathcal{R}}}]{} & \gamma_2(\mathcal{N}) \longrightarrow 1 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{A} & \xrightarrow{\mu} & \mathcal{R} & \xrightleftharpoons[\substack{s \\ \rho}]{} & \mathcal{N} \longrightarrow 1 \\
& & \downarrow F_{\mathcal{A}} & & \downarrow F_{\mathcal{R}} & & \downarrow F_{\mathcal{N}} \\
0 & \longrightarrow & \mathbb{D}^* \times \mathbb{D} & \xrightarrow{\mu^{ab}} & \mathcal{R}^{ab} & \xrightleftharpoons[\substack{s^{ab} \\ \rho^{(ab)}}]{} & \mathcal{N}^{ab} \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & 0 & & 0 & & 0
\end{array} \tag{2.10}$$

From the bottom exact sequence in 2.10, we obtain the abelianization of the Riordan group in terms of the abelianization  $\mathcal{N}^{ab}(\mathbb{D})$ . Let us summarize this discussion as

**Theorem 3.** *Let  $\mathbb{D}$  be an arbitrary commutative ring with identity.*

- For each  $n \in \mathbb{N}$ ,

$$T\mathcal{R}_n^{ab}(\mathbb{D}) \cong T\mathcal{A}_1(\mathbb{D}) \times T\mathcal{N}_n^{ab}(\mathbb{D}) \cong \mathbb{D}^* \times \mathbb{D} \times T\mathcal{N}_n^{ab}(\mathbb{D}).$$

- Considering  $\mathcal{R}(\mathbb{D})$  and  $\mathcal{N}(\mathbb{D})$  as abstract groups,

$$\mathcal{R}^{ab}(\mathbb{D}) \cong \mathbb{D}^* \times \mathbb{D} \times \mathcal{N}^{ab}(\mathbb{D}).$$

## 2.1 Abelianization of $S\mathcal{R}(\mathbb{Z})$

L. Shapiro asked about the commutator group of the subgroup of  $\mathcal{R}(\mathbb{Z})$ , that has all 1s on the main diagonal (see [18], Q10). He denoted such a subgroup by  $S\mathcal{R}$ . Using results of I. Babenko and S. Bogatyi [2], on the commutator subgroup  $[\mathcal{J}(\mathbb{Z}), \mathcal{J}(\mathbb{Z})]$ , of the group of f.p.s. under substitution  $\mathcal{J}(\mathbb{Z})$ , we answer Shapiro's question below in Corollary 4. For the topology on  $\mathcal{J}(\mathbb{Z})$  the reader is referred to [2], or §5.8 in Babenko's survey [1], and topology on  $S\mathcal{R}(\mathbb{Z})$  is naturally induced by the epimorphism  $S\mathcal{R}(\mathbb{Z}) \rightarrow \mathcal{J}(\mathbb{Z})$ . The kernel of this epimorphism is the subgroup  $S\mathcal{A}(\mathbb{Z})$ , and Theorem 3 in this case says  $S\mathcal{R}^{ab}(\mathbb{Z}) \cong \mathbb{Z} \times \mathcal{J}^{ab}(\mathbb{Z})$ . Thus, from Proposition 2.9 in [2] and Theorem 3, we obtain

**Corollary 4.** *The Riordan array with integer coefficients  $c_i, \alpha_j \in \mathbb{Z}$*

$$(g, f) = \left(1 + \sum_{i \geq 1} c_i t^i, t(1 + \alpha_1 t + \dots + \alpha_6 t^6 + \mathcal{O}(t^7))\right) \in [S\mathcal{R}(\mathbb{Z}), S\mathcal{R}(\mathbb{Z})],$$

if and only if  $c_1 = 0$ , and  $\alpha_1 = \alpha_2 = 0$  and  $\alpha_3 \equiv \alpha_4 \equiv \alpha_6 \pmod{2}$ .  
In particular,

$$S\mathcal{R}^{ab}(\mathbb{Z}) \cong (\mathbb{Z})^3 \times (\mathbb{Z}_2)^2.$$

### 3 When $\mathbb{D} = \mathbb{F}_2$

In this section we describe the lower central series  $\gamma_i(\mathcal{R}(\mathbb{F}_2))$ ,  $i \geq 1$  together with the lower central quotients of  $\mathcal{R}(\mathbb{F}_2)$ , in terms of the lower central series and the corresponding quotients of the Nottingham group  $\mathcal{N}(\mathbb{F}_2)$ . The latter were calculated in [13], Proposition 12.4.30 (c.f. [5], Theorem 14.). S. Bogataya and S. Bogatyi also obtained the commutator  $[\mathcal{N}(\mathbb{F}_2), \mathcal{N}(\mathbb{F}_2)]$  together with the abelianization homomorphism in [4].

For the rest of the paper all Riordan arrays will have coefficients in  $\mathbb{F}_2$ , so I will drop the ground field  $\mathbb{D} = \mathbb{F}_2$  from all the corresponding notations. Recall that the group  $\mathcal{A}_n$  consists of the infinite Riordan arrays

$$(1 + \alpha_{n+1}t^{n+1} + \alpha_{n+2}t^{n+2} + \dots, t),$$

and in particular,  $\mathcal{A}_0 \equiv \mathcal{A}$ . Truncated Appell subgroups  $T\mathcal{A}_n$  are finite abelian 2-groups, and we can classify them in terms of the decomposition

$$T\mathcal{A}_n \cong \mathbb{Z}_{2^{r_1}} \times \mathbb{Z}_{2^{r_2}} \times \dots \times \mathbb{Z}_{2^{r_k}}, \quad (3.1)$$

where  $2^{r_1}, 2^{r_2}, \dots, 2^{r_k}$  are the *invariant factors* of  $T\mathcal{A}_n$ , i.e.  $r_i \geq r_{i+1}$  for each  $i \geq 1$ . It is clear that the cardinality  $|T\mathcal{A}_n| = 2^n$ . As we saw above, the sequence

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow T\mathcal{A}_{n+1}(\mathbb{F}_2) \xrightarrow{P_n} T\mathcal{A}_n(\mathbb{F}_2) \longrightarrow 0 \quad (3.2)$$

doesn't split in general, since  $T\mathcal{A}_2(\mathbb{F}_2) \cong C_4$  is a cyclic group of order 4. It will follow from the proof below that 3.2 splits if and only if  $2 \mid n$ .

**Theorem 5.** *For each integer  $n \geq 1$ ,*

$$T\mathcal{A}_{n+1} \cong \mathbb{Z}_{2^{r_1}} \times \mathbb{Z}_{2^{r_2}} \times \dots \times \mathbb{Z}_{2^{r_k}}, \quad (3.3)$$

where

$$k = \left\lfloor \frac{n+1}{2} \right\rfloor, \text{ and } r_j = 1 + \left\lfloor \log_2 \left( \frac{n+1}{2j-1} \right) \right\rfloor, \forall j \in \{1, \dots, k\}.$$

*Proof.* A congruence of Lucas (see §6.5 in Sagan's textbook [16]) states that for a prime  $p$  and  $k \in \{0, \dots, m\}$ ,

$$\binom{m}{k} \equiv \prod_{i \geq 0} \binom{m_i}{k_i} \pmod{p}, \quad (3.4)$$

where

$$m = \sum_{i \geq 0} m_i p^i \quad \text{and} \quad k = \sum_{i \geq 0} k_i p^i$$

are the base  $p$  expansions of  $m$  and  $k$  respectively (i.e.  $0 \leq m_i, k_i < p$ ). For prime  $p = 2$ , this congruence implies that for any natural  $q \geq 1$ ,

$$(1 + t^m)^{2^q} = \sum_{k=0}^{2^q} \binom{2^q}{k} t^{mk} \equiv 1 + t^{m2^q} \pmod{2}. \quad (3.5)$$

Since, for an arbitrary even  $L \leq n+1$  we can find an odd  $l \geq 1$  such that  $L = l2^q$  for some  $q \geq 1$ , 3.5 implies that in  $T\mathcal{A}_{n+1}$ ,  $(1 + t^l)^{2^q} = 1 + t^L$ . Therefore, the Riordan arrays  $a_m = (1 + t^m, t)$  with odd  $m = 2k - 1$  where  $k \in \{1, \dots, \lfloor (n+1)/2 \rfloor\}$ , generate the group  $T\mathcal{A}_{n+1}$ . The minimality of this set of generators follows from the fact that the derivative of  $1 + t^m \in \mathbb{F}_2[t]$  doesn't have 1 as a root, while  $1 + t^{2k-1}$  has 1 as a root for every  $k \geq 1$ . This proves the statement about the number of factors in the direct product decomposition (3.3).

The kernel of  $P_n$  is determined by the single entry in the lower left corner of the arrays from  $T\mathcal{A}_{n+1}$ . It means that we can use  $a_{n+1} = (1 + t^{n+1}, t)$  to represent the generator of  $\ker(P_n) \cong \mathbb{Z}/2\mathbb{Z}$ . The identity  $(1 + t^{n+1})^2 \equiv 1 \pmod{t^{n+2}}$  in  $\mathbb{F}_2[t]$  implies  $a_{n+1}^2 = I$ , and the question is if  $a_{n+1}$  can be written as a product of matrices lifted to  $T\mathcal{A}_{n+1}$  from  $T\mathcal{A}_n$  (i.e. the  $(n+2) \times (n+2)$  matrices with zero in the lower left corner). Here, by the lifting  $s : T\mathcal{A}_n \rightarrow T\mathcal{A}_{n+1}$ , I mean a *function* (i.e. not a priori a homomorphism), which assigns to the  $(n+1) \times (n+1)$  array  $(p(t), t) \in T\mathcal{A}_n$  given by a polynomial  $p(t)$  of degree  $d \leq n+1$ , exactly the same pair  $(p(t), t)$  considered now as an  $(n+2) \times (n+2)$  array in  $T\mathcal{A}_{n+1}$ . In fact, as will be explained later, the lifting  $s$  will be a homomorphism if and only if  $n$  is even.

For each odd  $m = 2j - 1$  with  $j \in \{1, \dots, \lfloor (n+1)/2 \rfloor\}$  we can find the unique integer  $p_m \geq 1$  s.t.

$$m2^{(p_m-1)} \leq n+1 < m2^{p_m}. \quad (3.6)$$

Applying 3.5 we obtain the identities

$$(1 + t^m)^{2^{(p_m-1)}} = \begin{cases} 1 + t^{n+1} & \text{if } m2^{(p_m-1)} = n+1 \\ 1 + t^{m2^{(p_m-1)}} & \text{if } m2^{(p_m-1)} < n+1. \end{cases} \quad (3.7)$$

The first line in 3.7 says that when  $n$  is odd and  $m$  satisfies  $n+1 = m2^{(p_m-1)}$  with  $p_m \geq 2$ , then  $a_{n+1} = a_m^{2^{(p_m-1)}}$ , that is the array  $a_{n+1} = (1 + t^{n+1}, t)$  is not a generator in  $T\mathcal{A}_{n+1}$ . In other words, the short exact sequence 3.2 doesn't split. The identity 3.7 also implies that for every odd  $m \in \{1, \dots, n\}$ , the order of  $a_m$  in  $T\mathcal{A}_{n+1}$  will be exactly  $2^{p_m}$ , since

$$m2^{(p_m-1)} \leq n+1 < m2^{p_m}.$$

Dividing by  $m$ , and taking  $\log_2$  gives  $p_m - 1 \leq \log_2((n+1)/m) < p_m$ , i.e.

$$p_m = p_{2j-1} = 1 + \left\lfloor \log_2 \left( \frac{n+1}{2j-1} \right) \right\rfloor, \quad (3.8)$$

and the theorem is proven for odd  $n$ . To prove the case when  $n$  is even, we show that the splitting function  $s : T\mathcal{A}_n \rightarrow T\mathcal{A}_{n+1}$  respects all the defining relations in the group  $T\mathcal{A}_n$ , and hence is a homomorphism. Now we know that if  $n$  is even,  $T\mathcal{A}_n$  is generated by the arrays  $a_{2j-1}, j \in \{1, \dots, n/2\}$  with the corresponding orders  $2^{p_{2j-1}}$  given by 3.8. Therefore we need to check if  $a_{2j-1}^{2^{p_{2j-1}}} = (1, t)$  in  $T\mathcal{A}_{n+1}$  for all  $j \in \{1, \dots, n/2\}$ . Using 3.5 one more time, we see that in  $\mathbb{F}_2[t]$

$$(1 + t^{2j-1})^{2^{p_{2j-1}}} = 1 + t^{(2j-1)2^{p_{2j-1}}} \equiv 1 \pmod{t^{n+2}},$$

since  $(2j-1)2^{p_{2j-1}} = m2^{p_m} > n+1$ . Thus, the short exact sequence 3.2 splits, and  $T\mathcal{A}_{n+1} \cong T\mathcal{A}_n \times \mathbb{Z}/2\mathbb{Z}$ . Using these generators with their orders in  $T\mathcal{A}_n$ , we can write a group presentation for the abelian group  $T\mathcal{A}_{n+1}$  (skipping all the commutators, since the group is commutative) as

$$\langle a_{2j-1}, j \in \{1, \dots, (n+2)/2\} \mid a_{n+1}^2 = I, a_{2j-1}^{2^{p_{2j-1}}} = I, j \in \{1, \dots, n/2\} \rangle.$$

To prove the formula for  $r_k$  with  $j = k = \lfloor (n+1)/2 \rfloor$ , take any even  $n = 2w \geq 2$ , then  $2k-1 = 2w-1$ . If  $w \geq 2$ , then

$$p_{2k-1} = 1 + \left\lfloor \log_2 \left( \frac{2w+1}{2w-1} \right) \right\rfloor = 1,$$

and if  $w = 1$ , then we get  $p_1 = 2$ , as stated in the theorem.  $\square$

Table 1 lists the invariant factors for the first few truncated Appell subgroups.

**Corollary 6.**  $\mathcal{A}/\mathcal{A}_1 \cong \mathbb{Z}_2$ , and for all  $n \geq 1$ ,

$$\mathcal{A}_{2n-1}/\mathcal{A}_{2n+1} \cong \mathbb{Z}_2 \times \mathbb{Z}_2.$$

*Proof.* Isomorphism  $\mathcal{A}/\mathcal{A}_1 \cong \mathbb{Z}_2$  is obvious. As follows from our proof of Theorem 5, the kernel of  $P_{2n-1} \circ P_{2n} : T\mathcal{A}_{2n+1} \rightarrow T\mathcal{A}_{2n-1}$  is generated by the Riordan arrays  $(1 + t^{2n}, t)$ , and  $(1 + t^{2n+1}, t)$ . Since each of them has order 2 in  $T\mathcal{A}_{2n+1}$ , we have  $\ker(P_{2n-1} \circ P_{2n}) \cong (\mathbb{Z}_2)^2$ . Now apply the snake lemma to the diagram below.

n	Invariant factors of $T\mathcal{A}_n(\mathbb{F}_2)$
1	{2}
2	{4}
3	{4,2}
4	{8,2}
5	{8,2,2}
6	{8,4,2}
7	{8,4,2,2}
8	{16,4,2,2}
9	{16,4,2,2,2}
10	{16,4,4,2,2}

Table 1: Invariant factors of  $T\mathcal{A}_n(\mathbb{F}_2)$  for  $n \in \{1, \dots, 10\}$ .

$$\begin{array}{ccccccc}
& & 0 & & 0 & \longrightarrow & (\mathbb{Z}_2)^2 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{A}_{2n+1} & \longrightarrow & \mathcal{A} & \longrightarrow & T\mathcal{A}_{2n+1} \longrightarrow 0 \\
& & \downarrow \pi & & \downarrow \equiv & & \downarrow P_{2n-1} \circ P_{2n} \\
0 & \longrightarrow & \mathcal{A}_{2n-1} & \longrightarrow & \mathcal{A} & \longrightarrow & T\mathcal{A}_{2n-1} \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & \text{Coker}(\pi) & \longrightarrow & 0 & & 0
\end{array} \tag{3.9}$$

□

Next we prove the main theorem.

**Theorem 7.** *For all integers  $n \geq 2$ ,*

$$\gamma_n(\mathcal{R}) \cong \mathcal{A}_{2n-3} \ltimes \gamma_n(\mathcal{N}), \tag{3.10}$$

*and the lower central quotients are*

$$\gamma_i(\mathcal{R})/\gamma_{i+1}(\mathcal{R}) \cong \begin{cases} (\mathbb{Z}_2)^3 \times \mathbb{Z}_4 & \text{if } i = 1 \\ (\mathbb{Z}_2)^4 & \text{if } i \geq 2 \text{ is even} \\ (\mathbb{Z}_2)^6 & \text{if } i > 1 \text{ is odd.} \end{cases} \tag{3.11}$$

*The abelianization of  $\mathcal{R}$  has the following group presentation in generators and relations*

$$\langle a, b_1, b_1b_2, b_4 \mid a^2 = (b_1b_2)^2 = b_4^2 = b_1^4 = 1, [b_i, b_j] = 1, [a, b_i] = 1 \rangle, \tag{3.12}$$

where  $a$  and  $b_i$  are the images of the corresponding elements

$$a_1 = (1 + t, t), \quad e_i = (1, t + t^{i+1}).$$

by the abelianization homomorphism  $F_{\mathcal{R}} : \mathcal{R} \rightarrow \mathcal{R}^{ab}$ .

**Corollary 8.** *The Riordan group over the field  $\mathbb{F}_2$  has width 6.*

Recall that for all  $k \geq 1$  we denote the Riordan array  $(u + \alpha t^k, t)$  by  $a_k(u, \alpha)$ , and the array  $(1, t + t^{k+1})$  by  $e_k$ , with  $a_0 = e_0 = (1, t)$ . In this section  $u = 1$ , so we can simplify the notations, and use  $a_k[\alpha]$  for  $(1 + \alpha t^k, t)$ , and  $a_k$  for  $a_k[1]$  respectively. Following the Definition 3 of [5] and Definition 12.4.6 of [13], we will say that an element  $(1, t + t^{m+1} + \dots) \in \mathcal{N}_m$  has depth  $m$ . The identity has depth  $\infty$ . To prove Theorem 7 we will need

**Lemma 9.** *For all  $n \geq 1$  and  $i \geq 1$ ,  $[a_n, e_i^{-1}] \in \mathcal{A}$ , and*

$$[a_n, e_i^{-1}] \equiv a_{n+i}[\bar{n}] \pmod{t^{n+i+1}},$$

where  $\bar{n} \in \{0, 1\}$  and  $\bar{n} \equiv n \pmod{2}$ .

*Proof.* It is clear that  $[a_n, e_i^{-1}] \in \mathcal{A}$ . If we denote the compositional inverse of  $t + t^{i+1}$  by  $\overline{t + t^{i+1}}$ , then we have  $\pmod{2}$

$$\begin{aligned} [a_n, e_i^{-1}] &= \left( \frac{1}{1 + t^n}, t + t^{i+1} \right) \left( 1 + t^n, \overline{t + t^{i+1}} \right) \\ &= \left( \frac{1 + (t + t^{i+1})^n}{1 + t^n}, t \right) = \left( \frac{1 + t^n + t^n \left( \sum_{s=1}^n \binom{n}{s} t^{si} \right)}{1 + t^n}, t \right) \\ &= (1 + t^n(nt^i + \mathcal{O}(t^{2i}))(1 + t^n + \mathcal{O}(t^{2n})), t) \\ &\equiv (1 + nt^{n+i}, t) \pmod{t^{n+1+i}} \end{aligned}$$

as required.  $\square$

**Lemma 10.** *For all integers  $n \geq 1$ ,*

$$\mathcal{A} \cap \gamma_{n+1}(\mathcal{R}) = \mathcal{A}_{2n-1}.$$

*Proof.* We use induction on  $n$ . Repeating computations from Lemma 1, one easily finds that for any  $m \geq 2$ ,

$$\left[ (1 + t, t), (1, t + t^m + t^{m+1})^{-1} \right] = (1 + t^m, t) = a_m,$$

which implies that  $\mathcal{A}_1 \leq \mathcal{A} \cap \gamma_2(\mathcal{R})$ . Considering the projection onto a commutative group  $\Pi_1 : \mathcal{R} \rightarrow T\mathcal{R}_1 \cong \mathbb{Z}_2$ , we see that  $a_1 \notin \gamma_2(\mathcal{R})$ , and

hence  $\mathcal{A} \cap \gamma_2(\mathcal{R}) \leq \mathcal{A}_1$ . Assume now that the statement holds true for all  $n \in \{1, \dots, L-1\}$ , so in particular,  $a_{2L-1} \in \mathcal{A}_{2L-3} = \mathcal{A} \cap \gamma_L(\mathcal{R})$ . Applying Lemma 9 to the commutator

$$[a_{2L-1}, (1, t+t^2)^{-1}] \in [\gamma_L(\mathcal{R}), \mathcal{R}] = \gamma_{L+1}(\mathcal{R}),$$

we deduce that  $a_{2L} \in \gamma_{L+1}(\mathcal{R}) \pmod{t^{2L+1}}$ . If

$$[a_{2L-1}, (1, t+t^2)^{-1}] = (1+t^{2L}+t^{2L+k_1}+\dots, t),$$

for some  $k_1 > 0$ , take the commutator

$$[a_{2L-1}, (1, t+t^{2+k_1})^{-1}] = (1+t^{2L+k_1}+\dots, t),$$

and consider the product

$$\begin{aligned} [a_{2L-1}, (1, t+t^2)^{-1}] [a_{2L-1}, (1, t+t^{2+k_1})^{-1}] \\ \equiv (1+t^{2L}, t) \pmod{t^{2L+k_1+1}}. \end{aligned} \quad (3.13)$$

If the product in 3.13 equals  $(1+t^{2L}+t^{2L+k_2}+\dots, t)$  for some  $k_2 > k_1$ , then take the product of three commutators

$$\prod_{i \in \{0 < k_1 < k_2\}} [a_{2L-1}, (1, t+t^{2+i})^{-1}],$$

and so on. Since the group  $[\gamma_L(\mathcal{R}), \mathcal{R}]$  is closed in pro- $p$  topology, we can take the limit of the Cauchy sequence made by such products (see Exercise (13) in §1.6, [20] for the definition), and conclude that  $a_{2L} = (1+t^{2L}, t) \in \gamma_{L+1}(\mathcal{R})$ . Considering the commutators  $[a_{2L-1}, (1, t+t^m)]$  for  $m \geq 3$ , one can show similarly that  $a_n \in \gamma_{L+1}(\mathcal{R})$  for all  $n \geq 2L$ , so  $\mathcal{A}_{2L-1} \leq \mathcal{A} \cap \gamma_{L+1}(\mathcal{R})$ .

To show that  $\mathcal{A} \cap \gamma_{L+1}(\mathcal{R}) \leq \mathcal{A}_{2L-1}$ , it will be enough to prove that  $a_{2L-i} \notin [\mathcal{R}, \gamma_L(\mathcal{R})]$  for all  $i \geq 1$ . Suppose to the contrary that, for example  $a_{2L-1} \in [\mathcal{R}, \gamma_L(\mathcal{R})]$ . Then for any integer  $n > 2L$  there exists  $m \in \mathbb{N}$ , such that

$$a_{2L-1} \equiv \prod_{i=1}^m [x_i, y_i] \pmod{t^n},$$

where

$$x_i \in \mathcal{R}, y_i \in \gamma_L(\mathcal{R}), i \in \{1, \dots, m\}.$$

Since  $T\mathcal{A}_{2L-1}$  is abelian, there must exist a commutator  $[x, y] \in [\mathcal{R}, \gamma_L(\mathcal{R})]$ , such that

$$\rho_{2L-1} \circ \Pi_{2L-1}(x) \neq 1 \text{ or } \rho_{2L-1} \circ \Pi_{2L-1}(y) \neq 1$$

in the subgroup  $\gamma_L(T\mathcal{N}_{2L-1})$  of the truncated Nottingham group  $T\mathcal{N}_{2L-1}$ . But according to Proposition 12.4.30, [13] the lowest depth of an element in  $\gamma_L(\mathcal{N})$  equals  $2L-1$ , that is  $P_{2L-1}(\gamma_L(\mathcal{N})) \cap T\mathcal{N}_{2L-1} = (1, t)$ . This contradiction finishes the proof.  $\square$

Proposition 12.4.30 also gives a group presentation of the abelianization  $\mathcal{N}^{ab} \cong (\mathbb{Z}_2)^2 \times \mathbb{Z}_4$  as

$$\mathcal{N}^{ab} \cong \langle b_1, b_1 b_2, b_4 \mid (b_1 b_2)^2 = b_4^2 = b_1^4 = 1, [b_i, b_j] = 1 \rangle, \quad (3.14)$$

where  $b_i$  stands for the image of  $e_i$  in the quotient group, that is

$$b_i := F_{\mathcal{N}}((1, t + t^{i+1})) \in \mathcal{N}^{ab}.$$

Hence, if we take  $\mathbb{D} = \mathbb{F}_2$  in the commutative diagram 2.10, and use Corollary 6 together with 3.14, we immediately obtain a group presentation for  $\mathcal{R}^{ab}$

$$\langle a, b_1, b_1 b_2, b_4 \mid a^2 = (b_1 b_2)^2 = b_4^2 = b_1^4 = 1, [b_i, b_j] = 1, [a, b_i] = 1 \rangle,$$

where  $a$  and  $b_i$  are the corresponding images by  $F_{\mathcal{R}}$ , i.e.

$$a := F_{\mathcal{R}}((1 + t, t)), \quad b_i := F_{\mathcal{R}}((1, t + t^{i+1})).$$

In particular, we have

$$\mathcal{R}^{ab} \cong \mathbb{Z}_2 \times \mathcal{N}^{ab} \cong (\mathbb{Z}_2)^3 \times \mathbb{Z}_4.$$

That proves the first isomorphism in 3.11, and 3.12.

**Corollary 11.** *For all integers  $n \geq 2$ ,*

$$\mathcal{A}_{2n-3} \cap \gamma_{n+1}(\mathcal{R}) = \mathcal{A}_{2n-1}.$$

*Proof.* Since for any  $k \in \{2, \dots, 2n\}$ ,  $\mathcal{A}_{2n-1} \leq \mathcal{A}_{2n-k}$ , and from Lemma 10  $\mathcal{A}_{2n-1} \leq \gamma_{n+1}(\mathcal{R})$ , the required equality follows from

$$\mathcal{A}_{2n-1} \leq \mathcal{A}_{2n-k} \cap \gamma_{n+1}(\mathcal{R}) \leq \mathcal{A} \cap \gamma_{n+1}(\mathcal{R}) = \mathcal{A}_{2n-1}.$$

□

Starting with the top exact sequence in the diagram 2.10

$$0 \longrightarrow \mathcal{A}_1 \xrightarrow{\mu^{(2)}} \gamma_2(\mathcal{R}) \xrightarrow{\rho^{(2)}} \gamma_2(\mathcal{N}) \longrightarrow 1, \quad (3.15)$$

using Corollary 11, and repeating the steps we used to derive 2.10, one obtains in exactly the same way the following commutative diagram.

$$\begin{array}{ccccccc}
& 0 & & 1 & & 1 & \\
& \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & \mathcal{A}_3 & \xrightarrow{\mu^{(3)}} & \gamma_3(\mathcal{R}) & \xrightarrow{\rho^{(3)}} & \gamma_3(\mathcal{N}) \longrightarrow 1 \\
& & \downarrow & & \downarrow \iota_{\mathcal{R}} & \xleftarrow{s^{(3)}} & \downarrow \iota_{\mathcal{N}} \\
0 & \longrightarrow & \mathcal{A}_1 & \xrightarrow{\mu^{(2)}} & \gamma_2(\mathcal{R}) & \xrightarrow{\rho^{(2)}} & \gamma_2(\mathcal{N}) \longrightarrow 1 \\
& & \downarrow F_{\mathcal{A}} & & \downarrow F_{\mathcal{R}} & \xleftarrow{s^{(2)}} & \downarrow F_{\mathcal{N}} \\
0 & \longrightarrow & \mathcal{A}_1/\mathcal{A}_3 & \xrightarrow{\mu} & \mathcal{R}_{2/3} & \xrightarrow{\rho^{(2/3)}} & \mathcal{N}_{2/3} \longrightarrow 0 \\
& & \downarrow & & \downarrow & \xleftarrow{s^{(2/3)}} & \downarrow \\
& 0 & & 0 & & 0 &
\end{array} \tag{3.16}$$

Symbols  $\mathcal{R}_{n/(n+1)} := \gamma_n(\mathcal{R})/\gamma_{n+1}(\mathcal{R})$  and  $\mathcal{N}_{n/(n+1)} := \gamma_n(\mathcal{N})/\gamma_{n+1}(\mathcal{N})$  stand for the corresponding lower central quotients of  $\mathcal{R}$  and  $\mathcal{N}$  respectively. Corollary 6 gives  $\mathcal{A}_1/\mathcal{A}_3 = \langle a_2, a_3 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ , and since  $\mathcal{N}_{2/3}$  is elementary abelian of rank 2 (see again Proposition 12.4.30 in [13]) we obtain

$$\gamma_2(\mathcal{R})/\gamma_3(\mathcal{R}) \cong (\mathbb{Z}_2)^4.$$

Repeating this argument we finish the proof of Theorem 7 by induction.

*Proof of Theorem 7.* If  $n = 2$ , the isomorphism  $\gamma_n(\mathcal{R}) \cong \mathcal{A}_{2n-3} \ltimes \gamma_n(\mathcal{N})$  follows from the exact sequence 3.15. Assume next that for all  $k \in \{1, \dots, n\}$ ,  $\gamma_k(\mathcal{R}) \cong \mathcal{A}_{2k-3} \ltimes \gamma_k(\mathcal{N})$ . In particular, we have the splitting short exact sequence

$$0 \longrightarrow \mathcal{A}_{2n-3} \xrightarrow{\mu^{(n)}} \gamma_n(\mathcal{R}) \xrightarrow{\rho^{(n)}} \gamma_n(\mathcal{N}) \longrightarrow 1 \tag{3.17}$$

Using this sequence together with Corollary 11, one derives the following commutative diagram, which is completely analogous to 3.16.

$$\begin{array}{ccccccc}
& 0 & & 1 & & 1 & \\
& \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & \mathcal{A}_{2n-1} & \xrightarrow{\mu^{(n+1)}} & \gamma_{n+1}(\mathcal{R}) & \xrightarrow{\rho^{(n+1)}} & \gamma_{n+1}(\mathcal{N}) \longrightarrow 1 \\
& & \downarrow & & \downarrow \iota_{\mathcal{R}} & & \downarrow \iota_{\mathcal{N}} \\
0 & \longrightarrow & \mathcal{A}_{2n-3} & \xrightarrow{\mu^{(n)}} & \gamma_n(\mathcal{R}) & \xrightarrow{\rho^{(n)}} & \gamma_n(\mathcal{N}) \longrightarrow 1 \\
& & \downarrow F_{\mathcal{A}} & & \downarrow F_{\mathcal{R}} & & \downarrow F_{\mathcal{N}} \\
0 & \longrightarrow & \mathcal{A}_{2n-3}/\mathcal{A}_{2n-1} & \xrightarrow{\mu} & \mathcal{R}_{n/(n+1)} & \xrightarrow{\rho^{(n/(n+1))}} & \mathcal{N}_{n/(n+1)} \longrightarrow 0 \\
& & \downarrow & & \downarrow s^{(n/(n+1))} & & \downarrow \\
& 0 & & 0 & & 0 &
\end{array} \tag{3.18}$$

The top exact sequence in this diagram proves the induction step, and hence the isomorphism 3.10, for all  $n \geq 1$ . Furthermore, since by Proposition 12.4.30, [13] we have

$$\gamma_i(\mathcal{N})/\gamma_{i+1}(\mathcal{N}) \cong \begin{cases} (\mathbb{Z}_2)^2 & \text{if } i \text{ is even} \\ (\mathbb{Z}_2)^4 & \text{if } i \text{ is odd,} \end{cases}$$

taking the direct product of each these quotients with  $\mathcal{A}_{2n-3}/\mathcal{A}_{2n-1} \cong (\mathbb{Z}_2)^2$ , proves formula 3.11, and finishes the proof.  $\square$

### 3.1 Dihedral groups $D_{2^q}$ as subgroups of $T\mathcal{R}_n$

It is known that every finite 2-group can be embedded in  $\mathcal{N}(\mathbb{F}_2)$ , and hence in  $\mathcal{R}(\mathbb{F}_2)$  (Corollary 12.4.11, [13]). A simple argument shows that for any embedding of a finite group  $G \hookrightarrow \mathcal{R}(\mathbb{D})$ , there is an induced embedding  $G \hookrightarrow T\mathcal{R}_n(\mathbb{D})$  for large enough  $n$  (Proposition 2, [12]). Therefore, for every  $q \geq 1$  there exists  $n \in \mathbb{N}$ , and an embedding  $\mu_n : D_{2^q} \hookrightarrow T\mathcal{R}_n(\mathbb{F}_2)$ . Here are a few examples of such embeddings, for small  $q$ .

- If  $q = 0$ ,  $n = 1$ :  $T\mathcal{R}_1 \cong \mathbb{Z}_2 \cong D_1$
- If  $q \in \{1, 2\}$ ,  $n = 2$ : For any ring  $\mathbb{D}$ ,  $T\mathcal{R}_2(\mathbb{D})$  is isomorphic to the Heisenberg group of upper unitriangular  $3 \times 3$  matrices with coefficients in  $\mathbb{D}$ , and  $T\mathcal{R}_2(\mathbb{F}_2) \cong D_4 > D_2$ .
- If  $q = 3$ ,  $n = 4$ : Direct computations show that the group  $T\mathcal{R}_3$  has 19 nontrivial involutions, and 12 elements of order 4, which makes it isomorphic to the central product  $D_4 \circ_Z D_4$  of two copies of  $D_4$  over the

center. Since  $D_8$  has an element of order 8,  $D_8 \not\prec T\mathcal{R}_3$ . However, there exists an embedding  $D_8 \hookrightarrow T\mathcal{R}_4$ . Indeed, taking  $r := (1+t, t) \in T\mathcal{A}_4$  as the element of order 8, and  $s := (1, t+t^2+t^3+t^4)$  as the involution, one easily computes that the subgroup  $\langle r, s \rangle < T\mathcal{R}_4$  generated by  $r$  and  $s$  has 16 distinct elements  $\{1, r, r^2, \dots, r^7, s, sr, sr^2, \dots, sr^7\}$  and hence is isomorphic  $D_8$ . Here are the finite Riordan array representations of these two generators.

$$r = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \quad s = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

This embedding is hardly a surprise, since  $T\mathcal{A}_4$  is the first truncated Appell subgroup containing an element of order 8 (recall Table 1). The next theorem extends this example.

**Theorem 12.** *Dihedral group  $D_{2^{n+1}}$  is embeddable in  $T\mathcal{R}_{2^n}$  for any  $n \geq 1$ .*

*Proof.* As the rotation, take the Riordan array  $r := (1+t, t) \in T\mathcal{A}_{2^n}$ , which is an element of order  $2^{n+1}$ , due to Theorem 3.3. For the reflection, take  $s := (1, t+t^2+\dots+t^{2^n}) \in T\mathcal{N}_{2^n}$ . Since in  $\mathbb{F}_2[[t]]$  for

$$\begin{aligned} f(t) &= t + \dots + t^{2^n} = t \frac{1+t^{2^n}}{1+t}, \quad f(f(t)) = t \frac{1+t^{2^n}}{1+t} \cdot \frac{1 + \left(t \frac{1+t^{2^n}}{1+t}\right)^{2^n}}{1 + t \frac{1+t^{2^n}}{1+t}} \\ &= \frac{t + \mathcal{O}(t^{2^n+1})}{1 + t^{2^n+1}} = (t + \mathcal{O}(t^{2^n+1})) (1 + \mathcal{O}(t^{2^n+1})) \equiv t \pmod{t^{2^n+1}}, \end{aligned}$$

the elements  $r$  and  $s$  in  $T\mathcal{R}_{2^n}$  have orders  $2^{(n+1)}$  and 2 respectively. Furthermore, the congruence

$$\begin{aligned} (1+t, t)(1, t+t^2+\dots+t^{2^n})(1+t, t) &\equiv (1+t, t+t^2+\dots+t^{2^n})(1+t, t) \\ &\equiv (1, t+t^2+\dots+t^{2^n}) \pmod{t^{2^n+1}} \end{aligned}$$

together with a simple argument confirming that all  $2^{(n+2)}$  elements

$$\{1, r, r^2, \dots, r^{2^{(n+1)}-1}, s, sr, sr^2, \dots, sr^{2^{(n+1)}-1}\}$$

are distinct, proves that the subgroup of  $T\mathcal{R}_{2^n}$ , generated by  $r$  and  $s$ , is isomorphic to  $D_{2^{n+1}} \cong \langle r, s \mid r^{2^{n+1}} = s^2 = 1, rsr = s \rangle$ .  $\square$

**Note 1.** *Taking the infinite Riordan arrays*

$$r = (1 + t, t) \text{ and } s = \left(1, \frac{t}{1+t}\right),$$

one easily checks that they satisfy the relations  $s^2 = (1, t)$ , and  $rsr = s$ . Thus, we obtain an embedding of the infinite dihedral group  $D_\infty$  in the infinite Riordan group  $\mathcal{R}(\mathbb{F}_2)$  that induces all the embeddings  $D_{2^{n+1}} \hookrightarrow T\mathcal{R}_{2^n}$  from Theorem 12 by truncation.

Jointly with Tian-Xiao He, the author found explicit embeddings of an arbitrary dihedral group of order  $2n$  into the infinite Riordan group over the ring of integers modulo  $n$  (Theorem 7, [11]). It would be interesting to see explicit embeddings of the dihedral groups  $D_{2^n}$  into the infinite Riordan and Nottingham groups over  $\mathbb{F}_2$ .

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