

Infinite-Horizon Optimal Control of Jump-Diffusion Models for Pollution-Dependent Disasters

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Contents

1	Introduction	4
2	Modelling Framework	6
2.1	Baseline: Homogeneous Poisson Process (HPP)	9
2.2	Pollution-Driven Disaster Intensity via Nonhomogeneous Poisson Process (NHPP)	10
2.3	Jump-Diffusion Pollution with Intensity Feedback	12
2.4	Generalized Framework with Poisson Random Measures (PRMs) . . .	14
2.4.1	An intermediate model: PRMs with no diffusive term	17
3	The Hamilton-Jacobi-Bellman PDE via the Dynamic Programming Principle	18
3.1	Preliminaries	18
3.1.1	Assumptions on the Stochastic Basis	18
3.1.2	Functional Objective	19
3.1.3	Dynamic Programming Principle	20
3.2	Models of constant Jump-size	21
3.2.1	Standard Poisson Process	21
3.2.2	Nonhomogeneous Poisson	27
3.2.3	Brownian-driven pollution with nonhomogeneous Poisson jumps	29
3.3	Models of random Jumps-size (PRMs)	31
3.3.1	Equation for the intermediate model	31

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3.3.2	Extension to randomized pollution	35
4	Analysis of solutions to Hamilton–Jacobi–Bellman Equations	38
4.1	Preliminaries	38
4.1.1	Hamiltonian and Optimality Conditions	38
4.1.2	Candidate form for the value function	39
4.2	Disasters at a constant arrival rate: Poisson process	41
4.3	Disasters at a dynamic arrival rate: Nonhomogeneous Poisson process	44
4.4	Disasters at a stochastic arrival rate: Jump–diffusion models	46
4.4.1	Brownian–driven pollution stock	46
4.4.2	Randomized magnitude of the disasters	47
5	Verification theorems and viscosity solutions	49
5.1	Non–Diffusive models	49
5.1.1	Standard Poisson: Verification theorem for the candidate value function	49
5.1.2	Nonhomogeneous Poisson: Verification theorem for the model with state–dependent jump intensity	54
5.2	Diffusive Models	56
5.2.1	Nonhomogeneous Poisson process	56
5.2.2	General jump–diffusion model	61
6	Conclusion	66

Abstract

The paper develops a unified framework for stochastic growth models with environmental risk, in which rare but catastrophic shocks interact with capital accumulation and pollution. The analysis begins with a Poisson process formulation, leading to a Hamilton–Jacobi–Bellman (HJB) equation with jump terms that admits closed–form candidate solutions and yields a composite state variable capturing exposure to rare shocks. The framework is then extended by endogenizing disaster intensity via a nonhomogeneous Poisson process, showing how environmental degradation amplifies macroeconomic risk and strengthens incentives for abatement. A further extension introduces pollution diffusion alongside state–dependent jump intensity, yielding a tractable jump–diffusion HJB that decomposes naturally into capital and pollution components under power–type value functions. Finally, a formulation in terms of Poisson random measures unifies the dynamics, makes arrivals and compensators explicit, and accommodates state–dependent magnitudes. Together, these results establish rigorous verification theorems and viscosity–solution characterizations for the associated integro–differential HJB equations, highlight how vulnerability emerges endogenously from the joint evolution of capital and pollution, and show that the prospect of rare, state–dependent disasters fundamentally reshapes optimal intertemporal trade–offs.

Keywords: stochastic control, Hamilton–Jacobi–Bellman equation, viscosity solutions, forward–backward stochastic differential equations with jumps, Poisson random measures, jump–diffusion processes, integro–differential equations, growth–environment models, pollution–dependent intensity, rare disasters.

1 Introduction

The analysis of economic growth under environmental risk raises fundamental mathematical questions about stochastic control in systems with both continuous dynamics and discontinuous jumps. Classical growth–environment models typically treat pollution and damages as deterministic or smoothly evolving, with dynamics governed by ordinary or stochastic differential equations without discontinuities. However, a growing empirical record shows that environmental risks also manifest as rare, catastrophic events – such as abrupt climate disasters, ecosystem collapses, or large–scale technological failures. These events arrive unpredictably, are naturally modelled as jumps, and can dramatically alter the trajectory of the economy.

The mathematical challenge is to develop a rigorous and tractable framework that incorporates both continuous fluctuations (e.g. Brownian noise in pollution dynamics) and state–dependent jump risks (e.g. disaster intensities increasing with pollution). Doing so requires advancing the theory of dynamic programming with nonlocal terms, deriving and analyzing the associated Hamilton–Jacobi–Bellman (HJB) equations, and verifying the optimality of candidate value functions under such dynamics.

This paper extends the model framework of [4] and contributes to the literature on growth under environmental risk by organizing and progressively generalizing the stochastic framework for growth–environment problems:

1. Starting from the benchmark case of a *standard Poisson process*, formulated in [4], we consider the social planner’s problem in an economy subject to rare disasters that destroy a fraction of the capital stock at a constant arrival intensity. Within this setting we derive the dynamic programming equation and the associated Hamilton–Jacobi–Bellman (HJB) equation.
2. We then extend the analysis to a *nonhomogeneous Poisson process*, where the arrival intensity of disasters depends on the pollution stock. This extension endogenizes the feedback from environmental degradation to catastrophe risk and leads to an HJB equation with a pollution–dependent jump term. We also provide conditions under which the value function solves this HJB equation.
3. Next, we incorporate *Brownian motion in the pollution dynamics*, yielding a jump–diffusion system in which pollution is affected by both continuous fluctuations and discrete shocks. In this setting the planner’s problem gives rise to an integro–differential HJB equation that combines a second–order local diffusion operator with nonlocal jump terms. We establish a verification theorem for sufficiently regular value functions and show that, under mild assumptions, the value function is a viscosity solution of the HJB equation.
4. Finally, we generalize the entire framework using *Poisson random measures with marks* which allows for random disaster magnitudes and a broad class

of jump specifications for both capital and pollution. In this most general formulation we give a characterization of the value function as the solution of a forward–backward stochastic differential equation (FBSDE) with jumps and establish the correspondence between solutions of the FBSDE system and (viscosity) solutions of the HJB equation. This unifies the previous models in a single stochastic control framework and facilitates extensions with random disaster magnitudes.

Thus, the contribution of the paper is threefold. First, on the modelling side, we organize a broad class of pollution–driven disaster specifications – ranging from constant–intensity Poisson shocks to pollution–dependent intensities, jump–diffusion dynamics, and formulations based on Poisson random measures with marks – into a single coherent stochastic control framework. This framework clarifies how existing models relate to each other and how richer forms of environmental risk can be captured without compromising analytical structure. Second, on the analytical side, we derive the associated Hamilton–Jacobi–Bellman equations and provide verification and viscosity–solution results that rigorously characterize the value function of the planner’s problem under general conditions on preferences, technology, and the jump mechanism. Third, we establish a representation of the value function in terms of the forward–backward stochastic differential equations with jumps, which links the control problem to FBSDE methods.

While motivated by economic questions of growth, environment, and climate risk, the primary contribution of this paper lies in the mathematical development of stochastic control techniques with jump risks. The results demonstrate how dynamic programming, HJB equations, and verification methods can be extended to incorporate rare, state–dependent disasters and diffusion–driven uncertainty. The considered models achieve both tractability and generality, providing a rigorous analytical foundation and practical tools for characterizing and solving the stochastic control problems that govern the long–run interaction between the economy and the environment in the presence of rare but potentially catastrophic events.

Thus, the paper enriches both the applied theory of sustainable growth and the broader mathematical toolbox for stochastic control with jumps. Within this setting, sustainability in the presence of deep uncertainty requires not only gradual adjustments in consumption, investment, and abatement, but also resilience against low–probability, high–impact shocks.

2 Modelling Framework

We consider a stylized representation of the global economy, which produces a single composite good under constant returns to scale. Production relies on the aggregate capital stock at time t , denoted by K_t which includes physical capital, human capital, and intangible assets. The production process $(K_t)_{t \geq 0}$ generates pollution: at each instant t , greenhouse gas (GHG) emissions E_t are released into the atmosphere. These emissions accumulate in the atmospheric pollution stock, P_t , which increases with the flow E_t and decreases at a natural absorption rate $\alpha \in [0, 1)$, assumed to be small or negligible. While P_t is referred to as the pollution stock, it can more generally be interpreted as the inverse of environmental quality. Similarly, the emissions variable E_t , can be viewed more broadly as any environmentally damaging by-product of economic activity.

The model incorporates the possibility of natural disasters (we refer to it as an “event”) which occur randomly over time. When such an event occurs, it instantaneously destroys a fraction of the capital stock. The surviving share of capital is given by

$$\omega : \mathcal{S} \rightarrow (0, 1), \quad \mathcal{S} := \mathbb{R}_{>0} \times \mathbb{R}_{>0}.$$

which is determined endogenously and depends on the current levels of pollution and capital:

$$\omega(K, P) = e^{-\delta P^\xi K^\eta}, \quad (2.1)$$

where $\xi \geq 0$ and $\eta \geq 0$ are parameters that capture the sensitivity of damage intensity to pollution and capital, respectively.

The output, $Y : \mathbb{R}_{>0} \rightarrow \mathbb{R}_{\geq 0}$ defined as $Y := Y(K)$, can be allocated to consumption C , investment in capital, or environmental protection. A fraction $\theta \in [0, 1]$ of output is allocated to abatement, yielding abatement investment $I : [0, 1] \times \mathbb{R}_{>0} \rightarrow \mathbb{R}_{\geq 0}$, defined as

$$I(\theta, K) = \theta Y(K). \quad (2.2)$$

The remaining share, $(1-\theta)Y$, is split between consumption and capital accumulation. Abatement activities reduce emissions through a function $Z : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$, with $Z' > 0$ and defined as

$$Z(I) = \sigma I, \quad (2.3)$$

where $\sigma > 0$ stands for the efficiency of abatement. In modelling emissions control, we assume abatement activities exhibit constant returns to scale. Thus, the total abatement is proportional to the resources allocated to it. Emissions $E : \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0} \rightarrow \mathbb{R}_{\geq 0}$ are thus given by the net balance between gross emissions – proportional to output with emission intensity $\phi > 0$ – and abatement,

$$E(I, K) = \phi Y(K) - Z(I). \quad (2.4)$$

Preferences are represented by a utility function $U : \mathbb{R}_{>0} \times \mathbb{R}_{>0} \rightarrow \mathbb{R}$ given by

$$U(C_t, P_t) = \frac{C_t^{1-\varepsilon}}{1-\varepsilon} - \chi \frac{P_t^{1+\beta}}{1+\beta}, \quad \varepsilon > 0, \varepsilon \neq 1, \beta > 0, \chi > 0, \quad (2.5)$$

where ε measures relative risk aversion, β governs the curvature of disutility from pollution, and χ reflects the weight placed on pollution in the utility function. The utility function is twice continuously differentiable $U \in C^2(\mathcal{S})$.

To ensure the well-posedness of the optimization problem, we have to check the following properties of the utility function, defined in (2.5):

1. Monotonicity and concavity.

We check the properties of U with respect to C , treating P as fixed:

$$\begin{aligned} \frac{\partial U}{\partial C}(C, P) &= C^{-\varepsilon} > 0 \quad \text{for all } C > 0, \\ \frac{\partial^2 U}{\partial C^2}(C, P) &= -\varepsilon C^{-\varepsilon-1} < 0 \quad \text{for all } C > 0. \end{aligned} \quad (2.6)$$

We also consider the dependence on P , treating C as fixed:

$$\begin{aligned} \frac{\partial U}{\partial P}(C, P) &= -\chi P^\beta < 0 \quad \text{for } P > 0, \\ \frac{\partial^2 U}{\partial P^2}(C, P) &= -\chi \beta P^{\beta-1} < 0. \end{aligned} \quad (2.7)$$

Therefore, the utility function is strictly increasing (i.e., monotonic preference for more consumption) and concave in consumption C , and decreasing and concave in the pollution stock P .

2. Inada conditions in C .

We check the boundary behavior of the marginal utility of C for all $P > 0$:

$$\begin{aligned} \lim_{C \rightarrow 0^+} \frac{\partial U}{\partial C}(C, P) &= \lim_{C \rightarrow 0^+} C^{-\varepsilon} = +\infty, \\ \lim_{C \rightarrow \infty} \frac{\partial U}{\partial C}(C, P) &= \lim_{C \rightarrow \infty} C^{-\varepsilon} = 0. \end{aligned} \quad (2.8)$$

The utility function satisfies the Inada conditions with respect to consumption C at both the initial time $t = 0$ and in the limit as $t \rightarrow \infty$ for all $P > 0$. These conditions imply that the marginal utility of consumption becomes unbounded as consumption approaches zero and vanishes as consumption becomes arbitrarily large. Under standard regularity assumptions, this ensures the existence of interior optimal consumption paths.

3. Boundedness from above.

We observe that the map $C \mapsto \frac{C^{1-\varepsilon}}{1-\varepsilon}$ is strictly increasing and strictly concave on $\mathbb{R}_{>0}$. Its behavior as $C \rightarrow \infty$ depends on the parameter ε as follows:

- If $\varepsilon < 1$, then

$$\lim_{C \rightarrow \infty} \frac{C^{1-\varepsilon}}{1-\varepsilon} = \infty, \quad (2.9)$$

and hence $U(C, P) \rightarrow \infty$ as $C \rightarrow \infty$ for any fixed $P > 0$.

- If $\varepsilon = 1$, then the utility function takes the logarithmic form

$$U(C, P) = \log C - \chi \frac{P^{1+\beta}}{1+\beta}, \quad (2.10)$$

and $\lim_{C \rightarrow \infty} \log C = \infty$.

- If $\varepsilon > 1$, then

$$\lim_{C \rightarrow \infty} \frac{C^{1-\varepsilon}}{1-\varepsilon} = 0, \quad (2.11)$$

and the utility function is bounded above by 0 for each fixed $P > 0$.

Moreover, for each fixed $C > 0$, the disutility from pollution, $-\chi \frac{P^{1+\beta}}{1+\beta}$, satisfies

$$\lim_{P \rightarrow 0^+} -\chi \frac{P^{1+\beta}}{1+\beta} = 0 \quad \text{and} \quad \lim_{P \rightarrow \infty} -\chi \frac{P^{1+\beta}}{1+\beta} = -\infty. \quad (2.12)$$

Thus, the utility function $U(C, P)$ is unbounded below as $P \rightarrow \infty$, and it is bounded above on \mathcal{S} if and only if $\varepsilon > 1$. If $\varepsilon \leq 1$, then $U(C, P) \rightarrow \infty$ as $c \rightarrow \infty$, and the utility function is unbounded above in the consumption C .

Consequently, by [10], the utility function (2.5) satisfies all standard regularity conditions required in dynamic optimization and Hamilton–Jacobi–Bellman frameworks.

Given $(K_0, P_0) \in \mathcal{S}$, the social planner aims to maximize the expected discounted utility over an infinite horizon by choosing optimal paths for consumption C_t and abatement share θ_t ,

$$v(K_0, P_0) = \sup_{(C, \theta)} \mathbb{E} \left[\int_0^\infty e^{-\rho t} U(C_t, P_t) dt \right], \quad (2.13)$$

subject to (model-specific) jump terms and the common drift components

$$b_K(K_t, P_t, C_t, \theta_t) := (1 - \theta_t)Y(K_t) - C_t, \quad (2.14)$$

$$b_P(K_t, P_t, C_t, \theta_t) := \phi Y(K_t) - Z(\theta_t Y(K_t)) - \alpha P_t, \quad (2.15)$$

where $\rho > 0$ defines the constant rate of time preference.

The planner's problem is then to optimally allocate resources between consumption and abatement in order to balance economic growth, environmental quality, and resilience against environmentally driven disasters.

We analyze four increasingly rich variants of the model (2.13), which share the state (K, P) , the controls (C, θ) , the drift parts (2.14)–(2.15) and the preferences (2.5):

1. Constant arrival rate (homogeneous Poisson – HPP). Events arrive with constant intensity $\lambda > 0$; jumps are unit-sized and destroy a state-dependent fraction of capital via $\omega(K_t, P_t)$.
2. Pollution-driven intensity (NHPP). The intensity becomes state-dependent, $\lambda(P_t)$ (e.g., affine $\lambda_0 + \lambda_1 P_t$), introducing feedback from environmental quality to disaster risk.
3. Jump-diffusion with stochastic pollution. We keep $\lambda(P_t)$ and add Brownian fluctuations in P_t , yielding an integro-diffusion HJB.
4. Jump-diffusion with stochastic pollution and marked jumps (Poisson random measures). We generalize the previous model by incorporating a marked point process $q(dt, d\zeta)$ determined by the compensator $\lambda(P_t, \zeta)dt \otimes \nu(d\zeta)$. This change allows random disaster magnitudes to be considered in the model, therefore linking the problem to that of nonlocal operators and Partial Integro-Differential Equations (PIDEs).

We work on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ satisfying the usual conditions of right-continuity and \mathbb{P} -completeness. Each subsection below modifies only the jump mechanism and/or the law of P_t ; all other components remain the same.

2.1 Baseline: Homogeneous Poisson Process (HPP)

On a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we define a Poisson process $q = (q_t)_{t \geq 0}$ with intensity $\lambda > 0$, that is, an \mathbb{F} -adapted càdlàg process with values in \mathbb{N}_0 such that

- (i) $q_0 = 0$ almost surely,
- (ii) q is continuous in probability,
- (iii) the increments are stationary and independent, i.e., for all $0 \leq s < t$, the random variable $q_t - q_s$ is independent of \mathcal{F}_s and $q_t - q_s \sim \text{Poisson}(\lambda(t - s))$.

The parameter λ can be characterized as

$$\lambda := \lim_{h \rightarrow 0} \frac{1}{h} \mathbb{P}[q_t - q_{t-h} = 1]. \quad (2.16)$$

Over a small interval $[t-h, t]$, the probability of two or more jumps satisfies $\mathbb{P}[q_t - q_{t-h} \geq 2] = o(h^2)$ as $h \rightarrow 0$, therefore we consider only scenarios with 0 or 1 jump in that interval. Then for $t > 0$, we can define the jump size of the Poisson process q ,

$$\Delta q_h := q_h - q_{h-} \in \{0, 1\}, \quad h \geq 0, \quad (2.17)$$

where $q_{t-} := \lim_{s \nearrow t} q_s$, for all $t > 0$.

In the model, each unit jump of q represents the occurrence of a natural-disaster *event*. Thus, events arrive according to a Poisson process with mean arrival rate $\lambda > 0$. Given the initial $(K_0, P_0) \in \mathcal{S}$, capital and pollution dynamics are given by

$$dK_t = b_K(K_t, P_t, C_t, \theta_t) dt - (1 - \omega(K_{t-}, P_t)) K_{t-} dq_t, \quad (2.18)$$

$$dP_t = b_P(K_t, P_t, C_t, \theta_t) dt. \quad (2.19)$$

We next relax the constant-hazard assumption by letting the arrival rate respond to the contemporaneous pollution stock P_t .

2.2 Pollution-Driven Disaster Intensity via Nonhomogeneous Poisson Process (NHPP)

In the model described in Section 2.1, the arrival of natural disasters is modelled as a homogeneous Poisson process $(q_t)_{t \geq 0}$ with constant intensity $\lambda > 0$ defined in (2.16). The process is memoryless and the expected number of events up to time t is $\mathbb{E}[q_t] = \lambda t$. This means that the expected frequency of destructive events is independent of the state of the environment. However, this assumption neglects the substantial empirical evidence linking environmental degradation to increased disaster likelihood. Rising concentrations of greenhouse gases, for instance, have been associated with heightened frequency and severity of extreme climatic events, such as storms, floods, and droughts. To capture the empirically supported notion that environmental degradation amplifies the frequency of natural disasters, we generalize the arrival process to a nonhomogeneous Poisson process whose intensity depends on the current stock of pollution P_t . Specifically, we replace the constant intensity λ with a state-dependent stochastic intensity $\lambda_t := \lambda(P_t)$, where

$$\lambda : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}, \quad \lambda \in C^1, \quad \lambda'(P) > 0. \quad (2.20)$$

This modification reflects the empirical and theoretical insight that increased pollution – interpreted broadly as environmental degradation – raises the likelihood of

extreme climate-related events. As the pollution stock grows, the expected frequency of disasters increases, thereby introducing an additional endogenous channel through which environmental harm amplifies economic losses. This modification creates a two-way feedback mechanism: economic activity degrades the environment, which in turn raises the incidence of disasters, leading to capital destruction and further economic vulnerability.

Rare destructive events are now modelled by a nonhomogeneous Poisson process $\hat{q} = (\hat{q}_t)_{t \geq 0}$ with state-dependent intensity $\lambda(P_t)$, where P_t denotes the current level of pollution. That is, \hat{q} is an \mathbb{F} -adapted càdlàg process with values in \mathbb{N}_0 such that

1. $\hat{q}_0 = 0$ almost surely,
2. for all $0 \leq s < t$, conditional on \mathcal{F}_s , the increment $\hat{q}_t - \hat{q}_s$ is independent of the past and satisfies

$$\hat{q}_t - \hat{q}_s \sim \text{Poisson} \left(\int_s^t \lambda(P_u) du \right).$$

In this setting, the intensity function $\lambda : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{>0}$ governs the instantaneous probability of an event. Formally, for small $h > 0$, $\lambda(P_t) = \lim_{h \searrow 0} \frac{1}{h} \mathbb{E}[\hat{q}_t - \hat{q}_{t-h}]$, and

$$\mathbb{P}(\hat{q}_t - \hat{q}_{t-h} = 1 | \mathcal{F}_t) = \lambda(P_t)h + o(h), \quad \mathbb{P}(\hat{q}_t - \hat{q}_{t-h} \geq 2 | \mathcal{F}_t) = o(h^2). \quad (2.21)$$

Hence, the jump size

$$\Delta \hat{q}_t := \hat{q}_t - \hat{q}_{t-} \in \{0, 1\}, \quad t \geq 0, \quad (2.22)$$

where $\hat{q}_{t-} := \lim_{s \nearrow t} \hat{q}_s$, for all $t > 0$. Equivalently, there exists a nonnegative, (\mathcal{F}_t) -predictable process $\lambda(P_t)$ such that the compensator of \hat{q} is given by

$$\Lambda_t = \int_0^t \lambda(P_s) ds. \quad (2.23)$$

By the Doob–Meyer decomposition theorem [17], the nonhomogeneous Poisson process \hat{q} admits the representation

$$\hat{q}_t = M_t + \Lambda_t, \quad t \geq 0, \quad (2.24)$$

where $M_t := \hat{q}_t - \Lambda_t$ is an (\mathcal{F}_t) -martingale.

Equivalently, the jump integral can be written with the compensator as

$$\int_0^t H_s d\hat{q}_s = \int_0^t H_s dM_s + \int_0^t H_s \lambda(P_s) ds, \quad (2.25)$$

for any bounded predictable process H . In particular, the martingale property of M implies that

$$\mathbb{E} \left[\int_0^t H_s dM_s \right] = 0, \quad t \geq 0. \quad (2.26)$$

This property will allow us to simplify the expectation of the jump contribution.

We specify the pollution-dependent hazard rate as

$$\lambda(P_t) = \lambda_0 + \lambda_1 P_t, \quad \lambda_0 \geq 0, \lambda_1 \geq 0, \quad (2.27)$$

where λ_0 represents the baseline hazard rate unrelated to environmental conditions, and λ_1 measures the marginal increase in disaster risk per unit of pollution stock. This specification ensures that hazard rates rise linearly with environmental damage, allowing the model to reflect the dual role of abatement: reducing both the direct disutility from pollution and the frequency of capital-destroying events. From a technical perspective, the affine form of $\lambda(\cdot)$ preserves much of the tractability of the homogeneous case while introducing a meaningful state dependence in the jump intensity, thereby enriching the policy implications without introducing the analytical complexity of fully general Lévy jump structures.

In particular, the map $\lambda : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is continuous, locally Lipschitz, and of at most linear growth:

$$|\lambda(P)| \leq c_0 + c_1 P \quad \text{for some } c_0, c_1 \geq 0. \quad (2.28)$$

These ensure integrability of the compensator and well-posedness of the jump term.

Thus, at each jump time of \hat{q} , a fraction $\omega(K_{t-}, P_{t-}) \in (0, 1)$ of the capital stock survives, i.e.,

$$K_t = \omega(K_{t-}, P_{t-}) K_{t-} \quad \text{if } \Delta \hat{q}_t = 1, \quad (2.29)$$

with pollution unchanged contemporaneously, $P_t = P_{t-}$.

Hence, the dynamics of capital and pollution becomes

$$dK_t = b_K(K_t, P_t, C_t, \theta_t) dt - (1 - \omega(K_{t-}, P_{t-})) K_{t-} d\hat{q}_t, \quad (2.30)$$

$$dP_t = b_P(K_t, P_t, C_t, \theta_t) dt \quad (2.31)$$

Remark 2.1. It is worth noting that the model proposed above describes a weak form of self-exciting interaction: capital and pollution are coupled through bidirectional feedback. In contrast to the previous setting where the controller was fully subject to the exogenous timing of disasters, the present model allows control actions to delay the occurrence of adverse events that reduce capital. This coupling between the action of the planner and the occurrence of disasters is only strengthened in the models to come, as we will see further.

In the next section, we introduce continuous environmental uncertainty by adding diffusion to P_t while keeping $\lambda(P_t)$.

2.3 Jump-Diffusion Pollution with Intensity Feedback

We now extend the framework of Section 2.2 by allowing the pollution stock itself to be subject to stochastic fluctuations driven by Brownian motion, while

retaining the specification of rare destructive events through a nonhomogeneous Poisson process whose intensity depends on the current level of pollution. This enriches the dynamics of the economy by capturing both continuous environmental uncertainty and state-dependent disaster risk.

The economy is now subject to two sources of risk:

- rare, destructive events (disasters) which are captured by a nonhomogeneous Poisson process with intensity increasing in the pollution stock;
- continuous fluctuations in pollution, i.e. pollution evolves according to a diffusion process with multiplicative noise.

The dynamics of capital K_t follows according to (2.30), while the dynamics of pollution P_t with initial condition $P_0 > 0$ are given by

$$dP_t = b_P(K_t, P_t, C_t, \theta_t) dt + \sigma_P P_t dW_t, \quad (2.32)$$

where $W = (W_t)_{t \geq 0}$ is a Brownian motion and $\sigma_P > 0$ is the diffusion parameter. The multiplicative term $\sigma_P P_t dW_t$ captures proportional (log-normal-type) fluctuations in pollution (e.g., meteorological dispersion or natural absorption shocks).

First, from an economic perspective, stochastic pollution introduces time-varying disaster exposure: because the arrival intensity depends on P_t , abatement that lowers P_t reduces not only expected pollution damages but also the volatility of disaster risk through the intensity channel. Thus, the multiplicative term $\sigma_P P_t dW_t$, combined with the existing mean-reverting forces in b_P (natural decay $-\alpha P_t$ and abatement), yields a positive diffusion: starting from $P_0 > 0$, the process P_t remains strictly positive. This allows us to work on the natural state space $(K_t, P_t) \in \mathcal{S}$, which is exactly the domain on which preferences $U(C_t, P_t)$ and damage functions such as $\omega(K_t, P_t)$ are defined, and it avoids any need for artificial boundary behaviour at $P = 0$ in the HJB problem. Second, the geometric-type specification has a clear economic interpretation in terms of proportional environmental shocks. Conditionally on \mathcal{F}_t , the conditional variance of P_t over a short interval is

$$\text{Var}(dP_t \mid \mathcal{F}_t) = \sigma_P^2 P_t^2 dt, \quad (2.33)$$

so the relative volatility dP_t/P_t is driven by a constant parameter σ_P , while the absolute volatility of P_t increases with the level of the pollution stock. This matches the idea that, at an aggregate level, environmental and measurement uncertainties scale approximately in percentage terms: (i) when pollution is low, random fluctuations due to meteorological dispersion, natural sinks, or policy implementation are small in absolute terms; (ii) when pollution is high, similar percentage disturbances translate into larger absolute changes in P_t . This feature is particularly important in our framework because both the arrival intensity of disasters, $\lambda(P_t)$, and (in the marked setting) the distribution of disaster magnitudes depend on P_t . Multiplicative

noise implies that highly polluted states are not only more damaging in expectation, but also more uncertain: disaster risk becomes both higher and more volatile as P_t rises. In this stochastic setting, abatement has a dual role. By lowering the pollution stock P_t , it reduces the direct disutility from pollution and the expected size of environmentally driven losses. At the same time, because the disaster arrival intensity $\lambda(P_t)$ is increasing in P_t and P_t itself is stochastic, abatement also stabilizes the risk environment: it decreases both the level and the variability of the disaster intensity. In other words, abatement mitigates expected pollution damages and acts as a form of risk management by reducing the volatility of disaster risk. Third, the specification (2.32) is analytically convenient and in line with standard continuous-time macro-finance modelling. It leads to an integro-diffusion HJB equation whose second-order term in P has the form $\frac{1}{2}\sigma_P^2 P^2 v_{PP}(K, P)$, for the value function v , so that the resulting control problem falls within the well-developed theory of stochastic control with positive diffusions (see, e.g., [10]). Moreover, the deterministic benchmark of our model is recovered in the limit $\sigma_P \rightarrow 0$, so the stochastic specification is a natural enrichment of the baseline framework rather than a qualitatively different model. Thus, the geometric-type diffusion in (2.32) achieves a transparent balance: it respects the non-negativity of pollution, generates level-dependent proportional shocks with a natural economic meaning, and keeps the associated control problem in a tractable and widely used class.

Finally, to allow for random disaster magnitudes in a unified way, we move to a marked Poisson random measure representation.

2.4 Generalized Framework with Poisson Random Measures (PRMs)

The model formulation using Poisson Random Measures (PRMs) approach provides a unified framework for modelling jump processes: it makes explicit the random measure of disaster arrivals and their compensator, and allows us to write the dynamics of capital in terms of both drift and martingale components. This generalization not only gives a more rigorous mathematical foundation to the model, but also facilitates extensions, such as allowing for disaster magnitudes. In this way, the PRM formulation serves as a bridge between the models introduced in Sections 2.1 – 2.3 and richer specifications with marked jumps. Within this PRM framework, the social planner selects consumption and abatement paths to balance output growth, environmental quality, and resilience to pollution-driven disasters, whose arrivals are governed by a state-dependent intensity and whose magnitudes may be modelled as random marks.

Intuitively, the current PRM-based model constitutes an extension of the ones introduced earlier by incorporating an additional component into the jump process. This added component allows to model, for example, not only the frequency at which

disasters occur but also their magnitude as a function of the current pollution stock:

- the higher the pollution stock P_t , the more frequent disasters become.
- the higher the pollution stock P_t , the more destructive disasters become.

Mathematically, we now consider q as a Poisson random measure

$$q : (\Omega, \mathcal{F}, \mathbb{P}) \longrightarrow \mathcal{M}_c^*([0, \infty) \times [0, \infty)),$$

where $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a prescribed probability space and $\mathcal{M}_c^*(\mathcal{X})$ denotes the space of simple, counting Borel measures over \mathcal{X} , endowed with the topology of weak convergence, see [8]–[9]. This means that q is of the form

$$q(dt, d\zeta) = \sum_{n \geq 1} \delta_{(\tau_n, \Delta_n)}(dt, d\zeta), \quad (2.34)$$

where τ_n denotes the occurrence time of the n -th disaster, and Δ_n its magnitude. In fact, we have that Δ_n follows the conditional law

$$\mathbb{P}[\Delta_n \in B | \sigma\{(\tau_1, \Delta_1), \dots, (\tau_{n-1}, \Delta_{n-1})\} \vee \sigma\{\tau_n\}] \propto \int_B \lambda(P_{\tau_n}, \zeta) \nu(d\zeta), \quad (2.35)$$

for all $B \in \mathcal{B}(\mathbb{R}_{\geq 0})$, up to a normalizing constant.

Regarding the compensator of q , it is assumed to have the form

$$\Lambda(dt, d\zeta) := \lambda(P_t, \zeta) dt \otimes \nu(d\zeta), \quad (2.36)$$

where ν is a finite measure over $\mathbb{R}_{\geq 0}$ with finite second moment, and

$$\lambda : \mathbb{R}_{\geq 0}^2 \rightarrow \mathbb{R}_{\geq 0}$$

is a continuous differentiable function with

$$\frac{\partial}{\partial P} \lambda(P, \zeta) > 0, \quad (P, \zeta) \in \mathbb{R}_{>0} \times \mathbb{R}_{\geq 0}$$

and

$$\int_{[0, \infty)} (1 + \zeta^2) \lambda(P, \zeta) \nu(d\zeta) < \infty, \quad \forall P > 0.$$

Remark 2.2. To allow models in which the disaster arrival rate is endogenous (i.e., the intensity λ of the jump measure q may depend on P), we model q as a random measure with a (possibly random) intensity measure Λ . To overcome this technicality, it is necessary to ensure our stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is sufficiently rich. We will address this issue more precisely at the beginning of the next section.

With these assumptions in place, the controlled dynamics now become

$$dK_t = b_K(K_t, P_t, C_t, \theta_t) dt - \int_{[0, \infty)} (1 - \omega(K_{t-}, P_t, \zeta)) K_{t-} q(dt \times d\zeta), \quad (2.37)$$

$$dP_t = b_P(K_t, P_t, C_t, \theta_t) dt + \sigma_P P_t dW_t, \quad (2.38)$$

$$K_0 > 0, \quad P_0 > 0, \quad (2.39)$$

where b_K and b_P are as in (2.14) – (2.15), $(C_t)_{t \geq 0}$ and $(\theta_t)_{t \geq 0}$ are \mathbb{F} -adapted processes such that the system (2.37) – (2.39) is well-defined, and

$$(K, P, \zeta) \longmapsto \omega(K, P, \zeta) \in (0, 1)$$

represents the surviving proportion of capital after a disaster, which is dependent on the current levels capital K and of pollution P , as well as the magnitude of the disaster ζ .

The function ω is assumed to be continuous in all its coordinates, and to satisfy the integrability condition

$$\int_{[0, \infty)} \omega(K, P, \zeta)^2 \lambda(P, \zeta) \nu(d\zeta) < \infty \quad \forall (K, P) \in \mathcal{S}, \quad \forall t \geq 0.$$

Moreover, we assume ω is decreasing on K (resp. P and ζ), thereby reflecting a higher vulnerability when capital intensity (resp. level of pollution and magnitude of the disaster) increases.

As stated previously, this model is motivated by the notion that the level of pollution affects not only the frequency but also the magnitude of risk at play. For example, consider the following: let q be a Poisson random measure determined by its compensator

$$\Lambda(dt, d\zeta) := \frac{\lambda(P_{t-})}{\Gamma(P_{t-})} \zeta^{P_{t-}-1} e^{-\zeta} dt \otimes d\zeta, \quad (2.40)$$

where $\lambda : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is a measurable function; that is, the kernel Λ corresponds to $\lambda(P_t)$ times the density of a Gamma distribution of shape parameter P_t and scale parameter 1.

As a result, from (2.34) and (2.35) we have that the disasters q arrive as a Poisson point process of intensity

$$\mathbb{E}[q((0, t] \times \mathbb{R}_{\geq 0})] = \int_{(0, t]} \lambda(P_{t-}) \left(\int_{\mathbb{R}_{\geq 0}} \frac{1}{\Gamma(P_{t-})} \zeta^{P_{t-}-1} e^{-\zeta} d\zeta \right) dt = \int_{(0, t]} \lambda(P_{t-}) dt,$$

and their magnitude follows a conditional distribution $\text{Gamma}(P_t, 1)$,

$$\begin{aligned} \mathbb{P}[\zeta_n \in B | \sigma\{(\tau_1, \zeta_1), \dots, (\tau_{n-1}, \zeta_{n-1})\} \vee \sigma\{\tau_n\}] \\ = \frac{\Lambda(\{\tau_n\}, B)}{\Lambda(\{\tau_n\}, \mathbb{R}_{\geq 0})} = \int_B \frac{1}{\Gamma(P_{\tau_n})} \zeta^{P_{\tau_n}-1} e^{-\zeta} d\zeta. \end{aligned}$$

The reason behind the specific Λ in (2.40) comes from the use of the Gamma density in point processes for modelling natural disasters; see for example [19], where Hawkes processes with a Gamma density are considered for the modelling of insurance claims subjected to natural disasters. From these closed expressions planers can evaluate their position more accurately since now they know that, in this particular case, the *expected* destruction of any forthcoming catastrophe grows proportionally (in fact, linearly) to the state of pollution during the previous disaster.

2.4.1 An intermediate model: PRMs with no diffusive term

A quick inspection of this framework show that the current formulation indeed generalizes the previous models: the model in Section 2.3 can be recovered as the special case in which the mark space is trivial and the intensity depends only on the pollution stock.

Another interesting case can be recovered by suppressing the diffusive term in P :

$$dK_t = b_K(K_t, P_t, C_t, \theta_t) dt - \int_{[0, \infty)} (1 - \omega(K_{t-}, P_t, \zeta)) K_{t-} q(dt, d\zeta), \quad (2.41)$$

$$dP_t = b_P(K_t, P_t, C_t, \theta_t) dt \quad (2.42)$$

$$K_0 > 0, \quad P_0 > 0. \quad (2.43)$$

Intuitively, this model can be regarded as a natural extension to the one based on nonhomogeneous Poisson processes from Section 2.2. For the sake of presentation, we shall not focus on the interpretations of this model, as most of them can be derived from the dynamics at (2.37)–(2.39). We will, however, refer to it as an intermediate step when deriving the HJB for the general jump–diffusion case.

3 The Hamilton–Jacobi–Bellman PDE via the Dynamic Programming Principle

3.1 Preliminaries

Before proceeding, it is necessary to make a technical remark regarding the nature of the jumps under consideration, as well as the overall structure of the control problem.

3.1.1 Assumptions on the Stochastic Basis

As mentioned previously, by considering a state–dependent intensity on the model an implicit stochastic dependence is placed in the jumps in the form of a self–exciting interaction, see Remarks 2.1 and 2.2. This is possible because the jumps we are considering can all be derived from a Poisson point process, and more general, a Poisson random measure.

To be more precise, the *superposition* and *thinning* properties of Poisson point processes allow us to assume, without any loss of generality, that a marked process q can be obtained as the integral of a larger counting random measure N :

$$q(dt \times d\zeta) = \int \mathbf{1}_{[0, \lambda_t(\zeta)]}(r) N(dt \times d\zeta \times dr),$$

where λ is a suitable (integrable, predictable) non–negative random field; see [18], [5] and [9] for a reminder on general Poisson processes and random measures. Technically, this means we are working with jump processes of stochastic intensity.

Whenever the source of randomness consists only on the jumps themselves, i.e. Sections 2.1, 2.2 and 2.4.1, our approach is justified by the embedding theorems on the extended state space (K, P, λ) and the integration with respect to the Poisson random measure N of intensity $dt \times dr$ on $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$, see Chapter 5 of [5].

As we move to more complex specifications and introduce additional sources of randomness such as an independent Brownian motion, the stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ must be correspondingly enlarged. Importantly, this will not pose a problem in what follows, since the Wiener–Poisson structure of the model, together with the additive nature of the proposed intensity λ , falls within the framework of [16] and [15] for stochastic optimal control problems with environment–dependent jumps. Hence, the following assumption will be taken implicitly throughout the rest of the paper:

Assumptions 3.1. There exists an underlying Poisson random measure N on $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ of intensity measure $dt \otimes \nu(d\zeta) \otimes dr$ for some σ –finite measure ν with finite second moment, such that

$$q(dt, d\zeta) = \int \mathbf{1}_{[0, \lambda(P_t, \zeta)]}(r) N(dt, d\zeta, dr).$$

Additionally, we assume $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$ is (the complete, right-continuous augmentation of) the filtration

$$\mathcal{F}_t := \sigma\{q(B \times C \times D) \mid (B, C, D) \in \mathcal{B}([0, t]) \otimes \mathcal{B}(\mathbb{R}_{\geq 0}) \otimes \mathcal{B}(\mathbb{R}_{\geq 0})\} \vee \mathcal{G}_t,$$

where

$$\mathcal{G}_t = \sigma\{W_s \mid 0 \leq s \leq t\}$$

if a Brownian motion is present on the model, and

$$\mathcal{G}_t = \{\emptyset, \Omega\}$$

otherwise.

3.1.2 Functional Objective

We consider the set of admissible control processes $C = (C_t)_{t \geq 0}$ and $\theta = (\theta_t)_{t \geq 0}$. The state of the economy at time t is described by $(K_t, P_t) \in \mathcal{S}$. Let $\bar{\theta} := \min\{1, \phi/\sigma\}$ so that $E_t = \phi Y(K_t) - \sigma \theta_t Y(K_t) \geq 0$. We call a control pair $(C, \theta) = ((C_t)_{t \geq 0}, (\theta_t)_{t \geq 0})$ *admissible* from (k, p) if

1. (C_t, θ_t) are \mathbb{F} -progressively measurable, with $C_t \geq 0$ a.e. and $\theta_t \in [0, \bar{\theta}]$ a.s.;
2. under (C, θ) , the model's state equations admit a (pathwise) unique strong solution with nonnegative paths and initial condition $(K_0, P_0) = (k, p)$;
3. the discounted utility is integrable:

$$\mathbb{E} \left[\int_0^\infty e^{-\rho t} |U(C_t, P_t)| dt \right] < \infty. \quad (3.1)$$

We denote the set of admissible controls by $\mathcal{A}(k, p)$.

Remark 3.2. The expectation in (3.1) is conditional on the initial state (k, p) , and $\mathcal{A}(k, p) \neq \emptyset$ for all $(k, p) \in \mathcal{S}$.

For any admissible control $(C, \theta) \in \mathcal{A}(k, p)$, the associated gain function is defined by

$$J(k, p; C, \theta) := \mathbb{E} \left[\int_0^\infty e^{-\rho s} U(C_s, P_s) ds \right], \quad (3.2)$$

and the corresponding *value function* $v : \mathcal{S} \rightarrow \mathbb{R}$ is given by

$$v(k, p) := \sup_{(C, \theta) \in \mathcal{A}(k, p)} J(k, p; C, \theta) = \sup_{(C, \theta) \in \mathcal{A}(k, p)} \mathbb{E} \left[\int_0^\infty e^{-\rho s} U(C_s, P_s) ds \right]. \quad (3.3)$$

In order to ensure the well-posedness of the problem and to derive the associated Hamilton–Jacobi–Bellman (HJB) equation in classical form, we impose the following conditions on the value function $v : \mathcal{S} \rightarrow \mathbb{R}$:

1. Continuous differentiability: v is continuously differentiable with respect to both state variables k and p ,

$$v \in C^1(\mathcal{S}), \quad (3.4)$$

i.e. both partial derivatives $\frac{\partial v}{\partial K}$ and $\frac{\partial v}{\partial P}$ exist and are continuous on \mathcal{S} .

2. Sufficient regularity so that

$$v \in C^2(\mathcal{S}). \quad (3.5)$$

3. We assume the following monotonicity properties, which are consistent with the economic interpretation of the model:

- v is non-decreasing in capital:

$$\frac{\partial v}{\partial K}(K, P) \geq 0 \quad \text{for all } (K, P) \in \mathcal{S}, \quad (3.6)$$

reflecting that higher capital stock does not decrease the maximal attainable utility.

- v is non-increasing in pollution:

$$\frac{\partial v}{\partial P}(K, P) \leq 0 \quad \text{for all } (K, P) \in \mathcal{S}, \quad (3.7)$$

reflecting the intuition that capital accumulation increases utility, while pollution reduces it.

4. To ensure well-posedness of the stochastic control problem and integrability of the value function, we impose a polynomial growth bound on v . There exist constants $c > 0$, $\gamma, \delta \geq 0$ such that

$$|v(K, P)| \leq c(1 + K^\gamma + P^\delta), \quad \forall (K, P) \in \mathcal{S}. \quad (3.8)$$

3.1.3 Dynamic Programming Principle

Let $\mathcal{T}_{t,T}$ denote the set of stopping times with values in the interval $[t, T]$, and define $\mathcal{T} := \mathcal{T}_{0,\infty}$ as the set of admissible stopping times on the infinite horizon. The Dynamic Programming Principle (DPP) asserts that for any admissible initial condition $(k, p) \in \mathcal{S}$, we have:

$$v(k, p) = \sup_{(C, \theta) \in \mathcal{A}(k, p)} \sup_{h \in \mathcal{T}} \mathbb{E} \left[\int_0^h e^{-\rho s} U(C_s, P_s) ds + e^{-\rho h} v(K_h, P_h) \right]. \quad (3.9)$$

Equivalently, the value function also satisfies:

$$v(k, p) = \sup_{(C, \theta) \in \mathcal{A}(k, p)} \inf_{h \in \mathcal{T}} \mathbb{E} \left[\int_0^h e^{-\rho s} U(C_s, P_s) ds + e^{-\rho h} v(K_h, P_h) \right], \quad (3.10)$$

with the convention that $e^{-\rho s(\omega)} = 0$ whenever $s(\omega) = \infty$. This principle reflects the fact that optimal decision-making is time-consistent: the planner optimally balances immediate utility against the continuation value of the system's future state.

Suppose the candidate value function satisfies (3.5), (3.8) and

$$\lim_{T \rightarrow \infty} \mathbb{E}_{k,p} [e^{-\rho T} v(K_T, P_T)] = 0. \quad (3.11)$$

These conditions ensure that Itô's formula applies and that the transversality condition holds.

Let $(C, \theta) \in \mathcal{A}(k, p)$ be admissible controls. Consider the system over a small time interval $[0, h]$, where $h > 0$. From the dynamic programming principle (3.9), we have the inequality:

$$v(k, p) \geq \mathbb{E} \left[\int_0^h e^{-\rho s} U(C_s, P_s) ds + e^{-\rho h} v(K_h, P_h) \right]. \quad (3.12)$$

3.2 Models of constant Jump-size

The Hamilton–Jacobi–Bellman (HJB) equation provides the infinitesimal version of the dynamic programming principle and characterizes the value function via a nonlinear partial integro-differential equation. We derive this equation under the assumption that the value function is sufficiently smooth.

3.2.1 Standard Poisson Process

To compute $v(K_h, P_h)$, we apply Itô's formula for jump processes to the value function v , assuming that $v \in C^1(\mathcal{S})$ and that the state dynamics are given by a jump-diffusion process with jump times driven by Poisson process $(q_t)_{t \geq 0}$ of intensity $\lambda > 0$. Then

$$\begin{aligned} v(K_h, P_h) &= v(K_0, P_0) + \int_0^h \left(\frac{\partial v}{\partial K}(K_{s-}, P_{s-}) dK_s + \frac{\partial v}{\partial P}(K_{s-}, P_{s-}) dP_s \right) \\ &\quad + \sum_{s \leq h} \left(v(K_s, P_s) - v(K_{s-}, P_{s-}) - \frac{\partial v}{\partial K}(K_{s-}, P_{s-}) \Delta K_s \right. \\ &\quad \left. - \frac{\partial v}{\partial P}(K_{s-}, P_{s-}) \Delta P_s \right) \\ &= v(K_0, P_0) + \int_0^h \left(\frac{\partial v}{\partial K}(K_{s-}, P_{s-}) dK_s + \frac{\partial v}{\partial P}(K_{s-}, P_{s-}) dP_s \right) \\ &\quad + \sum_{s \leq h} \left(v(K_s, P_s) - v(K_{s-}, P_{s-}) - \frac{\partial v}{\partial K}(K_{s-}, P_{s-}) \Delta K_s \right), \end{aligned} \quad (3.13)$$

where we used the fact that P_t evolves continuously, i.e., $\Delta P_s = 0$, for all $s > 0$, and hence $P_{s-} = P_s$. The jumps in K_t correspond to the discrete losses in capital due to natural disasters.

Substituting (3.13) into (3.12), yields

$$\begin{aligned}
v(k, p) \geq \mathbb{E} \left[\int_0^h e^{-\rho s} U(C_s, P_s) ds \right. \\
+ e^{-\rho h} \left(v(k, p) + \int_0^h \left(\frac{\partial v}{\partial K}(K_{s-}, P_s) dK_s + \frac{\partial v}{\partial P}(K_{s-}, P_s) dP_s \right) \right. \\
\left. \left. + \sum_{0 < s \leq h} \left(v(K_s, P_s) - v(K_{s-}, P_{s-}) - \frac{\partial v}{\partial K}(K_{s-}, P_{s-}) \Delta K_s \right) \right) \right]. \quad (3.14)
\end{aligned}$$

Now, using the state dynamics of capital (2.18) and pollution (2.19), we obtain

$$\begin{aligned}
v(k, p) - e^{-\rho h} v(k, p) \geq \mathbb{E} \left[\int_0^h e^{-\rho s} U(C_s, P_s) ds \right. \\
+ e^{-\rho h} \left(\int_0^h \frac{\partial v}{\partial K}(K_{s-}, P_s) ((1 - \theta_s) Y_s(K_s) - C_s) ds \right. \\
- \int_0^h \frac{\partial v}{\partial K}(K_{s-}, P_s) (1 - \omega_s(P_s, K_s)) K_s dq_s \\
+ \int_0^h \frac{\partial v}{\partial P}(K_{s-}, P_s) (\phi Y_s(K_s) - \sigma \theta_s Y_s(K_s) - \alpha P_s) ds \\
\left. \left. + \sum_{s \leq h} \left(v(K_s, P_s) - v(K_{s-}, P_{s-}) - \frac{\partial v}{\partial K}(K_{s-}, P_{s-}) \Delta K_s \right) \right) \right], \quad (3.15)
\end{aligned}$$

or, equivalently by combining the terms, we get

$$\begin{aligned}
v(k, p) - e^{-\rho h} v(k, p) \geq \mathbb{E} \left[\int_0^h e^{-\rho s} U(C_s, P_s) ds \right. \\
+ e^{-\rho h} \left(\int_0^h \left[\frac{\partial v}{\partial K}(K_{s-}, P_s) ((1 - \theta_s) Y_s(K_s) - C_s) \right. \right. \\
+ \frac{\partial v}{\partial P}(K_{s-}, P_s) ((\phi - \sigma \theta_s) Y_s(K_s) - \alpha P_s) \left. \right] ds \\
- \int_0^h \frac{\partial v}{\partial K}(K_{s-}, P_s) (1 - \omega_s(P_s, K_s)) K_s dq_s \\
\left. \left. + \sum_{0 < s \leq h} \left(v(K_s, P_s) - v(K_{s-}, P_{s-}) \right) - \sum_{0 < s \leq h} \frac{\partial v}{\partial K}(K_{s-}, P_{s-}) \Delta K_s \right) \right]. \quad (3.16)
\end{aligned}$$

Let us now examine the jump process associated with the capital dynamics $(K_t)_{t \geq 0}$. A jump in the process occurs only at the arrival times of a natural disaster,

i.e., at the jump times of the counting process (q_t) . In such events, the capital stock experiences an instantaneous loss, modelled as

$$K_t = \omega(K_{t-}, P_{t-})K_{t-}, \quad (3.17)$$

where $\omega(K_{t-}, P_{t-}) \in (0, 1)$ represents the fraction of capital preserved upon the occurrence of a disaster. The corresponding jump in K at time t is given by

$$\Delta K_t := K_t - K_{t-} = -(1 - \omega(K_{t-}, P_{t-}))K_{t-}\Delta q_t. \quad (3.18)$$

Thus, the capital loss upon a jump is proportional to the current capital stock.

We now examine the impact of such jumps on the dynamic programming principle. In particular, we focus on the contribution of the jump component to the marginal variation of the value function, that is, the term representing the instantaneous change in the value function induced by the occurrence of a jump. Substituting the jump term from (3.18), we get:

$$\begin{aligned} \sum_{s \leq h} \frac{\partial v}{\partial K}(K_{s-}, P_s) \Delta K_s &= \sum_{s \leq h} \frac{\partial v}{\partial K}(-(1 - \omega(K_{s-}, P_s))K_{s-}\Delta q_s) \\ &= - \int_0^h \frac{\partial v}{\partial K}(1 - \omega(K_{s-}, P_s))K_{s-} dq_s, \end{aligned} \quad (3.19)$$

where the integral is understood in the sense of a stochastic integral with respect to the Poisson process.

We return to the integral formulation of the dynamic programming principle (3.16) and substitute in the expressions derived for the jump terms:

$$\begin{aligned} v(k, p) - e^{-\rho h}v(k, p) &\geq \mathbb{E} \left[\int_0^h e^{-\rho s} U(C_s, P_s) ds \right. \\ &\quad + e^{-\rho h} \left(\int_0^h \left[\frac{\partial v}{\partial K}(K_{s-}, P_s) ((1 - \theta_s)Y_s(K_s) - C_s) \right. \right. \\ &\quad + \frac{\partial v}{\partial P}(K_{s-}, P_s) ((\phi - \sigma\theta_s)Y_s(K_s) - \alpha P_s) \Big] ds \\ &\quad \left. \left. + \sum_{0 < s \leq h} (v(K_s, P_s) - v(K_{s-}, P_s)) \right) \right]. \end{aligned} \quad (3.20)$$

We simplify the last term in (3.20), representing the cumulative jump in the value function over $[0, h]$:

$$\begin{aligned} \sum_{0 < s \leq h} (v(K_s, P_s) - v(K_{s-}, P_s)) &= \sum_{s \leq h} (v(K_{s-} - (1 - \omega(K_{s-}, P_s))K_{s-}, P_s) - v(K_{s-}, P_s)) \\ &= \sum_{s \leq h} (v(\omega(K_{s-}, P_s)K_{s-}, P_s) - v(K_{s-}, P_s)) \\ &= \int_0^h (v(\omega(K_{s-}, P_s)K_{s-}, P_s) - v(K_{s-}, P_s)) dq_s. \end{aligned} \quad (3.21)$$

Substituting (3.21) back into (3.16), we obtain:

$$\begin{aligned}
v(k, p) - e^{-\rho h} v(k, p) \geq \mathbb{E} \Bigg[& \int_0^h e^{-\rho s} U(C_s, P_s) ds \\
& + e^{-\rho h} \left(\int_0^h \left[\frac{\partial v}{\partial K}(K_{s-}, P_s) ((1 - \theta_s) Y_s(K_s) - C_s) \right. \right. \\
& + \frac{\partial v}{\partial P}(K_{s-}, P_s) ((\phi - \sigma \theta_s) Y_s(K_s) - \alpha P_s) \Big] ds \\
& \left. \left. + \int_0^h (v(\omega_s K_{s-}, P_s) - v(K_{s-}, P_s)) dq_s \right) \right]. \tag{3.22}
\end{aligned}$$

The expression (3.22) represents the key inequality that leads to the HJB equation when dividing by h and letting $h \rightarrow 0$. In the next steps, we will compute the expectations using the properties of the Poisson process and derive the formal HJB equation:

$$\begin{aligned}
\lim_{h \rightarrow 0} \frac{v(k, p) - e^{-\rho h} v(k, p)}{h} \geq & \lim_{h \rightarrow 0} \mathbb{E} \left[\int_0^h e^{-\rho s} U(C_s, P_s) ds \right] \\
& + \lim_{h \rightarrow 0} \mathbb{E} \left[e^{-\rho h} \frac{1}{h} \int_0^h \frac{\partial v}{\partial K}(K_{s-}, P_s) ((1 - \theta_s) Y_s(K_s) - C_s) ds \right] \\
& + \lim_{h \rightarrow 0} \mathbb{E} \left[e^{-\rho h} \frac{1}{h} \int_0^h \frac{\partial v}{\partial P}(K_{s-}, P_s) ((\phi - \sigma \theta_s) Y_s(K_s) - \alpha P_s) ds \right] \\
& + \lim_{h \rightarrow 0} \mathbb{E} \left[e^{-\rho h} \frac{1}{h} \int_0^h (v(\omega_s K_{s-}, P_s) - v(K_{s-}, P_s)) dq_s \right]. \tag{3.23}
\end{aligned}$$

To derive the associated HJB equation, we study the infinitesimal generator of the controlled stochastic process. Starting from the dynamic programming inequality for the value function $v(k, p)$ in (3.23), we investigate the limiting behavior of the expression as the time increment $h \rightarrow 0$. The analysis proceeds term by term, applying L'Hôpital's rule, the mean value theorem, and properties of the Poisson process. We first consider the contribution from the discounting factor:

$$\lim_{h \rightarrow 0} \frac{v(k, p) - e^{-\rho h} v(k, p)}{h} = \rho v(k, p). \tag{3.24}$$

Next, we find the expected value of the integral of the utility over a short time interval. Assuming sufficient regularity of $U(C_t, P_t)$, and using the mean value

theorem inside the expectation, we obtain

$$\begin{aligned} \lim_{h \rightarrow 0} \mathbb{E} \left[\frac{1}{h} \int_0^h e^{-\rho s} U(C_s, P_s) ds \right] &= \mathbb{E} \left[\lim_{h \rightarrow 0} \frac{1}{h} \int_0^h e^{-\rho s} U(C_s, P_s) ds \right] \\ &= \mathbb{E} \left[e^{-\rho \cdot 0} U(C_0, P_0) \right] = U(C_0, P_0). \end{aligned} \quad (3.25)$$

Similarly, we analyze the expected contribution from the deterministic drift in the state variables K_t and P_t . For the capital variable:

$$\begin{aligned} &\lim_{h \rightarrow 0} \mathbb{E} \left[e^{-\rho h} \frac{1}{h} \int_0^h \frac{\partial v}{\partial K}(K_{s-}, P_s) ((1 - \theta_s) A K_s - C_s) ds \right] \\ &= \lim_{h \rightarrow 0} e^{-\rho h} \lim_{h \rightarrow 0} \mathbb{E} \left[\frac{1}{h} \int_0^h \frac{\partial v}{\partial K}(K_{s-}, P_s) ((1 - \theta_s) Y_s(K_s) - C_s) ds \right] \\ &= \mathbb{E} \left[\lim_{h \rightarrow 0} \frac{1}{h} \int_0^h \frac{\partial v}{\partial K}(K_{s-}, P_s) ((1 - \theta_s) Y_s(K_s) - C_s) ds \right] \\ &= \mathbb{E} \left[\frac{\partial v}{\partial K}(K_0, P_0) ((1 - \theta_0) Y_0(K_0) - C_0) \right] \\ &= \frac{\partial v}{\partial K}(k, p) ((1 - \theta_0) Y_0(k) - C_0), \end{aligned} \quad (3.26)$$

and analogously for the pollution variable:

$$\begin{aligned} &\lim_{h \rightarrow 0} \mathbb{E} \left[e^{-\rho h} \frac{1}{h} \int_0^h \frac{\partial v}{\partial P}(K_{s-}, P_s) ((\phi - \sigma \theta_s) A K_s - \alpha P_s) ds \right] \\ &= \lim_{h \rightarrow 0} e^{-\rho h} \lim_{h \rightarrow 0} \mathbb{E} \left[\frac{1}{h} \int_0^h \frac{\partial v}{\partial P}(K_{s-}, P_s) ((\phi - \sigma \theta_s) A K_s - \alpha P_s) ds \right] \\ &= \mathbb{E} \left[\lim_{h \rightarrow 0} \frac{1}{h} \int_0^h \frac{\partial v}{\partial P}(K_{s-}, P_s) ((\phi - \sigma \theta_s) A K_s - \alpha P_s) ds \right] \\ &= \mathbb{E} \left[\frac{\partial v}{\partial P}(K_0, P_0) ((\phi - \sigma \theta_0) Y_0(K_0) - \alpha P_0) \right] \\ &= \frac{\partial v}{\partial P}(k, p) ((\phi - \sigma \theta_0) Y_0(k) - \alpha p). \end{aligned} \quad (3.27)$$

These limits again rely on the mean value theorem and the continuity of the involved functions.

The effect of the Poisson jump process is captured by the integral involving the jump component dq_s . Using the Doob–Meyer decomposition, we write

$$dq_s = \tilde{q}_s + \lambda ds, \quad (3.28)$$

where $\tilde{q}_t := q_t - \mathbb{E}[q_t] = q_t - \lambda t$ denotes the compensated Poisson process. This process is a martingale with respect to its natural filtration \mathcal{F}_t , and hence its integral against a predictable integrand has zero expectation, i.e.,

$$\mathbb{E} \left[\int_0^h (v(\omega(K_{s-}, P_s)K_{s-}, P_s) - v(K_{s-}, P_s)) d\tilde{q}_s \right] = 0. \quad (3.29)$$

Thus, we compute

$$\begin{aligned} & \lim_{h \rightarrow 0} \mathbb{E} \left[e^{-\rho h} \frac{1}{h} \int_0^h (v(\omega(K_{s-}, P_s)K_{s-}, P_s) - v(K_{s-}, P_s)) dq_s \right] \\ &= \lim_{h \rightarrow 0} e^{-\rho h} \mathbb{E} \left[\frac{1}{h} \int_0^h (v(\omega(K_{s-}, P_s)K_{s-}, P_s) - v(K_{s-}, P_s)) dq_s \right] \\ &= \lim_{h \rightarrow 0} \mathbb{E} \left[\frac{1}{h} \left(\int_0^h (v(\omega(K_{s-}, P_s)K_{s-}, P_s) - v(K_{s-}, P_s)) d\tilde{q}_s \right. \right. \\ & \quad \left. \left. + \int_0^h (v(\omega(K_{s-}, P_s)K_{s-}, P_s) - v(K_{s-}, P_s)) d(\lambda s) \right) \right] \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \mathbb{E} \left[\int_0^h (v(\omega(K_{s-}, P_s)K_{s-}, P_s) - v(K_{s-}, P_s)) d\tilde{q}_s \right] \\ & \quad + \lim_{h \rightarrow 0} \lambda \mathbb{E} \left[\frac{1}{h} \int_0^h (v(\omega(K_{s-}, P_s)K_{s-}, P_s) - v(K_{s-}, P_s)) ds \right] \\ &= \lambda \mathbb{E} \left[\lim_{h \rightarrow 0} \frac{1}{h} \int_0^h (v(\omega(K_{s-}, P_s)K_{s-}, P_s) - v(K_{s-}, P_s)) ds \right] \\ &= \lambda (v(\omega(k, p)k, p) - v(k, p)). \end{aligned} \quad (3.30)$$

Using the obtained results, we arrive at the inequality (3.21):

$$\begin{aligned} \rho v(k, p) &\geq U(C, p) + \frac{\partial v}{\partial P}(k, p)((\phi - \sigma\theta)Y_0(k) - \alpha p) \\ & \quad + \frac{\partial v}{\partial K}(k, p)((1 - \theta)Y_0(k) - C) \\ & \quad + \lambda(v(\omega(k, p)k, p) - v(k, p)). \end{aligned} \quad (3.31)$$

Taking the supremum over admissible controls $(C, \theta) \in \mathcal{A}(k, p)$, we obtain the Hamilton–Jacobi–Bellman equation (see e.g. [13], [22] or [24]).

$$\begin{aligned} \rho v(k, p) &= \sup_{(C, \theta) \in \mathcal{A}(k, p)} \left\{ U(C, p) + \frac{\partial v}{\partial P}(k, p)((\phi - \sigma\theta)Y_0(k) - \alpha p) \right. \\ & \quad + \frac{\partial v}{\partial K}(k, p)((1 - \theta)Y_0(k) - C) \\ & \quad \left. + \lambda(v(\omega(k, p)k, p) - v(k, p)) \right\}. \end{aligned} \quad (3.32)$$

The equation (3.32) characterizes the value function $v(k, p)$ as the unique viscosity solution (under suitable regularity assumptions) to the associated stochastic optimal control problem involving continuous dynamics and Poisson-driven jump risk.

3.2.2 Nonhomogeneous Poisson

The derivation of the dynamic programming principle and the associated HJB equation proceeds in the same way as in the homogeneous case in Section 3.2.1, with one key modification: the constant intensity λ is replaced by the state-dependent intensity $\lambda(P_t)$ with the compensator (2.23).

For $h > 0$ sufficiently small, an application of Itô's formula to the value function v , followed by taking the expectation, yields

$$\begin{aligned} v(k, p) - e^{-\rho h} v(k, p) \geq \mathbb{E} \left[\int_0^h e^{-\rho s} U(C_s, P_s) ds \right. \\ \left. + e^{-\rho h} \left(\int_0^h \left[\frac{\partial v}{\partial K}(K_{s-}, P_s) ((1 - \theta_s) Y_s(K_s) - C_s) \right. \right. \right. \\ \left. \left. + \frac{\partial v}{\partial P}(K_{s-}, P_s) ((\phi - \sigma \theta_s) Y_s(K_s) - \alpha P_s) \right] ds \right. \\ \left. + \int_0^h (v(\omega_s K_{s-}, P_s) - v(K_{s-}, P_s)) d\hat{q}_s \right]. \end{aligned} \quad (3.33)$$

The last integral in (3.33) represents the contribution of the jump process. Using the Doob–Meyer decomposition (2.24),

$$d\hat{q}_s = dM_s + d\Lambda_s = dM_s + \lambda(P_s) ds, \quad (3.34)$$

we obtain

$$\begin{aligned} \int_0^h (v(\omega_s K_{s-}, P_s) - v(K_{s-}, P_s)) d\hat{q}_s &= \int_0^h (v(\omega_s K_{s-}, P_s) - v(K_{s-}, P_s)) dM_s \\ &\quad + \int_0^h (v(\omega_s K_{s-}, P_s) - v(K_{s-}, P_s)) \lambda(P_s) ds. \end{aligned} \quad (3.35)$$

By the martingale property (2.26),

$$\mathbb{E} \left[\int_0^h (v(\omega_s K_{s-}, P_s) - v(K_{s-}, P_s)) dM_s \right] = 0. \quad (3.36)$$

Hence, only the compensator contributes to the expectation. We now compute the

infinitesimal expectation of the jump contribution:

$$\begin{aligned}
& \lim_{h \rightarrow 0} \mathbb{E} \left[e^{-\rho h} \frac{1}{h} \int_0^h \left(v(\omega(K_{s-}, P_s) K_{s-}, P_s) - v(K_{s-}, P_s) \right) d\hat{q}_s \right] \\
&= \lim_{h \rightarrow 0} \mathbb{E} \left[\frac{1}{h} \int_0^h \left(v(\omega(K_{s-}, P_s) K_{s-}, P_s) - v(K_{s-}, P_s) \right) d(\hat{q}_s - \Lambda_s + \Lambda_s) \right] \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \mathbb{E} \left[\int_0^h \left(v(\omega(K_{s-}, P_s) K_{s-}, P_s) - v(K_{s-}, P_s) \right) dM_s \right] \\
&\quad + \lim_{h \rightarrow 0} \mathbb{E} \left[\frac{1}{h} \int_0^h \left(v(\omega(K_{s-}, P_s) K_{s-}, P_s) - v(K_{s-}, P_s) \right) \lambda(P_s) ds \right] \\
&= \lambda(p) (v(\omega(k, p)k, p) - v(k, p)).
\end{aligned} \tag{3.37}$$

Substituting this result back into (3.33), dividing by h , and taking the limit as $h \rightarrow 0$, we obtain the HJB inequality

$$\begin{aligned}
\rho v(k, p) &\geq U(C, p) + \frac{\partial v}{\partial P}(k, p) ((\phi - \sigma\theta)Ak - \alpha p) \\
&\quad + \frac{\partial v}{\partial K}(k, p) ((1 - \theta)Ak - C) \\
&\quad + \lambda(p) (v(\omega(k, p)k, p) - v(k, p)).
\end{aligned} \tag{3.38}$$

Finally, optimizing over admissible controls $(C, \theta) \in \mathcal{A}(k, p)$, we arrive at the Hamilton–Jacobi–Bellman equation for the nonhomogeneous Poisson case:

$$\begin{aligned}
\rho v(k, p) &= \sup_{(C, \theta) \in \mathcal{A}(k, p)} \left\{ U(C, p) + \frac{\partial v}{\partial P}(k, p) ((\phi - \sigma\theta)Ak - \alpha p) \right. \\
&\quad + \frac{\partial v}{\partial K}(k, p) ((1 - \theta)Ak - C) \\
&\quad \left. + \lambda(p) (v(\omega(k, p)k, p) - v(k, p)) \right\},
\end{aligned} \tag{3.39}$$

where $\lambda(p) = \lambda_0 + \lambda_1 p$. The equation (3.39) characterizes the value function v as the unique solution (under suitable regularity assumptions outlined before) to the associated stochastic optimal control problem involving continuous dynamics and Poisson-driven jump risk. The term $\lambda(p)(v(\omega(k, p)k, p) - v(k, p))$ captures the expected capital loss from stochastic disasters, whose intensity now increases with pollution.

3.2.3 Brownian-driven pollution with nonhomogeneous Poisson jumps

Applying Itô's formula for jump processes to the value function v on $[0, h]$:

$$\begin{aligned}
v(K_h, P_h) - v(k, p) = & \int_0^h v_k(K_{s-}, P_{s-})((1 - \theta_s)Y(K_s) - C_s) ds \\
& + \int_0^h v_p(K_{s-}, P_{s-})(\phi Y(K_s) - Z(\theta_s Y(K_s)) - \alpha P_s) ds \\
& + \int_0^h \frac{1}{2} \sigma_P^2 P_s^2 v_{pp}(K_{s-}, P_{s-}) ds \\
& + \int_0^h v_P(K_{s-}, P_{s-}) \sigma_P P_s dW_s \\
& + \int_0^h (v(\omega(P_s, K_{s-})K_{s-}, P_s) - v(K_{s-}, P_s)) d\hat{q}_s.
\end{aligned} \tag{3.40}$$

Substituting (2.30) and (2.32) into (3.40), yields

$$\begin{aligned}
v(K_h, P_h) - v(k, p) = & \int_0^h v_k((1 - \theta_s)Y(K_s) - C_s) ds \\
& + \int_0^h v_p(\phi Y(K_s) - Z(\theta_s Y(K_s)) - \alpha P_s) ds \\
& + \int_0^h \frac{1}{2} \sigma_P^2 P_s^2 v_{pp} ds + \int_0^h v_p \sigma_P P_s dW_s \\
& + \int_0^h (v(\omega(P_s, K_{s-})K_{s-}, P_s) - v(K_{s-}, P_s)) d\hat{q}_s.
\end{aligned} \tag{3.41}$$

Inserting (3.41) into (3.12) and considering a small time horizon $h > 0$, we obtain:

$$\begin{aligned}
v(k, p) - e^{-\rho h} v(k, p) \geq & \mathbb{E} \left[\int_0^h e^{-\rho s} U(C_s, P_s) ds \right. \\
& + e^{-\rho h} \int_0^h \left(\frac{\partial v}{\partial K}(K_{s-}, P_s)((1 - \theta_s)Y_s(K_s) - C_s) \right. \\
& + \frac{\partial v}{\partial P}(K_{s-}, P_s)((\phi - \sigma \theta_s)Y_s(K_s) - \alpha P_s) + \frac{1}{2} \sigma_P^2 P_s^2 v_{PP} \Big) ds \\
& + e^{-\rho h} \int_0^h v_P \sigma_P P_s dW_s \\
& \left. + e^{-\rho h} \int_0^h (v(\omega_s K_{s-}, P_s) - v(K_{s-}, P_s)) d\hat{q}_s \right].
\end{aligned} \tag{3.42}$$

The last term in (3.42) represents the contribution of the jump process. Using the Doob–Meyer decomposition (2.24), we may rewrite

$$\begin{aligned}
& \int_0^h \left(v(\omega(K_{s-}, P_s)K_{s-}, P_s) - v(K_{s-}, P_s) \right) d\hat{q}_s \\
&= \int_0^h \left(v(\omega(K_{s-}, P_s)K_{s-}, P_s) - v(K_{s-}, P_s) \right) dM_s \\
&+ \int_0^h \lambda(P_s) \left(v(\omega(K_{s-}, P_s)K_{s-}, P_s) - v(K_{s-}, P_s) \right) ds.
\end{aligned} \tag{3.43}$$

By the martingale property (2.26),

$$\mathbb{E} \left[\int_0^h \left(v(\omega_s K_{s-}, P_s) - v(K_{s-}, P_s) \right) dM_s \right] = 0. \tag{3.44}$$

Hence, only the compensator contributes to the expectation. The term

$$\int_0^h e^{-\rho h} v_p \sigma_P P_s dW_s \tag{3.45}$$

is a true martingale on $[0, h]$, and thus

$$\mathbb{E} \left[\int_0^h e^{-\rho h} v_p \sigma_P P_s dW_s \right] = 0. \tag{3.46}$$

Using (3.44) and (3.46), the (3.42) becomes

$$\begin{aligned}
v(k, p) - e^{-\rho h} v(k, p) &\geq \mathbb{E} \left[\int_0^h e^{-\rho s} U(C_s, P_s) ds \right. \\
&+ e^{-\rho h} \int_0^h \left(\frac{\partial v}{\partial K}(K_{s-}, P_s) ((1 - \theta_s) Y_s(K_s) - C_s) \right. \\
&+ \frac{\partial v}{\partial P}(K_{s-}, P_s) ((\phi - \sigma \theta_s) Y_s(K_s) - \alpha P_s) + \frac{1}{2} \sigma_P^2 P_s^2 v_{PP} \Big) ds \\
&\left. + e^{-\rho h} \int_0^h \lambda(P_s) \left(v(\omega_s(K_{s-}, P_s)K_{s-}, P_s) - v(K_{s-}, P_s) \right) ds \right].
\end{aligned} \tag{3.47}$$

By continuity of the coefficients and dominated convergence, we obtain the following limits as $h \downarrow 0$:

$$\lim_{h \downarrow 0} \frac{v(k, p) - e^{-\rho h} v(k, p)}{h} = \rho v(k, p), \tag{3.48}$$

and, for any continuous function $G : \mathbb{R}_{\geq 0}^2 \rightarrow \mathbb{R}$,

$$\lim_{h \downarrow 0} \mathbb{E} \left[\frac{1}{h} \int_0^h e^{-\rho s} G(K_{s-}, P_s) ds \right] = G(k, p). \tag{3.49}$$

Dividing (3.47) by h , letting $h \rightarrow 0$, and using the results in (3.48) and (3.49), we obtain the HJB inequality

$$\begin{aligned} \rho v(k, p) \geq & U(C, p) + \frac{\partial v}{\partial P}(k, p)((\phi - \sigma\theta)Y_0(k) - \alpha p) + \frac{1}{2}\sigma_P^2 P^2 v_{PP}(k, p) \\ & + \frac{\partial v}{\partial K}(k, p)((1 - \theta)Y_0(k) - C) + \lambda(p)(v(\omega(k, p)k, p) - v(k, p)), \end{aligned} \quad (3.50)$$

where $\lambda(p) = \lambda_0 + \lambda_1 p$.

Finally, optimizing over admissible controls $(C, \theta) \in \mathcal{A}(k, p)$, we arrive at the Hamilton–Jacobi–Bellman equation for the Brownian–driven pollution model with nonhomogeneous Poisson jumps

$$\begin{aligned} \rho v(k, p) = \sup_{(C, \theta) \in \mathcal{A}(k, p)} \bigg\{ & U(C, p) + \frac{\partial v}{\partial P}(k, p)((\phi - \sigma\theta)Y_0(k) - \alpha p) \\ & + \frac{1}{2}\sigma_P^2 P^2 v_{PP}(k, p) \\ & + \frac{\partial v}{\partial K}(k, p)((1 - \theta)Y_0(k) - C) \bigg\} \\ & + \lambda(p)(v(\omega(k, p)k, p) - v(k, p)). \end{aligned} \quad (3.51)$$

Equation (3.51) shows that the effect of disasters enters additively via the compensator–adjusted jump term. In contrast to the homogeneous Poisson case, where disaster risk is constant, here the intensity $\lambda(p)$ rises with the level of pollution. Thus the marginal damage of emissions is amplified through both the continuous deterioration of environmental quality (via the drift and diffusion of P_t) and the increased likelihood of discrete catastrophic events. The second–order term $\frac{1}{2}\sigma_P^2 p^2 v_{pp}$ captures the effect of Brownian pollution shocks. The combination of diffusion and jump risk implies that optimal policies (C, θ) must balance the trade–off between consumption, abatement, and the endogenous exposure to both continuous and discontinuous environmental risks.

3.3 Models of random Jumps–size (PRMs)

We now derive the HJB equation for the model introduced in Section 2.4. The argument proceeds in two steps: we first obtain the corresponding equation for the intermediate model from Section 2.4.1 (i.e. with the Gaussian component removed), and then we extend the result to the randomized pollution model.

3.3.1 Equation for the intermediate model

The overall strategy remains the same, up to some minor modifications necessitated by q being a Poisson random measure. First, note that instead of (2.17), size

of the jumps vary according to Δ :

$$\Delta q_t(d\zeta) = \lim_{h \downarrow 0} q((t-h, t], d\zeta) = q(\{t\}, d\zeta) = \sum_{n \geq 1} \mathbf{1}_{\{\tau_n = t\}} \delta_{\{\Delta_n\}}(d\zeta),$$

where the law of Δ_n is determined as in (2.35). Without loss of generality, assume $K_t = K_0$ for all $t < 0$; then, the jump in capital (3.18) is replaced by

$$\begin{aligned} \Delta K_t &= K_t - K_{t-} = - \lim_{h \downarrow 0} \int_{(t-h, t] \times [0, \infty)} (1 - \omega(K_{s-}, P_s, \zeta)) K_{s-} q(ds, d\zeta) \\ &= - \int_{(0, \infty)} (1 - \omega(K_{t-}, P_t, \zeta)) K_{t-} q(\{t\}, d\zeta) \\ &= \begin{cases} -(1 - \omega(K_{t-}, P_t, \Delta_n)) K_{t-} & \text{on the event } \{\tau_n = t\}, \\ 0 & \text{in any other case,} \end{cases} \end{aligned} \quad (3.52)$$

for every $t \geq 0$.

With these considerations in mind, applying Itô's rule (see Chapter 14 in [6]) on $v(K_t, P_t)$ for the prescribed control (C_t, θ_t) yields the following equality

$$\begin{aligned} v(K_t, P_t) - v(k, p) &= \int_0^t \frac{\partial v}{\partial K}(K_{s-}, P_{s-}) dK_s + \int_0^t \frac{\partial v}{\partial P}(K_{s-}, P_{s-}) dP_s \\ &\quad + \sum_{0 < s \leq t} (v(K_s, P_s) - v(K_{s-}, P_{s-})) \\ &\quad - \sum_{0 < s \leq t} \left(\frac{\partial v}{\partial K}(K_{s-}, P_{s-}) \Delta K_s + \frac{\partial v}{\partial P}(K_{s-}, P_{s-}) \Delta P_s \right) \\ &= \int_0^t \frac{\partial v}{\partial K}(K_{s-}, P_s) dK_s + \int_0^t \frac{\partial v}{\partial P}(K_{s-}, P_s) dP_s \\ &\quad + \sum_{0 < s \leq t} \left(v(K_s, P_s) - v(K_{s-}, P_s) - \frac{\partial v}{\partial K}(K_{s-}, P_s) \Delta K_s \right). \end{aligned}$$

From the dynamic programming principle (see e.g. [22] or [20]), equation (3.12) holds, and as a consequence we get that for any small $h > 0$,

$$\begin{aligned} \frac{v(k, p) - e^{-\rho h} v(k, p)}{h} &\geq \frac{1}{h} \mathbb{E} \left[\int_0^h e^{-\rho s} U(C_s, P_s) ds + e^{-\rho h} \int_0^h \frac{\partial v}{\partial K}(K_{s-}, P_s) dK_s \right. \\ &\quad + e^{-\rho h} \int_0^h \frac{\partial v}{\partial P}(K_{s-}, P_s) dP_s \\ &\quad + e^{-\rho h} \sum_{0 < s \leq h} (v(K_s, P_s) - v(K_{s-}, P_s)) \\ &\quad \left. - e^{-\rho h} \sum_{0 < s \leq h} \frac{\partial v}{\partial K}(K_{s-}, P_s) \Delta K_s \right]. \end{aligned} \quad (3.53)$$

We now analyze each term individually when h goes to zero. First, using the same arguments as above – namely, (3.24), (3.25) and (3.27), – we obtain that

$$\lim_{h \rightarrow 0} \frac{v(k, p) - e^{-\rho h} v(k, p)}{h} = \rho v(k, p), \quad (3.54)$$

$$\lim_{h \rightarrow 0} \mathbb{E} \left[\frac{1}{h} \int_0^h e^{-\rho s} U(C_s, P_s) ds \right] = U(C_0, p), \quad (3.55)$$

and

$$\lim_{h \rightarrow 0} e^{-\rho h} \mathbb{E} \left[\frac{1}{h} \int_0^h \frac{\partial v}{\partial P}(K_{s-}, P_s) dP_s \right] = \frac{\partial v}{\partial P}(k, p) b_P(k, p, C, \theta), \quad (3.56)$$

respectively.

For the integral with respect to the capital, observe that (2.41) can be rewritten in terms of the compensated martingale measure \tilde{q} :

$$\begin{aligned} dK_t = & \left(b_K(K_t, P_t, C_t, \theta_t) - \int_{(0, \infty)} (1 - \omega(K_{t-}, P_t, z)) K_{t-} \lambda(P_t, \zeta) \nu(d\zeta) \right) dt \\ & - \int_{(0, \cdot] \times (0, \infty)} (1 - \omega(K_{t-}, P_t, \zeta)) K_{t-} \tilde{q}(dt, d\zeta), \end{aligned}$$

where

$$\tilde{q}(dt, d\zeta) := q(dt, d\zeta) - \Lambda(dt, d\zeta).$$

Then,

$$\begin{aligned} & \mathbb{E} \left[\int_0^h \frac{\partial v}{\partial K}(K_{s-}, P_s) dK_s \right] \\ &= \mathbb{E} \left[\int_0^h \frac{\partial v}{\partial K}(K_{s-}, P_s) b_K(K_s, P_s, C_s, \theta_s) ds \right] \\ & - \mathbb{E} \left[\int_{(0, h] \times (0, \infty)} \frac{\partial v}{\partial K}(K_{s-}, P_s) (1 - \omega(K_{s-}, P_s, \zeta)) K_{s-} \Lambda(ds, d\zeta) \right]. \end{aligned}$$

On the one hand, from (2.41) we have

$$\begin{aligned} & \lim_{h \rightarrow 0} e^{-\rho h} \mathbb{E} \left[\frac{1}{h} \int_0^h \frac{\partial v}{\partial K}(K_{s-}, P_s) b_K(K_s, P_s, C_s, \theta_s) ds \right] \\ &= \frac{\partial v}{\partial K}(k, p) b_K(k, p, C_0, \theta_0). \end{aligned} \quad (3.57)$$

On the other hand, from (2.36), (3.52) and the definition of integral with respect to Poisson random measures,

$$\begin{aligned}
& \mathbb{E} \left[\sum_{0 < s \leq h} \frac{\partial v}{\partial K}(K_{s-}, P_s) \Delta K_s \right] \\
&= -\mathbb{E} \left[\int_{(0,h] \times (0,\infty)} \frac{\partial v}{\partial K}(K_{s-}, P_s) (1 - \omega(K_{s-}, P_s, \zeta)) K_{s-} q(ds, d\zeta) \right] \\
&= -\mathbb{E} \left[\int_{(0,h] \times (0,\infty)} \frac{\partial v}{\partial K}(K_{s-}, P_s) (1 - \omega(K_{s-}, P_s, \zeta)) K_{s-} \Lambda(ds, d\zeta) \right],
\end{aligned} \tag{3.58}$$

where the last equality is due to Campbell's theorem and the definition of compensator, see [5].

Lastly, observe that

$$\begin{aligned}
& \mathbb{E} \left[\sum_{0 < s \leq h} v(K_s, P_s) - v(K_{s-}, P_s) \right] = \mathbb{E} \left[\sum_{0 < s \leq h} v(\Delta K_s + K_{s-}, P_s) - v(K_{s-}, P_s) \right] \\
&= \mathbb{E} \left[\sum_{0 < s \leq h} \sum_{n \geq 1} (v(\omega(K_{s-}, P_s, \zeta_n) K_{s-}, P_s) - v(K_{s-}, P_s)) \mathbf{1}_{\{\tau_n = s\}} \right] \\
&= \mathbb{E} \left[\int_{(0,h] \times (0,\infty)} (v(\omega(K_{s-}, P_s, \zeta) K_{s-}, P_s) - v(K_{s-}, P_s)) q(ds, d\zeta) \right] \\
&= \mathbb{E} \left[\int_{(0,h] \times (0,\infty)} (v(\omega_s(K_{s-}, P_s, \zeta) K_{s-}, P_s) - v(K_{s-}, P_s)) \Lambda(ds, d\zeta) \right],
\end{aligned}$$

where we have again used Campbell's theorem. Then, by Lebesgue differentiation theorem [12],

$$\begin{aligned}
& \lim_{h \downarrow 0} \frac{1}{h} \mathbb{E} \left[\int_{(0,h] \times (0,\infty)} (v(\omega(K_{s-}, P_s, \zeta) K_{s-}, P_s) - v(K_{s-}, P_s)) \Lambda(ds, d\zeta) \right] \\
&= \int_{(0,\infty)} (v(\omega(k, p, \zeta) k, p) - v(k, p)) \lambda(p, \zeta) \nu(d\zeta). \\
\implies & \lim_{h \downarrow 0} \frac{e^{-\rho h}}{h} \mathbb{E} \left[\int_{(0,h] \times (0,\infty)} (v(\omega(K_{s-}, P_s, \zeta) K_{s-}, P_s) - v(K_{s-}, P_s)) \Lambda(ds, d\zeta) \right] \\
&= \lim_{h \downarrow 0} e^{-\rho h} \cdot \int_{(0,\infty)} (v(\omega(k, p, \zeta) k, p) - v(k, p)) \lambda(p, \zeta) \nu(d\zeta) \\
&= 1 \cdot \int_{(0,\infty)} (v(\omega(k, p, \zeta) k, p) - v(k, p)) \lambda(p, \zeta) \nu(d\zeta).
\end{aligned} \tag{3.59}$$

Taking limits at both sides of (3.53) and plugging in the estimates from (3.54)

to (3.59), we obtain that for any admissible control such that $(C_0, \theta_0) = (C, \theta)$,

$$\begin{aligned} \rho v(k, p) &\geq U(C, p) + \frac{\partial v}{\partial K}(k, p) b_K(k, p, C, \theta) + \frac{\partial v}{\partial P}(k, p) b_P(k, p, C, \theta) \\ &\quad + \int_{(0, \infty)} (v(\omega(k, p, \zeta)k, p) - v(k, p)) \lambda(p, \zeta) \nu(d\zeta). \end{aligned}$$

Thus, under suitable regularity conditions, the value function v solves (in an adequate, possibly viscosity sense) the Hamilton–Jacobi–Bellman partial integro–differential equation

$$\begin{aligned} \rho v(k, p) &= \sup_{(C, \theta) \in [0, \infty) \times [0, 1)} \left\{ U(C, p) + \frac{\partial v}{\partial K}(k, p) b_K(k, p, C, \theta) \right. \\ &\quad \left. + \frac{\partial v}{\partial P}(k, p) b_P(k, p, C, \theta) \right. \\ &\quad \left. + \int_{(0, \infty)} (v(\omega(k, p, \zeta)k, p) - v(k, p)) \lambda(p, \zeta) \nu(d\zeta) \right\} \\ &= \sup_{(C, \theta) \in [0, \infty) \times [0, 1)} \left\{ U(C, p) + \frac{\partial v}{\partial K}(k, p) b_K(k, p, C, \theta) \right. \\ &\quad \left. + \frac{\partial v}{\partial P}(k, p) b_P(k, p, C, \theta) \right\} \\ &\quad + \int_{(0, \infty)} (v(\omega(k, p, \zeta)k, p) - v(k, p)) \lambda(p, \zeta) \nu(d\zeta). \end{aligned} \tag{3.60}$$

3.3.2 Extension to randomized pollution

We now present another extension of the model to include a randomized pollution:

$$dK_t = b_K(K_t, P_t, C_t, \theta_t) dt - \int_{[0, \infty)} (1 - \omega(K_{t-}, P_t, \zeta)) K_{t-} q(dt, d\zeta), \tag{2.37}$$

$$dP_t = b_P(K_t, P_t, C_t, \theta_t) dt + \sigma_P P_t dW_t, \tag{2.38}$$

$$K_0 > 0, \quad P_0 > 0, \tag{2.39}$$

for some given constant $\sigma_P > 0$.

We now apply Itô's rule [6] on $e^{-\rho t} v(K_t, P_t)$ for the prescribed control (C_t, θ_t) on

the new dynamics:

$$\begin{aligned}
e^{-\rho t}v(K_t, P_t) - v(k, p) &= - \int_0^t \rho e^{-\rho s} v(K_{s-}, P_{s-}) ds \\
&+ \int_0^t e^{-\rho s} \left\{ \frac{\partial v}{\partial K}(K_{s-}, P_{s-}) dK_s + \frac{\partial v}{\partial P}(K_{s-}, P_{s-}) dP_s \right\} \\
&+ \frac{1}{2} \int_0^t e^{-\rho s} \left(\frac{\partial^2 v}{\partial K^2}(K_{s-}, P_{s-}) d\langle K^c, K^c \rangle_s + \frac{\partial^2 v}{\partial K \partial P}(K_{s-}, P_{s-}) d\langle K^c, P^c \rangle_s \right. \\
&\quad \left. + \frac{\partial^2 v}{\partial P \partial K}(K_{s-}, P_{s-}) d\langle P^c, K^c \rangle_s + \frac{\partial^2 v}{\partial P^2}(K_{s-}, P_{s-}) d\langle P^c, P^c \rangle_s \right) \\
&+ \sum_{0 < s \leq t} e^{-\rho s} \left(v(K_s, P_s) - v(K_{s-}, P_{s-}) \right. \\
&\quad \left. - \frac{\partial v}{\partial K}(K_{s-}, P_{s-}) \Delta K_s - \frac{\partial v}{\partial P}(K_{s-}, P_{s-}) \Delta P_s \right),
\end{aligned}$$

where K^c and P^c denote the continuous components of K and P , respectively. Then, using the same arguments as before and including the Brownian component of P , we have that

$$\begin{aligned}
e^{-\rho t}v(K_t, P_t) - v(k, p) &= - \int_0^t \rho e^{-\rho s} v(K_{s-}, P_s) ds \\
&+ \int_0^t e^{-\rho s} \left(\frac{\partial v}{\partial K}(K_{s-}, P_s) dK_s + \frac{\partial v}{\partial P}(K_{s-}, P_s) dP_s \right) \\
&+ \frac{1}{2} \int_0^t e^{-\rho s} \frac{\partial^2 v}{\partial P^2}(K_{s-}, P_s) d\langle P^c, P^c \rangle_s \\
&+ \sum_{0 < s \leq t} e^{-\rho s} \left(v(K_s, P_s) - v(K_{s-}, P_s) - \frac{\partial v}{\partial K}(K_{s-}, P_{s-}) \Delta K_s \right).
\end{aligned}$$

As in the previous cases, from the dynamic programming principle equation (3.12) holds, and as a result

$$\begin{aligned}
\frac{1}{h} \mathbb{E} \left[\rho \int_0^h e^{-\rho s} v(K_{s-}, P_s) ds - v(k, p) \right] &\geq \frac{1}{h} \mathbb{E} \left[\int_0^h e^{-\rho s} U(C_s, P_s) ds \right] \\
&+ \frac{1}{h} \mathbb{E} \left[\int_0^h e^{-\rho s} \frac{\partial v}{\partial K}(K_{s-}, P_s) dK_s \right] + \frac{1}{h} \mathbb{E} \left[\int_0^h e^{-\rho s} \frac{\partial v}{\partial P}(K_{s-}, P_s) dP_s \right] \\
&+ \frac{1}{2h} \mathbb{E} \left[\int_0^h e^{-\rho s} \frac{\partial^2 v}{\partial P^2}(K_{s-}, P_s) d\langle P^c, P^c \rangle_s \right] \\
&+ \frac{1}{h} \mathbb{E} \left[\sum_{0 < s \leq t} e^{-\rho s} \left(v(K_s, P_s) - v(K_{s-}, P_s) - \frac{\partial v}{\partial K}(K_{s-}, P_{s-}) \Delta K_s \right) \right].
\end{aligned}$$

Observe that the only difference from the previous case is in the inclusion of a Brownian motion in the dynamics of P and in the integral with respect to $\langle P^c \rangle$; however, from the properties of W we have that

$$\begin{aligned} & \lim_{h \downarrow 0} \frac{1}{h} \mathbb{E} \left[\int_0^h e^{-\rho s} \frac{\partial v}{\partial P}(K_{s-}, P_s) dP_s \right] \\ &= \lim_{h \downarrow 0} \frac{1}{h} \mathbb{E} \left[\int_0^h e^{-\rho s} \frac{\partial v}{\partial P}(K_{s-}, P_s) b_P(K_s, P_s, C_s, \theta_s) ds \right] \\ &= \frac{\partial v}{\partial P}(k, p) b_P(k, p, C, \theta), \end{aligned}$$

and

$$\begin{aligned} & \lim_{h \downarrow 0} \frac{1}{2h} \mathbb{E} \left[\int_0^h e^{-\rho s} \frac{\partial^2 v}{\partial P^2}(K_{s-}, P_s) d\langle P^c, P^c \rangle_s \right] \\ &= \lim_{h \downarrow 0} \frac{1}{2h} \mathbb{E} \left[\int_0^h e^{-\rho s} \frac{\partial^2 v}{\partial P^2}(K_{s-}, P_s) \sigma_P^2 ds \right] = \frac{1}{2} \frac{\partial^2 v}{\partial P^2}(k, p) \sigma_P^2 p^2. \end{aligned}$$

Adding this estimates to the ones presented in the previous section, i.e. equations (3.54) through (3.59), yields the following HJB equation:

$$\begin{aligned} \rho v(k, p) &= \sup_{(C, \theta) \in [0, \infty) \times [0, \bar{\theta}]} \left\{ U(C, p) + \frac{\partial v}{\partial K}(k, p) b_K(k, p, C, \theta) \right. \\ &\quad + \frac{\partial v}{\partial P}(k, p) b_P(k, p, C, \theta) + \frac{1}{2} \frac{\partial^2 v}{\partial P^2}(k, p) \sigma_P^2 p^2 \\ &\quad \left. + \int_{(0, \infty)} (v(\omega(k, p, \zeta)k, p) - v(k, p)) \lambda(p, \zeta) \nu(d\zeta) \right\} \\ &= \sup_{(C, \theta) \in [0, \infty) \times [0, \bar{\theta}]} \left\{ U(C, p) + \frac{\partial v}{\partial K}(k, p) b_K(k, p, C, \theta) + \frac{\partial v}{\partial P}(k, p) b_P(k, p, C, \theta) \right\} \\ &\quad + \frac{1}{2} \frac{\partial^2 v}{\partial P^2}(k, p) \sigma_P^2 p^2 + \int_{(0, \infty)} (v(\omega(k, p, \zeta)k, p) - v(k, p)) \lambda(p, \zeta) \nu(d\zeta). \end{aligned} \tag{3.61}$$

4 Analysis of solutions to Hamilton–Jacobi–Bellman Equations

Having derived the HJB equation for each of the models under consideration, we now proceed to the analysis of their solutions under specific model–inspired assumptions. In particular, we work under the notion of classical solutions to the equations from Section 3, and defer the discussion of viscosity solutions to Section 5. Since the framework encompasses models with similar characteristics, we follow the same road map as in the previous sections, starting from the simplest specification and mentioning only the changes that arise as the complexity increases.

It is important to note that under the standing assumptions, the structure of the control problem cannot be altered substantially (in particular, the control of the agent is restricted to the trend or drift of the system and never on the noise or jump component). As a consequence, many of the results obtained for the base model extend naturally to the subsequent specifications. This is due to the fact that the Hamiltonian is preserved, which in turn maintains most of the structure of the optimality conditions for the controls.

4.1 Preliminaries

4.1.1 Hamiltonian and Optimality Conditions

Observe that, in each case, the control variables (C, θ) never directly act outside the drift term in the dynamics of the state (K, P) , i.e. the jumps in capital and the diffusion in pollution are allowed to evolve uncontrolled. When translated to the HJB, this means that both the nonlocal and the second–order terms can be placed outside the supremum. This provides a cleaner representation of the dependence on the controls, since we can define the Hamiltonian (i.e., the part that depends on the controls) as

$$\begin{aligned} H(C, \theta; K, P, v_K, v_P) &:= U(C, P) + v_K b_K(K_t, P_t, C_t, \theta_t) + v_P b_P(K_t, P_t, C_t, \theta_t), \\ &= U(C, P) + v_K \left((1 - \theta_t) Y(K_t) - C_t \right) \\ &\quad + v_P \left(\phi Y(K_t) - Z(\theta_t Y(K_t)) - \alpha P_t \right). \end{aligned} \quad (4.1)$$

Furthermore, when referring to the maximized Hamiltonian we shall use the notation

$$\hat{H}(K, P) := \sup_{(C, \theta) \in \mathcal{A}} H(C, \theta; K, P, v_K, v_P), \quad (4.2)$$

where v_K and v_P are the partial derivatives of v with respect to K and P , respectively.

Let $(\hat{C}, \hat{\theta})$ be the maximizers of H over $\mathcal{A}(k, p)$. On the one hand, observe that the optimal consumption \hat{C} solves the interior first-order conditions:

$$\frac{\partial H}{\partial C}(\hat{C}, \hat{\theta}) = \frac{\partial U}{\partial C}(\hat{C}, P) - v_K(K, P) = 0 \quad \Longleftrightarrow \quad U_C(\hat{C}, P) = v_K(K, P). \quad (4.3)$$

On the other hand, observe that the optimal abatement share $\hat{\theta}$ satisfies the interior first-order condition

$$\frac{\partial H}{\partial \theta}(\hat{C}, \hat{\theta}) = -v_K Y(K) - v_P Z'(\theta Y(K)) Y(K) = 0 \iff -v_K = v_P Z'(\theta Y(K)). \quad (4.4)$$

Moreover, since Z itself is linear in the investment, see (2.3), the Hamiltonian in (4.1) is linear in θ as well. Consequently, the maximizer $\hat{\theta}$ is determined by the sign of $v_K + \sigma v_P$:

$$\hat{\theta} = \begin{cases} 0, & v_K + \sigma v_P > 0, \\ \text{any } \theta \in [0, \bar{\theta}], & v_K + \sigma v_P = 0, \\ \bar{\theta}, & v_K + \sigma v_P < 0, \end{cases} \quad (4.5)$$

where $\bar{\theta} := \min\{1, \phi/\sigma\}$.

4.1.2 Candidate form for the value function

In this section, we derive a candidate closed-form expression for the value function v which solves the dynamic programming equation (3.9). Let us fix the constants

$$\psi \in \mathbb{R}_{>0}, \quad x \in \mathbb{R}_{>0}. \quad (4.6)$$

Motivated by the first-order conditions (4.3)–(4.5) and the dynamic programming principle (3.9), we postulate that for all $(K, P) \in \mathcal{S}$ the partial derivatives of v satisfy

$$\frac{\partial v}{\partial K}(K, P) = (\psi K)^{-\varepsilon}, \quad (4.7)$$

$$\frac{\partial v}{\partial P}(K, P) = -x P^\beta. \quad (4.8)$$

Integrating (4.7) with respect to K (for fixed P) yields

$$v(K, P) = \psi^{-\varepsilon} \frac{K^{1-\varepsilon}}{1-\varepsilon} + c_1(P), \quad (4.9)$$

for some function $c_1 : \mathbb{R}_{>0} \rightarrow \mathbb{R}$. Thus, for all $(K, P) \in \mathcal{S}$ we can write

$$v(K, P) = \psi^{-\varepsilon} \frac{K^{1-\varepsilon}}{1-\varepsilon} + c_1(P). \quad (4.10)$$

Similarly, integrating (4.8) with respect to P (for fixed K) yields

$$v(K, P) = -x \frac{P^{1+\beta}}{1+\beta} + c_2(K), \quad (4.11)$$

for some function $c_2 : \mathbb{R}_{>0} \rightarrow \mathbb{R}$. Hence, for all $(K, P) \in \mathcal{S}$ we have

$$v(K, P) = -x \frac{P^{1+\beta}}{1+\beta} + c_2(K). \quad (4.12)$$

We introduce the functions $f, g : \mathcal{S} \rightarrow \mathbb{R}$ by

$$f(K, P) := \psi^{-\varepsilon} \frac{K^{1-\varepsilon}}{1-\varepsilon} + c_1(P), \quad (4.13)$$

$$g(K, P) := -x \frac{P^{1+\beta}}{1+\beta} + c_2(K). \quad (4.14)$$

Therefore, we aim to find the candidate solution v such that for all $(K, P) \in \mathcal{S}$ we have

$$v(K, P) = f(K, P) = g(K, P). \quad (4.15)$$

Since v is continuously differentiable on \mathcal{S} , the same holds for f and g , and their partial derivatives must coincide with those of v :

$$\frac{\partial f}{\partial P}(K, P) = \frac{\partial v}{\partial P}(K, P), \quad (4.16)$$

$$\frac{\partial g}{\partial K}(K, P) = \frac{\partial v}{\partial K}(K, P), \quad (4.17)$$

for all $(K, P) \in \mathcal{S}$. We now use (4.16) and (4.17) together with (4.13) and (4.14) to determine the unknown functions c_1 and c_2 . From (4.10) we obtain

$$\frac{\partial f}{\partial P}(K, P) = \frac{\partial}{\partial P} \left(\psi^{-\varepsilon} \frac{K^{1-\varepsilon}}{1-\varepsilon} + c_1(P) \right) = \frac{dc_1}{dP}, \quad (4.18)$$

for all $(K, P) \in \mathcal{S}$. Combining this with (4.8) and (4.16) gives

$$\frac{dc_1}{dP} = \frac{\partial v}{\partial P}(K, P) = -xP^\beta \quad \text{for all } P > 0. \quad (4.19)$$

Thus c_1 satisfies the ordinary differential equation

$$\frac{dc_1}{dP} = -xP^\beta, \quad P > 0, \quad (4.20)$$

whose general C^1 solution is

$$c_1(P) = -x \frac{P^{1+\beta}}{1+\beta} + C_1, \quad (4.21)$$

for some constant $C_1 \in \mathbb{R}$. Similarly, from (4.12) we obtain

$$\frac{\partial g}{\partial K}(K, P) = \frac{\partial}{\partial K} \left(-x \frac{P^{1+\beta}}{1+\beta} + c_2(K) \right) = \frac{dc_2}{dK}, \quad (4.22)$$

for all $(K, P) \in \mathcal{S}$. Combining this with (4.7) and (4.17) yields

$$\frac{dc_2}{dK} = \frac{\partial v}{\partial K}(K, P) = (\psi K)^{-\varepsilon} \quad \text{for all } K > 0. \quad (4.23)$$

Therefore c_2 satisfies

$$\frac{dc_2}{dK} = (\psi K)^{-\varepsilon}, \quad K > 0, \quad (4.24)$$

whose general C^1 solution is

$$c_2(K) = \psi^{-\varepsilon} \frac{K^{1-\varepsilon}}{1-\varepsilon} + C_2, \quad (4.25)$$

for some constant $C_2 \in \mathbb{R}$. Substituting these expressions into (4.10) and (4.12), we find that for all $(K, P) \in \mathcal{S}$,

$$v(K, P) = \psi^{-\varepsilon} \frac{K^{1-\varepsilon}}{1-\varepsilon} - x \frac{P^{1+\beta}}{1+\beta} + C_1 = \psi^{-\varepsilon} \frac{K^{1-\varepsilon}}{1-\varepsilon} - x \frac{P^{1+\beta}}{1+\beta} + C_2. \quad (4.26)$$

Hence, $C_1 = C_2 =: c$ and v must be of the form

$$v(K, P) = \psi^{-\varepsilon} \frac{K^{1-\varepsilon}}{1-\varepsilon} - x \frac{P^{1+\beta}}{1+\beta} + c. \quad (4.27)$$

The system (4.7)–(4.8), and hence the HJB equation and the associated optimal policies, are invariant under adding a constant to v . Therefore $c \in \mathbb{R}$ is an arbitrary constant which we may fix by normalization. Without loss of generality, we set $c = 0$, and obtain the candidate

$$v(K, P) = \psi^{-\varepsilon} \frac{K^{1-\varepsilon}}{1-\varepsilon} - x \frac{P^{1+\beta}}{1+\beta}, \quad (K, P) \in \mathcal{S}. \quad (4.28)$$

Conversely, it is immediate to verify that the function v defined in (4.28) satisfies (4.7) and (4.8). This justifies (4.28) as a natural candidate form for the value function.

Taking into account (2.5), (4.7), (4.8), (4.28), as well as the first order conditions (4.3) and (4.4), we obtain

$$U_C = v_K \iff C^{-\varepsilon} = (\psi K)^{-\varepsilon} \iff C = \psi K. \quad (4.29)$$

Remark 4.1. In dynamic economic models, the equations in (4.7) and (4.8) admit a natural interpretation as shadow prices: v_K is the shadow value of an additional unit of capital and v_P is the shadow cost of an additional unit of pollution. They are the dynamic-programming analogue of the adjoint equations from Pontryagin's Maximum Principle; see the well-known [26] for a in-depth discussion in the continuous-time stochastic case.

4.2 Disasters at a constant arrival rate: Poisson process

We begin with the base model introduced in Section 2.1, in which disasters are driven by a standard Poisson process of intensity λ . From the derivation presented

in Section 3.2.1 (see equation (3.32)), the associated HJB equation takes the form

$$\begin{aligned} \rho v(k, p) = \sup_{(C, \theta) \in \mathcal{A}(k, p)} & \left\{ U(C, p) + v_K(k, p)((1 - \theta)Ak - C) \right. \\ & \left. + v_P(k, p)((\phi - \sigma\theta)Ak - \alpha p) \right\} \\ & + \lambda(v(\omega(k, p)k, p) - v(k, p)), \end{aligned} \quad (4.30)$$

where $v_P := \frac{\partial v}{\partial P}$, $v_K := \frac{\partial v}{\partial K}$.

Envelope identities

We now examine the so-called *envelope identities* related to (4.30). These relations connect the derivatives of the value function with respect to the state (or parameters) to the gradients of the Hamiltonian (or Lagrangian) evaluated at the optimal control. They are useful for establishing differentiability of the value function, deriving optimality conditions for the controls, and characterizing the feedback control law.

In this particular case, the envelope identities obtained by differentiating the HJB with respect to the state variables ensure that the derivatives of the optimal controls do not appear in the value function, so that only the direct partial derivatives of the Hamiltonian and the jump term remain. Furthermore, the structure of the Hamiltonian presented in Section 4.1.1 allows these identities to be naturally extended to the remaining models, as will be shown later.

To see these explicit relations in our current model, let $\widehat{H}(K, P)$ denote the maximized Hamiltonian from (4.2). Then (4.30) is equivalent to

$$\rho v(K, P) = \widehat{H}(K, P) + \lambda(v(\omega(K, P)K, P) - v(K, P)). \quad (4.31)$$

Proposition 4.2 (Envelope identities). *Assume $v \in C^2$ and that $(\hat{C}, \hat{\theta})$ is an optimizer of H at (K, P) . Then*

$$\begin{aligned} \rho v_K(K, P) &= \frac{\partial \widehat{H}}{\partial K}(K, P) + \lambda(v_K(\omega(K, P)K, P) \partial_K(\omega(K, P))K - v_K(K, P)), \\ \rho v_P(K, P) &= \frac{\partial \widehat{H}}{\partial P}(K, P) + \lambda(v_K(\omega(K, P)K, P) \partial_P(\omega(K, P)K) \\ &\quad + v_P(\omega(K, P)K, P) - v_P(K, P)). \end{aligned} \quad (4.32)$$

Proof. Let us differentiate (4.31) with respect to K (respectively, P). Since $\widehat{H} = \sup_{C, \theta} H(C, \theta; \cdot)$ and H is C^1 , the envelope theorem (see [21]; alternatively, recall

the Newton-Leibniz rule from measure theory, see [3]) gives

$$\frac{\partial \widehat{H}}{\partial K}(K, P) = \frac{\partial H}{\partial K}(C, \theta; K, P, v_K, v_P) \Big|_{(C, \theta) = (\hat{C}, \hat{\theta})}. \quad (4.33)$$

For the jump term, we need only apply the chain rule on $v(\omega(P, K)K, P)$ for the corresponding variable:

$$\begin{aligned} \partial_K(v(\omega(P, K)K, P)) &= \frac{\partial v}{\partial \omega(K, P)K}(\omega(K, P)K, P) \frac{\partial \omega(K, P)K}{\partial K} \\ &= v_K(\omega(K, P)K, P) \partial_K(\omega(K, P)K) \\ &= v_K(\omega(K, P)K, P)(\omega(K, P) + K\omega_K(K, P)). \end{aligned} \quad (4.34)$$

where $\omega_K := \partial \omega / \partial K$. Similarly,

$$\begin{aligned} \partial_P(v(\omega(P, K)K, P)) &= \frac{\partial v}{\partial P}(\omega(K, P)K, P) \\ &= v_K(\omega(K, P)K, P) \partial_P(\omega(K, P)K) + v_P(\omega(K, P)K, P) \\ &= v_K(\omega(K, P)K, P)K\omega_P(K, P) + v_P(\omega(K, P)K, P). \end{aligned} \quad (4.35)$$

where $\omega_P := \partial \omega / \partial P$. Substituting (4.34) and (4.35) into (4.31) gives (4.32). \square

Let us now compute $\partial_K \widehat{H}$ and $\partial_P \widehat{H}$ explicitly. Since $(\hat{C}, \hat{\theta})$ are the maximizers of H over $\mathcal{A}(k, p)$, we can write

$$\widehat{H}(K, P) = U(\hat{C}, P) + v_K((1 - \hat{\theta})AK - \hat{C}) + v_P((\phi - \hat{\theta}\sigma)AK - \alpha P). \quad (4.36)$$

Differentiating with respect to K and using the envelope property (4.32), we obtain

$$\begin{aligned} \partial_K \widehat{H} &= v_{KK}((1 - \hat{\theta})AK - \hat{C}) + v_K(1 - \hat{\theta})A \\ &\quad + v_{PK}((\phi - \hat{\theta}\sigma)AK - \alpha P) + v_P(\phi - \hat{\theta}\sigma)A. \end{aligned} \quad (4.37)$$

Substituting (4.37) into (4.32) for v_K yields

$$\begin{aligned} \rho v_K(K, P) &= ((1 - \hat{\theta})AK - \hat{C})v_{KK}(K, P) + (1 - \hat{\theta})Av_K(K, P) \\ &\quad + (\phi - \hat{\theta}\sigma)Av_P(K, P) + ((\phi - \hat{\theta}\sigma)AK - \alpha P)v_{PK}(K, P) \\ &\quad + \lambda(v_K(\omega(K, P)K, P)(\omega(K, P) + K\omega_K(K, P)) - v_K(K, P)). \end{aligned} \quad (4.38)$$

Similarly,

$$\partial_P \widehat{H} = U_P(\hat{C}, P) + ((1 - \hat{\theta})AK - \hat{C})v_{KP} + ((\phi - \hat{\theta}\sigma)AK - \alpha P)v_{PP} - \alpha v_P. \quad (4.39)$$

Substituting (4.39) into (4.32) for v_P gives

$$\begin{aligned} \rho v_P(K, P) &= U_P(\hat{C}, P) + ((1 - \hat{\theta})AK - \hat{C})v_{KP}(K, P) \\ &\quad + ((\phi - \hat{\theta}\sigma)AK - \alpha P)v_{PP}(K, P) - \alpha v_P(K, P) \\ &\quad + \lambda(v_K(\omega(K, P)K, P)K\omega_P(K, P) + v_P(\omega(K, P)K, P) - v_P(K, P)). \end{aligned} \quad (4.40)$$

4.3 Disasters at a dynamic arrival rate: Nonhomogeneous Poisson process

We now consider the next model and characterize the optimal controls in the nonhomogeneous Poisson framework with pollution-dependent disaster intensity. Recall that the Hamilton–Jacobi–Bellman equation (3.39) naturally decomposes into two components:

- (i) the Hamiltonian term depending on the continuous dynamics and the controls, and
- (ii) the jump contribution, which depends on the state variables but not directly on the controls.

Moreover, recall that the Hamiltonian part remains identical as in Section 4.2 since the jump term does not depend on the controls (C, θ) . Thus, both the first-order conditions for C and θ and the envelope identities maintain a similar form. Indeed, let $(\hat{C}, \hat{\theta})$ denote the maximizers of H over the admissible control set $\mathcal{A}(k, p)$. Since the jump component

$$\lambda(P)(v(\omega(K, P)K, P) - v(K, P)), \quad (4.41)$$

depends only on the state variables (K, P) , it is placed outside the Hamiltonian and does not affect the first-order conditions for the controls (compare with (4.30)).

In fact, given that the first-order conditions (4.3) – (4.4) characterize the feedback rules for optimal controls, by coupling them together with the pollution-dependent jump term in the HJB equation (4.41) we obtain a system that determines the value function and the associated optimal policies. Relative to the homogeneous Poisson case, the only structural change is the replacement of the constant intensity λ by the state-dependent intensity $\lambda(P) = \lambda_0 + \lambda_1 P$, which introduces an additional endogenous feedback from pollution into the disaster hazard.

Envelope identities

Taking into account that the maximized Hamiltonian \hat{H} for this model coincides with the one defined in (4.2), the Hamilton–Jacobi–Bellman equation (3.39) can also be written in the compact form

$$\rho v(K, P) = \hat{H}(K, P) + \lambda(P)(v(\omega(P, K)K, P) - v(K, P)). \quad (4.42)$$

As a result, the next result can be deduced:

Proposition 4.3 (Envelope identities with state-dependent hazard). *Assume $v \in C^2$ and that $(\hat{C}, \hat{\theta})$ maximizes H at (K, P) . Then*

$$\begin{aligned}\rho v_K(K, P) &= \frac{\partial \hat{H}}{\partial K}(K, P) + \lambda(P) \left(v_K(\omega(K, P)K, P) \partial_K(\omega(K, P))K - v_K(K, P) \right), \\ \rho v_P(K, P) &= \frac{\partial \hat{H}}{\partial P}(K, P) + \lambda'(P) \left(v(\omega(K, P)K, P) - v(K, P) \right) \\ &\quad + \lambda(P) \left(v_K(\omega(K, P)K, P) \partial_P(\omega(K, P)K) \right. \\ &\quad \left. + v_P(\omega(K, P)K, P) - v_P(K, P) \right).\end{aligned}\tag{4.43}$$

Proof. The proof follows the same steps as in Proposition 4.2, albeit with some minor adjustments due to the hazard rate being state-dependent. In particular, applying the chain rule gives

$$\partial_P \left(\lambda(P) v(\omega(K, P)K, P) \right) = \lambda'(P) v(\omega(K, P)K, P) + \lambda(P) \partial_P v_P(\omega(K, P)K, P); \tag{4.44}$$

consequently we obtain

$$\lambda(P) v(K, P) = \lambda'(P) v(K, P) + \lambda(P) \partial_P v_P(K, P). \tag{4.45}$$

Finally, collecting the terms in (4.44) and (4.45) yields (4.43). \square

Now, observe that substituting into (4.43) gives the K -envelope identity

$$\begin{aligned}\rho v_K(K, P) &= ((1 - \hat{\theta})AK - \hat{C})v_{KK}(K, P) + (1 - \hat{\theta})Av_K(K, P) \\ &\quad + (\phi - \hat{\theta}\sigma)Av_P(K, P) + ((\phi - \hat{\theta}\sigma)AK - \alpha P)v_{PK}(K, P) \\ &\quad + \lambda(P) \left(v_K(\omega(P, K)K, P) (\omega(P, K) + K\omega_K(P, K)) - v_K(K, P) \right),\end{aligned}\tag{4.46}$$

where as for the pollution hazard, substituting into (4.43) yields the P -envelope identity

$$\begin{aligned}\rho v_P(K, P) &= U_P(\hat{C}, P) + ((1 - \hat{\theta})AK - \hat{C})v_{KP}(K, P) \\ &\quad + ((\phi - \hat{\theta}\sigma)AK - \alpha P)v_{PP}(K, P) - \alpha v_P(K, P) \\ &\quad + \lambda'(P) \left(v(\omega(K, P)K, P) - v(K, P) \right) \\ &\quad + \lambda(P) \left(v_K(\omega(K, P)K, P) K\omega_P(K, P) \right. \\ &\quad \left. + v_P(\omega(K, P)K, P) - v_P(K, P) \right).\end{aligned}\tag{4.47}$$

Equations (4.46) and (4.47) provide the exact envelope identities for the non-homogeneous Poisson case. Relative to the homogeneous case, see equations (4.34) and (4.35) above, the key new feature is the appearance of the additional term proportional to $\lambda'(P)$, reflecting the feedback of pollution on the hazard rate of disasters. Indeed, differentiating the HJB with respect to K , and applying the envelope property together with the chain rule for the jump term, yields

$$\begin{aligned} \rho v_K(K, P) = & ((1 - \theta)AK - C)v_{KK}(K, P) + (1 - \theta)Av_K(K, P) \\ & + (\phi - \theta\sigma)Av_P(K, P) + ((\phi - \theta\sigma)AK - \alpha P)v_{PK}(K, P) \\ & + \lambda(P)\left(v_K(\omega(K, P)K, P)(\omega(K, P) + K\omega_K(K, P)) - v_K(K, P)\right). \end{aligned} \quad (4.48)$$

Similarly, differentiating the HJB with respect to P , we obtain

$$\begin{aligned} \rho v_P(K, P) = & U_P(C, P) + ((1 - \theta)AK - C)v_{KP}(K, P) \\ & + ((\phi - \theta\sigma)AK - \alpha P)v_{PP}(K, P) - \alpha v_P(K, P) \\ & + \lambda'(P)\left(v(\omega(K, P)K, P) - v(K, P)\right) \\ & + \lambda(P)\left(v_K(\omega(K, P)K, P)K\omega_P(K, P) + v_P(\omega(K, P)K, P) - v_P(K, P)\right). \end{aligned} \quad (4.49)$$

Equations (4.48) and (4.49) generalize the envelope identities of the homogeneous Poisson case. The term

$$\lambda'(P)\left(v(\omega(K, P)K, P) - v(K, P)\right) \quad (4.50)$$

captures the effect of pollution on the disaster intensity, and hence on the marginal value of pollution in the planner's problem.

4.4 Disasters at a stochastic arrival rate: Jump–diffusion models

4.4.1 Brownian–driven pollution stock

We continue with the model presented in Section 2.3, building on the discussions in Sections 4.2 and 4.3. In this setting, the Hamiltonian again coincides with the one presented in equation (4.1) since the diffusion term and the jump term are both independent of the controls (C, θ) . Consequently, the first–order conditions with respect to C and θ remain unchanged and are not repeated here. In particular, the optimal consumption \hat{C} is characterized by

$$U_C(\hat{C}, P) = v_K(K, P), \quad (4.51)$$

and the optimal abatement $\hat{\theta}$ is determined by the same conditions as in equations (4.4) and (4.5).

For the corresponding envelope identities, a similar situation occurs. Let $\widehat{H}(K, P)$ be the maximized Hamiltonian from (4.2). Then the HJB equation (3.51) can be written as

$$\rho v(K, P) = \widehat{H}(K, P) + \frac{1}{2}\sigma_P^2 P^2 v_{PP}(K, P) + \lambda(P) \left(v(\omega(K, P)K, P) - v(K, P) \right). \quad (4.52)$$

Applying the same procedure as in the proofs for Propositions 4.2 and 4.3 yields

$$\begin{aligned} \rho v_K(K, P) &= \partial_K \widehat{H}(K, P) + \frac{1}{2}\sigma_P^2 P^2 v_{PPK}(K, P) \\ &\quad + \lambda(P) \left(v_K(\omega(K, P)K, P) (\omega(K, P) + K\omega_K(K, P)) - v_K(K, P) \right). \\ \rho v_P(K, P) &= \partial_P \widehat{H}(K, P) + \sigma_P^2 P v_{PP}(K, P) + \frac{1}{2}\sigma_P^2 P^2 v_{PPP}(K, P) \\ &\quad + \lambda'(P) \left(v(\omega(K, P)K, P) - v(K, P) \right) \\ &\quad + \lambda(P) \left(v_K(\omega(K, P)K, P) K\omega_P(K, P) + v_P(\omega(K, P)K, P) - v_P(K, P) \right). \end{aligned} \quad (4.53)$$

Differentiating \widehat{H} with respect to K and P , we obtain

$$\begin{aligned} \partial_K \widehat{H} &= v_{KK}((1 - \hat{\theta})AK - \hat{C}) + (1 - \hat{\theta})A v_K \\ &\quad + v_{PK}((\phi - \hat{\theta}\sigma)AK - \alpha P) + (\phi - \hat{\theta}\sigma)A v_P, \\ \partial_P \widehat{H} &= U_P(\hat{C}, P) + ((1 - \hat{\theta})AK - \hat{C}) v_{KP} \\ &\quad + ((\phi - \hat{\theta}\sigma)AK - \alpha P) v_{PP} - \alpha v_P. \end{aligned} \quad (4.54)$$

Substituting these expressions into (4.53) yields the full system of envelope identities.

4.4.2 Randomized magnitude of the disasters

In the general model introduced in Section 3.3.2, the optimal consumption \hat{C} and abatement share $\hat{\theta}$ are determined by the same conditions as in (4.4) and (4.5), since the controls do not act on either the diffusion or the jump components. In fact, plugging the maximized Hamiltonian from (4.2) into (3.61) yields

$$\rho v(K, P) = \widehat{H}(K, P) + \frac{1}{2} \frac{\partial^2 v}{\partial P^2}(K, P) \sigma_P^2 P^2 \quad (4.55)$$

$$+ \int_{(0, \infty)} (v(\omega(K, P, \zeta)K, P) - v(K, P)) \lambda(P, \zeta) \nu(d\zeta). \quad (4.56)$$

To find the corresponding envelope identities it is sufficient to invoke the dominated convergence theorem along with the mean value theorem in order to handle

the differentiation under the integral of ν :

$$\begin{aligned}
\rho v_K(K, P) &= \partial_K \widehat{H}(K, P) + \frac{1}{2} \sigma_P^2 P^2 v_{PPK}(K, P) \\
&\quad + \int_{(0, \infty)} \lambda(P, \zeta) v_K(\omega(K, P, \zeta) K, P) (\omega(K, P, \zeta) + K \omega_K(K, P, \zeta)) \nu(d\zeta) \\
&\quad - \int_{(0, \infty)} \lambda(P, \zeta) v_K(K, P) \nu(d\zeta), \\
\rho v_P(K, P) &= \partial_P \widehat{H}(K, P) + \sigma_P^2 P v_{PP}(K, P) + \frac{1}{2} \sigma_P^2 P^2 v_{PPP}(K, P) \\
&\quad + \int_{(0, \infty)} \partial_P \lambda(P, \zeta) \left(v(\omega(K, P) K, P) - v(K, P) \right) \nu(d\zeta) \\
&\quad + \int_{(0, \infty)} \lambda(P, \zeta) \left(v_K(\omega(K, P, \zeta) K, P) K \omega_P(K, P, \zeta) + v_P(\omega(K, P, \zeta) K, P) \right) \nu(d\zeta) \\
&\quad - \int_{(0, \infty)} \lambda(P, \zeta) v_P(K, P) \nu(d\zeta),
\end{aligned} \tag{4.57}$$

with the derivatives of \widehat{H} exactly as in (4.54).

5 Verification theorems and viscosity solutions

In this section, we consider the problem of *verification* for the solutions to HJB equations, which concerns identifying when a solution to a given Partial Integro–Differential Equation (PIDE) coincides with the value function of an optimal control problem. However, it is well-known in the literature that the value function of an optimal control problem need not be smooth. Thus, before we can state any verification result, we introduce two notions of solutions for the class of PIDEs under study: *classical* and *viscosity* solutions.

As the name suggests, a classical solution u is a *sufficiently differentiable function* such that the PIDE is satisfied by u and its derivatives. In contrast, a viscosity solution v is a function for which the *notion of differentiability itself is weakened*, and these “generalized derivatives” are precisely those that satisfy the PIDE together with v . For a detailed exposition of viscosity theory, see [7].

In our framework, it is reasonable to consider both classical and viscosity solutions, as the nonlocal terms induced by jumps in the HJB equations affect the regularity of possible solutions. This effect becomes more apparent as the complexity of the jump processes grow.

5.1 Non–Diffusive models

5.1.1 Standard Poisson: Verification theorem for the candidate value function

The central step in the classical dynamic programming approach is to establish that, whenever a sufficiently smooth solution to the Hamilton–Jacobi–Bellman equation exists, this solution coincides with the value function of the control problem. This result, known as the verification theorem, further yields the existence of an optimal Markovian control as a direct corollary. The proof relies fundamentally on the application of Itô’s formula.

The planner maximizes

$$J(K, P; C, \theta) = \mathbb{E}_x \left[\int_0^\infty e^{-\rho t} \left(\frac{C_t^{1-\varepsilon}}{1-\varepsilon} - \chi \frac{P_t^{1+\beta}}{1+\beta} \right) dt \right], \quad (K, P) \in \mathcal{S}.$$

with $\rho > 0$, $A, \phi, \sigma, \alpha, \chi > 0$ and $\varepsilon, \beta > 0$.

For a test function $f \in C^2(\mathcal{S})$, we define the controlled generator

$$(\mathcal{L}^{C,\theta} f)(K, P) = f_K(K, P)((1-\theta)AK - C) + f_P(K, P)((\phi - \sigma\theta)AK - \alpha P) + \lambda(f(\omega(K, P)K, P) - f(K, P)). \quad (5.1)$$

The value function

$$v(x) = \sup_{(C,\theta)} J(K, P; C, \theta). \quad (5.2)$$

satisfies the Hamilton–Jacobi–Bellman equation

$$\rho v(K, P) = \sup_{(C, \theta) \in \mathcal{A}(K, P)} \left\{ \frac{C^{1-\varepsilon}}{1-\varepsilon} - \chi \frac{P^{1+\beta}}{1+\beta} + (\mathcal{L}^{C, \theta} v)(K, P) \right\}. \quad (5.3)$$

Let us now formulate and prove the verification theorem for the jump–control growth–environment model based on the results from [13] on the second–order integro–differential problems and [14] on general SDEs with jumps; for an in–depth treatment on the Brownian case we refer to [24].

Theorem 5.1 (Verification – Jump–control growth–environment model). *Suppose $v \in C^2(\mathcal{S})$ is nonnegative, has at most polynomial growth, and satisfies the HJB (5.3) pointwise.*

(i) (Upper bound) *Assume the transversality condition*

$$\lim_{T \rightarrow \infty} \mathbb{E} \left[e^{-\rho T} v(K_T, P_T) \right] = 0 \quad (5.4)$$

for every admissible $(C, \theta) \in \mathcal{A}(K, P)$.

Then for any admissible $(C, \theta) \in \mathcal{A}(K, P)$,

$$v(K, P) \geq J(K, P; C, \theta). \quad (5.5)$$

(ii) (Optimality) *If there exist Borel functions $(\hat{C}, \hat{\theta}) : \mathcal{S} \rightarrow [0, \infty) \times [0, 1]$ such that for all $(K, P) \in (0, \infty)^2$*

$$U(\hat{C}, P) + (\mathcal{L}^{\hat{C}, \hat{\theta}} v)(K, P) = \sup_{(C, \theta) \in \mathcal{A}(K, P)} \left\{ U(C, P) + (\mathcal{L}^{C, \theta} v)(K, P) \right\}, \quad (5.6)$$

then $(\hat{C}, \hat{\theta})$ is optimal and

$$v(K, P) = J(K, P; \hat{C}, \hat{\theta}) = \sup_{(C, \theta) \in \mathcal{A}(K, P)} J(K, P; C, \theta). \quad (5.7)$$

Proof.

(i) Fix any admissible (C_t, θ_t) with state (K_t, P_t) . Define $Y_t := e^{-\rho t} v(K_t, P_t)$. Applying Itô’s formula for jump processes to Y_t on $[0, T]$ gives

$$e^{-\rho T} v(K_T, P_T) - v(K_0, P_0) = \int_0^T e^{-\rho s} \left((\mathcal{L}^{C_s, \theta_s} v)(K_s, P_s) - \rho v(K_s, P_s) \right) ds + M_T, \quad (5.8)$$

where

$$M_T := \int_0^T e^{-\rho s} \left(v(\omega(P_{s-}, K_{s-}) K_{s-}, P_{s-}) - v(K_{s-}, P_{s-}) \right) d\tilde{q}_s \quad (5.9)$$

is a martingale. Equivalently, (5.9) can be written as

$$M_T = \sum_{0 < \tau_n \leq T} e^{-\rho \tau_n} \left(v(\omega(P_{\tau_n-}, K_{\tau_n-}) K_{\tau_n-}, P_{\tau_n-}) - v(K_{\tau_n-}, P_{\tau_n-}) \right) - \lambda \int_0^T e^{-\rho s} \left(v(\omega(P_s, K_s) K_s, P_s) - v(K_s, P_s) \right) ds. \quad (5.10)$$

Taking expectations and using $\mathbb{E}[M_T] = 0$,

$$\mathbb{E} \left[e^{-\rho T} v(K_T, P_T) \right] - v(K_0, P_0) = \mathbb{E} \left[\int_0^T e^{-\rho s} \left((\mathcal{L}^{C_s, \theta_s} v)(K_s, P_s) - \rho v(K_s, P_s) \right) ds \right]. \quad (5.11)$$

By (5.3), we obtain

$$\rho v \geq U(C_t, P_t) + \mathcal{L}^{C_t, \theta_t} v \iff \mathcal{L}^{C_t, \theta_t} v - \rho v \leq -U(C_t, P_t). \quad (5.12)$$

For every T ,

$$\mathbb{E} \left[e^{-\rho T} v(K_T, P_T) \right] - v(K_0, P_0) \leq -\mathbb{E} \left[\int_0^T e^{-\rho t} U(C_t, P_t) dt \right]. \quad (5.13)$$

Then, through a localization argument (see Section 7.2 of [13], Theorem 7.2.1, for the solution of the discounted control problem for $(t, (K, P))$ on the localized domain $[0, T] \times [-n, n]^2$ for any $n \geq 1$), we can use standard results on the asymptotic stability of Wiener-Poisson semigroups to extend the (sub-optimally) controlled generator from (5.1) up to $[0, \infty) \times \mathcal{S}$ (see Theorem 1 of III.13 from [14]); as a result, we can let $T \rightarrow \infty$ so that (5.4) becomes

$$-v(K, P) \leq -J(K, P; C, \theta), \quad \text{i.e.} \quad v(K, P) \geq J(K, P; C, \theta). \quad (5.14)$$

Therefore, $v(K, P) \geq \mathbb{E} \left[\int_0^\infty e^{-\rho t} U(C_t, P_t) dt \right]$.

(ii) Let $(\hat{C}, \hat{\theta})$ satisfy (5.6). Repeating the above with $(\hat{C}, \hat{\theta})$, the HJB inequality becomes an equality:

$$\rho v = U(\hat{C}, P) + \mathcal{L}^{\hat{C}, \hat{\theta}} v. \quad (5.15)$$

So for each T ,

$$\mathbb{E} \left[e^{-\rho T} v(\hat{K}_T, \hat{P}_T) \right] - v(K, P) = -\mathbb{E} \left[\int_0^T e^{-\rho t} U(\hat{C}_t, \hat{P}_t) dt \right]. \quad (5.16)$$

As before, we use a localization argument to let $T \rightarrow \infty$ (this time for the optimally controlled generator); combining this with (5.4) for $(\hat{C}, \hat{\theta})$ yields $v(K, P) = J(K, P; \hat{C}, \hat{\theta})$. Combined with (i) this implies optimality. \square

Remark 5.2 (Transversality condition). A standard sufficient condition for (5.4) to hold is the following:

- (a) the value function v grows at most polynomially;
- (b) under any admissible control, there exists $m > 0$ such that

$$\sup_{t \geq 0} \mathbb{E}[|K_t|^m + |P_t|^m] < \infty; \quad (5.17)$$

- (c) the discount rate ρ dominates the growth rate of these moments.

Under these assumptions, the dominated convergence theorem ensures that

$$\lim_{T \rightarrow \infty} \mathbb{E}[e^{-\rho T} v(K_T, P_T)] = 0.$$

Corollary 5.3 (Verification for the candidate power-separable value function). *Let v be given by*

$$v(K, P) = \psi^{-\varepsilon} \frac{K^{1-\varepsilon}}{1-\varepsilon} - x \frac{P^{1+\beta}}{1+\beta}, \quad (4.28)$$

with $x > 0$ and $\psi > 0$. Let \hat{C} and $\hat{\theta}$ be the feedback controls defined by (4.29) and (4.5), respectively. Then for all $(K, P) \in \mathcal{S}$,

$$v(K, P) = \sup_{(C, \theta) \in \mathcal{A}} J(K, P; C, \theta) = J(K, P; \hat{C}, \hat{\theta}). \quad (5.18)$$

Proof. We divide the argument into several steps in order to verify that the candidate function v is indeed the value function of the control problem:

1. Regularity and growth of v . For $\varepsilon, \beta > 0$, $v \in C^2(\mathcal{S})$ with

$$v_K = (\psi K)^{-\varepsilon}, \quad v_P = -x P^\beta, \quad v_{KK} = -\varepsilon \psi^{-\varepsilon} K^{-\varepsilon-1}, \quad v_{KP} = 0, \quad v_{PP} = -x \beta P^{\beta-1}. \quad (5.19)$$

Hence, v has at most polynomial growth in (K, P) . Thus, it satisfies the regularity requirements in the Verification Theorem 5.1.

2. Although the feedback rule $\hat{\theta}$ is of bang-bang type and introduces a discontinuity in the drift across the switching surface $\{(K, P) : v_K(K, P) + \sigma v_P(K, P) = 0\}$, existence and uniqueness of strong solutions remain valid. Indeed, the drift is piecewise affine and thus of bounded variation on compacts. Recent results on SDEs with discontinuous drifts (see [25]) show that pathwise uniqueness and strong existence hold for such dynamics under nondegeneracy and bounded-variation conditions on the drift.

3. Attainment of the Hamiltonian. For fixed (K, P) , the map $C \mapsto H$ is strictly concave; the first-order condition $U_C = v_K$ gives $\hat{C} = \psi K$. The map $\theta \mapsto H$ is affine with slope $-AK(v_K + \sigma v_P)$, so θ defined above attains the supremum on $[0, \bar{\theta}]$. Therefore, for all (K, P) ,

$$\begin{aligned} \sup_{(C, \theta) \in \mathcal{A}(K, P)} \left\{ \frac{C^{1-\varepsilon}}{1-\varepsilon} - \chi \frac{P^{1+\beta}}{1+\beta} + (\mathcal{L}^{C, \theta} v)(K, P) \right\} \\ = \frac{(\hat{C})^{1-\varepsilon}}{1-\varepsilon} - \chi \frac{P^{1+\beta}}{1+\beta} + (\mathcal{L}^{\hat{C}, \hat{\theta}} v)(K, P), \end{aligned} \quad (5.20)$$

i.e. the maximizer $(\hat{C}, \hat{\theta})$ attains the supremum in the HJB (4.1).

4. Transversality. Since v has polynomial growth

$$v(K, P) \leq c_1(1 + K^{1-\varepsilon} + P^{1+\beta}) \quad (5.21)$$

or some c_1 and (K_t, P_t) has moments with at most exponential growth rate strictly smaller than ρ (this holds under linear drift/jump structure and $\theta \in [0, \bar{\theta}]$), we obtain that the condition

$$\lim_{T \rightarrow \infty} \mathbb{E}_{(k, p)}[e^{-\rho T} v(K_T, P_T)] = 0 \quad (5.22)$$

holds for all initial states $(k, p) := (K_0, P_0) \in \mathcal{S}$.

5. Upper bound via verification. For any admissible (C, θ) , applying Itô's formula for jump processes to $e^{-\rho t} v(K_t, P_t)$, taking expectation, using the HJB inequality and the transversality condition, we obtain

$$v(K, P) \geq J(K, P; C, \theta). \quad (5.23)$$

With $(\hat{C}, \hat{\theta})$ attaining the Hamiltonian, the previous inequality becomes equality, yielding

$$v(K, P) = J(K, P; \hat{C}, \hat{\theta}). \quad (5.24)$$

Combining with Step 4 gives the stated identity.

Thus, for every initial state $(k, p) \in \mathcal{S}$,

$$v(k, p) = \sup_{(C, \theta) \in \mathcal{A}(k, p)} \mathbb{E}_{(k, p)} \left[\int_0^\infty e^{-\rho t} \left(\frac{C_t^{1-\varepsilon}}{1-\varepsilon} - \chi \frac{P_t^{1+\beta}}{1+\beta} \right) dt \right] = J(k, p; \hat{C}, \hat{\theta}). \quad (5.25)$$

□

We have established that the candidate power-separable function v satisfies all the conditions of the verification theorem. The feedback controls $(\hat{C}, \hat{\theta})$ are admissible, attain the supremum in the Hamiltonian, and yield a state process for which the transversality condition holds. Therefore, v is indeed the value function of the optimization problem, and $(\hat{C}, \hat{\theta})$ constitute an optimal control policy.

5.1.2 Nonhomogeneous Poisson: Verification theorem for the model with state-dependent jump intensity

We now state and prove a verification theorem for the planner's problem when the arrival of destructive events is governed by a nonhomogeneous Poisson process with state-dependent intensity (2.27).

For a twice continuously differentiable test function $v \in C^2(\mathcal{S})$, the (controlled) infinitesimal generator is

$$\begin{aligned} (\mathcal{L}^{C,\theta}v)(K, P) = & v_K(K, P)((1-\theta)AK - C) + v_P(K, P)((\phi - \sigma\theta)AK - \alpha P) \\ & + \lambda(P)(v(\omega(P, K)K, P) - v(K, P)). \end{aligned} \quad (5.26)$$

The HJB equation reads

$$\rho v(K, P) = \sup_{(C,\theta) \in \mathcal{A}(K,P)} \left\{ U(C, P) + (\mathcal{L}^{C,\theta}v)(K, P) \right\}. \quad (5.27)$$

Theorem 5.4 (Verification – jump-control growth-environment model with state-dependent intensity). *Suppose $v \in C^2(\mathcal{S})$ is nonnegative, has at most polynomial growth, and satisfies the HJB (5.27) pointwise.*

(i) (Upper bound) *Assume the transversality condition*

$$\lim_{T \rightarrow \infty} \mathbb{E} \left[e^{-\rho T} v(K_T, P_T) \right] = 0 \quad (5.28)$$

for every admissible $(C, \theta) \in \mathcal{A}(K, P)$ and the associated state process (K_t, P_t) solving (2.30)-(2.31). Then, for any admissible (C, θ) ,

$$v(K, P) \geq J(K, P; C, \theta). \quad (5.29)$$

(ii) (Optimality) *If there exist Borel functions $(\hat{C}, \hat{\theta}) : \mathcal{S} \rightarrow [0, \infty) \times [0, 1]$ such that, for all $(K, P) \in \mathcal{S}$,*

$$U(\hat{C}, P) + (\mathcal{L}^{\hat{C}, \hat{\theta}}v)(K, P) = \sup_{(C,\theta) \in \mathcal{A}(K,P)} \left\{ U(C, P) + (\mathcal{L}^{C,\theta}v)(K, P) \right\}, \quad (5.30)$$

then $(\hat{C}, \hat{\theta})$ is optimal and

$$v(K, P) = J(K, P; \hat{C}, \hat{\theta}) = \sup_{(C,\theta) \in \mathcal{A}(K,P)} J(K, P; C, \theta). \quad (5.31)$$

Proof. (i) Fix any admissible $(C_t, \theta_t) \in \mathcal{A}(K, P)$ and let (K_t, P_t) be the corresponding state. Set $Y_t := e^{-\rho t} v(K_t, P_t)$. By the Itô's formula for processes with

jumps driven by a counting process with (predictable) compensator (2.23), on $[0, T]$ we obtain

$$\begin{aligned} e^{-\rho T} v(K_T, P_T) - v(K, P) &= \int_0^T e^{-\rho s} (\mathcal{L}^{C_s, \theta_s} v(K_s, P_s) - \rho v(K_s, P_s)) ds + M_T \\ &= \int_0^T e^{-\rho s} (\mathcal{L}^{C_s, \theta_s} v - \rho v)(K_s, P_s) ds + M_T. \end{aligned} \quad (5.32)$$

where

$$M_T := \int_0^T e^{-\rho s} \left(v(\omega(K_{s-}, P_{s-})K_{s-}, P_{s-}) - v(K_{s-}, P_{s-}) \right) d\tilde{q}_s. \quad (5.33)$$

Under the polynomial growth of v and admissibility (linear growth of the drift; boundedness of $\omega \in (0, 1)$; integrability of $\int_0^T \lambda(P_s) ds$), the stochastic integral M_T is a true martingale. Taking expectations in (5.32) and using $\mathbb{E}[M_T] = 0$ gives

$$\mathbb{E} \left[e^{-\rho T} v(K_T, P_T) \right] - v(K, P) = \mathbb{E} \left[\int_0^T e^{-\rho s} (\mathcal{L}^{C_s, \theta_s} v - \rho v)(K_s, P_s) ds \right]. \quad (5.34)$$

Since v solves the HJB (5.27), we have, pointwise,

$$\rho v \geq U(C_s, P_s) + \mathcal{L}^{C_s, \theta_s} v \iff \mathcal{L}^{C_s, \theta_s} v - \rho v \leq -U(C_s, P_s). \quad (5.35)$$

Hence, for all T ,

$$\mathbb{E} \left[e^{-\rho T} v(K_T, P_T) \right] - v(K_0, P_0) \leq -\mathbb{E} \left[\int_0^T e^{-\rho s} U(C_s, P_s) ds \right]. \quad (5.36)$$

Using a localization argument as before (see the proof of Theorem 5.1), we let $T \rightarrow \infty$ and, using the transversality condition (5.28), we obtain

$$-v(K, P) \leq -J(K, P; C, \theta) \iff v(K, P) \geq J(K, P; C, \theta), \quad (5.37)$$

which proves the upper bound.

- (ii) Let $(\hat{C}, \hat{\theta})$ satisfy the attainment condition (5.30). Repeating the argument above with $(\hat{C}, \hat{\theta})$,

$$\rho v = U(\hat{C}, P) + \mathcal{L}^{\hat{C}, \hat{\theta}} v. \quad (5.38)$$

Therefore, for each T ,

$$\mathbb{E} \left[e^{-\rho T} v(\hat{K}_T, \hat{P}_T) \right] - v(K, P) = -\mathbb{E} \left[\int_0^T e^{-\rho s} U(\hat{C}_s, \hat{P}_s) ds \right]. \quad (5.39)$$

Letting $T \rightarrow \infty$ and using (5.28) for $(\hat{C}, \hat{\theta})$ gives $v(K, P) = J(K, P; \hat{C}, \hat{\theta})$. Together with part (i), this implies $v(K, P) = \sup_{C, \theta \in \mathcal{A}(K, P)} J(K, P; \hat{C}, \hat{\theta})$ and thus the optimality of $(\hat{C}, \hat{\theta})$. \square

Remark 5.5 (On transversality condition (5.28) and integrability). A sufficient set of conditions for the martingale property of M_T and for (5.28):

- (a) v has at most polynomial growth;
- (b) $\omega \in C^1(\mathcal{S})$;
- (c) $\lambda(\cdot)$ is locally Lipschitz with at most linear growth, so $\mathbb{E}[\int_0^T \lambda(P_s) ds] < \infty$ for each T ;
- (d) under any admissible (C, θ) , moments of K_t and P_t grow at most exponentially with rate strictly smaller than ρ .

Under these assumptions, by dominated convergence theorem

$$\lim_{T \rightarrow \infty} \mathbb{E}[e^{-\rho T} v(K_T, P_T)] = 0. \quad (5.40)$$

5.2 Diffusive Models

5.2.1 Nonhomogeneous Poisson process

Verification theorem

The verification theorem (Theorem 5.6 below) provides a general criterion: any sufficiently smooth function v that solves the HJB (5.44), with admissible feedback controls $(\hat{C}, \hat{\theta})$ attaining the Hamiltonian, coincides with the value function and makes $(\hat{C}, \hat{\theta})$ optimal. In the present jump–diffusion setting with pollution–dependent intensity, the Hamiltonian retains the same (C, θ) dependence as in the constant-intensity case, so the first–order conditions deliver the feedback rules $\hat{C} = \psi K$ and the θ –projection in (4.5).

The planner maximizes

$$J(K, P; C, \theta) = \mathbb{E} \left[\int_0^\infty e^{-\rho t} \left(\frac{C_t^{1-\varepsilon}}{1-\varepsilon} - \chi \frac{P_t^{1+\beta}}{1+\beta} \right) dt \right], \quad (K, P) \in \mathcal{S}, \quad (5.41)$$

with parameters $\rho > 0$, $A, \phi, \sigma, \alpha, \chi > 0$, $\varepsilon, \beta > 0$.

For $f \in C^2(\mathcal{S})$, the generator under controls (C, θ) is

$$\begin{aligned} (\mathcal{L}^{C, \theta} f)(K, P) = & f_K(K, P)((1 - \theta)AK - C) + f_P(K, P)((\phi - \sigma\theta)AK - \alpha P) \\ & + \frac{1}{2} \sigma_P^2 P^2 f_{PP}(K, P) \\ & + \lambda(P) \left(f(\omega(K, P)K, P) - f(K, P) \right). \end{aligned} \quad (5.42)$$

The value function

$$v(K, P) = \sup_{(C, \theta) \in \mathcal{A}(K, P)} J(K, P; C, \theta) \quad (5.43)$$

satisfies the Hamilton–Jacobi–Bellman equation:

$$\rho v(K, P) = \sup_{(C, \theta) \in \mathcal{A}(K, P)} \left\{ \frac{C^{1-\varepsilon}}{1-\varepsilon} - \chi \frac{P^{1+\beta}}{1+\beta} + (\mathcal{L}^{C, \theta} v)(K, P) \right\}. \quad (5.44)$$

We now state and prove the verification result. As in the previous cases we shall be using a localization argument when taking $T \rightarrow \infty$, which we left implicit in the text.

Theorem 5.6 (Verification – nonhomogeneous Poisson with Brownian pollution). *Assume $v \in C^2(\mathcal{S})$ is nonnegative, has at most polynomial growth, and satisfies the HJB (5.44) pointwise.*

(i) (Upper bound) Assume the transversality condition

$$\lim_{T \rightarrow \infty} \mathbb{E}[e^{-\rho T} v(K_T, P_T)] = 0 \quad (5.45)$$

holds for every admissible $(C, \theta) \in \mathcal{A}(K, P)$. Then, for any admissible $(C, \theta) \in \mathcal{A}(K, P)$,

$$v(K, P) \geq J(K, P; C, \theta). \quad (5.46)$$

(ii) (Optimality) If there exist Borel measurable functions $(\hat{C}, \hat{\theta}) : \mathcal{S} \rightarrow [0, \infty) \times [0, 1]$ such that, for all $(K, P) \in \mathcal{S}$,

$$U(\hat{C}, P) + (\mathcal{L}^{\hat{C}, \hat{\theta}} v)(K, P) = \sup_{(C, \theta) \in \mathcal{A}(K, P)} \left\{ U(C, P) + (\mathcal{L}^{C, \theta} v)(K, P) \right\}, \quad (5.47)$$

then $(\hat{C}, \hat{\theta})$ is optimal and

$$v(K, P) = J(K, P; \hat{C}, \hat{\theta}) = \sup_{(C, \theta) \in \mathcal{A}(K, P)} J(K, P; C, \theta). \quad (5.48)$$

Proof. (i) Fix (C_t, θ_t) admissible with state (K_t, P_t) . Define $Y_t := e^{-\rho t} v(K_t, P_t)$. By Itô's formula for jump–diffusions,

$$\begin{aligned} e^{-\rho T} v(K_T, P_T) - v(K_0, P_0) &= \int_0^T e^{-\rho s} \left((\mathcal{L}^{C_s, \theta_s} v)(K_s, P_s) - \rho v(K_s, P_s) \right) ds \\ &\quad + \underbrace{\int_0^T e^{-\rho s} v_P(K_s, P_s) \sigma_P P_s dW_s}_{=: M_T^{(W)}} \\ &\quad + \underbrace{\int_0^T e^{-\rho s} \Delta v_s dM_s}_{=: M_T^{(\hat{q})}}, \end{aligned} \quad (5.49)$$

where M is the compensated jump martingale of \hat{q} ,

$$\Delta v_s := v(\omega(K_{s-}, P_s)K_{s-}, P_s) - v(K_{s-}, P_s), \quad (5.50)$$

and we have used

$$\int_0^T e^{-\rho s} \Delta v_s d\hat{q}_s = \int_0^T e^{-\rho s} \Delta v_s dM_s + \int_0^T e^{-\rho s} \lambda(P_s) \Delta v_s ds. \quad (5.51)$$

Rearranging (5.49) gives

$$\mathbb{E}[e^{-\rho T} v(K_T, P_T)] - v(K, P) = \mathbb{E}\left[\int_0^T e^{-\rho s} ((\mathcal{L}^{C_s, \theta_s} v) - \rho v)(K_s, P_s) ds\right], \quad (5.52)$$

since $\mathbb{E}[M_T^{(W)}] = \mathbb{E}[M_T^{(\hat{q})}] = 0$.

By the HJB inequality,

$$\rho v \geq U(C_s, P_s) + (\mathcal{L}^{C_s, \theta_s} v) \iff (\mathcal{L}^{C_s, \theta_s} v) - \rho v \leq -U(C_s, P_s). \quad (5.53)$$

Thus, for every T ,

$$\mathbb{E}[e^{-\rho T} v(K_T, P_T)] - v(K_0, P_0) \leq -\mathbb{E}\left[\int_0^T e^{-\rho s} U(C_s, P_s) ds\right]. \quad (5.54)$$

Letting $T \rightarrow \infty$ and using (5.45), yields

$$v(K, P) \geq \mathbb{E}\left[\int_0^\infty e^{-\rho s} U(C_s, P_s) ds\right] = J(K, P; C, \theta). \quad (5.55)$$

(i) If $(\hat{C}, \hat{\theta})$ attains the supremum (5.47), then

$$\rho v = U(\hat{C}, P) + (\mathcal{L}^{\hat{C}, \hat{\theta}} v). \quad (5.56)$$

Then for each T ,

$$\mathbb{E}[e^{-\rho T} v(\hat{K}_T, \hat{P}_T)] - v(K, P) = -\mathbb{E}\left[\int_0^T e^{-\rho s} U(\hat{C}_s, \hat{P}_s) ds\right]. \quad (5.57)$$

Letting $T \rightarrow \infty$ and using (5.45) for $(\hat{C}, \hat{\theta})$ yields $v(K, P) = J(K, P; \hat{C}, \hat{\theta})$. Combined with part (i) this proves optimality. \square

Remark 5.7 (On the transversality condition and integrability). A sufficient set of conditions ensuring (5.45) and the martingale properties used in the proof is

(a) $v \in C^2(\mathcal{S})$ with at most polynomial growth so

$$|v| + |v_K| + |v_P| + |v_{PP}| \leq C(1 + K^m + P^m); \quad (5.58)$$

(b) for any admissible (C, θ) there exists $m' > 0$ such that

$$\sup_{t \geq 0} \mathbb{E}[|K_t|^{m'} + |P_t|^{m'}] < \infty; \quad (5.59)$$

(c) $\sup_{t \in [0, T]} \mathbb{E}[\lambda(P_t)] < \infty$ for each finite T ;

(d) ρ dominates the growth rate of the moments in (b).

Then both

$$M_T^{(W)} = \int_0^T e^{-\rho s} v_P \sigma_P P_s dW_s \quad \text{and} \quad M_T^{(\hat{q})} = \int_0^T e^{-\rho s} \Delta v_s dM_s \quad (5.60)$$

are square-integrable martingales with zero mean, and dominated convergence implies

$$\lim_{T \rightarrow \infty} \mathbb{E}[e^{-\rho T} v(K_T, P_T)] = 0. \quad (5.61)$$

It remains to check that our candidate value function v from (4.28) indeed solves the HJB and that $(\hat{C}, \hat{\theta})$ attains the supremum. Substituting the candidate function into (5.44) separates the K - and P -blocks. The resulting v is C^2 with polynomial growth, and the induced feedback is admissible and attains the Hamiltonian pointwise. Hence, by Theorem 5.6, the candidate is the value function and the feedback is optimal.

Viscosity solutions

In this section, we develop the corresponding viscosity solution framework. The setting involves a jump–diffusion system with endogenous, pollution–dependent jump intensity and stochastic pollution evolution, leading to an integro–differential HJB equation that may not admit classical smooth solutions. The viscosity approach therefore provides a general and robust concept of solution that ensures existence, stability, and uniqueness within the class of continuous functions.

The controlled state process (K_t, P_t) evolves according to

$$dK_t = b_K(K_t, P_t, C_t, \theta_t) dt - (1 - \omega(K_{t-}, P_t)) K_{t-} d\hat{q}_t, \quad (5.62)$$

$$dP_t = b_P(K_t, P_t, \theta_t) dt + \sigma_P P_t dW_t, \quad (5.63)$$

where the drift coefficients are given by

$$\begin{aligned} b_K(K, P, C, \theta) &:= (1 - \theta)Y(K) - C, \\ b_P(K, P, C, \theta) &:= \phi Y(K) - Z(\theta Y(K)) - \alpha P. \end{aligned} \quad (5.64)$$

Given a continuous function $u : \mathcal{S} \rightarrow \mathbb{R}$, we write u^* for its *upper semicontinuous envelope* and u_* for its *lower semicontinuous envelope*, defined respectively by

$$u^*(x) = \limsup_{y \rightarrow x} u(y), \quad u_*(x) = \liminf_{y \rightarrow x} u(y). \quad (5.65)$$

According to the literature on viscosity solutions (see [7], or [2] and [24] for the discontinuous and continuous settings, respectively), we introduce the following *test-function comparison principle*:

Definition 5.8 (Local test functions). Let u be a locally bounded function on \mathcal{S} .

- (i) A function $\varphi \in C^2(\mathcal{S})$ is said to *touch u from above at (k, p)* if $(u - \varphi)(k, p) = 0$ and $(u - \varphi)(k', p') \leq 0$ for (k', p') in a neighborhood of (k, p) .
- (ii) Analogously, φ *touches u from below at (k, p)* if $(u - \varphi)(k, p) = 0$ and $(u - \varphi)(k', p') \geq 0$ in a neighborhood of (k, p) .

For any smooth test function $f \in C^2(\mathcal{S})$, the infinitesimal generator of the process (K_t, P_t) under a fixed control (C, θ) is

$$\begin{aligned} (\mathcal{L}^{C, \theta} f)(K, P) &= f_K(K, P) b_K(K, P, C, \theta) + f_P(K, P) b_P(K, P, \theta) \\ &\quad + \frac{1}{2} \sigma_P^2 P^2 f_{PP}(K, P) + \lambda(P) (f(\omega(K, P)K, P) - f(K, P)), \end{aligned} \quad (5.66)$$

where the last term is the *nonlocal jump operator* representing the expected instantaneous effect of disasters on the function f . By applying the DPP over a short time horizon and using Itô's formula for jump-diffusions, one obtains that v formally satisfies

$$\rho v(K, P) = \sup_{(C, \theta) \in \mathcal{A}(K, P)} \left\{ U(C, P) + (\mathcal{L}^{C, \theta} v)(K, P) \right\}. \quad (5.67)$$

Since v need not be smooth, we proceed to a weak formulation in the sense of viscosity solutions.

Definition 5.9 (Viscosity sub- and supersolutions). Let u be a continuous function on \mathcal{S} .

- (a) u is a *viscosity subsolution* of (5.67) if, for all $\varphi \in C^2(\mathcal{S})$ and all points (k, p) where $u - \varphi$ attains a local maximum,

$$\rho u(k, p) \leq \sup_{(C, \theta) \in \mathcal{A}(k, p)} \left\{ U(C, p) + (\mathcal{L}^{C, \theta} \varphi)(k, p) \right\}. \quad (5.68)$$

- (b) u is a *viscosity supersolution* if, for all $\varphi \in C^2(\mathcal{S})$ and all points (k, p) where $u - \varphi$ attains a local minimum,

$$\rho u(k, p) \geq \sup_{(C, \theta) \in \mathcal{A}(k, p)} \left\{ U(C, p) + (\mathcal{L}^{C, \theta} \varphi)(k, p) \right\}. \quad (5.69)$$

(c) u is a *viscosity solution* if it is both a subsolution and a supersolution.

The next result states that the value function of the control problem is indeed a viscosity solution of the HJB equation.

Theorem 5.10 (Value function as viscosity solution). *Assume that:*

- (a) $U(C, P)$ is continuous and concave in C , with polynomial growth;
- (b) Y, Z are locally Lipschitz and have linear growth;
- (c) $\omega(K, P) \in (0, 1)$ and $\lambda(P)$ are continuous with λ locally Lipschitz;
- (d) for every $(C, \theta) \in \mathcal{A}$, the SDEs (5.62)–(5.63) admit a unique strong solution.

Then the value function v is continuous and is a viscosity solution of the HJB equation.

Proof. By the dynamic programming principle, for any stopping time τ and any admissible control (C, θ) ,

$$v(k, p) \geq \mathbb{E} \left[\int_0^\tau e^{-\rho s} U(C_s, P_s) ds + e^{-\rho \tau} v(K_\tau, P_\tau) \right]. \quad (5.70)$$

Fix (K_0, P_0) and let φ be a C^2 test function such that $v - \varphi$ attains a local maximum at (K_0, P_0) . Applying Itô's formula for jump-diffusions to $e^{-\rho t} \varphi(K_t, P_t)$ on $[0, \tau]$, taking expectations, and using the local optimality of (K_0, P_0) , we obtain the inequality (5.68). The argument for supersolutions is analogous. \square

5.2.2 General jump–diffusion model

Unlike our previous models, the presence of a random measure with a stochastic intensity kernel $\lambda(P_t, \zeta)$ requires some modifications in the verification theorems for viscosity solutions of the value function v , due to the stronger interaction between the controlled states and the sizes of the jumps. To overcome this issue, we now make use of the recent developments in infinite–horizon recursive control provided in [20]. We focus on a modification of the model (2.37)–(2.39) containing both a Poissonian and a Brownian component and impose an additional set of suitable assumptions, to be described next:

Assumptions 5.11.

- (1) The set of admissible actions $A \subseteq \mathbb{R}_{\geq 0} \times [0, 1]$ is compact.

(2) Let $\tilde{\omega} : \mathcal{S} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ be a measurable function such that

$$0 \leq \tilde{\omega}(K, P, \zeta) \leq K, \quad \forall (K, P) \in \mathcal{S}, \quad \forall \zeta \in \mathbb{R}_{\geq 0}.$$

Furthermore, there exists a measurable mapping $\ell_\gamma : [0, \infty) \rightarrow [0, 1]$ satisfying

$$|\ell_\gamma|_{P,2} + |\ell_\gamma|_{P,m} < \infty$$

for some $m \geq 2$ with

$$|\ell_\gamma|_{P,m} := \left(\int_{[0,\infty)^2} |\ell_\gamma(\zeta, r)|^m \nu(d\zeta) \otimes dr \right)^{\frac{1}{m}}, \quad \forall P \in \mathbb{R}_{>0},$$

such that for all $(K, P), (K', P') \in \mathcal{S}$, $\zeta, r \in [0, \infty)$, $(C, \theta) \in A$,

$$\begin{aligned} & |\tilde{\omega}(K, P, \zeta) \mathbf{1}_{[0, \lambda(P, \zeta)]}(r) - \tilde{\omega}(K', P', \zeta) \mathbf{1}_{[0, \lambda(P', \zeta)]}(r)| \\ & \leq \ell_\gamma(\zeta, r) (|K - K'| + |P - P'|). \end{aligned}$$

(3) Let b_K and b_P be as in (5.64) for some given functions Y and Z , representing the production and efficiency of abatement, respectively. There exist constants $\ell_b \geq 0$ and $\alpha_b > 0$ such that for all $(K, P), (K', P') \in \mathcal{S}$, $(C, \theta) \in A$,

$$\begin{aligned} & |\tilde{b}_K(K, P, C, \theta) - \tilde{b}_K(K', P', C, \theta)| + |b_P(K, P, C, \theta) - b_P(K', P', C, \theta)| \\ & \leq \ell_b (|K - K'| + |P - P'|), \\ & (\tilde{b}_K(K, P, C, \theta) - \tilde{b}_K(K', P', C, \theta))(K - K') \\ & \quad + (b_P(K, P, C, \theta) - b_P(K', P', C, \theta))(P - P') \\ & \leq -\alpha_b (|K - K'|^2 + |P - P'|^2), \end{aligned}$$

where

$$\tilde{b}_K(K, P, C, \theta) := b_K(K, P, C, \theta) - \int_{(0,\infty)} \tilde{\omega}(K, P, \zeta) \lambda(P, \zeta) \nu(d\zeta).$$

(4) The utility function $U(C, P)$ is Lipschitz on P .

(5) Let $m \geq 2$ be as in the previous point. The following inequality holds

$$2\alpha_b - (m-1)\sigma - \frac{2c_m}{m} |\ell_\gamma|_{P,2}^2 - c_m |\ell_\gamma|_{P,m}^m > 0,$$

with

$$c_m := \begin{cases} \frac{m(m-1)}{2} - 1, & \text{if } 2 < m < 3, \\ 2m - 4, & \text{if } m = 2 \text{ or } m \geq 3. \end{cases}$$

Some comments regarding the previous assumptions: 5.11.(1) is a well-known sufficient condition for the minimization of the Hamiltonian at any given instant t ; Assumptions 5.11.(2), 5.11.(3) and 5.11.(4) correspond to standard regularity conditions from the theory of forward-backward SDEs, with 5.11.(2) being the Lipschitz regularity of the jump component in terms of the extended Poisson space, i.e. the underlying random measure N from Assumption 3.1, while also maintaining the economic interpretation that disasters destroy part of the capital; lastly, 5.11.(5) is a technical assumption linking the convexity of the value function to the regularity of the jump sizes. This last point is explored with more detail in [20].

Definition 5.12. Let $A \subseteq \mathbb{R}_{\geq 0} \times [0, 1]$ be compact. For $m \geq 2$ and $t \geq 0$, the set of admissible controls at time t is defined as

$$\mathcal{A}_t^m := \left\{ (C, \theta) : \Omega \times [t, \infty) \rightarrow A \mid (C, \theta) \text{ is } \mathbb{F}\text{-predictable, and} \right. \\ \left. \mathbb{E} \left[\int_t^\infty \max\{C_s^2, C_s^m\} ds \mid \mathcal{F}_t \right] < \infty \right\}.$$

With the assumptions set, we are ready to present the main result of this section. In what follows, we use integral notation on the SDEs in order to clarify the forward and backward components.

Theorem 5.13. *Under our standing assumptions, the following hold:*

- (a) *For every $(C, \theta) \in \mathcal{A}_0^m$, there exists a strong unique solution to the system of forward-backward SDEs in infinite time horizon:*

$$\begin{aligned} K_t &= K_0 + \int_0^t \tilde{b}_K(K_s, P_s, C_s, \theta_s) dt \\ &\quad - \int_{(0,t] \times (0,\infty) \times (0,\infty)} \tilde{\omega}(K_s, P_s, \zeta) \mathbf{1}_{[0, \lambda(P_s, \zeta)]}(r) \tilde{N}(ds, d\zeta, dr), \\ P_t &= P_0 + \int_0^t b_P(K_s, P_s, C_s, \theta_s) dt + \int_0^t \sigma_P P_s dW_s, \end{aligned}$$

with $K_0 = k, P_0 = p > 0$ for all $t \geq 0$, and

$$\begin{aligned} \mathcal{V}_t &= \mathcal{V}_T + \int_t^T (U(C_s, P_s) - \rho \mathcal{V}_s) ds - \int_t^T \mathcal{Z}_s dW_s \\ &\quad - \int_{(t,T] \times (0,\infty) \times (0,\infty)} \mathcal{U}_s(\zeta, r) \tilde{N}(ds, d\zeta, dr), \end{aligned} \tag{5.71}$$

for all $0 \leq t \leq T < \infty$.

(b) *There exists a unique viscosity solution $v(\cdot)$ to the HJB (3.61). Moreover:*

$$\mathcal{V}_0 = \sup_{(C, \theta) \in \mathcal{A}_0^m} J(k, p; C, \theta) = J(k, p; \hat{C}, \hat{\theta}) = v(k, p), \quad (5.72)$$

where

$$J(k, p; C, \theta) = \mathbb{E} \left[\int_0^\infty e^{-\rho t} U(C_t, P_t) dt \mid K_0 = k, P_0 = p \right].$$

(c) *Conversely, if v is a classical (respectively, viscosity) solution to the HJB (3.61), such that for the maximizer of the Hamiltonian at time t there exists an admissible control $(\hat{C}, \hat{\theta}) \in \mathcal{A}_t^m$ for all $t \geq 0$, and the previous forward-backward SDE has a strong solution for the control $(\hat{C}, \hat{\theta})$; then $(\hat{C}, \hat{\theta})$ is an optimal control and v is the corresponding value function.*

Before presenting the proof of the previous theorem, we briefly illustrate how the BSDE in infinite time horizon (5.71) encodes the information corresponding to the original objective functional J from (2.13). Let $(\mathcal{V}, \mathcal{Z}, \mathcal{U})$ be a solution to the infinite-horizon BSDE (5.71). An application of Itô's formula to $g(t, \mathcal{V}_t)$ with $g(t, x) = e^{-\rho t} x$ yields

$$\begin{aligned} e^{-\rho T} \mathcal{V}_T - e^{-\rho t} \mathcal{V}_t &= - \int_t^T e^{-\rho s} U(C_s, P_s) ds + \int_t^T e^{-\rho s} \mathcal{Z}_s dW_s \\ &\quad + \int_{(t, T] \times (0, \infty) \times (0, \infty)} \mathcal{U}_s(\zeta, r) \tilde{N}(ds, d\zeta, dr), \end{aligned} \quad \forall 0 \leq t \leq T < \infty.$$

Then, applying conditional expectations with respect to \mathcal{F}_t , the following recursive relation for the running utility is obtained:

$$\mathcal{V}_t = \mathbb{E} \left[e^{-\rho(T-t)} \mathcal{V}_T + \int_t^T e^{-\rho(s-t)} U(C_s, P_s) ds \mid \mathcal{F}_t \right], \quad \forall 0 \leq t \leq T < \infty. \quad (5.73)$$

In other words,

$$\mathcal{V}_t = \liminf_{T \rightarrow \infty} \mathbb{E} \left[e^{-\rho(T-t)} \mathcal{V}_T + \int_t^T e^{-\rho(s-t)} U(C_s, P_s) ds \mid \mathcal{F}_t \right], \quad \forall t \geq 0,$$

from where we can deduce the necessity of the usual transversality condition

$$\liminf_{T \rightarrow \infty} \mathbb{E} \left[e^{-\rho T} \mathcal{V}_T \right] = 0$$

in order to obtain

$$\mathcal{V}_0 = \liminf_{T \rightarrow \infty} \mathbb{E} \left[\int_0^T e^{-\rho s} U(C_s, P_s) ds \right],$$

see Lemma 3.2 in [20]. It is worth noting that the type of recursive relation presented in (5.73) – hence the name *recursive utility* – has been known in literature; in the realm of randomized settings, we would like to point out to the pioneer work Duffie and Epstein [11] on stochastic differential utility.

Once the relation between BSDEs and the objective functional has been established, the proof of the main result reduces to verifying the conditions required in [20]:

Proof of Theorem 5.13. Let

$$f(K, P, C, \theta, \mathcal{V}) = U(C, P) - \rho \mathcal{V},$$

and observe that for all $t \geq 0$, $(C, \theta) \in A$, $(K, P), (K', P') \in \mathcal{S}$ and $\mathcal{V}, \mathcal{V}' \in \mathbb{R}$,

$$\begin{aligned} |f(K, P, C, \theta, \mathcal{V}) - f(K', P', C, \theta, \mathcal{V}')| &\leq \ell_U |P - P'| + \rho |\mathcal{V} - \mathcal{V}'|, \\ (f(K, P, C, \theta, \mathcal{V}) - f(K, P, C, \theta, \mathcal{V}')) \cdot (\mathcal{V} - \mathcal{V}') &\leq -\rho |\mathcal{V} - \mathcal{V}'|^2, \end{aligned}$$

for some nonnegative Lipschitz constant ℓ_U . This, coupled with Assumptions 5.11, means that the main conditions (C1)_p, (C2), (C3) and (C4)' of [20] hold true, with p and ρ in their paper corresponding to m and 1 in our current setting, respectively. Consequently, most of the results of Theorem 5.13 are obtained from propositions in [20]: 5.13.(a) is due to Lemmas 3.1 and 3.2; the equality of (5.72) in 5.13.(b) corresponds to Theorem 4.7; and the verification results from 5.13.(c) are obtained through Theorems 5.2 and 5.7 for the classical and viscosity solution cases, respectively. □

Before closing this section, we point out that, although Assumptions 5.11 may exclude some of the models considered before, the present framework could be extended to include them. More precisely, $\tilde{\omega}$ and U are now assumed to be *globally* Lipschitz in the state variables, rather than merely *locally* Lipschitz as in previous sections, see Theorem 5.10.

This is a technical restriction due to the approach taken by [20], which requires the existence of strong solutions to BSDEs with infinite time horizon in order to proof the existence of a unique viscosity solution v to the HJB equation (3.61). This does not contradict our previous statements, since points 5.13.(a) and 5.13.(b) of Theorem 5.13 fall within the scope of 5.10, but it does affect the verification result from the theorem, i.e. point 5.13.(c). We conjecture that these conditions of global Lipschitz regularity can be weakened, for example, by extending the results in [1] that ensure the existence of strong solutions to BSDEs with locally Lipschitz coefficients from a finite to an infinite time horizon setting, or the ones found in [23] to include locally Lipschitz drivers.

6 Conclusion

In this paper we have developed a unified stochastic control framework for growth-environment models in which the intensity and severity of rare disasters are endogenously linked to the state of pollution. Building on the setup of [4], we formulated the social planner's problem on an infinite horizon with capital and pollution evolving according to controlled jump(–diffusion) dynamics, where disasters destroy a state-dependent fraction of the capital stock while their arrival intensity may depend on pollution (and, in the most general specifications, on additional marks and sources of randomness). Within this framework we defined the value function and characterized optimal trade-offs between consumption, investment, and abatement under environmentally driven catastrophe risk.

From a modelling perspective, our contribution has been to organize several specifications that have appeared in the economics and climate literature into a single, mathematically coherent hierarchy. We began with the benchmark case of a standard Poisson process with constant arrival intensity. We then allowed the intensity to depend on pollution via a nonhomogeneous Poisson process, thereby capturing feedback from environmental degradation to catastrophe risk. Next, we introduced Brownian noise into the pollution dynamics, leading to jump–diffusion models and integro–differential HJB equations with both local (diffusive) and nonlocal (jump) terms. At the most general level, we showed how all these cases can be embedded in a formulation driven by a Poisson random measure with state-dependent compensator and random marks, which accommodates both random disaster magnitudes and randomized pollution dynamics within a coherent stochastic framework. This nesting clarifies the precise sense in which models with pollution-driven disaster intensity extend the constant-intensity benchmark and provides a common language for comparing them.

On the analytical side, we derived the associated Hamilton–Jacobi–Bellman equations from the dynamic programming principle, both for the simpler pure-jump models and for the jump–diffusion and Poisson random measure specifications. For the sufficiently regular case we have stated a verification theorem for classical solutions (Theorem 5.1), identifying conditions under which a C^2 solution to the HJB equation indeed coincides with the value function and yields optimal feedback controls, and we showed how the functional form of the HJB equation and the structure of the optimality conditions evolve as the jump mechanism becomes richer. When such regularity cannot be guaranteed, we have shown that the value function is a viscosity solution of the HJB equation (Theorem 5.10) under natural assumptions on preferences, production, abatement, and the jump mechanism. In the most general setting, where pollution is affected by both Brownian motion and Poisson random measures, we have further characterized the value function by means of forward-backward stochastic differential equations with jumps. Theorem 5.13 shows that, under the standing assumptions, the value function can be represented as the

solution of a suitable infinite-horizon FBSDE, and conversely, that solutions of the associated FBSDE system correspond to (classical or viscosity) solutions of the HJB equation. This closes the loop between the stochastic control formulation, its PIDE representation, and the FBSDE approach, and it provides a flexible analytical and numerical toolbox for studying pollution-driven disaster models.

The flexibility of the random-measure approach opens the way for a wide range of extensions: richer damage functions, nonlinear or threshold intensities, endogenous technological progress in abatement, spatial diffusion of pollution, and etc. Each of these avenues promises deeper insights into the design of robust climate and environmental policies under uncertainty.

While our focus has been on the mathematical structure of pollution-driven disaster models, the framework is sufficiently general to accommodate a variety of economically relevant extensions. For instance, one can incorporate more detailed damage functions linking pollution to the distribution of disaster sizes, allow for multiple types of pollutants and capital stocks, let technological and policy parameters evolve according to additional diffusive or jump components, or strategic interaction among multiple agents (e.g. regions), leading to game-theoretic control problems. On the analytical side, natural extensions include studying state-constraint problems (e.g. at zero capital or pollution) and analyzing models with learning about the disaster intensity, which would lead to robust control formulations and HJB equations with additional nonlinearity.

Thus, the paper develops a unified framework for analyzing the interplay between economic growth, environmental policy, and the risk of rare but destructive events. In particular, it highlights that sustainability in the presence of deep uncertainty requires not only gradual adjustments but also resilience against rare disasters. The models developed here achieve both tractability and generality, providing a rigorous analytical foundation and constructive tools to characterize and solve the stochastic control problems governing the long-run interaction between the economy and the environment.

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