

Sigmoid-FTRL: Design-Based Adaptive Neyman Allocation for AIPW Estimators

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Abstract

We consider the problem of Adaptive Neyman Allocation for the class of AIPW estimators in a design-based setting, where potential outcomes and covariates are deterministic. As each subject arrives, an adaptive procedure must select both a treatment assignment probability and a linear predictor to be used in the AIPW estimator. Our goal is to construct an adaptive procedure that minimizes the Neyman Regret, which is the difference between the variance of the adaptive procedure and an oracle variance which uses the optimal non-adaptive choice of assignment probability and linear predictors. While previous work has drawn insightful connections between Neyman Regret and online convex optimization for the Horvitz–Thompson estimator, one of the central challenges for AIPW estimator is that the underlying optimization is non-convex. In this paper, we propose Sigmoid-FTRL, an adaptive experimental design which addresses the non-convexity via simultaneous minimization of two convex regrets. We prove that under standard regularity conditions, the Neyman Regret of Sigmoid-FTRL converges at a $T^{-1/2}R^2$ rate, where T is the number of subjects in the experiment and R is the maximum norm of covariate vectors. Moreover, we show that no adaptive design can improve upon the $T^{-1/2}$ rate under our regularity conditions. Finally, we establish a central limit theorem and a consistently conservative variance estimator which facilitate the construction of asymptotically valid Wald-type confidence intervals.

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1 Introduction

Randomized experiments are used to investigate causal effects in virtually all of the social sciences, from economics and political science to sociology and public health. In a classical randomized experiment, the experimental design (i.e. the randomized treatment assignment mechanism) does not depend on the observed outcomes. In other words, the subjects enter the study, treatment is assigned, and only afterwards are the outcomes observed and the treatment effects consequently estimated. In recent years, there has been a growing interest in adaptive randomized experiments, where subjects arrive sequentially and the experimenter can incorporate previously observed outcomes in the experimental design.

In this paper, we study the problem of Adaptive Neyman Allocation for AIPW estimators in the design-based framework. In this context, an adaptive experimental design must select both the treatment assignment probability and the linear predictors used in the AIPW estimator. Roughly speaking, the goal of Adaptive Neyman Allocation is to construct an adaptive experiment design under which the variance of an effect estimator is nearly equal to its optimal variance under the best non-adaptive design that has oracle access to all potential outcomes. We focus specifically on constructing an adaptive experiment design which minimizes this difference, which is known as the Neyman Regret (Dai, Gradu, and Harshaw, 2023; Kato et al., 2025). A formal description of the problem is deferred to Section 3.

We focus on the design-based framework, where potential outcomes and covariates of each subject are considered to be deterministic and treatment assignment is the sole source of randomness. The design-based framework stands in contrast to a super-population framework where subjects are assumed to be independent and identical draws from an unknown distribution. The sampling assumption may be difficult to interpret and justify in settings where subjects were not literally randomly selected into the study; moreover, it precludes the possibility of drift or otherwise systematic change in the potential outcomes of subjects over time. For these reasons, the design-based framework is sometimes seen as more robust and assumption-lean (Harshaw, 2025).

The paper which is most closely related to ours is Dai, Gradu, and Harshaw (2023), who consider Adaptive Neyman Allocation for the unadjusted Horvitz–Thompson estimator. They introduce Clip-OGD, an experimental design based on online gradient descent with probability clipping, which guarantees that the Neyman Regret converges at a rate of $T^{-1/2} \exp(\sqrt{\log(T)})$. Their results are based upon an insightful connection between Adaptive Neyman Allocation and online convex optimization, which we further explore here.

One of the pressing questions left open by this work is how to extend the results to AIPW estimators, which are known to be more efficient in the non-adaptive setting when covariate information is available (Lin, 2013; Lei and Ding, 2020). As we show in this paper, the optimization problem underlying Neyman Regret for AIPW is non-convex, which precludes the possibility of directly using techniques from online convex optimization. A secondary question is whether the convergence rate of $T^{-1/2} \exp(\sqrt{\log(T)})$ is optimal.

In this paper, we make the following contributions which resolve these open questions:

- **Optimal Rates and Experimental Design:** We present SIGMOID-FTRL, a new adaptive experimental design under which the AIPW Neyman Regret converges at a rate of $T^{-1/2} R^2$, where T is the number of subjects and R is the maximum covariate norm. To overcome the issue of non-convexity, the design simultaneously minimizes two convex objectives corresponding to the selection of treatment assignment probability and linear predictors, respectively. We derive a matching $T^{-1/2}$ lower bound which demonstrates that the design is rate optimal under our regularity assumptions. In order to obtain optimal rates, SIGMOID-FTRL employs a sigmoidal transformation of the domain, which

may likely be of independent interest to the online optimization community.

- **Inferential Methods:** We provide a central limit theorem for the AIPW estimator under SIGMOID-FTRL, which requires further technical developments. We also construct a consistent estimator for Neyman’s variance bound. Together, these enable the development of Wald-type intervals that asymptotically cover at the nominal level.

An interesting conclusion of this work is the distinction between Adaptive Neyman Allocation in design-based and super-population frameworks. We show that $T^{-1/2}$ is the optimal rate of Neyman Regret in a design-based setting, whereas prior work has shown that $T^{-1} \log(T)$ is the optimal rate in a super-population setting (Neopane, Ramdas, and Singh, 2025a; Neopane, Ramdas, and Singh, 2025b). This difference mirrors results in the bandit literature, where adversarial and stochastic settings have minimax regret $T^{1/2}$ and $\log(T)$, respectively (Lattimore and Szepesvári, 2020). In both literatures, treating data as being deterministic offers more robustness, but at the cost of slower convergence.

1.1 Related Work

The foundations of Adaptive Neyman Allocation go back nearly a century. Neyman (1934) was the first to consider optimal allocation strategies, demonstrating that sampling from treatments proportional to the within-treatment outcome variance will minimize the variance of standard estimators. To the best of our knowledge, Robbins (1952) was the first to propose the sequential problem of constructing adaptive procedures which attain nearly the same variance as their optimal non-adaptive counterparts.

Adaptive experiments have seen a resurgence of interest from the causal inference community in the last twenty years. We focus our attention on the potential outcomes framework (Neyman, 1923; Rubin, 1980). From the causal inference perspective, early work focused on estimation and inference under adaptive treatment assignment (Laan, 2008; Hahn, Hirano, and Karlan, 2011). The focus in these papers is in estimating parameters (e.g. treatment probabilities) of the optimal design, but this does not directly guarantee that the resulting effect estimator obtains the correspondingly optimal variance. More recent work has focused on adaptive experiment design for obtaining efficiency bounds (Laan and Lendle, 2014; Cook, Mishler, and Ramdas, 2024; Kato et al., 2024). These results are for the asymptotic variance and do not provide an analysis of how close the finite sample variance is to the efficiency bound.

The Neyman Regret is a non-asymptotic quantification of the gap between the actual variance and the optimal variance, which has only been investigated more recently (Dai, Gradu, and Harshaw, 2023; Kato et al., 2025). In a series of work, Neopane, Ramdas, and Singh (2025a) and Neopane, Ramdas, and Singh (2025b) show that $T^{-1} \log(T)$ Neyman regret can be attained for Horvitz-Thompson and AIPW estimators in a super-population setting. Li, Simchi-Levi, and Zhao (2024) have shown that vanishing Neyman regret is achievable in these settings even when the design parameters can only be adapted a few number of times, i.e. low-switching designs.

In the context of design-based inference, Blackwell, Pashley, and Valentino (2022) propose a two-stage approach to reduce the variance of the difference-in-means estimator. As previously discussed, Dai, Gradu, and Harshaw (2023) use techniques from online optimization to construct an experimental design which attains $T^{-1/2} \exp(\log(\sqrt{T}))$ Neyman Regret for the Horvitz-Thompson estimator. Noarov et al. (2025) obtain $T^{-1} \log(T)$ Neyman Regret for the Horvitz-Thompson estimator, but under non-standard assumptions where the experimenter knows a constant lower bound on the absolute value of each individual potential outcomes¹.

¹In the context of AIPW estimators, this assumption would translate into a constant and known lower bound on each subject’s absolute residual under the optimal regression.

Outside of Adaptive Neyman Allocation, other aspects of adaptive experiments have been studied from a number of perspectives. A recent line of work has provided methods for estimation and inference of causal effects when treatment is assigned via a bandit algorithm (Hadad et al., 2021; Zhang, Janson, and Murphy, 2020; Zhang, Janson, and Murphy, 2021). Offer-Westort, Coppock, and Green (2021) and Chen and Andrews (2023) focus on selective inference under adaptive treatment assignment. Finally, any-time inference and data-dependent stopping are central areas of study in sequential analysis (Wald, 1945; Howard et al., 2021; Waudby-Smith et al., 2024).

2 Design-based Adaptive Experiments

2.1 Design-Based Potential Outcomes Framework

We consider a sequential experiment with T experimental subjects denoted by integers $t \in [T]$. The experimenter must assign each unit to exactly one of two treatment conditions. For each subject $t \in [T]$, we denote their treatment assignment as $Z_t \in \{0, 1\}$. Each experimental subject $t \in [T]$ is presumed to have two potential outcomes, $y_t(1)$ and $y_t(0)$, which correspond to the outcomes that would be measured under treatment ($Z_t = 1$) and control ($Z_t = 0$), respectively. The causal estimand of interest is the *average treatment effect* (ATE) which is defined as

$$\tau = \frac{1}{T} \sum_{t=1}^T y_t(1) - y_t(0) .$$

The experimenter also measures a vector of covariates $\mathbf{x}_t \in \mathbb{R}^d$ for each subject $t \in [T]$, which are not affected by treatment assignment and may be used to improve estimates of the average treatment effect. Implicit in the above is the standard Stable Unit Treatment Value Assumption (SUTVA) which posits that there are not hidden versions of treatment and that subjects do not interfere with each other (Holland, 1986; Imbens and Rubin, 2015; Hernán and Robins, 2020).

In this paper, we work in a design-based framework where the subjects, their potential outcomes, and their covariates are all considered to be deterministic and treatment is the sole source of randomness. In such a setting, randomization of treatment serves as the sole basis for statistical inference, e.g. no i.i.d. assumptions are placed on the subjects.

2.2 Adaptive Experiment Designs

The sequential experimental procedure proceeds in T rounds. At each round $t \in [T]$, the experimenter observed the covariates $\mathbf{x}_t \in \mathbb{R}^d$, then assigns treatment $Z_t \in \{0, 1\}$ and consequently observes the outcome

$$Y_t = \mathbf{1}[Z_t = 1]y_t(1) + \mathbf{1}[Z_t = 0]y_t(0) .$$

The experiment may be adaptive in the sense that the randomization of treatment assignment Z_t can depend on previously observed outcomes $Y_1 \dots Y_{t-1}$ and treatment assignments $Z_1 \dots Z_{t-1}$. Formally, we represent an *adaptive experiment design* by a sequence of mappings $\boldsymbol{\Pi} = \{\Pi_t\}_{t=1}^T$ with signature $\Pi_t : (\{0, 1\} \times \mathbb{R})^{t-1} \rightarrow [0, 1] \times \mathbb{R}^d \times \mathbb{R}^d$ that encode the conditional treatment assignment probability and linear predictors, i.e.

$$\Pr(Z_t = 1 \mid \mathcal{F}_{t-1}), \boldsymbol{\beta}_t(1), \boldsymbol{\beta}_t(0) = \Pi_t(Z_1, Y_1, \dots, Z_{t-1}, Y_{t-1}) ,$$

where \mathcal{F}_{t-1} conditions on past observations, i.e. formally \mathcal{F}_{t-1} is the σ -algebra generated by $Z_1 \dots Z_{t-1}$. The linear predictors $\boldsymbol{\beta}_t(1)$ and $\boldsymbol{\beta}_t(0)$ are used in the estimator, as described in Section 2.4. We focus on the setting where the order of subjects in the sequence is fixed and

arbitrary (i.e. cannot be chosen by the experimenter), which reflects the conditions often arising in practice. For this reason, we refer to the *sequence* of potential outcomes $\{(y_t(1), y_t(0))\}_{t=1}^T$. We will not assume that the sequence of potential outcomes satisfies any type of stationary condition, i.e. the outcomes are allowed to arrive in an arbitrary order.

2.3 Technical Assumptions

Our main technical assumptions on the potential outcomes and covariates are given below. We emphasize that all constants in the assumptions are presumed to exist but are not known to the experimenter. Indeed, the fact that these constants are not known before the experiment is part of what makes the problem more challenging.

Assumption 1 (Bounded Moments). There exists constants $0 < c_0 \leq c_1$ such that for both treatments $k \in \{0, 1\}$,

$$c_0 \leq \min_{\beta \in \mathbb{R}^d} \left(\frac{1}{T} \sum_{t=1}^T \left\{ y_t(k) - \langle \mathbf{x}_t, \beta \rangle \right\}^2 \right)^{1/2} \leq \left(\frac{1}{T} \sum_{t=1}^T y_t(k)^4 \right)^{1/4} \leq c_1 .$$

The moment conditions in Assumption 1 ensures two things. First, the second moments of the OLS residuals are assumed to be bounded from below. This assumption is generally plausible, unless the outcomes are suspected to be exactly a linear function of the covariates, which is rarely (if ever) the case in practice. The assumption also places a bound on the fourth moments of the potential outcomes. Because the problem of Adaptive Neyman Allocation essentially requires the estimation of squared residuals, it is unlikely that either of these moment assumptions can be weakened.

Assumption 2 (Covariate Regularity). There exists constants $c_2 > 0$ and $\gamma_0 > 0$ such that for all $t \geq T^{1/2} \cdot \gamma_0$, the covariate matrix is well-invertible:

$$\frac{1}{c_2} \leq \sigma_{\min} \left(\frac{1}{t} \sum_{s \leq t} \mathbf{x}_s \mathbf{x}_s^\top \right) .$$

Assumption 2 ensures that after the “early iterations” (i.e. iterations $t = \mathcal{O}(T^{1/2})$), the empirical covariance matrix $\frac{1}{t} \sum_{s \leq t} \mathbf{x}_s \mathbf{x}_s^\top$ is well-invertible. This invertability condition guarantees that adaptively estimated regression coefficients are well-behaved. Assumption 2 requires that the dimension of the covariates is bounded as $d = \mathcal{O}(T^{1/2})$. We will not require that the covariate matrices in the sequence are well-conditioned, in the sense that the largest singular value may at times be much larger than its smallest one. Moreover, our asymptotic analyses will not assume that this covariate matrix converges to any limiting quantity.

The next assumption bounds the maximum radius, defined as $R = \max_{t \in [T]} \|x_t\|$. We do not presume that the maximum radius R is known a priori to the experimenter.

Assumption 3 (Maximum Radius). The maximum radius is bounded as $R = o(T^{1/4})$.

When each of the entry of the covariate vectors is viewed as being of constant order, then the maximum radius is on the order $R = \mathcal{O}(d^{1/2})$. In this case, Assumption 3 places a slightly stronger assumption on the dimension of the covariates, i.e. $d = o(T^{1/2})$.

The most salient aspect of Assumptions 1-3 is that they allow for non-stationarity in the sequence of potential outcomes and covariates. Nearly all quantities are allowed to drift arbitrarily throughout the experiment, including individual treatment effects and the residuals of best linear predictors. The only substantive restriction on the order of the subjects is the well-invertible condition of Assumption 2, which is fairly mild.

To reason about the practicality of the assumptions above, it may help to view them through the lens of a super-population. If the outcomes and covariates were sampled i.i.d., then Assumptions 1 and 2 would hold with probability tending to 1 under the conditions that (i) the fourth moments of the outcomes existed, (ii) the covariates were sampled according to a subgaussian distribution with $d = o(T^{1/2})$, and (iii) the conditional variance of the outcomes is positive almost surely.

Triangular Array Asymptotics Although the majority of our results are finite-sample in nature, we will also consider asymptotic analyses. We follow the convention in the design-based literature of using triangular array asymptotics. In the triangular array asymptotics, we consider a sequence of experiments indexed by $T \in \mathbb{N}$. For each T , there is a sequence of potential outcomes and covariates $\{y_t^{(T)}(1), y_t^{(T)}(0), \mathbf{x}_t^{(T)}\}_{t=1}^T$ and an adaptive experimental design $\boldsymbol{\Pi}^{(T)}$. This yields a sequence of (deterministic) average treatment effects $\tau^{(T)}$ and estimators $\hat{\tau}^{(T)}$. All limiting statements, e.g. $\tau^{(T)} - \hat{\tau}^{(T)} \xrightarrow{P} 0$, are made with respect to this sequence. For notational clarity, we drop the superscript T when in the asymptotic statements.

2.4 Adaptive AIPW Estimators

We focus on the class of adaptive Augmented Inverse Propensity Weighted (AIPW) estimators for the average treatment effect. The AIPW estimator is widespread in causal inference, where a regression model is used to improve the efficiency of the standard IPW estimator. We focus on adaptive AIPW estimators with linear regression models, though we expect that the extension to more flexible kernel methods should be immediate.

At each round, the experimenter updates a linear regression model of the outcomes under treatment and control using the observed history. Let $\beta_t(1)$ and $\beta_t(0) \in \mathbb{R}^d$ denote the coefficients of the linear regression models for the outcomes under treatment and control at time t , respectively, i.e. $\beta_t(1)$ and $\beta_t(0)$ are determined completely by the history \mathcal{F}_{t-1} . At each round $t \in [T]$, the adaptive AIPW estimator proceeds by estimating the individual treatment effect using the regression models then correcting via an IPW estimate of the residuals. Formally, the adaptive AIPW estimator is given as

$$\hat{\tau} = \frac{1}{T} \sum_{t=1}^T \left\{ \langle \mathbf{x}_t, \beta_t(1) \rangle - \langle \mathbf{x}_t, \beta_t(0) \rangle + \frac{\mathbf{1}[Z_t = 1]}{p_t} (Y_t - \langle \mathbf{x}_t, \beta_t(1) \rangle) - \frac{\mathbf{1}[Z_t = 0]}{1-p_t} (Y_t - \langle \mathbf{x}_t, \beta_t(0) \rangle) \right\} .$$

Proposition 2.1 (AIPW Bias). *If $p_t \in (0, 1)$ for all $t \in [T]$ a.s. then the adaptive AIPW estimator is unbiased: $E[\hat{\tau}] = \tau$.*

Proposition 2.1 shows that the adaptive AIPW estimator is unbiased, regardless of how well the regression model fits the data. The next proposition shows that better fitted models typically correspond to smaller variance, so long as the conditional treatment probabilities are not too extreme.

Proposition 2.2 (AIPW Variance). *The normalized variance of the AIPW estimator is given as*

$$T \cdot \text{Var}(\hat{\tau}) = E \left[\frac{1}{T} \sum_{t=1}^T \left(\{y_t(1) - \langle \mathbf{x}_t, \beta_t(1) \rangle\} \cdot \sqrt{\frac{1-p_t}{p_t}} + \{y_t(0) - \langle \mathbf{x}_t, \beta_t(0) \rangle\} \cdot \sqrt{\frac{p_t}{1-p_t}} \right)^2 \right] .$$

3 Adaptive Neyman Allocation

3.1 Formulation of the Neyman Regret

Introduced by Robbins (1952), the problem of Adaptive Neyman Allocation is to design an adaptive protocol which has nearly the same performance as the optimal non-adaptive protocol which has access to all of the data. In the recent literature on Adaptive Neyman Allocation, the performance of an experimental design is measured by the Neyman Regret, which is the gap between the adaptive variance and the non-adaptive oracle variance.

We begin by deriving the oracle variance which serves as the relevant comparator when constructing an adaptive design. In this context, the *oracle variance* V^* is defined as the minimal variance of the AIPW estimator when selecting the best fixed linear predictors and assignment probability:

$$V^* = \arg \min_{\beta(1), \beta(0), p} \text{Var}(\hat{\tau}; p, \beta(1), \beta(0)) .$$

The *Neyman Allocation* refers to the optimal choice of linear predictors $\beta^*(1)$, $\beta^*(0)$ and assignment probability p^* which attain this oracle variance. Note that the oracle variance V^* and the Neyman Allocation $\beta^*(1)$, $\beta^*(0)$, and p depend on the potential outcomes and the covariates. For each treatment $k \in \{0, 1\}$, define the *optimal squared residuals* as

$$\mathcal{E}(k) = \min_{\beta \in \mathbb{R}^d} \left(\frac{1}{T} \sum_{t=1}^T (y_t(k) - \langle \mathbf{x}_t, \beta \rangle)^2 \right)^{1/2} ,$$

and denote the corresponding OLS predictor as $\beta_{\text{OLS}}(k)$. Define the *residual correlation* ρ as

$$\rho = \frac{\frac{1}{T} \sum_{t=1}^T (y_t(1) - \langle \mathbf{x}_t, \beta_{\text{OLS}}(1) \rangle)(y_t(0) - \langle \mathbf{x}_t, \beta_{\text{OLS}}(0) \rangle)}{\mathcal{E}(0)\mathcal{E}(1)} .$$

The following proposition derives the oracle variance and the Neyman Allocation in terms of these parameters. We include a proof sketch because it contains ideas which are essential to understanding non-convexity of the Neyman Regret.

Proposition 3.1. *The oracle variance is given by $T \cdot V^* = 2(1 + \rho)\mathcal{E}(1)\mathcal{E}(0)$ and the Neyman allocation is given by the least squares predictors $\beta^*(k) = \beta_{\text{OLS}}(k)$ and assignment probability $p = (1 + \mathcal{E}(0)/\mathcal{E}(1))^{-1}$.*

Proof Sketch. We begin by writing the variance as a function of the fixed parameters, $g(p, \beta(1), \beta(0)) = \text{Var}(\hat{\tau}; p, \beta(1), \beta(0))$. Using Proposition 2.2 and expanding terms, we write g as

$$\begin{aligned} g(p, \beta(1), \beta(0)) &= \frac{1}{T} \sum_{t=1}^T \{y_t(1) - \langle \mathbf{x}_t, \beta(1) \rangle\}^2 \cdot \left(\frac{1}{p} - 1 \right) \\ &\quad + \frac{1}{T} \sum_{t=1}^T \{y_t(0) - \langle \mathbf{x}_t, \beta(0) \rangle\}^2 \cdot \left(\frac{1}{1-p} - 1 \right) \\ &\quad + \frac{2}{T} \sum_{t=1}^T \{y_t(1) - \langle \mathbf{x}_t, \beta(1) \rangle\} \{y_t(0) - \langle \mathbf{x}_t, \beta(0) \rangle\} , \end{aligned}$$

where the expectations do not appear because the parameters are fixed. Observe that the function $g(p, \beta(1), \beta(0))$ is separately convex in $(\beta(1), \beta(0))$ and p , but not jointly convex in both. Nevertheless, we can analytically obtain the minimizers.

First, we fix the linear predictors $\beta(1)$ and $\beta(0)$ and minimize with respect to p . In a slight abuse of notation, let us write $\mathcal{E}(k) = ((1/T) \sum_{t=1}^T \{y_t(k) - \langle \mathbf{x}_t, \beta(k) \rangle\}^2)^{1/2}$ for the fixed $\beta(k)$. Because the mapping $p \mapsto g(p, \beta(1), \beta(0))$ is convex, any local minimizer is global and thus we may solve the first order condition to obtain $p^* = (1 + \mathcal{E}(0)/\mathcal{E}(1))^{-1}$. Plugging this in, we have that the variance for fixed $\beta(1), \beta(0)$ with optimal p^* is

$$\begin{aligned} g(p^*, \beta(1), \beta(0)) &= 2 \left(\frac{1}{T} \sum_{t=1}^T \{y_t(1) - \langle \mathbf{x}_t, \beta(1) \rangle\}^2 \right)^{1/2} \left(\frac{1}{T} \sum_{t=1}^T \{y_t(0) - \langle \mathbf{x}_t, \beta(0) \rangle\}^2 \right)^{1/2} \\ &\quad + \frac{2}{T} \sum_{t=1}^T \{y_t(1) - \langle \mathbf{x}_t, \beta(1) \rangle\} \{y_t(0) - \langle \mathbf{x}_t, \beta(0) \rangle\} . \end{aligned}$$

At first glance, it may appear that the OLS predictors cannot be the minimizers of the variance due to the presence of the residual correlation term. However, this intuition is false—it turns out that the minimizers are indeed given by the OLS predictors. Some care is needed to establish this fact because the function $(\beta(1), \beta(0)) \mapsto g(p^*, \beta(1), \beta(0))$ is non-convex. In Section B.3 of the appendix, we give the full proof which uses orthogonality properties of the OLS predictors as well as several judicious applications of AM-GM and Cauchy Schwarz.

Once the Neyman Allocation p^* , $\beta^*(1)$, and $\beta^*(0)$ have been derived, the oracle variance is obtained by substituting these parameters into the variance and using the definition of the residual correlation ρ . \square

Definition 1. Given an adaptive design, the *Neyman Regret* $\mathcal{R}_T^{\text{Neyman}}$ is defined as the difference between the (normalized) adaptive and oracle variances:

$$\mathcal{R}_T^{\text{Neyman}} = T \cdot \text{Var}(\hat{\tau}) - T \cdot V^* .$$

We use a subscript T in the Neyman Regret $\mathcal{R}_T^{\text{Neyman}}$ to reflect the dependence on the sample size T . If the Neyman Regret is decreasing to 0 with the sample size, then the true adaptive variance is essentially upper bounded by the oracle variance. Roughly speaking, this means that the experimenter will be able to use estimates of the oracle variance $T \cdot V^*$ in their inferential procedures. Our goal in this paper is to construct an adaptive experimental design so that the Neyman Regret $\mathcal{R}_T^{\text{Neyman}}$ converges to 0 as fast as possible. The following theorem shows that under our assumptions, the Neyman regret cannot be converging faster than $T^{-1/2}$.

Theorem 3.2 (Lower Bound). *For all integers T and any adaptive experimental design Π , there exists a sequence $\{y_t(1), y_t(0), \mathbf{x}_t\}_{t=1}^T$ satisfying Assumptions 1-3 and a constant $c > 0$, such that the corresponding Neyman regret is at least $\mathcal{R}_T^{\text{Neyman}} \geq c \cdot T^{-1/2}$.*

Theorem 3.2 demonstrates a $T^{-1/2}$ lower bound under the conditions of bounded moments (Assumption 1) and covariate regularity (Assumption 2). In Section B.1 of the appendix, we show that both of these assumptions are required in order to achieve a Neyman regret which converges to zero; in other words, if either of them does not hold, then the Neyman regret of any experimental design will be bounded away from zero.

3.2 Online Optimization and Technical Challenges

Following Dai, Gradu, and Harshaw (2023), we view the problem of minimizing the Neyman Regret through the lens of online optimization. Define the objective functions $g_t : (0, 1) \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}_+$ as

$$g_t(p, \beta(1), \beta(0)) = \left(\{y_t(1) - \langle \mathbf{x}_t, \beta(1) \rangle\} \cdot \sqrt{\frac{1-p}{p}} + \{y_t(0) - \langle \mathbf{x}_t, \beta(0) \rangle\} \cdot \sqrt{\frac{p}{1-p}} \right)^2 .$$

Then, using Propositions 2.2 and 3.1, we can write the Neyman regret in terms of online optimization, i.e. as the difference between the expected objective under the adaptive design and the objective for the optimal set of parameters:

$$\mathcal{R}_T^{\text{Neyman}} = \mathbb{E} \left[\frac{1}{T} \sum_{t=1}^T g_t(p_t, \boldsymbol{\beta}_t(1), \boldsymbol{\beta}_t(0)) \right] - \arg \min_{p, \boldsymbol{\beta}(1), \boldsymbol{\beta}(0)} \frac{1}{T} \sum_{t=1}^T g_t(p, \boldsymbol{\beta}(1), \boldsymbol{\beta}(0)) .$$

The underlying online optimization problem will guide our construction of an adaptive experimental design. However, the online optimization itself presents two challenges which require new technical developments:

1. **Nonconvexity:** The objective functions g_t are non-convex. Broadly speaking, convexity is considered to be the watershed for whether a problem can be efficiently solved. In order to guarantee convergence of the Neyman Regret, we need to overcome the issues presented by non-convexity in the objective.
2. **Ill-Conditioned:** The gradients of g_t become arbitrarily large as p approaches the boundary of $(0, 1)$. Analysis of conventional optimization methods (e.g. gradient descent) often require that the gradients are bounded over the entire domain. The tension is that the optimal p^* may be close to the boundary $(0, 1)$ and so we should allow the adaptive p_t to get close to the boundary, albeit in a controlled manner. In order to get clean $T^{-1/2}$ rates of Neyman Regret, we must also address the fundamental ill-conditioning in the objective functions.

3.3 Decomposition of the Neyman Regret

In this section, we show how to overcome the non-convexity underlying the Neyman Regret. Perhaps surprisingly, we show that the non-convex Neyman Regret can be decomposed into the sum of two convex regrets: probability regret and prediction regret. Roughly speaking, the probability regret measures the extent to which the adaptively chosen probability balances the residuals while the prediction regret measures the performance of the adaptively chosen linear predictors .

We begin by showing how to evaluate the performance of the adaptively chosen treatment probability. For each iteration t , we define the *probability loss* $f_t : (0, 1) \rightarrow \mathbb{R}_+$ as the convex function

$$f_t(p) = \left(y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle \right)^2 \cdot \frac{1}{p} + \left(y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle \right)^2 \cdot \frac{1}{1-p} .$$

The probability loss measures how well a treatment assignment probability p balances the squared error of the predictions made at iteration t . We define the *probability regret* as

$$\mathcal{R}_T^{\text{prob}} = \sum_{t=1}^T f_t(p_t) - \sum_{t=1}^T f_t(p^*) .$$

The probability regret measures how well the adaptively chosen assignment probabilities $p_1 \dots p_T$ balances the online residuals via comparison to the Neyman Allocation assignment probability p^* . Because the adaptively chosen probabilities p_t and linear predictors $\boldsymbol{\beta}_t(1)$ and $\boldsymbol{\beta}_t(0)$ are random, the prediction regret $\mathcal{R}_T^{(k), \text{pred}}$ is also a random variable. Note that the scaling of the probability regret is such that $o(T)$ is considered to be asymptotically vanishing.

We now focus on how to evaluate the performance of the adaptively chosen linear predictors. For each iteration t , we define the *prediction loss* function $\ell_t : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}_+$ as the convex

function

$$\ell_t(\beta(1), \beta(0)) = \left(\{y_t(1) - \langle \mathbf{x}_t, \beta(1) \rangle\} \cdot \sqrt{\frac{\mathcal{E}(0)}{\mathcal{E}(1)}} + \{y_t(0) - \langle \mathbf{x}_t, \beta(0) \rangle\} \cdot \sqrt{\frac{\mathcal{E}(1)}{\mathcal{E}(0)}} \right)^2.$$

The prediction loss measures the error of linear predictions of the potential outcomes at a given iteration, weighted by the ratio of the optimal residuals. We define the *prediction regret* as

$$\mathcal{R}_T^{\text{pred}} = \sum_{t=1}^T \ell_t(\beta_t(1), \beta_t(0)) - \sum_{t=1}^T \ell_t(\beta^*(1), \beta^*(0)) .$$

The prediction regret measures the overall prediction error of the adaptively chosen predictors by comparing to the prediction error under the optimal least squares predictors $\beta^*(1), \beta^*(0)$. Because the linear predictors are random, the prediction regret $\mathcal{R}_T^{\text{pred}}$ is also a random variable. The prediction regret is on the same scale as the probability regret, i.e. a sublinear $o(T)$ regret is considered to be asymptotically vanishing.

Lemma 3.3. *Under Assumption 1, the Neyman Regret can be decomposed as the T -normalized sum of the probability regret and the prediction regret:*

$$\mathcal{R}_T^{\text{Neyman}} = \frac{1}{T} \mathbb{E}[\mathcal{R}_T^{\text{prob}}] + \frac{1}{T} \mathbb{E}[\mathcal{R}_T^{\text{pred}}] .$$

The decomposition of the Neyman regret in Lemma 3.3 effectively replaces the issue of non-convexity with the issue of multiple objectives; that is, we still need the expected probability regret and the expected prediction regret to be small *simultaneously*. The main benefit of Lemma 3.3 is that it identifies individual convex objectives on which techniques from convex analysis may be used.

4 The Sigmoid-FTRL Design

4.1 Formal Description of The Sigmoid-FTRL Design

The SIGMOID-FTRL design is formally described in Algorithm 1. The design can be understood as applying the *follow-the-regularized leader* (FTRL) principle separately to the probability regret and the prediction regret. FTRL is a commonly used technique in online convex optimization which is an alternative to gradient-descent based methods (Hazan, 2016). As each subject arrives in the experiment, the SIGMOID-FTRL design proceeds in two main steps: first, the linear predictors $\beta_t(1), \beta_t(0)$ are computed and then the treatment assignment probability p_t is computed.

The linear predictors $\beta_t(1)$ and $\beta_t(0)$ are chosen to minimize the estimated squared prediction errors on the previously observed units. The squared prediction errors on previously observed units are estimated using adaptive IPW weighting, which is necessary because we observe only one outcome for each unit. To ensure that that predictors are sufficiently regularized, we add a ridge regularizer with penalty term η_t^{-1} , which depends on the number of units T and the largest norm of any covariate vector seen so far. As we demonstrate in Section 4.4, this step aims to minimize the prediction regret.

The selection of treatment probabilities ser The treatment probability p_t is chosen so that the subject is more likely to be assigned to the treatment where the online predictions have so far resulted in larger errors. For each treatment $k \in \{0, 1\}$, define the *online residuals up to time t* as $A_{t-1}(k) = \sum_{s \leq t-1} (y_s(k) - \langle \mathbf{x}_s, \beta_s(k) \rangle)^2$. The online residuals are not directly observable

Algorithm 1: SIGMOID-FTRL

Input: Sigmoid $\phi : \mathbb{R} \rightarrow (0, 1)$ satisfying Condition 1.

Define the sigmoid regularizer $\Psi = \psi \circ \phi^{-1}$ with $\psi(u) = \frac{1}{2}u^2 + |u|^3$.

Initialize max radius $R_0 = 1$.

for $t = 1, 2, \dots, T$ **do**

Observe covariate $\mathbf{x}_t \in \mathbb{R}^d$ and update $R_t = \max(R_{t-1}, \|\mathbf{x}_t\|)$ and $\eta_t = T^{-1/2}R_t^{-2}$.

Construct the regression coefficients $\beta_1(t)$ and $\beta_0(t)$ as

$$\begin{aligned}\beta_1(t) &= \arg \min_{\beta \in \mathbb{R}^d} \sum_{s=1}^{t-1} \left(Y_s \cdot \frac{\mathbf{1}[Z_s = 1]}{p_s} - \langle \mathbf{x}_s, \beta \rangle \right)^2 + \frac{1}{\eta_t} \cdot \|\beta\|^2 \\ \beta_0(t) &= \arg \min_{\beta \in \mathbb{R}^d} \sum_{s=1}^{t-1} \left(Y_s \cdot \frac{\mathbf{1}[Z_s = 0]}{1-p_s} - \langle \mathbf{x}_s, \beta \rangle \right)^2 + \frac{1}{\eta_t} \cdot \|\beta\|^2\end{aligned}$$

Construct estimates of the online squared residuals $\hat{A}_{t-1}(1)$ and $\hat{A}_{t-1}(0)$ as

$$\begin{aligned}\hat{A}_{t-1}(1) &= \sum_{s=1}^{t-1} \frac{\mathbf{1}[Z_s = 1]}{p_s} \left(Y_s - \langle \mathbf{x}_s, \beta_1(s) \rangle \right)^2 \\ \hat{A}_{t-1}(0) &= \sum_{s=1}^{t-1} \frac{\mathbf{1}[Z_s = 0]}{1-p_s} \left(Y_s - \langle \mathbf{x}_s, \beta_0(s) \rangle \right)^2\end{aligned}$$

Select assignment probability p_t as

$$p_t = \arg \min_{p \in (0,1)} \frac{\hat{A}_{t-1}(1)}{p} + \frac{\hat{A}_{t-1}(0)}{1-p} + \frac{1}{\eta_t} \cdot \Psi(p).$$

Sample treatment assignment $Z_t = 1$ with probability p_t , and $Z_t = 0$ otherwise.

Observe outcome $Y_t = \mathbf{1}[Z_t = 1] \cdot y_t(1) + \mathbf{1}[Z_t = 0] \cdot y_t(0)$.

end

because in all previous iterations $s < t$, we have seen either $y_s(1)$ or $y_s(0)$, but not both. For this reason, we use adaptive IPW weighting to obtain estimates $\hat{A}_{t-1}(k)$. The treatment probability p_t is then chosen to minimize the weighted sum $\hat{A}_{t-1}(1)/p + \hat{A}_{t-1}(0)/(1-p)$ with an additional regularization term $\eta_t^{-1} \cdot \Psi(p)$. This step seeks As we demonstrate in Section 4.3, this step aims to minimize the probability regret.

We use a *sigmoid regularizer* $\Psi : (0, 1) \rightarrow \mathbb{R}_+$ to regularize the selected treatment probability p_t . The purpose of the sigmoidal regularizer is to ensure that the treatment probability does not get too close to the boundary of the interval $[0, 1]$, which would increase the variance of the AIPW estimator. We refer to this penalty function as sigmoidal because it takes the form $\Psi = \psi \circ \phi^{-1}$, where $\phi : \mathbb{R} \rightarrow (0, 1)$ is a sigmoid function and $\psi(u) = \frac{1}{2}u^2 + |u|^3$. In other words, the penalty Ψ can be understood as using a quadratic + cubic penalty on the transformed variable u_t which yields the treatment probability, i.e. $p_t = \phi(u_t)$.

The sigmoidal penalty is one of the novel techniques introduced in our work and is the namesake of the SIGMOID-FTRL design. The sigmoid penalty is a key ingredient to facilitating

the clean $T^{1/2}$ term in our regret analysis. This improves upon the probability clipping design of Dai, Gradu, and Harshaw (2023), which features an additional sub-polynomial $\exp(\sqrt{\log(T)})$ factor as a result of the probability clipping. One can interpret the sigmoidal regularizer as a gentler version of the harsh regularization implicitly introduced by probability clipping.

The choice of an appropriate sigmoid function ϕ is crucial for obtaining the performance guarantees of the SIGMOID-FTRL design. Many choices of sigmoid will not work, including the logistic function, the hyperbolic tangent, and the error function. Fortunately, there are several good choices, such as the arctangent function $\phi(u) = \frac{1}{\pi}(\arctan(u) + \pi/2)$ and the algebraic sigmoid $\phi(u) = \frac{1}{2}(\frac{u}{1+|u|} + 1)$. Our theoretical results will hold for any sigmoid satisfying the following condition:

Condition 1 (Sigmoid Condition). The sigmoid $\phi : \mathbb{R} \rightarrow (0, 1)$ satisfies the following:

1. $\phi(u)$ is a strictly monotone increasing with $\phi(u) + \phi(-u) = 1$ and $\phi(+\infty) = 1$.
2. $u \mapsto 1/\phi(u)$ and $u \mapsto 1/(1 - \phi(u))$ are convex functions.
3. There exists constants $b_1, b_2, b_3 > 0$ such that
 - (a) $-\left(\frac{1}{\phi(u)}\right)' \leq b_1$ for all $u \in \mathbb{R}$
 - (b) $\left(\frac{1}{\phi(u)}\right)'' \leq b_2 \cdot \frac{1}{(1+|u|)^3}$ for all $u \in \mathbb{R}$
 - (c) $\left(\frac{1}{\phi(u)}\right)'' \geq b_3 \cdot \frac{1}{(1+u)^3}$ for all $u \geq 0$.

Another key aspect of the SIGMOID-FTRL design is the introduction of an adaptive penalty term, η_t . The penalty term is set adaptively as $\eta_t = (T^{1/2}R_t^2)^{-1}$ where R_t is the largest covariate norm seen so far. The variable is initialized as $R_0 = 1$ and updated iteratively as $R_t = \max(R_{t-1}, \|\mathbf{x}_t\|)$. From a theoretical point of view, the adaptively chosen penalty term η_t ensures that the regularization appropriately scales with the magnitude of the covariates. From a practical point of view, the primary benefit of an adaptive penalty term η_t is that the experimenter is not required to correctly specify the magnitude of the covariates a priori. In keeping with the conventions of online optimization, we define $\eta_{T+1} = \eta_T$.

SIGMOID-FTRL is computationally practical to implement. In particular, the number of arithmetic operations per iteration scales like $\mathcal{O}(d^3)$ and the total storage required for the algorithm is $\mathcal{O}(d^2)$. The dominant computational cost is solving the ridge regression. By maintaining and updating a few intermediate variables, the solution to the ridge regression may be obtained by solving a d -by- d linear system, requiring $\mathcal{O}(d^3)$ arithmetic operations and $\mathcal{O}(d^2)$ storage. The estimated online squared residuals $\hat{A}_t(1)$ and $\hat{A}_t(0)$ can be updated using $\mathcal{O}(d)$ operations. The minimization required to obtain p_t is convex and one-dimensional, so simple root finding algorithms may be used for this purpose.

4.2 Neyman Regret Guarantee

The first main result of this paper is the following theorem, which shows that the Neyman regret converges to zero at a $T^{-1/2}R^2$ rate under SIGMOID-FTRL.

Theorem 4.1. *Under Assumptions 1-3 and Condition 1, there exists a constant $C > 0$ such that the Neyman Regret under SIGMOID-FTRL is bounded as*

$$\mathcal{R}_T^{\text{Neyman}} \leq (C + o(1)) \cdot T^{-1/2}R^2 .$$

In the theorem above, C is a constant depending only on the constants c_0, c_1, c_2 and b_1, b_2, b_3 appearing in Assumptions 1-3 and Condition 1, respectively. We remark that C is a small

polynomial function of these constants, unlike the exponential dependence in the analysis of Dai, Gradu, and Harshaw (2023). In a slight abuse of notation, we use C throughout the paper to express any constant which depends only on these constants in the assumptions; however, all constants are made explicit in the appendix.

In the context of our lower bound (Theorem 3.2), Theorem 4.1 establishes the optimal T dependence on the Neyman regret. In particular, this removes the sub-polynomial factors in the probability clipping approach of Dai, Gradu, and Harshaw (2023).

Although the proof of Theorem 4.1 is lengthy and technical, the central ideas of the proof can be conveyed by several key lemmas. In the remainder of Section 4, we will present certain lemmas with the aim of providing the intuition underlying our approach. The full proof of Theorem 4.1 appears in Section B of the appendix.

4.3 Probability Regret

We begin by showing how to bound the expected probability regret $\mathbb{E}[\mathcal{R}_T^{\text{prob}}]$ under the SIGMOID-FTRL design. The key idea is the use of a *sigmoidal transformation* of the regret problem. Rather than selecting a probability $p_t \in (0, 1)$, we consider the equivalent selection of a decision variable $u_t \in \mathbb{R}$ through the sigmoid function $\phi : \mathbb{R} \rightarrow (0, 1)$, i.e. $p_t = \phi(u_t)$. For each iteration $t \in [T]$, we define the *sigmoid probability loss* function $h_t : \mathbb{R} \rightarrow \mathbb{R}_+$ as the composition $h_t = f_t \circ \phi$ which may be expressed as:

$$h_t(u) = \left(y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle \right)^2 \cdot \frac{1}{\phi(u)} + \left(y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle \right)^2 \cdot \frac{1}{1 - \phi(u)} .$$

Using the equivalence $p_t = \phi(u_t)$, we can re-express the probability regret as

$$\mathcal{R}_T^{\text{prob}} = \sum_{t=1}^T h_t(u_t) - \sum_{t=1}^T h_t(u^*) ,$$

where $u^* = \phi^{-1}(p^*)$ is the transformed Neyman Allocation. Parts 1-3 of Condition 1 ensure that the sigmoid probability loss functions h_t are convex and have uniformly bounded gradients. In contrast, the original probability loss functions had gradients which blew up at the boundary of the $[0, 1]$ interval. In this way, the benefit of the sigmoidal transformation is to transform an ill-conditioned constrained problem into a well-conditioned unconstrained problem.

The sigmoid probability loss functions h_t depend on both of the potential outcomes, so they cannot be directly observed. In light of this fact, the SIGMOID-FTRL design uses the adaptive IPW weighting technique to construct *estimated sigmoid loss functions* \hat{h}_t as

$$\hat{h}_t(u) = \left(Y_t - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle \right)^2 \cdot \frac{\mathbf{1}[Z_t = 1]}{p_t} \cdot \frac{1}{\phi(u)} + \left(Y_t - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle \right)^2 \cdot \frac{\mathbf{1}[Z_t = 0]}{1 - p_t} \cdot \frac{1}{1 - \phi(u)}$$

and then selects the treatment assignment probability $p_t = \phi(u_t)$ where u_t is the minimizer of the estimated sigmoid loss functions

$$u_t = \arg \min_{u \in \mathbb{R}} \sum_{s < t} \hat{h}_s(u) + \frac{1}{\eta_t} \psi(u) ,$$

with regularization $\psi(u) = \frac{1}{2}u^2 + |u|^3$. In this way, the SIGMOID-FTRL design is working directly in the sigmoidal transformation. The following lemma shows that these estimated sigmoid loss functions \hat{h}_t are conditionally unbiased for the sigmoid loss functions h_t .

Lemma 4.2. *The estimated sigmoid loss functions are conditionally unbiased: $\mathbb{E}[\hat{h}_t(u) | \mathcal{F}_{t-1}] = h_t(u)$ for all \mathcal{F}_{t-1} -measurable random variables u .*

While these estimated sigmoidal loss functions are unbiased, their magnitude now depends on the inverse probabilities. Because the inverse probabilities may grow close to the boundary, the gradients $\nabla \hat{h}_t$ may become large. When the gradients $\nabla h_t(u_t)$ are not bounded, a traditional FTRL algorithm using the squared regularizer $\psi(u) = u^2$ would not attain a $T^{1/2}$ probability regret. The issue becomes more pronounced when the maximum inverse probability $\max\{1/p_t, 1/(1-p_t)\}$ is large, which happens exactly when $|u_t|$ is large.

To overcome this issue, SIGMOID-FTRL introduces a modified regularization function $\psi(u) = \frac{1}{2}u^2 + |u|^3$, which features an additional cubic term. The role of this cubic term is to regularize more aggressively in regions where the estimated sigmoid loss functions have larger gradients, i.e. when $|u|$ is large. In fact, the cubic term is chosen specifically so that the regularization “offsets” the magnitude of the squared gradient.

To formalize this intuition, we need to use the geometry of Bregman divergences. Bregman divergences play a crucial role in the analysis of optimization algorithms because they express the relevant curvature of the objective functions (Hazan, 2016; Orabona, 2023). Given a differentiable convex function $f : \mathcal{X} \rightarrow \mathbb{R}$ on a convex set \mathcal{X} , the *Bregman divergence* $\mathcal{B}_f : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$ is defined as $\mathcal{B}_f(u|v) = f(u) - f(v) - \langle \nabla f(v), u - v \rangle$. The Bregman divergence measures local geometry of the function f at different pairs of points through the gap between $f(u)$ and the convex lower bound $f(v) + \langle \nabla f(v), u - v \rangle$. The following lemma demonstrates that the Bregman divergence of the custom regularizer ψ depends not only on the distance between u and v , but also on their magnitudes:

Lemma 4.3. *The Bregman divergence is bounded: $\mathcal{B}_\psi(v|u) \geq \frac{1}{2}(v-u)^2(1 + \frac{1}{2}|v| + |u|)$.*

Lemma 4.3 is a key technical result which facilitates the $T^{-1/2}$ rate of SIGMOID-FTRL. At first sight, SIGMOID-FTRL may appear similar to imposing a heavy regularizer such as

$$\Psi(p) = p^{-2} + (1-p)^{-2} + p^{-3} + (1-p)^{-3}.$$

However, the key advantage of the sigmoid transformation is different: Lemma 4.3 shows that the Bregman divergence $\mathcal{B}_\psi(u_{t+1}|u_t)$ in the transformed space can be controlled *globally*, even when the corresponding probabilities $p_t = \phi(u_t)$ and $p_{t+1} = \phi(u_{t+1})$ are far apart. In contrast, working with such a regularizer Ψ directly in probability space would force us to rely on $\mathcal{B}_\Psi(p_{t+1}|p_t)$, which could only be controlled *locally*, due to the classical localization constraint of FTRL: the divergence can only be well-approximated by its second-order expansion when p_{t+1} remains close to p_t . The sigmoid geometry circumvents this limitation by expressing potentially large moves in probability space as well-behaved movements in the unconstrained u -space, enabling a clean regret analysis and ultimately the optimal $T^{-1/2}$ rate.

Using Lemma 4.3 together with well-known techniques from online convex optimization, we derive the following bound on the expected probability regret.

Proposition 4.4. *The expected probability regret can be bounded as*

$$E[\mathcal{R}_T^{\text{prob}}] \leq \frac{1}{\eta_{T+1}} \psi(u^*) + \sum_{t=1}^T \eta_t E \left[\frac{(\nabla \hat{h}_t(u_t))^2}{2(1 + |u_t|)} \right].$$

The upper bound in Proposition 4.4 has two parts. The first part is easily analyzed, as it involves the final step size η_T and the regularization $\psi(u^*)$, which is of constant order by Assumption 1. The second part is the sum of the expected squared gradients $(\nabla h_t(u_t))^2$ normalized by $(1 + |u_t|)$. This normalization term is a result of our choice of modified regularization ψ and Lemma 4.3. Roughly speaking, this normalization appears because it is equal to $\psi''(u) = 1 + 6|u|$, up to constants. The following lemma calculates this expectation, formally showing that the normalization “offsets” the inverse probability terms in the squared gradient $(\nabla h_t(u_t))^2$:

Lemma 4.5. *Under Condition 1, the conditional expectation of the squared gradient term is at most*

$$\mathbb{E} \left[\frac{(\nabla \hat{h}_t(u_t))^2}{1 + |u_t|} \mid \mathcal{F}_{t-1} \right] \leq b_1^2 \max\{b_1, 2\} \left(\{y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle\}^4 + \{y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle\}^4 \right).$$

Thus, applying the law of iterated expectation yields

$$\sum_{t=1}^T \eta_t \mathbb{E} \left[\frac{(\nabla \hat{h}_t(u_t))^2}{2(1 + |u_t|)} \right] \leq \frac{1}{2} b_1^2 \max\{b_1, 2\} \sum_{k \in \{0, 1\}} \mathbb{E} \left[\sum_{t=1}^T \eta_t \{y_t(k) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(k) \rangle\}^4 \right].$$

The first part of Lemma 4.5 is a straightforward calculation. Overall, Lemma 4.5 shows that the expected squared gradient term in the probability regret is upper bounded by the fourth moment of the online residuals, weighted by the adaptively chosen step size.

In order to bound the probability regret, the last step is to ensure that the fourth moment of the online residuals is bounded. This will also be the last step in bounding the prediction regret. The following lemma contains such a bound, but its proof sketch is delayed until Section 4.5.

Lemma 4.6. *Under Assumptions 1-3 and Condition 1, there exists a constant $C > 0$ such that for each treatment $k \in \{0, 1\}$, the fourth moment of the online residuals can be bounded as*

$$\mathbb{E} \left[\sum_{t=1}^T \eta_t \{y_t(k) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(k) \rangle\}^4 \right] \leq (C + o(1)) \cdot T^{1/2} R^2.$$

We are now ready to provide a bound on the probability regret. Using Proposition 4.4 to upper bound the expected probability regret in terms of the regularization and squared gradient terms, Lemma 4.5 to bound the squared gradient term via the fourth moment of online residuals, and Lemma 4.6 to bound the fourth moments, we arrive at the following bound on the expected probability regret:

Proposition 4.7. *Under Assumptions 1-3 and Condition 1, there exists a constant $C > 0$ such that the expected probability regret is bounded as $\mathbb{E}[\mathcal{R}_T^{\text{prob}}] \leq (C + o(1))T^{1/2}R^2$.*

4.4 Prediction Regret

In this section, we show how to bound the prediction regret. Throughout this section, we sometimes use the shorthand $\boldsymbol{\beta}_t = (\boldsymbol{\beta}_t(1), \boldsymbol{\beta}_t(0))$ to refer to the concatenation of the linear predictors.

The prediction loss ℓ_t depends on both of the potential outcomes, which are not simultaneously observed. Using the adaptive inverse probability weighting technique, we define the *estimated prediction loss* $\hat{\ell}_t : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}_+$ as

$$\hat{\ell}_t(\boldsymbol{\beta}(1), \boldsymbol{\beta}(0)) = \left(\left\{ Y_t \cdot \frac{\mathbf{1}[Z_t = 1]}{p_t} - \langle \mathbf{x}_t, \boldsymbol{\beta}(1) \rangle \right\} \cdot \sqrt{\frac{\mathcal{E}(0)}{\mathcal{E}(1)}} + \left\{ Y_t \cdot \frac{\mathbf{1}[Z_t = 0]}{1-p_t} - \langle \mathbf{x}_t, \boldsymbol{\beta}(0) \rangle \right\} \cdot \sqrt{\frac{\mathcal{E}(1)}{\mathcal{E}(0)}} \right)^2.$$

Define also the regularizer function $m : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^+$ as

$$m(\boldsymbol{\beta}(1), \boldsymbol{\beta}(0)) = \left\| \sqrt{\frac{\mathcal{E}(0)}{\mathcal{E}(1)}} \cdot \boldsymbol{\beta}(1) + \sqrt{\frac{\mathcal{E}(1)}{\mathcal{E}(0)}} \cdot \boldsymbol{\beta}(0) \right\|^2.$$

The SIGMOID-FTRL can be understood as selecting the linear predictors via the FTRL principle

$$\boldsymbol{\beta}_t(1), \boldsymbol{\beta}_t(0) = \arg \min_{\boldsymbol{\beta}(1), \boldsymbol{\beta}(0)} \sum_{s=1}^{t-1} \hat{\ell}_s(\boldsymbol{\beta}(1), \boldsymbol{\beta}(0)) + \frac{1}{\eta_t} \cdot m(\boldsymbol{\beta}(1), \boldsymbol{\beta}(0)) .$$

It is not obvious that the individual ridge predictors $\boldsymbol{\beta}_t(1), \boldsymbol{\beta}_t(0)$ used in SIGMOID-FTRL are also the joint minimizers of the program above. Indeed, the program above includes both a crossing term and the optimal residuals $\mathcal{E}(1)$ and $\mathcal{E}(0)$ in its definition, neither of which do not appear in the usual ridge loss. In the same way that the individual OLS predictors jointly minimize the AIPW variance (Proposition 3.1), it is also true that the individual ridge predictors jointly minimize this program above.

The following lemma demonstrates that from the perspective of regret minimization, working with the estimated prediction loss functions is, in expectation, equivalent to working with the actual prediction loss functions:

Lemma 4.8. *For each iteration t , the following conditional unbiasedness holds:*

$$E[\hat{\ell}_t(\boldsymbol{\beta}_t(1), \boldsymbol{\beta}_t(0)) - \hat{\ell}_t(\boldsymbol{\beta}^*(1), \boldsymbol{\beta}^*(0)) | \mathcal{F}_{t-1}] = \ell_t(\boldsymbol{\beta}_t(1), \boldsymbol{\beta}_t(0)) - \ell_t(\boldsymbol{\beta}^*(1), \boldsymbol{\beta}^*(0)) .$$

The SIGMOID-FTRL design uses an adaptive step size η_t which decreases over time. A standard way to analyze online algorithms with adaptive step sizes is to introduce auxiliary objective functions and decision variables, e.g. see Chapter 7 of (Orabona, 2023). Following this conventional analysis, we introduce

$$\tilde{L}_t(\boldsymbol{\beta}(1), \boldsymbol{\beta}(0)) = \sum_{s=1}^{t-1} \hat{\ell}_s(\boldsymbol{\beta}(1), \boldsymbol{\beta}(0)) + \eta_{t-1}^{-1} m(\boldsymbol{\beta}(1), \boldsymbol{\beta}(0)) \quad \text{and} \quad \tilde{\boldsymbol{\beta}}_t(1), \tilde{\boldsymbol{\beta}}_t(0) = \arg \min_{\boldsymbol{\beta}(1), \boldsymbol{\beta}(0)} \tilde{L}_t(\boldsymbol{\beta}(1), \boldsymbol{\beta}(0)) ,$$

where \tilde{L}_t is the regularized loss function which uses the previous iteration's step size η_{t-1}^{-1} instead than the current step size η_t^{-1} and $\tilde{\boldsymbol{\beta}}_t(1)$ and $\tilde{\boldsymbol{\beta}}_t(0)$ are the linear predictors which are obtained by minimizing \tilde{L}_t . Standard FTRL arguments yield the following regret bound:

Lemma 4.9. *The expected prediction regret is bounded as*

$$E[\mathcal{R}_T^{\text{pred}}] \leq \frac{m(\boldsymbol{\beta}^*(1), \boldsymbol{\beta}^*(0))}{\eta_{T+1}} + E\left[\sum_{t=1}^T \tilde{L}_{t+1}(\boldsymbol{\beta}_t(1), \boldsymbol{\beta}_t(0)) - \tilde{L}_{t+1}(\tilde{\boldsymbol{\beta}}_{t+1}(1), \tilde{\boldsymbol{\beta}}_{t+1}(0)) \right] .$$

The upper bound in Lemma 4.9 has two parts. The first part involves the final step size and the norm of the OLS predictors. This term will be simple to bound because the step size is controlled throughout the algorithm and the OLS solution is easily bounded via Assumptions 1 and 2. The second part involves the sum of the successive differences of regularized objective functions, which measures the optimality gap between the selected $\boldsymbol{\beta}_t(k)$ and the minimal $\tilde{\boldsymbol{\beta}}_{t+1}(k)$ on the objective function \tilde{L}_{t+1} . By the definition of $\tilde{\boldsymbol{\beta}}_t$, this term is non-negative and so it must be upper bounded.

Such analyses usually proceed by bounding these successive differences by first order convexity, i.e. using the shorthand $\boldsymbol{\beta}_t = (\boldsymbol{\beta}_t(1), \boldsymbol{\beta}_t(0))$ for concatenation:

$$\tilde{L}_{t+1}(\boldsymbol{\beta}_t) - \tilde{L}_{t+1}(\tilde{\boldsymbol{\beta}}_{t+1}) \leq \langle \nabla \tilde{L}_{t+1}(\tilde{\boldsymbol{\beta}}_{t+1}), \boldsymbol{\beta}_t - \tilde{\boldsymbol{\beta}}_{t+1} \rangle .$$

However, this will not suffice for our analysis because it will not allow us to leverage the regularity in the covariates. Indeed, using the first order convexity bound will result in each covariate vector \mathbf{x}_t being considered separately, whereas the covariates are only assumed to be

regular in aggregate as measured by the invertibility of the covariate matrix (Assumption 2). For this reason, we must analyze this difference of terms appearing in Lemma 4.9 directly. To this end, define the *per iteration leverage scores* as

$$\Pi_{t,s} = \mathbf{x}_t^\top \left(\mathbf{X}_{t-1}^\top \mathbf{X}_{t-1} + \eta_t^{-1} \mathbf{I} \right)^{-1} \mathbf{x}_s ,$$

where $\mathbf{X}_{t-1} = \sum_{s < t-1} \mathbf{x}_s \mathbf{x}_s^\top$. Unlike the usual leverage score, the per iteration leverage score $\Pi_{t,s}$ is not symmetric because the matrix \mathbf{X}_{t-1} appearing inside depends on the iteration t . In usual regression analysis, the leverage score appears when adding a new (y_t, \mathbf{x}_t) pair and studying how the estimated $\hat{\beta}$ changes. This per-iteration leverage score is playing a subtly different role in our analysis. Roughly speaking, it appears when we ask how the addition of a new (y_t, \mathbf{x}_t) pair affects the ridge loss on the entire data, including the newly added pair. The following lemma shows precisely how the per-iteration leverage score arises in analyzing the successive difference terms appearing in Lemma 4.9.

Lemma 4.10. *The successive difference can be written as:*

$$\tilde{L}_{t+1}(\beta_t(1), \beta_t(0)) - \tilde{L}_{t+1}(\tilde{\beta}_{t+1}(1), \tilde{\beta}_{t+1}(0)) = \frac{\Pi_{t,t}}{1 + \Pi_{t,t}} \cdot \hat{\ell}_t(\beta_t(1), \beta_t(0)) .$$

The benefit of Lemma 4.10 is that it allows us to make use regularity of the entire sequence of covariates (Assumption 2) through the per-iteration leverage scores. The remainder of the proof for bounding the prediction regret can be carried out via direct calculation, using properties of the per-iteration leverage scores and the bound on the fourth moment of the online residuals (Lemma 4.6). We defer to Section B.4 of the appendix for details and summarize the main result here:

Proposition 4.11. *Under Assumptions 1-3 and Condition 1, there exists a constant $C > 0$ such that the expected prediction regret is bounded as $E[\mathcal{R}_T^{\text{pred}}] \leq (C + o(1))T^{1/2}R^2$.*

The proof of Theorem 4.1 is completed by applying the Neyman Regret decomposition (Lemma 3.3) with the bounds on the probability regret (Proposition 4.7) and prediction regret (Proposition 4.11).

4.5 Fourth Moments: Prediction Tracking

In this section, we provide a proof sketch for Lemma 4.6, which bounds the fourth moments of the online residuals. Our high level approach will be as follows: we introduce a sequence of deterministic linear predictors and argue that (1) the online residuals of this deterministic predictor sequence is bounded (2) the difference between the deterministic and the adaptive residuals is a low order term. We refer to this proof technique as *prediction tracking*.

For each treatment $k \in \{0, 1\}$, we define the sequence of *full information predictors* $\{\beta_t^*(k)\}_{t=1}^T$ as the solutions to the ridge regression:

$$\beta_t^*(k) = \arg \min_{\beta \in \mathbb{R}^d} \sum_{s < t} \left(y_s(k) - \langle \mathbf{x}_s, \beta \rangle \right)^2 + \eta_t^{-1} \|\beta\|^2 ,$$

where the penalty term η_t is as chosen in SIGMOID-FTRL. We remark that the sequence of full-information predictors $\{\beta_t^*(k)\}_{t=1}^T$ is deterministic because it depends on the potential outcomes, not the observed outcomes, hence the name “full-information”. As such, we cannot hope to compute the full-information predictors directly—they are defined merely for the purposes of analysis. The purpose of introducing these full-information predictors is that they

are a deterministic sequence around which the adaptively chosen predictors will concentrate. The following conditional unbiasedness property is a direct consequence of the adaptive IPW weighting used in the ridge regression step of SIGMOID-FTRL:

Lemma 4.12. *For each iteration $t \in [T]$ and treatment $k \in \{0, 1\}$, the adaptive linear predictors are conditionally unbiased for the full-information predictors: $E[\beta_t(k) | \mathcal{F}_{t-1}] = \beta_t^*(k)$ a.s.*

In Section B.2 of the appendix, we show that the fourth moments of the full-information residuals are bounded as $\sum_{t=1}^T \eta_t \{y_t(k) - \langle \mathbf{x}_t, \beta_t^*(k) \rangle\}^4 = \mathcal{O}(T^{1/2} R^2)$ using the moment conditions and covariate regularity, i.e. Assumptions 1 and 2. What remains to be shown is that the adaptively selected predictors' residuals "tracks" that of the deterministic full-information predictors, in the sense that they have roughly the same fourth moment. To this end, the following proposition calculates what we refer to as the "prediction tracking error". We focus our exposition on the treatment $k = 1$:

Lemma 4.13. *The expected prediction tracking error is computed as*

$$E\left[\sum_{t=1}^T \eta_t \langle \mathbf{x}_t, \beta_t(1) - \beta_t^*(1) \rangle^4\right] = E\left[\sum_{t=1}^T \eta_t \left(\sum_{s=1}^{t-1} \Pi_{t,s} y_s(1) \left\{\frac{1[Z_s=1]}{p_s} - 1\right\}\right)^4\right].$$

Lemma 4.13 demonstrates that the expected prediction tracking error depend on three things: the per-iteration leverage scores, the potential outcomes, and the fourth moments of the inverse probabilities. Throughout the remainder of the section, we focus on the role of the inverse probabilities moments; in the appendix, all three of these quantities are carefully dealt with.

The inverse probabilities are controlled, in part, by our choice of sigmoidal regularization. The following lemma illustrates the effect of the sigmoidal regularization for any sigmoid satisfying Condition 1 on the resulting inverse probabilities.

Lemma 4.14. *Consider $A, B \geq 0$ and define p^* as the minimizer of the following program:*

$$p^* = \arg \min_{p \in (0,1)} \frac{A}{p} + \frac{B}{1-p} + \eta^{-1} \Psi(p).$$

Then, the minimizer p^ is bounded away from 0 and 1 in the following sense: there exists a universal constant $C > 0$ depending only on b_1, b_2, b_3 in Condition 1 such that*

$$\frac{1}{p^*} \leq 2 + C \cdot \min\left((\eta B)^{1/4}, \left(\frac{B}{A}\right)^{1/2}\right) \quad \text{and} \quad \frac{1}{1-p^*} \leq 2 + C \cdot \min\left((\eta A)^{1/4}, \left(\frac{A}{B}\right)^{1/2}\right).$$

Lemma 4.14 provides two types of upper bounds on the inverse probabilities based on the quantities A and B in the objective and the regularization parameter η^{-1} . If the regularization parameter η^{-1} is large relative to A and B , then the inverse probabilities are bounded. Alternatively, if A and B are of the same magnitude so that their ratio is small, then the inverse probabilities are also bounded. Lemma 4.14 relies on the curvature conditions of the sigmoidal regularizer (Condition 1).

In each iteration of the SIGMOID-FTRL design, the treatment assignment probability p_t is chosen according to the minimization in Lemma 4.14 where η is the step size η_t and A and B are the estimated online residuals $\hat{A}_{t-1}(1)$ and $\hat{A}_{t-1}(0)$ under treatment and control, respectively. The first type of bounds in Lemma 4.14 imply that if we want $E[1/p_t]$ and $E[1/(1-p_t)]$ to be small, we need that the estimated residuals are small in expectation:

Corollary 4.15. *There exists a universal constant $C > 0$ depending on b_1 and b_2 in Condition 1 such that for each iteration $t \in [T]$ and any $0 \leq k \leq 4$, the k th moment of the inverse probabilities are bounded as*

$$\begin{aligned} \mathbb{E}\left[\left(\frac{1}{p_t}\right)^k\right] &\leq \left\{2 + C \cdot \left(\eta_t \mathbb{E}[\hat{A}_{t-1}(0)]\right)^{1/4}\right\}^k \\ \mathbb{E}\left[\left(\frac{1}{1-p_t}\right)^k\right] &\leq \left\{2 + C \cdot \left(\eta_t \mathbb{E}[\hat{A}_{t-1}(1)]\right)^{1/4}\right\}^k. \end{aligned}$$

Corollary 4.15 demonstrates that in order to bound the inverse probabilities, we must bound the expectation of the estimated online residuals $\mathbb{E}[\hat{A}_{t-1}(1)]$ and $\mathbb{E}[\hat{A}_{t-1}(0)]$. The following lemma computes these expectation directly. For each treatment $k \in \{0, 1\}$, we define the full-information online residuals as $A_t^*(k) = \sum_{s \leq t} (y_t(k) - \langle \mathbf{x}_t, \beta_t^*(k) \rangle)^2$.

Lemma 4.16. *For any $t \in [T]$, the expectation of the estimated squared residuals is equal to*

$$\begin{aligned} \mathbb{E}[\hat{A}_t(1)] &= A_t^*(1) + \sum_{s=1}^t \sum_{r=1}^{s-1} \Pi_{s,r}^2 y_r^2(1) \mathbb{E}\left[\frac{1}{p_r} - 1\right] \\ \mathbb{E}[\hat{A}_t(0)] &= A_t^*(0) + \sum_{s=1}^t \sum_{r=1}^{s-1} \Pi_{s,r}^2 y_r^2(0) \mathbb{E}\left[\frac{1}{1-p_r} - 1\right]. \end{aligned}$$

The magnitude of the estimated online residuals depends on all of the inverse probabilities in previous iterations. The reason for this is that $\hat{A}_t(1)$ and $\hat{A}_t(0)$ are estimated using the adaptive IPW technique. At first glance, it appears that we have run into a circular problem: in order to bound the inverse probabilities, we must bound the estimated online residuals, but doing so requires bounding the inverse probabilities.

Fortunately, both the inverse probabilities and the estimated online residuals can be bounded through an inductive method. At the first iteration, the estimated online residuals are $\hat{A}_0(1) = \hat{A}_0(0) = 0$ so the treatment probability is $p_1 = 1/2$, and thus both of these quantities are well-controlled. At later iterations, we use the inductive hypothesis that $\mathbb{E}[\hat{A}_{t-1}(1)]$ and $\mathbb{E}[\hat{A}_{t-1}(0)]$ cannot be too large together with Corollary 4.15 to establish that the inverse probabilities $\mathbb{E}[1/p_t]$ and $\mathbb{E}[1/(1-p_t)]$ are bounded. Using this together with the inductive hypothesis that previous inverse probabilities $\mathbb{E}[1/p_s]$ and $\mathbb{E}[1/(1-p_s)]$ are bounded for $s < t$, we are able to bound $\mathbb{E}[\hat{A}_t(1)]$ and $\mathbb{E}[\hat{A}_t(0)]$, thereby completing the inductive proof. This method of argument, described in full detail in Section B.4 of the appendix, establishes the bound on the prediction tracking error:

Proposition 4.17. *Under Assumptions 1-3 and Condition 1, the prediction tracking error for both treatments $k \in \{0, 1\}$ can be bounded as $\mathbb{E}[\sum_{t=1}^T \eta_t \langle \mathbf{x}_t, \beta_t(k) - \beta_t^*(k) \rangle^4] = o(T^{1/2} R^2)$.*

5 Asymptotically Valid Inference

In this section, we propose methods for asymptotically valid confidence intervals using SIGMOID-FTRL. While the results on Neyman regret in Section 4 were stated in terms of finite sample analysis, the results in this section will be solely focused on asymptotic analysis using the triangular array asymptotics discussed in Section 2.3.

5.1 Non-Superefficiency

In design-based inference, it is necessary to place assumptions on the potential outcomes which ensure that the variance of an effect estimator is not superefficient, i.e. the variance does not

decay faster than the usual n^{-1} rate (Aronow and Samii, 2017; Leung, 2022; Harshaw, Sävje, and Wang, 2022). For example, if the potential outcomes are perfectly correlated so that the AIPW estimator is constant, then the variance can be equal to zero. When potential outcomes are presumed to be sampled i.i.d. from a superpopulation, such an event would occur with probability zero; however, it has so far been permissible under our assumptions. In particular, note that the convergence of the Neyman regret ensures only an upper bound the variance, not a lower bound.

Throughout most of the design-based literature which focuses on complex designs, superefficiency is guaranteed by direct assumption—that is, by assuming that $\liminf n \cdot \text{Var}(\hat{\tau}) > 0$. Perhaps surprisingly, we are able to identify precise conditions when the adaptive AIPW estimator will be non-superefficient under SIGMOID-FTRL. Recall that the oracle variance is given by $T \cdot V^* = 2(1 + \rho)\mathcal{E}(1)\mathcal{E}(0)$, where $\mathcal{E}(1)$ and $\mathcal{E}(0)$ are the minimal residuals in each treatment condition and ρ is the correlation of residuals between the conditions. The following theorem shows that the asymptotic variance of the adaptive estimator is exactly the oracle variance.

Theorem 5.1. *Under Assumptions 1-3 and Condition 1, the asymptotic variance of the adaptive AIPW estimator under SIGMOID-FTRL is the oracle variance:*

$$T \cdot \text{Var}(\hat{\tau}) = 2(1 + \rho)\mathcal{E}(1)\mathcal{E}(0) + o(1).$$

While Theorem 5.1 derives the exact asymptotic variance of the adaptive AIPW estimator under SIGMOID-FTRL, it does not imply the Neyman regret guarantee of Theorem 4.1. Theorem 4.1 shows that the difference between the (standardized) adaptive and oracle variances goes to zero at the rate $T^{-1/2}R^2$. On the other hand, Theorem 5.1 establishes the exact asymptotic variance, albeit at slower rates of convergence hidden in the $o(1)$ term. In this sense, Theorems 4.1 and 5.1 are complimentary but ultimately incomparable.

Theorem 5.1 shows that the exact asymptotic variance depends only on the optimal residuals $\mathcal{E}(1)$ and $\mathcal{E}(0)$ as well as the residual correlation ρ . Assumption 1 ensures that these $\mathcal{E}(1)$ and $\mathcal{E}(0)$ are bounded from below. The following assumption ensures that the correlation is bounded away from -1 , which implies that the estimator is not superefficient.

Assumption 4 (Bounded Correlation). $\liminf_{T \rightarrow \infty} \rho > -1$.

Corollary 5.2 (Non-Superefficiency). *Under Assumptions 1-4 and Condition 1, $\liminf_{T \rightarrow \infty} T \cdot \text{Var}(\hat{\tau}) > 0$.*

5.2 Central Limit Theorem

To facilitate the development of asymptotically valid inference, we derive a central limit theorem for the adaptive AIPW estimator under SIGMOID-FTRL. Because the estimator has a martingale structure with respect to iterations in the adaptive design, we will use a standard martingale CLT. While the martingale CLT itself is standard, verifying the necessary conditions will require new technical developments.

Lemma 5.3 (Helland, 1982). *If $X_{t,T}$ is a triangular array of martingale difference sequences with respect to filtrations $\mathcal{F}_{t,T}$, i.e. $E[X_{t,T} | \mathcal{F}_{t-1,T}] = 0$, then if*

1. *There exists $\delta > 0$ such that $\sum_{t=1}^T E[X_{t,T}^{2+\delta} | \mathcal{F}_{t-1,T}] \xrightarrow{p} 0$*
2. *$V_T^2 \triangleq \sum_{t=1}^T E[X_{t,T}^2 | \mathcal{F}_{t-1,T}] \xrightarrow{p} 1$*

then the sum $S_T = \sum_{t=1}^T X_{t,T}$ converges to a standard normal in distribution, $S_T \xrightarrow{d} \mathcal{N}(0, 1)$.

In order to use the martingale CLT, we will consider the martingale difference sequence given by

$$X_{t,T} = \frac{\hat{\tau}_t - \tau_t}{T\sqrt{\text{Var}(\hat{\tau})}} ,$$

where $\tau_t = y_t(1) - y_t(0)$ in the individual treatment effect and

$$\hat{\tau}_t = \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle + (Y_t - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle) \cdot \frac{\mathbf{1}[Z_t = 1]}{p_t} - (Y_t - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle) \cdot \frac{\mathbf{1}[Z_t = 0]}{1 - p_t}$$

is the individual effect estimator. One can readily verify that $\sum_{t=1}^T X_{t,T} = (\hat{\tau} - \tau)/\sqrt{\text{Var}(\hat{\tau})}$, so that it remains to argue for the two conditions in Lemma 5.3: the Lyapunov condition and the stable variance condition. Due to space considerations, we defer the full proof to Section C.2 in the appendix. Instead, we highlight two technical developments in the proof which involve controlling the inverse probabilities.

Almost Sure Bounds on Inverse Probabilities In the proof of the central limit theorem, we bound the inverse probabilities almost surely. Because Assumption 1 bounds only the fourth moments of outcomes, an almost sure bound on the inverse probabilities is necessary to verify the Lyapunov condition. The following proposition contains our almost sure bound:

Proposition 5.4. *Under Assumption 1 and Condition 1, there exists a constant $K > 0$ so that*

$$\Pr\left(\max\left\{\frac{1}{p_t}, \frac{1}{1-p_t}\right\} \leq K \cdot T^{7/26} R_t^{-4/11} \text{ for all } t \in [T]\right) = 1 .$$

The curvature conditions on the sigmoidal regularizer (Condition 1) play a crucial role in proving Proposition 5.4. In particular, we apply an inductive and iterative argument based on the probability regularization lemma (Lemma 4.14). It is crucial to our argument to use not only the bound based on the magnitude of the estimated residuals $\hat{A}_t(1)$ and $\hat{A}_t(0)$, but also the bound based on the magnitude of their ratios $\hat{A}_t(1)/\hat{A}_t(0)$ and $\hat{A}_t(0)/\hat{A}_t(1)$.

Stability of Inverse Probabilities In proving the variance stabilization condition, we need to go beyond almost sure bounds and establish the stability of the inverse probabilities; stability of the linear predictors is also required, but this is established through the prediction tracking technique described in Section 4.5. For this purpose, we introduce the *stabilized probability* \bar{p}_t , which we define as

$$\bar{p}_t = \arg \min_{p \in (0,1)} \frac{\mathbb{E}[\hat{A}_{t-1}(1)]}{p} + \frac{\mathbb{E}[\hat{A}_{t-1}(0)]}{1-p} + \frac{1}{\eta_t} \cdot \Psi(p) . \quad (1)$$

The stabilized probability \bar{p}_t is a deterministic quantity which is defined similarly to the random adaptively chosen probability p_t , except that the stabilized probability \bar{p}_t replaces the estimated squared residuals with their expected values.

In the proof of the central limit theorem, we investigate the expected absolute differences of the inverse probabilities to the inverse stabilized probabilities: $\mathbb{E}[|1/p_t - 1/\bar{p}_t|]$ and $\mathbb{E}[|1/(1-p_t) - 1/(1-\bar{p}_t)|]$. These differences can be expressed in terms of the differences in the first order optimality conditions of (1), which can further be expressed as differences $\hat{A}_{t-1}(1) - \mathbb{E}[\hat{A}_{t-1}(1)]$ and $\hat{A}_{t-1}(0) - \mathbb{E}[\hat{A}_{t-1}(0)]$. The subtle aspect is that variance $\text{Var}(\hat{A}_{t-1}(1))$ depends on the expectation $\mathbb{E}[\hat{A}_{t-1}(0)]$ in the following way: when $\mathbb{E}[\hat{A}_{t-1}(0)]$ is large, then $1/p$ is small and thus $\text{Var}(\hat{A}_{t-1}(1))$ is large due to the favorable inverse probability weighting. Likewise, the

variance $\text{Var}(\hat{A}_{t-1}(0))$ depends on the expectation $E[\hat{A}_{t-1}(1)]$. In essence, the terms $\hat{A}_{t-1}(1)$ and $\hat{A}_{t-1}(0)$ have a *mutually normalizing* property. This mutually normalizing property is used to prove several delicate aspects of the central limit theorem, including the following proposition:

Proposition 5.5. *Under Assumptions 1-3,*

$$\frac{1}{T} \sum_{t=1}^T (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^2 E\left[\left|\frac{1}{p_t} - \frac{1}{\bar{p}_t}\right|\right] \rightarrow 0 .$$

Proving Proposition 5.5 is challenging because the residuals and the difference of the inverse probabilities cannot be analyzed separately, e.g. using Hölder's inequality. The issue is that under SIGMOID-FTRL, a small proportion of the inverse probabilities $1/p_t$ could be large; in fact, some inverse probability terms $1/p_t$ will not concentrate around the inverse stabilized probability $1/\bar{p}_t$. This means that some of the terms $E[|1/p_t - 1/\bar{p}_t|]$ can be large. The saving grace is that when the difference of these probabilities are large, the corresponding residuals must be small, so that the product in the sum is controlled. The formal proof of Proposition 5.5 relies heavily on the mutually normalizing property described above. Given its central role in the proof, we believe that the mutual normalizing property might be of independent interest.

5.3 Variance Estimation and Confidence Intervals

In order to construct confidence intervals, we need to first construct an estimator for the oracle variance $T \cdot V^* = 2(1 + \rho)\mathcal{E}(1)\mathcal{E}(0)$. However, this oracle variance cannot be consistently estimated because the residual correlation ρ does not admit a consistent estimator. The impossibility of consistent variance estimation is a well-known problem in the design-based framework (see e.g. Harshaw, Middleton, and Sävje, 2021; Imbens and Rubin, 2015). In light of this issue, experimenters tend to opt for conservative (i.e. upwardly biased) estimates of the variance, which in turn yield conservative inferential procedures. Conservative variance estimates are typically constructed by first identifying an estimable upper bound on the variance, or *variance bound* for short.

The most common variance bound is due to Neyman (1923) and was originally derived in the setting of a (non-adaptive) completely randomized experiment. We proceed by adapting his variance bound to the current setting. Because the correlation is at most $\rho \leq 1$, we have that $T \cdot V^* \leq T \cdot VB \triangleq 4\mathcal{E}(1)\mathcal{E}(0)$, which is referred to as the Neyman variance bound. Unlike the variance itself, the Neyman variance bound is estimable because it depends only on the squared residuals $\mathcal{E}(1)$ and $\mathcal{E}(0)$, each of which may be estimated to high precision. Interestingly, selecting the Neyman allocation p^* , $\boldsymbol{\beta}^*(1)$, and $\boldsymbol{\beta}^*(0)$ which minimizes the variance and then applying Neyman's bound is equivalent to first applying Neyman's bound and then selecting the allocation to minimize the bound. In other words, by constructing an adaptive design which minimizes the variance of the point estimator, we are also minimizing the expected width of the resulting confidence interval.

In order to construct a consistent estimator for the variance bound VB , we focus first on constructing consistent estimates of the squared residuals. For each treatment $k \in \{0, 1\}$, the squared residual may be given as

$$\mathcal{E}^2(k) = \arg \min_{\boldsymbol{\beta} \in \mathbb{R}^d} \frac{1}{T} \sum_{t=1}^T (y_t(k) - \langle \mathbf{x}_t, \boldsymbol{\beta} \rangle)^2 = \frac{1}{T} \mathbf{y}(k)^\top Q \mathbf{y}(k) ,$$

where Q is the orthogonal projection matrix $I - \mathbf{X}(\mathbf{X}^\top \mathbf{X})^{-1} \mathbf{X}^\top$, \mathbf{X} is the T -by- d matrix whose rows are covariate vectors, and $\mathbf{y}(k)$ is the vector of potential outcomes $\mathbf{y}(k) = (y_1(k) \dots y_T(k))$.

To simplify the notation, we introduce $A(k) = \mathcal{E}^2(k)$. In order to estimate the optimal residuals, we use an adaptive IPW estimator applied to the quadratic form,

$$\begin{aligned}\hat{A}(1) &= \frac{1}{T} \sum_{t=1}^T Q_{t,t} Y_t^2 \frac{\mathbf{1}[Z_t = 1]}{p_t} + \frac{1}{T} \sum_{t=1}^T \sum_{s \neq t} Q_{t,s} Y_t Y_s \frac{\mathbf{1}[Z_t = 1, Z_s = 1]}{p_t p_s} \quad \text{and} \\ \hat{A}(0) &= \frac{1}{T} \sum_{t=1}^T Q_{t,t} Y_t^2 \frac{\mathbf{1}[Z_t = 0]}{1 - p_t} + \frac{1}{T} \sum_{t=1}^T \sum_{s \neq t} Q_{t,s} Y_t Y_s \frac{\mathbf{1}[Z_t = 0, Z_s = 0]}{(1 - p_t)(1 - p_s)} .\end{aligned}$$

The adaptive inverse probability weighting is chosen so that $\hat{A}(k)$ is an unbiased estimate of $A(k)$. Even though this estimate $\hat{A}(k)$ is unbiased for the positive value $A(k)$, there is a small chance that it will take negative values. To account for this possibility, we define the estimated OLS residuals as $\hat{\mathcal{E}}(k) = \max(\hat{A}(k), 0)^{1/2}$. However, the following Theorem shows that $\hat{A}(k)$ is a consistent estimator, and so it takes negative values with vanishingly small probability.

Theorem 5.6. *Under Assumptions 1-3 and Condition 1, for each treatment $k \in \{0, 1\}$, we have that the estimated squared OLS residuals satisfy*

$$E[\hat{A}(k)] = A(k) \quad \text{and} \quad \text{Var}(\hat{A}(k)^2) = \mathcal{O}(\{T^{-5/12} R^{2/3}\}^2) .$$

Theorem 5.6 shows that $\hat{\mathcal{E}}(k)$ is consistent at a rate $T^{-5/12} R^{2/3}$, which depends on both the number of samples T and the maximum radius of the covariates R . Under Assumption 3, we have that $R = o(T^{1/4})$ so that the consistency is always at least $\hat{\mathcal{E}}(k) - \mathcal{E}(k) = o_p(T^{-1/4})$. On the other hand, if $R = \mathcal{O}(1)$, then the estimator $\hat{\mathcal{E}}(k)$ converges at an improved rate of $\mathcal{O}_p(T^{-5/12})$.

It's interesting to note that SIGMOID-FTRL uses adaptive experimentation to ensure maximal efficiency of the point estimator, but this seems to come at the cost of slightly worsened rates of consistency of the variance bound estimator. We find this trade-off to be unproblematic because efficiency of the point estimator is the first order concern for confidence intervals. In contrast, the estimated variance bound is required only to be consistent, and the precise rate is inconsequential for asymptotic coverage. It is an interesting question whether this trade-off is necessary, or merely an artifact of our approach.

The proof of Theorem 5.6 proceeds by analyzing the covariance of the individual terms in the quadratic estimator. This requires specialized case analysis of subject pairs (s, t) and (k, ℓ) . For example, if the subject pairs are distinct, i.e. $\{s, t\} \cap \{k, \ell\} = \emptyset$, then the corresponding covariance term is zero. We refer the reader to Section C.4 of the appendix for the full case analysis.

Given the estimates of the OLS residuals, we construct an estimator of the Neyman variance bound as $T \cdot \widehat{VB} = 4\hat{\mathcal{E}}(1)\hat{\mathcal{E}}(0)$. The estimated variance bound inherits the same rates of consistency:

Corollary 5.7. *Under Assumptions 1-3 and Condition 1, $T \cdot \widehat{VB} - T \cdot VB = \mathcal{O}_p(T^{-5/12} R^{2/3})$.*

The estimator of the variance bound, together with the central limit theorem, yields the following Wald-type confidence intervals: for a given $\alpha \in (0, 1)$ level, the confidence interval \widehat{CI}_α is

$$\widehat{CI}_\alpha = \hat{\tau} \pm \Phi^{-1}(1 - \alpha/2) \cdot \sqrt{\widehat{VB}} ,$$

where $\Phi^{-1} : (0, 1) \rightarrow \mathbb{R}_+$ is the quantile function of a standard normal.

Corollary 5.8. *Under Assumptions 1-4 and Condition 1, the Wald-type intervals cover at the nominal level: $\liminf_{T \rightarrow \infty} \Pr(\tau \in \widehat{CI}_\alpha) \geq 1 - \alpha$.*

6 Conclusion

In this paper, we have investigated the problem of Adaptive Neyman Allocation for AIPW estimators in the design-based framework. We have presented SIGMOID-FTRL, an adaptive experimental design which selects treatment probabilities and linear predictors together. Assuming mild regularity conditions, the Neyman Regret under SIGMOID-FTRL converges at a rate of $T^{-1/2}R^2$. This rate is optimal in its T dependence as evidenced by our lower bound of $T^{-1/2}$. A central limit theorem and variance estimator facilitate the construction of asymptotically valid Wald-type confidence intervals.

There are several avenues for future work. One of the technical questions raised by this paper is whether the dependence of R^2 in the Neyman Regret is necessary. We conjecture that the answer is yes, but establishing this would require a more refined lower bound construction. It is also an interesting direction to explore anytime valid confidence sequences in conjunction with Adaptive Neyman Allocation in the design-based setting. For example, it would be interesting to understand the extent to which minimizing the width of the Wald-type interval (i.e. the central goal of Adaptive Neyman Allocation) and minimizing the width of the anytime valid intervals may be similar or conflicting goals.

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Appendix

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A Preliminary Results

A.1 Properties of AIPW Estimator

Proposition 2.1. *If $p_t \in (0, 1)$ for all $t \in [T]$ a.s. then the adaptive AIPW estimator is unbiased: $E[\hat{\tau}] = \tau$.*

Proof. By the construction method, $\beta_t(1)$ and $\beta_t(0)$ are measurable with respect to \mathcal{F}_{t-1} . Hence by the law of iterated expectations, we have

$$\begin{aligned} & E[\hat{\tau}] \\ &= E\left[\frac{1}{T} \sum_{t=1}^T \left\{ \langle \mathbf{x}_t, \beta_t(1) \rangle - \langle \mathbf{x}_t, \beta_t(0) \rangle + \frac{\mathbf{1}[Z_t = 1]}{p_t} (Y_t - \langle \mathbf{x}_t, \beta_t(1) \rangle) - \frac{\mathbf{1}[Z_t = 0]}{1-p_t} (Y_t - \langle \mathbf{x}_t, \beta_t(0) \rangle) \right\}\right] \\ &= \frac{1}{T} \sum_{t=1}^T E\left[E\left[\langle \mathbf{x}_t, \beta_t(1) \rangle - \langle \mathbf{x}_t, \beta_t(0) \rangle + \frac{\mathbf{1}[Z_t = 1]}{p_t} (Y_t - \langle \mathbf{x}_t, \beta_t(1) \rangle) - \frac{\mathbf{1}[Z_t = 0]}{1-p_t} (Y_t - \langle \mathbf{x}_t, \beta_t(0) \rangle) \mid \mathcal{F}_{t-1} \right] \right] \\ &= \frac{1}{T} \sum_{t=1}^T E[\langle \mathbf{x}_t, \beta_t(1) \rangle - \langle \mathbf{x}_t, \beta_t(0) \rangle + (y_t(1) - \langle \mathbf{x}_t, \beta_t(1) \rangle) - (y_t(0) - \langle \mathbf{x}_t, \beta_t(0) \rangle)] \\ &= \frac{1}{T} \sum_{t=1}^T (y_t(1) - y_t(0)) \\ &= \tau . \end{aligned}$$

□

Proposition 2.2. *The normalized variance of the AIPW estimator is given as*

$$T \cdot \text{Var}(\hat{\tau}) = E\left[\frac{1}{T} \sum_{t=1}^T \left(\{y_t(1) - \langle \mathbf{x}_t, \beta_t(1) \rangle\} \cdot \sqrt{\frac{1-p_t}{p_t}} + \{y_t(0) - \langle \mathbf{x}_t, \beta_t(0) \rangle\} \cdot \sqrt{\frac{p_t}{1-p_t}} \right)^2\right].$$

Proof. By the proof in Proposition 2.1, it is easy to see that $\hat{\tau} - \tau$ can be written as the sum of a martingale difference sequence, whose variance can be calculated by summing up the variance of each individual term. Hence by the law of total variance, the variance of $\hat{\tau}$ can be calculated by:

$$\begin{aligned} & T \cdot \text{Var}(\hat{\tau}) \\ &= T \cdot \text{Var}(\hat{\tau} - \tau) \\ &= T \cdot \text{Var}\left[\frac{1}{T} \sum_{t=1}^T \left\{ \left(\frac{\mathbf{1}[Z_t = 1]}{p_t} - 1 \right) (y_t(1) - \langle \mathbf{x}_t, \beta_t(1) \rangle) - \left(\frac{\mathbf{1}[Z_t = 0]}{1-p_t} - 1 \right) (y_t(0) - \langle \mathbf{x}_t, \beta_t(0) \rangle) \right\}\right] \\ &= \frac{1}{T} \sum_{t=1}^T \text{Var}\left[\left(\frac{\mathbf{1}[Z_t = 1]}{p_t} - 1 \right) (y_t(1) - \langle \mathbf{x}_t, \beta_t(1) \rangle) - \left(\frac{\mathbf{1}[Z_t = 0]}{1-p_t} - 1 \right) (y_t(0) - \langle \mathbf{x}_t, \beta_t(0) \rangle) \right] \\ &= \frac{1}{T} \sum_{t=1}^T \text{Var}\left[E\left[\left(\frac{\mathbf{1}[Z_t = 1]}{p_t} - 1 \right) (y_t(1) - \langle \mathbf{x}_t, \beta_t(1) \rangle) - \left(\frac{\mathbf{1}[Z_t = 0]}{1-p_t} - 1 \right) (y_t(0) - \langle \mathbf{x}_t, \beta_t(0) \rangle) \mid \mathcal{F}_{t-1} \right] \right] \\ &\quad + \frac{1}{T} \sum_{t=1}^T E\left[\text{Var}\left[\left(\frac{\mathbf{1}[Z_t = 1]}{p_t} - 1 \right) (y_t(1) - \langle \mathbf{x}_t, \beta_t(1) \rangle) - \left(\frac{\mathbf{1}[Z_t = 0]}{1-p_t} - 1 \right) (y_t(0) - \langle \mathbf{x}_t, \beta_t(0) \rangle) \mid \mathcal{F}_{t-1} \right] \right] \\ &= \frac{1}{T} \sum_{t=1}^T E\left[\text{Var}\left[\left(\frac{1}{p_t} (y_t(1) - \langle \mathbf{x}_t, \beta_t(1) \rangle) + \frac{1}{1-p_t} (y_t(0) - \langle \mathbf{x}_t, \beta_t(0) \rangle) \right) \mathbf{1}[Z_t = 1] \mid \mathcal{F}_{t-1} \right] \right] \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{T} \sum_{t=1}^T \mathbb{E} \left[\left(\frac{1}{p_t} (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle) + \frac{1}{1-p_t} (y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle) \right)^2 p_t(1-p_t) \right] \\
&= \mathbb{E} \left[\frac{1}{T} \sum_{t=1}^T \left(\{y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle\} \cdot \sqrt{\frac{1-p_t}{p_t}} + \{y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle\} \cdot \sqrt{\frac{p_t}{1-p_t}} \right)^2 \right].
\end{aligned}$$

□

A.2 Verification of the Sigmoidal Condition

In this section, we verify that two commonly used sigmoid functions: $\phi(u) = \frac{1}{\pi}(\arctan(u) + \pi/2)$ and $\phi(u) = \frac{1}{2}(\frac{u}{1+|u|} + 1)$ satisfy all the required conditions.

Lemma A.1. *The choice of sigmoid function: $\phi(u) = \frac{1}{\pi}(\arctan(u) + \pi/2)$ satisfies Condition 1 with constants $b_1 = \pi$, $b_2 = \frac{2^{5/2}\pi}{3}$ and $b_3 = \frac{2}{\pi}$.*

Proof. We verify the requirements in Condition 1 one by one.

(1) Condition 1 (1) is obvious.

(2) The derivative of $1/\phi(u)$ and $1/(1-\phi(u))$ are calculated as:

$$\begin{aligned}
\left(\frac{1}{\phi(u)} \right)' &= -\frac{\pi}{(1+u^2)(\arctan(u) + \frac{1}{2}\pi)^2}, \\
\left(\frac{1}{1-\phi(u)} \right)' &= \frac{\pi}{(1+u^2)(-\arctan(u) + \frac{1}{2}\pi)^2}.
\end{aligned}$$

The second derivative of $1/\phi(u)$ and $1/(1-\phi(u))$ are calculated as:

$$\begin{aligned}
\left(\frac{1}{\phi(u)} \right)'' &= \frac{2u}{\pi(1+u^2)^2 \left(\frac{\arctan(u)}{\pi} + \frac{1}{2} \right)^2} + \frac{2}{\pi^2(1+u^2)^2 \left(\frac{\arctan(u)}{\pi} + \frac{1}{2} \right)^3} \\
&= \frac{2}{\pi^2(1+u^2)^2 \left(\frac{\arctan(u)}{\pi} + \frac{1}{2} \right)^3} \left[1 + u \left(\arctan(u) + \frac{1}{2}\pi \right) \right] \\
\left(\frac{1}{1-\phi(u)} \right)'' &= -\frac{2u}{\pi(1+u^2)^2 \left(-\frac{\arctan(u)}{\pi} + \frac{1}{2} \right)^2} + \frac{2}{\pi^2(1+u^2)^2 \left(-\frac{\arctan(u)}{\pi} + \frac{1}{2} \right)^3} \\
&= \frac{2}{\pi^2(1+u^2)^2 \left(-\frac{\arctan(u)}{\pi} + \frac{1}{2} \right)^3} \left[1 - u \left(-\arctan(u) + \frac{1}{2}\pi \right) \right].
\end{aligned}$$

When $u \geq 0$, it is easy to see that $(1/\phi(u))'' > 0$. When $u < 0$, we have

$$\begin{aligned}
u \left(\arctan(u) + \frac{1}{2}\pi \right) &= -(-u) \left(-\arctan(-u) + \frac{1}{2}\pi \right) \\
&= -(-u) \arctan(-1/u) \\
&> -(-u) \cdot (-1/u) \\
&= -1.
\end{aligned}$$

Hence $1/\phi(u)$ is convex. Similarly, we can prove that $1/(1-\phi(u))$ is also convex by the symmetry.

(3a) By L'Hopital's rule, we have

$$\begin{aligned}
\lim_{u \rightarrow -\infty} \sqrt{1+u^2} \left(\arctan(u) + \frac{1}{2}\pi \right) &= \lim_{u \rightarrow -\infty} \frac{(\arctan(u) + \frac{1}{2}\pi)}{1/\sqrt{1+u^2}} \\
&= \lim_{u \rightarrow -\infty} \frac{1/(1+u^2)}{-u(1+u^2)^{-3/2}} \\
&= \lim_{u \rightarrow -\infty} \frac{(1+u^2)^{1/2}}{-u} \\
&= 1 .
\end{aligned}$$

Hence we have

$$-\pi \leq \left(\frac{1}{\phi(u)} \right)' \leq 0 .$$

(3b) Let $\theta = \arctan(u)$. Then $1+u^2 = \sec^2(\theta)$. Hence

$$\begin{aligned}
\left(\frac{1}{\phi(u)} \right)'' &= \frac{2}{\pi^2(1+u^2)^2 \left(\frac{\arctan(u)}{\pi} + \frac{1}{2}\pi \right)^3} \left[1 + u \left(\arctan(u) + \frac{1}{2}\pi \right) \right] \\
&= \frac{2}{\pi^2 \sec^4(\theta) \left(\frac{\theta}{\pi} + \frac{1}{2}\pi \right)^3} \left[1 + \tan(\theta) \left(\theta + \frac{1}{2}\pi \right) \right] \\
&= \frac{2\pi \cos^3(\theta)}{\left(\theta + \frac{1}{2}\pi \right)^3} \left[\cos(\theta) + \sin(\theta) \left(\theta + \frac{1}{2}\pi \right) \right] .
\end{aligned}$$

Since $(1+u^2)^{3/2} = \sec^3(\theta)$, we have

$$(1+u^2)^{3/2} \left(\frac{1}{\phi(u)} \right)'' = \frac{2\pi}{\left(\theta + \frac{1}{2}\pi \right)^3} \left[\cos(\theta) + \sin(\theta) \left(\theta + \frac{1}{2}\pi \right) \right] .$$

Let $t = \theta + \frac{1}{2}\pi \in (0, \pi)$ and $f(t) = \frac{\sin t - t \cos t}{t^3}$. By direct calculation, we have

$$f'(t) = \frac{\sin t(t^2 - 3) + 3t \cos t}{t^4} \triangleq \frac{k(t)}{t^4} .$$

By direct calculation, we have

$$k'(t) = \cos t(t^2 - 3) + 2t \sin t + 3 \cos t - 3t \sin t = t(t \cos t - \sin t) .$$

Since when $t = 0$, $t \cos t - \sin t = 0$ and for any $t \in (0, 2\pi)$, we have

$$(t \cos t - \sin t)' = \cos t - t \sin t - \cos t = -t \sin t < 0 .$$

Hence we have $k'(t) < 0$ for any $t \in (0, \pi)$. Since $k(0) = 0$, this implies that $k(t) < 0$ for any $t \in (0, \pi)$. This further indicates that $f'(t) < 0$ for any $t \in (0, \pi)$. By L'Hopital's rule, we have

$$\lim_{t \downarrow 0} f(t) = \lim_{t \downarrow 0} \frac{\sin t - t \cos t}{t^3} = \lim_{t \downarrow 0} \frac{\cos t - \cos t + t \sin t}{3t^2} = \frac{1}{3} .$$

we have proved that for any $t \in (0, 2\pi)$, $f(t) \leq 1/3$. This implies that for any $\theta \in (-\pi/2, \pi/2)$,

$$(1+u^2)^{3/2} \left(\frac{1}{\phi(u)} \right)'' = \frac{2\pi}{\left(\theta + \frac{1}{2}\pi \right)^3} \left[\cos(\theta) + \sin(\theta) \left(\theta + \frac{1}{2}\pi \right) \right] \leq \frac{2\pi}{3} .$$

Hence for any $u \in \mathbb{R}$, we have

$$\left(\frac{1}{\phi(u)} \right)'' \leq \frac{2\pi}{3} (1+u^2)^{-3/2} \leq \frac{2\pi}{3} \cdot 2^{3/2} (1+|u|)^{-3} = \frac{2^{5/2}\pi}{3} (1+|u|)^{-3} .$$

(3c) By the same argument as in the previous part, since $f(\pi) = 1/\pi^2$, we have proved that for any $t \in (\pi/2, \pi)$, $f(t) \geq 1/\pi^2$. This implies that for any $\theta \in (0, \pi/2)$,

$$(1+u^2)^{3/2} \left(\frac{1}{\phi(u)} \right)^{''} = \frac{2\pi}{(\theta + \frac{1}{2}\pi)^3} \left[\cos(\theta) + \sin(\theta) \left(\theta + \frac{1}{2}\pi \right) \right] \geq \frac{2}{\pi} .$$

Hence for any $u \geq 0$, we have

$$\left(\frac{1}{\phi(u)} \right)^{''} \geq \frac{2}{\pi} (1+u^2)^{-3/2} \geq \frac{2}{\pi} (1+u)^{-3} .$$

□

Lemma A.2. *The choice of sigmoid function: $\phi(u) = \frac{1}{2}(\frac{u}{1+|u|} + 1)$ satisfies Condition 1 with constants $b_1 = 2$, $b_2 = 8$ and $b_3 = 1$.*

Proof. We verify the requirements in Condition 1 one by one.

(1) Condition 1 (1) is obvious.

(2) The derivative of $1/\phi(u)$ and $1/(1-\phi(u))$ are calculated as:

$$\begin{aligned} \left(\frac{1}{\phi(u)} \right)' &= \begin{cases} -\frac{2}{(2u+1)^2} & \text{if } u \geq 0 \\ -2 & \text{if } u \leq 0 \end{cases}, \\ \left(\frac{1}{1-\phi(u)} \right)' &= \begin{cases} 2 & \text{if } u \geq 0 \\ \frac{2}{(1-2u)^2} & \text{if } u \leq 0 \end{cases}. \end{aligned}$$

The second derivative of $1/\phi(u)$ and $1/(1-\phi(u))$ are calculated as:

$$\begin{aligned} \left(\frac{1}{\phi(u)} \right)^{''} &= \begin{cases} \frac{8}{(2u+1)^3} & \text{if } u \geq 0 \\ 0 & \text{if } u \leq 0 \end{cases}, \\ \left(\frac{1}{1-\phi(u)} \right)^{''} &= \begin{cases} 0 & \text{if } u \geq 0 \\ \frac{8}{(-2u+1)^3} & \text{if } u \leq 0 \end{cases}. \end{aligned}$$

Hence it is easy to see that $1/\phi(u)$ and $1/(1-\phi(u))$ are convex.

(3a) By the calculations in part (2), it is easy to see that for any $u \in \mathbb{R}$,

$$-\left(\frac{1}{\phi(u)} \right)' \leq 2 .$$

(3b) By the calculations in part (2), it is easy to see that for any $u \in \mathbb{R}$,

$$\left(\frac{1}{\phi(u)} \right)^{''} \leq \frac{8}{(1+2|u|)^3} \leq \frac{8}{(1+|u|)^3} .$$

(3c) By the calculations in part (2), it is easy to see that for any $u \in \mathbb{R}$,

$$\left(\frac{1}{\phi(u)} \right)^{''} = \frac{8}{(2u+1)^3} \geq \frac{1}{(1+u)^3} .$$

□

B Neyman Regret Analysis

In this section, we aim to perform a comprehensive analysis of the Neyman regret. Section B.1 derives a lower bound for the Neyman Regret. The rest sections derive an upper bound for the Neyman regret. Section B.2 presents the technical lemmas used in the subsequent proofs. In Section B.3, we prove Lemma 3.3, which decomposes the Neyman regret into two components: the probability regret and the prediction regret. Sections B.4 and B.5 provide upper bounds for each of these two components in Proposition 4.7 and Proposition 4.11, respectively. By combining the results from these sections, we establish the main theorem on Neyman regret (Theorem 4.1) in Section B.6.

B.1 Neyman Regret Lower Bound

In this section, we first prove the lower bound on the Neyman regret stated in Theorem 3.2. We then present Corollary B.1 and Lemma B.2, which together show that Assumptions 1–3 are all necessary for obtaining a regret upper bound of order $\mathcal{O}(T^{1/2})$, or at least sub-linear order.

Theorem 3.2 (Lower Bound). *For all integers T and any adaptive experimental design Π , there exists a sequence $\{y_t(1), y_t(0), \mathbf{x}_t\}_{t=1}^T$ satisfying Assumptions 1–3 and a constant $c > 0$, such that the corresponding Neyman regret is at least $\mathcal{R}_T^{\text{Neyman}} \geq c \cdot T^{-1/2}$.*

Proof. For simplicity, we only prove the case where $d = 1$ and $R = 1$. Let $\mathbf{x}_t = 1$ for any $t \in [T]$. We generate the random sequence $\{y_t(1), y_t(0)\}$ by the following: Generate $\epsilon_1, \dots, \epsilon_T$ as i.i.d. rademacher random variables. Generate an independent random vector $\mathbf{D} = (D_1, D_2)$ that takes value $(2, 4)$ with probability $1/2$ and takes value $(-4, -2)$ with probability $1/2$. Let $y_1(1) = D_1(T^{1/4} + \epsilon_1(1))$, $y_1(0) = D_2(T^{1/4} + \epsilon_1(0))$, $y_t(1) = D_1(1 + \epsilon_t(1))$ and $y_t(0) = D_2(1 + \epsilon_t(0))$ for any $t \geq 2$. Define the filtration $\mathcal{F}_t = \{y_s(0), y_s(1) : 1 \leq s \leq t-1\}$. It is easy to show that $y_1(1), y_1(0)$ are independent of \mathcal{F}_1 and $E[y_1(1)] = -T^{1/4}$ and $E[y_1(0)] = T^{1/4}$. Hence by AM-GM inequality and law of iterated expectation, we have

$$\begin{aligned}
& E[T \cdot \text{Var}(\hat{\tau})] \\
&= \frac{1}{T} \sum_{t=1}^T E \left[\{y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle\}^2 \cdot \frac{1-p_t}{p_t} + \{y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle\}^2 \cdot \frac{p_t}{1-p_t} \right] \\
&\quad + \frac{2}{T} \sum_{t=1}^T E \left[\{y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle\} \cdot \{y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle\} \right] \\
&= \frac{1}{T} E \left[\{y_1(1) + T^{1/4} - T^{1/4} - \langle \mathbf{x}_1, \boldsymbol{\beta}_1(1) \rangle\}^2 \cdot \frac{1-p_1}{p_1} + \{y_1(0) - T^{1/4} + T^{1/4} - \langle \mathbf{x}_1, \boldsymbol{\beta}_1(0) \rangle\}^2 \cdot \frac{p_1}{1-p_1} \right] \\
&\quad + \frac{2}{T} E \left[\{y_1(1) + T^{1/4} - T^{1/4} - \langle \mathbf{x}_1, \boldsymbol{\beta}_1(1) \rangle\} \cdot \{y_1(0) - T^{1/4} + T^{1/4} - \langle \mathbf{x}_1, \boldsymbol{\beta}_1(0) \rangle\} \right] \\
&\quad + \frac{1}{T} \sum_{t=2}^T E \left[\{D_1 \epsilon_t + D_1 - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle\}^2 \cdot \frac{1-p_t}{p_t} + \{D_2 \epsilon_t + D_2 - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle\}^2 \cdot \frac{p_t}{1-p_t} \right] \\
&\quad + \frac{2}{T} \sum_{t=2}^T E \left[\{D_1 \epsilon_t + D_1 - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle\} \{D_2 \epsilon_t + D_2 - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle\} \right] \\
&= \frac{1}{T} E \left[(y_1(1) + T^{1/4})^2 \cdot \frac{1-p_1}{p_1} \right] + \frac{1}{T} E \left[(T^{1/4} + \langle \mathbf{x}_1, \boldsymbol{\beta}_1(1) \rangle)^2 \cdot \frac{1-p_1}{p_1} \right] \\
&\quad + \frac{1}{T} E \left[(y_1(0) - T^{1/4})^2 \cdot \frac{p_1}{1-p_1} \right] + \frac{1}{T} E \left[(-T^{1/4} + \langle \mathbf{x}_1, \boldsymbol{\beta}_1(0) \rangle)^2 \cdot \frac{p_1}{1-p_1} \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{2}{T} \mathbb{E} \left[\{T^{1/4} + \langle \mathbf{x}_1, \boldsymbol{\beta}_1(1) \rangle\} \cdot \{-T^{1/4} + \langle \mathbf{x}_1, \boldsymbol{\beta}_1(0) \rangle\} \right] \\
& + \frac{1}{T} \sum_{t=2}^T \mathbb{E} \left[\mathbb{E} \left[\{D_1 \epsilon_t + D_1 - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle\}^2 \cdot \frac{1-p_t}{p_t} + \{D_2 \epsilon_t + D_2 - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle\}^2 \cdot \frac{p_t}{1-p_t} \middle| \mathcal{F}_{t-1}, \mathbf{D} \right] \right] \\
& + \frac{2}{T} \sum_{t=2}^T \mathbb{E} \left[\mathbb{E} \left[\{D_1 \epsilon_t + D_1 - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle\} \{D_2 \epsilon_t + D_2 - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle\} \middle| \mathcal{F}_{t-1}, \mathbf{D} \right] \right] \\
& \geq \frac{1}{T} (18T^{1/2} + 1) \mathbb{E} \left[\frac{1-p_t}{p_t} \right] + \frac{1}{T} (18T^{1/2} + 1) \mathbb{E} \left[\frac{p_t}{1-p_t} \right] + \frac{1}{T} \sum_{t=2}^T \mathbb{E} \left[D_1^2 \frac{1-p_t}{p_t} + D_2^2 \frac{p_t}{1-p_t} + 2D_1 D_2 \middle| \mathcal{F}_{t-1}, \mathbf{D} \right] \\
& + \frac{1}{T} \sum_{t=2}^T \mathbb{E} \left[\mathbb{E} \left[\{y_t^*(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle\}^2 \cdot \frac{1-p_t}{p_t} + \{y_t^*(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle\}^2 \cdot \frac{p_t}{1-p_t} \middle| \mathcal{F}_{t-1}, \mathbf{D} \right] \right] \\
& + \frac{2}{T} \sum_{t=2}^T \mathbb{E} \left[\mathbb{E} \left[\{y_t^*(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle\} \{y_t^*(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle\} \middle| \mathcal{F}_{t-1}, \mathbf{D} \right] \right] \\
& \geq \frac{2}{T} (18T^{1/2} + 1) + \frac{32(T-1)}{T} \\
& = \frac{36T^{1/2} + 32T - 30}{T}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\mathbb{E} [T \cdot V^*] &= \frac{2}{T} \mathbb{E} \left(\left[\sum_{t=1}^T (y_t(1) - \boldsymbol{\beta}^*(1))^2 \right]^{1/2} \left[\sum_{t=1}^T (y_t(0) - \boldsymbol{\beta}^*(0))^2 \right]^{1/2} + \sum_{t=1}^T (y_t(1) - \boldsymbol{\beta}^*(1)) (y_t(0) - \boldsymbol{\beta}^*(0)) \right) \\
&\leq \frac{4}{T} \mathbb{E} \left(\left[\sum_{t=1}^T (y_t(1) - \boldsymbol{\beta}^*(1))^2 \right]^{1/2} \left[\sum_{t=1}^T (y_t(0) - \boldsymbol{\beta}^*(0))^2 \right]^{1/2} \right) \\
&= \frac{2}{T} \mathbb{E} \left[\left[\sum_{t=1}^T (y_t(1) - \boldsymbol{\beta}^*(1))^2 \right]^{1/2} \left[\sum_{t=1}^T (y_t(0) - \boldsymbol{\beta}^*(0))^2 \right]^{1/2} \middle| \mathbf{D} = (2, 4) \right] \\
&\quad + \frac{2}{T} \mathbb{E} \left[\left[\sum_{t=1}^T (y_t(1) - \boldsymbol{\beta}^*(1))^2 \right]^{1/2} \left[\sum_{t=1}^T (y_t(0) - \boldsymbol{\beta}^*(0))^2 \right]^{1/2} \middle| \mathbf{D} = (-4, -2) \right] \\
&= \frac{4}{T} \mathbb{E} \left[\left[\sum_{t=1}^T (y_t(1) - \boldsymbol{\beta}^*(1))^2 \right]^{1/2} \left[\sum_{t=1}^T (y_t(0) - \boldsymbol{\beta}^*(0))^2 \right]^{1/2} \middle| \mathbf{D} = (-4, -2) \right] \\
&\leq \frac{2}{T} \left[\frac{1}{2} \mathbb{E} \left[\sum_{t=1}^T (y_t(1) - \bar{y}_t(1))^2 \middle| \mathbf{D} = (-4, -2) \right] + 2 \mathbb{E} \left[\sum_{t=1}^T (y_t(0) - \bar{y}_t(0))^2 \middle| \mathbf{D} = (-4, -2) \right] \right] \\
&= \frac{2}{T} \sum_{t=1}^T \left[\frac{1}{2} \text{Var}[y_t(1) | \mathbf{D} = (-4, -2)] + \frac{1}{2} \mathbb{E}^2[y_t(1) | \mathbf{D} = (-4, -2)] \right. \\
&\quad \left. + 2 \text{Var}[y_t(0) | \mathbf{D} = (-4, -2)] + 2 \mathbb{E}^2[y_t(0) | \mathbf{D} = (-4, -2)] \right] \\
&\quad - 2 \left[\frac{1}{2} \text{Var}[\bar{y}_t(1) | \mathbf{D} = (-4, -2)] + \frac{1}{2} \mathbb{E}^2[\bar{y}_t(1) | \mathbf{D} = (-4, -2)] \right. \\
&\quad \left. + 2 \text{Var}[\bar{y}_t(0) | \mathbf{D} = (-4, -2)] + 2 \mathbb{E}^2[\bar{y}_t(0) | \mathbf{D} = (-4, -2)] \right] \\
&= \frac{2}{T} \left[8T + 8T^{1/2} + 8(T-1) + 8T + 8T^{1/2} + 8(T-1) \right] \\
&\quad - 2 \left[\frac{16}{2T} + \frac{1}{2} \left(\frac{4T^{1/4} + 4(T-1)}{T} \right)^2 + \frac{8}{T} + 2 \left(\frac{2T^{1/4} + 2(T-1)}{T} \right)^2 \right]
\end{aligned}$$

$$= \frac{32T + 32T^{1/2} + o(T^{1/2})}{T} .$$

Hence we have

$$\mathbb{E}[T \cdot \text{Var}(\hat{\tau}) - T \cdot V^*] \geq 4T^{-1/2} + o(T^{-1/2}) .$$

We then prove that there exist a certain configuration of $\{y_t(1), y_t(0), \mathbf{x}_t\}$ which satisfies Assumption 1-3 and the rate of the lower bound is attained. By construction method, we only need to verify that the squared loss under the OLS is of order $\mathcal{O}(T)$. By symmetry, it suffices to prove the result in the treatment group $k = 1$ when $\mathbf{D} = (2, 4)$. Since $\mathbf{x}_t = 1$, we have

$$\begin{aligned} \sum_{t=1}^T (y_t(1) - \beta^*(1))^2 &\geq \sum_{t=2}^T (y_t(1) - \beta^*(1))^2 \\ &\geq \min_{\beta \in \mathbb{R}} \sum_{t=2}^T (y_t(1) - \beta)^2 \\ &= \min_{\beta \in \mathbb{R}} \sum_{t=2}^T (2 + \epsilon_t - \beta)^2 \\ &= \min_{\beta \in \mathbb{R}} \sum_{t=2}^T (\epsilon_t - \beta)^2 \\ &\gtrsim \min \left\{ \sum_{t=2}^T \mathbf{1}[\epsilon_t = 1], \sum_{t=2}^T \mathbf{1}[\epsilon_t = -1] \right\} . \end{aligned}$$

By Hoeffding's inequality, there exists constant $r_1, r_2 > 0$ such that the probability of

$$\min \left\{ \sum_{t=2}^T \mathbf{1}[\epsilon_t = 1], \sum_{t=2}^T \mathbf{1}[\epsilon_t = -1] \right\} \leq \frac{T}{4}$$

is smaller than $r_1 \exp(-r_2 T)$. On the other hand, the regret is lower bounded by $-\mathbb{E}[T \cdot V^*]$, which is of order $\mathcal{O}(T)$. Since $T \cdot r_1 \exp(-r_2 T) = o(T^{-1/2})$, there exists a configuration such that $\min \left\{ \sum_{t=2}^T \mathbf{1}[\epsilon_t = 1], \sum_{t=2}^T \mathbf{1}[\epsilon_t = -1] \right\} \geq \frac{T}{4}$. Moreover, its Neyman regret is positive and of order $\mathcal{O}(T^{-1/2})$. Hence the result is proved. \square

The lower bound argument can also be used to demonstrate the necessity of the bounded fourth moment condition in Assumption 1 and the covariate regularity condition in Assumption 2. By similar construction as in Theorem 3.2, we can show the necessity of the bounded fourth moment condition in Assumption 1. This is formally stated in the following corollary.

Corollary B.1. *For all integers T and for any adaptive experimental design Π , there exists a sequence $\{y_t(1), y_t(0), \mathbf{x}_t\}_{t=1}^T$ satisfying Assumptions 1-3 except for the bounded fourth moment condition in Assumption 1, and a constant $c > 0$, such that the corresponding Neyman regret is at least $\mathcal{R}_T^{\text{Neyman}} \geq c$.*

Proof. For simplicity, we only prove the case where $d = 1$ and $R = 1$. The proof is similar to Theorem 3.2. Let $\mathbf{x}_t = 1$ for any $t \in [T]$. We generate the random sequence $\{y_t(1), y_t(0)\}$ by the following: Generate $\epsilon_1, \dots, \epsilon_T$ as i.i.d. rademacher random variables. Generate an independent random vector $\mathbf{D} = (D_1, D_2)$ that takes value $(2, 4)$ with probability $1/2$ and takes value $(-4, -2)$ with probability $1/2$. Let $y_1(1) = D_1(T^{1/2} + \epsilon_1(1))$, $y_1(0) = D_2(T^{1/2} + \epsilon_1(0))$, $y_t(1) = D_1(1 + \epsilon_t(1))$ and $y_t(0) = D_2(1 + \epsilon_t(0))$ for any $t \geq 2$. This construction differs

from that in Theorem 3.2 only in the specification of $y_1(1)$ and $y_1(0)$, where the order $\mathcal{O}(T^{1/4})$ is replaced by $\mathcal{O}(T^{1/2})$. By the same proof as in Theorem 3.2, we can prove that $E[T \cdot \text{Var}(\hat{\tau}) - T \cdot V^*] \geq 4 + o(1)$. Moreover, one can similarly verify that Assumption 1 is satisfied, except for the bounded fourth-moment condition, for a sequence $\{y_t(1), y_t(0), \mathbf{x}_t\}_{t=1}^T$ such that the Neyman regret is of order $\Omega(1)$. Hence the result is proved. \square

The following lemma states that the lower bound of the Neyman regret can also attain order $\Omega(1)$ if the covariate regularity condition in Assumption 2 is not placed.

Lemma B.2. *For all integers T and for any adaptive experimental design Π , there exists a sequence $\{y_t(1), y_t(0), \mathbf{x}_t\}_{t=1}^T$ satisfying Assumptions 1 and 3, and a constant $c > 0$, such that the corresponding Neyman regret is at least $\mathcal{R}_T^{\text{Neyman}} \geq c$.*

Proof. For simplicity, we assume that $c_0 = 1$, $c_1 = 2$ and $R \geq 1$ (R is shown to be at least order $\Theta(1)$ in Lemma B.5). Suppose $\mathbf{x}_t \in \mathbb{R}^d$, where $d = T/2$. We generate the random sequence $\{y_t(1), y_t(0), \mathbf{x}_t\}$ by the following: $\mathbf{x}_t = \mathbf{e}_t$ for $t = 1, \dots, T/2$ and $\mathbf{x}_{T/2+1} = \dots = \mathbf{x}_T = \mathbf{0}$. $y_1(1), \dots, y_T(1)$ are independently sampled from the uniform distribution on $\{\pm 1\}$. Let $y_t(0) = y_t(1)$ for $t = 1, \dots, T$. It is easy to see that the y 's have their marginal expectations as 0. Now we prove that

$$E[T \cdot \text{Var}(\hat{\tau}) - T \cdot V^*] = \Omega(1).$$

Note that in such construction, the natural filtration \mathcal{F}_t is generated by $\{y_s(1), y_s(0), Z_s : 1 \leq s \leq t-1\}$. By Proposition 2.2 and law of iterated expectation, we have

$$\begin{aligned} & E[T \cdot \text{Var}(\hat{\tau})] \\ &= \frac{1}{T} \sum_{t=1}^T E \left[\{y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle\}^2 \cdot \frac{1-p_t}{p_t} + \{y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle\}^2 \cdot \frac{p_t}{1-p_t} \right] \\ &\quad + \frac{2}{T} \sum_{t=1}^T E \left[\{y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle\} \{y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle\} \right] \\ &= \frac{1}{T} \sum_{t=1}^T E \left[y_t^2(1) \cdot \frac{1-p_t}{p_t} + y_t^2(0) \cdot \frac{p_t}{1-p_t} \right] + \frac{1}{T} \sum_{t=1}^T E \left[y_t(1) \cdot \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle \cdot \frac{1-p_t}{p_t} \right] \\ &\quad + \frac{1}{T} \sum_{t=1}^T E \left[y_t(0) \cdot \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle \cdot \frac{p_t}{1-p_t} \right] + \frac{1}{T} \sum_{t=1}^T E \left[\langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle^2 \cdot \frac{1-p_t}{p_t} + \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle^2 \cdot \frac{p_t}{1-p_t} \right] \\ &\quad + \frac{2}{T} \sum_{t=1}^T E[y_t(1)y_t(0)] - \frac{2}{T} \sum_{t=1}^T E[y_t(1) \cdot \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle] - \frac{2}{T} \sum_{t=1}^T E[y_t(0) \cdot \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle] \\ &\quad + \frac{2}{T} \sum_{t=1}^T E[\langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle \cdot \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle] \\ &\geq \frac{1}{T} \sum_{t=1}^T E \left[\frac{1-p_t}{p_t} + \frac{p_t}{1-p_t} \right] + \frac{1}{T} \sum_{t=1}^T E \left[E \left[y_t(1) \cdot \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle \cdot \frac{1-p_t}{p_t} \mid \mathcal{F}_{t-1} \right] \right] \\ &\quad + \frac{1}{T} \sum_{t=1}^T E \left[E \left[y_t(0) \cdot \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle \cdot \frac{p_t}{1-p_t} \mid \mathcal{F}_{t-1} \right] \right] + \frac{2}{T} \sum_{t=1}^T 1 - \frac{2}{T} \sum_{t=1}^T E \left[y_t(0) \cdot \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle + y_t(1) \cdot \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle \mid \mathcal{F}_{t-1} \right] \\ &= \frac{1}{T} \sum_{t=1}^T E \left[\frac{1-p_t}{p_t} + \frac{p_t}{1-p_t} \right] + \frac{2}{T} \sum_{t=1}^T E[y_t(1)y_t(0)] \\ &\geq 4. \end{aligned}$$

Denote $\mathbf{Y}_T = (y_1(1), \dots, y_T(1)) = (y_1(0), \dots, y_T(0))$. It is easy to verify that matrix $\mathbf{X}_T^\top \mathbf{X}_T$ is invertible. Hence we have

$$\begin{aligned}
\mathbb{E} [T \cdot V^*] &= \frac{2}{T} \mathbb{E} (\|\mathbf{Y}_T - \mathbf{X}_T \boldsymbol{\beta}^*(1)\|_2 \|\mathbf{Y}_T - \mathbf{X}_T \boldsymbol{\beta}^*(0)\|_2 + \langle \mathbf{Y}_T - \mathbf{X}_T \boldsymbol{\beta}^*(1), \mathbf{Y}_T - \mathbf{X}_T \boldsymbol{\beta}^*(0) \rangle) \\
&= \frac{4}{T} \mathbb{E} (\|\mathbf{Y}_T - \mathbf{X}_T \boldsymbol{\beta}^*(1)\|_2^2) \\
&= 4 \mathbb{E} (\mathbf{Y}_T^\top (\mathbf{I}_T - \mathbf{X}_T (\mathbf{X}_T^\top \mathbf{X}_T)^{-1} \mathbf{X}_T^\top) \mathbf{Y}_N) \\
&= \frac{4}{T} \mathbb{E} \left[\sum_{t=1}^T y_t^2(1) \right] - \frac{4}{T} \mathbb{E}_{\mathbf{x}, \mathbf{y}} [\mathbf{Y}_T^\top \mathbf{X}_T (\mathbf{X}_T^\top \mathbf{X}_T)^{-1} \mathbf{X}_T^\top \mathbf{Y}_T] \\
&= 4 - \frac{4}{T} \mathbb{E} [\text{tr} (\mathbf{Y}_T^\top \mathbf{X}_T (\mathbf{X}_T^\top \mathbf{X}_T)^{-1} \mathbf{X}_T^\top \mathbf{Y}_T)] \\
&= 4 - \frac{4}{T} \text{tr} (\mathbf{X}_T (\mathbf{X}_T^\top \mathbf{X}_T)^{-1} \mathbf{X}_T^\top \mathbb{E}_{\mathbf{y}} [\mathbf{Y}_T \mathbf{Y}_T^\top]) \\
&= 4 - \frac{4}{T} \text{tr} (\mathbf{X}_T (\mathbf{X}_T^\top \mathbf{X}_T)^{-1} \mathbf{X}_T^\top \cdot \mathbf{I}_T) \\
&= 4 - 4T^{-1} \text{tr} (\mathbf{X}_T^\top \mathbf{X}_T (\mathbf{X}_T^\top \mathbf{X}_T)^{-1}) \\
&= 4 - 4T^{-1} \cdot T/2 \\
&= 2 .
\end{aligned}$$

Hence we have proved that

$$\mathbb{E} [T \cdot \text{Var}(\hat{\tau}) - T \cdot V^*] \geq 2 .$$

It is easy to verify that Assumption 1 and 3 are satisfied for any sequence $\{y_t(1), y_t(0), \mathbf{x}_t\}$ generated by this mechanism. Hence there exists $c > 0$ and one specific choice of $\{y_t(1), y_t(0), \mathbf{x}_t\}$ such that

$$\mathcal{R}_T^{\text{Neyman}} \geq c.$$

□

B.2 Technical Lemmas

In this section, we present several technical lemmas that will be used to establish the main results in the subsequent sections.

Lemma B.3 provides upper bounds for the spectral norms of the matrices involved in bounding the regression coefficients.

Lemma B.3. *Under Assumption 2, for any $t \in [T]$, there holds:*

$$\begin{aligned}
\|(\mathbf{X}_{t-1}^\top \mathbf{X}_{t-1} + \eta_t^{-1} \mathbf{I}_d)^{-1}\|_2 &\leq (\gamma_0 \vee c_2 \vee 1)((t-1) \vee \eta_t^{-1})^{-1} , \\
\|\mathbf{X}_{t-1} (\mathbf{X}_{t-1}^\top \mathbf{X}_{t-1} + \eta_t^{-1} \mathbf{I}_d)^{-2} \mathbf{X}_{t-1}^\top\|_2 &\leq \left(\frac{\gamma_0}{4} \vee c_2\right) (t-1)^{-1} .
\end{aligned}$$

Proof. For $t \leq \gamma_0 \cdot \eta_t^{-1}$, there holds $\|(\mathbf{X}_{t-1}^\top \mathbf{X}_{t-1} + \eta_t^{-1} \mathbf{I}_d)^{-1}\|_2 \leq \eta_t$. For $t \geq \gamma_0 \cdot \eta_t^{-1} + 1 \geq \gamma_0 \cdot T^{1/2} + 1$, by Assumption 2 we have $\|(\mathbf{X}_{t-1}^\top \mathbf{X}_{t-1} + \eta_t^{-1} \mathbf{I}_d)^{-1}\|_2 \leq c_2(t-1)^{-1}$, hence for any $t \in [T]$, we have $\|(\mathbf{X}_{t-1}^\top \mathbf{X}_{t-1} + \eta_t^{-1} \mathbf{I}_d)^{-1}\|_2 \leq (\gamma_0 \vee c_2 \vee 1)((t-1) \vee \eta_t^{-1})^{-1}$. Suppose the d eigenvalues of matrix $\mathbf{X}_{t-1}^\top \mathbf{X}_{t-1}$ are as $\lambda_1 \geq \dots \geq \lambda_d$. Then by Assumption 2, for any $t \in [T]$, we have

$$\|\mathbf{X}_{t-1} (\mathbf{X}_{t-1}^\top \mathbf{X}_{t-1} + \eta_t^{-1} \mathbf{I}_d)^{-2} \mathbf{X}_{t-1}^\top\|_2 = \|(\mathbf{X}_{t-1}^\top \mathbf{X}_{t-1} + \eta_t^{-1} \mathbf{I}_d)^{-2} \mathbf{X}_{t-1}^\top \mathbf{X}_{t-1}\|_2$$

$$\begin{aligned}
&\leq \max_{k=1,\dots,d} \frac{\lambda_k}{(\lambda_k + \eta_t^{-1})^2} \\
&\leq \begin{cases} \frac{1}{4}\eta_t & \text{if } t-1 < \gamma_0 \cdot \eta_t^{-1} \\ c_2(t-1)^{-1} & \text{if } t-1 \geq \gamma_0 \cdot \eta_t^{-1} \end{cases} \\
&\leq \left(\frac{\gamma_0}{4} \vee c_2 \right) (t-1)^{-1}.
\end{aligned}$$

□

Based on Lemma B.3, we provide the following upper bound on $\Pi_{t,s}$.

Corollary B.4. *Under Assumption 2, for any $1 \leq s \leq t \leq T$, there holds $|\Pi_{t,s}| \leq (\gamma_0 \vee c_2 \vee 1)R_t R_s((t-1) \vee \eta_t^{-1})^{-1}$.*

Proof. By Lemma B.3 and the definition of R_t , we have

$$\begin{aligned}
|\Pi_{t,s}| &= \left| \mathbf{x}_t^\top (\mathbf{X}_{t-1}^\top \mathbf{X}_{t-1} + \eta_t^{-1} \mathbf{I}_d)^{-1} \mathbf{x}_s \right| \\
&\leq \|\mathbf{x}_t\|_2 \|\mathbf{x}_s\|_2 \|(\mathbf{X}_{t-1}^\top \mathbf{X}_{t-1} + \eta_t^{-1} \mathbf{I}_d)^{-1}\|_2 \\
&\leq (\gamma_0 \vee c_2 \vee 1)R_t R_s((t-1) \vee \eta_t^{-1})^{-1}.
\end{aligned}$$

□

Our proposed algorithm does not guarantee that $R_T = R$. The following lemma shows that $R_T = \mathcal{O}(R)$, which allows us to use R_T as a proxy of R in several established upper bounds.

Lemma B.5. *Under Assumption 2, the radius satisfies: $R_T \leq \max^{1/2}\{c_1, 1\}R$.*

Proof. Since we have

$$\lambda_{\min}(\mathbf{X}_T^\top \mathbf{X}_T) \leq \text{tr}(\mathbf{X}_T^\top \mathbf{X}_T) = \text{tr} \left(\sum_{t=1}^T \mathbf{x}_t \mathbf{x}_t^\top \right) = \sum_{t=1}^T \text{tr}(\mathbf{x}_t \mathbf{x}_t^\top) = \sum_{t=1}^T \|\mathbf{x}_t\|_2^2 \leq TR^2,$$

which implies that $R \geq c_2^{-1/2}$ by Assumption 2. If $R \geq 1$, then R_T is guaranteed to be bounded by R . If $c_2^{-1/2} \leq R \leq 1$, then $R_T = 1$. Hence we have $R_T \leq \max^{1/2}\{c_1, 1\}R$. □

Lemma 4.12. *For each iteration $t \in [T]$ and treatment $k \in \{0, 1\}$, the adaptive linear predictors are conditionally unbiased for the full-information predictors: $E[\beta_t(k) | \mathcal{F}_{t-1}] = \beta_t^*(k)$ a.s.*

Proof. For simplicity, we only prove the result for $k = 1$. By direct calculation, we have

$$\begin{aligned}
E[\beta_t(1) | \mathcal{F}_{t-1}] &= E \left[(\mathbf{X}_{t-1}^\top \mathbf{X}_{t-1} + \eta_t^{-1} \mathbf{I}_d) \sum_{s=1}^{t-1} y_s(1) \frac{\mathbf{1}[Z_s = 1]}{p_s} \cdot \mathbf{x}_s \middle| \mathcal{F}_{t-1} \right] \\
&= (\mathbf{X}_{t-1}^\top \mathbf{X}_{t-1} + \eta_t^{-1} \mathbf{I}_d) \sum_{s=1}^{t-1} y_s(1) \cdot \mathbf{x}_s \\
&= \beta_t^*(1).
\end{aligned}$$

□

We now state the well-known Hardy's inequality, which will be used to bound the fourth moment of the online residuals in Lemma B.7.

Proposition B.6 (Hardy's inequality). *If a_1, \dots, a_n is a sequence of non-negative real numbers, then for every real number $p > 1$ one has*

$$\sum_{k=1}^n \left(\frac{1}{k} \sum_{r=1}^k a_r \right)^p \leq \left(\frac{p}{p-1} \right)^p \sum_{k=1}^n a_k^p .$$

Lemma B.7. *Under Assumption 1-2, for $k \in \{0, 1\}$, there holds:*

$$\sum_{t=1}^T \eta_t (y_t(k) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(k) \rangle)^4 \leq c_1^4 (1 + 2^{-1/2} (\gamma_0 \vee 4c_2)^{1/2})^4 R_T^2 T^{1/2} .$$

Proof. Without loss of generality, we only prove the result for $k = 1$. By Hölder's inequality, we have

$$\begin{aligned} & \sum_{t=1}^T \eta_t (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^4 \\ &= \sum_{t=1}^T \eta_t [y_t^4(1) - 4y_t^3(1)\langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle + 6y_t^2(1)\langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle^2 - 4y_t(1)\langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle^3 + \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle^4]^4 \\ &\leq \sum_{t=1}^T \eta_t y_t^4(1) + 4 \left(\sum_{t=1}^T \eta_t y_t^4(1) \right)^{3/4} \left(\sum_{t=1}^T \eta_t \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle^4 \right)^{1/4} + 6 \left(\sum_{t=1}^T \eta_t y_t^4(1) \right)^{1/2} \left(\sum_{t=1}^T \eta_t \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle^4 \right)^{1/2} \\ &\quad + 4 \left(\sum_{t=1}^T \eta_t y_t^4(1) \right)^{1/4} \left(\sum_{t=1}^T \eta_t \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle^4 \right)^{3/4} + \left(\sum_{t=1}^T \eta_t \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle^4 \right) \\ &\leq \left[\left(\sum_{t=1}^T \eta_t y_t^4(1) \right)^{1/4} + \left(\sum_{t=1}^T \eta_t \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle^4 \right)^{1/4} \right]^4 \quad (\text{Hölder's inequality}) . \end{aligned} \tag{2}$$

Denote $\mathbf{Y}_{t-1} = (y_1(1), \dots, y_{t-1}(1))^\top$. By Lemma B.3, for any $t \in [T]$, we have

$$\begin{aligned} \|\boldsymbol{\beta}_t^*(1)\|_2^2 &= \mathbf{Y}_{t-1}^\top \mathbf{X}_{t-1} (\mathbf{X}_{t-1}^\top \mathbf{X}_{t-1} + \eta_t^{-1} \mathbf{I}_d)^{-2} \mathbf{X}_{t-1}^\top \mathbf{Y}_{t-1} \\ &\leq \|\mathbf{Y}_{t-1}\|_2^2 \left\| \mathbf{X}_{t-1} (\mathbf{X}_{t-1}^\top \mathbf{X}_{t-1} + \eta_t^{-1} \mathbf{I}_d)^{-2} \mathbf{X}_{t-1}^\top \right\|_2 \\ &\leq \left(\frac{\gamma_0}{4} \vee c_2 \right) (t-1)^{-1} \left(\sum_{s=1}^{t-1} y_s^2(1) \right) \quad (\text{Lemma B.3}) . \end{aligned} \tag{3}$$

By (3), Hardy's inequality (Proposition B.6) and Assumption 1, we have

$$\begin{aligned} \sum_{t=1}^T \eta_t \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle^4 &\leq T^{-1/2} \sum_{t=1}^T R_t^{-2} \|\mathbf{x}_t\|_2^4 \|\boldsymbol{\beta}_t^*(1)\|_2^4 \\ &\leq T^{-1/2} \sum_{t=1}^T R_t^2 \|\boldsymbol{\beta}_t^*(1)\|_2^4 \\ &\leq \left(\frac{\gamma_0}{4} \vee c_2 \right)^2 R_T^2 T^{-1/2} \sum_{t=2}^T \left(\frac{1}{t-1} \sum_{s=1}^{t-1} y_s^2(1) \right)^2 \quad (\text{by (3) and } R_t \leq R_T) \\ &\leq \left(\frac{\gamma_0}{4} \vee c_2 \right)^2 R_T^2 T^{-1/2} \left(\frac{2}{2-1} \right)^2 \sum_{t=1}^{T-1} y_t^4(1) \quad (\text{Hardy's inequality}) \\ &\leq \frac{1}{4} c_1^4 (\gamma_0 \vee 4c_2)^2 R_T^2 T^{1/2} \quad (\text{Assumption 1}) . \end{aligned} \tag{4}$$

By Assumption 1, it is easy to see that

$$\sum_{t=1}^T \eta_t y_t^4(1) \leq T^{-1/2} \cdot c_1^4 T = c_1^4 T^{1/2}. \quad (5)$$

Hence by (2), (4) and (5), we can prove that

$$\begin{aligned} \sum_{t=1}^T \eta_t (y_t(1) - \langle \mathbf{x}_t, \beta_t^*(1) \rangle)^4 &\leq \left(c_1 T^{1/8} + 2^{-1/2} c_1 (\gamma_0 \vee 4c_2)^{1/2} R_T^{1/2} T^{1/8} \right)^4 \\ &= \left(1 + 2^{-1/2} (\gamma_0 \vee 4c_2)^{1/2} R_T^{1/2} \right)^4 c_1^4 T^{1/2} \\ &= \left(R_T^{-1/2} + 2^{-1/2} (\gamma_0 \vee 4c_2)^{1/2} \right)^4 c_1^4 R_T^2 T^{1/2} \\ &\leq c_1^4 \left(1 + 2^{-1/2} (\gamma_0 \vee 4c_2)^{1/2} \right)^4 R_T^2 T^{1/2}. \end{aligned}$$

□

The following corollary is a direct consequence of Lemma B.7.

Corollary B.8. *Under Assumption 1-2, for $k \in \{0, 1\}$, there holds:*

$$\sum_{t=1}^T (y_t(k) - \langle \mathbf{x}_t, \beta_t^*(k) \rangle)^4 \leq c_1^4 \left(1 + 2^{-1/2} (\gamma_0 \vee 4c_2)^{1/2} \right)^4 R_T^4 T.$$

Proof. For any $t \in [T]$, we have $\eta_t^{-1} \leq \eta_T^{-1} = T^{1/2} R_T^2$. Hence by Lemma B.7, we have

$$\begin{aligned} \sum_{t=1}^T (y_t(k) - \langle \mathbf{x}_t, \beta_t^*(k) \rangle)^4 &\leq \eta_T^{-1} \sum_{t=1}^T \eta_t (y_t(k) - \langle \mathbf{x}_t, \beta_t^*(k) \rangle)^4 \\ &\leq c_1^4 \left(1 + 2^{-1/2} (\gamma_0 \vee 4c_2)^{1/2} \right)^4 R_T^4 T. \end{aligned}$$

□

Lemmas B.9 and B.10 provide upper bounds for deterministic p -series summations, which play a crucial role in establishing Lemma B.15.

Lemma B.9. *For any $k > 1$ and fixed $t, r \in [T]$, we have $\sum_{s=t+1}^T ((s-1) \vee \eta_r^{-1})^{-k} \leq \xi_k (t \vee \eta_r^{-1})^{-(k-1)}$, where $\xi_k \triangleq \frac{2^{k-1}}{k-1} + 1 > 0$ is a constant.*

Proof. For $t \geq \lceil \eta_r^{-1} \rceil + 1 \geq T^{1/2} + 1 \geq 2$, we have $(t-1) \geq t/2$. Hence we have

$$\begin{aligned} \sum_{s=t+1}^T ((s-1) \vee \eta_r^{-1})^{-k} &\leq \sum_{s=t}^{T-1} s^{-k} \\ &\leq \int_{t-1}^{T-2} s^{-k} ds \\ &= \frac{1}{k-1} \left[(t-1)^{-(k-1)} - (T-2)^{-(k-1)} \right] \\ &\leq \frac{(t-1)^{-(k-1)}}{k-1} \\ &\leq \frac{2^{k-1} t^{-(k-1)}}{k-1}. \end{aligned}$$

For $t < \lceil \eta_r^{-1} \rceil + 1$, we have

$$\begin{aligned} \sum_{s=t+1}^T ((s-1) \vee \eta_r^{-1})^{-k} &\leq \sum_{s=\lceil \eta_r^{-1} \rceil + 1}^T ((s-1) \vee \eta_r^{-1})^{-k} + \sum_{s=2}^{\lceil \eta_r^{-1} \rceil} \eta_r^k \\ &\leq \frac{2^{k-1} \eta_r^{k-1}}{k-1} + \eta_r^{k-1} \\ &\leq \left(\frac{2^{k-1}}{k-1} + 1 \right) \eta_r^{k-1}. \end{aligned}$$

Hence for any fixed $t, r \in [T]$, we have

$$\sum_{s=t+1}^T ((s-1) \vee \eta_r^{-1})^{-k} \leq \xi_k(t \vee \eta_r^{-1})^{-(k-1)}.$$

□

Lemma B.10. *For any $k, \nu \geq 0$ such that $0 < 2k - \nu < 2$, when T is large enough, there holds:*

$$\sum_{t=1}^T R_t^\nu((t-1) \vee \eta_t^{-1})^{-k} \leq \frac{T^{1-k+\nu/4}}{1 - (k - \nu/2)}.$$

Proof. For any $t \in [T]$, a natural upper bound for $R_t^\nu((t-1) \vee \eta_t^{-1})^{-k}$ is $R_t^\nu((t-1) \vee \eta_t^{-1})^{-k} \leq R_t^\nu \eta_t^k = T^{-k/2} R_t^{-(2k-\nu)} \leq T^{-k/2}$. We then derive another upper bound for $R_t^\nu((t-1) \vee \eta_t^{-1})^{-k}$:

(1) If $(t-1) \geq T^{1/2} R_t^2$, i.e., $1 \leq R_t \leq ((t-1)T^{-1/2})^{1/2}$, then we have

$$\begin{aligned} R_t^\nu((t-1) \vee T^{1/2} R_t^2)^{-k} &= R_t^\nu(t-1)^{-k} \\ &\leq ((t-1)T^{-1/2})^{\nu/2} (t-1)^{-k} \\ &= (t-1)^{-(k-\nu/2)} T^{-\nu/4}. \end{aligned}$$

(2) If $(t-1) \leq T^{1/2} R_t^2$, i.e., $R_t \geq ((t-1)T^{-1/2})^{1/2}$, then we have

$$\begin{aligned} R_t^\nu((t-1) \vee T^{1/2} R_t^2)^{-k} &= R_t^\nu(T^{1/2} R_t^2)^{-k} \\ &= T^{-k/2} R_t^{-(2k-\nu)} \\ &\leq T^{-k/2} ((t-1)T^{-1/2})^{-(2k-\nu)/2} \\ &= (t-1)^{-(k-\nu/2)} T^{-\nu/4}. \end{aligned}$$

Since $0 < 1 - (k - \nu/2) < 1$ by assumption, when T is large enough, we have

$$\frac{(\lfloor T^{1/2} \rfloor - 1)^{1-(k-\nu/2)}}{1 - (k - \nu/2)} = \frac{1}{1 - (k - \nu/2)} \left(\frac{\lfloor T^{1/2} \rfloor - 1}{T^{1/2}} \right)^{1-(k-\nu/2)} \cdot T^{1/2 \cdot (1-(k-\nu/2))} \geq T^{1/2 \cdot (1-(k-\nu/2))}.$$

This implies that

$$\begin{aligned} \sum_{t=1}^T R_t^\nu((t-1) \vee \eta_t^{-1})^{-k} &\leq \sum_{t=1}^T T^{-k/2} \wedge (t-1)^{-(k-\nu/2)} T^{-\nu/4} \\ &\leq \sum_{t=1}^{\lfloor T^{1/2} \rfloor} T^{-k/2} + \sum_{\lfloor T^{1/2} \rfloor + 1}^T (t-1)^{-(k-\nu/2)} T^{-\nu/4} \\ &\leq T^{-k/2} \cdot \lfloor T^{1/2} \rfloor + T^{-\nu/4} \int_{\lfloor T^{1/2} \rfloor - 1}^{T-2} t^{-(k-\nu/2)} dt \end{aligned}$$

$$\begin{aligned}
&\leq T^{-k/2} \cdot [T^{1/2}] + \frac{T^{-\nu/4}}{1 - (k - \nu/2)} \left[(T - 2)^{1-(k-\nu/2)} - \left([T^{1/2}] - 1 \right)^{1-(k-\nu/2)} \right] \\
&\leq T^{-(k-1)/2} - \frac{T^{-\nu/4} ([T^{1/2}] - 1)^{1-(k-\nu/2)}}{1 - (k - \nu/2)} + \frac{T^{-\nu/4+1-(k-\nu/2)}}{1 - (k - \nu/2)} \\
&\leq T^{-(k-1)/2} - T^{-\nu/4} \cdot T^{1/2 \cdot (1-(k-\nu/2))} + \frac{T^{-\nu/4+1-(k-\nu/2)}}{1 - (k - \nu/2)} \\
&= \frac{T^{1-k+\nu/4}}{1 - (k - \nu/2)} .
\end{aligned}$$

□

The following Lemma B.11 has similar proof as in Lemma 4.9, Lemma 4.10 and Proposition 4.11. However, we present an alternative proof that does not rely on the fourth-moment condition in Assumption 1. Throughout the proofs of Lemma B.11, Corollary B.12 and Lemma B.13, we instead impose only the second-moment assumption (1*):

$$\sum_{t=1}^T y_t^2(k) \leq c_1^2 T .$$

This assumption is weaker than the Assumption 1, which plays a crucial role in establishing Lemma B.13.

Lemma B.11. *Under Assumption 1-3, for $k \in \{0, 1\}$, there holds $A_T^*(k) \leq 2c_1^2 T$ for T large enough.*

Proof. Without loss of generality, we only prove the result for $k = 1$. By the second moment assumption (1*), we can easily see that $\sum_{t=1}^T (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^2 \leq \sum_{t=1}^T y_t^2(1) \leq c_1^2 T$. By Corollary B.4, Assumption 2, and similar proofs as in Lemma 4.9 and Lemma 4.10, we have

$$\begin{aligned}
&\sum_{t=1}^T (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^2 - \sum_{t=1}^T (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^2 \\
&\leq \eta_{T+1}^{-1} \|\boldsymbol{\beta}^*(1)\|_2^2 + \sum_{t=1}^T (\tilde{L}_{t+1}^{(1)}(\boldsymbol{\beta}_t^*(1)) - \tilde{L}_{t+1}^{(1)}(\tilde{\boldsymbol{\beta}}_{t+1}^*(1))) \quad (\text{similar proof as in Lemma 4.9}) \\
&\lesssim \eta_{T+1}^{-1} \|\mathbf{Y}_T(1)\|_2^2 \left\| \mathbf{X}_T (\mathbf{X}_T^\top \mathbf{X}_T)^{-2} \mathbf{X}_T^\top \right\|_2 + \sum_{t=1}^T \Pi_{t,t} (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^2 \quad (\text{similar proof as in Lemma 4.10}) \\
&\lesssim \eta_T^{-1} + R_t^2 \eta_t \sum_{t=1}^T (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^2 \quad (\text{Corollary B.4 and Assumption 2}) . \tag{6}
\end{aligned}$$

Note that $R_t^2 \eta_t = T^{-1/2} = o(1)$ and $\eta_T^{-1} = T^{1/2} R_T^2 = o(T)$ by Assumption 3. Hence (6) implies that $A_T^*(1) = \sum_{t=1}^T (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^2 \leq 2 \sum_{t=1}^T (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^2 \leq 2c_1^2 T$ when T is large enough. □

The following corollary is a direct result of Lemma B.11.

Corollary B.12. *Under Assumption 1-3, for $k \in \{0, 1\}$, there holds: $\sum_{t=1}^T \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle^2 \leq (2^{1/2} + 1)^2 c_1^2 T$ for T large enough.*

Proof. Without loss of generality, we only prove the result for $k = 1$. By Lemma B.11 and Cauchy-Schwarz inequality, for large enough T , we have

$$2c_1^2 T \geq \sum_{t=1}^T (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^2 \quad (\text{Lemma B.11})$$

$$\begin{aligned}
&= \sum_{t=1}^T y_t^2 - 2 \sum_{t=1}^T y_t(1) \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle + \sum_{t=1}^T \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle^2 \\
&\geq \sum_{t=1}^T y_t^2 - 2 \left(\sum_{t=1}^T y_t^2 \right)^{1/2} \left(\sum_{t=1}^T \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle^2 \right)^{1/2} + \sum_{t=1}^T \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle^2 \quad (\text{Cauchy-Schwarz inequality}) \\
&= \left[\left(\sum_{t=1}^T y_t^2 \right)^{1/2} - \left(\sum_{t=1}^T \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle^2 \right)^{1/2} \right]^2.
\end{aligned}$$

By the second moment assumption (1*), we have $\left(\sum_{t=1}^T y_t^2\right)^{1/2} \leq c_1 T^{1/2}$. Hence it is easy to see that $\left(\sum_{t=1}^T \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle^2\right)^{1/2} \leq (2^{1/2} + 1)c_1 T^{1/2}$, which indicates that $\sum_{t=1}^T \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle^2 \leq (2^{1/2} + 1)^2 c_1^2 T$ for T large enough. \square

Based on Corollary B.12, we can establish the following lemma.

Lemma B.13. *Suppose T is large enough. Under Assumption 1-3, for any $2 \leq t \leq T$, let the entries in matrix $\check{\mathbf{Q}}^{(t)} = (\check{Q}_{i,j}^{(t)}) \in \mathbb{R}_{(t-1) \times (t-1)}$ be $\check{Q}_{i,j}^{(t)} = \sum_{s=i \vee j+1}^t \Pi_{s,i} \Pi_{s,j}$ for any $1 \leq i, j \leq t-1$. Then $\check{\mathbf{Q}}^{(t)}$ is a semi-definite positive matrix and there holds: $\|\check{\mathbf{Q}}^{(t)}\|_2 \leq (2^{1/2} + 1)^2$.*

Proof. By direct calculation, we have

$$\begin{aligned}
\sum_{s=1}^T \langle \mathbf{x}_s, \boldsymbol{\beta}_s^*(1) \rangle^2 &\geq \sum_{s=1}^t \langle \mathbf{x}_s, \boldsymbol{\beta}_s^*(1) \rangle^2 \\
&= \sum_{s=1}^t \left(\sum_{r=1}^{s-1} \Pi_{s,r} y_r(1) \right)^2 \\
&= \sum_{s=1}^t \sum_{1 \leq r_1, r_2 \leq s-1} \Pi_{s,r_1} \Pi_{s,r_2} y_{r_1}(1) y_{r_2}(1) \\
&= \sum_{1 \leq r_1, r_2 \leq t-1} \left(\sum_{s=r_1 \vee r_2+1}^t \Pi_{s,r_1} \Pi_{s,r_2} \right) y_{r_1}(1) y_{r_2}(1) \\
&= \sum_{r_1=1}^{t-1} \sum_{r_2=1}^{t-1} \check{Q}_{r_1, r_2}^{(t)} y_{r_1}(1) y_{r_2}(1).
\end{aligned}$$

Note that the proof in Lemma B.11 and Corollary B.12 only rely on the second moment assumption (1*): $\sum_{t=1}^T y_t^2(1) \leq c_1^2 T$ (we do not use the fourth moment assumption). Hence for T large enough, we have

- (1) $\sum_{s_1=1}^{t-1} \sum_{s_2=1}^{t-1} \check{Q}_{s_1, s_2}^{(t)} y_{s_1}(1) y_{s_2}(1) \geq 0$, which implies that $\check{\mathbf{Q}}^{(t)}$ is semi-definite positive.
- (2) $\sum_{s_1=1}^{t-1} \sum_{s_2=1}^{t-1} \check{Q}_{s_1, s_2}^{(t)} y_{s_1}(1) y_{s_2}(1) \leq (2^{1/2} + 1)^2 c_1^2 T$, which implies that $\|\check{\mathbf{Q}}^{(t)}\|_2 \leq (2^{1/2} + 1)^2$. \square

Based on Lemma B.13, we can derive the following corollary.

Corollary B.14. *Suppose T is large enough. Under Assumption 1-3, for a fixed $t \in [T]$, and any $1 \leq s \leq t$, let the entries in matrix $\mathbf{Q}^{(s,t)} = (Q_{i,j}^{(s,t)}) \in \mathbb{R}_{(t-s) \times (t-s)}$ be $Q_{i,j}^{(s,t)} = (\sum_{r=i \vee j+s}^t \Pi_{r,i+s-1} \Pi_{r,j+s-1})^2 \left(\frac{R_t}{R_{i+s-1}} \cdot \frac{R_t}{R_{j+s-1}} \right)^{1/2}$ for any $1 \leq i, j \leq t-s$. Then $\mathbf{Q}^{(s,t)}$ is a semi-definite positive matrix and there holds: $\|\mathbf{Q}^{(s,t)}\|_2 \leq 2(2^{1/2} + 1)^2 (\gamma_0 \vee c_2 \vee 1)^2 R_t^4 (s \vee \eta_t^{-1})^{-1}$.*

Proof. For any $1 \leq s \leq t \leq T$, let the entries in matrix $\tilde{\mathbf{Q}}^{(s,t)} = (\tilde{Q}_{i,j}^{(s,t)}) \in \mathbb{R}_{(t-s) \times (t-s)}$ be $\tilde{Q}_{i,j}^{(s,t)} = (\sum_{r=i \vee j+s}^t \Pi_{r,i+s-1} \Pi_{r,j+s-1})$ for any $1 \leq i, j \leq t-s$ and let the entries in matrix $\bar{\mathbf{Q}}^{(s,t)} = (\bar{Q}_{i,j}^{(s,t)}) \in \mathbb{R}_{(t-s) \times (t-s)}$ be $\bar{Q}_{i,j}^{(s,t)} = (\sum_{r=i \vee j+s}^t \Pi_{r,i+s-1} \Pi_{r,j+s-1}) \left(\frac{R_t}{R_{i+s-1}} \cdot \frac{R_t}{R_{j+s-1}} \right)^{1/2}$ for any $1 \leq i, j \leq t-s$. Since $\tilde{\mathbf{Q}}^{(s,t)}$ is a principal submatrix of semi-definite positive matrix $\check{\mathbf{Q}}^{(t)}, \tilde{\mathbf{Q}}^{(s,t)}$ is semi-definite positive. By the definition of $\bar{\mathbf{Q}}^{(s,t)}$, we have

$$\bar{\mathbf{Q}}^{(s,t)} = \text{diag} \left\{ \left(\frac{R_t}{R_s} \right)^{1/2}, \dots, \left(\frac{R_t}{R_{t-1}} \right)^{1/2} \right\} \tilde{\mathbf{Q}}^{(s,t)} \text{diag} \left\{ \left(\frac{R_t}{R_{t_1}} \right)^{1/2}, \dots, \left(\frac{R_t}{R_{t-1}} \right)^{1/2} \right\} .$$

Hence $\bar{\mathbf{Q}}^{(s,t)}$ is also semi-definite positive. This implies that $\mathbf{Q}^{(s,t)} = \tilde{\mathbf{Q}}^{(s,t)} \circ \bar{\mathbf{Q}}^{(s,t)}$ is semi-definite positive. Now we prove the second part. For any $1 \leq k \leq t-1$, we aim to bound the k -th diagonal element of $\bar{\mathbf{Q}}^{(1,t)}$. We derive the upper in two different cases. If $1 \leq k \leq \eta_t^{-1}$, we have

$$\begin{aligned} \bar{Q}_{k,k}^{(1,t)} &= \frac{R_t}{R_k} \cdot \sum_{s=k+1}^t \Pi_{s,k}^2 \\ &\leq (\gamma_0 \vee c_2 \vee 1)^2 \frac{R_t}{R_k} \cdot \sum_{s=k+1}^t R_s^2 R_k^2 ((s-1) \vee \eta_s^{-1})^{-2} \quad (\text{Corollary B.4}) \\ &\leq (\gamma_0 \vee c_2 \vee 1)^2 R_t R_k \sum_{s=k+1}^{\lfloor \eta_t^{-1} \rfloor + 2} R_s^2 ((s-1) \vee \eta_s^{-1})^{-2} + (\gamma_0 \vee c_2 \vee 1)^2 R_t \sum_{s=\lfloor \eta_t^{-1} \rfloor + 3}^t R_s^3 ((s-1) \vee \eta_s^{-1})^{-2} \\ &\triangleq S_1 + S_2 . \end{aligned} \tag{7}$$

By calculation, it is easy to see that function $x^{-1/2}(2 + \log x)$ has maximum value 2 on $[1, \infty)$. Hence we have

$$\begin{aligned} S_1 &\leq (\gamma_0 \vee c_2 \vee 1)^2 T^{-1/2} R_t R_k \sum_{s=k+1}^{\lfloor \eta_t^{-1} \rfloor + 2} \eta_s^{-1} ((s-1) \vee \eta_s^{-1})^{-2} \\ &\leq (\gamma_0 \vee c_2 \vee 1)^2 R_t^2 \cdot \frac{R_k}{R_t} T^{-1/2} \sum_{s=k+1}^{\lfloor \eta_t^{-1} \rfloor + 2} ((s-1) \vee \eta_s^{-1})^{-1} \\ &\leq (\gamma_0 \vee c_2 \vee 1)^2 R_t^2 \cdot \frac{R_k}{R_t} T^{-1/2} \left(\sum_{s=1}^{\lfloor \eta_{k+1}^{-1} \rfloor + 2} \eta_{k+1} + \sum_{s=\lfloor \eta_{k+1}^{-1} \rfloor + 3}^{\lfloor \eta_t^{-1} \rfloor + 2} (s-1)^{-1} \right) \\ &\leq (\gamma_0 \vee c_2 \vee 1)^2 R_t^2 \cdot \frac{R_k}{R_t} T^{-1/2} \left(\sum_{s=1}^{\lfloor \eta_{k+1}^{-1} \rfloor + 2} \eta_{k+1} + \int_{\lfloor \eta_{k+1}^{-1} \rfloor + 1}^{\lfloor \eta_t^{-1} \rfloor} s^{-1} ds \right) \\ &\leq (\gamma_0 \vee c_2 \vee 1)^2 R_t^2 \cdot \frac{R_k}{R_t} T^{-1/2} (\eta_{k+1} (\lfloor \eta_{k+1}^{-1} \rfloor + 2) + \log(\lfloor \eta_t^{-1} \rfloor) - \log(\lfloor \eta_{k+1}^{-1} \rfloor + 1)) \\ &\leq (\gamma_0 \vee c_2 \vee 1)^2 R_t^2 \cdot \left(\frac{\eta_t}{\eta_k} \right)^{1/2} T^{-1/2} (2 + \log(\lfloor \eta_t^{-1} \rfloor \eta_t)) \quad (\text{since } T \text{ large enough and } \eta_{k+1} \leq \eta_k) \\ &= (\gamma_0 \vee c_2 \vee 1)^2 R_t^2 \cdot \left(\frac{\eta_t}{\eta_k} \right)^{1/2} T^{-1/2} \left(2 + \log(\lfloor \eta_t^{-1} \rfloor \eta_t) + \log \left(\frac{\eta_k}{\eta_t} \right) \right) \\ &\leq (\gamma_0 \vee c_2 \vee 1)^2 R_t^2 \cdot \left(\frac{\eta_t}{\eta_k} \right)^{1/2} T^{-1/2} \left(2 + \log \left(\frac{\eta_k}{\eta_t} \right) \right) \\ &\leq 2(\gamma_0 \vee c_2 \vee 1)^2 R_t^2 T^{-1/2} \quad (\text{since } \eta_k/\eta_t \geq 1) \end{aligned}$$

$$= 2(\gamma_0 \vee c_2 \vee 1)^2 R_t^4 \eta_t . \quad (8)$$

By direct calculation, we have

$$\begin{aligned} S_2 &\leq (\gamma_0 \vee c_2 \vee 1)^2 R_t \sum_{s=\lceil \eta_t^{-1} \rceil + 3}^t ((s-1)R_s^{-2} \vee T^{1/2})^{-3/2} ((s-1) \vee \eta_s^{-1})^{-1/2} \\ &\leq (\gamma_0 \vee c_2 \vee 1)^2 R_t \sum_{s=\lceil \eta_t^{-1} \rceil + 3}^t ((s-1)R_t^{-2} \vee T^{1/2})^{-3/2} \cdot (s-1)^{-1/2} \\ &\leq (\gamma_0 \vee c_2 \vee 1)^2 R_t^4 \eta_t^{1/2} \sum_{s=\lceil \eta_t^{-1} \rceil + 3}^t ((s-1) \vee \eta_t^{-1})^{-3/2} \\ &\leq (\gamma_0 \vee c_2 \vee 1)^2 R_t^4 \eta_t^{1/2} \sum_{s=\lceil \eta_t^{-1} \rceil + 3}^t (s-1)^{-3/2} \\ &\leq (\gamma_0 \vee c_2 \vee 1)^2 R_t^4 \eta_t^{1/2} \int_{\lceil \eta_t^{-1} \rceil + 1}^{t-2} (s-1)^{-3/2} ds \\ &\leq 2(\gamma_0 \vee c_2 \vee 1)^2 R_t^4 \eta_t^{1/2} (\lceil \eta_t^{-1} \rceil + 1)^{-1/2} \\ &\leq 2(\gamma_0 \vee c_2 \vee 1)^2 R_t^4 \eta_t . \end{aligned} \quad (9)$$

Hence by (7), (8) and (9), for any $1 \leq k \leq \lceil \eta_t^{-1} \rceil$, we can derive the following upper bound:

$$\bar{Q}_{k,k}^{(1,t)} \leq S_1 + S_2 \leq 4(\gamma_0 \vee c_2 \vee 1)^2 R_t^4 \eta_t . \quad (10)$$

For any $\eta_t^{-1} \leq k \leq t$, by Corollary B.4 we have

$$\begin{aligned} \bar{Q}_{k,k}^{(1,t)} &= \frac{R_t}{R_k} \cdot \sum_{s=k+1}^t \Pi_{s,k}^2 \\ &\leq (\gamma_0 \vee c_2 \vee 1)^2 \frac{R_t}{R_k} \cdot \sum_{s=k+1}^t R_s^2 R_k^2 ((s-1) \vee \eta_s^{-1})^{-2} \quad (\text{Corollary B.4}) \\ &\leq (\gamma_0 \vee c_2 \vee 1)^2 \frac{R_t}{R_k} \cdot R_t^2 R_k^2 \cdot \sum_{s=k+1}^t (s-1)^{-2} \\ &\leq (\gamma_0 \vee c_2 \vee 1)^2 R_t^4 \int_{k-1}^{t-2} s^{-2} ds \\ &\leq 2(\gamma_0 \vee c_2 \vee 1)^2 R_t^4 k^{-1} \quad (\text{since } T \text{ is large enough}) . \end{aligned} \quad (11)$$

By (10) and (11), we have $\bar{Q}_{k,k}^{(1,t)} \leq 4(\gamma_0 \vee c_2 \vee 1)^2 R_t^4 (k \vee \eta_t^{-1})^{-1}$. Since $\tilde{\mathbf{Q}}^{(s,t)}$ is a principal submatrix of $\check{\mathbf{Q}}^{(t)}$, we have $\|\tilde{\mathbf{Q}}^{(s,t)}\|_2 \leq \|\check{\mathbf{Q}}^{(t)}\|_2 \leq (2^{1/2} + 1)^2$ by Lemma B.13. Then by Theorem 5.3.4 in Horn and Johnson, 2012, we have

$$\begin{aligned} \|\mathbf{Q}^{(s,t)}\|_2 &= \|\tilde{\mathbf{Q}}^{(s,t)} \circ \bar{\mathbf{Q}}^{(s,t)}\|_2 \\ &\leq \|\tilde{\mathbf{Q}}^{(s,t)}\|_2 \max_{s \leq k \leq t} \bar{Q}_{k,k}^{(1,t)} \\ &\leq 4(2^{1/2} + 1)^2 (\gamma_0 \vee c_2 \vee 1)^2 R_t^4 (s \vee \eta_t^{-1})^{-1} . \end{aligned}$$

□

The following lemma provides upper bounds for several deterministic summations that will be used in the proofs throughout the subsequent sections.

Lemma B.15. Under Assumption 1-3, for any $t \in [T]$ and $k \in \{0, 1\}$, we have

- (1) $\sum_{s=1}^{t-1} |\Pi_{t,s}| |y_s(k)| \leq (\gamma_0 \vee c_2 \vee 1)^{1/2} R_t((t-1) \vee \eta_t^{-1})^{-1/2} \left(\sum_{s=1}^{t-1} y_s^2(1) \right)^{1/2}.$
- (2) $\sum_{s=1}^{t-1} |\Pi_{t,s}| y_s^2(k) \leq c_1^2 (\gamma_0 \vee c_2 \vee 1)^{1/2} R_t((t-1) \vee \eta_t^{-1})^{-1/2} T^{1/2}.$
- (3) $\sum_{s=1}^t \sum_{r=1}^{s-1} \Pi_{s,r}^2 y_r^2(k) \leq c_1^2 (\gamma_0 \vee c_2 \vee 1)^{3/2} \xi_{3/2} R_t^3 \eta_t^{1/2} T^{1/2}.$
- (4) $\sum_{s=1}^t R_s^{-\nu_1} \sum_{r=1}^{s-1} R_r^{-\nu_2} \Pi_{s,r}^2 y_r^2(k) \leq c_1^2 (\gamma_0 \vee c_2 \vee 1)^{3/2} (1 - (\nu_1 + \nu_2)/2)^{-1} T^{3/4 - (\nu_1 + \nu_2)/4}$ for any ν_1, ν_2 such that $0 \leq \nu_1, 0 \leq \nu_2 \leq 1$ and $\nu_1 + \nu_2 < 2$. Suppose T is large enough.
- (5) $\sum_{s=1}^t \sum_{r=1}^{s-1} \Pi_{s,r}^4 y_r^4(k) \leq c_1^4 (\gamma_0 \vee c_2 \vee 1)^4 \xi_4 R_t^8 \eta_t^3 T.$
- (6) $\sum_{s=1}^t \sum_{1 \leq r_1 \neq r_2 \leq s-1} |\Pi_{s,r_1}|^3 |\Pi_{s,r_2}| |y_{r_1}(k)|^3 |y_{r_2}(k)| \leq c_1^4 (\gamma_0 \vee c_2 \vee 1)^{7/2} \xi_3 \xi_8^{1/4} R_t^{13/2} \eta_t^{7/4} T^{7/8}.$
- (7) $\sum_{s=1}^t \sum_{1 \leq r_1 \neq r_2 \leq s-1} \Pi_{s,r_1}^2 \Pi_{s,r_2}^2 y_{r_1}^2(k) y_{r_2}^2(k) \leq c_1^4 (\gamma_0 \vee c_2 \vee 1)^3 \xi_3 R_t^6 \eta_t^2 T.$
- (8) $\sum_{s=1}^t R_s^{-1} \sum_{1 \leq r_1 \neq r_2 \neq r_3 \leq s-1} \Pi_{s,r_1}^2 |\Pi_{s,r_2}| |\Pi_{s,r_3}| y_{r_1}^2(k) |y_{r_2}(k)| |y_{r_3}(k)| \leq c_1^4 \xi_2 (\gamma_0 \vee c_2 \vee 1)^{5/2} R_t^4 \eta_t T.$
- (9) $\sum_{s=1}^{t-1} \left(\sum_{r=s+1}^t \Pi_{r,s}^2 \right)^2 y_s^4(k) \leq c_1^4 (\gamma_0 \vee c_2 \vee 1)^4 \xi_2^2 R_t^8 \eta_t^2 T.$
- (10) $\sum_{s=1}^{t-1} \left(\sum_{r=s+1}^t \Pi_{r,s} (y_r(k) - \langle \mathbf{x}_r, \boldsymbol{\beta}_r^*(k) \rangle) \right)^2 R_s^{-1/2} y_s^2(k) \leq 2c_1^4 (\gamma_0 \vee c_2 \vee 1)^{3/2} \xi_{5/4} T^{11/8} R_t^{5/2} \eta_t^{1/4}$ for T large enough.
- (11) $\sum_{1 \leq t_2, t_3 < t_1 \leq t-1} \left| \sum_{s=t_1+1}^t \Pi_{s,t_1} \Pi_{s,t_2} \right| \left| \sum_{s=t_1+1}^t \Pi_{s,t_1} \Pi_{s,t_3} \right| y_{t_1}^2(k) |y_{t_2}(k)| |y_{t_3}(k)| \left(\frac{R_{t_1}^2}{R_{t_2}^2} \cdot \frac{R_{t_2}}{R_{t_3}} \right)^{1/4} \leq 18c_1^4 (\gamma_0 \vee c_2 \vee 1)^2 R_t^4 \eta_t^{1/2} T^{3/2} \log^2(\eta_t T)$ for T large enough.

Proof. Without loss of generality, we only prove the result for $k = 1$.

- (1) By Cauchy-Schwarz inequality and Lemma B.3, we have

$$\begin{aligned}
& \left(\sum_{s=1}^{t-1} |\Pi_{t,s}| |y_s(1)| \right)^2 \\
& \leq \left(\sum_{s=1}^{t-1} y_s^2(1) \right) \left(\sum_{s=1}^{t-1} \Pi_{t,s}^2 \right) \quad (\text{Cauchy-Schwarz inequality}) \\
& = \left(\sum_{s=1}^{t-1} y_s^2(1) \right) \left(\sum_{s=1}^{t-1} \mathbf{x}_t^\top (\mathbf{X}_{t-1}^\top \mathbf{X}_{t-1} + \eta_t^{-1} \mathbf{I}_d)^{-1} \mathbf{x}_s \mathbf{x}_s^\top (\mathbf{X}_{t-1}^\top \mathbf{X}_{t-1} + \eta_t^{-1} \mathbf{I}_d)^{-1} \mathbf{x}_t \right) \\
& = \left(\sum_{s=1}^{t-1} y_s^2(1) \right) \left(\mathbf{x}_t^\top (\mathbf{X}_{t-1}^\top \mathbf{X}_{t-1} + \eta_t^{-1} \mathbf{I}_d)^{-1} \mathbf{X}_{t-1}^\top \mathbf{X}_{t-1} (\mathbf{X}_{t-1}^\top \mathbf{X}_{t-1} + \eta_t^{-1} \mathbf{I}_d)^{-1} \mathbf{x}_t \right) \\
& \leq R_t^2 \left(\sum_{s=1}^{t-1} y_s^2(1) \right) \| (\mathbf{X}_{t-1}^\top \mathbf{X}_{t-1} + \eta_t^{-1} \mathbf{I}_d)^{-1} \mathbf{X}_{t-1}^\top \mathbf{X}_{t-1} (\mathbf{X}_{t-1}^\top \mathbf{X}_{t-1} + \eta_t^{-1} \mathbf{I}_d)^{-1} \|_2 \\
& \leq R_t^2 \left(\sum_{s=1}^{t-1} y_s^2(1) \right) \| (\mathbf{X}_{t-1}^\top \mathbf{X}_{t-1} + \eta_t^{-1} \mathbf{I}_d)^{-1} \|_2 \\
& \leq (\gamma_0 \vee c_2 \vee 1) R_t^2 ((t-1) \vee \eta_t^{-1})^{-1} \left(\sum_{s=1}^{t-1} y_s^2(1) \right) \quad (\text{Lemma B.3}) .
\end{aligned}$$

- (2) By Assumption 1 and similar method as in (1), we have

$$\left(\sum_{s=1}^{t-1} |\Pi_{t,s}| y_s^2(1) \right)^2 \leq (\gamma_0 \vee c_2 \vee 1) R_t^2 ((t-1) \vee \eta_t^{-1})^{-1} \left(\sum_{s=1}^{t-1} y_s^4(1) \right) \leq c_1^4 (\gamma_0 \vee c_2 \vee 1) R_t^2 ((t-1) \vee \eta_t^{-1})^{-1} T .$$

(3) By Corollary B.4, Lemma B.9 and the result in (2), we have

$$\begin{aligned}
\sum_{s=1}^t \sum_{r=1}^{s-1} \Pi_{s,r}^2 y_r^2(1) &\leq (\gamma_0 \vee c_2 \vee 1) \sum_{s=1}^t R_s^2 ((s-1) \vee \eta_s^{-1})^{-1} \left(\sum_{r=1}^{s-1} |\Pi_{s,r}| y_r^2(1) \right) \quad (\text{by } R_r \leq R_s \text{ and Corollary B.4}) \\
&\leq c_1^2 (\gamma_0 \vee c_2 \vee 1)^{3/2} T^{1/2} \sum_{s=1}^t R_s^3 ((s-1) \vee \eta_s^{-1})^{-3/2} \quad (\text{by the result in (2)}) \\
&= c_1^2 (\gamma_0 \vee c_2 \vee 1)^{3/2} T^{1/2} \sum_{s=1}^t ((s-1) R_s^{-2} \vee T^{1/2})^{-3/2} \\
&\leq c_1^2 (\gamma_0 \vee c_2 \vee 1)^{3/2} T^{1/2} \sum_{s=1}^t ((s-1) R_t^{-2} \vee T^{1/2})^{-3/2} \quad (\text{by } R_s \leq R_t) \\
&= c_1^2 (\gamma_0 \vee c_2 \vee 1)^{3/2} R_t^3 T^{1/2} \sum_{s=1}^t ((s-1) \vee \eta_t^{-1})^{-3/2} \\
&\leq c_1^2 (\gamma_0 \vee c_2 \vee 1)^{3/2} \xi_{3/2} R_t^3 \eta_t^{1/2} T^{1/2} \quad (\text{Lemma B.9}) .
\end{aligned}$$

(4) When T is large enough, by Cauchy-Schwarz inequality, Corollary B.4, Lemma B.10 and Assumption 1, we have

$$\begin{aligned}
&\sum_{s=1}^t R_s^{-\nu_1} \sum_{r=1}^{s-1} R_r^{-\nu_2} \Pi_{s,r}^2 y_r^2(1) \\
&\leq (\gamma_0 \vee c_2 \vee 1) \sum_{s=1}^t R_s^{1-\nu_1} ((s-1) \vee \eta_s^{-1})^{-1} \sum_{r=1}^{s-1} R_r^{1-\nu_2} |\Pi_{s,r}| y_r^2(1) \quad (\text{Corollary B.4}) \\
&\leq (\gamma_0 \vee c_2 \vee 1) \sum_{s=1}^t R_s^{1-\nu_1+1-\nu_2} ((s-1) \vee \eta_s^{-1})^{-1} \sum_{r=1}^{s-1} |\Pi_{s,r}| y_r^2(1) \quad (\text{by } R_r \leq R_s \text{ and } 1-\nu_2 \geq 0) \\
&\leq (\gamma_0 \vee c_2 \vee 1) \sum_{s=1}^t R_s^{2-\nu_1-\nu_2} ((s-1) \vee \eta_s^{-1})^{-1} \left(\sum_{r=1}^{s-1} \Pi_{s,r}^2 \right)^{1/2} \left(\sum_{r=1}^{s-1} y_r^4(1) \right)^{1/2} \quad (\text{Cauchy-Schwarz inequality}) \\
&\leq c_1^2 (\gamma_0 \vee c_2 \vee 1) T^{1/2} \sum_{s=1}^t R_s^{2-\nu_1-\nu_2} ((s-1) \vee \eta_s^{-1})^{-1} \\
&\quad \times \left(\sum_{r=1}^{s-1} \mathbf{x}_s^\top (\mathbf{X}_{s-1}^\top \mathbf{X}_{s-1} + \eta_s^{-1} \mathbf{I}_d)^{-1} \mathbf{x}_r \mathbf{x}_r^\top (\mathbf{X}_{s-1}^\top \mathbf{X}_{s-1} + \eta_s^{-1} \mathbf{I}_d)^{-1} \mathbf{x}_s \right)^{1/2} \quad (\text{Assumption 1}) \\
&= c_1^2 (\gamma_0 \vee c_2 \vee 1) T^{1/2} \sum_{s=1}^t R_s^{2-\nu_1-\nu_2} ((s-1) \vee \eta_s^{-1})^{-1} \\
&\quad \times \left(\mathbf{x}_s^\top (\mathbf{X}_{s-1}^\top \mathbf{X}_{s-1} + \eta_s^{-1} \mathbf{I}_d)^{-1} \mathbf{X}_{s-1}^\top \mathbf{X}_{s-1} (\mathbf{X}_{s-1}^\top \mathbf{X}_{s-1} + \eta_s^{-1} \mathbf{I}_d)^{-1} \mathbf{x}_s \right)^{1/2} \\
&\leq c_1^2 (\gamma_0 \vee c_2 \vee 1) T^{1/2} \sum_{s=1}^t R_s^{2-\nu_1-\nu_2} ((s-1) \vee \eta_s^{-1})^{-1} \left(\mathbf{x}_s^\top (\mathbf{X}_{s-1}^\top \mathbf{X}_{s-1} + \eta_s^{-1} \mathbf{I}_d)^{-1} \mathbf{x}_s \right)^{1/2} \\
&\leq c_1^2 (\gamma_0 \vee c_2 \vee 1)^{3/2} T^{1/2} \sum_{s=1}^t R_s^{2-\nu_1-\nu_2+1} ((s-1) \vee \eta_s^{-1})^{-3/2} \quad (\text{Corollary B.4}) \\
&\leq c_1^2 (\gamma_0 \vee c_2 \vee 1)^{3/2} T^{1/2} \frac{T^{1-3/2+(3-\nu_1-\nu_2)/4}}{1-(3/2-(3-\nu_1-\nu_2)/2)} \quad (\text{Lemma B.10}) \\
&\leq c_1^2 (\gamma_0 \vee c_2 \vee 1)^{3/2} T^{1/2+1/4-(\nu_1+\nu_2)/4} (1-(\nu_1+\nu_2)/2)^{-1} \\
&= c_1^2 (\gamma_0 \vee c_2 \vee 1)^{3/2} (1-(\nu_1+\nu_2)/2)^{-1} T^{3/4-(\nu_1+\nu_2)/4} .
\end{aligned}$$

(5) By Corollary B.4, Lemma B.9 and Assumption 1, we have

$$\begin{aligned}
\sum_{s=1}^t \sum_{r=1}^{s-1} \Pi_{s,r}^4 y_r^4(1) &= \sum_{r=1}^{t-1} \left(\sum_{s=r+1}^t \Pi_{s,r}^4 \right) y_r^4(1) \quad (\text{rewrite the summation}) \\
&\leq \sum_{r=1}^{t-1} \left(\sum_{s=r+1}^t (\gamma_0 \vee c_2 \vee 1)^4 R_s^8 ((s-1) \vee \eta_s^{-1})^{-4} \right) y_r^4(1) \quad (\text{by } R_r \leq R_s \text{ and Corollary B.4}) \\
&= \sum_{r=1}^{t-1} \left(\sum_{s=r+1}^t (\gamma_0 \vee c_2 \vee 1)^4 ((s-1) R_s^{-2} \vee T^{1/2})^{-4} \right) y_r^4(1) \\
&\leq \sum_{r=1}^{t-1} \left(\sum_{s=r+1}^t (\gamma_0 \vee c_2 \vee 1)^4 ((s-1) R_t^{-2} \vee T^{1/2})^{-4} \right) y_r^4(1) \quad (\text{by } R_s \leq R_t) \\
&= (\gamma_0 \vee c_2 \vee 1)^4 R_t^8 \sum_{r=1}^{t-1} \left(\sum_{s=r+1}^t ((s-1) \vee \eta_t^{-1})^{-4} \right) y_r^4(1) \\
&\leq (\gamma_0 \vee c_2 \vee 1)^4 \xi_4 R_t^8 \sum_{r=1}^{t-1} \eta_t^3 y_r^4(1) \quad (\text{Lemma B.9}) \\
&\leq c_1^4 (\gamma_0 \vee c_2 \vee 1)^4 \xi_4 R_t^8 \eta_t^3 T \quad (\text{Assumption 1}) .
\end{aligned}$$

(6) By Corollary B.4, Lemma B.9, Cauchy-Schwarz inequality, Hölder's inequality and Assumption 1, we have

$$\begin{aligned}
&\sum_{s=1}^t \sum_{1 \leq r_1 \neq r_2 \leq s-1} |\Pi_{s,r_1}|^3 |\Pi_{s,r_2}| |y_{r_1}(1)|^3 |y_{r_2}(1)| \\
&\leq \sum_{s=1}^t \left(\sum_{r=1}^{s-1} |\Pi_{s,r}|^3 |y_r(1)|^3 \right) \left(\sum_{r=1}^{s-1} |\Pi_{s,r}| |y_r(1)| \right) \\
&\leq (\gamma_0 \vee c_2 \vee 1)^{1/2} \sum_{s=1}^t \left(\sum_{r=1}^{s-1} |\Pi_{s,r}|^3 |y_r(1)|^3 \right) R_s ((s-1) \vee \eta_s^{-1})^{-1/2} \left(\sum_{r=1}^{s-1} y_r^2(1) \right)^{1/2} \quad (\text{by the result in (1)}) \\
&\leq (\gamma_0 \vee c_2 \vee 1)^{1/2} \sum_{s=1}^t \left(\sum_{r=1}^{s-1} |\Pi_{s,r}|^3 |y_r(1)|^3 \right) R_s ((s-1) \vee \eta_s^{-1})^{-1/2} (s-1)^{1/4} \left(\sum_{r=1}^{s-1} y_r^4(1) \right)^{1/4} \quad (\text{Cauchy-Schwarz}) \\
&\leq c_1 (\gamma_0 \vee c_2 \vee 1)^{1/2} T^{1/4} \sum_{s=1}^t \left(\sum_{r=1}^{s-1} |\Pi_{s,r}|^3 |y_r(1)|^3 \right) R_s \eta_s^{1/4} \quad (\text{Assumption 1}) \\
&\leq c_1 (\gamma_0 \vee c_2 \vee 1)^{1/2} R_t^{1/2} T^{1/8} \sum_{r=1}^{t-1} \left(\sum_{s=r+1}^t |\Pi_{s,r}|^3 \right) |y_r(1)|^3 \quad (\text{by } R_s \leq R_t) \\
&\leq c_1 (\gamma_0 \vee c_2 \vee 1)^{7/2} R_t^{1/2} T^{1/8} \sum_{r=1}^{t-1} \left(\sum_{s=r+1}^t R_s^6 ((s-1) \vee \eta_s^{-1})^{-3} \right) |y_r(1)|^3 \quad (\text{Corollary B.4}) \\
&= c_1 (\gamma_0 \vee c_2 \vee 1)^{7/2} R_t^{1/2} T^{1/8} \sum_{r=1}^{t-1} \left(\sum_{s=r+1}^t ((s-1) R_s^{-2} \vee T^{1/2})^{-3} \right) |y_r(1)|^3 \\
&\leq c_1 (\gamma_0 \vee c_2 \vee 1)^{7/2} R_t^{1/2} T^{1/8} \sum_{r=1}^{t-1} \left(\sum_{s=r+1}^t ((s-1) R_t^{-2} \vee T^{1/2})^{-3} \right) |y_r(1)|^3 \quad (\text{by } R_s \leq R_t) \\
&= c_1 (\gamma_0 \vee c_2 \vee 1)^{7/2} R_t^{1/2} T^{1/8} \sum_{r=1}^{t-1} \left(\sum_{s=r+1}^t R_t^6 ((s-1) \vee \eta_t^{-1})^{-3} \right) |y_r(1)|^3 \\
&\leq c_1 (\gamma_0 \vee c_2 \vee 1)^{7/2} \xi_3 R_t^{13/2} T^{1/8} \sum_{r=1}^{t-1} ((r-1) \vee \eta_t^{-1})^{-2} |y_r(1)|^3 \quad (\text{Lemma B.9})
\end{aligned}$$

$$\begin{aligned}
&\leq c_1(\gamma_0 \vee c_2 \vee 1)^{7/2} \xi_3 R_t^{13/2} T^{1/8} \left(\sum_{r=1}^{t-1} ((r-1) \vee \eta_t^{-1})^{-8} \right)^{1/4} \left(\sum_{r=1}^{t-1} y_r^4(1) \right)^{3/4} \quad (\text{Hölder's inequality}) \\
&\leq c_1^4 (\gamma_0 \vee c_2 \vee 1)^{7/2} \xi_3 \xi_8^{1/4} R_t^{13/2} T^{1/8+3/4} \eta_t^{7/4} \quad (\text{Lemma B.9 and Assumption 1}) \\
&= c_1^4 (\gamma_0 \vee c_2 \vee 1)^{7/2} \xi_3 \xi_8^{1/4} R_t^{13/2} T^{7/8} \eta_t^{7/4}.
\end{aligned}$$

(7) By Corollary B.4, Lemma B.9, Cauchy-Schwarz inequality and Assumption 1, we have

$$\begin{aligned}
&\sum_{s=1}^t \sum_{1 \leq r_1 \neq r_2 \leq s-1} \Pi_{s,r_1}^2 \Pi_{s,r_2}^2 y_{r_1}^2(1) y_{r_2}^2(1) \\
&\leq \sum_{s=1}^t \left(\sum_{r=1}^{s-1} \Pi_{s,r}^2 y_r^2(1) \right)^2 \\
&\leq (\gamma_0 \vee c_2 \vee 1)^2 \sum_{s=1}^t R_s^4 ((s-1) \vee \eta_s^{-1})^{-2} \left(\sum_{r=1}^{s-1} |\Pi_{s,r}| y_r^2(1) \right)^2 \quad (\text{by } R_r \leq R_s \text{ and Corollary B.4}) \\
&\leq c_1^4 (\gamma_0 \vee c_2 \vee 1)^3 T \sum_{s=1}^t R_s^6 ((s-1) \vee \eta_s^{-1})^{-3} \quad (\text{by the result in (2)}) \\
&= c_1^4 (\gamma_0 \vee c_2 \vee 1)^3 T \sum_{s=1}^t ((s-1) R_s^{-2} \vee T^{1/2})^{-3} \\
&\leq c_1^4 (\gamma_0 \vee c_2 \vee 1)^3 T \sum_{s=1}^t ((s-1) R_t^{-2} \vee T^{1/2})^{-3} \quad (\text{by } R_s \leq R_t) \\
&= c_1^4 (\gamma_0 \vee c_2 \vee 1)^3 T \sum_{s=1}^t R_t^6 ((s-1) \vee \eta_t^{-1})^{-3} \\
&\leq c_1^4 (\gamma_0 \vee c_2 \vee 1)^3 \xi_3 R_t^6 T \eta_t^2 \quad (\text{Lemma B.9}).
\end{aligned}$$

(8) By Corollary B.4, Lemma B.9, Cauchy-Schwarz inequality, Assumption 1 and the results in (1), (2), we have

$$\begin{aligned}
&\sum_{s=1}^t R_s^{-1} \sum_{1 \leq r_1 \neq r_2 \neq r_3 \leq s-1} \Pi_{s,r_1}^2 |\Pi_{s,r_2}| |\Pi_{s,r_3}| y_{r_1}^2(1) |y_{r_2}(1)| |y_{r_3}(1)| \\
&\leq \sum_{s=1}^t R_s^{-1} \left(\sum_{r=1}^{s-1} \Pi_{s,r}^2 y_r^2(1) \right) \left(\sum_{r=1}^{s-1} |\Pi_{s,r}| |y_r(1)| \right)^2 \\
&\leq (\gamma_0 \vee c_2 \vee 1) \sum_{s=1}^t R_s^{-1+2} ((s-1) \vee \eta_s^{-1})^{-1} \left(\sum_{r=1}^{s-1} |\Pi_{s,r}| y_r^2(1) \right) \left(\sum_{r=1}^{s-1} |\Pi_{s,r}| |y_r(1)| \right)^2 \quad (\text{by } R_r \leq R_s, \text{ Corollary B.4}) \\
&\leq c_1^2 (\gamma_0 \vee c_2 \vee 1)^{5/2} T^{1/2} \sum_{s=1}^t R_s^4 ((s-1) \vee \eta_s^{-1})^{-5/2} \left(\sum_{r=1}^{s-1} y_r^2(1) \right) \quad (\text{results in (1) and (2)}) \\
&\leq c_1^2 (\gamma_0 \vee c_2 \vee 1)^{5/2} T^{1/2} \sum_{s=1}^t R_s^4 ((s-1) \vee \eta_s^{-1})^{-5/2} (s-1)^{1/2} \left(\sum_{r=1}^{s-1} y_r^4(1) \right)^{1/2} \quad (\text{Cauchy-Schwarz inequality}) \\
&\leq c_1^4 (\gamma_0 \vee c_2 \vee 1)^{5/2} T \sum_{s=1}^t R_s^4 ((s-1) \vee \eta_s^{-1})^{-2} \quad (\text{Assumption 1}) \\
&= c_1^4 (\gamma_0 \vee c_2 \vee 1)^{5/2} T \sum_{s=1}^t ((s-1) R_s^{-2} \vee T^{1/2})^{-2} \\
&\leq c_1^4 (\gamma_0 \vee c_2 \vee 1)^{5/2} T \sum_{s=1}^t ((s-1) R_t^{-2} \vee T^{1/2})^{-2} \quad (\text{by } R_s \leq R_t)
\end{aligned}$$

$$\begin{aligned}
&= c_1^4 (\gamma_0 \vee c_2 \vee 1)^{5/2} R_t^4 T \sum_{s=1}^t ((s-1) \vee \eta_t^{-1})^{-2} \\
&\leq c_1^4 \xi_2 (\gamma_0 \vee c_2 \vee 1)^{5/2} R_t^4 \eta_t T \quad (\text{Lemma B.9}) .
\end{aligned}$$

(9) By Corollary B.4, Lemma B.9 and Assumption 1, we have

$$\begin{aligned}
&\sum_{s=1}^{t-1} \left(\sum_{r=s+1}^t \Pi_{r,s}^2 \right)^2 y_s^4(1) \\
&\leq (\gamma_0 \vee c_2 \vee 1)^4 \sum_{s=1}^{t-1} \left(\sum_{r=s+1}^t R_r^4 ((r-1) \vee \eta_r^{-1})^{-2} \right)^2 y_s^4(1) \quad (\text{by } R_r \leq R_s \text{ and Corollary B.4}) \\
&= (\gamma_0 \vee c_2 \vee 1)^4 \sum_{s=1}^{t-1} \left(\sum_{r=s+1}^t ((r-1) R_r^{-2} \vee T^{1/2})^{-2} \right)^2 y_s^4(1) \\
&\leq (\gamma_0 \vee c_2 \vee 1)^4 \sum_{s=1}^{t-1} \left(\sum_{r=s+1}^t ((r-1) R_t^{-2} \vee T^{1/2})^{-2} \right)^2 y_s^4(1) \quad (\text{by } R_r \leq R_t) \\
&= (\gamma_0 \vee c_2 \vee 1)^4 \sum_{s=1}^{t-1} \left(\sum_{r=s+1}^t R_t^4 ((r-1) \vee \eta_t^{-1})^{-2} \right)^2 y_s^4(1) \\
&\leq (\gamma_0 \vee c_2 \vee 1)^4 \xi_2^2 R_t^8 \eta_t^2 \sum_{s=1}^{t-1} y_s^4(1) \quad (\text{Lemma B.9}) \\
&\leq c_1^4 (\gamma_0 \vee c_2 \vee 1)^4 \xi_2^2 R_t^8 \eta_t^2 T \quad (\text{Assumption 1}) .
\end{aligned}$$

(10) By Cauchy-Schwarz inequality, Corollary B.4, Lemma B.9, Lemma B.11 and Assumption 1, we have

$$\begin{aligned}
&\sum_{s=1}^{t-1} \left(\sum_{r=s+1}^t \Pi_{r,s} (y_r(1) - \langle x_r, \beta_r^*(1) \rangle) \right)^2 R_s^{-1/2} y_s^2(1) \\
&\leq \sum_{s=1}^{t-1} \left(\sum_{r=s+1}^t \Pi_{r,s}^2 \right) \left(\sum_{r=s+1}^t (y_r(1) - \langle x_r, \beta_r^*(1) \rangle)^2 \right) R_s^{-1/2} y_s^2(1) \quad (\text{Cauchy-Schwarz inequality}) \\
&\leq 2c_1^2 T \sum_{s=1}^{t-1} \left(\sum_{r=s+1}^t \Pi_{r,s}^2 \right) R_s^{-1/2} y_s^2(1) \quad (\text{Lemma B.11}) \\
&= 2c_1^2 T \sum_{s=1}^t \left(\sum_{r=1}^{s-1} \Pi_{s,r}^2 R_r^{-1/2} y_r^2(1) \right) \quad (\text{change the summation order}) \\
&\leq 2(\gamma_0 \vee c_2 \vee 1) c_1^2 T \sum_{s=1}^t \left(\sum_{r=1}^{s-1} ((s-1) \vee \eta_s^{-1})^{-1} R_s R_r^{1-1/2} |\Pi_{s,r}| y_r^2(1) \right) \quad (\text{Corollary B.4}) \\
&\leq 2(\gamma_0 \vee c_2 \vee 1) c_1^2 T \sum_{s=1}^t ((s-1) \vee \eta_s^{-1})^{-1} R_s^{3/2} \left(\sum_{r=1}^{s-1} |\Pi_{s,r}| y_r^2(1) \right) \quad (\text{by } R_r \leq R_s) \\
&\leq 2(\gamma_0 \vee c_2 \vee 1) c_1^2 T \sum_{s=1}^t ((s-1) \vee \eta_s^{-1})^{-1} R_s^{3/2} \left(\sum_{r=1}^{s-1} \Pi_{s,r}^2 \right)^{1/2} \left(\sum_{r=1}^{s-1} y_r^4(1) \right)^{1/2} \quad (\text{Cauchy-Schwarz inequality}) \\
&\leq 2(\gamma_0 \vee c_2 \vee 1) c_1^4 T^{1+1/2} \sum_{s=1}^t R_s^{3/2} ((s-1) \vee \eta_s^{-1})^{-1} \Pi_{s,s}^{1/2} \quad (\text{Assumption 1}) \\
&\leq 2(\gamma_0 \vee c_2 \vee 1)^{3/2} c_1^4 T^{1+1/2} \sum_{s=1}^t R_s^{3/2} ((s-1) \vee \eta_s^{-1})^{-1} R_s ((s-1) \vee \eta_s^{-1})^{-1/2} \quad (\text{Corollary B.4})
\end{aligned}$$

$$\begin{aligned}
&\leq 2(\gamma_0 \vee c_2 \vee 1)^{3/2} c_1^4 T^{3/2} \sum_{s=1}^t ((s-1)R_s^{-2} \vee T^{1/2})^{-3/4} ((s-1) \vee \eta_s^{-1})^{-1/4} ((s-1)R_s^{-2} \vee T^{1/2})^{-1/2} \\
&\leq 2(\gamma_0 \vee c_2 \vee 1)^{3/2} c_1^4 T^{3/2} \eta_1^{1/4} \sum_{s=1}^t ((s-1)R_s^{-2} \vee T^{1/2})^{-3/4-1/2} \quad (\text{since } \eta_s \leq \eta_1) \\
&\leq 2(\gamma_0 \vee c_2 \vee 1)^{3/2} c_1^4 T^{3/2-1/8} \sum_{s=1}^t ((s-1)R_t^{-2} \vee T^{1/2})^{-5/4} \quad (\text{by } R_s \leq R_t) \\
&\leq 2(\gamma_0 \vee c_2 \vee 1)^{3/2} c_1^4 T^{11/8} R_t^{5/2} \sum_{s=1}^t ((s-1) \vee \eta_t^{-1})^{-5/4} \\
&\leq 2c_1^4 (\gamma_0 \vee c_2 \vee 1)^{3/2} \xi_{5/4} T^{11/8} R_t^{5/2} \eta_t^{1/4} \quad (\text{Lemma B.9}) .
\end{aligned}$$

(11) By AM-GM inequality and the definition of matrix $\mathbf{Q}^{(1,t)}$ in Corollary B.14, we have

$$\begin{aligned}
&\sum_{1 \leq t_2, t_3 < t_1 \leq t-1} \left| \sum_{s=t_1+1}^t \Pi_{s,t_1} \Pi_{s,t_2} \right| \left| \sum_{s=t_1+1}^t \Pi_{s,t_1} \Pi_{s,t_3} \right| y_{t_1}^2(1) |y_{t_2}(1)| |y_{t_3}(1)| \left(\frac{R_t^2}{R_{t_1}^2} \cdot \frac{R_t}{R_{t_2}} \cdot \frac{R_t}{R_{t_3}} \right)^{1/4} \\
&\leq \frac{1}{2} \sum_{1 \leq t_2, t_3 < t_1 \leq t-1} \left[\left(\sum_{s=t_1+1}^t \Pi_{s,t_1} \Pi_{s,t_2} \right)^2 y_{t_1}^2(1) y_{t_2}^2(1) \left(\frac{R_t^2}{R_{t_1}^2} \cdot \frac{R_t^2}{R_{t_2}^2} \right)^{1/4} \right. \\
&\quad \left. + \left(\sum_{s=t_1+1}^t \Pi_{s,t_1} \Pi_{s,t_3} \right)^2 y_{t_1}^2(1) y_{t_3}^2(1) \left(\frac{R_t^2}{R_{t_1}^2} \cdot \frac{R_t^2}{R_{t_3}^2} \right)^{1/4} \right] \quad (\text{AM-GM inequality}) \\
&\leq \sum_{1 \leq t_2 < t_1 \leq t-1} t_1 \left(\sum_{s=t_1+1}^t \Pi_{s,t_1} \Pi_{s,t_2} \right)^2 y_{t_1}^2(1) y_{t_2}^2(1) \left(\frac{R_t}{R_{t_1}} \cdot \frac{R_t}{R_{t_2}} \right)^{1/2} \\
&\leq \frac{1}{2} \sum_{1 \leq t_1, t_2 \leq t-1} (t_1 \vee t_2) \left(\sum_{s=t_1 \vee t_2+1}^t \Pi_{s,t_1} \Pi_{s,t_2} \right)^2 y_{t_1}^2(1) y_{t_2}^2(1) \left(\frac{R_t}{R_{t_1}} \cdot \frac{R_t}{R_{t_2}} \right)^{1/2} \\
&= \frac{1}{2} \sum_{1 \leq t_1, t_2 \leq t-1} (t_1 \vee t_2) Q_{t_1, t_2}^{(1,t)} y_{t_1}^2(1) y_{t_2}^2(1) .
\end{aligned}$$

Let $K = \lfloor \log^2(\eta_t T) \rfloor \in \mathbb{N}$. Since $\eta_t^{-1} = \mathcal{O}(T^{1/2} R_t^2) = o(T)$ by Assumption 3, K tends to infinity when T grows. For $k = 1, \dots, K+1$, let $C_k = \lfloor \eta_t^{-1+(k-1)/K} T^{(k-1)/K} \rfloor$ and let $C_0 = 0$. For $k = 0, \dots, K$, denote $B_k = \{C_k + 1, \dots, C_{k+1}\} \cap \{1, \dots, t-1\}$. Then we have

$$\begin{aligned}
&\sum_{1 \leq t_2 < t_1, t_3 < t_1 \leq t-1} \left| \sum_{s=t_1+1}^t \Pi_{s,t_1} \Pi_{s,t_2} \right| \left| \sum_{s=t_1+1}^t \Pi_{s,t_1} \Pi_{s,t_3} \right| y_{t_1}^2(1) |y_{t_2}(1)| |y_{t_3}(1)| \left(\frac{R_t^2}{R_{t_1}^2} \cdot \frac{R_t}{R_{t_2}} \cdot \frac{R_t}{R_{t_3}} \right)^{1/4} \\
&\leq \frac{1}{2} \sum_{1 \leq t_1, t_2 \leq t-1} (t_1 \vee t_2) Q_{t_1, t_2}^{(1,t)} y_{t_1}^2(1) y_{t_2}^2(1) \\
&\leq \frac{1}{2} \sum_{k_1=0}^K \sum_{k_2=0}^K \sum_{t_1 \in B_{k_1}} \sum_{t_2 \in B_{k_2}} (t_1 \vee t_2) Q_{t_1, t_2}^{(1,t)} y_{t_1}^2(1) y_{t_2}^2(1) \\
&\triangleq \frac{1}{2} \sum_{k_1=0}^K \sum_{k_2=0}^K S_{k_1, k_2} .
\end{aligned}$$

For $S_{0,0}$, by Corollary B.14, we have

$$\begin{aligned}
S_{0,0} &= \sum_{t_1 \in B_0} \sum_{t_2 \in B_0} (t_1 \vee t_2) Q_{t_1, t_2}^{(1,t)} y_{t_1}^2(1) y_{t_2}^2(1) \\
&\leq C_1 \sum_{t_1 \in B_0} \sum_{t_2 \in B_0} Q_{t_1, t_2}^{(1,t)} y_{t_1}^2(1) y_{t_2}^2(1)
\end{aligned}$$

$$\begin{aligned}
&\leq C_1 \|\mathbf{Q}^{(1,t)}\|_2 \sum_{s \in B_0} y_s^4(1) \\
&\leq 4(2^{1/2} + 1)^2 (\gamma_0 \vee c_2 \vee 1)^2 R_t^4 \eta_t \cdot \eta_t^{-1} \sum_{t \in B_0} y_t^4(1) \quad (\text{Corollary B.14}) \\
&\leq 4(2^{1/2} + 1)^2 (\gamma_0 \vee c_2 \vee 1)^2 R_t^4 \sum_{s \in B_0} y_s^4(1) .
\end{aligned}$$

For $k = 1, \dots, K$, by Corollary B.14 and Cauchy-Schwarz inequality on semi-definite positive matrix, we have

$$\begin{aligned}
&S_{k,0} \\
&= \sum_{t_1 \in B_k} \sum_{t_2 \in B_0} (t_1 \vee t_2) Q_{t_1, t_2}^{(1,t)} y_{t_1}^2(1) y_{t_2}^2(1) \\
&\leq C_{k+1} \sum_{t_1 \in B_k} \sum_{t_2 \in B_0} Q_{t_1, t_2}^{(1,t)} y_{t_1}^2(1) y_{t_2}^2(1) \\
&\leq C_{k+1} \left(\sum_{t_1, t_2 \in B_k} Q_{t_1, t_2}^{(t \wedge C_k + 1, t)} y_{t_1}^2(1) y_{t_2}^2(1) \right)^{1/2} \left(\sum_{t_1, t_2 \in B_0} Q_{t_1, t_2}^{(1,t)} y_{t_1}^2(1) y_{t_2}^2(1) \right)^{1/2} \quad (\text{Cauchy-Schwarz}) \\
&\leq C_{k+1} \left(\left\| \mathbf{Q}^{(t \wedge C_k + 1, t)} \right\|_2 \sum_{s \in B_k} y_s^4(1) \right)^{1/2} \left(\left\| \mathbf{Q}^{(1,t)} \right\|_2 \sum_{s \in B_0} y_s^4(1) \right)^{1/2} \\
&\leq C_{k+1} \left(4(2^{1/2} + 1)^2 (\gamma_0 \vee c_2 \vee 1)^2 R_t^4 C_k^{-1} \sum_{s \in B_k} y_s^4(1) \right)^{1/2} \\
&\quad \times \left(4(2^{1/2} + 1)^2 (\gamma_0 \vee c_2 \vee 1)^2 R_t^4 \eta_t \sum_{s \in B_0} y_s^4(1) \right)^{1/2} \quad (\text{Corollary B.14}) \\
&= 4(2^{1/2} + 1)^2 (\gamma_0 \vee c_2 \vee 1)^2 R_t^4 \left(\frac{C_{k+1}}{C_k} \frac{C_{k+1}}{\eta_t^{-1}} \right)^{1/2} \left(\sum_{s \in B_k} y_s^4(1) \right)^{1/2} \left(\sum_{s \in B_0} y_s^4(1) \right)^{1/2} .
\end{aligned}$$

For $k_1, k_2 = 1, \dots, K$, by Corollary B.14 and Cauchy-Schwarz inequality on semi-positive definite matrix, we have

$$\begin{aligned}
&S_{k_1, k_2} \\
&= \sum_{t_1 \in B_{k_1}} \sum_{t_2 \in B_{k_2}} (t_1 \vee t_2) Q_{t_1, t_2}^{(1,t)} y_{t_1}^2(1) y_{t_2}^2(1) \\
&\leq C_{k_1 \vee k_2 + 1} \left(\sum_{t_1, t_2 \in B_{k_1}} Q_{t_1, t_2}^{(1,t)} y_{t_1}^2(1) y_{t_2}^2(1) \right)^{1/2} \left(\sum_{t_1, t_2 \in B_{k_2}} Q_{t_1, t_2}^{(1,t)} y_{t_1}^2(1) y_{t_2}^2(1) \right)^{1/2} \quad (\text{Cauchy-Schwarz}) \\
&\leq C_{k_1 \vee k_2 + 1} \left(\left\| \mathbf{Q}^{(t \wedge C_{k_1} + 1, t)} \right\|_2 \sum_{s \in B_{k_1}} y_s^4(1) \right)^{1/2} \left(\left\| \mathbf{Q}^{(t \wedge C_{k_2} + 1, t)} \right\|_2 \sum_{s \in B_{k_2}} y_s^4(1) \right)^{1/2} \\
&\leq C_{k_1 \vee k_2 + 1} \left(4(2^{1/2} + 1)^2 \xi_2 R_t^4 C_{k_1}^{-1} \sum_{t_1 \in B_{k_1}} y_t^4(1) \right)^{1/2} \\
&\quad \times \left(4(2^{1/2} + 1)^2 (\gamma_0 \vee c_2 \vee 1)^2 R_t^4 C_{k_2}^{-1} \sum_{t_2 \in B_{k_2}} y_t^4(1) \right)^{1/2} \quad (\text{Corollary B.14})
\end{aligned}$$

$$\leq 4(2^{1/2} + 1)^2 (\gamma_0 \vee c_2 \vee 1)^2 R_t^4 \left(\frac{C_{k_1 \vee k_2 + 1}}{C_{k_1}} \frac{C_{k_1 \vee k_2 + 1}}{C_{k_2}} \right)^{1/2} \left(\sum_{s \in B_{k_1}} y_s^4(1) \right)^{1/2} \left(\sum_{s \in B_{k_2}} y_s^4(1) \right)^{1/2}.$$

Since K tends to infinity when T grows to infinity, when T is large enough, it is easy to see that

$$\begin{aligned} \max_{k=1,\dots,K} \frac{\eta_t^{-1+((k+1)-1)/K} T^{((k+1)-1)/K}}{\eta_t^{-1+(k-1)/K} T^{(k-1)/K}} &= \exp \left(\log(T\eta_t)^{\frac{1}{K}} \right) \\ &= \exp \left(\frac{1}{K} \log(T\eta_t) \right) \\ &\leq \exp \left(\frac{2}{\log^2(T\eta_t)} \log(T\eta_t) \right) \\ &\rightarrow 1. \end{aligned}$$

Since $C_1 = \lfloor \eta_t^{-1} \rfloor \rightarrow \infty$, this implies that for T large enough, we have

$$\max_{k=1,\dots,K} \frac{C_{k+1}}{C_k} \leq 2^{1/2}.$$

Then when T is large enough, by the construction method of C_1, \dots, C_{K+1} , we have

$$\begin{aligned} \max_{k,k_1,k_2=1,\dots,K} &\left\{ 1, \left(\frac{C_{k+1}}{C_k} \frac{C_{k+1}}{\eta_t^{-1}} \right)^{1/2}, \left(\frac{C_{k_1 \vee k_2 + 1}}{C_{k_1}} \frac{C_{k_1 \vee k_2 + 1}}{C_{k_2}} \right)^{1/2} \right\} \\ &\leq \left(2^{1/2} \cdot \frac{T}{\lfloor \eta_t^{-1} \rfloor} \right)^{1/2} \\ &\leq 2^{1/2} (T\eta_t)^{1/2}. \end{aligned}$$

Hence by Cauchy-Schwarz inequality and Assumption 1, for T large enough we have

$$\begin{aligned} &\sum_{1 \leq t_2, t_3 < t_1 \leq t-1} \left| \sum_{s=t_1+1}^t \Pi_{s,t_1} \Pi_{s,t_2} \right| \left| \sum_{s=t_1+1}^t \Pi_{s,t_1} \Pi_{s,t_3} \right| |y_{t_1}^2(1)| |y_{t_2}(1)| |y_{t_3}(1)| \\ &\leq \frac{1}{2} \sum_{k_1=0}^K \sum_{k_2=0}^K S_{k_1, k_2} \\ &\leq 2^{3/2} (2^{1/2} + 1)^2 (\gamma_0 \vee c_2 \vee 1)^2 R_t^4 (T\eta_t)^{1/2} \left[\sum_{k_1=0}^K \sum_{k_2=0}^K \left(\sum_{s \in B_{k_1}} y_s^4(1) \right)^{1/2} \left(\sum_{s \in B_{k_2}} y_s^4(1) \right)^{1/2} \right] \\ &\leq 18(\gamma_0 \vee c_2 \vee 1)^2 R_t^4 (T\eta_t)^{1/2} \left(\sum_{k=0}^K \left(\sum_{s \in B_k} y_s^4(1) \right)^{1/2} \right)^2 \\ &\leq 18(\gamma_0 \vee c_2 \vee 1)^2 R_t^4 (T\eta_t)^{1/2} K \left(\sum_{k=0}^K \left(\sum_{C_k \leq t \leq C_{k+1}-1} y_t^4(1) \right) \right) \quad (\text{Cauchy-Schwarz inequality}) \\ &= 18(\gamma_0 \vee c_2 \vee 1)^2 R_t^4 (T\eta_t)^{1/2} K \sum_{t=1}^{T-1} y_t^4(1) \\ &\leq 18c_1^4 (\gamma_0 \vee c_2 \vee 1)^2 R_t^4 (T\eta_t)^{1/2} KT \\ &\leq 18c_1^4 (\gamma_0 \vee c_2 \vee 1)^2 R_t^4 \eta_t^{1/2} T^{3/2} \log^2(\eta_t T) \quad (\text{Assumption 1}). \end{aligned}$$

□

The following lemma characterizes the relationship between the original p -space and the transformed u -space induced by the sigmoid function ϕ .

Lemma B.16. *Under Condition 1, for any $u, \tilde{u} \geq 0$, $p = \phi(u)$ and $\tilde{p} = \phi(\tilde{u})$, we have*

$$(1) \left(\frac{1}{1-\phi(0)} \right)' \geq \frac{b_3}{2}.$$

$$(2) \left| \left(\frac{1}{\phi(u)} \right)' \right| \leq \frac{b_2}{2} \cdot \frac{1}{(1+u)^2}.$$

$$(3) \frac{b_3}{2} \cdot |u - \tilde{u}| \leq \left| \frac{1}{1-p} - \frac{1}{1-\tilde{p}} \right| \leq b_1 \cdot |u - \tilde{u}|.$$

$$(4) \left| \frac{1}{p} - \frac{1}{\tilde{p}} \right| \leq \frac{b_2}{2} \cdot \frac{|u - \tilde{u}|}{(1+u)(1+\tilde{u})}.$$

$$(5) \left| \left(\frac{1}{\phi(u)} \right)' - \left(\frac{1}{\phi(\tilde{u})} \right)' \right| \geq \frac{b_3}{2} \cdot \frac{|u - \tilde{u}|}{(1+u)(1+\tilde{u})(1+u \wedge \tilde{u})}.$$

$$(6) \max \left\{ \frac{1}{\phi(u)} - \frac{1}{\phi(\tilde{u})} - \left(\frac{1}{\phi(\tilde{u})} \right)' (u - \tilde{u}), \frac{1}{1-\phi(u)} - \frac{1}{1-\phi(\tilde{u})} - \left(\frac{1}{1-\phi(\tilde{u})} \right)' (u - \tilde{u}) \right\} \leq \frac{b_2}{2} \cdot \frac{\tilde{u} - u}{(1+u)(1+\tilde{u})} \text{ for } \tilde{u} \geq u.$$

Proof. (1) By Condition 1(1) and Condition 1(3c), we have

$$\left(\frac{1}{1-\phi(0)} \right)' \geq \int_{-\infty}^0 \left(\frac{1}{1-\phi(u)} \right)'' du \geq b_3 \int_{-\infty}^0 \frac{1}{(1-u)^3} du \geq \frac{b_3}{2}.$$

(2) By Condition 1(1) and Condition 1(3b), we have

$$\left| \left(\frac{1}{\phi(u)} \right)' \right| \leq \left| \int_u^\infty \left(\frac{1}{\phi(t)} \right)'' dt \right| \leq b_2 \left| \int_u^\infty \frac{1}{(1+t)^3} dt \right| \leq \frac{b_2}{2(1+u)^2}.$$

(3) By Condition 1(1) and Condition 1(3a), we have

$$\left| \frac{1}{1-p} - \frac{1}{1-\tilde{p}} \right| = \left| \frac{1}{1-\phi(u)} - \frac{1}{1-\phi(\tilde{u})} \right| = \left| \left(\frac{1}{1-\phi(\tilde{u})} \right)' \right| |u - \tilde{u}| = \left| \left(\frac{1}{\phi(-\tilde{u})} \right)' \right| |u - \tilde{u}| \leq b_1 |u - \tilde{u}|.$$

By the result in (1) and Condition 1(2), we have

$$\begin{aligned} \left| \frac{1}{1-p} - \frac{1}{1-\tilde{p}} \right| &= \left| \frac{1}{1-\phi(u)} - \frac{1}{1-\phi(\tilde{u})} \right| \\ &= \left| \int_{\tilde{u}}^u \left(\frac{1}{1-\phi(t)} \right)' dt \right| \\ &\geq \left| \int_{\tilde{u}}^u \left(\frac{1}{1-\phi(0)} \right)' dt \right| \\ &\geq \frac{b_3}{2} |u - \tilde{u}|. \end{aligned}$$

(4) By the result in (2), we have

$$\left| \frac{1}{p} - \frac{1}{\tilde{p}} \right| = \left| \frac{1}{\phi(u)} - \frac{1}{\phi(\tilde{u})} \right| \leq \left| \int_u^{\tilde{u}} \left(\frac{1}{\phi(t)} \right)' dt \right| \leq \frac{b_2}{2} \left| \int_u^{\tilde{u}} \frac{1}{(1+t)^2} dt \right| = \frac{b_2 |u - \tilde{u}|}{2(1+u)(1+\tilde{u})}.$$

(5) By Condition 1(3c), we have

$$\left| \left(\frac{1}{\phi(u)} \right)' - \left(\frac{1}{\phi(\tilde{u})} \right)' \right| \geq b_3 \left| \int_{\tilde{u}}^u \frac{1}{(1+t)^3} dt \right|$$

$$\begin{aligned}
&= \frac{b_3}{2} \left| \frac{1}{(1+\tilde{u})^2} - \frac{1}{(1+u)^2} \right| \\
&= \frac{b_3}{2} \frac{(1+\tilde{u}+u)}{(1+\tilde{u})^2(1+u)^2} |u-\tilde{u}| \\
&\geq \frac{b_3}{2} \frac{1}{(1+\tilde{u})(1+u)(1+\tilde{u}\wedge u)} |u-\tilde{u}| .
\end{aligned}$$

(6) By Condition 1(1), Condition 1(2) and the result in (4), we have

$$\frac{1}{\phi(u)} - \frac{1}{\phi(\tilde{u})} - \left(\frac{1}{\phi(\tilde{u})} \right)' (u - \tilde{u}) \leq \frac{1}{\phi(u)} - \frac{1}{\phi(\tilde{u})} \leq \frac{b_2(\tilde{u}-u)}{2(1+u)(1+\tilde{u})} .$$

By Condition 1(2) and Condition 1(3b), we have

$$\begin{aligned}
&\frac{1}{1-\phi(u)} - \frac{1}{1-\phi(\tilde{u})} - \left(\frac{1}{1-\phi(\tilde{u})} \right)' (u - \tilde{u}) \\
&= \int_u^{\tilde{u}} \int_t^{\tilde{u}} \left(\frac{1}{1-\phi(s)} \right)'' ds dt \\
&= \int_u^{\tilde{u}} \int_t^{\tilde{u}} \left(\frac{1}{\phi(-s)} \right)'' ds dt \\
&\leq b_2 \int_u^{\tilde{u}} \int_t^{\tilde{u}} \frac{1}{(1+s)^3} ds dt \\
&= \frac{b_2}{2} \int_u^{\tilde{u}} \left[\frac{1}{(1+t)^2} - \frac{1}{(1+\tilde{u})^2} \right] dt \\
&\leq \frac{b_2}{2} \left[\frac{1}{1+u} - \frac{1}{1+\tilde{u}} - \frac{1}{(1+\tilde{u})^2} (\tilde{u}-u) \right] \\
&= \frac{b_2}{2} \cdot \frac{(1+\tilde{u})(\tilde{u}-u) - (1+u)(\tilde{u}-u)}{(1+u)(1+\tilde{u})^2} \\
&= \frac{b_2}{2} \cdot \frac{(\tilde{u}-u)^2}{(1+u)(1+\tilde{u})^2} \\
&\leq \frac{b_2}{2} \cdot \frac{\tilde{u}-u}{(1+u)(1+\tilde{u})} .
\end{aligned}$$

□

B.3 Regret Decomposition

In this subsection, we derive a decomposition of the Neyman regret into two components: the probability regret and the prediction regret. We start by deriving the oracle variance in the following proposition.

Proposition 3.1. *The oracle variance is given by $T \cdot V^* = 2(1+\rho)\mathcal{E}(1)\mathcal{E}(0)$ and the Neyman allocation is given by the least squares predictors $\beta^*(k) = \beta_{OLS}(k)$ and assignment probability $p = (1 + \mathcal{E}(0)/\mathcal{E}(1))^{-1}$.*

Proof. By direct calculation, $\text{Var}(\hat{\tau}; \beta(1), \beta(0), p)$ is given by:

$$\begin{aligned}
&\text{Var}(\hat{\tau}; \beta(1), \beta(0), p) \\
&= \left(\sum_{t=1}^T \{y_t(1) - \langle \mathbf{x}_t, \beta(1) \rangle\}^2 \right) \cdot \left(\frac{1}{p} - 1 \right) + \left(\sum_{t=1}^T \{y_t(0) - \langle \mathbf{x}_t, \beta(0) \rangle\}^2 \right) \cdot \left(\frac{1}{1-p} - 1 \right)
\end{aligned}$$

$$+ 2 \sum_{t=1}^T \{y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}(1) \rangle\} \cdot \{y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}(0) \rangle\} .$$

By fixing $\boldsymbol{\beta}(1)$ and $\boldsymbol{\beta}(0)$, it is easy to see that the optimal choice of p is $p^* = (1 + (\sum_{t=1}^T (y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}(0) \rangle)^2)^{1/2} / (\sum_{t=1}^T (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}(1) \rangle)^2)^{1/2})^{-1}$. By plugging the form of p^* into the variance, we can obtain

$$\begin{aligned} \text{Var}(\hat{\tau}; \boldsymbol{\beta}(1), \boldsymbol{\beta}(0), p^*) &= 2 \left(\sum_{t=1}^T \{y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}(1) \rangle\}^2 \right)^{1/2} \left(\sum_{t=1}^T \{y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}(0) \rangle\}^2 \right)^{1/2} \\ &\quad + 2 \sum_{t=1}^T \{y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}(1) \rangle\} \cdot \{y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}(0) \rangle\} . \end{aligned}$$

Now we prove that $\text{Var}(\hat{\tau}; \boldsymbol{\beta}(1), \boldsymbol{\beta}(0), p^*) \geq \text{Var}(\hat{\tau}; \boldsymbol{\beta}^*(1), \boldsymbol{\beta}^*(0), p^*)$ for any $\boldsymbol{\beta}(1), \boldsymbol{\beta}(0)$. Let $\boldsymbol{\beta}(1) = \boldsymbol{\beta}^*(1) + \boldsymbol{\delta}_1$ and $\boldsymbol{\beta}(0) = \boldsymbol{\beta}^*(0) + \boldsymbol{\delta}_0$. We further denote $\boldsymbol{\alpha}_1 = \mathbf{Y}_T(1) - \mathbf{X}_T \boldsymbol{\beta}^*(1)$, $\boldsymbol{\alpha}_0 = \mathbf{Y}_T(0) - \mathbf{X}_T \boldsymbol{\beta}^*(0)$, $\mathbf{z}_1 = \mathbf{X}_T \boldsymbol{\delta}_1$ and $\mathbf{z}_0 = \mathbf{X}_T \boldsymbol{\delta}_0$. By the explicit form of $\boldsymbol{\beta}^*(1)$ and $\boldsymbol{\beta}^*(0)$, we can easily prove that $\langle \boldsymbol{\alpha}_1, \mathbf{z}_1 \rangle = \langle \boldsymbol{\alpha}_1, \mathbf{z}_0 \rangle = \langle \boldsymbol{\alpha}_0, \mathbf{z}_1 \rangle = \langle \boldsymbol{\alpha}_0, \mathbf{z}_0 \rangle = 0$. Hence we have

$$\begin{aligned} \text{Var}(\hat{\tau}; \boldsymbol{\beta}(1), \boldsymbol{\beta}(0), p^*) &= 2 (\|\boldsymbol{\alpha}_1 - \mathbf{z}_1\|_2^2)^{1/2} (\|\boldsymbol{\alpha}_0 - \mathbf{z}_0\|_2^2)^{1/2} + 2 \langle \boldsymbol{\alpha}_1 - \mathbf{z}_1, \boldsymbol{\alpha}_0 - \mathbf{z}_0 \rangle \\ &= 2 (\|\boldsymbol{\alpha}_1\|_2^2 + \|\mathbf{z}_1\|_2^2)^{1/2} (\|\boldsymbol{\alpha}_0\|_2^2 + \|\mathbf{z}_0\|_2^2)^{1/2} + 2 \langle \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_0 \rangle + 2 \langle \mathbf{z}_1, \mathbf{z}_0 \rangle , \\ \text{Var}(\hat{\tau}; \boldsymbol{\beta}^*(1), \boldsymbol{\beta}^*(0), p^*) &= 2 (\|\boldsymbol{\alpha}_1\|_2^2)^{1/2} (\|\boldsymbol{\alpha}_0\|_2^2)^{1/2} + 2 \langle \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_0 \rangle . \end{aligned}$$

Hence it suffices to prove that

$$(\|\boldsymbol{\alpha}_1\|_2^2 + \|\mathbf{z}_1\|_2^2)^{1/2} (\|\boldsymbol{\alpha}_0\|_2^2 + \|\mathbf{z}_0\|_2^2)^{1/2} + \langle \mathbf{z}_1, \mathbf{z}_0 \rangle \geq (\|\boldsymbol{\alpha}_1\|_2^2)^{1/2} (\|\boldsymbol{\alpha}_0\|_2^2)^{1/2} .$$

Note that we have $\langle \mathbf{z}_1, \mathbf{z}_0 \rangle \geq -\|\mathbf{z}_1\|_2 \|\mathbf{z}_0\|_2$ by Cauchy-Schwarz inequality. Then we only need to show that

$$(\|\boldsymbol{\alpha}_1\|_2^2 + \|\mathbf{z}_1\|_2^2)^{1/2} (\|\boldsymbol{\alpha}_0\|_2^2 + \|\mathbf{z}_0\|_2^2)^{1/2} \geq (\|\boldsymbol{\alpha}_1\|_2^2)^{1/2} (\|\boldsymbol{\alpha}_0\|_2^2)^{1/2} + \|\mathbf{z}_1\|_2 \|\mathbf{z}_0\|_2 . \quad (12)$$

Since both sides of (12) are nonnegative, it is equivalent to

$$(\|\boldsymbol{\alpha}_1\|_2^2 + \|\mathbf{z}_1\|_2^2) (\|\boldsymbol{\alpha}_0\|_2^2 + \|\mathbf{z}_0\|_2^2) \geq (\|\boldsymbol{\alpha}_1\|_2^2) (\|\boldsymbol{\alpha}_0\|_2^2) + 2(\|\boldsymbol{\alpha}_1\|_2^2 \|\boldsymbol{\alpha}_0\|_2^2 \|\mathbf{z}_1\|_2^2 \|\mathbf{z}_0\|_2^2)^{1/2} + (\|\mathbf{z}_1\|_2^2) (\|\mathbf{z}_0\|_2^2) ,$$

which can be further simplified as

$$\|\boldsymbol{\alpha}_1\|_2^2 \|\mathbf{z}_0\|_2^2 + \|\boldsymbol{\alpha}_0\|_2^2 \|\mathbf{z}_1\|_2^2 \geq 2(\|\boldsymbol{\alpha}_1\|_2^2 \|\boldsymbol{\alpha}_0\|_2^2 \|\mathbf{z}_1\|_2^2 \|\mathbf{z}_0\|_2^2)^{1/2} .$$

This is immediately verified by AM-GM inequality. Hence $\boldsymbol{\beta}^*(1)$ and $\boldsymbol{\beta}^*(0)$ minimize $\text{Var}(\hat{\tau}; \boldsymbol{\beta}(1), \boldsymbol{\beta}(0), p^*)$. By plugging in $\boldsymbol{\beta}^*(1)$ and $\boldsymbol{\beta}^*(0)$, we can derive the minimizer for p as $p^* = (1 + \mathcal{E}(0)/\mathcal{E}(1))^{-1}$. \square

Based on Proposition 3.1, we can derive the decomposition of the Neyman regret as stated in the following lemma.

Lemma 3.3. *Under Assumption 1, the Neyman Regret can be decomposed as the T -normalized sum of the probability regret and the prediction regret:*

$$\mathcal{R}_T^{\text{Neyman}} = \frac{1}{T} \mathbb{E}[\mathcal{R}_T^{\text{prob}}] + \frac{1}{T} \mathbb{E}[\mathcal{R}_T^{\text{pred}}] .$$

Proof. Let us recall the expression for the variance of the AIPW estimator from Proposition 2.2:

$$T \cdot \text{Var}(\hat{\tau})$$

$$\begin{aligned}
&= \mathbb{E} \left[\frac{1}{T} \sum_{t=1}^T \{y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle\}^2 \cdot \left(\frac{1}{p_t} - 1 \right) \right] + \mathbb{E} \left[\frac{1}{T} \sum_{t=1}^T \{y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle\}^2 \cdot \left(\frac{1}{1-p_t} - 1 \right) \right] \\
&\quad + 2\mathbb{E} \left[\frac{1}{T} \sum_{t=1}^T \{y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle\} \cdot \{y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle\} \right] \\
&= \mathbb{E} \left[\frac{1}{T} \sum_{t=1}^T f_t(p_t) \right] + 2\mathbb{E} \left[\frac{1}{T} \sum_{t=1}^T \{y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle\} \cdot \{y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle\} \right] \\
&= \frac{1}{T} \mathbb{E}[\mathcal{R}_T^{\text{prob}}] + \mathbb{E} \left[\frac{1}{T} \sum_{t=1}^T f_t(p^*) \right] + 2\mathbb{E} \left[\frac{1}{T} \sum_{t=1}^T \{y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle\} \cdot \{y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle\} \right] \\
&\triangleq \frac{1}{T} \mathbb{E}[\mathcal{R}_T^{\text{prob}}] + S_1.
\end{aligned}$$

By the definition of prediction regret, we can further decompose term S_1 as:

$$\begin{aligned}
S_1 &= \mathbb{E} \left[\frac{1}{T} \sum_{t=1}^T \{y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle\}^2 \right] \cdot \left(\frac{1}{p^*} - 1 \right) + \mathbb{E} \left[\frac{1}{T} \sum_{t=1}^T \{y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle\}^2 \right] \cdot \left(\frac{1}{1-p^*} - 1 \right) \\
&\quad + 2\mathbb{E} \left[\frac{1}{T} \sum_{t=1}^T \{y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle\} \cdot \{y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle\} \right] \\
&= \frac{1}{T} \sum_{t=1}^T \{y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}^*(1) \rangle\}^2 \cdot \left(\frac{1}{p^*} - 1 \right) + \frac{1}{T} \sum_{t=1}^T \{y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}^*(0) \rangle\}^2 \cdot \left(\frac{1}{1-p^*} - 1 \right) \\
&\quad + \frac{2}{T} \sum_{t=1}^T \{y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}^*(1) \rangle\} \cdot \{y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}^*(0) \rangle\} \\
&\quad + \left(\frac{1}{p^*} - 1 \right) \cdot \mathbb{E} \left[\frac{1}{T} \sum_{t=1}^T \{y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle\}^2 - \frac{1}{T} \sum_{t=1}^T \{y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}^*(1) \rangle\}^2 \right] \\
&\quad + \left(\frac{1}{1-p^*} - 1 \right) \cdot \mathbb{E} \left[\frac{1}{T} \sum_{t=1}^T \{y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle\}^2 - \frac{1}{T} \sum_{t=1}^T \{y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}^*(0) \rangle\}^2 \right] \\
&\quad + 2\mathbb{E} \left[\frac{1}{T} \sum_{t=1}^T \{y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle\} \cdot \{y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle\} - \frac{1}{T} \sum_{t=1}^T \{y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}^*(1) \rangle\} \cdot \{y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}^*(0) \rangle\} \right] \\
&= \mathcal{E}^2(1) \frac{\mathcal{E}(0)}{\mathcal{E}(1)} + \mathcal{E}^2(0) \frac{\mathcal{E}(1)}{\mathcal{E}(0)} + 2\rho \mathcal{E}(1) \mathcal{E}(0) + \frac{\mathcal{E}(0)}{\mathcal{E}(1)} \mathbb{E} \left[\frac{1}{T} \sum_{t=1}^T \{y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle\}^2 - \frac{1}{T} \sum_{t=1}^T \{y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}^*(1) \rangle\}^2 \right] \\
&\quad + \frac{\mathcal{E}(1)}{\mathcal{E}(0)} \mathbb{E} \left[\frac{1}{T} \sum_{t=1}^T \{y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle\}^2 - \frac{1}{T} \sum_{t=1}^T \{y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}^*(0) \rangle\}^2 \right] \\
&\quad + 2\mathbb{E} \left[\frac{1}{T} \sum_{t=1}^T \{y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle\} \cdot \{y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle\} - \frac{1}{T} \sum_{t=1}^T \{y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}^*(1) \rangle\} \cdot \{y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}^*(0) \rangle\} \right] \\
&= 2(1+\rho) \mathcal{E}(1) \mathcal{E}(0) + \frac{1}{T} \mathbb{E}[\mathcal{R}_T^{\text{pred}}].
\end{aligned}$$

Hence the Neyman regret can be finally decomposed by:

$$\begin{aligned}
\mathcal{R}_T^{\text{Neyman}} &= T \cdot \text{Var}(\hat{\tau}) - T \cdot V^* \\
&= \frac{1}{T} \mathbb{E}[\mathcal{R}_T^{\text{prob}}] + 2(1+\rho) \mathcal{E}(1) \mathcal{E}(0) + \frac{1}{T} \mathbb{E}[\mathcal{R}_T^{\text{pred}}] - 2(1+\rho) \mathcal{E}(1) \mathcal{E}(0) \\
&= \frac{1}{T} \mathbb{E}[\mathcal{R}_T^{\text{prob}}] + \frac{1}{T} \mathbb{E}[\mathcal{R}_T^{\text{pred}}].
\end{aligned}$$

□

B.4 Probability Regret

In this subsection, we aim to analyze the inverse weighting terms that are appearing in the procedure and derive an upper bound for the probability regret. Throughout the analysis, we find it important to control the expectation of the inverse probability. This bound is established through a sequence of intermediate results, including Lemma 4.14, Corollary 4.15, Lemma 4.16, and Corollary B.17.

In particular, Lemma 4.14 provides an upper bound for the inverse probability under the given optimization program.

Lemma 4.14. *Consider $A, B \geq 0$ and define p^* as the minimizer of the following program:*

$$p^* = \arg \min_{p \in (0,1)} \frac{A}{p} + \frac{B}{1-p} + \eta^{-1} \Psi(p) .$$

Under Condition 1, the minimizer p^ is bounded away from 0 and 1 in the following sense:*

$$\begin{aligned} \frac{1}{p^*} &\leq 2 + b_1(b_2/6)^{1/4} \eta^{1/4} B^{1/4} \wedge b_1(b_2/b_3)^{1/2} (B/A)^{1/2} , \\ \frac{1}{1-p^*} &\leq 2 + b_1(b_2/6)^{1/4} \eta^{1/4} A^{1/4} \wedge b_1(b_2/b_3)^{1/2} (A/B)^{1/2} . \end{aligned}$$

Proof. If $A \geq B$, then we have $p^* \geq 1/2$ and $1/p^* \leq 2$. By definition, $u^* = \phi(p^*) \geq 0$ should satisfy the following first-order equation:

$$A \left(\frac{1}{\phi(u^*)} \right)' + B \left(\frac{1}{1-\phi(u^*)} \right)' + \eta^{-1}(u^* + 3(u^*)^2) = 0 .$$

By Lemma B.16 and Condition 1, we have

$$3\eta^{-1}(u^*)^2 \leq \eta^{-1}(u^* + 3(u^*)^2) + B \left(\frac{1}{1-\phi(u^*)} \right)' = -A \left(\frac{1}{\phi(u^*)} \right)' \leq \frac{b_2}{2} \cdot \frac{A}{(1+u^*)^2} \leq \frac{b_2 A}{2(u^*)^2} .$$

Hence we have $u^* \leq (b_2/6)^{1/4} \eta^{1/4} A^{1/4}$. Moreover, by Lemma B.16, we have

$$\frac{b_3 B}{2} \leq B \left(\frac{1}{1-\phi(0)} \right)' \leq B \left(\frac{1}{1-\phi(u^*)} \right)' \leq \eta^{-1}(u^* + 3(u^*)^3) + B \left(\frac{1}{1-\phi(u^*)} \right)' \leq \frac{b_2 A}{2(1+u^*)^2} .$$

This implies that $u^* \leq (b_2/b_3)^{1/2} (A/B)^{1/2} - 1$. Then by Condition 1 we have

$$\frac{1}{1-p^*} = \frac{1}{1-\phi(u^*)} \leq \frac{1}{1-\phi(0)} + b_1 u^* \leq 2 + b_1(b_2/6)^{1/4} \eta^{1/4} A^{1/4} \wedge b_1(b_2/b_3)^{1/2} (A/B)^{1/2} .$$

We can derive similar results when $A < B$, i.e., $\frac{1}{p^*} \leq 2 + b_1(b_2/6)^{1/4} \eta^{1/4} B^{1/4} \wedge b_1(b_2/b_3)^{1/2} (B/A)^{1/2}$. \square

Based Lemma 4.14 and Jensen's inequality, we can upper bound the moments of the inverse probabilities in terms of the expectation of the estimated squared residuals. The result is summarized in the following corollary.

Corollary 4.15. *Under Condition 1, for each iteration $t \in [T]$ and any $0 \leq k \leq 4$, the k th moment of the inverse probabilities are bounded as*

$$\begin{aligned} \mathbb{E} \left[\left(\frac{1}{p_t} \right)^k \right] &\leq \left(2 + b_1(b_2/6)^{1/4} \eta_t^{1/4} \left[\mathbb{E} \hat{A}_{t-1}(0) \right]^{1/4} \right)^k , \\ \mathbb{E} \left[\left(\frac{1}{1-p_t} \right)^k \right] &\leq \left(2 + b_1(b_2/6)^{1/4} \eta_t^{1/4} \left[\mathbb{E} \hat{A}_{t-1}(1) \right]^{1/4} \right)^k . \end{aligned}$$

Proof. By the definition of p_t , Lemma 4.14 and Jensen's inequality, for any $0 \leq k \leq 4$ we have

$$\mathbb{E} \left[\frac{1}{p_t^k} \right] \leq \mathbb{E} \left[\left(2 + b_1(b_2/6)^{1/4} \eta_t^{1/4} \hat{A}_{t-1}^{1/4}(0) \right)^k \right] \leq \left(2 + b_1(b_2/6)^{1/4} \eta_t^{1/4} \left[\mathbb{E} \hat{A}_{t-1}(0) \right]^{1/4} \right)^k.$$

Similarly, we can prove the result for the moments of $1/(1 - p_t)$. \square

Corollary 4.15 implies that we need to further bound the expectation of the estimated squared residuals, which is explicitly derived in the following lemma.

Lemma 4.16. *For any $t \in [T]$, the expectation of the estimated squared residuals is equal to*

$$\begin{aligned} \mathbb{E}[\hat{A}_t(1)] &= A_t^*(1) + \sum_{s=1}^t \sum_{r=1}^{s-1} \Pi_{s,r}^2 y_r^2(1) \mathbb{E} \left[\frac{1}{p_r} - 1 \right] \\ \mathbb{E}[\hat{A}_t(0)] &= A_t^*(0) + \sum_{s=1}^t \sum_{r=1}^{s-1} \Pi_{s,r}^2 y_r^2(0) \mathbb{E} \left[\frac{1}{1-p_r} - 1 \right]. \end{aligned}$$

Proof. By Lemma 4.12 and the law of iterated expectation, the expectation of $\hat{A}_t(1)$ is calculated by:

$$\begin{aligned} &\mathbb{E}[\hat{A}_t(1)] \\ &= \mathbb{E} \left[\sum_{s=1}^t \frac{\mathbf{1}[Z_s = 1]}{p_s} \cdot (y_s(1) - \langle \mathbf{x}_s, \boldsymbol{\beta}_s(1) \rangle)^2 \right] \\ &= \sum_{s=1}^t \mathbb{E} \left[\mathbb{E} \left[\frac{\mathbf{1}[Z_s = 1]}{p_s} \cdot (y_s(1) - \langle \mathbf{x}_s, \boldsymbol{\beta}_s(1) \rangle)^2 \middle| \mathcal{F}_{s-1} \right] \right] \\ &= \sum_{s=1}^t \mathbb{E} \left[(y_s(1) - \langle \mathbf{x}_s, \boldsymbol{\beta}_s(1) \rangle)^2 \right] \\ &= \sum_{s=1}^t \mathbb{E} \left[y_s^2(1) - 2y_s(1)\langle \mathbf{x}_s, \boldsymbol{\beta}_s(1) \rangle + \sum_{1 \leq r_1, r_2 \leq s-1} \Pi_{s,r_1} \Pi_{s,r_2} y_{r_1}(1) y_{r_2}(1) \frac{\mathbf{1}[Z_{r_1} = 1]}{p_{r_1}} \cdot \frac{\mathbf{1}[Z_{r_2} = 1]}{p_{r_2}} \right] \\ &= \sum_{s=1}^t y_s^2(1) - 2y_s(1)\langle \mathbf{x}_s, \boldsymbol{\beta}_s^*(1) \rangle + \sum_{s=1}^t \sum_{r=1}^{s-1} \Pi_{s,r}^2 y_r^2(1) \mathbb{E} \left[\left(\frac{\mathbf{1}[Z_r = 1]}{p_r} \right)^2 \middle| \mathcal{F}_{r-1} \right] \\ &\quad + 2 \sum_{s=1}^t \sum_{1 \leq r_1 < r_2 \leq s-1} \mathbb{E} \left[\mathbb{E} \left[\Pi_{s,r_1} \Pi_{s,r_2} y_{r_1}(1) y_{r_2}(1) \frac{\mathbf{1}[Z_{r_1} = 1]}{p_{r_1}} \cdot \frac{\mathbf{1}[Z_{r_2} = 1]}{p_{r_2}} \middle| \mathcal{F}_{r_2-1} \right] \right] \\ &= \sum_{s=1}^t y_s^2(1) - 2y_s(1)\langle \mathbf{x}_s, \boldsymbol{\beta}_s^*(1) \rangle + \sum_{s=1}^t \sum_{r=1}^{s-1} \Pi_{s,r}^2 y_r^2(1) \mathbb{E} \left[\frac{1}{p_r} \right] + 2 \sum_{s=1}^t \sum_{1 \leq r_1 < r_2 \leq s-1} \Pi_{s,r_1} \Pi_{s,r_2} y_{r_1}(1) y_{r_2}(1) \\ &= \sum_{s=1}^t (y_s(1) - \langle \mathbf{x}_s, \boldsymbol{\beta}_s^*(1) \rangle)^2 + \sum_{s=1}^t \sum_{r=1}^{s-1} \Pi_{s,r}^2 y_r^2(1) \mathbb{E} \left[\frac{1}{p_r} - 1 \right] \\ &= A_t^*(1) + \sum_{s=1}^t \sum_{r=1}^{s-1} \Pi_{s,r}^2 y_r^2(1) \mathbb{E} \left[\frac{1}{p_r} - 1 \right]. \end{aligned}$$

Similarly, we can prove that

$$\mathbb{E}[\hat{A}_t(0)] = A_t^*(0) + \sum_{s=1}^t \sum_{r=1}^{s-1} \Pi_{s,r}^2 y_r^2(0) \mathbb{E} \left[\frac{1}{1-p_r} - 1 \right].$$

\square

The following corollary characterized the rate of the expectation of the estimated squared residuals.

Corollary B.17. *Under Assumptions 1-3 and Condition 1, for any $t \in [T]$, the expectation of the estimated squared residuals can be bounded as:*

$$\max \left\{ \mathbb{E}[\hat{A}_t(1)], \mathbb{E}[\hat{A}_t(0)] \right\} \leq \kappa = (2c_1^2 + o(1))T ,$$

where $\kappa > 2c_1^2 T$ is the largest solution to the following equation:

$$\kappa = 2c_1^2 T + c_1^2 (\gamma_0 \vee c_2 \vee 1)^{3/2} \xi_{3/2} \left(b_1 (b_2/6)^{1/4} T^{-1/8} \kappa^{1/4} + 1 \right) R_T^2 T^{1/4} .$$

Proof. We use induction method to prove the result. For $t = 1$, we have $\mathbb{E}\hat{A}_t(1) = A_t^*(1) = y_t(1)^2 \leq c_1^2 T^{1/2} \leq 2c_1^2 T \leq \kappa$ and $\mathbb{E}\hat{A}_t(0) = A_t^*(0) = y_t(0)^2 \leq c_1^2 T^{1/2} \leq 2c_1^2 T \leq \kappa$. Hence the result holds. If the result is proved for $1, \dots, t-1$, then by Lemma B.11, Lemma B.15, Lemma 4.16 and Corollary 4.15, we have

$$\begin{aligned} \mathbb{E}[\hat{A}_t(1)] &= A_t^*(1) + \sum_{s=1}^t \sum_{r=1}^{s-1} \Pi_{s,r}^2 y_r^2(1) \mathbb{E} \left[\frac{1}{p_r} - 1 \right] \quad (\text{Lemma 4.16}) \\ &\leq 2c_1^2 T + \sum_{s=1}^t \sum_{r=1}^{s-1} \Pi_{s,r}^2 y_r^2(1) \left(1 + b_1 (b_2/6)^{1/4} \eta_r^{1/4} \left[\mathbb{E}\hat{A}_{r-1}(0) \right]^{1/4} \right) \quad (\text{Corollary 4.15, Lemma B.11}) \\ &\leq 2c_1^2 T + b_1 (b_2/6)^{1/4} \kappa^{1/4} \sum_{s=1}^t \sum_{r=1}^{s-1} \Pi_{s,r}^2 \eta_r^{1/4} y_r^2(1) + \sum_{s=1}^t \sum_{r=1}^{s-1} \Pi_{s,r}^2 y_r^2(1) \quad (\text{induction assumption}) \\ &= 2c_1^2 T + b_1 (b_2/6)^{1/4} T^{-1/8} \kappa^{1/4} \sum_{s=1}^t R_s^{-1/2} \sum_{r=1}^{s-1} \Pi_{s,r}^2 y_r^2(1) + \sum_{s=1}^t \sum_{r=1}^{s-1} \Pi_{s,r}^2 y_r^2(1) \\ &\leq 2c_1^2 T + \left(b_1 (b_2/6)^{1/4} T^{-1/8} \kappa^{1/4} + 1 \right) \sum_{s=1}^t \sum_{r=1}^{s-1} \Pi_{s,r}^2 y_r^2(1) \\ &\leq 2c_1^2 T + c_1^2 (\gamma_0 \vee c_2 \vee 1)^{3/2} \xi_{3/2} \left(b_1 (b_2/6)^{1/4} T^{-1/8} \kappa^{1/4} + 1 \right) R_t^3 \eta_t^{1/2} T^{1/2} \quad (\text{Lemma B.15}) \\ &\leq 2c_1^2 T + c_1^2 (\gamma_0 \vee c_2 \vee 1)^{3/2} \xi_{3/2} \left(b_1 (b_2/6)^{1/4} T^{-1/8} \kappa^{1/4} + 1 \right) R_T^2 T^{1/4} \quad (\text{since } R_t \leq R_T) \\ &= \kappa . \end{aligned}$$

Similarly, we can prove that $\mathbb{E}[\hat{A}_t(0)] \leq \kappa$. The result is thus proved by induction method. By Assumption 3, we have $R_T^2 T^{1/4} = o(T^{3/4})$. Hence it is easy to see that $\kappa = (2c_1^2 + o(1))T$. \square

Under Corollary 4.15 and Corollary B.17, we are now able to control the expectations of powers of inverse probabilities. The following lemma extends these results by providing upper bounds for moments of the inverse weighting terms that arise in the subsequent proofs, beyond the simple power-type functions.

Lemma B.18. *Under Assumptions 1-3 and Condition 1, there holds:*

- (1) $\max_{1 \leq s_1 \leq T} \mathbb{E} \left[\left(\frac{\mathbf{1}[Z_{s_1}=1]}{p_{s_1}} - 1 \right)^4 \right] = \mathcal{O}(T^{3/8})$.
- (2) $\max_{1 \leq s_1 \neq s_2 \leq T} \left| \mathbb{E} \left[\left(\frac{\mathbf{1}[Z_{s_1}=1]}{p_{s_1}} - 1 \right)^3 \left(\frac{\mathbf{1}[Z_{s_2}=1]}{p_{s_2}} - 1 \right) \right] \right| = \mathcal{O}(T^{5/16})$.
- (3) $\max_{1 \leq s_1 \neq s_2 \leq T} \mathbb{E} \left[\left(\frac{\mathbf{1}[Z_{s_1}=1]}{p_{s_1}} - 1 \right)^2 \left(\frac{\mathbf{1}[Z_{s_2}=1]}{p_{s_2}} - 1 \right)^2 \right] = \mathcal{O}(T^{9/32})$.
- (4) $\max_{1 \leq s_1 \neq s_2 \neq s_3 \leq T} \left| \mathbb{E} \left[\left(\frac{\mathbf{1}[Z_{s_1}=1]}{p_{s_1}} - 1 \right)^2 \left(\frac{\mathbf{1}[Z_{s_2}=1]}{p_{s_2}} - 1 \right) \left(\frac{\mathbf{1}[Z_{s_3}=1]}{p_{s_3}} - 1 \right) \right] \right| = \mathcal{O}(T^{25/128})$.

$$\begin{aligned}
(5) \quad & \text{For } s_3 \neq s_2 < s_1, \left| \text{Cov} \left(\left(\frac{\mathbf{1}[Z_{s_1}=1]}{p_{s_1}} - 1 \right) \left(\frac{\mathbf{1}[Z_{s_2}=1]}{p_{s_2}} - 1 \right), \left(\frac{\mathbf{1}[Z_{s_1}=1]}{p_{s_1}} - 1 \right) \left(\frac{\mathbf{1}[Z_{s_3}=1]}{p_{s_3}} - 1 \right) \right) \right| \\
& \leq 4 \left(2 + b_1 (b_2/6)^{1/4} \eta_{s_1}^{1/4} \left[\text{E} \hat{A}_{s_1}(0) \right]^{1/4} \right)^{25/16} \left(\frac{R_{s_1}}{R_{s_2}} \right)^{15/96} \left(\frac{R_{s_1}}{R_{s_3}} \right)^{15/96}. \\
(6) \quad & \max_{1 \leq s_1 \leq T} \text{E} \left[\left(\frac{\mathbf{1}[Z_{s_1}=0]}{1-p_{s_1}} - 1 \right)^4 \right] = \mathcal{O}(T^{3/8}). \\
(7) \quad & \max_{1 \leq s_1 \neq s_2 \leq T} \left| \text{E} \left[\left(\frac{\mathbf{1}[Z_{s_1}=0]}{1-p_{s_1}} - 1 \right)^3 \left(\frac{\mathbf{1}[Z_{s_2}=0]}{1-p_{s_2}} - 1 \right) \right] \right| = \mathcal{O}(T^{5/16}). \\
(8) \quad & \max_{1 \leq s_1 \neq s_2 \leq T} \text{E} \left[\left(\frac{\mathbf{1}[Z_{s_1}=0]}{1-p_{s_1}} - 1 \right)^2 \left(\frac{\mathbf{1}[Z_{s_2}=0]}{1-p_{s_2}} - 1 \right)^2 \right] = \mathcal{O}(T^{9/32}). \\
(9) \quad & \max_{1 \leq s_1 \neq s_2 \neq s_3 \leq T} \left| \text{E} \left[\left(\frac{\mathbf{1}[Z_{s_1}=0]}{1-p_{s_1}} - 1 \right)^2 \left(\frac{\mathbf{1}[Z_{s_2}=0]}{1-p_{s_2}} - 1 \right) \left(\frac{\mathbf{1}[Z_{s_3}=0]}{1-p_{s_3}} - 1 \right) \right] \right| = \mathcal{O}(T^{25/128}). \\
(10) \quad & \text{For } s_3 \neq s_2 < s_1, \left| \text{Cov} \left(\left(\frac{\mathbf{1}[Z_{s_1}=0]}{1-p_{s_1}} - 1 \right) \left(\frac{\mathbf{1}[Z_{s_2}=0]}{1-p_{s_2}} - 1 \right), \left(\frac{\mathbf{1}[Z_{s_1}=0]}{1-p_{s_1}} - 1 \right) \left(\frac{\mathbf{1}[Z_{s_3}=0]}{1-p_{s_3}} - 1 \right) \right) \right| \\
& \leq 4 \left(2 + b_1 (b_2/6)^{1/4} \eta_{s_1}^{1/4} \left[\text{E} \hat{A}_{s_1}(1) \right]^{1/4} \right)^{25/16} \left(\frac{R_{s_1}}{R_{s_2}} \right)^{15/96} \left(\frac{R_{s_1}}{R_{s_3}} \right)^{15/96}.
\end{aligned}$$

Proof. Without loss of generality, we only prove (1)-(5).

(1) For any $s_1 \in [T]$, by Corollary 4.15 and Corollary B.17, we have

$$\begin{aligned}
\max_{1 \leq s_1 \leq T} \text{E} \left[\left(\frac{\mathbf{1}[Z_{s_1}=1]}{p_{s_1}} - 1 \right)^4 \right] & \leq \max_{1 \leq s \leq T} \text{E} \left[\left(\frac{\mathbf{1}[Z_s=1]}{p_s} \right)^4 \right] + 1 \\
& = \max_{1 \leq s \leq T} \text{E} \left[\frac{1}{p_s^3} \right] + 1 \\
& \leq \left(2 + b_1 (b_2/6)^{1/4} \eta_1^{1/4} \kappa^{1/4} \right)^3 + 1 \\
& \leq \left(2 + b_1 (b_2/6)^{1/4} T^{-1/8} \kappa^{1/4} \right)^3 + 1 \\
& = \mathcal{O}(T^{3/8}) .
\end{aligned}$$

(2) If $s_1 < s_2$, then by law of iterated expectation, it is easy to see that

$$\text{E} \left[\left(\frac{\mathbf{1}[Z_{s_1}=1]}{p_{s_1}} - 1 \right)^3 \left(\frac{\mathbf{1}[Z_{s_2}=1]}{p_{s_2}} - 1 \right) \right] = 0 .$$

If $s_2 < s_1$, then by law of iterated expectation, Corollary 4.15, Corollary B.17 and Hölder's inequality, we have

$$\begin{aligned}
& \max_{1 \leq s_2 < s_1 \leq T} \left| \text{E} \left[\left(\frac{\mathbf{1}[Z_{s_1}=1]}{p_{s_1}} - 1 \right)^3 \left(\frac{\mathbf{1}[Z_{s_2}=1]}{p_{s_2}} - 1 \right) \right] \right| \\
& \leq \max_{1 \leq s_2 < s_1 \leq T} \left| \text{E} \left[\left(\frac{\mathbf{1}[Z_{s_1}=1]}{p_{s_1}^3} + 1 \right) \left(\frac{\mathbf{1}[Z_{s_2}=1]}{p_{s_2}} + 1 \right) \right] \right| \\
& \leq \max_{1 \leq s_2 < s_1 \leq T} \left(\text{E} \left[\frac{\mathbf{1}[Z_{s_1}=1]}{p_{s_1}^3} \frac{\mathbf{1}[Z_{s_2}=1]}{p_{s_2}} \right] + \text{E} \left[\frac{\mathbf{1}[Z_{s_1}=1]}{p_{s_1}^3} \right] + \text{E} \left[\frac{\mathbf{1}[Z_{s_2}=1]}{p_{s_2}} \right] + 1 \right) \\
& \leq \max_{1 \leq s_2 < s_1 \leq T} \left(2 + \text{E} \left[\frac{1}{p_{s_1}^2} \frac{\mathbf{1}[Z_{s_2}=1]}{p_{s_2}} \right] + \text{E} \left[\frac{1}{p_{s_1}^2} \right] \right) \\
& \leq \max_{1 \leq s_2 < s_1 \leq T} \left(2 + \left(\text{E} \left[\frac{1}{p_{s_1}^4} \right] \right)^{1/2} \left(\text{E} \left[\frac{\mathbf{1}[Z_{s_2}=1]}{p_{s_2}^2} \right] \right)^{1/2} + \left(2 + b_1 (b_2/6)^{1/4} \eta_{s_1}^{1/4} \kappa^{1/4} \right)^2 \right)
\end{aligned}$$

$$\begin{aligned}
&\leq \max_{1 \leq s_2 < s_1 \leq T} \left(2 + \left(\mathbb{E} \left[\frac{1}{p_{s_1}^4} \right] \right)^{1/2} \left(\mathbb{E} \left[\frac{1}{p_{s_2}} \right] \right)^{1/2} + \left(2 + b_1(b_2/6)^{1/4} \eta_1^{1/4} \kappa^{1/4} \right)^2 \right) \\
&\leq \max_{1 \leq s_2 < s_1 \leq T} \left(2 + \left(2 + b_1(b_2/6)^{1/4} \eta_1^{1/4} \kappa^{1/4} \right)^{4 \cdot \frac{1}{2} + \frac{1}{2}} + \left(2 + b_1(b_2/6)^{1/4} \eta_1^{1/4} \kappa^{1/4} \right)^2 \right) \\
&= \mathcal{O}(T^{5/16}) .
\end{aligned}$$

- (3) For any $1 \leq s_2 < s_1 \leq T$, by law of iterated expectation, Corollary 4.15, Corollary B.17 and Hölder's inequality, we have

$$\begin{aligned}
&\max_{1 \leq s_2 < s_1 \leq T} \mathbb{E} \left[\left(\frac{\mathbf{1}[Z_{s_1} = 1]}{p_{s_1}} - 1 \right)^2 \left(\frac{\mathbf{1}[Z_{s_2} = 1]}{p_{s_2}} - 1 \right)^2 \right] \\
&\leq \max_{1 \leq s_2 < s_1 \leq T} \mathbb{E} \left[\left(\frac{\mathbf{1}[Z_{s_1} = 1]}{p_{s_1}^2} + 1 \right) \left(\frac{\mathbf{1}[Z_{s_2} = 1]}{p_{s_2}^2} + 1 \right) \right] \\
&= \max_{1 \leq s_2 < s_1 \leq T} \left(\mathbb{E} \left[\frac{\mathbf{1}[Z_{s_1} = 1]}{p_{s_1}^2} \frac{\mathbf{1}[Z_{s_2} = 1]}{p_{s_2}^2} \right] + \mathbb{E} \left[\frac{\mathbf{1}[Z_{s_1} = 1]}{p_{s_1}^2} \right] + \mathbb{E} \left[\frac{\mathbf{1}[Z_{s_2} = 1]}{p_{s_2}^2} \right] + 1 \right) \\
&= \max_{1 \leq s_2 < s_1 \leq T} \left(\mathbb{E} \left[\frac{1}{p_{s_1}} \frac{\mathbf{1}[Z_{s_2} = 1]}{p_{s_2}^2} \right] + \mathbb{E} \left[\frac{1}{p_{s_1}} \right] + \mathbb{E} \left[\frac{1}{p_{s_2}} \right] + 1 \right) \\
&\leq \max_{1 \leq s_2 < s_1 \leq T} \left(\left(\mathbb{E} \left[\frac{1}{p_{s_1}^4} \right] \right)^{1/4} \left(\mathbb{E} \left[\frac{\mathbf{1}[Z_{s_2} = 1]}{p_{s_2}^{2 \cdot \frac{4}{3}}} \right] \right)^{3/4} + \mathbb{E} \left[\frac{1}{p_{s_1}} \right] + \mathbb{E} \left[\frac{1}{p_{s_2}} \right] + 1 \right) \\
&\leq \max_{1 \leq s_2 < s_1 \leq T} \left(\left(\mathbb{E} \left[\frac{1}{p_{s_1}^4} \right] \right)^{1/4} \left(\mathbb{E} \left[\frac{1}{p_{s_2}^{5/3}} \right] \right)^{3/4} + 2 \left(2 + b_1(b_2/6)^{1/4} \eta_1^{1/4} \kappa^{1/4} \right) + 1 \right) \\
&\leq \max_{1 \leq s_2 < s_1 \leq T} \left(\left(2 + b_1(b_2/6)^{1/4} \eta_1^{1/4} \kappa^{1/4} \right)^{4 \cdot \frac{1}{4} + \frac{5}{3} \cdot \frac{3}{4}} + 2 \left(2 + b_1(b_2/6)^{1/4} \eta_1^{1/4} \kappa^{1/4} \right) + 1 \right) \\
&= \mathcal{O}(T^{9/32}) .
\end{aligned}$$

- (4) If s_2 or s_3 is the largest number among $\{s_1, s_2, s_3\}$, then by law of iterated expectation, it is easy to see that

$$\mathbb{E} \left[\left(\frac{\mathbf{1}[Z_{s_1} = 1]}{p_{s_1}} - 1 \right)^2 \left(\frac{\mathbf{1}[Z_{s_2} = 1]}{p_{s_2}} - 1 \right) \left(\frac{\mathbf{1}[Z_{s_3} = 1]}{p_{s_3}} - 1 \right) \right] = 0 .$$

If s_1 is the largest number among $\{s_1, s_2, s_3\}$, we assume WLOG that $s_3 < s_2$. By law of iterated expectation, Cauchy-Schwarz inequality and the result in (3), we have

$$\begin{aligned}
&\max_{1 \leq s_3 < s_2 < s_1 \leq T} \left| \mathbb{E} \left[\left(\frac{\mathbf{1}[Z_{s_1} = 1]}{p_{s_1}} - 1 \right)^2 \left(\frac{\mathbf{1}[Z_{s_2} = 1]}{p_{s_2}} - 1 \right) \left(\frac{\mathbf{1}[Z_{s_3} = 1]}{p_{s_3}} - 1 \right) \right] \right| \\
&= \max_{1 \leq s_3 < s_2 < s_1 \leq T} \left| \mathbb{E} \left[\frac{1-p_{s_1}}{p_{s_1}} \left(\frac{\mathbf{1}[Z_{s_2} = 1]}{p_{s_2}} - 1 \right) \left(\frac{\mathbf{1}[Z_{s_3} = 1]}{p_{s_3}} - 1 \right) \right] \right| \\
&= \max_{1 \leq s_3 < s_2 < s_1 \leq T} \left| \mathbb{E} \left[\frac{1}{p_{s_1}} \left(\frac{\mathbf{1}[Z_{s_2} = 1]}{p_{s_2}} - 1 \right) \left(\frac{\mathbf{1}[Z_{s_3} = 1]}{p_{s_3}} - 1 \right) \right] \right| \\
&\leq \max_{1 \leq s_3 < s_2 < s_1 \leq T} \mathbb{E} \left[\frac{1}{p_{s_1}} \left(\frac{\mathbf{1}[Z_{s_2} = 1]}{p_{s_2}} + 1 \right) \left(\frac{\mathbf{1}[Z_{s_3} = 1]}{p_{s_3}} + 1 \right) \right] \\
&\leq \max_{1 \leq s_3 < s_2 < s_1 \leq T} \left(\mathbb{E} \left[\frac{1}{p_{s_1}} \right] + \mathbb{E} \left[\frac{1}{p_{s_1}} \frac{\mathbf{1}[Z_{s_2} = 1]}{p_{s_2}} \right] + \mathbb{E} \left[\frac{1}{p_{s_1}} \frac{\mathbf{1}[Z_{s_3} = 1]}{p_{s_3}} \right] + \mathbb{E} \left[\frac{1}{p_{s_1}} \frac{\mathbf{1}[Z_{s_2} = 1]}{p_{s_2}} \frac{\mathbf{1}[Z_{s_3} = 1]}{p_{s_3}} \right] \right) \\
&\triangleq S_1 + S_2 + S_3 + S_4 .
\end{aligned}$$

By Corollary 4.15, Corollary B.17, we can bound S_1 as:

$$S_1 \leq \left(2 + b_1(b_2/6)^{1/4} \eta_1^{1/4} \kappa^{1/4} \right) .$$

By law of iterated expectation and Hölder's inequality, we can bound S_2 as:

$$\begin{aligned} S_2 &\leq \max_{1 \leq s_3 < s_2 < s_1 \leq T} \left(\mathbb{E} \left[\frac{1}{p_{s_1}^4} \right] \right)^{1/4} \left(\mathbb{E} \left[\frac{\mathbf{1}[Z_{s_2} = 1]}{p_{s_2}^{4/3}} \right] \right)^{3/4} \\ &= \max_{1 \leq s_3 < s_2 < s_1 \leq T} \left(\mathbb{E} \left[\frac{1}{p_{s_1}^4} \right] \right)^{1/4} \left(\mathbb{E} \left[\frac{1}{p_{s_2}^{1/3}} \right] \right)^{3/4} \\ &\leq \left(2 + b_1(b_2/6)^{1/4} \eta_1^{1/4} \kappa^{1/4} \right)^{4 \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{3}{4}} \\ &\leq \left(2 + b_1(b_2/6)^{1/4} \eta_1^{1/4} \kappa^{1/4} \right)^{5/4} . \end{aligned}$$

Similarly, S_3 can be bounded as:

$$S_3 \leq \left(2 + b_1(b_2/6)^{1/4} \eta_1^{1/4} \kappa^{1/4} \right)^{5/4} .$$

By Hölder's inequality, we can bound S_4 as:

$$\begin{aligned} S_4 &\leq \max_{1 \leq s_3 < s_2 < s_1 \leq T} \left(\mathbb{E} \left[\frac{1}{p_{s_1}^4} \right] \right)^{1/4} \left(\mathbb{E} \left[\frac{\mathbf{1}[Z_{s_2} = 1] \mathbf{1}[Z_{s_3} = 1]}{p_{s_2}^{4/3} p_{s_3}^{4/3}} \right] \right)^{3/4} \\ &\leq \max_{1 \leq s_3 < s_2 < s_1 \leq T} \left(\mathbb{E} \left[\frac{1}{p_{s_1}^4} \right] \right)^{1/4} \left(\mathbb{E} \left[\frac{1}{p_{s_2}^{1/3}} \frac{\mathbf{1}[Z_{s_3} = 1]}{p_{s_3}^{4/3}} \right] \right)^{3/4} \\ &\leq \max_{1 \leq s_3 < s_2 < s_1 \leq T} \left(\mathbb{E} \left[\frac{1}{p_{s_1}^4} \right] \right)^{1/4} \left(\mathbb{E} \left[\frac{1}{p_{s_2}^4} \right] \right)^{\frac{3}{4} \cdot \frac{1}{12}} \left(\mathbb{E} \left[\frac{\mathbf{1}[Z_{s_3} = 1]}{p_{s_3}^{\frac{4}{3} \cdot \frac{12}{11}}} \right] \right)^{\frac{3}{4} \cdot \frac{11}{12}} \\ &= \max_{1 \leq s_3 < s_2 < s_1 \leq T} \left(\mathbb{E} \left[\frac{1}{p_{s_1}^4} \right] \right)^{1/4} \left(\mathbb{E} \left[\frac{1}{p_{s_2}^4} \right] \right)^{\frac{3}{4} \cdot \frac{1}{12}} \left(\mathbb{E} \left[\frac{1}{p_{s_3}^{\frac{5}{11}}} \right] \right)^{\frac{3}{4} \cdot \frac{11}{12}} \\ &\leq \left(2 + b_1(b_2/6)^{1/4} \eta_1^{1/4} \kappa^{1/4} \right)^{4 \cdot \frac{1}{4} + 4 \cdot \frac{3}{4} \cdot \frac{1}{12} + \frac{5}{11} \cdot \frac{3}{4} \cdot \frac{11}{12}} \\ &= \left(2 + b_1(b_2/6)^{1/4} \eta_1^{1/4} \kappa^{1/4} \right)^{25/16} . \end{aligned}$$

Hence we have

$$\max_{1 \leq s_3 \neq s_2 \neq s_1 \leq T} \left| \mathbb{E} \left[\left(\frac{\mathbf{1}[Z_{s_1} = 1]}{p_{s_1}} - 1 \right)^2 \left(\frac{\mathbf{1}[Z_{s_2} = 1]}{p_{s_2}} - 1 \right) \left(\frac{\mathbf{1}[Z_{s_3} = 1]}{p_{s_3}} - 1 \right) \right] \right| = \mathcal{O}(T^{25/128}) .$$

(5) For any $1 \leq s_3 < s_2 < s_1 \leq T$, by law of total variance we have

$$\begin{aligned} &\left| \text{Cov} \left(\left(\frac{\mathbf{1}[Z_{s_1} = 1]}{p_{s_1}} - 1 \right) \left(\frac{\mathbf{1}[Z_{s_2} = 1]}{p_{s_2}} - 1 \right), \left(\frac{\mathbf{1}[Z_{s_1} = 1]}{p_{s_1}} - 1 \right) \left(\frac{\mathbf{1}[Z_{s_3} = 1]}{p_{s_3}} - 1 \right) \right) \right| \\ &= \left| \mathbb{E} \left[\text{Cov} \left(\left(\frac{\mathbf{1}[Z_{s_1} = 1]}{p_{s_1}} - 1 \right) \left(\frac{\mathbf{1}[Z_{s_2} = 1]}{p_{s_2}} - 1 \right), \left(\frac{\mathbf{1}[Z_{s_1} = 1]}{p_{s_1}} - 1 \right) \left(\frac{\mathbf{1}[Z_{s_3} = 1]}{p_{s_3}} - 1 \right) \middle| \mathcal{F}_{s_1-1} \right) \right] \right| \\ &\quad + \text{Cov} \left[\mathbb{E} \left(\left(\frac{\mathbf{1}[Z_{s_1} = 1]}{p_{s_1}} - 1 \right) \left(\frac{\mathbf{1}[Z_{s_2} = 1]}{p_{s_2}} - 1 \right) \middle| \mathcal{F}_{s_1-1} \right), \right. \end{aligned}$$

$$\begin{aligned}
& \left| \mathbb{E} \left(\left(\frac{\mathbf{1}[Z_{s_1} = 1]}{p_{s_1}} - 1 \right) \left(\frac{\mathbf{1}[Z_{s_3} = 1]}{p_{s_3}} - 1 \right) \middle| \mathcal{F}_{s_1-1} \right) \right| \\
&= \left| \mathbb{E} \left[\frac{1-p_{s_1}}{p_{s_1}} \left(\frac{\mathbf{1}[Z_{s_2} = 1]}{p_{s_2}} - 1 \right) \left(\frac{\mathbf{1}[Z_{s_3} = 1]}{p_{s_3}} - 1 \right) \right] \right| \\
&= \left| \mathbb{E} \left[\frac{1}{p_{s_1}} \left(\frac{\mathbf{1}[Z_{s_2} = 1]}{p_{s_2}} - 1 \right) \left(\frac{\mathbf{1}[Z_{s_3} = 1]}{p_{s_3}} - 1 \right) \right] \right| \\
&\leq \left(\mathbb{E} \left[\frac{1}{p_{s_1}} \right] + \mathbb{E} \left[\frac{1}{p_{s_1}} \frac{\mathbf{1}[Z_{s_2} = 1]}{p_{s_2}} \right] + \mathbb{E} \left[\frac{1}{p_{s_1}} \frac{\mathbf{1}[Z_{s_3} = 1]}{p_{s_3}} \right] + \mathbb{E} \left[\frac{1}{p_{s_1}} \frac{\mathbf{1}[Z_{s_2} = 1]}{p_{s_2}} \frac{\mathbf{1}[Z_{s_3} = 1]}{p_{s_3}} \right] \right) .
\end{aligned}$$

By similar method as in (4), we can show that

$$\begin{aligned}
& \left| \text{Cov} \left(\left(\frac{\mathbf{1}[Z_{s_1} = 1]}{p_{s_1}} - 1 \right) \left(\frac{\mathbf{1}[Z_{s_2} = 1]}{p_{s_2}} - 1 \right), \left(\frac{\mathbf{1}[Z_{s_1} = 1]}{p_{s_1}} - 1 \right) \left(\frac{\mathbf{1}[Z_{s_3} = 1]}{p_{s_3}} - 1 \right) \right) \right| \\
&\leq \left(2 + b_1(b_2/6)^{1/4} \eta_{s_1}^{1/4} \mathbb{E}^{1/4} \hat{A}_{s_1}(0) \right) + \left(2 + b_1(b_2/6)^{1/4} \eta_{s_1}^{1/4} \mathbb{E}^{1/4} \hat{A}_{s_1}(0) \right) \left(2 + b_1(b_2/6)^{1/4} \eta_{s_2}^{1/4} \mathbb{E}^{1/4} \hat{A}_{s_1}(0) \right)^{1/4} \\
&\quad + \left(2 + b_1(b_2/6)^{1/4} \eta_{s_1}^{1/4} \mathbb{E}^{1/4} \hat{A}_{s_1}(0) \right) \left(2 + b_1(b_2/6)^{1/4} \eta_{s_3}^{1/4} \mathbb{E}^{1/4} \hat{A}_{s_1}(0) \right)^{1/4} \\
&\quad + \left(2 + b_1(b_2/6)^{1/4} \eta_{s_1}^{1/4} \mathbb{E}^{1/4} \hat{A}_{s_1}(0) \right) \left(2 + b_1(b_2/6)^{1/4} \eta_{s_2}^{1/4} \mathbb{E}^{1/4} \hat{A}_{s_1}(0) \right)^{1/4} \left(2 + b_1(b_2/6)^{1/4} \eta_{s_3}^{1/4} \mathbb{E}^{1/4} \hat{A}_{s_1}(0) \right)^{15/48} \\
&\leq \left(2 + b_1(b_2/6)^{1/4} \eta_{s_1}^{1/4} \mathbb{E}^{1/4} \hat{A}_{s_1}(0) \right) \\
&\quad + \left(2 + b_1(b_2/6)^{1/4} \eta_{s_1}^{1/4} \mathbb{E}^{1/4} \hat{A}_{s_1}(0) \right) \left(2 + b_1(b_2/6)^{1/4} \eta_{s_1}^{1/4} \mathbb{E}^{1/4} \hat{A}_{s_1}(0) \right)^{1/4} \left(\frac{R_{s_1}}{R_{s_2}} \right)^{1/8} \\
&\quad + \left(2 + b_1(b_2/6)^{1/4} \eta_{s_1}^{1/4} \mathbb{E}^{1/4} \hat{A}_{s_1}(0) \right) \left(2 + b_1(b_2/6)^{1/4} \eta_{s_1}^{1/4} \mathbb{E}^{1/4} \hat{A}_{s_1}(0) \right)^{1/4} \left(\frac{R_{s_1}}{R_{s_3}} \right)^{1/8} \\
&\quad + \left(2 + b_1(b_2/6)^{1/4} \eta_{s_1}^{1/4} \mathbb{E}^{1/4} \hat{A}_{s_1}(0) \right)^{1+1/4+15/48} \left(\frac{R_{s_1}}{R_{s_2}} \right)^{1/8} \left(\frac{R_{s_1}}{R_{s_3}} \right)^{15/96} \\
&\leq 4 \left(2 + b_1(b_2/6)^{1/4} \eta_{s_1}^{1/4} \mathbb{E}^{1/4} \hat{A}_{s_1}(0) \right)^{25/16} \left(\frac{R_{s_1}}{R_{s_2}} \right)^{15/96} \left(\frac{R_{s_1}}{R_{s_3}} \right)^{15/96} .
\end{aligned}$$

□

With the help of Lemma B.18, we can establish an upper bound for the fourth moment of the random online OLS residuals, similar to the bound for the deterministic online OLS residuals given in Lemma B.7 except for an additional smaller order term. We first give the upper bound on the tracking error term in the following proposition.

Proposition 4.17. *Under Assumptions 1-3 and Condition 1, the prediction tracking error for both treatments $k \in \{0, 1\}$ can be bounded as $\mathbb{E}[\sum_{t=1}^T \eta_t \langle \mathbf{x}_t, \boldsymbol{\beta}_t(k) - \boldsymbol{\beta}_t^*(k) \rangle^4] = o(T^{1/2} R^2)$.*

Proof. By explicit calculation, we have

$$\begin{aligned}
& \mathbb{E} \left[\eta_t \sum_{t=1}^T \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) - \boldsymbol{\beta}_t^*(1) \rangle^4 \right] \\
&= T^{-1/2} \mathbb{E} \left[\sum_{t=1}^T R_t^{-2} \left(\sum_{s=1}^{t-1} \Pi_{t,s} y_s(1) \left[\frac{\mathbf{1}[Z_s = 1]}{p_s} - 1 \right] \right)^4 \right] \\
&\lesssim T^{-1/2} \sum_{t=1}^T \sum_{s=1}^{t-1} \Pi_{t,s}^4 y_s^4(1) \mathbb{E} \left[\left(\frac{\mathbf{1}[Z_s = 1]}{p_s} - 1 \right)^4 \right]
\end{aligned}$$

$$\begin{aligned}
& + T^{-1/2} \sum_{t=1}^T \sum_{1 \leq s_1 \neq s_2 \leq t-1} |\Pi_{t,s_1}|^3 |\Pi_{t,s_2}| |y_{s_1}(1)|^3 |y_{s_2}(1)| \left| \mathbb{E} \left[\left(\frac{\mathbf{1}[Z_{s_1} = 1]}{p_{s_1}} - 1 \right)^3 \left(\frac{\mathbf{1}[Z_{s_2} = 1]}{p_{s_2}} - 1 \right) \right] \right| \\
& + T^{-1/2} \sum_{t=1}^T \sum_{1 \leq s_1 \neq s_2 \leq t-1} \Pi_{t,s_1}^2 \Pi_{t,s_2}^2 y_{s_1}^2(1) y_{s_2}^2(1) \mathbb{E} \left[\left(\frac{\mathbf{1}[Z_{s_1} = 1]}{p_{s_1}} - 1 \right)^2 \left(\frac{\mathbf{1}[Z_{s_2} = 1]}{p_{s_2}} - 1 \right)^2 \right] \\
& + T^{-1/2} \sum_{t=1}^T R_t^{-1} \sum_{1 \leq s_1 \neq s_2 \neq s_3 \leq t-1} \Pi_{t,s_1}^2 |\Pi_{t,s_2}| |\Pi_{t,s_3}| |y_{s_1}^2(1)| |y_{s_2}(1)| |y_{s_3}(1)| \\
& \times \left| \mathbb{E} \left[\left(\frac{\mathbf{1}[Z_{s_1} = 1]}{p_{s_1}} - 1 \right)^2 \left(\frac{\mathbf{1}[Z_{s_2} = 1]}{p_{s_2}} - 1 \right) \left(\frac{\mathbf{1}[Z_{s_3} = 1]}{p_{s_3}} - 1 \right) \right] \right| \\
& \triangleq T^{-1/2} S_1 + T^{-1/2} S_2 + T^{-1/2} S_3 + T^{-1/2} S_4 .
\end{aligned}$$

By Assumption 3, Lemma B.15 and Lemma B.18, we have

$$\begin{aligned}
S_1 & \lesssim R_T^8 \eta_T^3 T \cdot T^{3/8} = T R_T^2 \cdot T^{-9/8} = o(T R_T^2) , \\
S_2 & \lesssim R_t^{13/2} \eta_t^{7/4} T^{7/8} \cdot T^{5/16} = T R_T^2 \cdot (T R_T^{-4})^{-11/16} \cdot R_T^{-7/4} = o(T R_T^2) , \\
S_3 & \lesssim R_T^6 \eta_T^2 T \cdot T^{9/32} = T R_T^2 \cdot (T R_T^{-4})^{-23/32} \cdot R_T^{-23/8} = o(T R_T^2) , \\
S_4 & \lesssim R_T^4 \eta_T T \cdot T^{25/128} = T R_T^2 \cdot (T R_T^{-4})^{-39/128} \cdot R_T^{-39/32} = o(T R_T^2) .
\end{aligned}$$

Hence by Lemma B.5 we have

$$\mathbb{E} \left[\sum_{t=1}^T \eta_t \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) - \boldsymbol{\beta}_t^*(1) \rangle^4 \right] \leq T^{-1/2} (S_1 + S_2 + S_3 + S_4) = o(T^{1/2} R_T^2) = o(T^{1/2} R^2) .$$

□

The following corollary is a direct consequence of the proof in Proposition 4.17 and will be used in the proofs of Lemma C.3 and Lemma C.18.

Corollary B.19. *Under Assumptions 1-3 and Condition 1, for $k \in \{0, 1\}$, there holds:*

$$\mathbb{E} \left[\sum_{s=1}^t R_s^{-1} \langle \mathbf{x}_s, \boldsymbol{\beta}_s^*(1) - \boldsymbol{\beta}_s(1) \rangle^4 \right] = \mathcal{O}(T^{89/128} R_t^2) .$$

Proof. By similar proof as in Proposition 4.17 and Assumption 3, we have

$$\begin{aligned}
& \mathbb{E} \left[\sum_{s=1}^t R_s^{-1} \langle \mathbf{x}_s, \boldsymbol{\beta}_s^*(1) - \boldsymbol{\beta}_s(1) \rangle^4 \right] \\
& \lesssim R_t^8 \eta_t^3 T \cdot T^{3/8} + R_t^{13/2} \eta_t^{7/4} T^{7/8} \cdot T^{5/16} + R_t^6 \eta_t^2 T \cdot T^{9/32} + R_t^4 \eta_t T \cdot T^{25/128} \\
& = T^{89/128} R_t^2 \left(T^{-105/128} + R_t T^{-49/128} + T^{-53/128} + 1 \right) \\
& = T^{89/128} R_t^2 \left(T^{-105/128} + (T R_t^{-4})^{-49/128} R_t^{-17/32} + T^{-53/128} + 1 \right) \\
& \lesssim T^{89/128} R_t^2 .
\end{aligned}$$

□

By combining the results in Lemma B.7 and Proposition 4.17, we can obtain the upper bound on the fourth moment of the random online OLS residuals in the following lemma.

Lemma 4.6*. Under Assumptions 1-3 and Condition 1, for each treatment $k \in \{0, 1\}$, the fourth moment of the online residuals is bounded as:

$$\mathbb{E} \left[\sum_{t=1}^T \eta_t (y_t(k) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(k) \rangle)^4 \right] \leq \left[c_1^4 \max\{c_2, 1\} \left(1 + 2^{-1/2} (\gamma_0 \vee 4c_2)^{1/2} \right)^4 + o(1) \right] T^{1/2} R^2 .$$

Proof. Without loss of generality, we only prove the result for $k = 1$. By Hölder's inequality, we have

$$\begin{aligned} & \mathbb{E} \left[\sum_{t=1}^T \eta_t (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle)^4 \right] \\ &= \mathbb{E} \left[\sum_{t=1}^T \eta_t (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle + \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) - \boldsymbol{\beta}_t(1) \rangle)^4 \right] \\ &= \mathbb{E} \left[\sum_{t=1}^T \eta_t (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^4 \right] + 4\mathbb{E} \left[\sum_{t=1}^T \eta_t (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^3 \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) - \boldsymbol{\beta}_t(1) \rangle \right] \\ &\quad + 6\mathbb{E} \left[\sum_{t=1}^T \eta_t (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^2 \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) - \boldsymbol{\beta}_t(1) \rangle^2 \right] \\ &\quad + 4\mathbb{E} \left[\sum_{t=1}^T \eta_t (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle) \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) - \boldsymbol{\beta}_t(1) \rangle^3 \right] + \mathbb{E} \left[\sum_{t=1}^T \eta_t \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) - \boldsymbol{\beta}_t(1) \rangle^4 \right] \\ &\leq \sum_{t=1}^T \eta_t (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^4 + 4 \left(\sum_{t=1}^T \eta_t (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^4 \right)^{3/4} \left(\mathbb{E} \left[\sum_{t=1}^T \eta_t \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) - \boldsymbol{\beta}_t(1) \rangle^4 \right] \right)^{1/4} \\ &\quad + 6 \left(\sum_{t=1}^T \eta_t (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^4 \right)^{1/2} \left(\mathbb{E} \left[\sum_{t=1}^T \eta_t \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) - \boldsymbol{\beta}_t(1) \rangle^4 \right] \right)^{1/2} \\ &\quad + 4 \left(\sum_{t=1}^T \eta_t (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^4 \right)^{1/4} \left(\mathbb{E} \left[\sum_{t=1}^T \eta_t \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) - \boldsymbol{\beta}_t(1) \rangle^4 \right] \right)^{3/4} + \mathbb{E} \left[\sum_{t=1}^T \eta_t \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) - \boldsymbol{\beta}_t(1) \rangle^4 \right] \\ &\leq \left[\left(\sum_{t=1}^T \eta_t (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^4 \right)^{1/4} + \left(\mathbb{E} \left[\sum_{t=1}^T \eta_t \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) - \boldsymbol{\beta}_t(1) \rangle^4 \right] \right)^{1/4} \right]^4 . \end{aligned}$$

Hence by Lemma B.5, Lemma B.7 and Proposition 4.17, we have

$$\begin{aligned} \mathbb{E} \left[\sum_{t=1}^T \eta_t (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle)^4 \right] &\leq c_1^4 \left(1 + 2^{-1/2} (\gamma_0 \vee 4c_2)^{1/2} \right)^4 R_T^2 T^{1/2} + o(R^2 T^{1/2}) \\ &\leq \left[c_1^4 \max\{c_2, 1\} \left(1 + 2^{-1/2} (\gamma_0 \vee 4c_2)^{1/2} \right)^4 + o(1) \right] T^{1/2} R^2 . \end{aligned}$$

□

We now turn to the analysis of the probability regret. The following lemma is a direct implication of the adaptive sequential design.

Lemma 4.2. The estimated sigmoid loss functions are conditionally unbiased: $\mathbb{E}[\hat{h}_t(u) | \mathcal{F}_{t-1}] = h_t(u)$ for all \mathcal{F}_{t-1} -measurable random variables u .

Proof. Since β_t , p_t and u are measurable with respect to \mathcal{F}_{t-1} , we have

$$\mathbb{E}[\hat{h}_t(u) | \mathcal{F}_{t-1}]$$

$$\begin{aligned}
&= \mathbb{E} \left[\left(y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle \right)^2 \cdot \frac{\mathbf{1}[Z_t = 1]}{p_t} \cdot \frac{1}{\phi(u)} + \left(y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle \right)^2 \cdot \frac{\mathbf{1}[Z_t = 0]}{1-p_t} \cdot \frac{1}{1-\phi(u)} \middle| \mathcal{F}_{t-1} \right] \\
&= (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle)^2 \cdot \frac{1}{\phi(u)} + (y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle)^2 \cdot \frac{1}{1-\phi(u)} \\
&= h_t(u) .
\end{aligned}$$

□

In Lemma 4.3 and Lemma B.20, we establish the lower and upper bounds of the utilized Bregman divergence, respectively.

Lemma 4.3. *The Bregman divergence is bounded: $\mathcal{B}_\psi(v|u) \geq \frac{1}{2}(v-u)^2(1 + \frac{1}{2}|v| + |u|)$.*

Proof. Recall that $\psi(u) = \frac{1}{2}u^2 + |u|^3$. Using the shorthand $h(u) = \frac{1}{2}u^2$ and $g(u) = |u|^3$, we have that $\psi(u) = h(u) + g(u)$. Using the properties of Bregman divergence, we can separate the Bregman divergence as

$$\mathcal{B}_\psi(v|u) = \mathcal{B}_{h+g}(v|u) = \mathcal{B}_h(v|u) + \mathcal{B}_g(v|u) = \frac{1}{2}(v-u)^2 + \mathcal{B}_g(v|u) .$$

To complete the proof, it suffices to lower bound the second Bregman divergence as

$$\mathcal{B}_g(v|u) \geq \frac{1}{4}(v-u)^2 \cdot (|v| + 2|u|) . \quad (13)$$

To this end, we derive the Bregman divergence corresponding to g as

$$\mathcal{B}_g(v|u) = |v|^3 - |u|^3 - (3|u|u) \cdot (v-u) = |v|^3 + 2|u|^3 - 3uv|u| .$$

To show the inequality (13) holds, we will consider two cases.

Case 1. In the first case, we will suppose that u and v have the same sign, i.e., $uv \geq 0$. In this case, it is true that $|u-v| = |u| + |v|$. Thus, we have that the Bregman divergence is given by

$$B_g(v|u) = |v|^3 + 2|u|^3 - 3|u|^2|v| .$$

On the other hand, we have

$$\begin{aligned}
(v-u)^2(|v| + 2|u|) &= (v^2 + u^2 - 2vu)(|v| + 2|u|) \\
&= |v|^3 + 2v^2|u| + u^2|v| + 2|u|^3 - 2vu(|v| + 2|u|) \\
&= |v|^3 + 2v^2|u| + u^2|v| + 2|u|^3 - 2|v||u|(|v| + 2|u|) \quad (\text{because } uv > 0) \\
&= |v|^3 + 2|u|^3 - 3u^2|v| .
\end{aligned}$$

Thus, this establishes that inequality (13) holds in this case, as we have

$$B_g(v|u) = (v-u)^2(|v| + 2|u|) \geq \frac{1}{4}(v-u)^2(|v| + 2|u|) .$$

Case 2. In the second case, we will suppose that v and u have different signs, i.e., $vu < 0$. This means that $vu = -|v||u|$. Before continuing, we establish a handy upper bound, which is that

$$4v^2|u| \leq 2(|v|^3 + 2|u|^3 + |u|^2v) . \quad (14)$$

This upper bound may be established using the AM-GM inequality, as

$$4v^2|u| = 4|v|^{3/2} \cdot (|v|^{1/2}|u|) \leq 2(|v|^3 + |v||u|^2) \leq 2(|v|^3 + 2|u|^3 + |v||u|^2) .$$

With that inequality established, let us turn our attention back to the Bregman divergence. In this case, the Bregman divergence is

$$\begin{aligned} B_g(v|u) &= |v|^3 + 2|u|^3 - 3uv|u| \\ &= |v|^3 + 2|u|^3 + 3|u||v||u| \quad (\text{because } uv < 0) \\ &= |v|^3 + 2|u|^3 + 3u^2|v|. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \frac{1}{4}(v-u)^2(|v|+2|u|) &= \frac{1}{4}(v^2+u^2-2vu)(|v|+2|u|) \\ &= \frac{1}{4}(v^2+u^2+2|v||u|)(|v|+2|u|) \\ &= \frac{1}{4}\{|v|^3+2v^2|u|+2|u|^3+u^2|v|+2v^2|u|+4|v||u|^2\} \\ &= \frac{1}{4}\{|v|^3+2|u|^3+4v^2|u|+5|v||u|^2\} \\ &\leq \frac{1}{4}\{|v|^3+2|u|^3+2(|v|^3+2|u|^3+|v||u|^2)+6|v||u|^2\} \quad (\text{use (14)}) \\ &= \frac{1}{4}\{3|v|^3+5|u|^3+7|v||u|^2\} \\ &\leq |v|^3+2|u|^3+3|v||u|^2 \\ &= B_g(v|u). \end{aligned}$$

Thus, we have established that the inequality (13) holds for all $v, u \in \mathbb{R}$, which completes the proof. \square

The upper bound of the probability is derived through standard FTRL analysis. We first state one key property of the Bregman divergence.

Lemma B.20. *Suppose that f and ψ are both differentiable and $f - \psi$ is convex. Then, for all $x, y \in \mathcal{X}$, we have that*

$$B_f(x|y) \leq \sup_{z \in \mathcal{X}} \{\langle \nabla f(y) - \nabla f(x), z - x \rangle - B_\psi(z|x)\}.$$

Proof. We begin the proof by showing that the the following inequality is true:

$$f(z) - f(x) - \langle \nabla f(y), z - x \rangle \geq \langle \nabla f(x) - \nabla f(y), z - x \rangle + B_\psi(z|x) \quad \text{for all } z \in \mathcal{X}. \quad (15)$$

To this end,

$$\begin{aligned} &f(z) - f(x) - \langle \nabla f(y), z - x \rangle \\ &= \langle \nabla f(x) - \nabla f(y), z - x \rangle + B_\psi(z|x) \\ &\quad + \{f(z) - f(x) - \langle \nabla f(x), z - x \rangle - B_\psi(z|x)\} \quad (\text{add + subtract}) \\ &= \langle \nabla f(x) - \nabla f(y), z - x \rangle + B_\psi(z|x) + \{B_f(z|x) - B_\psi(z|x)\} \quad (\text{definition of } B_f) \\ &= \langle \nabla f(x) - \nabla f(y), z - x \rangle + B_\psi(z|x) + B_{f-\psi}(z|x) \quad (\text{linearity of Bregman divergence}) \\ &\geq \langle \nabla f(x) - \nabla f(y), z - x \rangle + B_\psi(z|x) \quad (f - \psi \text{ is convex}). \end{aligned}$$

We are now ready to proceed with the proof of the main lemma. Define the auxiliary function $\phi_y : \mathcal{X} \rightarrow \mathbb{R}$ as

$$\phi_y(z) = f(z) - \langle \nabla f(y), z \rangle.$$

Observe that ϕ_y is convex and differentiable. Thus, we have that

$$\begin{aligned}
\phi_y(y) &\geq \inf_{z \in \mathcal{X}} \phi_y(z) \\
&= \phi_y(x) + \inf_{z \in \mathcal{X}} \{\phi_y(z) - \phi_y(x)\} \\
&= \phi_y(x) + \inf_{z \in \mathcal{X}} \left\{ f(z) - f(x) - \langle \nabla f(y), z - x \rangle \right\} \quad (\text{definition of } \phi_y) \\
&\geq \phi_y(x) + \inf_{z \in \mathcal{X}} \left\{ \langle \nabla f(x) - \nabla f(y), z - x \rangle + B_\psi(z|x) \right\} \quad (\text{Using (15)}) .
\end{aligned}$$

Rearranging terms and using the definition of $\phi_y(x)$ and B_f , we have that

$$\begin{aligned}
\sup_{z \in \mathcal{X}} \left\{ \langle \nabla f(y) - \nabla f(x), z - x \rangle - B_\psi(z|x) \right\} &\geq \phi_y(x) - \phi_y(y) \\
&= f(x) - f(y) - \langle \nabla f(y), x - y \rangle \\
&= B_f(x|y) .
\end{aligned}$$

□

Based on Lemma 4.3, Lemma B.20 and the sigmoid transformation, we can derive the following upper bound for the probability regret.

Proposition 4.4. *The expected probability regret can be bounded as*

$$\mathbb{E}[\mathcal{R}_T^{\text{prob}}] \leq \frac{1}{\eta_T} \psi(u^*) + \sum_{t=1}^T \eta_t \mathbb{E} \left[\frac{(\nabla \hat{h}_t(u_t))^2}{2(1+|u_t|)} \right] .$$

Proof. We introduce the following notations:

$$\begin{aligned}
\hat{H}_t(u) &= \sum_{s=1}^{t-1} \hat{h}_s(u) + \frac{1}{\eta_t} \psi(u) , \\
\tilde{H}_t(u) &= \sum_{s=1}^{t-1} \hat{h}_s(u) + \frac{1}{\eta_{t-1}} \psi(u) .
\end{aligned}$$

Let u_t, \tilde{u}_t be the minimizer of functions \hat{H}_t and \tilde{H}_t , respectively. By Lemma 4.2, law of iterated expectation and similar proof as in of Lemma 4.9, we have

$$\begin{aligned}
\mathbb{E}[\mathcal{R}_T^{\text{prob}}] &= \sum_{t=1}^T \mathbb{E}[f_t(p_t) - f_t(p^*)] \\
&= \sum_{t=1}^T \mathbb{E}[h_t(u_t) - h_t(u^*)] \quad (\text{sigmoid transformation}) \\
&= \sum_{t=1}^T \mathbb{E} \left[\mathbb{E} \left[\hat{h}_t(u_t) - \hat{h}_t(u^*) \mid \mathcal{F}_{t-1} \right] \right] \quad (\text{Lemma 4.2}) \\
&= \sum_{t=1}^T \mathbb{E} \left[\hat{h}_t(u_t) - \hat{h}_t(u^*) \right] \quad (\text{law of iterated expectation}) \\
&\leq \eta_{T+1}^{-1} \psi(u^*) + \mathbb{E} \left[\sum_{t=1}^T \left(\tilde{H}_{t+1}(u_t) - \tilde{H}_{t+1}(\tilde{u}_{t+1}) \right) \right] \quad (\text{similar proof as Lemma 4.9}) . \tag{16}
\end{aligned}$$

Then by Lemma B.20, we have

$$\sum_{t=1}^T \mathbb{E} \left[\left(\tilde{H}_{t+1}(u_t) - \tilde{H}_{t+1}(\tilde{u}_{t+1}) \right) \right]$$

$$\begin{aligned}
&= \mathbb{E} \left[\sum_{t=1}^T \left[\langle \nabla \tilde{H}_{t+1}(\tilde{u}_{t+1}), u_t - \tilde{u}_{t+1} \rangle + \mathcal{B}_{\tilde{H}_{t+1}}(u_t | \tilde{u}_{t+1}) \right] \right] \\
&= \mathbb{E} \left[\sum_{t=1}^T \mathcal{B}_{\tilde{H}_{t+1}}(u_t | \tilde{u}_{t+1}) \right] \quad (\text{optimality of } \tilde{u}_{t+1}) \\
&\leq \mathbb{E} \left[\sum_{t=1}^T \sup_{z \in \mathbb{R}} \left\{ \langle \nabla \tilde{H}_{t+1}(\tilde{u}_{t+1}) - \nabla \tilde{H}_{t+1}(u_t), z - u_t \rangle - \eta_t^{-1} \mathcal{B}_\psi(z | u_t) \right\} \right] \\
&= \sum_{t=1}^T \mathbb{E} \left[\sup_{z \in \mathbb{R}} \left\{ \langle \nabla \hat{h}_t(u_t), u_t - z \rangle - \eta_t^{-1} \mathcal{B}_\psi(z | u_t) \right\} \right] \quad (\text{optimality of } \tilde{u}_{t+1} \text{ and } u_t) . \quad (17)
\end{aligned}$$

For any $t \in [T]$ and any fixed $z \in \mathbb{R}$, by Lemma 4.3, we have

$$\begin{aligned}
\langle \nabla \hat{h}_t(u_t), u_t - z \rangle - \eta_t^{-1} \mathcal{B}_\psi(z | u_t) &\leq \langle \nabla \hat{h}_t(u_t), u_t - z \rangle - \frac{\eta_t^{-1}}{2} (z - u_t)^2 \left(1 + \frac{1}{2} |z| + |u_t| \right) \\
&\leq \langle \nabla \hat{h}_t(u_t), u_t - z \rangle - \frac{\eta_t^{-1}}{2} (z - u_t)^2 (1 + |u_t|) \\
&\leq \frac{\eta_t [\nabla \hat{h}_t(u_t)]^2}{2(1 + |u_t|)} .
\end{aligned} \quad (18)$$

By (16), (17) and (18), we can derive

$$\mathbb{E}[\mathcal{R}_T^{\text{prob}}] \leq \frac{1}{\eta_T} \psi(u^*) + \sum_{t=1}^T \eta_t \mathbb{E} \left[\frac{(\nabla \hat{h}_t(u_t))^2}{2(1 + |u_t|)} \right] .$$

□

Given the upper bound of the probability regret established in Proposition 4.4, we further upper bound the two corresponding terms in Lemma 4.5 and Lemma B.21, respectively.

Lemma 4.5. *Under Condition 1, the conditional expectation of the squared gradient term is at most*

$$\mathbb{E} \left[\frac{(\nabla \hat{h}_t(u_t))^2}{1 + |u_t|} \mid \mathcal{F}_{t-1} \right] \leq b_1^2 \max\{b_1, 2\} \left(\{y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle\}^4 + \{y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle\}^4 \right) .$$

Thus, applying the law of iterated expectation yields

$$\sum_{t=1}^T \eta_t \mathbb{E} \left[\frac{(\nabla \hat{h}_t(u_t))^2}{2(1 + |u_t|)} \right] \leq \frac{1}{2} b_1^2 \max\{b_1, 2\} \sum_{k \in \{0, 1\}} \mathbb{E} \left[\sum_{t=1}^T \eta_t \{y_t(k) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(k) \rangle\}^4 \right] .$$

Proof. The gradient of the estimated loss function can be computed as

$$\begin{aligned}
\nabla \hat{h}_t(u) &= - \frac{\mathbf{1}[Z_t = 1]}{p_t} (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle)^2 \cdot \left(\frac{1}{\phi(u)} \right)' \\
&\quad + \frac{\mathbf{1}[Z_t = 0]}{1 - p_t} (y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle)^2 \cdot \left(\frac{1}{1 - \phi(u)} \right)' .
\end{aligned}$$

Using Assumption 1, Condition 1 and the fact that the cross term is zero, we can upper bound the square of the gradient at u_t as

$$(\nabla \hat{h}_t(u_t))^2$$

$$\begin{aligned}
&= \frac{\mathbf{1}[Z_t = 1]}{p_t^2} (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle)^4 \cdot \left[\left(\frac{1}{\phi(u_t)} \right)' \right]^2 + \frac{\mathbf{1}[Z_t = 0]}{(1-p_t)^2} (y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle)^4 \cdot \left[\left(\frac{1}{1-\phi(u_t)} \right)' \right]^2 \\
&\leq b_1^2 \left[\frac{\mathbf{1}[Z_t = 1]}{p_t} (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle)^4 \cdot \frac{1}{\phi(u_t)} + \frac{\mathbf{1}[Z_t = 0]}{1-p_t} (y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle)^4 \cdot \frac{1}{1-\phi(u_t)} \right] \\
&\leq b_1^2 \left[\frac{\mathbf{1}[Z_t = 1]}{p_t} (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle)^4 \cdot \left(\frac{1}{\phi(0)} + b_1|u_t| \right) \right. \\
&\quad \left. + \frac{\mathbf{1}[Z_t = 0]}{1-p_t} (y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle)^4 \cdot \left(\frac{1}{1-\phi(0)} + b_1|u_t| \right) \right] \\
&\leq b_1^2 (2 + b_1|u_t|) \left(\frac{\mathbf{1}[Z_t = 1]}{p_t} (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle)^4 + \frac{\mathbf{1}[Z_t = 0]}{1-p_t} (y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle)^4 \right) .
\end{aligned}$$

Hence we have

$$\begin{aligned}
&\mathbb{E} \left[\frac{(\nabla \hat{h}_t(u_t))^2}{1+|u_t|} \middle| \mathcal{F}_{t-1} \right] \\
&\leq \mathbb{E} \left[\frac{b_1^2(2+b_1|u_t|)}{1+|u_t|} \left(\frac{\mathbf{1}[Z_t = 1]}{p_t} (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle)^4 + \frac{\mathbf{1}[Z_t = 0]}{1-p_t} (y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle)^4 \right) \middle| \mathcal{F}_{t-1} \right] \\
&\leq b_1^2 \max\{b_1, 2\} \mathbb{E} \left[\frac{\mathbf{1}[Z_t = 1]}{p_t} (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle)^4 + \frac{\mathbf{1}[Z_t = 0]}{1-p_t} (y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle)^4 \middle| \mathcal{F}_{t-1} \right] \\
&= b_1^2 \max\{b_1, 2\} \left[(y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle)^4 + (y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle)^4 \right] .
\end{aligned}$$

□

Lemma B.21. Under Assumption 1 and Condition 1, there holds:

$$\psi(u^*) \leq \frac{8}{b_3^3} \left[\left(\frac{c_1}{c_0} - 1 \right)^3 + \frac{b_3}{4} \left(\frac{c_1}{c_0} - 1 \right)^2 \right] .$$

Proof. Since $p^* = (1 + \mathcal{E}(0)/\mathcal{E}(1))^{-1}$, by Assumption 1 we have $p^* \in [\frac{1}{1+c_1/c_0}, 1 - \frac{1}{1+c_1/c_0}]$. For $p^* \geq 1/2$, we have

$$\frac{1}{1-p^*} - \frac{1}{1-1/2} \leq \frac{1}{1 - \left(1 - \frac{1}{1+c_1/c_0}\right)} - 2 = \frac{c_1}{c_0} - 1 .$$

Hence by Lemma B.16, we have

$$\frac{c_1}{c_0} - 1 \geq \frac{1}{1-p^*} - \frac{1}{1-1/2} \geq \frac{b_3}{2}(u^* - 0) ,$$

which implies that $0 \leq u^* \leq \frac{2}{b_3}(\frac{c_1}{c_0} - 1)$. Since $\psi(u)$ is monotone increasing on $[0, \infty)$, we can derive the following upper bound:

$$\psi(u^*) \leq \frac{1}{2} \left[\frac{2}{b_3} \left(\frac{c_1}{c_0} - 1 \right) \right]^2 + \left[\frac{2}{b_3} \left(\frac{c_1}{c_0} - 1 \right) \right]^3 .$$

For $p^* \leq 1/2$, we can derive the same upper bound by symmetry. □

Based on Proposition 4.4, Lemma 4.5, and Lemma B.21, we can finally establish the upper bound for the probability regret in the following proposition.

Proposition 4.7. Under Assumptions 1-3 and Condition 1, the expected probability regret is bounded in expectation as

$$\mathbb{E}[\mathcal{R}_T^{\text{prob}}] \leq \max\{c_2, 1\} \left(\frac{8}{b_3^3} \left[\left(\frac{c_1}{c_0} - 1 \right)^3 + \frac{b_3}{4} \left(\frac{c_1}{c_0} - 1 \right)^2 \right] \right)$$

$$+ b_1^2 \max\{b_1, 2\} c_1^4 \left(1 + 2^{-1/2} (\gamma_0 \vee 4c_2)^{1/2}\right)^4 + o(1) \Bigg) T^{1/2} R^2 .$$

Proof. By Lemma B.5, Lemma 4.6*, Proposition 4.4, Lemma 4.5 and Lemma B.21, we can derive the following upper bound

$$\begin{aligned} & \mathbb{E}[\mathcal{R}_T^{\text{prob}}] \\ & \leq \frac{1}{\eta_T} \psi(u^*) + \sum_{t=1}^T \eta_t \mathbb{E} \left[\frac{(\nabla \hat{h}_t(u_t))^2}{2(1 + |u_t|)} \right] \\ & \leq \frac{1}{\eta_T} \cdot \frac{8}{b_3^3} \left[\left(\frac{c_1}{c_0} - 1 \right)^3 + \frac{b_3}{4} \left(\frac{c_1}{c_0} - 1 \right)^2 \right] \\ & \quad + \frac{b_1^2}{2} \max\{b_1, 2\} \mathbb{E} \left[\sum_{t=1}^T \eta_t \left[(y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle)^4 + (y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle)^4 \right] \right] \\ & \leq R_T^2 T^{1/2} \cdot \frac{8}{b_3^3} \left[\left(\frac{c_1}{c_0} - 1 \right)^3 + \frac{b_3}{4} \left(\frac{c_1}{c_0} - 1 \right)^2 \right] \\ & \quad + b_1^2 \max\{b_1, 2\} c_1^4 \max\{c_2, 1\} \left(1 + 2^{-1/2} (\gamma_0 \vee 4c_2)^{1/2}\right)^4 R^2 T^{1/2} + o(R^2 T^{1/2}) \\ & \leq \max\{c_2, 1\} \left(\frac{8}{b_3^3} \left[\left(\frac{c_1}{c_0} - 1 \right)^3 + \frac{b_3}{4} \left(\frac{c_1}{c_0} - 1 \right)^2 \right] + b_1^2 \max\{b_1, 2\} c_1^4 \left(1 + 2^{-1/2} (\gamma_0 \vee 4c_2)^{1/2}\right)^4 + o(1) \right) T^{1/2} R^2 . \end{aligned}$$

□

B.5 Prediction Regret

In this subsection, we derive an upper bound for the expected prediction regret. The following lemma establish the connection between the defined prediction regret with the corresponding terms that are actually utilized in proposed algorithm.

Lemma 4.8. *For each iteration t , the following conditional unbiasedness holds:*

$$\mathbb{E}[\hat{\ell}_t(\boldsymbol{\beta}_t(1), \boldsymbol{\beta}_t(0)) - \hat{\ell}_t(\boldsymbol{\beta}^*(1), \boldsymbol{\beta}^*(0)) \mid \mathcal{F}_{t-1}] = \ell_t(\boldsymbol{\beta}_t(1), \boldsymbol{\beta}_t(0)) - \ell_t(\boldsymbol{\beta}^*(1), \boldsymbol{\beta}^*(0)) .$$

Proof. By the definition of $\hat{\ell}_t$ and ℓ_t , since $\beta_t(1)$ and $\beta_t(0)$ are measurable with respect to \mathcal{F}_{t-1} , we have

$$\begin{aligned} & \mathbb{E}[\hat{\ell}_t(\boldsymbol{\beta}_t(1), \boldsymbol{\beta}_t(0)) - \hat{\ell}_t(\boldsymbol{\beta}^*(1), \boldsymbol{\beta}^*(0)) \mid \mathcal{F}_{t-1}] \\ & = \mathbb{E} \left[\frac{\mathcal{E}(0)}{\mathcal{E}(1)} \left(y_t(1) \cdot \frac{\mathbf{1}[Z_t = 1]}{p_t} - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle \right)^2 + \frac{\mathcal{E}(1)}{\mathcal{E}(0)} \left(y_t(0) \cdot \frac{\mathbf{1}[Z_t = 0]}{1-p_t} - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle \right)^2 \right. \\ & \quad \left. + 2 \left(y_t(1) \cdot \frac{\mathbf{1}[Z_t = 1]}{p_t} - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle \right) \cdot \left(y_t(0) \cdot \frac{\mathbf{1}[Z_t = 0]}{1-p_t} - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle \right) \Big| \mathcal{F}_{t-1} \right] \\ & \quad - \mathbb{E} \left[\frac{\mathcal{E}(0)}{\mathcal{E}(1)} \left(y_t(1) \cdot \frac{\mathbf{1}[Z_t = 1]}{p_t} - \langle \mathbf{x}_t, \boldsymbol{\beta}^*(1) \rangle \right)^2 + \frac{\mathcal{E}(1)}{\mathcal{E}(0)} \left(y_t(0) \cdot \frac{\mathbf{1}[Z_t = 0]}{1-p_t} - \langle \mathbf{x}_t, \boldsymbol{\beta}^*(0) \rangle \right)^2 \right. \\ & \quad \left. + 2 \left(y_t(1) \cdot \frac{\mathbf{1}[Z_t = 1]}{p_t} - \langle \mathbf{x}_t, \boldsymbol{\beta}^*(1) \rangle \right) \cdot \left(y_t(0) \cdot \frac{\mathbf{1}[Z_t = 0]}{1-p_t} - \langle \mathbf{x}_t, \boldsymbol{\beta}^*(0) \rangle \right) \Big| \mathcal{F}_{t-1} \right] \\ & = \mathbb{E} \left[\frac{\mathcal{E}(0)}{\mathcal{E}(1)} (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle)^2 + \frac{\mathcal{E}(1)}{\mathcal{E}(0)} (y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle)^2 + 2(y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle)(y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle) \Big| \mathcal{F}_{t-1} \right] \end{aligned}$$

$$\begin{aligned}
& + \mathbb{E} \left[\frac{\mathcal{E}(0)}{\mathcal{E}(1)} y_t^2(1) \left(\frac{\mathbf{1}[Z_t = 1]}{p_t} - 1 \right)^2 + \frac{\mathcal{E}(1)}{\mathcal{E}(0)} y_t^2(0) \left(\frac{\mathbf{1}[Z_t = 0]}{1-p_t} - 1 \right)^2 \middle| \mathcal{F}_{t-1} \right] \\
& + 2\mathbb{E} \left[\frac{\mathcal{E}(0)}{\mathcal{E}(1)} y_t(1)(y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle) \left(\frac{\mathbf{1}[Z_t = 1]}{p_t} - 1 \right) + \frac{\mathcal{E}(1)}{\mathcal{E}(0)} y_t(0)(y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle) \left(\frac{\mathbf{1}[Z_t = 0]}{1-p_t} - 1 \right) \middle| \mathcal{F}_{t-1} \right] \\
& - \mathbb{E} \left[\frac{\mathcal{E}(0)}{\mathcal{E}(1)} (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}^*(1) \rangle)^2 + \frac{\mathcal{E}(1)}{\mathcal{E}(0)} (y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}^*(0) \rangle)^2 + 2(y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}^*(1) \rangle)(y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}^*(0) \rangle) \middle| \mathcal{F}_{t-1} \right] \\
& - 2\mathbb{E} \left[\frac{\mathcal{E}(0)}{\mathcal{E}(1)} y_t(1)(y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}^*(1) \rangle) \left(\frac{\mathbf{1}[Z_t = 1]}{p_t} - 1 \right) + \frac{\mathcal{E}(1)}{\mathcal{E}(0)} y_t(0)(y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}^*(0) \rangle) \left(\frac{\mathbf{1}[Z_t = 0]}{1-p_t} - 1 \right) \middle| \mathcal{F}_{t-1} \right] \\
& = \frac{\mathcal{E}(0)}{\mathcal{E}(1)} (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle)^2 + \frac{\mathcal{E}(1)}{\mathcal{E}(0)} (y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle)^2 + 2(y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle) \cdot (y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle) \\
& \quad - \frac{\mathcal{E}(0)}{\mathcal{E}(1)} (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}^*(1) \rangle)^2 + \frac{\mathcal{E}(1)}{\mathcal{E}(0)} (y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}^*(0) \rangle)^2 + 2(y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}^*(1) \rangle) \cdot (y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}^*(0) \rangle) \\
& = \ell_t(\boldsymbol{\beta}_t(1), \boldsymbol{\beta}_t(0)) - \ell_t(\boldsymbol{\beta}^*(1), \boldsymbol{\beta}^*(0)) .
\end{aligned}$$

□

Under Lemma 4.8, the following lemma establishes that the regret can be bounded by a regularization term and the expectation of the sum of a sequence of successive differences. The derivation follows the standard FTRL analysis, but is adapted to our setting where the step size varies over time.

Lemma 4.9. *The expected prediction regret is bounded as*

$$\mathbb{E}[\mathcal{R}_T^{\text{pred}}] \leq \frac{m(\boldsymbol{\beta}^*(1), \boldsymbol{\beta}^*(0))}{\eta_{T+1}} + \mathbb{E} \left[\sum_{t=1}^T \tilde{L}_{t+1}(\boldsymbol{\beta}_t(1), \boldsymbol{\beta}_t(0)) - \tilde{L}_{t+1}(\tilde{\boldsymbol{\beta}}_{t+1}(1), \tilde{\boldsymbol{\beta}}_{t+1}(0)) \right].$$

Proof. We first prove that $(\boldsymbol{\beta}_t(1), \boldsymbol{\beta}_t(0))$ is the minimizer of $\hat{L}_t(\boldsymbol{\beta}(1), \boldsymbol{\beta}(0))$. For simplicity, for any $t \in [T]$, we denote $\mathbf{Y}_t(1) = \left(y_1(1) \cdot \frac{\mathbf{1}[Z_1=1]}{p_1}, \dots, y_t(1) \cdot \frac{\mathbf{1}[Z_t=1]}{p_t} \right)$ and $\mathbf{Y}_t(0) = \left(y_1(0) \cdot \frac{\mathbf{1}[Z_1=0]}{1-p_1}, \dots, y_t(0) \cdot \frac{\mathbf{1}[Z_t=0]}{1-p_t} \right)$. Let $\boldsymbol{\beta}_1 = \boldsymbol{\beta}_t(1) + \boldsymbol{\delta}_1$ and $\boldsymbol{\beta}_0 = \boldsymbol{\beta}_t(0) + \boldsymbol{\delta}_0$. We further denote $\boldsymbol{\alpha}_1 = \mathbf{Y}_{t-1}(1) - \mathbf{X}_{t-1}\boldsymbol{\beta}_t(1)$, $\boldsymbol{\alpha}_0 = \mathbf{Y}_{t-1}(0) - \mathbf{X}_{t-1}\boldsymbol{\beta}_t(0)$, $\mathbf{z}_1 = \mathbf{X}_{t-1}\boldsymbol{\delta}_1$ and $\mathbf{z}_0 = \mathbf{X}_{t-1}\boldsymbol{\delta}_0$. Hence we have

$$\begin{aligned}
\hat{L}_t(\boldsymbol{\beta}_1, \boldsymbol{\beta}_0) &= \frac{\mathcal{E}(0)}{\mathcal{E}(1)} (\|\boldsymbol{\alpha}_1 - \mathbf{z}_1\|^2 + \eta_t^{-1}\|\boldsymbol{\beta}_t(1) + \boldsymbol{\delta}_1\|^2) + \frac{\mathcal{E}(1)}{\mathcal{E}(0)} (\|\boldsymbol{\alpha}_0 - \mathbf{z}_0\|^2 + \eta_t^{-1}\|\boldsymbol{\beta}_t(0) + \boldsymbol{\delta}_0\|^2) \\
&\quad + 2(\langle \boldsymbol{\alpha}_1 - \mathbf{z}_1, \boldsymbol{\alpha}_0 - \mathbf{z}_0 \rangle + \eta_t^{-1}\langle \boldsymbol{\beta}_t(1) + \boldsymbol{\delta}_1, \boldsymbol{\beta}_t(0) + \boldsymbol{\delta}_0 \rangle), \\
\hat{L}_t(\boldsymbol{\beta}_t(1), \boldsymbol{\beta}_t(0)) &= \frac{\mathcal{E}(0)}{\mathcal{E}(1)} (\|\boldsymbol{\alpha}_1\|^2 + \eta_t^{-1}\|\boldsymbol{\beta}_t(1)\|^2) + \frac{\mathcal{E}(1)}{\mathcal{E}(0)} (\|\boldsymbol{\alpha}_0\|^2 + \eta_t^{-1}\|\boldsymbol{\beta}_t(0)\|^2) \\
&\quad + 2(\langle \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_0 \rangle + \eta_t^{-1}\langle \boldsymbol{\beta}_t(1), \boldsymbol{\beta}_t(0) \rangle).
\end{aligned}$$

This implies that

$$\begin{aligned}
& \hat{L}_t(\boldsymbol{\beta}_1, \boldsymbol{\beta}_0) - \hat{L}_t(\boldsymbol{\beta}_t(1), \boldsymbol{\beta}_t(0)) \\
&= \frac{\mathcal{E}(0)}{\mathcal{E}(1)} (\|\mathbf{z}_1\|^2 - 2\langle \boldsymbol{\alpha}_1, \mathbf{z}_1 \rangle + \eta_t^{-1}\|\boldsymbol{\delta}_1\|^2 + 2\eta_t^{-1}\langle \boldsymbol{\beta}_t(1), \boldsymbol{\delta}_1 \rangle) \\
&\quad + \frac{\mathcal{E}(1)}{\mathcal{E}(0)} (\|\mathbf{z}_0\|^2 - 2\langle \boldsymbol{\alpha}_0, \mathbf{z}_0 \rangle + \eta_t^{-1}\|\boldsymbol{\delta}_0\|^2 + 2\eta_t^{-1}\langle \boldsymbol{\beta}_t(0), \boldsymbol{\delta}_0 \rangle)
\end{aligned}$$

$$\begin{aligned}
& + 2(-\langle \mathbf{z}_1, \boldsymbol{\alpha}_0 \rangle - \langle \boldsymbol{\alpha}_1, \mathbf{z}_0 \rangle + \langle \mathbf{z}_1, \mathbf{z}_0 \rangle + \eta_t^{-1} \langle \boldsymbol{\delta}_1, \boldsymbol{\beta}_t(0) \rangle + \eta_t^{-1} \langle \boldsymbol{\delta}_0, \boldsymbol{\beta}_t(1) \rangle + \eta_t^{-1} \langle \boldsymbol{\delta}_1, \boldsymbol{\delta}_0 \rangle) \\
& = \frac{\mathcal{E}(0)}{\mathcal{E}(1)} \left(-2(\mathbf{Y}_{t-1}(1) - \mathbf{X}_{t-1}(\mathbf{X}_{t-1}^\top \mathbf{X}_{t-1} + \eta_t^{-1} \mathbf{I}_d)^{-1} \mathbf{X}_{t-1}^\top \mathbf{Y}_{t-1}(1))^\top \mathbf{X}_{t-1} \boldsymbol{\delta}_1 \right. \\
& \quad \left. + 2\eta_t^{-1} \mathbf{Y}_{t-1}^\top(1) \mathbf{X}_{t-1}(\mathbf{X}_{t-1}^\top \mathbf{X}_{t-1} + \eta_t^{-1} \mathbf{I}_d)^{-1} \boldsymbol{\delta}_1 + \|\mathbf{z}_1\|^2 + \eta_t^{-1} \|\boldsymbol{\delta}_1\|^2 \right) \\
& \quad + \frac{\mathcal{E}(1)}{\mathcal{E}(0)} \left(-2(\mathbf{Y}_{t-1}(0) - \mathbf{X}_{t-1}(\mathbf{X}_{t-1}^\top \mathbf{X}_{t-1} + \eta_t^{-1} \mathbf{I}_d)^{-1} \mathbf{X}_{t-1}^\top \mathbf{Y}_{t-1}(0))^\top \mathbf{X}_{t-1} \boldsymbol{\delta}_0 \right. \\
& \quad \left. + 2\eta_t^{-1} \mathbf{Y}_{t-1}^\top(0) \mathbf{X}_{t-1}(\mathbf{X}_{t-1}^\top \mathbf{X}_{t-1} + \eta_t^{-1} \mathbf{I}_d)^{-1} \boldsymbol{\delta}_0 + \|\mathbf{z}_0\|^2 + \eta_t^{-1} \|\boldsymbol{\delta}_0\|^2 \right) \\
& \quad + 2 \left(-(\mathbf{Y}_{t-1}(1) - \mathbf{X}_{t-1}(\mathbf{X}_{t-1}^\top \mathbf{X}_{t-1} + \eta_t^{-1} \mathbf{I}_d)^{-1} \mathbf{X}_{t-1}^\top \mathbf{Y}_{t-1}(1))^\top \mathbf{X}_{t-1} \boldsymbol{\delta}_0 \right) \\
& \quad + 2 \left(\eta_t^{-1} \mathbf{Y}_{t-1}^\top(1) \mathbf{X}_{t-1}(\mathbf{X}_{t-1}^\top \mathbf{X}_{t-1} + \eta_t^{-1} \mathbf{I}_d)^{-1} \boldsymbol{\delta}_0 \right) \\
& \quad + 2 \left(-(\mathbf{Y}_{t-1}(0) - \mathbf{X}_{t-1}(\mathbf{X}_{t-1}^\top \mathbf{X}_{t-1} + \eta_t^{-1} \mathbf{I}_d)^{-1} \mathbf{X}_{t-1}^\top \mathbf{Y}_{t-1}(0))^\top \mathbf{X}_{t-1} \boldsymbol{\delta}_1 \right) \\
& \quad + 2 \left(\eta_t^{-1} \mathbf{Y}_{t-1}^\top(0) \mathbf{X}_{t-1}(\mathbf{X}_{t-1}^\top \mathbf{X}_{t-1} + \eta_t^{-1} \mathbf{I}_d)^{-1} \boldsymbol{\delta}_1 \right) \\
& \quad + 2(\langle \mathbf{z}_1, \mathbf{z}_0 \rangle + \eta_t^{-1} \langle \boldsymbol{\delta}_1, \boldsymbol{\delta}_0 \rangle) . \tag{19}
\end{aligned}$$

Note that

$$\begin{aligned}
& -(\mathbf{I} - \mathbf{X}_{t-1}(\mathbf{X}_{t-1}^\top \mathbf{X}_{t-1} + \eta_t^{-1} \mathbf{I}_d)^{-1} \mathbf{X}_{t-1}^\top)^\top \mathbf{X}_{t-1} + \eta_t^{-1} \mathbf{X}_{t-1}(\mathbf{X}_{t-1}^\top \mathbf{X}_{t-1} + \eta_t^{-1} \mathbf{I}_d)^{-1} \\
& = -\mathbf{X}_{t-1} + \mathbf{X}_{t-1}(\mathbf{X}_{t-1}^\top \mathbf{X}_{t-1} + \eta_t^{-1} \mathbf{I}_d)^{-1} \mathbf{X}_{t-1}^\top \mathbf{X}_{t-1} + \eta_t^{-1} \mathbf{X}_{t-1}(\mathbf{X}_{t-1}^\top \mathbf{X}_{t-1} + \eta_t^{-1} \mathbf{I}_d)^{-1} \\
& = -\mathbf{X}_{t-1} + \mathbf{X}_{t-1}(\mathbf{X}_{t-1}^\top \mathbf{X}_{t-1} + \eta_t^{-1} \mathbf{I}_d)^{-1} (\mathbf{X}_{t-1}^\top \mathbf{X}_{t-1} + \eta_t^{-1} \mathbf{I}_d) - \eta_t^{-1} \mathbf{X}_{t-1}(\mathbf{X}_{t-1}^\top \mathbf{X}_{t-1} + \eta_t^{-1} \mathbf{I}_d)^{-1} \\
& \quad + \eta_t^{-1} \mathbf{X}_{t-1}(\mathbf{X}_{t-1}^\top \mathbf{X}_{t-1} + \eta_t^{-1} \mathbf{I}_d)^{-1} \\
& = \mathbf{0}.
\end{aligned}$$

Hence (19) can be further simplified and lower bounded by:

$$\begin{aligned}
& \hat{L}_t(\boldsymbol{\beta}_1, \boldsymbol{\beta}_0) - \hat{L}_t(\boldsymbol{\beta}_t(1), \boldsymbol{\beta}_t(0)) \\
& = \frac{\mathcal{E}(0)}{\mathcal{E}(1)} (\|\mathbf{z}_1\|^2 + \eta_t^{-1} \|\boldsymbol{\delta}_1\|^2) + \frac{\mathcal{E}(1)}{\mathcal{E}(0)} (\|\mathbf{z}_0\|^2 + \eta_t^{-1} \|\boldsymbol{\delta}_0\|^2) + 2(\langle \mathbf{z}_1, \mathbf{z}_0 \rangle + \eta_t^{-1} \langle \boldsymbol{\delta}_1, \boldsymbol{\delta}_0 \rangle) \\
& = \left\| \left(\frac{\mathcal{E}(0)}{\mathcal{E}(1)} \right)^{1/2} \mathbf{z}_1 + \left(\frac{\mathcal{E}(1)}{\mathcal{E}(0)} \right)^{1/2} \mathbf{z}_0 \right\|_2^2 + \eta_t^{-1} \left\| \left(\frac{\mathcal{E}(0)}{\mathcal{E}(1)} \right)^{1/2} \boldsymbol{\delta}_1 + \left(\frac{\mathcal{E}(1)}{\mathcal{E}(0)} \right)^{1/2} \boldsymbol{\delta}_0 \right\|_2^2 \\
& \geq 0.
\end{aligned}$$

Hence we proved that $(\boldsymbol{\beta}_t(1), \boldsymbol{\beta}_t(0))$ is the minimizer of $\hat{L}_t(\boldsymbol{\beta}(1), \boldsymbol{\beta}(0))$. Similarly we can prove that $(\tilde{\boldsymbol{\beta}}_t(1), \tilde{\boldsymbol{\beta}}_t(0))$ is the minimizer of $\tilde{L}_t(\boldsymbol{\beta}(1), \boldsymbol{\beta}(0))$. Furthermore, we can see that the penalty term is nonnegative:

$$m(\boldsymbol{\beta}(1), \boldsymbol{\beta}(0)) = \frac{\mathcal{E}(0)}{\mathcal{E}(1)} \|\boldsymbol{\beta}(1)\|^2 + \frac{\mathcal{E}(1)}{\mathcal{E}(0)} \|\boldsymbol{\beta}(0)\|^2 + 2\langle \boldsymbol{\beta}(1), \boldsymbol{\beta}(0) \rangle = \left\| \left(\frac{\mathcal{E}(0)}{\mathcal{E}(1)} \right)^{1/2} \boldsymbol{\beta}(1) + \left(\frac{\mathcal{E}(1)}{\mathcal{E}(0)} \right)^{1/2} \boldsymbol{\beta}(0) \right\|_2^2 \geq 0.$$

By the construction method, we can easily see that the step size η_t is decreasing with respect to t . By Lemma 4.8, we have

$$\begin{aligned}
& \mathbb{E}[\mathcal{R}_T^{\text{pred}}] \\
& = \mathbb{E} \left[\sum_{t=1}^T \ell_t(\boldsymbol{\beta}_t(1), \boldsymbol{\beta}_t(0)) - \sum_{t=1}^T \ell_t(\boldsymbol{\beta}^*(1), \boldsymbol{\beta}^*(0)) \right] \\
& = \sum_{t=1}^T \mathbb{E} \left[\mathbb{E} \left[\hat{\ell}_t(\boldsymbol{\beta}_t(1), \boldsymbol{\beta}_t(0)) - \hat{\ell}_t(\boldsymbol{\beta}^*(1), \boldsymbol{\beta}^*(0)) | \mathcal{F}_{t-1} \right] \right] \quad (\text{Lemma 4.8})
\end{aligned}$$

$$\begin{aligned}
&= \mathbb{E} \left[\sum_{t=1}^T \hat{\ell}_t(\boldsymbol{\beta}_t(1), \boldsymbol{\beta}_t(0)) - \sum_{t=1}^T \hat{\ell}_t(\boldsymbol{\beta}^*(1), \boldsymbol{\beta}^*(0)) \right] \\
&= \eta_{T+1}^{-1} m(\boldsymbol{\beta}^*(1), \boldsymbol{\beta}^*(0)) + \mathbb{E} \left[\sum_{t=1}^T \hat{\ell}_t(\boldsymbol{\beta}_t(1), \boldsymbol{\beta}_t(0)) - \hat{L}_{T+1}(\boldsymbol{\beta}^*(1), \boldsymbol{\beta}^*(0)) \right] \\
&= \eta_{T+1}^{-1} m(\boldsymbol{\beta}^*(1), \boldsymbol{\beta}^*(0)) + \mathbb{E} \left[\sum_{t=1}^T \hat{\ell}_t(\boldsymbol{\beta}_t(1), \boldsymbol{\beta}_t(0)) + \hat{L}_1(\boldsymbol{\beta}_1(1), \boldsymbol{\beta}_1(0)) - \hat{L}_{T+1}(\boldsymbol{\beta}_{T+1}(1), \boldsymbol{\beta}_{T+1}(0)) \right] \\
&\quad + \mathbb{E} \left[\hat{L}_{T+1}(\boldsymbol{\beta}_{T+1}(1), \boldsymbol{\beta}_{T+1}(0)) - \hat{L}_{T+1}(\boldsymbol{\beta}^*(1), \boldsymbol{\beta}^*(0)) \right] \\
&\leq \eta_{T+1}^{-1} m(\boldsymbol{\beta}^*(1), \boldsymbol{\beta}^*(0)) + \mathbb{E} \left[\sum_{t=1}^T \hat{\ell}_t(\boldsymbol{\beta}_t(1), \boldsymbol{\beta}_t(0)) + \sum_{t=1}^T (\hat{L}_t(\boldsymbol{\beta}_t(1), \boldsymbol{\beta}_t(0)) - \hat{L}_{t+1}(\boldsymbol{\beta}_{t+1}(1), \boldsymbol{\beta}_{t+1}(0))) \right]
\end{aligned}$$

(The last inequality is due to the optimality of $(\boldsymbol{\beta}_{T+1}(1), \boldsymbol{\beta}_{T+1}(0))$.)

$$\begin{aligned}
&= \eta_{T+1}^{-1} m(\boldsymbol{\beta}^*(1), \boldsymbol{\beta}^*(0)) + \mathbb{E} \left[\sum_{t=1}^T (\tilde{L}_{t+1}(\boldsymbol{\beta}_t(1), \boldsymbol{\beta}_t(0)) - \tilde{L}_{t+1}(\boldsymbol{\beta}_{t+1}(1), \boldsymbol{\beta}_{t+1}(0))) \right] \\
&\quad + \mathbb{E} \left[\sum_{t=1}^T (\tilde{L}_{t+1}(\boldsymbol{\beta}_{t+1}(1), \boldsymbol{\beta}_{t+1}(0)) - \hat{L}_{t+1}(\boldsymbol{\beta}_{t+1}(1), \boldsymbol{\beta}_{t+1}(0))) \right] \\
&\leq \eta_{T+1}^{-1} m(\boldsymbol{\beta}^*(1), \boldsymbol{\beta}^*(0)) + \mathbb{E} \left[\sum_{t=1}^T (\tilde{L}_{t+1}(\boldsymbol{\beta}_t(1), \boldsymbol{\beta}_t(0)) - \tilde{L}_{t+1}(\tilde{\boldsymbol{\beta}}_{t+1}(1), \tilde{\boldsymbol{\beta}}_{t+1}(0))) \right] \\
&\quad + \sum_{t=1}^T (\eta_t^{-1} - \eta_{t+1}^{-1}) \mathbb{E} [m(\boldsymbol{\beta}_{t+1}(1), \boldsymbol{\beta}_{t+1}(0))] \quad (\text{optimality of } (\tilde{\boldsymbol{\beta}}_{t+1}(1), \tilde{\boldsymbol{\beta}}_{t+1}(0))) \\
&\leq \eta_{T+1}^{-1} m(\boldsymbol{\beta}^*(1), \boldsymbol{\beta}^*(0)) + \mathbb{E} \left[\sum_{t=1}^T (\tilde{L}_{t+1}(\boldsymbol{\beta}_t(1), \boldsymbol{\beta}_t(0)) - \tilde{L}_{t+1}(\tilde{\boldsymbol{\beta}}_{t+1}(1), \tilde{\boldsymbol{\beta}}_{t+1}(0))) \right],
\end{aligned}$$

where the last inequality is due to the monotonicity of η_t and the nonnegativity of $m(\boldsymbol{\beta}(1), \boldsymbol{\beta}(0))$.

□

Lemma 4.10. *The successive difference can be written as:*

$$\tilde{L}_{t+1}(\boldsymbol{\beta}_t(1), \boldsymbol{\beta}_t(0)) - \tilde{L}_{t+1}(\tilde{\boldsymbol{\beta}}_{t+1}(1), \tilde{\boldsymbol{\beta}}_{t+1}(0)) = \frac{\Pi_{t,t}}{1 + \Pi_{t,t}} \cdot \hat{\ell}_t(\boldsymbol{\beta}_t(1), \boldsymbol{\beta}_t(0)) .$$

Proof. We first derive the explicit form of:

$$\left[\sum_{s=1}^t \hat{\ell}_s(\boldsymbol{\beta}_s(1)) + \eta_t^{-1} \|\boldsymbol{\beta}_t(1)\|_2^2 \right] - \left[\sum_{s=1}^t \hat{\ell}_s(\tilde{\boldsymbol{\beta}}_{s+1}(1)) + \eta_t^{-1} \|\tilde{\boldsymbol{\beta}}_{t+1}(1)\|_2^2 \right] .$$

For notation simplicity, we slightly abuse the notation and suppress $y_t(1) \cdot \frac{\mathbf{1}[Z_t=1]}{p_t}$ as y_t . Let $\mathbf{Y}_{t-1} = (y_1, \dots, y_{t-1})^\top$ and $\mathbf{Y}_t = (y_1, \dots, y_t)^\top$. Then we have the explicit form for the regression coefficient: $\boldsymbol{\beta}_t(1) = (\mathbf{X}_{t-1}^\top \mathbf{X}_{t-1} + \eta_t^{-1} \mathbf{I}_d)^{-1} \mathbf{X}_{t-1}^\top \mathbf{Y}_{t-1}$ and $\tilde{\boldsymbol{\beta}}_{t+1}(1) = (\mathbf{X}_t^\top \mathbf{X}_t + \eta_t^{-1} \mathbf{I}_d)^{-1} \mathbf{X}_t^\top \mathbf{Y}_t$. Now we calculate the explicit form for $\tilde{L}_{t+1}^{(1)}(\boldsymbol{\beta}_t(1)) - \tilde{L}_{t+1}^{(1)}(\tilde{\boldsymbol{\beta}}_{t+1}(1))$.

$$\begin{aligned}
\mathbf{W} &= \mathbf{X}_{t-1}^\top \mathbf{X}_{t-1} + \eta_t^{-1} \mathbf{I}_d , \\
h &= \mathbf{x}_t^\top \mathbf{W}^{-1} \mathbf{x}_t , \\
\boldsymbol{\alpha} &= \mathbf{X}_{t-1} \mathbf{W}^{-1} \mathbf{x}_t , \\
\mathbf{H} &= \mathbf{X}_{t-1} \mathbf{W}^{-1} \mathbf{X}_{t-1}^\top .
\end{aligned}$$

Step 1: Calculate $\sum_{s=1}^t \hat{\ell}_s(\tilde{\beta}_{t+1}(1)) + \eta_t^{-1} \|\tilde{\beta}_{t+1}(1)\|_2^2$.

By the explicit form of $\tilde{\beta}_{t+1}(1)$, we have

$$\begin{aligned}
& \sum_{s=1}^t \hat{\ell}_s(\tilde{\beta}_{t+1}(1)) + \eta_t^{-1} \|\tilde{\beta}_{t+1}(1)\|_2^2 \\
&= \sum_{s=1}^t \left(y_s(1) \cdot \frac{\mathbf{1}[Z_s = 1]}{p_s} - \langle \mathbf{x}_s, \tilde{\beta}_{t+1}(1) \rangle \right)^2 + \eta_t^{-1} \|\tilde{\beta}_{t+1}(1)\|_2^2 \\
&= \|\mathbf{Y}_t - \mathbf{X}_t(\mathbf{X}_t^\top \mathbf{X}_t + \eta_t^{-1} \mathbf{I}_d)^{-1} \mathbf{X}_t^\top \mathbf{Y}_t\|^2 + \eta_t^{-1} \|(\mathbf{X}_t^\top \mathbf{X}_t + \eta_t^{-1} \mathbf{I}_d)^{-1} \mathbf{X}_t^\top \mathbf{Y}_t\|^2 \\
&= \mathbf{Y}_t^\top (\mathbf{I}_t - \mathbf{X}_t(\mathbf{X}_t^\top \mathbf{X}_t + \eta_t^{-1} \mathbf{I}_d)^{-1} \mathbf{X}_t^\top) \mathbf{Y}_t + \eta_t^{-1} \mathbf{Y}_t^\top \mathbf{X}_t (\mathbf{X}_t^\top \mathbf{X}_t + \eta_t^{-1} \mathbf{I}_d)^{-2} \mathbf{X}_t^\top \mathbf{Y}_t \\
&= \mathbf{Y}_t^\top \mathbf{Y}_t - 2 \mathbf{Y}_t^\top \mathbf{X}_t (\mathbf{X}_t^\top \mathbf{X}_t + \eta_t^{-1} \mathbf{I}_d)^{-1} \mathbf{X}_t^\top \mathbf{Y}_t \\
&\quad + \mathbf{Y}_t^\top \mathbf{X}_t (\mathbf{X}_t^\top \mathbf{X}_t + \eta_t^{-1} \mathbf{I}_d)^{-1} \mathbf{X}_t^\top \mathbf{X}_t (\mathbf{X}_t^\top \mathbf{X}_t + \eta_t^{-1} \mathbf{I}_d)^{-1} \mathbf{X}_t^\top \mathbf{Y}_t + \eta_t^{-1} \mathbf{Y}_t^\top \mathbf{X}_t (\mathbf{X}_t^\top \mathbf{X}_t + \eta_t^{-1} \mathbf{I}_d)^{-2} \mathbf{X}_t^\top \mathbf{Y}_t \\
&= \mathbf{Y}_t^\top \mathbf{Y}_t - 2 \mathbf{Y}_t^\top \mathbf{X}_t (\mathbf{X}_t^\top \mathbf{X}_t + \eta_t^{-1} \mathbf{I}_d)^{-1} \mathbf{X}_t^\top \mathbf{Y}_t \\
&\quad + \mathbf{Y}_t^\top \mathbf{X}_t (\mathbf{X}_t^\top \mathbf{X}_t + \eta_t^{-1} \mathbf{I}_d)^{-1} (\mathbf{X}_t^\top \mathbf{X}_t + \eta_t^{-1} \mathbf{I}_d) (\mathbf{X}_t^\top \mathbf{X}_t + \eta_t^{-1} \mathbf{I}_d)^{-1} \mathbf{X}_t^\top \mathbf{Y}_t \\
&= \mathbf{Y}_t^\top (\mathbf{I}_t - \mathbf{X}_t(\mathbf{X}_t^\top \mathbf{X}_t + \eta_t^{-1} \mathbf{I}_d)^{-1} \mathbf{X}_t^\top) \mathbf{Y}_t \\
&= [\mathbf{Y}_{t-1}^\top \quad y_t] \left(\mathbf{I}_t - \begin{bmatrix} \mathbf{X}_{t-1} \\ \mathbf{x}_t^\top \end{bmatrix} (\mathbf{W} + \mathbf{x}_t \mathbf{x}_t^\top)^{-1} \begin{bmatrix} \mathbf{X}_{t-1}^\top & \mathbf{x}_t \end{bmatrix} \right) \begin{bmatrix} \mathbf{Y}_{t-1} \\ y_t \end{bmatrix}. \tag{20}
\end{aligned}$$

By Sherman-Morrison formula, we have

$$(\mathbf{W} + \mathbf{x}_t \mathbf{x}_t^\top)^{-1} = \mathbf{W}^{-1} - \frac{\mathbf{W}^{-1} \mathbf{x}_t \mathbf{x}_t^\top \mathbf{W}^{-1}}{1 + \mathbf{x}_t^\top \mathbf{W}^{-1} \mathbf{x}_t} = \mathbf{W}^{-1} - \frac{\mathbf{W}^{-1} \mathbf{x}_t \mathbf{x}_t^\top \mathbf{W}^{-1}}{1 + h}. \tag{21}$$

For simplicity, suppose we have the following block form for the matrix:

$$\begin{bmatrix} \mathbf{X}_{t-1} \\ \mathbf{x}_t^\top \end{bmatrix} (\mathbf{W} + \mathbf{x}_t \mathbf{x}_t^\top)^{-1} \begin{bmatrix} \mathbf{X}_{t-1}^\top & \mathbf{x}_t \end{bmatrix} = \begin{bmatrix} \mathbf{R}_{1,1} & \mathbf{R}_{1,2} \\ \mathbf{R}_{1,2}^\top & R_{2,2} \end{bmatrix}.$$

Now we calculate the explicit form for each block:

(1) By (21), $\mathbf{R}_{1,1}$ can calculated as:

$$\mathbf{R}_{1,1} = \mathbf{X}_{t-1} \mathbf{W}^{-1} \mathbf{X}_{t-1}^\top - \frac{\mathbf{X}_{t-1} \mathbf{W}^{-1} \mathbf{x}_t \mathbf{x}_t^\top \mathbf{W}^{-1} \mathbf{X}_{t-1}^\top}{1 + h} = \mathbf{H} - \frac{\alpha \alpha^\top}{1 + h}.$$

(2) By (21), $\mathbf{R}_{1,2}$ can be calculated as:

$$\mathbf{R}_{1,2} = \mathbf{X}_{t-1} \mathbf{W}^{-1} \mathbf{x}_t - \frac{\mathbf{X}_{t-1} \mathbf{W}^{-1} \mathbf{x}_t \mathbf{x}_t^\top \mathbf{W}^{-1} \mathbf{x}_t}{1 + h} = \alpha - \frac{h}{1 + h} \alpha = \frac{1}{1 + h} \alpha.$$

(3) By (21), $R_{2,2}$ can be calculated as:

$$R_{2,2} = \mathbf{x}_t^\top \mathbf{W}^{-1} \mathbf{x}_t - \frac{\mathbf{x}_t^\top \mathbf{W}^{-1} \mathbf{x}_t \mathbf{x}_t^\top \mathbf{W}^{-1} \mathbf{x}_t}{1 + h} = h - \frac{h^2}{1 + h} = \frac{h}{1 + h}.$$

Hence by (20), we have

$$\begin{aligned}
& \sum_{s=1}^t \hat{\ell}_s(\tilde{\beta}_{t+1}(1)) + \eta_t^{-1} \|\tilde{\beta}_{t+1}(1)\|_2^2 \\
&= [\mathbf{Y}_{t-1}^\top \quad y_t] \left(\mathbf{I}_t - \begin{bmatrix} \mathbf{X}_{t-1} \\ \mathbf{x}_t^\top \end{bmatrix} (\mathbf{W} + \mathbf{x}_t \mathbf{x}_t^\top)^{-1} \begin{bmatrix} \mathbf{X}_{t-1}^\top & \mathbf{x}_t \end{bmatrix} \right) \begin{bmatrix} \mathbf{Y}_{t-1} \\ y_t \end{bmatrix} \\
&= [\mathbf{Y}_{t-1}^\top \quad y_t] \left(\mathbf{I}_t - \begin{bmatrix} \mathbf{R}_{1,1} & \mathbf{R}_{1,2} \\ \mathbf{R}_{1,2}^\top & R_{2,2} \end{bmatrix} \right) \begin{bmatrix} \mathbf{Y}_{t-1} \\ y_t \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
&= [\mathbf{Y}_{t-1}^\top \quad y_t] \begin{bmatrix} \mathbf{I}_{t-1} - \mathbf{R}_{1,1} & -\mathbf{R}_{1,2} \\ -\mathbf{R}_{1,2}^\top & 1 - R_{2,2} \end{bmatrix} \begin{bmatrix} \mathbf{Y}_{t-1} \\ y_t \end{bmatrix} \\
&= [\mathbf{Y}_{t-1}^\top \quad y_t] \begin{bmatrix} (\mathbf{I}_{t-1} - \mathbf{H}) + \frac{\alpha\alpha^\top}{1+h} & -\frac{1}{1+h}\alpha \\ -\frac{1}{1+h}\alpha^\top & \frac{1}{1+h} \end{bmatrix} \begin{bmatrix} \mathbf{Y}_{t-1} \\ y_t \end{bmatrix} \\
&= \mathbf{Y}_{t-1}^\top (\mathbf{I}_{t-1} - \mathbf{H}) \mathbf{Y}_{t-1} + \frac{1}{1+h} (\alpha^\top \mathbf{Y}_{t-1})^2 - \frac{2y_t}{1+h} \alpha^\top \mathbf{Y}_{t-1} + \frac{1}{1+h} y_t^2 \\
&= \mathbf{Y}_{t-1}^\top (\mathbf{I}_{t-1} - \mathbf{H}) \mathbf{Y}_{t-1} + \frac{1}{1+h} (y_t - \alpha^\top \mathbf{Y}_{t-1})^2. \tag{22}
\end{aligned}$$

Step 2: Calculate $\sum_{s=1}^t \hat{\ell}_s(\beta_t(1)) + \eta_t^{-1} \|\beta_t(1)\|_2^2$.

Since $\beta_t(1) = (\mathbf{X}_{t-1}^\top \mathbf{X}_{t-1} + \eta_t^{-1} \mathbf{I}_d)^{-1} \mathbf{X}_{t-1}^\top \mathbf{Y}_{t-1} = \mathbf{W}^{-1} \mathbf{X}_{t-1}^\top \mathbf{Y}_{t-1}$, we have

$$\begin{aligned}
&\sum_{s=1}^t \hat{\ell}_s(\beta_t(1)) + \eta_t^{-1} \|\beta_t(1)\|_2^2 \\
&= \|\mathbf{Y}_{t-1} - \mathbf{X}_{t-1} \mathbf{W}^{-1} \mathbf{X}_{t-1}^\top \mathbf{Y}_{t-1}\|_2^2 + (y_t - \mathbf{x}_t^\top \mathbf{W}^{-1} \mathbf{X}_{t-1}^\top \mathbf{Y}_{t-1})^2 + \eta_t^{-1} \|\mathbf{Y}_{t-1}^\top \mathbf{X}_{t-1} \mathbf{W}^{-2} \mathbf{X}_{t-1}^\top \mathbf{Y}_{t-1}\|_2^2 \\
&= \mathbf{Y}_{t-1}^\top (\mathbf{I}_{t-1} - \mathbf{H})^2 \mathbf{Y}_{t-1} + (y_t - \alpha^\top \mathbf{Y}_{t-1})^2 + \eta_t^{-1} \|\mathbf{Y}_{t-1}^\top \mathbf{X}_{t-1} \mathbf{W}^{-2} \mathbf{X}_{t-1}^\top \mathbf{Y}_{t-1}\|_2^2 \\
&= \mathbf{Y}_{t-1}^\top [(\mathbf{I}_{t-1} - \mathbf{H})^2 + \eta_t^{-1} \mathbf{X}_{t-1} \mathbf{W}^{-2} \mathbf{X}_{t-1}^\top] \mathbf{Y}_{t-1} + (y_t - \alpha^\top \mathbf{Y}_{t-1})^2 \\
&= \mathbf{Y}_{t-1}^\top [(\mathbf{I}_{t-1} - \mathbf{X}_{t-1} (\mathbf{X}_{t-1}^\top \mathbf{X}_{t-1} + \eta_t^{-1} \mathbf{I}_d)^{-1} \mathbf{X}_{t-1}^\top)^2 + \eta_t^{-1} \mathbf{X}_{t-1} (\mathbf{X}_{t-1}^\top \mathbf{X}_{t-1} + \eta_t^{-1} \mathbf{I}_d)^{-2} \mathbf{X}_{t-1}^\top] \mathbf{Y}_{t-1} \\
&\quad + (y_t - \alpha^\top \mathbf{Y}_{t-1})^2 \\
&= \mathbf{Y}_{t-1}^\top [\mathbf{I}_{t-1} - 2\mathbf{X}_{t-1} (\mathbf{X}_{t-1}^\top \mathbf{X}_{t-1} + \eta_t^{-1} \mathbf{I}_d)^{-1} \mathbf{X}_{t-1}^\top + \eta_t^{-1} \mathbf{X}_{t-1} (\mathbf{X}_{t-1}^\top \mathbf{X}_{t-1} + \eta_t^{-1} \mathbf{I}_d)^{-2} \mathbf{X}_{t-1}^\top \\
&\quad + \mathbf{X}_{t-1} (\mathbf{X}_{t-1}^\top \mathbf{X}_{t-1} + \eta_t^{-1} \mathbf{I}_d)^{-1} \mathbf{X}_{t-1}^\top \mathbf{X}_{t-1} (\mathbf{X}_{t-1}^\top \mathbf{X}_{t-1} + \eta_t^{-1} \mathbf{I}_d)^{-1} \mathbf{X}_{t-1}^\top] \mathbf{Y}_{t-1} + (y_t - \alpha^\top \mathbf{Y}_{t-1})^2 \\
&= \mathbf{Y}_{t-1}^\top [\mathbf{I}_{t-1} - 2\mathbf{X}_{t-1} (\mathbf{X}_{t-1}^\top \mathbf{X}_{t-1} + \eta_t^{-1} \mathbf{I}_d)^{-1} \mathbf{X}_{t-1}^\top \\
&\quad + \mathbf{X}_{t-1} (\mathbf{X}_{t-1}^\top \mathbf{X}_{t-1} + \eta_t^{-1} \mathbf{I}_d)^{-1} (\mathbf{X}_{t-1}^\top \mathbf{X}_{t-1} + \eta_t^{-1} \mathbf{I}_d) (\mathbf{X}_{t-1}^\top \mathbf{X}_{t-1} + \eta_t^{-1} \mathbf{I}_d)^{-1} \mathbf{X}_{t-1}^\top] \mathbf{Y}_{t-1} + (y_t - \alpha^\top \mathbf{Y}_{t-1})^2 \\
&= \mathbf{Y}_{t-1}^\top [\mathbf{I}_{t-1} - \mathbf{X}_{t-1} (\mathbf{X}_{t-1}^\top \mathbf{X}_{t-1} + \eta_t^{-1} \mathbf{I}_d)^{-1} \mathbf{X}_{t-1}^\top] \mathbf{Y}_{t-1} + (y_t - \alpha^\top \mathbf{Y}_{t-1})^2 \\
&= \mathbf{Y}_{t-1}^\top (\mathbf{I}_{t-1} - \mathbf{H}) \mathbf{Y}_{t-1} + (y_t - \alpha^\top \mathbf{Y}_{t-1})^2. \tag{23}
\end{aligned}$$

Step 3: Combine the results in the previous steps.

By (22) and (23), we have

$$\begin{aligned}
&\left[\sum_{s=1}^t \left(y_s(1) \cdot \frac{\mathbf{1}[Z_s = 1]}{p_s} - \langle \mathbf{x}_s, \beta_t(1) \rangle \right)^2 + \eta_t^{-1} \|\beta_t(1)\|_2^2 \right] \\
&\quad - \left[\sum_{s=1}^t \left(y_s(1) \cdot \frac{\mathbf{1}[Z_s = 1]}{p_s} - \langle \mathbf{x}_s, \tilde{\beta}_{t+1}(1) \rangle \right)^2 + \eta_t^{-1} \|\tilde{\beta}_{t+1}(1)\|_2^2 \right] \\
&= \mathbf{Y}_{t-1}^\top (\mathbf{I}_{t-1} - \mathbf{H}) \mathbf{Y}_{t-1} + (y_t - \alpha^\top \mathbf{Y}_{t-1})^2 - \mathbf{Y}_{t-1}^\top (\mathbf{I}_{t-1} - \mathbf{H}) \mathbf{Y}_{t-1} - \frac{1}{1+h} (y_t - \alpha^\top \mathbf{Y}_{t-1})^2 \\
&= \frac{h}{1+h} (y_t - \alpha^\top \mathbf{Y}_{t-1})^2 \\
&= \frac{\mathbf{x}_t^\top (\mathbf{X}_{t-1}^\top \mathbf{X}_{t-1} + \eta_t^{-1} \mathbf{I}_d)^{-1} \mathbf{x}_t}{1 + \mathbf{x}_t^\top (\mathbf{X}_{t-1}^\top \mathbf{X}_{t-1} + \eta_t^{-1} \mathbf{I}_d)^{-1} \mathbf{x}_t} \left(y_t(1) \cdot \frac{\mathbf{1}[Z_t = 1]}{p_t} - \langle \mathbf{x}_t, \beta_t(1) \rangle \right)^2 \\
&= \frac{\Pi_{t,t}}{1 + \Pi_{t,t}} \left(y_t(1) \cdot \frac{\mathbf{1}[Z_t = 1]}{p_t} - \langle \mathbf{x}_t, \beta_t(1) \rangle \right)^2. \tag{24}
\end{aligned}$$

By the same method, we can also prove that

$$\left[\sum_{s=1}^t \left(y_s(0) \cdot \frac{\mathbf{1}[Z_s = 0]}{1-p_s} - \langle \mathbf{x}_s, \beta_t(0) \rangle \right)^2 + \eta_t^{-1} \|\beta_t(0)\|_2^2 \right]$$

$$\begin{aligned}
& - \left[\sum_{s=1}^t \left(y_s(0) \cdot \frac{\mathbf{1}[Z_s = 0]}{1-p_s} - \langle \mathbf{x}_s, \tilde{\boldsymbol{\beta}}_{t+1}(0) \rangle \right) + \eta_t^{-1} \|\tilde{\boldsymbol{\beta}}_{t+1}(0)\|_2^2 \right] \\
& = \frac{\Pi_{t,t}}{1 + \Pi_{t,t}} \left(y_t(0) \cdot \frac{\mathbf{1}[Z_t = 0]}{1-p_t} - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle \right)^2 . \tag{25}
\end{aligned}$$

and

$$\begin{aligned}
& \left[\sum_{s=1}^t \left(y_s(1) \cdot \frac{\mathbf{1}[Z_s = 1]}{p_s} - \langle \mathbf{x}_s, \boldsymbol{\beta}_t(1) \rangle \right) \cdot \left(y_s(0) \cdot \frac{\mathbf{1}[Z_s = 0]}{1-p_s} - \langle \mathbf{x}_s, \boldsymbol{\beta}_t(0) \rangle \right) + \eta_t^{-1} \langle \boldsymbol{\beta}_t(1), \boldsymbol{\beta}_t(0) \rangle \right] \\
& - \left[\sum_{s=1}^t \left(y_s(1) \cdot \frac{\mathbf{1}[Z_s = 1]}{p_s} - \langle \mathbf{x}_s, \tilde{\boldsymbol{\beta}}_{t+1}(1) \rangle \right) \cdot \left(y_s(0) \cdot \frac{\mathbf{1}[Z_s = 0]}{1-p_s} - \langle \mathbf{x}_s, \tilde{\boldsymbol{\beta}}_{t+1}(1) \rangle \right) + \eta_t^{-1} \langle \tilde{\boldsymbol{\beta}}_{t+1}(1), \tilde{\boldsymbol{\beta}}_{t+1}(0) \rangle \right] \\
& = \frac{\Pi_{t,t}}{1 + \Pi_{t,t}} \left(y_t(1) \cdot \frac{\mathbf{1}[Z_t = 1]}{p_t} - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle \right) \left(y_t(0) \cdot \frac{\mathbf{1}[Z_t = 0]}{1-p_t} - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle \right) . \tag{26}
\end{aligned}$$

By (24), (25) and (26), we can finally prove that

$$\begin{aligned}
& \tilde{L}_{t+1}(\boldsymbol{\beta}_t(1), \boldsymbol{\beta}_t(0)) - \tilde{L}_{t+1}(\tilde{\boldsymbol{\beta}}_{t+1}(1), \tilde{\boldsymbol{\beta}}_{t+1}(0)) \\
& = \frac{\mathcal{E}(0)}{\mathcal{E}(1)} \cdot \frac{\Pi_{t,t}}{1 + \Pi_{t,t}} \left(y_t(1) \cdot \frac{\mathbf{1}[Z_t = 1]}{p_t} - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle \right)^2 + \frac{\mathcal{E}(1)}{\mathcal{E}(0)} \cdot \frac{\Pi_{t,t}}{1 + \Pi_{t,t}} \left(y_t(0) \cdot \frac{\mathbf{1}[Z_t = 0]}{1-p_t} - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle \right)^2 \\
& \quad + 2 \cdot \frac{\Pi_{t,t}}{1 + \Pi_{t,t}} \left(y_t(1) \cdot \frac{\mathbf{1}[Z_t = 1]}{p_t} - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle \right) \left(y_t(0) \cdot \frac{\mathbf{1}[Z_t = 0]}{1-p_t} - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle \right) \\
& = \frac{\Pi_{t,t}}{1 + \Pi_{t,t}} \hat{\ell}_t(\boldsymbol{\beta}_t(1), \boldsymbol{\beta}_t(0)) .
\end{aligned}$$

□

Proposition 4.11*. Under Assumptions 1-3 and Condition 1, the expected prediction regret is bounded as

$$E[\mathcal{R}_T^{pred}] \leq \left[(c_1^2 c_2 \max\{c_2, 1\}) \left(\frac{c_1}{c_0} + \frac{c_0}{c_1} + 2 \right) + o(1) \right] T^{1/2} R^2 .$$

Proof. By the definition of $\hat{\ell}_t$, we have the following decomposition:

$$\begin{aligned}
& \hat{\ell}_t(\boldsymbol{\beta}_t(1), \boldsymbol{\beta}_t(0)) \\
& = \frac{\mathcal{E}(0)}{\mathcal{E}(1)} \left(y_t(1) \cdot \frac{\mathbf{1}[Z_t = 1]}{p_t} - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle \right)^2 + \frac{\mathcal{E}(1)}{\mathcal{E}(0)} \left(y_t(0) \cdot \frac{\mathbf{1}[Z_t = 0]}{1-p_t} - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle \right)^2 \\
& \quad + 2 \left(y_t(1) \cdot \frac{\mathbf{1}[Z_t = 1]}{p_t} - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle \right) \left(y_t(0) \cdot \frac{\mathbf{1}[Z_t = 0]}{1-p_t} - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle \right) \\
& = \frac{\mathcal{E}(0)}{\mathcal{E}(1)} (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle)^2 + \frac{\mathcal{E}(1)}{\mathcal{E}(0)} (y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle)^2 + 2 (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle) (y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle) \\
& \quad + \frac{\mathcal{E}(0)}{\mathcal{E}(1)} y_t^2(1) \left(\frac{\mathbf{1}[Z_t = 1]}{p_t} - 1 \right)^2 + \frac{2\mathcal{E}(0)}{\mathcal{E}(1)} y_t(1) (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle) \left(\frac{\mathbf{1}[Z_t = 1]}{p_t} - 1 \right) \\
& \quad + \frac{\mathcal{E}(0)}{\mathcal{E}(1)} y_t^2(0) \left(\frac{\mathbf{1}[Z_t = 0]}{1-p_t} - 1 \right)^2 + \frac{2\mathcal{E}(1)}{\mathcal{E}(0)} y_t(0) (y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle) \left(\frac{\mathbf{1}[Z_t = 0]}{1-p_t} - 1 \right) \\
& \quad + 2y_t(1)y_t(0) \left(\frac{\mathbf{1}[Z_t = 1]}{p_t} - 1 \right) \left(\frac{\mathbf{1}[Z_t = 0]}{1-p_t} - 1 \right) + 2y_t(1) \left(\frac{\mathbf{1}[Z_t = 1]}{p_t} - 1 \right) (y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle) \\
& \quad + 2y_t(0) \left(\frac{\mathbf{1}[Z_t = 0]}{1-p_t} - 1 \right) (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle) .
\end{aligned}$$

It is easy to see that among all terms, term 5, 7, 9, 10 have mean zero by law of iterated expectation. Hence by Lemma 4.10, we have

$$\begin{aligned}
& \mathbb{E} \left[\sum_{t=1}^T (\tilde{L}_{t+1}(\boldsymbol{\beta}_t(1), \boldsymbol{\beta}_t(0)) - \tilde{L}_{t+1}(\tilde{\boldsymbol{\beta}}_{t+1}(1), \tilde{\boldsymbol{\beta}}_{t+1}(0))) \right] \\
& \leq \mathbb{E} \left[\sum_{t=1}^T \Pi_{t,t} \hat{\ell}_t(\boldsymbol{\beta}_t(1), \boldsymbol{\beta}_t(0)) \right] \quad (\text{Lemma 4.10}) \\
& = \frac{\mathcal{E}(0)}{\mathcal{E}(1)} \mathbb{E} \left[\sum_{t=1}^T \Pi_{t,t} (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle)^2 \right] + \frac{\mathcal{E}(1)}{\mathcal{E}(0)} \mathbb{E} \left[\sum_{t=1}^T \Pi_{t,t} (y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle)^2 \right] \\
& \quad + 2 \mathbb{E} \left[\sum_{t=1}^T \Pi_{t,t} (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle) (y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle) \right] + \frac{\mathcal{E}(0)}{\mathcal{E}(1)} \mathbb{E} \left[\sum_{t=1}^T \Pi_{t,t} y_t^2(1) \left(\frac{\mathbf{1}[Z_t = 1]}{p_t} - 1 \right)^2 \right] \\
& \quad + \frac{\mathcal{E}(1)}{\mathcal{E}(0)} \mathbb{E} \left[\sum_{t=1}^T \Pi_{t,t} y_t^2(0) \left(\frac{\mathbf{1}[Z_t = 0]}{1-p_t} - 1 \right)^2 \right] + 2 \mathbb{E} \left[\sum_{t=1}^T \Pi_{t,t} y_t(1) y_t(0) \left(\frac{\mathbf{1}[Z_t = 1]}{p_t} - 1 \right) \left(\frac{\mathbf{1}[Z_t = 0]}{1-p_t} - 1 \right) \right] \\
& \triangleq \frac{\mathcal{E}(0)}{\mathcal{E}(1)} S_1 + \frac{\mathcal{E}(1)}{\mathcal{E}(0)} S_2 + 2S_3 + \frac{\mathcal{E}(0)}{\mathcal{E}(1)} S_4 + \frac{\mathcal{E}(1)}{\mathcal{E}(0)} S_5 + 2S_6. \tag{27}
\end{aligned}$$

By Cauchy-Schwarz inequality, Corollary B.4, Corollary B.8, Lemma 4.6* and Assumption 3, we have the following upper bound on S_1 :

$$\begin{aligned}
S_1 & \leq \left(\sum_{t=1}^T \Pi_{t,t}^2 \right)^{1/2} \left(\mathbb{E} \left[\sum_{t=1}^T (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle)^4 \right] \right)^{1/2} \quad (\text{Cauchy-Schwarz inequality}) \\
& \leq \left(\sum_{t=1}^T \Pi_{t,t}^2 \right)^{1/2} \left(\mathbb{E} \left[\eta_T^{-1} \sum_{t=1}^T \eta_t (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle)^4 \right] \right)^{1/2} \quad (\eta_t \text{ is decreasing}) \\
& \lesssim \left(\sum_{t=1}^T R_t^4 ((t-1) \vee \eta_t^{-1})^{-2} \right)^{1/2} R_T^2 T^{1/2} \quad (\text{Corollary B.4 and Lemma 4.6*}) \\
& = \eta_T^{-1} \left(\sum_{t=1}^T ((t-1) R_t^{-2} \vee T^{1/2})^{-2} \right)^{1/2} \\
& \leq \eta_T^{-1} \left(\sum_{t=1}^T ((t-1) R_T^{-2} \vee T^{1/2})^{-2} \right)^{1/2} \quad (R_t \text{ is increasing}) \\
& = \eta_T^{-1} \left(\sum_{t=1}^T R_T^4 ((t-1) \vee \eta_T^{-1})^{-2} \right)^{1/2} \\
& \lesssim \eta_T^{-1} \cdot R_T^2 \eta_T^{1/2} \quad (\text{Lemma B.9}) \\
& = \eta_T^{-1} \cdot (T R_T^{-4})^{-1/4} \\
& = o(\eta_T^{-1}) \quad (\text{Assumption 3}). \tag{28}
\end{aligned}$$

Similar, we can prove that

$$S_2 = o(\eta_T^{-1}), \quad S_3 = o(\eta_T^{-1}). \tag{29}$$

By Lemma B.9, Corollary 4.15, Corollary B.17, Cauchy-Schwarz inequality and Assumption 1, we have

$$S_4 = \mathbb{E} \left[\sum_{t=1}^T \Pi_{t,t} y_t^2(1) \left(\frac{\mathbf{1}[Z_t = 1]}{p_t} - 1 \right)^2 \right]$$

$$\begin{aligned}
&= \sum_{t=1}^T \Pi_{t,t} y_t^2(1) \mathbb{E} \left[\frac{1}{p_t} - 1 \right] \\
&\lesssim \sum_{t=1}^T R_t^2((t-1) \vee \eta_t^{-1})^{-1} y_t^2(1) \cdot \eta_t^{1/4} T^{1/4} \quad (\text{Corollary 4.15 and B.17}) \\
&\lesssim T^{1/8} \sum_{t=1}^T R_t^2((t-1) \vee \eta_t^{-1})^{-1} y_t^2(1) \\
&\leq T^{1/8} \left(\sum_{t=1}^T R_t^4((t-1) \vee \eta_t^{-1})^{-2} \right)^{1/2} \left(\sum_{t=1}^T y_t^4(1) \right)^{1/2} \quad (\text{Cauchy-Schwarz inequality}) \\
&\lesssim T^{5/8} \left(\sum_{t=1}^T R_t^4((t-1) \vee \eta_t^{-1})^{-2} \right)^{1/2} \quad (\text{Assumption 1}) \\
&\lesssim T^{5/8} R_T^2 \eta_T^{1/2} \quad (\text{Lemma B.9}) \\
&= \eta_T^{-1} \cdot T^{-1/8} R_T^{-1} \\
&= o(\eta_T^{-1}) . \tag{30}
\end{aligned}$$

Similar, we can prove that

$$S_5 = o(\eta_T^{-1}), \quad S_6 = o(\eta_T^{-1}) . \tag{31}$$

Hence by Assumption 1, (27), (28), (29), (30) and (31), we have proved that

$$\mathbb{E} \left[\sum_{t=1}^T (\tilde{L}_{t+1}(\beta_t(1), \beta_t(0)) - \tilde{L}_{t+1}(\tilde{\beta}_{t+1}(1), \tilde{\beta}_{t+1}(0))) \right] \lesssim S_1 + S_2 + S_3 + S_4 + S_5 + S_6 = o(\eta_T^{-1}) . \tag{32}$$

On the other hand, by Lemma B.5, Cauchy-Schwarz inequality, Assumption 1 and Assumption 2, we have the following upper bound:

$$\begin{aligned}
&\eta_{T+1}^{-1} m(\beta^*(1), \beta^*(0)) \\
&= \eta_T^{-1} \left[\frac{\mathcal{E}(0)}{\mathcal{E}(1)} \|\beta^*(1)\|^2 + \frac{\mathcal{E}(1)}{\mathcal{E}(0)} \|\beta^*(0)\|^2 + 2 \langle \beta^*(1), \beta^*(0) \rangle \right] \\
&\leq \eta_T^{-1} \left[\left(\frac{\mathcal{E}(0)}{\mathcal{E}(1)} + 1 \right) \|\beta^*(1)\|^2 + \left(\frac{\mathcal{E}(1)}{\mathcal{E}(0)} + 1 \right) \|\beta^*(0)\|^2 \right] \quad (\text{Cauchy-Schwarz inequality}) \\
&= \eta_T^{-1} \left[\left(\frac{\mathcal{E}(0)}{\mathcal{E}(1)} + 1 \right) \mathbf{Y}_T^\top(1) \mathbf{X}_T (\mathbf{X}_T^\top \mathbf{X}_T)^{-2} \mathbf{X}_T^\top \mathbf{Y}_T(1) + \left(\frac{\mathcal{E}(1)}{\mathcal{E}(0)} + 1 \right) \mathbf{Y}_T^\top(0) \mathbf{X}_T (\mathbf{X}_T^\top \mathbf{X}_T)^{-2} \mathbf{X}_T^\top \mathbf{Y}_T(0) \right] \\
&\leq \eta_T^{-1} \left[\left(\frac{\mathcal{E}(0)}{\mathcal{E}(1)} + 1 \right) \|\mathbf{Y}_T(1)\|_2^2 + \left(\frac{\mathcal{E}(1)}{\mathcal{E}(0)} + 1 \right) \|\mathbf{Y}_T(0)\|_2^2 \right] \|\mathbf{X}_T (\mathbf{X}_T^\top \mathbf{X}_T)^{-2} \mathbf{X}_T^\top\|_2 \\
&\leq \eta_T^{-1} T^{1/2} \left[\left(\frac{\mathcal{E}(0)}{\mathcal{E}(1)} + 1 \right) \left(\sum_{t=1}^T y_t^4(1) \right)^{1/2} + \left(\frac{\mathcal{E}(1)}{\mathcal{E}(0)} + 1 \right) \left(\sum_{t=1}^T y_t^4(0) \right)^{1/2} \right] \|\mathbf{X}_T^\top \mathbf{X}_T (\mathbf{X}_T^\top \mathbf{X}_T)^{-2}\|_2 \quad (\text{Cauchy-Schwarz}) \\
&\leq \eta_T^{-1} c_1^2 T \left(\frac{\mathcal{E}(0)}{\mathcal{E}(1)} + \frac{\mathcal{E}(1)}{\mathcal{E}(0)} + 2 \right) \|(\mathbf{X}_T^\top \mathbf{X}_T)^{-1}\|_2 \quad (\text{Assumption 1}) \\
&\leq \eta_T^{-1} c_1^2 \left(\frac{\mathcal{E}(0)}{\mathcal{E}(1)} + \frac{\mathcal{E}(1)}{\mathcal{E}(0)} + 2 \right) T \cdot \frac{c_2}{T} \quad (\text{Assumption 2}) \\
&= c_1^2 c_2 \left(\frac{c_1}{c_0} + \frac{c_0}{c_1} + 2 \right) T^{1/2} R_T^2 \quad (\text{Assumption 1}) \\
&\leq (c_1^2 c_2 \max\{c_2, 1\}) \left(\frac{c_1}{c_0} + \frac{c_0}{c_1} + 2 \right) T^{1/2} R^2 \quad (\text{Lemma B.5}) . \tag{33}
\end{aligned}$$

Hence by Lemma 4.9, (32) and (33), we have

$$\begin{aligned} \mathbb{E}[\mathcal{R}_T^{\text{pred}}] &\leq \frac{m(\boldsymbol{\beta}^*(1), \boldsymbol{\beta}^*(0))}{\eta_{T+1}} + \mathbb{E}\left[\sum_{t=1}^T \tilde{L}_{t+1}(\boldsymbol{\beta}_t(1), \boldsymbol{\beta}_t(0)) - \tilde{L}_{t+1}(\tilde{\boldsymbol{\beta}}_{t+1}(1), \tilde{\boldsymbol{\beta}}_{t+1}(0))\right] \\ &\leq \left[(c_1^2 c_2 \max\{c_2, 1\}) \left(\frac{c_1}{c_0} + \frac{c_0}{c_1} + 2\right) + o(1)\right] T^{1/2} R^2 . \end{aligned}$$

□

B.6 Neyman Regret Analysis (Theorem 4.1)

Based on the results proved in the previous sections, we can finally prove Theorem 4.1.

Theorem 4.1*. *Under Assumptions 1-3 and Condition 1, the Neyman Regret is bounded as*

$$\begin{aligned} \mathcal{R}_T^{\text{Neyman}} &\leq \max\{c_2, 1\} \left(\frac{8}{b_3^3} \left[\left(\frac{c_1}{c_0} - 1\right)^3 + \frac{b_3}{4} \left(\frac{c_1}{c_0} - 1\right)^2 \right] \right. \\ &\quad \left. + b_1^2 \max\{b_1, 2\} c_1^4 \left(1 + 2^{-1/2} (\gamma_0 \vee 4c_2)^{1/2}\right)^4 + c_1^2 c_2 \left(\frac{c_1}{c_0} + \frac{c_0}{c_1} + 2\right) + o(1) \right) T^{-1/2} R^2 . \end{aligned}$$

Proof. By Proposition 3.3, Proposition 4.11, Proposition 4.7 and Assumption 3, we have

$$\begin{aligned} \mathcal{R}_T^{\text{Neyman}} &= \frac{1}{T} \mathbb{E}[\mathcal{R}_T^{\text{prob}}] + \frac{1}{T} \mathbb{E}[\mathcal{R}_T^{\text{pred}}] \\ &\leq \max\{c_2, 1\} \left(\frac{8}{b_3^3} \left[\left(\frac{c_1}{c_0} - 1\right)^3 + \frac{b_3}{4} \left(\frac{c_1}{c_0} - 1\right)^2 \right] + b_1^2 \max\{b_1, 2\} c_1^4 \left(1 + 2^{-1/2} (\gamma_0 \vee 4c_2)^{1/2}\right)^4 + o(1) \right) T^{-1/2} R^2 \\ &\quad + \left[(c_1^2 c_2 \max\{c_2, 1\}) \left(\frac{c_1}{c_0} + \frac{c_0}{c_1} + 2\right) + o(1) \right] T^{-1/2} R^2 \\ &= \max\{c_2, 1\} \left[\frac{8}{b_3^3} \left[\left(\frac{c_1}{c_0} - 1\right)^3 + \frac{b_3}{4} \left(\frac{c_1}{c_0} - 1\right)^2 \right] + b_1^2 \max\{b_1, 2\} c_1^4 \left(1 + 2^{-1/2} (\gamma_0 \vee 4c_2)^{1/2}\right)^4 \right. \\ &\quad \left. + c_1^2 c_2 \left(\frac{c_1}{c_0} + \frac{c_0}{c_1} + 2\right) \right] T^{-1/2} R^2 . \end{aligned}$$

□

C Inference Analysis

In this section, we aim to verify the inference procedure based on the Horvitz–Thompson estimator and the proposed variance estimator. Section C.1 establishes an almost sure upper bound for the inverse probabilities, which play a crucial role in verifying the central limit theorem. Section C.2 provides a detailed proof of the central limit theorem. Section C.3 demonstrates the equivalence between Assumption 4 and the well-known non-superefficiency condition in design-based causal inference. Section C.4 verifies the consistency of the proposed variance estimator.

C.1 Almost Sure Bounds on Inverse Probabilities

The following proposition derives the almost sure bounds for the inverse probabilities.

Proposition 5.4. *Under Assumption 1 and Condition 1, there exists a constant $K > 0$ so that*

$$\Pr\left(\max\left\{\frac{1}{p_t}, \frac{1}{1-p_t}\right\} \leq K \cdot T^{7/26} R_t^{-4/11} \text{ for all } t \in [T]\right) = 1 .$$

Proof. The proof contains three parts. First, we obtain an initial uniform bound for $1/p_t$ and $1/(1-p_t)$. Then we refine the uniform bound in a sequential manner. Finally, we derive an almost sure upper bound for each $t \in [T]$ based on the obtained uniform bound. Let $\mu_1 > 2$ denote the largest solution to equation: $2 + b_1 b_2^{1/4} c_1^{1/2} \mu_1^{3/4} T^{1/8} = \mu_1$. We then use induction method to prove that $\max_{t \in [T]} \left(\frac{1}{p_t} \vee \frac{1}{1-p_t} \right) \leq \mu_1$, which serves as the initial coarse uniform bound. When $t = 1$, we have $p_t = 1/2$. Hence the result is proved. If the result is proved for $1, \dots, t$, now we derive the result for $t+1$. Without loss of generality, we only derive the upper bound for $\hat{A}_t(0)$. By induction assumption and AM-GM inequality, we have

$$\begin{aligned} \hat{A}_t(0) &= \sum_{s=1}^t \frac{\mathbf{1}[Z_s = 0]}{1-p_s} \cdot (y_s(0) - \langle \mathbf{x}_s, \boldsymbol{\beta}_s(0) \rangle)^2 \\ &\leq \mu_1 \sum_{s=1}^t (y_s(0) - \langle \mathbf{x}_s, \boldsymbol{\beta}_s(0) \rangle)^2 \quad (\text{induction assumption}) \\ &= \mu_1 \sum_{s=1}^t \left[\left(y_s(0) \cdot \frac{\mathbf{1}[Z_s = 0]}{1-p_s} - \langle \mathbf{x}_s, \boldsymbol{\beta}_s(0) \rangle \right) - y_s(0) \left(\frac{\mathbf{1}[Z_s = 0]}{1-p_s} - 1 \right) \right]^2 \\ &\leq 2\mu_1 \sum_{s=1}^t \left(y_s(0) \cdot \frac{\mathbf{1}[Z_s = 0]}{1-p_s} - \langle \mathbf{x}_s, \boldsymbol{\beta}_s(0) \rangle \right)^2 + 2\mu_1 \sum_{s=1}^t y_s^2(0) \left(\frac{\mathbf{1}[Z_s = 0]}{1-p_s} - 1 \right)^2 \quad (\text{AM-GM inequality}) . \end{aligned}$$

We introduce the following notations:

$$\begin{aligned} \hat{\ell}_t^{(0)}(\boldsymbol{\beta}) &= \left(y_t(0) \cdot \frac{\mathbf{1}[Z_t = 0]}{1-p_t} - \langle \mathbf{x}_t, \boldsymbol{\beta} \rangle \right)^2 , \\ \hat{G}_t^{(k)}(\boldsymbol{\beta}) &= \sum_{s=1}^{t-1} \hat{\ell}_s^{(k)}(\boldsymbol{\beta}) + \eta_t^{-1} \|\boldsymbol{\beta}\|_2^2 . \\ \bar{G}_t^{(k)}(\boldsymbol{\beta}) &= \sum_{s=1}^{t-1} \hat{\ell}_s^{(k)}(\boldsymbol{\beta}) + \eta_{t-1}^{-1} \|\boldsymbol{\beta}\|_2^2 . \end{aligned}$$

Denote $\bar{\boldsymbol{\beta}}_t(0)$ be the minimizer of $\bar{G}_t^{(0)}(\boldsymbol{\beta})$. By similar methods as in the proofs of Lemma 4.9 and Lemma 4.10, we have

$$\begin{aligned} &\sum_{s=1}^t \left(y_s(0) \cdot \frac{\mathbf{1}[Z_s = 0]}{1-p_s} - \langle \mathbf{x}_s, \boldsymbol{\beta}_s(0) \rangle \right)^2 - \sum_{s=1}^t \left(y_s(0) \cdot \frac{\mathbf{1}[Z_s = 0]}{1-p_s} - \langle \mathbf{x}_s, \boldsymbol{\beta}_{t+1}(0) \rangle \right)^2 \\ &= \sum_{s=1}^t \hat{\ell}_s^{(0)}(\boldsymbol{\beta}_s(0)) - \sum_{s=1}^t \hat{\ell}_s^{(0)}(\boldsymbol{\beta}_{t+1}(0)) \\ &\leq \eta_{t+1}^{-1} \|\boldsymbol{\beta}_{t+1}(0)\|_2^2 + \sum_{s=1}^t (\bar{G}_{s+1}^{(0)}(\boldsymbol{\beta}_s(0)) - \bar{G}_{s+1}^{(0)}(\bar{\boldsymbol{\beta}}_{s+1}(0))) \quad (\text{by similar method as in Lemma 4.9}) \\ &= \eta_{t+1}^{-1} \|\boldsymbol{\beta}_{t+1}(0)\|_2^2 + \sum_{s=1}^t \frac{\Pi_{s,s}}{1+\Pi_{s,s}} \left(y_s(0) \cdot \frac{\mathbf{1}[Z_s = 0]}{1-p_s} - \langle \mathbf{x}_s, \boldsymbol{\beta}_s(0) \rangle \right)^2 \quad (\text{by similar proof as in Lemma 4.10}) \end{aligned}$$

$$\begin{aligned}
&\leq \eta_{t+1}^{-1} \|\beta_{t+1}(0)\|_2^2 + \sum_{s=1}^t \mathbf{x}_s^\top (\mathbf{X}_{s-1}^\top \mathbf{X}_{s-1} + \eta_s^{-1} \mathbf{I}_d)^{-1} \mathbf{x}_s \left(y_s(0) \cdot \frac{\mathbf{1}[Z_s = 0]}{1-p_s} - \langle \mathbf{x}_s, \beta_s(0) \rangle \right)^2 \\
&\leq \eta_{t+1}^{-1} \|\beta_{t+1}(0)\|_2^2 + \sum_{s=1}^t R_s^2 \eta_s \left(y_s(0) \cdot \frac{\mathbf{1}[Z_s = 0]}{1-p_s} - \langle \mathbf{x}_s, \beta_s(0) \rangle \right)^2 \\
&= \eta_{t+1}^{-1} \|\beta_{t+1}(0)\|_2^2 + T^{-1/2} \sum_{s=1}^t \left(y_s(0) \cdot \frac{\mathbf{1}[Z_s = 0]}{1-p_s} - \langle \mathbf{x}_s, \beta_s(0) \rangle \right)^2. \tag{34}
\end{aligned}$$

By induction assumption, the definition of $\beta_{t+1}(0)$ and Assumption 1, we have

$$\begin{aligned}
&\sum_{s=1}^t \left(y_s(0) \cdot \frac{\mathbf{1}[Z_s = 0]}{1-p_s} - \langle \mathbf{x}_s, \beta_{t+1}(0) \rangle \right)^2 + \eta_{t+1}^{-1} \|\beta_{t+1}(0)\|_2^2 \\
&= \min_{\beta \in \mathbb{R}^d} \sum_{s=1}^t \left(y_s(0) \cdot \frac{\mathbf{1}[Z_s = 0]}{1-p_s} - \langle \mathbf{x}_s, \beta \rangle \right)^2 + \eta_{t+1}^{-1} \|\beta\|_2^2 \\
&\leq \sum_{s=1}^t \left(y_s(0) \cdot \frac{\mathbf{1}[Z_s = 0]}{1-p_s} \right)^2 \quad (\text{optimality of } \beta_{t+1}(0)) \\
&\leq \left(\sum_{s=1}^t y_s^2(0) \right) \left(\max_{s=1,\dots,t} \frac{1}{1-p_s} \right)^2 \\
&\leq c_1^2 \mu_1^2 T \quad (\text{induction assumption, Cauchy-Schwarz inequality and Assumption 1}) . \tag{35}
\end{aligned}$$

Note that $T^{-1/2}$ should be smaller than $1/2$ for T large enough. Hence by (34) and (35), we have

$$\begin{aligned}
&\sum_{s=1}^t \left(y_s(0) \cdot \frac{\mathbf{1}[Z_s = 0]}{1-p_s} - \langle \mathbf{x}_s, \beta_s(0) \rangle \right)^2 \\
&\leq \frac{1}{1-T^{-1/2}} \left(\sum_{s=1}^t \left(y_s(0) \cdot \frac{\mathbf{1}[Z_s = 0]}{1-p_s} - \langle \mathbf{x}_s, \beta_{t+1}(0) \rangle \right)^2 + \eta_{t+1}^{-1} \|\beta_{t+1}(0)\|_2^2 \right) \\
&\leq 2 \left(\sum_{s=1}^t \left(y_s(0) \cdot \frac{\mathbf{1}[Z_s = 0]}{1-p_s} - \langle \mathbf{x}_s, \beta_{t+1}(0) \rangle \right)^2 + \eta_{t+1}^{-1} \|\beta_{t+1}(0)\|_2^2 \right) \\
&\leq 2c_1^2 \mu_1^2 T . \tag{36}
\end{aligned}$$

By induction assumption and Assumption 1, we also have

$$\sum_{s=1}^t y_s^2(0) \left(\frac{\mathbf{1}[Z_s = 0]}{1-p_s} - 1 \right)^2 \leq \mu_1^2 \sum_{s=1}^t y_s^2(0) \leq c_1^2 \mu_1^2 T . \tag{37}$$

Hence by (36), (37), we have

$$\hat{A}_t(0) \leq 2\mu_1 \sum_{s=1}^t \left(y_s(0) \cdot \frac{\mathbf{1}[Z_s = 0]}{1-p_s} - \langle \mathbf{x}_s, \beta_s(0) \rangle \right)^2 + 2\mu_1 \sum_{s=1}^t y_s^2(0) \left[\frac{\mathbf{1}[Z_s = 0]}{1-p_s} - 1 \right]^2 \leq 6c_1^2 \mu_1^3 T .$$

Then by Lemma 4.14, we have

$$\frac{1}{p_{t+1}} \leq 2 + b_1 (b_2/6)^{1/4} \eta_{t+1}^{1/4} \hat{A}_t^{1/4}(0) \leq 2 + b_1 b_2^{1/4} \eta_1^{1/4} c_1^{1/2} \mu_1^{3/4} T^{1/4} = 2 + b_1 b_2^{1/4} c_1^{1/2} \mu_1^{3/4} T^{1/8} = \mu_1 .$$

Similarly we can prove that $\frac{1}{1-p_{t+1}}$ is bounded by μ_1 . Then by induction method, we have

$$\max_{t \in [T]} \left(\frac{1}{p_t} \vee \frac{1}{1-p_t} \right) \leq \mu_1 .$$

Now we refine the initial uniform bound μ_1 in sequential steps. Suppose the upper bound obtained in the r -th step is μ_r , then we refine this bound to μ_{r+1} in $r+1$ -th step through the following procedure. By similar steps, we can prove that $\hat{A}_T(1) \leq 6c_1^2\mu_r^3 T$ and $\hat{A}_T(0) \leq 6c_1^2\mu_r^3 T$. We fix a T -dependent constant $\delta > 1$ (determined later). For any $t \in [T]$, if $\hat{A}_{t-1}(1) \geq 6c_1^2\mu_r^3 T \delta^{-2}$, then by Lemma 4.14, we have

$$\frac{1}{p_t} \leq 2 + b_1(b_2/b_3)^{1/2} \left(\frac{\hat{A}_{t-1}(0)}{\hat{A}_{t-1}(1)} \right)^{1/2} \leq 2 + b_1(b_2/b_3)^{1/2} \left(\frac{6c_1^2\mu_r^3 T}{6c_1^2\mu_r^3 T \delta^{-2}} \right)^{1/2} = 2 + b_1(b_2/b_3)^{1/2} \delta .$$

If $\hat{A}_{t-1}(1) \leq 6c_1^2\mu_r^3 T \delta^{-2}$, then by Lemma 4.14, we have

$$\begin{aligned} \frac{1}{p_t} &\leq 2 + b_1(b_2/6)^{1/4} \eta_t^{1/4} \hat{A}_{t-1}^{1/4}(0) \\ &\leq 2 + b_1(b_2/6)^{1/4} \eta_t^{1/4} \cdot \left(6c_1^2 T \cdot \left[\max_{s=1,\dots,t-1} \frac{1}{1-p_s} \right]^3 \right)^{1/4} \\ &\leq 2 + b_1 b_2^{1/4} c_1^{1/2} \eta_1^{1/4} T^{1/4} \max_{s=1,\dots,t-1} \left(2 + b_1(b_2/6)^{1/4} \eta_s^{1/4} \hat{A}_{s-1}^{1/4}(1) \right)^{3/4} \\ &\leq 2 + b_1 b_2^{1/4} c_1^{1/2} \eta_1^{1/4} T^{1/4} \left(2 + b_1(b_2/6)^{1/4} \eta_1^{1/4} \hat{A}_{t-1}^{1/4}(1) \right)^{3/4} \\ &\leq 2 + b_1 b_2^{1/4} c_1^{1/2} \eta_1^{1/4} T^{1/4} \left[2 + b_1(b_2/6)^{1/4} \eta_1^{1/4} (6c_1^2\mu_r^3 T \delta^{-2})^{1/4} \right]^{3/4} \\ &\leq 2 + b_1 b_2^{1/4} c_1^{1/2} \eta_1^{1/4} T^{1/4} \left(2 + b_1 b_2^{1/4} \eta_1^{1/4} c_1^{1/2} \mu_r^{3/4} T^{1/4} \delta^{-1/2} \right)^{3/4} \\ &= 2 + b_1 b_2^{1/4} c_1^{1/2} T^{1/8} \left(2 + b_1 b_2^{1/4} c_1^{1/2} \mu_r^{3/4} T^{1/8} \delta^{-1/2} \right)^{3/4} . \end{aligned}$$

Now we choose $\delta > 0$ to be the largest solution to equation:

$$2 + b_1(b_2/b_3)^{1/2} \delta = 2 + b_1 b_2^{1/4} c_1^{1/2} T^{1/8} \left(2 + b_1 b_2^{1/4} c_1^{1/2} \mu_r^{3/4} T^{1/8} \delta^{-1/2} \right)^{3/4} . \quad (38)$$

We first prove the existence of such δ . The left hand side of equation (38) attains value 2 at $\delta = 0$ and tends to $+\infty$ when $\delta \rightarrow \infty$. The right hand side of equation (38) tends to infinity when $\delta \downarrow 0$ and tends to 2 when $\delta \rightarrow \infty$. Hence the solution to (38) exists by intermediate value theorem for continuous functions and such δ is well defined due to the continuity on both sides. Then for any $t \in [T]$, we have $1/p_t \leq 2 + b_1(b_2/b_3)^{1/2} \delta = \mu_{r+1}$. Similarly, we can prove that for any $t \in [T]$, we have $1/(1-p_t) \leq 2 + b_1(b_2/b_3)^{1/2} \delta = \mu_{r+1}$. Hence we obtain a new uniform bound μ_{r+1} for $1/p_t$ and $1/(1-p_t)$. By such sequential steps, we obtain a sequence of upper bounds: $\{\mu_r : r = 1, 2, \dots\}$. Suppose $\mu > 2$ is the largest solution to the following equation:

$$\mu = 2 + b_1 b_2^{1/4} c_1^{1/2} T^{1/8} \left(2 + b_1 b_2^{1/4} c_1^{1/2} \mu^{3/4} T^{1/8} (b_1^{-1}(b_2/b_3)^{-1/2}(\mu - 2))^{-1/2} \right)^{3/4} . \quad (39)$$

We first prove that existence of such μ . The left hand side of (39) attains value 2 at $\mu = 2$, while the right hand side tends to infinity when $\mu \downarrow 2$. Meanwhile, for any fixed T , the right hand side is of order $\mathcal{O}(\mu^{3/16})$ while the left hand side is of order $\mathcal{O}(\mu)$. Hence by intermediate value theorem for continuous functions, there exists solution to (39) on $(2, \infty)$ for any given T . Hence μ is well-defined due to the continuity on both sides. Then we have

$$\begin{aligned} \mu &= 2 + b_1 b_2^{1/4} c_1^{1/2} T^{1/8} \left(2 + b_1 b_2^{1/4} c_1^{1/2} \mu^{3/4} T^{1/8} (b_1^{-1}(b_2/b_3)^{-1/2}(\mu - 2))^{-1/2} \right)^{3/4} \\ &\approx 1 + T^{1/8} \left[1 + (\mu^{1/4} T^{1/8})^{3/4} \right] \end{aligned}$$

$$\asymp 1 + T^{1/8} \left(1 + T^{3/32} \mu^{3/16} \right) \\ \asymp T^{7/32} \mu^{3/16} .$$

It is easy to see that $\mu_1 = \Theta(T^{1/2})$ and $\mu = \mathcal{O}(T^{7/26})$, which indicates that $\mu = o(\mu_1)$. Hence for T large enough, we can assume WLOG that $\mu_1 > \mu$. Now we prove that the sequence $\{\mu_r : r = 1, 2, \dots\}$ is monotone decreasing and $\mu_r > \mu$ for any r by induction method. When $r = 1$, the result is proved. Suppose the result is proved for $1, \dots, r$, by the definition of μ_{r+1} , there holds:

$$\mu_{r+1} = 2 + b_1 b_2^{1/4} c_1^{1/2} T^{1/8} \left(2 + b_1 b_2^{1/4} c_1^{1/2} \mu_r^{3/4} T^{1/8} (b_1^{-1} (b_2/b_3)^{-1/2} (\mu_{r+1} - 2))^{-1/2} \right)^{3/4} . \quad (40)$$

Since $\mu > 2$ is the largest solution to equation (39), then either there holds

$$\delta < 2 + b_1 b_2^{1/4} c_1^{1/2} T^{1/8} \left(2 + b_1 b_2^{1/4} c_1^{1/2} \delta^{3/4} T^{1/8} (b_1^{-1} (b_2/b_3)^{-1/2} (\delta - 2))^{-1/2} \right)^{3/4}$$

for any $\delta > \mu$ or there holds

$$\delta > 2 + b_1 b_2^{1/4} c_1^{1/2} T^{1/8} \left(2 + b_1 b_2^{1/4} c_1^{1/2} \delta^{3/4} T^{1/8} (b_1^{-1} (b_2/b_3)^{-1/2} (\delta - 2))^{-1/2} \right)^{3/4}$$

for any $\delta > \mu$ due to the continuity on both sides. For any fixed T , the left hand side has order $\mathcal{O}(\delta)$ while the right hand side has order $\mathcal{O}(\delta^{3/16})$, which indicates that the only possibility is that the second case holds. Since $\mu_r > \mu$ is proved by induction assumption, we have

$$\mu_r > 2 + b_1 b_2^{1/4} c_1^{1/2} T^{1/8} \left(2 + b_1 b_2^{1/4} c_1^{1/2} \mu_r^{3/4} T^{1/8} (b_1^{-1} (b_2/b_3)^{-1/2} (\mu_r - 2))^{-1/2} \right)^{3/4} .$$

Since the left hand side of (38) is monotone increasing while the right hand side is monotone decreasing, this together with (40) imply that $\mu_{r+1} < \mu_r$. Then by induction method, we prove that $\{\mu_r\}$ is monotone decreasing and $\mu_r > \mu$ for any r . Hence the limit of the sequence exists, which should also be a solution to (39) by continuity. Since $\mu > 2$ is the largest solution of equation (39) by definition, it is easy to see that $\lim_{r \rightarrow \infty} \mu_r = \mu$. Hence we can prove that

$$\max_{t \in [T]} \left(\frac{1}{p_t} \vee \frac{1}{1 - p_t} \right) \leq \mu = \mathcal{O}(T^{7/26}) .$$

Finally, we obtain the upper bound for each individual $t \in [T]$. If $\hat{A}_{t-1}(1) \geq 6c_1^2 \mu^3 T \delta^{-2}$, by Lemma 4.14 we have

$$\frac{1}{p_t} \leq 2 + b_1 (b_2/b_3)^{1/2} \left(\frac{\hat{A}_{t-1}(0)}{\hat{A}_{t-1}(1)} \right)^{1/2} \leq 2 + b_1 (b_2/b_3)^{1/2} \left(\frac{6c_1^2 \mu^3 T}{6c_1^2 \mu^3 T \delta^{-2}} \right)^{1/2} = 2 + b_1 (b_2/b_3)^{1/2} \delta .$$

If $\hat{A}_{t-1}(1) \leq 6c_1^2 \mu^3 T \delta^{-2}$, by Lemma 4.14 we have

$$\begin{aligned} \frac{1}{p_t} &\leq 2 + b_1 (b_2/6)^{1/4} \eta_t^{1/4} \hat{A}_{t-1}^{1/4}(0) \\ &\leq 2 + b_1 (b_2/6)^{1/4} \eta_t^{1/4} \cdot \left(6c_1^2 T \cdot \left[\max_{s=1, \dots, t-1} \frac{1}{1 - p_s} \right]^3 \right)^{1/4} \\ &\leq 2 + b_1 b_2^{1/4} c_1^{1/2} \eta_t^{1/4} T^{1/4} \max_{s=1, \dots, t-1} \left(2 + b_1 (b_2/6)^{1/4} \eta_s^{1/4} \hat{A}_{s-1}^{1/4}(1) \right)^{3/4} \\ &\leq 2 + b_1 b_2^{1/4} c_1^{1/2} \eta_t^{1/4} T^{1/4} \left(2 + b_1 (b_2/6)^{1/4} \eta_1^{1/4} \hat{A}_{t-1}^{1/4}(1) \right)^{3/4} \end{aligned}$$

$$\begin{aligned} &\leq 2 + b_1 b_2^{1/4} c_1^{1/2} \eta_t^{1/4} T^{1/4} \left[2 + b_1 (b_2/6)^{1/4} \eta_1^{1/4} (6c_1^2 \mu^3 T \delta^{-2})^{1/4} \right]^{3/4} \\ &\leq 2 + b_1 b_2^{1/4} c_1^{1/2} \eta_t^{1/4} T^{1/4} \left(2 + b_1 b_2^{1/4} c_1^{1/2} \mu^{3/4} T^{1/8} \delta^{-1/2} \right)^{3/4}. \end{aligned}$$

Let $\delta_t > 0$ be the largest solution to equation:

$$2 + b_1 (b_2/b_3)^{1/2} \delta = 2 + b_1 b_2^{1/4} c_1^{1/2} \eta_t^{1/4} T^{1/4} \left(2 + b_1 b_2^{1/4} c_1^{1/2} \mu^{3/4} T^{1/8} \delta^{-1/2} \right)^{3/4}. \quad (41)$$

Choose $\delta = (2(b_2/b_3)^{-1/2} b_1^{3/4} b_2^{7/16} c_1^{7/8} \eta_t^{1/4} T^{11/32} \mu^{9/16})^{8/11}$, for T large enough it is easy to see that

$$\begin{aligned} &b_1 (b_2/b_3)^{1/2} \delta - b_1 b_2^{1/4} c_1^{1/2} \eta_t^{1/4} T^{1/4} \left(2 + b_1 b_2^{1/4} c_1^{1/2} \mu^{3/4} T^{1/8} \delta^{-1/2} \right)^{3/4} \\ &\geq b_1 (b_2/b_3)^{1/2} \delta - 2 b_1 b_2^{1/4} c_1^{1/2} \eta_t^{1/4} T^{1/4} \left(b_1 b_2^{1/4} c_1^{1/2} \mu^{3/4} T^{1/8} \delta^{-1/2} \right)^{3/4} \\ &= b_1 (b_2/b_3)^{1/2} \delta - 2 b_1^{7/4} b_2^{7/16} c_1^{7/8} \eta_t^{1/4} T^{11/32} \mu^{9/16} \delta^{-3/8} \\ &= 0. \end{aligned}$$

Since the left hand side of (41) is monotone increasing in δ while the right hand side is monotone decreasing in δ , this implies that

$$\begin{aligned} \max \left\{ \frac{1}{p_t}, \frac{1}{1-p_t} \right\} &\leq \delta_t \leq \delta = (2(b_2/b_3)^{-1/2} b_2^{7/16} c_1^{7/8} \eta_t^{1/4} T^{11/32} \mu^{9/16})^{8/11} \\ &= (2(b_2/b_3)^{-1/2} b_2^{7/16} c_1^{7/8} T^{7/32} R_t^{-1/2} \mu^{9/16})^{8/11} \\ &= (2(b_2/b_3)^{-1/2} b_2^{7/16} c_1^{7/8})^{8/11} T^{7/44} \mu^{9/22} R_t^{-4/11}. \end{aligned}$$

Since $\mu = \mathcal{O}(T^{7/26})$, we have shown that there exists a constant $K > 0$ (independent of t and T) such that

$$\Pr \left(\max \left\{ \frac{1}{p_t}, \frac{1}{1-p_t} \right\} \leq K \cdot T^{7/26} R_t^{-4/11} \text{ for all } t \in [T] \right) = 1.$$

□

C.2 Central Limit Theorem

In this section, we prove the central limit theorem for the Horvitz-Thompson estimator. We begin by stating the martingale central limit theorem, which serves as the main tool for our proof.

Lemma 5.3 (Helland, 1982). *If $X_{t,T}$ is a triangular array of martingale difference sequences with respect to filtrations $\mathcal{F}_{t,T}$, i.e. $E[X_{t,T} | \mathcal{F}_{t-1,T}] = 0$, then if*

1. *There exists $\delta > 0$ such that $\sum_{t=1}^T E[X_{t,T}^{2+\delta} | \mathcal{F}_{t-1,T}] \xrightarrow{P} 0$*
2. *$V_T^2 \triangleq \sum_{t=1}^T E[X_{t,T}^2 | \mathcal{F}_{t-1,T}] \xrightarrow{P} 1$*

then the sum $S_T = \sum_{t=1}^T X_{t,T}$ converges to a standard normal in distribution, $S_T \xrightarrow{d} \mathcal{N}(0, 1)$.

In Section 5, we have already defined the martingale difference sequence:

$$X_{t,T} = \frac{\hat{\tau}_t - \tau_t}{T \sqrt{\text{Var}(\hat{\tau})}}$$

$$= \frac{1}{T\sqrt{\text{Var}(\hat{\tau})}} \left[(y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle) \cdot \left(\frac{\mathbf{1}[Z_t = 1]}{p_t} - 1 \right) \right. \\ \left. - (y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle) \cdot \left(\frac{\mathbf{1}[Z_t = 0]}{1-p_t} - 1 \right) \right].$$

In the following two subsections, we separately verify that the martingale difference sequence $\{X_{t,T}\}$ satisfy the two conditions in Lemma 5.3. We first state, without proof, the non-superefficiency condition in the following corollary. Its detailed proof is deferred to Section C.3.

Corollary 5.2 (Non-Superefficiency). *Under Assumptions 1-4 and Condition 1, $\liminf_{T \rightarrow \infty} T \cdot \text{Var}(\hat{\tau}) > 0$.*

C.2.1 Stable Variance Condition

We first derive a simplified form of the stable variance condition. From the expression of $\text{Var}(\hat{\tau})$, the term V_T^2 can be simplified as:

$$V_T^2 = T^{-2} \text{Var}^{-1}(\hat{\tau}) \left[\sum_{t=1}^T \{y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle\}^2 \cdot \left(\frac{1}{p_t} - 1 \right) + \sum_{t=1}^T \{y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle\}^2 \cdot \left(\frac{1}{1-p_t} - 1 \right) \right. \\ \left. + 2 \sum_{t=1}^T \{y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle\} \cdot \{y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle\} \right] \\ = 1 + T^{-2} \text{Var}^{-1}(\hat{\tau}) \left[\sum_{t=1}^T \{y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle\}^2 \cdot \left(\frac{1}{p_t} - 1 \right) + \sum_{t=1}^T \{y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle\}^2 \cdot \left(\frac{1}{1-p_t} - 1 \right) \right. \\ \left. - \sum_{t=1}^T \mathbb{E} \left(\{y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle\}^2 \cdot \left(\frac{1}{p_t} - 1 \right) \right) - \sum_{t=1}^T \mathbb{E} \left(\{y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle\}^2 \cdot \left(\frac{1}{1-p_t} - 1 \right) \right) \right. \\ \left. + 2 \sum_{t=1}^T \{y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle\} \{y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle\} - 2 \sum_{t=1}^T \mathbb{E} (\{y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle\} \{y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle\}) \right].$$

Under the non-superefficiency condition (Corollary 5.2), to prove that $V_T^2 \xrightarrow{P} 1$, it suffices to establish the following three convergence results:

$$\frac{1}{T} \left[\sum_{t=1}^T \{y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle\}^2 \cdot \left(\frac{1}{p_t} - 1 \right) - \mathbb{E} \left[\sum_{t=1}^T \{y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle\}^2 \cdot \left(\frac{1}{p_t} - 1 \right) \right] \right] \xrightarrow{P} 0, \\ \frac{1}{T} \left[\sum_{t=1}^T \{y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle\}^2 \cdot \left(\frac{1}{1-p_t} - 1 \right) - \mathbb{E} \left[\sum_{t=1}^T \{y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle\}^2 \cdot \left(\frac{1}{1-p_t} - 1 \right) \right] \right] \xrightarrow{P} 0, \\ \frac{1}{T} \left[\sum_{t=1}^T \{y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle\} \{y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle\} - \mathbb{E} \left[\sum_{t=1}^T \{y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle\} \{y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle\} \right] \right] \xrightarrow{P} 0. \quad (42)$$

We begin with the first convergence result in (42). By direct calculation, we obtain the following decomposition:

$$\frac{1}{T} \sum_{t=1}^T (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle)^2 \cdot \left(\frac{1}{p_t} - 1 \right) \\ = \frac{1}{T} \sum_{t=1}^T (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^2 \cdot \left(\frac{1}{p_t} - 1 \right) + \frac{1}{T} \sum_{t=1}^T \left[(y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle)^2 - (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^2 \right] \left(\frac{1}{p_t} - 1 \right)$$

$$\begin{aligned}
&= \frac{1}{T} \sum_{t=1}^T (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^2 \cdot \left(\frac{1}{p_t} - \frac{1}{\bar{p}_t} \right) + \frac{1}{T} \sum_{t=1}^T (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^2 \cdot \left(\frac{1}{\bar{p}_t} - 1 \right) \\
&\quad - \frac{1}{T} \sum_{t=1}^T [2(y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) - \boldsymbol{\beta}_t^*(1) \rangle] \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) - \boldsymbol{\beta}_t^*(1) \rangle \left(\frac{1}{p_t} - 1 \right) \\
&= \frac{1}{T} \sum_{t=1}^T (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^2 \cdot \left(\frac{1}{p_t} - \frac{1}{\bar{p}_t} \right) + \frac{1}{T} \sum_{t=1}^T (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^2 \cdot \left(\frac{1}{\bar{p}_t} - 1 \right) \\
&\quad - \frac{2}{T} \sum_{t=1}^T (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle) \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) - \boldsymbol{\beta}_t^*(1) \rangle \left(\frac{1}{p_t} - \frac{1}{\bar{p}_t} \right) \\
&\quad - \frac{2}{T} \sum_{t=1}^T (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle) \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) - \boldsymbol{\beta}_t^*(1) \rangle \left(\frac{1}{\bar{p}_t} - 1 \right) \\
&\quad + \frac{1}{T} \sum_{t=1}^T \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) - \boldsymbol{\beta}_t^*(1) \rangle^2 \left(\frac{1}{p_t} - \frac{1}{\bar{p}_t} \right) + \frac{1}{T} \sum_{t=1}^T \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) - \boldsymbol{\beta}_t^*(1) \rangle^2 \left(\frac{1}{\bar{p}_t} - 1 \right) .
\end{aligned}$$

Hence, it suffices to show that each term concentrates around its respective expectation, i.e.,

$$\begin{aligned}
&\frac{1}{T} \sum_{t=1}^T (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^2 \cdot \left(\frac{1}{p_t} - \frac{1}{\bar{p}_t} \right) - \mathbb{E} \left[\frac{1}{T} \sum_{t=1}^T (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^2 \cdot \left(\frac{1}{p_t} - \frac{1}{\bar{p}_t} \right) \right] \xrightarrow{p} 0 , \\
&\frac{1}{T} \sum_{t=1}^T (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^2 \cdot \left(\frac{1}{\bar{p}_t} - 1 \right) - \mathbb{E} \left[\frac{1}{T} \sum_{t=1}^T (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^2 \cdot \left(\frac{1}{\bar{p}_t} - 1 \right) \right] \xrightarrow{p} 0 , \\
&\frac{1}{T} \sum_{t=1}^T (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle) \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) - \boldsymbol{\beta}_t^*(1) \rangle \left(\frac{1}{p_t} - \frac{1}{\bar{p}_t} \right) \\
&\quad - \mathbb{E} \left[\frac{1}{T} \sum_{t=1}^T (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle) \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) - \boldsymbol{\beta}_t^*(1) \rangle \left(\frac{1}{p_t} - \frac{1}{\bar{p}_t} \right) \right] \xrightarrow{p} 0 , \\
&\frac{1}{T} \sum_{t=1}^T (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle) \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) - \boldsymbol{\beta}_t^*(1) \rangle \left(\frac{1}{\bar{p}_t} - 1 \right) \\
&\quad - \mathbb{E} \left[\frac{1}{T} \sum_{t=1}^T (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle) \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) - \boldsymbol{\beta}_t^*(1) \rangle \left(\frac{1}{\bar{p}_t} - 1 \right) \right] \xrightarrow{p} 0 , \\
&\frac{1}{T} \sum_{t=1}^T \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) - \boldsymbol{\beta}_t^*(1) \rangle^2 \left(\frac{1}{p_t} - \frac{1}{\bar{p}_t} \right) - \mathbb{E} \left[\frac{1}{T} \sum_{t=1}^T \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) - \boldsymbol{\beta}_t^*(1) \rangle^2 \left(\frac{1}{p_t} - \frac{1}{\bar{p}_t} \right) \right] \xrightarrow{p} 0 , \\
&\frac{1}{T} \sum_{t=1}^T \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) - \boldsymbol{\beta}_t^*(1) \rangle^2 \left(\frac{1}{\bar{p}_t} - 1 \right) - \mathbb{E} \left[\frac{1}{T} \sum_{t=1}^T \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) - \boldsymbol{\beta}_t^*(1) \rangle^2 \left(\frac{1}{\bar{p}_t} - 1 \right) \right] \xrightarrow{p} 0 . \quad (43)
\end{aligned}$$

The second convergence result in (43) automatically holds since no randomness is involved. Under Markov's inequality, we only need to prove the following convergence results:

$$\begin{aligned}
&\mathbb{E} \left[\frac{1}{T} \sum_{t=1}^T (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^2 \cdot \left| \frac{1}{p_t} - \frac{1}{\bar{p}_t} \right| \right] \rightarrow 0 , \\
&\mathbb{E} \left[\frac{1}{T} \sum_{t=1}^T (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle) \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) - \boldsymbol{\beta}_t^*(1) \rangle \left| \frac{1}{p_t} - \frac{1}{\bar{p}_t} \right| \right] \rightarrow 0 , \\
&\mathbb{E} \left[\frac{1}{T} \sum_{t=1}^T (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle) \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) - \boldsymbol{\beta}_t^*(1) \rangle \left| \frac{1}{\bar{p}_t} - 1 \right| \right] \rightarrow 0 ,
\end{aligned}$$

$$\begin{aligned} \mathbb{E} \left[\frac{1}{T} \sum_{t=1}^T \langle \boldsymbol{x}_t, \boldsymbol{\beta}_t(1) - \boldsymbol{\beta}_t^*(1) \rangle^2 \left| \frac{1}{p_t} - \frac{1}{\bar{p}_t} \right| \right] &\rightarrow 0 , \\ \mathbb{E} \left[\frac{1}{T} \sum_{t=1}^T \langle \boldsymbol{x}_t, \boldsymbol{\beta}_t(1) - \boldsymbol{\beta}_t^*(1) \rangle^2 \left| \frac{1}{\bar{p}_t} - 1 \right| \right] &\rightarrow 0 . \end{aligned} \quad (44)$$

Before proving the first convergence result in (44), we establish several supporting lemmas that control the difference between the inverse probabilities associated with p_t and \bar{p}_t . The following two lemmas bound the difference between the inverse probabilities in terms of the differences between the corresponding first-order equations. Lemma C.1 discusses the two cases where both probabilities are greater than 1/2 or both are less than 1/2.

Lemma C.1. *Let $A, \tilde{A}, B, \tilde{B}$ be positive constants. Suppose p, \tilde{p} satisfy*

$$\begin{aligned} -\frac{A}{p^2} + \frac{B}{(1-p)^2} + \Psi'(p) &= 0 , \\ -\frac{\tilde{A}}{\tilde{p}^2} + \frac{\tilde{B}}{(1-\tilde{p})^2} + \Psi'(\tilde{p}) &= 0 . \end{aligned}$$

(1) If $A \geq B, \tilde{A} \geq \tilde{B}$, under Condition 1 we can attain the following upper bound:

$$\begin{aligned} \left| \frac{1}{1-p} - \frac{1}{1-\tilde{p}} \right| &\leq \frac{b_1 b_2}{b_3} \left(1 + \frac{2}{b_3} \right) \frac{|A-\tilde{A}|}{A} \frac{p}{1-p} + \frac{2b_1^2 b_2}{b_3^2} \left(1 + \frac{2}{b_3} \right) \frac{|B-\tilde{B}|}{B} \frac{\tilde{p}}{1-\tilde{p}} , \\ \left| \frac{1}{p} - \frac{1}{\tilde{p}} \right| &\leq \frac{b_2^2}{2b_3} \frac{|A-\tilde{A}|}{A} + \frac{b_1 b_2}{b_3} \left(1 + \frac{2}{b_3} \right) \frac{|B-\tilde{B}|}{B} \frac{p}{1-p} . \end{aligned}$$

(2) If $A \leq B, \tilde{A} \leq \tilde{B}$, under Condition 1 we can attain the following upper bound:

$$\begin{aligned} \left| \frac{1}{p} - \frac{1}{\tilde{p}} \right| &\leq \frac{b_1 b_2}{b_3} \left(1 + \frac{2}{b_3} \right) \frac{|B-\tilde{B}|}{B} \frac{1-p}{p} + \frac{2b_1^2 b_2}{b_3^2} \left(1 + \frac{2}{b_3} \right) \frac{|A-\tilde{A}|}{A} \frac{1-\tilde{p}}{\tilde{p}} , \\ \left| \frac{1}{1-p} - \frac{1}{1-\tilde{p}} \right| &\leq \frac{b_2^2}{2b_3} \frac{|B-\tilde{B}|}{B} + \frac{b_1 b_2}{b_3} \left(1 + \frac{2}{b_3} \right) \frac{|A-\tilde{A}|}{A} \frac{1-p}{p} . \end{aligned}$$

Proof. (1) For $u = \phi^{-1}(p)$ and $\tilde{u} = \phi^{-1}(\tilde{p})$, it is easy to see that u and \tilde{u} satisfy

$$\begin{aligned} A \left(\frac{1}{\phi(u)} \right)' + B \left(\frac{1}{1-\phi(u)} \right)' + u + 3u|u| &= 0 , \\ \tilde{A} \left(\frac{1}{\phi(\tilde{u})} \right)' + \tilde{B} \left(\frac{1}{1-\phi(\tilde{u})} \right)' + \tilde{u} + 3\tilde{u}|\tilde{u}| &= 0 . \end{aligned} \quad (45)$$

Since $A \geq B, \tilde{A} \geq \tilde{B}$, it is easy to see that $u, \tilde{u} \geq 0$. By subtracting the second equation in (45) from the first equation, we have

$$\begin{aligned} A \left[\left(\frac{1}{\phi(u)} \right)' - \left(\frac{1}{\phi(\tilde{u})} \right)' \right] + (A-\tilde{A}) \left(\frac{1}{\phi(\tilde{u})} \right)' + B \left[\left(\frac{1}{1-\phi(u)} \right)' - \left(\frac{1}{1-\phi(\tilde{u})} \right)' \right] \\ + (B-\tilde{B}) \left(\frac{1}{1-\phi(\tilde{u})} \right)' + (u-\tilde{u})(1+3u+3\tilde{u}) = 0 . \end{aligned} \quad (46)$$

By the convexity assumption in Condition 1, $A \left[\left(\frac{1}{\phi(u)} \right)' - \left(\frac{1}{\phi(\tilde{u})} \right)' \right], B \left[\left(\frac{1}{1-\phi(u)} \right)' - \left(\frac{1}{1-\phi(\tilde{u})} \right)' \right]$ and $(u-\tilde{u})(1+3u+3\tilde{u})$ always have the same sign. Hence by Lemma B.16, we have

$$\left| A \left[\left(\frac{1}{\phi(u)} \right)' - \left(\frac{1}{\phi(\tilde{u})} \right)' \right] + B \left[\left(\frac{1}{1-\phi(u)} \right)' - \left(\frac{1}{1-\phi(\tilde{u})} \right)' \right] + \eta^{-1}(u-\tilde{u})(1+3u+3\tilde{u}) \right|$$

$$\begin{aligned}
&\geq A \left| \left(\frac{1}{\phi(u)} \right)' - \left(\frac{1}{\phi(\tilde{u})} \right)' \right| \\
&\geq \frac{b_3}{2} \frac{A}{(1+\tilde{u})(1+u)(1+\tilde{u}\wedge u)} \cdot |u-\tilde{u}| .
\end{aligned} \tag{47}$$

By monotonicity assumption in Condition 1 and (45), we also have

$$A = \frac{B \left(\frac{1}{1-\phi(u)} \right)' + u + 3u^2}{\left(\frac{1}{\phi(u)} \right)'} \geq B \frac{\left(\frac{1}{1-\phi(u)} \right)'}{\left(\frac{1}{\phi(u)} \right)'} . \tag{48}$$

Hence by (46), (47), (48), Condition 1 and Lemma B.16, we have

$$\begin{aligned}
&|u-\tilde{u}| \\
&\leq \left(\frac{b_3}{2} \frac{A}{(1+\tilde{u})(1+u)(1+\tilde{u}\wedge u)} \right)^{-1} \left| (A-\tilde{A}) \left(\frac{1}{\phi(\tilde{u})} \right)' + (B-\tilde{B}) \left(\frac{1}{1-\phi(\tilde{u})} \right)' \right| \quad (\text{by (46) and (47)}) \\
&\leq \frac{2}{b_3} \frac{|A-\tilde{A}|}{A} (1+\tilde{u})^2 (1+u) \left| \left(\frac{1}{\phi(\tilde{u})} \right)' \right| + \frac{2}{b_3} \frac{|B-\tilde{B}|}{A} (1+\tilde{u}) (1+u)^2 \left| \left(\frac{1}{1-\phi(\tilde{u})} \right)' \right| \\
&\leq \frac{b_2}{b_3} \frac{|A-\tilde{A}|}{A} (1+\tilde{u})^2 (1+u) (1+\tilde{u})^{-2} + \frac{2b_1}{b_3} \frac{|B-\tilde{B}|}{B} (1+\tilde{u}) (1+u)^2 \frac{\left(\frac{1}{\phi(u)} \right)'}{\left(\frac{1}{1-\phi(u)} \right)'} \quad (\text{by Condition 1 and (48)}) \\
&\leq \frac{b_2}{b_3} \frac{|A-\tilde{A}|}{A} (1+u) + \frac{b_1 b_2}{b_3} \frac{|B-\tilde{B}|}{B} (1+\tilde{u}) (1+u)^2 (1+u)^{-2} \frac{1}{\left(\frac{1}{1-\phi(0)} \right)'} \quad (\text{by Lemma B.16 and Condition 1}) \\
&\leq \frac{b_2}{b_3} \frac{|A-\tilde{A}|}{A} (1+u) + \frac{2b_1 b_2}{b_3^2} \frac{|B-\tilde{B}|}{B} (1+\tilde{u}) \quad (\text{by Lemma B.16}) .
\end{aligned} \tag{49}$$

Hence by Lemma B.16, we have

$$\left| \frac{1}{1-p} - \frac{1}{1-\tilde{p}} \right| \leq b_1 |u-\tilde{u}| \leq \frac{b_1 b_2}{b_3} \frac{|A-\tilde{A}|}{A} (1+u) + \frac{2b_1^2 b_2}{b_3^2} \frac{|B-\tilde{B}|}{B} (1+\tilde{u})$$

and

$$\frac{1}{1-p} - \frac{1}{1-1/2} \geq \frac{b_3}{2} (u-0) . \tag{50}$$

We can finally obtain from (49) that

$$\begin{aligned}
\left| \frac{1}{1-p} - \frac{1}{1-\tilde{p}} \right| &\leq \frac{b_1 b_2}{b_3} \frac{|A-\tilde{A}|}{A} (1+u) + \frac{2b_1^2 b_2}{b_3^2} \frac{|B-\tilde{B}|}{B} (1+\tilde{u}) \\
&\leq \frac{b_1 b_2}{b_3} \frac{|A-\tilde{A}|}{A} \left[1 + \frac{2}{b_3} \left(\frac{1}{1-p} - 2 \right) \right] + \frac{2b_1^2 b_2}{b_3^2} \frac{|B-\tilde{B}|}{B} \left[1 + \frac{2}{b_3} \left(\frac{1}{1-\tilde{p}} - 2 \right) \right] \quad (\text{by (50)}) \\
&= \frac{b_1 b_2}{b_3} \frac{|A-\tilde{A}|}{A} \left[1 + \frac{2}{b_3} \left(\frac{p}{1-p} - 1 \right) \right] + \frac{2b_1^2 b_2}{b_3^2} \frac{|B-\tilde{B}|}{B} \left[1 + \frac{2}{b_3} \left(\frac{\tilde{p}}{1-\tilde{p}} - 1 \right) \right] \\
&\leq \frac{b_1 b_2}{b_3} \left(1 + \frac{2}{b_3} \right) \frac{|A-\tilde{A}|}{A} \frac{p}{1-p} + \frac{2b_1^2 b_2}{b_3^2} \left(1 + \frac{2}{b_3} \right) \frac{|B-\tilde{B}|}{B} \frac{\tilde{p}}{1-\tilde{p}} .
\end{aligned}$$

By similar method as in (49) and Lemma B.16, we can also derive

$$\left| \frac{1}{p} - \frac{1}{\tilde{p}} \right|$$

$$\begin{aligned}
&\leq \frac{b_2}{2} \cdot \frac{|u - \tilde{u}|}{(1+u)(1+\tilde{u})} \quad (\text{Lemma B.16}) \\
&\leq \frac{b_2}{2(1+u)(1+\tilde{u})} \left[\frac{b_2}{b_3} \frac{|A - \tilde{A}|}{A} (1+u) + \frac{2}{b_3} \frac{|B - \tilde{B}|}{A} (1+\tilde{u})(1+u)^2 \left| \left(\frac{1}{1-\phi(\tilde{u})} \right)' \right| \right] \quad (\text{similar proof as in (49)}) \\
&\leq \frac{b_2^2}{2b_3} \frac{|A - \tilde{A}|}{A} + \frac{b_1 b_2}{b_3} \frac{|B - \tilde{B}|}{A} (1+u) \quad (\text{Lemma B.16}) \\
&\leq \frac{b_2^2}{2b_3} \frac{|A - \tilde{A}|}{A} + \frac{b_1 b_2}{b_3} \left(1 + \frac{2}{b_3} \right) \frac{|B - \tilde{B}|}{A} \frac{p}{1-p} \quad (\text{by (50)}) .
\end{aligned}$$

(2) Let $q = 1 - p$ and $\tilde{q} = 1 - \tilde{p}$. Then q and \tilde{q} satisfy:

$$\begin{aligned}
&-\frac{A}{(1-q)^2} + \frac{B}{q^2} + \Psi'(1-q) = 0 , \\
&-\frac{\tilde{A}}{(1-\tilde{q})^2} + \frac{\tilde{B}}{\tilde{q}^2} + \Psi'(1-\tilde{q}) = 0 .
\end{aligned}$$

Since by Condition 1, $\phi^{-1}(p) = -\phi^{-1}(1-p)$, we have $\Psi(p) = \Psi(1-p)$. This implies that $\Psi'(1-q) = -\Psi'(q)$ and $\Psi'(1-\tilde{q}) = -\Psi'(\tilde{q})$. Hence by the proof in (1), the result is verified. \square

When p_t and \bar{p}_t are not both greater than $1/2$ or both smaller than $1/2$, it is not possible to directly bound the difference between their inverse probabilities. However, when the two first-order equations are close, both probabilities should be close to $1/2$. Therefore, we use $1/2$ as an intermediate comparator and bound their difference in the following lemma.

Lemma C.2. *Let A, B be positive constants. Suppose p satisfy:*

$$-\frac{A}{p^2} + \frac{B}{(1-p)^2} + \Psi'(p) = 0 .$$

- (1) If $B \leq A$, then under Condition 1, p satisfies $0 < \frac{1}{1-p} - 2 \leq \frac{b_1 b_2}{b_3} \cdot \frac{A-B}{B}$.
- (2) If $A \leq B$, then under Condition 1, p satisfies $0 < \frac{1}{p} - 2 \leq \frac{b_1 b_2}{b_3} \cdot \frac{B-A}{A}$.

Proof. (1) For $u = \phi^{-1}(p) \geq 0$ and $\tilde{u} = \phi^{-1}(1/2) = 0$, by Condition 1 it is easy to see that u and \tilde{u} satisfy:

$$\begin{aligned}
&A \left(\frac{1}{\phi(u)} \right)' + B \left(\frac{1}{1-\phi(u)} \right)' + u + 3u^2 = 0 , \\
&B \left(\frac{1}{\phi(\tilde{u})} \right)' + B \left(\frac{1}{1-\phi(\tilde{u})} \right)' + \tilde{u} + 3\tilde{u}^2 = 0 .
\end{aligned} \tag{51}$$

By subtracting the first equation in (51) from the second equation, we have

$$\begin{aligned}
&A \left[\left(\frac{1}{\phi(u)} \right)' - \left(\frac{1}{\phi(\tilde{u})} \right)' \right] + (A-B) \left(\frac{1}{\phi(\tilde{u})} \right)' + B \left[\left(\frac{1}{1-\phi(u)} \right)' - \left(\frac{1}{1-\phi(\tilde{u})} \right)' \right] \\
&+ (u - \tilde{u})(1 + 3u + 3\tilde{u}) = 0 .
\end{aligned}$$

Since $A \left[\left(\frac{1}{\phi(u)} \right)' - \left(\frac{1}{\phi(\tilde{u})} \right)' \right] \geq 0$, $B \left[\left(\frac{1}{1-\phi(u)} \right)' - \left(\frac{1}{1-\phi(\tilde{u})} \right)' \right] \geq 0$ and $(u - \tilde{u})(1 + 3u + 3\tilde{u}) \geq 0$, by Lemma B.16 we have

$$\left| A \left[\left(\frac{1}{\phi(u)} \right)' - \left(\frac{1}{\phi(\tilde{u})} \right)' \right] + B \left[\left(\frac{1}{1-\phi(u)} \right)' - \left(\frac{1}{1-\phi(\tilde{u})} \right)' \right] + \eta^{-1}(u - \tilde{u})(1 + 3u + 3\tilde{u}) \right|$$

$$\begin{aligned} &\geq A \left| \left(\frac{1}{\phi(u)} \right)' - \left(\frac{1}{\phi(\tilde{u})} \right)' \right| \\ &\geq \frac{b_3}{2} \frac{A}{(1+\tilde{u})(1+u)(1+\tilde{u} \wedge u)} \cdot |u - \tilde{u}| . \end{aligned}$$

Hence by Lemma B.16, we have

$$\begin{aligned} \frac{1}{1-p} - \frac{1}{1-1/2} &\leq b_1(u - \tilde{u}) \quad (\text{Lemma B.16}) \\ &\leq b_1 \left(\frac{b_3}{2} \frac{B}{(1+\tilde{u})(1+u)(1+\tilde{u} \wedge u)} \right)^{-1} (A-B) \left| \left(\frac{1}{\phi(u)} \right)' \right| \\ &\leq \frac{b_1 b_2}{b_3} \frac{A-B}{B} (1+u)^2 (1+\tilde{u})(1+u)^{-2} \quad (\text{Lemma B.16}) \\ &= \frac{b_1 b_2}{b_3} \cdot \frac{A-B}{B} \quad (\text{since } \tilde{u} = 0) . \end{aligned}$$

(2) Let $q = 1-p$. Then q satisfies

$$-\frac{B}{q^2} + \frac{A}{(1-q)^2} + \Psi'(q) = 0 .$$

By the same method as in (1) we can prove that

$$\frac{1}{p} - 2 = \frac{1}{1-q} - 2 \leq \frac{b_1 b_2}{b_3} \cdot \frac{B-A}{A} .$$

□

Lemma C.1 and Lemma C.2 together imply that controlling the difference between the inverse probabilities is equivalent to bounding the difference between the corresponding first-order equations. By the definitions of p_t and \bar{p}_t , this reduces to bounding the difference between the estimated squared residuals and their expectations, which can be achieved by controlling the variance of the estimated squared residuals. The following lemma establishes such variance bounds. We denote $\hat{a}_t(k) = \eta_t \hat{A}_t(k)$ and $a_t(k) = \eta_t E \hat{A}_t(k)$ for $k \in \{0, 1\}$ and any $t \in [T]$.

Lemma C.3. *Suppose T is large enough. Under Assumptions 1-3 and Condition 1, for any $t \in [T]$, the variance of $\hat{a}_t(1)$ and $\hat{a}_t(0)$ can be bounded by:*

$$\begin{aligned} \text{Var}(\hat{a}_t(1)) &\leq C(\eta_t T)^{1/2} \log^2(\eta_t T) (2 + b_1(b_2/6)^{1/4} a_t^{1/4}(0))^{25/16} , \\ \text{Var}(\hat{a}_t(0)) &\leq C(\eta_t T)^{1/2} \log^2(\eta_t T) (2 + b_1(b_2/6)^{1/4} a_t^{1/4}(1))^{25/16} , \end{aligned}$$

where $C \triangleq (64\xi_{5/4} + 5248) c_1^4 (\gamma_0 \vee c_2 \vee 1)^2 > 0$ is a constant.

Proof. Without loss of generality, we only prove the first inequality. By Cauchy-Schwarz inequality and AM-GM inequality, the variance of $\hat{A}_t(1)$ can be bounded by:

$$\begin{aligned} &\text{Var}(\hat{A}_t(1)) \\ &= \text{Var} \left[\sum_{s=1}^t \frac{\mathbf{1}[Z_s = 1]}{p_s} \cdot (y_s(1) - \langle \mathbf{x}_s, \boldsymbol{\beta}_s(1) \rangle)^2 \right] \\ &= \text{Var} \left[\sum_{s=1}^t \left(\frac{\mathbf{1}[Z_s = 1]}{p_s} - 1 \right) \cdot (y_s(1) - \langle \mathbf{x}_s, \boldsymbol{\beta}_s(1) \rangle)^2 + \sum_{s=1}^t (y_s(1) - \langle \mathbf{x}_s, \boldsymbol{\beta}_s(1) \rangle)^2 \right] \end{aligned}$$

$$\begin{aligned} &\leq 2 \operatorname{Var} \left[\sum_{s=1}^t \left(\frac{\mathbf{1}[Z_s = 1]}{p_s} - 1 \right) \cdot (y_s(1) - \langle \mathbf{x}_s, \boldsymbol{\beta}_s(1) \rangle)^2 \right] + 2 \operatorname{Var} \left[\sum_{s=1}^t (y_s(1) - \langle \mathbf{x}_s, \boldsymbol{\beta}_s(1) \rangle)^2 \right] \\ &\triangleq 2S_1 + 2S_2 . \end{aligned} \quad (52)$$

For term S_1 , using inequality $(x + y + z)^4 \leq 27x^4 + 27y^4 + 27z^4$ and the law of total variance, we have

$$\begin{aligned} S_1 &= \operatorname{Var} \left[\sum_{s=1}^t \left(\frac{\mathbf{1}[Z_s = 1]}{p_s} - 1 \right) \cdot (y_s(1) - \langle \mathbf{x}_s, \boldsymbol{\beta}_s(1) \rangle)^2 \right] \\ &= \sum_{s=1}^t \operatorname{Var} \left[\left(\frac{\mathbf{1}[Z_s = 1]}{p_s} - 1 \right) \cdot (y_s(1) - \langle \mathbf{x}_s, \boldsymbol{\beta}_s(1) \rangle)^2 \right] \\ &\quad + 2 \sum_{1 \leq s_1 < s_2 \leq t} \operatorname{Cov} \left(\left[\frac{\mathbf{1}[Z_{s_1} = 1]}{p_{s_1}} - 1 \right] \cdot (y_{s_1}(1) - \langle \mathbf{x}_{s_1}, \boldsymbol{\beta}_{s_1}(1) \rangle)^2, \left[\frac{\mathbf{1}[Z_{s_2} = 1]}{p_{s_2}} - 1 \right] \cdot (y_{s_2}(1) - \langle \mathbf{x}_{s_2}, \boldsymbol{\beta}_{s_2}(1) \rangle)^2 \right) \\ &= \sum_{s=1}^t \operatorname{Var} \left[\operatorname{E} \left[\left(\frac{\mathbf{1}[Z_s = 1]}{p_s} - 1 \right) \cdot (y_s(1) - \langle \mathbf{x}_s, \boldsymbol{\beta}_s(1) \rangle)^2 \middle| \mathcal{F}_{s-1} \right] \right] \\ &\quad + \sum_{s=1}^t \operatorname{E} \left[\operatorname{Var} \left[\left(\frac{\mathbf{1}[Z_s = 1]}{p_s} - 1 \right) \cdot (y_s(1) - \langle \mathbf{x}_s, \boldsymbol{\beta}_s(1) \rangle)^2 \middle| \mathcal{F}_{s-1} \right] \right] \\ &\quad + 2 \sum_{1 \leq s_1 < s_2 \leq t} \operatorname{Cov} \left(\operatorname{E} \left[\left(\frac{\mathbf{1}[Z_{s_1} = 1]}{p_{s_1}} - 1 \right) \cdot (y_{s_1}(1) - \langle \mathbf{x}_{s_1}, \boldsymbol{\beta}_{s_1}(1) \rangle)^2 \middle| \mathcal{F}_{s_2-1} \right], \right. \\ &\quad \quad \left. \operatorname{E} \left[\left(\frac{\mathbf{1}[Z_{s_2} = 1]}{p_{s_2}} - 1 \right) \cdot (y_{s_2}(1) - \langle \mathbf{x}_{s_2}, \boldsymbol{\beta}_{s_2}(1) \rangle)^2 \middle| \mathcal{F}_{s_2-1} \right] \right) \\ &\quad + 2 \sum_{1 \leq s_1 < s_2 \leq t} \operatorname{E} \left[\operatorname{Cov} \left(\left(\frac{\mathbf{1}[Z_{s_1} = 1]}{p_{s_1}} - 1 \right) \cdot (y_{s_1}(1) - \langle \mathbf{x}_{s_1}, \boldsymbol{\beta}_{s_1}(1) \rangle)^2, \right. \right. \\ &\quad \quad \left. \left. \left(\frac{\mathbf{1}[Z_{s_2} = 1]}{p_{s_2}} - 1 \right) \cdot (y_{s_2}(1) - \langle \mathbf{x}_{s_2}, \boldsymbol{\beta}_{s_2}(1) \rangle)^2 \middle| \mathcal{F}_{s_2-1} \right) \right] \\ &= \sum_{s=1}^t \operatorname{E} \left[\frac{1-p_s}{p_s} (y_s(1) - \langle \mathbf{x}_s, \boldsymbol{\beta}_s(1) \rangle)^4 \right] \\ &\leq \sum_{s=1}^t \operatorname{E} \left[\frac{1}{p_s} (y_s(1) - \langle \mathbf{x}_s, \boldsymbol{\beta}_s(1) \rangle)^4 \right] \\ &= \sum_{s=1}^t \operatorname{E} \left[\frac{1}{p_s} (y_s(1) - \langle \mathbf{x}_s, \boldsymbol{\beta}_s^*(1) \rangle - \langle \mathbf{x}_s, \boldsymbol{\beta}_s(1) - \boldsymbol{\beta}_s^*(1) \rangle)^4 \right] \\ &\leq 27 \sum_{s=1}^t y_s^4(1) \operatorname{E} \left[\frac{1}{p_s} \right] + 27 \sum_{s=1}^t \langle \mathbf{x}_s, \boldsymbol{\beta}_s^*(1) \rangle^4 \operatorname{E} \left[\frac{1}{p_s} \right] + 27 \sum_{s=1}^t \operatorname{E} \left[\frac{1}{p_s} \langle \mathbf{x}_s, \boldsymbol{\beta}_s(1) - \boldsymbol{\beta}_s^*(1) \rangle^4 \right] \\ &\triangleq 27S_{1,1} + 27S_{1,2} + 27S_{1,3} . \end{aligned} \quad (53)$$

By Corollary 4.15, Corollary B.17 and Assumption 1, $S_{1,1}$ can be bounded as:

$$S_{1,1} \lesssim T \cdot T^{1/8} = T^{9/8} = o(T^{5/4} R_t^3) . \quad (54)$$

By Cauchy-Schwarz inequality, Corollary B.4 and Assumption 1, for any $t \in [T]$ we have

$$\langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle^2 \leq \left(\sum_{s=1}^{t-1} \Pi_{t,s} y_s(1) \right)^2$$

$$\begin{aligned}
&\leq \left(\sum_{s=1}^{t-1} \Pi_{t,s}^2 \right) \left(\sum_{s=1}^{t-1} y_s^2(1) \right) \quad (\text{Cauchy-Schwarz inequality}) \\
&= \left(\sum_{s=1}^{t-1} \mathbf{x}_s^\top (\mathbf{X}_{t-1}^\top \mathbf{X}_{t-1} + \eta_t^{-1} \mathbf{I}_d)^{-1} \mathbf{x}_s \mathbf{x}_s^\top (\mathbf{X}_{t-1}^\top \mathbf{X}_{t-1} + \eta_t^{-1} \mathbf{I}_d)^{-1} \mathbf{x}_t \right) \left(\sum_{s=1}^{t-1} y_s^2(1) \right) \\
&= \left(\mathbf{x}_t^\top (\mathbf{X}_{t-1}^\top \mathbf{X}_{t-1} + \eta_t^{-1} \mathbf{I}_d)^{-1} \mathbf{X}_{t-1} \mathbf{X}_{t-1}^\top (\mathbf{X}_{t-1}^\top \mathbf{X}_{t-1} + \eta_t^{-1} \mathbf{I}_d)^{-1} \mathbf{x}_t \right) \left(\sum_{s=1}^{t-1} y_s^2(1) \right) \\
&\leq \Pi_{t,t}(t-1)^{1/2} \left(\sum_{s=1}^{t-1} y_s^4(1) \right)^{1/2} \quad (\text{Cauchy-Schwarz inequality}) \\
&\leq c_1^2 (\gamma_0 \vee c_2 \vee 1) R_t^2 ((t-1) \vee \eta_t^{-1})^{-1} (t-1)^{1/2} T^{1/2} \quad (\text{Corollary B.4 and Assumption 1}) \\
&\leq c_1^2 (\gamma_0 \vee c_2 \vee 1) R_t^2 ((t-1) \vee \eta_t^{-1})^{-1/2} T^{1/2} \\
&\leq c_1^2 (\gamma_0 \vee c_2 \vee 1) R_t^2 \eta_t^{1/2} T^{1/2} \\
&= c_1^2 (\gamma_0 \vee c_2 \vee 1) R_t T^{1/4}. \tag{55}
\end{aligned}$$

Hence by (55), Corollary B.12 and Lemma 4.14, $S_{1,2}$ can be bounded as:

$$\begin{aligned}
S_{1,2} &\leq \sum_{s=1}^t \langle \mathbf{x}_s, \boldsymbol{\beta}_s^*(1) \rangle^4 (2 + b_1(b_2/6)^{1/4} \eta_s^{1/4} \mathbb{E}^{1/4} \hat{A}_{t-1}(0)) \quad (\text{Lemma 4.14}) \\
&\leq \sum_{s=1}^t \langle \mathbf{x}_s, \boldsymbol{\beta}_s^*(1) \rangle^4 \left(\frac{R_t}{R_s} \right)^{1/2} (2 + b_1(b_2/6)^{1/4} \eta_t^{1/4} \mathbb{E}^{1/4} \hat{A}_t(0)) \\
&\leq \sum_{s=1}^t \langle \mathbf{x}_s, \boldsymbol{\beta}_s^*(1) \rangle^2 \cdot c_1^2 (\gamma_0 \vee c_2 \vee 1) R_s T^{1/4} \left(\frac{R_t}{R_s} \right)^{1/2} (2 + b_1(b_2/6)^{1/4} a_t^{1/4}(0)) \quad (\text{by (55)}) \\
&\leq c_1^2 (\gamma_0 \vee c_2 \vee 1) T^{1/4} \sum_{s=1}^t \langle \mathbf{x}_s, \boldsymbol{\beta}_s^*(1) \rangle^2 R_s^{1/2} R_t^{1/2} (2 + b_1(b_2/6)^{1/4} a_t^{1/4}(0)) \\
&\leq c_1^2 (\gamma_0 \vee c_2 \vee 1) T^{1/4} R_t (2 + b_1(b_2/6)^{1/4} a_t^{1/4}(0)) \sum_{s=1}^t \langle \mathbf{x}_s, \boldsymbol{\beta}_s^*(1) \rangle^2 \quad (\text{by } R_s \leq R_t) \\
&\leq (2^{1/2} + 1)^2 c_1^4 (\gamma_0 \vee c_2 \vee 1) T^{5/4} R_t (2 + b_1(b_2/6)^{1/4} a_t^{1/4}(0)) \quad (\text{Corollary B.12}). \tag{56}
\end{aligned}$$

By Proposition 5.4 and Corollary B.19, we have

$$\begin{aligned}
S_{1,3} &= \sum_{s=1}^t \mathbb{E} \left[\frac{1}{p_s} \langle \mathbf{x}_s, \boldsymbol{\beta}_s(1) - \boldsymbol{\beta}_s^*(1) \rangle^4 \right] \\
&\lesssim T^{7/26} \sum_{s=1}^t R_s \cdot R_s^{-1} \mathbb{E} [\langle \mathbf{x}_s, \boldsymbol{\beta}_s(1) - \boldsymbol{\beta}_s^*(1) \rangle^4] \quad (\text{Proposition 5.4}) \\
&\lesssim T^{7/26} R_t \cdot T^{89/128} R_t^2 \quad (\text{by } R_s \leq R_t \text{ and Corollary B.19}) \\
&= T^{5/4} R_t^3 \cdot T^{-475/1664} \\
&= o(T^{5/4} R_t^3). \tag{57}
\end{aligned}$$

Hence by (53), (54), (56) and (57), we have

$$\begin{aligned}
S_1 &\leq 27 S_{1,1} + 27 S_{1,2} + 27 S_{1,3} \\
&\leq 27 (2^{1/2} + 1)^2 c_1^4 (\gamma_0 \vee c_2 \vee 1) T^{5/4} R_t (2 + b_1(b_2/6)^{1/4} a_t^{1/4}(0)) + o(T^{5/4} R_t^3) \\
&\leq 27 (2^{1/2} + 1)^2 c_1^4 (\gamma_0 \vee c_2 \vee 1) T^{5/4} R_t (2 + b_1(b_2/6)^{1/4} a_t^{1/4}(0)) + o(\eta_t^{-2} \cdot (\eta_t T)^{1/2}). \tag{58}
\end{aligned}$$

For S_2 , by Cauchy-Schwarz inequality and AM-GM inequality, we have

$$\begin{aligned}
S_2 &= \text{Var} \left[\sum_{s=1}^t (y_s(1) - \langle \mathbf{x}_s, \boldsymbol{\beta}_s(1) \rangle)^2 \right] \\
&= \text{Var} \left[\sum_{s=1}^t \left[(y_s(1) - \langle \mathbf{x}_s, \boldsymbol{\beta}_s(1) \rangle)^2 - (y_s(1) - \langle \mathbf{x}_s, \boldsymbol{\beta}_s^*(1) \rangle)^2 \right] \right] \\
&= \text{Var} \left[\sum_{s=1}^t (y_s(1) - \langle \mathbf{x}_s, \boldsymbol{\beta}_s(1) \rangle + y_s(1) - \langle \mathbf{x}_s, \boldsymbol{\beta}_s^*(1) \rangle) \langle \mathbf{x}_s, \boldsymbol{\beta}_s^*(1) - \boldsymbol{\beta}_s(1) \rangle \right] \\
&= \text{Var} \left[\sum_{s=1}^t (2y_s(1) - 2\langle \mathbf{x}_s, \boldsymbol{\beta}_s^*(1) \rangle - \langle \mathbf{x}_s, \boldsymbol{\beta}_s(1) - \boldsymbol{\beta}_s^*(1) \rangle) \langle \mathbf{x}_s, \boldsymbol{\beta}_s(1) - \boldsymbol{\beta}_s^*(1) \rangle \right] \\
&= \text{Var} \left[2 \sum_{s=1}^t (y_s(1) - \langle \mathbf{x}_s, \boldsymbol{\beta}_s^*(1) \rangle) \langle \mathbf{x}_s, \boldsymbol{\beta}_s(1) - \boldsymbol{\beta}_s^*(1) \rangle - \sum_{s=1}^t \langle \mathbf{x}_s, \boldsymbol{\beta}_s(1) - \boldsymbol{\beta}_s^*(1) \rangle^2 \right] \\
&\leq 8 \text{Var} \left[\sum_{s=1}^t (y_s(1) - \langle \mathbf{x}_s, \boldsymbol{\beta}_s^*(1) \rangle) \langle \mathbf{x}_s, \boldsymbol{\beta}_s(1) - \boldsymbol{\beta}_s^*(1) \rangle \right] + 2 \text{Var} \left[\sum_{s=1}^t \langle \mathbf{x}_s, \boldsymbol{\beta}_s(1) - \boldsymbol{\beta}_s^*(1) \rangle^2 \right] \\
&\triangleq 8S_{2,1} + 2S_{2,2}. \tag{59}
\end{aligned}$$

For $S_{2,1}$, by Lemma B.15 and Corollary 4.15, we have

$$\begin{aligned}
S_{2,1} &= \text{Var} \left[\sum_{s=1}^t (y_s(1) - \langle \mathbf{x}_s, \boldsymbol{\beta}_s^*(1) \rangle) \langle \mathbf{x}_s, \boldsymbol{\beta}_s(1) - \boldsymbol{\beta}_s^*(1) \rangle \right] \\
&= \text{Var} \left[\sum_{s=1}^t (y_s(1) - \langle \mathbf{x}_s, \boldsymbol{\beta}_s^*(1) \rangle) \sum_{r=1}^{s-1} \Pi_{s,r} y_r(1) \left(\frac{\mathbf{1}[Z_r = 1]}{p_r} - 1 \right) \right] \\
&= \text{Var} \left[\sum_{r=1}^{t-1} \left(\sum_{s=r+1}^t \Pi_{s,r} (y_s(1) - \langle \mathbf{x}_s, \boldsymbol{\beta}_s^*(1) \rangle) \right) y_r(1) \left(\frac{\mathbf{1}[Z_r = 1]}{p_r} - 1 \right) \right] \quad (\text{rewrite the summation}) \\
&= \sum_{r=1}^{t-1} \text{Var} \left[\left(\sum_{s=r+1}^t \Pi_{s,r} (y_s(1) - \langle \mathbf{x}_s, \boldsymbol{\beta}_s^*(1) \rangle) \right) y_r(1) \left(\frac{\mathbf{1}[Z_r = 1]}{p_r} - 1 \right) \right] \quad (\text{variance of the sum of MDS}) \\
&= \sum_{r=1}^{t-1} \left(\sum_{s=r+1}^t \Pi_{s,r} (y_s(1) - \langle \mathbf{x}_s, \boldsymbol{\beta}_s^*(1) \rangle) \right)^2 y_r^2(1) \text{Var} \left(\frac{\mathbf{1}[Z_r = 1]}{p_r} - 1 \right) \\
&= \sum_{r=1}^{t-1} \left(\sum_{s=r+1}^t \Pi_{s,r} (y_s(1) - \langle \mathbf{x}_s, \boldsymbol{\beta}_s^*(1) \rangle) \right)^2 y_r^2(1) \mathbb{E} \left[\frac{1}{p_r} - 1 \right] \\
&\leq \sum_{r=1}^{t-1} \left(\sum_{s=r+1}^t \Pi_{s,r} (y_s(1) - \langle \mathbf{x}_s, \boldsymbol{\beta}_s^*(1) \rangle) \right)^2 y_r^2(1) \left(2 + b_1(b_2/6)^{1/4} \eta_r^{1/4} \mathbb{E}^{1/4} \hat{A}_{r-1}(0) \right) \quad (\text{Corollary 4.15}) \\
&\leq \sum_{r=1}^{t-1} \left(\sum_{s=r+1}^t \Pi_{s,r} (y_s(1) - \langle \mathbf{x}_s, \boldsymbol{\beta}_s^*(1) \rangle) \right)^2 y_r^2(1) \left(\frac{R_t}{R_r} \right)^{1/2} \left(2 + b_1(b_2/6)^{1/4} \eta_t^{1/4} \mathbb{E}^{1/4} \hat{A}_t(0) \right) \\
&\leq R_t^{1/2} \sum_{r=1}^{t-1} \left(\sum_{s=r+1}^t \Pi_{s,r} (y_s(1) - \langle \mathbf{x}_s, \boldsymbol{\beta}_s^*(1) \rangle) \right)^2 R_r^{-1/2} y_r^2(1) \left(2 + b_1(b_2/6)^{1/4} a_t^{1/4}(0) \right) \\
&\leq 2c_1^4 (\gamma_0 \vee c_2 \vee 1)^{3/2} \xi_{5/4} T^{11/8} R_t^{5/2+1/2} \eta_t^{1/4} \left(2 + b_1(b_2/6)^{1/4} a_t^{1/4}(0) \right) \quad (\text{Lemma B.15}) \\
&= 2c_1^4 (\gamma_0 \vee c_2 \vee 1)^{3/2} \xi_{5/4} T^{11/8} R_t^3 \eta_t^{1/4} \left(2 + b_1(b_2/6)^{1/4} a_t^{1/4}(0) \right) \\
&= 2c_1^4 (\gamma_0 \vee c_2 \vee 1)^{3/2} \xi_{5/4} T^{5/4} R_t^{5/2} \left(2 + b_1(b_2/6)^{1/4} a_t^{1/4}(0) \right). \tag{60}
\end{aligned}$$

For $S_{2,2}$, by Cauchy-Schwarz inequality and AM-GM inequality, we have

$$\begin{aligned}
& S_{2,2} \\
&= \text{Var} \left[\sum_{s=1}^t \left(\sum_{r=1}^{s-1} \Pi_{s,r} y_r(1) \left[\frac{\mathbf{1}[Z_r = 1]}{p_r} - 1 \right] \right)^2 \right] \\
&\leq \mathbb{E} \left[\left(\sum_{s=1}^t \left(\sum_{r=1}^{s-1} \Pi_{s,r} y_r(1) \left[\frac{\mathbf{1}[Z_r = 1]}{p_r} - 1 \right] \right)^2 \right)^2 \right] \\
&= \mathbb{E} \left[\left(\sum_{s=1}^t \sum_{1 \leq t_1, t_2 \leq s-1} \Pi_{s,t_1} \Pi_{s,t_2} y_{t_1}(1) y_{t_2}(1) \left(\frac{\mathbf{1}[Z_{t_1} = 1]}{p_{t_1}} - 1 \right) \left(\frac{\mathbf{1}[Z_{t_2} = 1]}{p_{t_2}} - 1 \right) \right)^2 \right] \\
&= \mathbb{E} \left[\left(\sum_{s=1}^t \sum_{1 \leq r \leq s-1} \Pi_{s,r}^2 y_r^2(1) \left(\frac{\mathbf{1}[Z_r = 1]}{p_r} - 1 \right)^2 \right. \right. \\
&\quad \left. \left. + \sum_{s=1}^t \sum_{1 \leq t_1 \neq t_2 \leq s-1} \Pi_{s,t_1} \Pi_{s,t_2} y_{t_1}(1) y_{t_2}(1) \left(\frac{\mathbf{1}[Z_{t_1} = 1]}{p_{t_1}} - 1 \right) \left(\frac{\mathbf{1}[Z_{t_2} = 1]}{p_{t_2}} - 1 \right) \right)^2 \right] \\
&\leq 2 \mathbb{E} \left[\left(\sum_{s=1}^t \sum_{r=1}^{s-1} \Pi_{s,r}^2 y_r^2(1) \left(\frac{\mathbf{1}[Z_r = 1]}{p_r} - 1 \right)^2 \right)^2 \right] \\
&\quad + 2 \mathbb{E} \left[\left(\sum_{s=1}^t \sum_{1 \leq t_1 \neq t_2 \leq s-1} \Pi_{s,t_1} \Pi_{s,t_2} y_{t_1}(1) y_{t_2}(1) \left(\frac{\mathbf{1}[Z_{t_1} = 1]}{p_{t_1}} - 1 \right) \left(\frac{\mathbf{1}[Z_{t_2} = 1]}{p_{t_2}} - 1 \right) \right)^2 \right] \quad (\text{Cauchy+AM-GM}) \\
&\leq 2 \mathbb{E} \left[\left(\sum_{r=1}^{t-1} \left[\sum_{s=r+1}^t \Pi_{s,r}^2 \right] y_r^2(1) \left(\frac{\mathbf{1}[Z_r = 1]}{p_r} - 1 \right)^2 \right)^2 \right] \quad (\text{rewrite the summation}) \\
&\quad + 2 \text{Var} \left[\sum_{1 \leq t_1 \neq t_2 \leq t-1} \left(\sum_{s=t_1 \vee t_2+1}^t \Pi_{s,t_1} \Pi_{s,t_2} \right) y_{t_1}(1) y_{t_2}(1) \left(\frac{\mathbf{1}[Z_{t_1} = 1]}{p_{t_1}} - 1 \right) \left(\frac{\mathbf{1}[Z_{t_2} = 1]}{p_{t_2}} - 1 \right) \right] \\
&\leq 2 \sum_{r=1}^{t-1} \left(\sum_{s=r+1}^t \Pi_{s,r}^2 \right)^2 y_r^4(1) \mathbb{E} \left[\left(\frac{\mathbf{1}[Z_r = 1]}{p_r} - 1 \right)^4 \right] \\
&\quad + 2 \sum_{1 \leq t_1 \neq t_2 \leq t-1} \left(\sum_{s=t_1+1}^t \Pi_{s,t_1}^2 \right) \left(\sum_{s=t_2+1}^t \Pi_{s,t_2}^2 \right) y_{t_1}^2(1) y_{t_2}^2(1) \mathbb{E} \left[\left(\frac{\mathbf{1}[Z_{t_1} = 1]}{p_{t_1}} - 1 \right)^2 \left(\frac{\mathbf{1}[Z_{t_2} = 1]}{p_{t_2}} - 1 \right)^2 \right] \\
&\quad + 2 \text{Var} \left[\sum_{1 \leq t_1 \neq t_2 \leq t-1} \left(\sum_{s=t_1 \vee t_2+1}^t \Pi_{s,t_1} \Pi_{s,t_2} \right) y_{t_1}(1) y_{t_2}(1) \left(\frac{\mathbf{1}[Z_{t_1} = 1]}{p_{t_1}} - 1 \right) \left(\frac{\mathbf{1}[Z_{t_2} = 1]}{p_{t_2}} - 1 \right) \right] \\
&\triangleq 2B_1 + 2B_2 + 2B_3 . \tag{61}
\end{aligned}$$

By Lemma B.15, Lemma B.18 and Assumption 3, we can bound B_1 by:

$$\begin{aligned}
B_1 &= \sum_{r=1}^{t-1} \left(\sum_{s=r+1}^t \Pi_{s,r}^2 \right)^2 y_r^4(1) \mathbb{E} \left[\left(\frac{\mathbf{1}[Z_r = 1]}{p_r} - 1 \right)^4 \right] \\
&\lesssim \left(\sum_{r=1}^{t-1} \left(\sum_{s=r+1}^t \Pi_{s,r}^2 \right)^2 y_r^4(1) \right) \cdot T^{3/8} \quad (\text{Lemma B.18}) \\
&\lesssim R_t^8 \eta_t^2 T \cdot T^{3/8} \quad (\text{Lemma B.15}) \\
&\lesssim R_t^8 T \cdot T^{-1} R_t^{-4} \cdot T^{3/8}
\end{aligned}$$

$$\begin{aligned}
&= T^{3/8} R_t^4 \\
&= T^{5/4} R_t^3 \cdot (T R_t^{-4})^{-7/8} \cdot R_t^{-5/2} \quad (\text{Assumption 3}) \\
&= o(T^{5/4} R_t^3) . \tag{62}
\end{aligned}$$

By Lemma B.15, Lemma B.18 and Assumption 3, we can bound B_2 by:

$$\begin{aligned}
B_2 &= \sum_{1 \leq t_1 \neq t_2 \leq t-1} \left(\sum_{s=t_1+1}^t \Pi_{s,t_1}^2 \right) \left(\sum_{s=t_2+1}^t \Pi_{s,t_2}^2 \right) y_{t_1}^2(1) y_{t_2}^2(1) \mathbb{E} \left[\left(\frac{\mathbf{1}[Z_{t_1} = 1]}{p_{t_1}} - 1 \right)^2 \left(\frac{\mathbf{1}[Z_{t_2} = 1]}{p_{t_2}} - 1 \right)^2 \right] \\
&\lesssim \left(\sum_{1 \leq t_1 \neq t_2 \leq t-1} \left(\sum_{s=t_1+1}^t \Pi_{s,t_1}^2 \right) \left(\sum_{s=t_2+1}^t \Pi_{s,t_2}^2 \right) y_{t_1}^2(1) y_{t_2}^2(1) \right) \cdot T^{9/32} \quad (\text{Lemma B.18}) \\
&\leq \left(\sum_{s=1}^{t-1} \left(\sum_{r=s+1}^t \Pi_{r,s}^2 \right) y_s^2(1) \right)^2 \cdot T^{9/32} \\
&= \left(\sum_{s=1}^t \left(\sum_{r=1}^{s-1} \Pi_{s,r}^2 \right) y_r^2(1) \right)^2 \cdot T^{9/32} \quad (\text{rewrite the summation}) \\
&\lesssim R_t^6 \eta_t T \cdot T^{9/32} \quad (\text{Lemma B.15}) \\
&\lesssim R_t^6 \cdot T^{-1/2} R_t^{-2} \cdot T^{1+9/32} \\
&= T^{5/4} R_t^3 \cdot (T R_t^{-4})^{-15/32} \cdot R_t^{-7/8} \\
&= o(T^{5/4} R_t^3) \quad (\text{Assumption 3}) . \tag{63}
\end{aligned}$$

For any $1 \leq t_1 \neq t_2, t_3 \neq t_4 \leq t-1$, if the maximum among $\{t_1, t_2, t_3, t_4\}$ is unique (assume without loss of generality that t_1 is the unique maximum), by the law of total variance, there holds:

$$\begin{aligned}
&\text{Cov} \left(\left(\frac{\mathbf{1}[Z_{t_1} = 1]}{p_{t_1}} - 1 \right) \left(\frac{\mathbf{1}[Z_{t_2} = 1]}{p_{t_2}} - 1 \right), \left(\frac{\mathbf{1}[Z_{t_3} = 1]}{p_{t_3}} - 1 \right) \left(\frac{\mathbf{1}[Z_{t_4} = 1]}{p_{t_4}} - 1 \right) \right) \\
&= \mathbb{E} \left[\text{Cov} \left(\left(\frac{\mathbf{1}[Z_{t_1} = 1]}{p_{t_1}} - 1 \right) \left(\frac{\mathbf{1}[Z_{t_2} = 1]}{p_{t_2}} - 1 \right), \left(\frac{\mathbf{1}[Z_{t_3} = 1]}{p_{t_3}} - 1 \right) \left(\frac{\mathbf{1}[Z_{t_4} = 1]}{p_{t_4}} - 1 \right) \middle| \mathcal{F}_{t_1-1} \right) \right] \\
&\quad + \text{Cov} \left[\mathbb{E} \left(\left(\frac{\mathbf{1}[Z_{t_1} = 1]}{p_{t_1}} - 1 \right) \left(\frac{\mathbf{1}[Z_{t_2} = 1]}{p_{t_2}} - 1 \right) \middle| \mathcal{F}_{t_1-1} \right), \mathbb{E} \left(\left(\frac{\mathbf{1}[Z_{t_3} = 1]}{p_{t_3}} - 1 \right) \left(\frac{\mathbf{1}[Z_{t_4} = 1]}{p_{t_4}} - 1 \right) \middle| \mathcal{F}_{t_1-1} \right) \right] \\
&= 0 .
\end{aligned}$$

Hence the only nonzero terms in the expansion of B_3 have four indices where exactly two numbers attain the maximum number among the four indices. Then we can rewrite B_3 as

$$\begin{aligned}
B_3 &\leq 2 \sum_{1 \leq t_1 \neq t_2 \leq t-1} \left(\sum_{s=t_1 \vee t_2+1}^t \Pi_{s,t_1} \Pi_{s,t_2} \right)^2 y_{t_1}^2(1) y_{t_2}^2(1) \text{Var} \left[\left(\frac{\mathbf{1}[Z_{t_1} = 1]}{p_{t_1}} - 1 \right) \left(\frac{\mathbf{1}[Z_{t_2} = 1]}{p_{t_2}} - 1 \right) \right] \\
&\quad + 4 \sum_{1 \leq t_2 \neq t_3 < t_1 \leq t-1} \left| \left(\sum_{s=t_1+1}^t \Pi_{s,t_1} \Pi_{s,t_2} \right) \left(\sum_{s=t_1+1}^t \Pi_{s,t_1} \Pi_{s,t_3} \right) y_{t_1}^2(1) y_{t_2}(1) y_{t_3}(1) \right| \\
&\quad \times \left| \text{Cov} \left(\left(\frac{\mathbf{1}[Z_{t_1} = 1]}{p_{t_1}} - 1 \right) \left(\frac{\mathbf{1}[Z_{t_2} = 1]}{p_{t_2}} - 1 \right), \left(\frac{\mathbf{1}[Z_{t_1} = 1]}{p_{t_1}} - 1 \right) \left(\frac{\mathbf{1}[Z_{t_3} = 1]}{p_{t_3}} - 1 \right) \right) \right| \\
&\triangleq 2B_{3,1} + 4B_{3,2} . \tag{64}
\end{aligned}$$

By Corollary B.4 and Lemma B.9, for any $1 \leq s \leq t-1$, we have

$$\sum_{r=s+1}^t \Pi_{r,s}^2 \leq (\gamma_0 \vee c_2 \vee 1)^2 \sum_{r=s+1}^t R_r^2 R_s^2 ((r-1) \vee \eta_r^{-1})^{-2} \quad (\text{Corollary B.4})$$

$$\begin{aligned}
&\leq (\gamma_0 \vee c_2 \vee 1)^2 \sum_{r=s+1}^t R_r^4 ((r-1) \vee \eta_r^{-1})^{-2} \quad (\text{by } R_s \leq R_r) \\
&= (\gamma_0 \vee c_2 \vee 1)^2 \sum_{r=s+1}^t ((r-1) R_r^{-2} \vee T^{1/2})^{-2} \\
&\leq (\gamma_0 \vee c_2 \vee 1)^2 \sum_{r=s+1}^t ((r-1) R_t^{-2} \vee T^{1/2})^{-2} \quad (\text{by } R_r \leq R_t) \\
&= (\gamma_0 \vee c_2 \vee 1)^2 \sum_{r=s+1}^t R_t^4 ((r-1) \vee \eta_t^{-1})^{-2} \\
&\leq (\gamma_0 \vee c_2 \vee 1)^2 \xi_2 R_t^4 \eta_t \quad (\text{Lemma B.9}) .
\end{aligned}$$

Then by Lemma B.15, Lemma B.18 and Assumption 3, $B_{3,1}$ can be bounded by

$$\begin{aligned}
&B_{3,1} \\
&\leq \sum_{1 \leq t_1 \neq t_2 \leq t-1} \left(\sum_{s=t_1 \vee t_2+1}^t \Pi_{s,t_1}^2 \right)^2 \left(\sum_{s=t_1 \vee t_2+1}^t \Pi_{s,t_2}^2 \right)^2 y_{t_1}^2(1) y_{t_2}^2(1) \text{Var} \left[\left(\frac{\mathbf{1}[Z_{t_1}=1]}{p_{t_1}} - 1 \right) \left(\frac{\mathbf{1}[Z_{t_2}=1]}{p_{t_2}} - 1 \right) \right] \\
&\leq \sum_{1 \leq t_1 \neq t_2 \leq t-1} \left(\sum_{s=t_1 \vee t_2+1}^t \Pi_{s,t_1}^2 \right)^2 \left(\sum_{s=t_1 \vee t_2+1}^t \Pi_{s,t_2}^2 \right)^2 y_{t_1}^2(1) y_{t_2}^2(1) \mathbb{E} \left[\left(\frac{\mathbf{1}[Z_{t_1}=1]}{p_{t_1}} - 1 \right)^2 \left(\frac{\mathbf{1}[Z_{t_2}=1]}{p_{t_2}} - 1 \right)^2 \right] \\
&\lesssim \left(\sum_{1 \leq t_1 \neq t_2 \leq t-1} \left(\sum_{s=t_1+1}^t \Pi_{s,t_1}^2 \right)^2 \left(\sum_{s=t_2+1}^t \Pi_{s,t_2}^2 \right)^2 y_{t_1}^2(1) y_{t_2}^2(1) \right) \cdot T^{9/32} \quad (\text{Lemma B.18}) \\
&\leq \left(\sum_{s=1}^{t-1} \left(\sum_{r=s+1}^t \Pi_{r,s}^2 \right)^2 y_s^2(1) \right)^2 \cdot T^{9/32} \\
&\lesssim R_t^8 \eta_t^2 \left(\sum_{s=1}^{t-1} \left(\sum_{r=s+1}^t \Pi_{r,s}^2 \right) y_s^2(1) \right)^2 \cdot T^{9/32} \\
&= R_t^8 (T^{-1/2} R_t^{-2})^2 \left(\sum_{s=1}^t \left(\sum_{r=1}^{s-1} \Pi_{s,r}^2 \right) y_r^2(1) \right)^2 \cdot T^{9/32} \\
&\lesssim R_t^4 T^{-1} \cdot R_t^6 \eta_t T \cdot T^{9/32} \quad (\text{Lemma B.15}) \\
&= T^{5/4} R_t^3 \cdot (T R_t^{-4})^{-47/32} \cdot R_t^{-7/8} \\
&= o(T^{5/4} R_t^3) \quad (\text{Assumption 3}) . \tag{65}
\end{aligned}$$

By Lemma B.15 and Lemma B.18, we can bound $B_{3,2}$ as:

$$\begin{aligned}
B_{3,2} &= \sum_{1 \leq t_2 \neq t_3 < t_1 \leq t-1} \left| \left(\sum_{s=t_1+1}^t \Pi_{s,t_1} \Pi_{s,t_2} \right) \left(\sum_{s=t_1+1}^t \Pi_{s,t_1} \Pi_{s,t_3} \right) y_{t_1}^2(1) y_{t_2}(1) y_{t_3}(1) \right| \\
&\quad \times \left| \text{Cov} \left(\left(\frac{\mathbf{1}[Z_{t_1}=1]}{p_{t_1}} - 1 \right), \left(\frac{\mathbf{1}[Z_{t_2}=1]}{p_{t_2}} - 1 \right), \left(\frac{\mathbf{1}[Z_{t_1}=1]}{p_{t_1}} - 1 \right), \left(\frac{\mathbf{1}[Z_{t_3}=1]}{p_{t_3}} - 1 \right) \right) \right| \\
&\leq 4 \sum_{1 \leq t_2 \neq t_3 < t_1 \leq t-1} \left| \left(\sum_{s=t_1+1}^t \Pi_{s,t_1} \Pi_{s,t_2} \right) \left(\sum_{s=t_1+1}^t \Pi_{s,t_1} \Pi_{s,t_3} \right) y_{t_1}^2(1) y_{t_2}(1) y_{t_3}(1) \right| \\
&\quad \times \left(2 + b_1 (b_2/6)^{1/4} \eta_{t_1}^{1/4} \mathbb{E}^{1/4} \hat{A}_{t_1}(0) \right)^{25/16} \left(\frac{R_{t_1}}{R_{t_2}} \right)^{15/96} \left(\frac{R_{t_1}}{R_{t_3}} \right)^{15/96} \quad (\text{Lemma B.18}) \\
&\leq 4 \sum_{1 \leq t_2 \neq t_3 < t_1 \leq t-1} \left| \left(\sum_{s=t_1+1}^t \Pi_{s,t_1} \Pi_{s,t_2} \right) \left(\sum_{s=t_1+1}^t \Pi_{s,t_1} \Pi_{s,t_3} \right) y_{t_1}^2(1) y_{t_2}(1) y_{t_3}(1) \right|
\end{aligned}$$

$$\begin{aligned}
& \times \left(2 + b_1(b_2/6)^{1/4} \eta_t^{1/4} E^{1/4} \hat{A}_t(0) \right)^{25/16} \left(\frac{R_t}{R_{t_1}} \right)^{25/32} \left(\frac{R_{t_1}}{R_{t_2}} \right)^{15/96} \left(\frac{R_{t_1}}{R_{t_3}} \right)^{15/96} \quad (\text{by } R_{t_1} \leq R_t) \\
= & 4 \sum_{1 \leq t_2 \neq t_3 \leq t-1} \left| \left(\sum_{s=t_1+1}^t \Pi_{s,t_1} \Pi_{s,t_2} \right) \left(\sum_{s=t_1+1}^t \Pi_{s,t_1} \Pi_{s,t_3} \right) y_{t_1}^2(1) y_{t_2}(1) y_{t_3}(1) \right| \\
& \times \left(2 + b_1(b_2/6)^{1/4} a_t^{1/4}(0) \right)^{25/16} \left(\frac{R_t}{R_{t_1}} \right)^{15/32} \left(\frac{R_t}{R_{t_2}} \right)^{15/96} \left(\frac{R_t}{R_{t_3}} \right)^{15/96} \\
\leq & 4 \sum_{1 \leq t_2, t_3 < t-1} \left| \left(\sum_{s=t_1+1}^t \Pi_{s,t_1} \Pi_{s,t_2} \right) \left(\sum_{s=t_1+1}^t \Pi_{s,t_1} \Pi_{s,t_3} \right) y_{t_1}^2(1) y_{t_2}(1) y_{t_3}(1) \left(\frac{R_t^2}{R_{t_1}^2} \cdot \frac{R_t}{R_{t_2}} \cdot \frac{R_t}{R_{t_3}} \right)^{1/4} \right| \\
& \times \left(2 + b_1(b_2/6)^{1/4} a_t^{1/4}(0) \right)^{25/16} \quad (\text{by } R_{t_1}, R_{t_2}, R_{t_3} \leq R_t) \\
\leq & 72 c_1^4 (\gamma_0 \vee c_2 \vee 1)^2 R_t^4 \eta_t^{1/2} T^{3/2} \log^2(\eta_t T) \cdot \left(2 + b_1(b_2/6)^{1/4} a_t^{1/4}(0) \right)^{25/16} \\
= & 72 c_1^4 (\gamma_0 \vee c_2 \vee 1)^2 R_t^3 T^{5/4} \log^2(\eta_t T) \cdot \left(2 + b_1(b_2/6)^{1/4} a_t^{1/4}(0) \right)^{25/16}. \tag{66}
\end{aligned}$$

Then by (59), (60), (61), (62), (63), (64), (65) and (66), we can bound S_2 by:

$$\begin{aligned}
S_2 \leq & 8S_{2,1} + 2S_{2,2} \\
\leq & 8S_{2,1} + 4B_1 + 4B_2 + 8B_{3,1} + 16B_{3,2} \\
\leq & 16c_1^4 (\gamma_0 \vee c_2 \vee 1)^{3/2} \xi_{5/4} T^{5/4} R_t^{5/2} \left(2 + b_1(b_2/6)^{1/4} a_t^{1/4}(0) \right) \\
& + 1152c_1^4 (\gamma_0 \vee c_2 \vee 1)^2 R_t^3 T^{5/4} \log^2(\eta_t T) \cdot \left(2 + b_1(b_2/6)^{1/4} a_t^{1/4}(0) \right)^{25/16} + o(T^{5/4} R_t^3) \\
\leq & \left(16c_1^4 (\gamma_0 \vee c_2 \vee 1)^{3/2} \xi_{5/4} + 1152c_1^4 (\gamma_0 \vee c_2 \vee 1)^2 \right) R_t^3 T^{5/4} \log^2(\eta_t T) \cdot \left(2 + b_1(b_2/6)^{1/4} a_t^{1/4}(0) \right)^{25/16} + o(T^{5/4} R_t^3) \\
= & (16\xi_{5/4} + 1152) c_1^4 (\gamma_0 \vee c_2 \vee 1)^2 \eta_t^{-2} (\eta_t T)^{1/2} \log^2(\eta_t T) \left(2 + b_1(b_2/6)^{1/4} a_t^{1/4}(0) \right)^{25/16} + o(\eta_t^{-2} (\eta_t T)^{1/2}). \tag{67}
\end{aligned}$$

Then for T large enough, by (52), (58) and (67), we have

$$\begin{aligned}
& \text{Var}(\hat{A}_t(1)) \\
\leq & 2S_1 + 2S_2 \\
\leq & 54(2^{1/2} + 1)^2 c_1^4 (\gamma_0 \vee c_2 \vee 1) T^{5/4} R_t (2 + b_1(b_2/6)^{1/4} a_t^{1/4}(0)) + o(\eta_t^{-2} (\eta_t T)^{1/2}) \\
& + (32\xi_{5/4} + 2304) c_1^4 (\gamma_0 \vee c_2 \vee 1)^2 \eta_t^{-2} (\eta_t T)^{1/2} \log^2(\eta_t T) \left(2 + b_1(b_2/6)^{1/4} a_t^{1/4}(0) \right)^{25/16} \\
\leq & (320 + 32\xi_{5/4} + 2304) c_1^4 (\gamma_0 \vee c_2 \vee 1)^2 \eta_t^{-2} (\eta_t T)^{1/2} \log^2(\eta_t T) \left(2 + b_1(b_2/6)^{1/4} a_t^{1/4}(0) \right)^{25/16} + o(\eta_t^{-2} (\eta_t T)^{1/2}) \\
\leq & (64\xi_{5/4} + 5248) c_1^4 (\gamma_0 \vee c_2 \vee 1)^2 \eta_t^{-2} (\eta_t T)^{1/2} \log^2(\eta_t T) \left(2 + b_1(b_2/6)^{1/4} a_t^{1/4}(0) \right)^{25/16} \\
= & C \eta_t^{-2} (\eta_t T)^{1/2} \log^2(\eta_t T) \left(2 + b_1(b_2/6)^{1/4} a_t^{1/4}(0) \right)^{25/16},
\end{aligned}$$

which indicates that

$$\text{Var}(\hat{A}_t(1)) = \text{Var}(\eta_t \hat{A}_t(1)) \leq C(\eta_t T)^{1/2} \log^2(\eta_t T) \left(2 + b_1(b_2/6)^{1/4} a_t^{1/4}(0) \right)^{25/16}.$$

□

Lemma C.3 establishes that the variance of $\hat{A}_t(1)$ (or $\hat{A}_t(1)$) and $\hat{A}_t(0)$ (or $\hat{A}_t(0)$) are upper bounded in a mutually normalizing manner. Such dependence can be undesirable, as the

estimated squared residuals for each group may fail to concentrate around their respective expectations when the treatment group and controlled group are imbalanced. Nevertheless, as implied by Corollary B.17 and Lemma C.1, once the expected squared residuals exceed a certain threshold, the maximal (positive) deviation between the two groups is controlled, thereby ensuring a degree of concentration despite the mutual normalization. The result is formally stated in the following lemma.

Lemma C.4. *Suppose T is large enough. Suppose $0 < \delta < \frac{1}{4}$ is a fixed constant and $\tilde{C} > 0$ is a fixed constant given by*

$$\begin{aligned}\tilde{C} = \max \left\{ \left[\frac{C^{1/2} b_1^{25/32} b_2^{281/128}}{b_3} + \frac{2^{57/32} C^{1/2} b_1^{89/32} b_2^{185/128}}{b_3} \left(1 + \frac{2}{b_3}\right) \right], \right. \\ 2^{283/64} C^{3/4} b_1^{139/64} b_2^{139/256}, \frac{2^{57/32} C^{1/2} b_1^{89/32} b_2^{185/128}}{b_3} \left(1 + \frac{2}{b_3}\right), \\ \frac{2^{89/32} C^{1/2} b_1^{121/32} b_2^{217/128}}{b_3^{5/2}} \left(1 + \frac{2}{b_3}\right), 2^{89/16} C b_1^{41/16} b_2^{41/64}, \\ \left. \frac{8b_1 b_2}{b_3}, 33 C^{3/4} b_1^{139/64} b_2^{139/256} \right\}.\end{aligned}$$

Then under Assumptions 1-3 and Condition 1, we have the following upper bound:

(1) For any $t \in [T]$ such that $a_t(1) \geq 5b_1^{-4}b_2^{-1}$, there holds

$$\begin{aligned}& \mathbb{E} \left[\left| \frac{1}{p_{t+1}} - \frac{1}{\bar{p}_{t+1}} \right| \right] \\ & \leq \tilde{C} \max \left\{ (\eta_t T)^{1/4} \log(\eta_t T) a_t^{-71/128}(1) + (\eta_t T)^{3/8} \log^{3/2}(\eta_t T) \delta^{-3/2} a_t^{-245/256}(1), \right. \\ & \quad (\eta_t T)^{1/4} \log(\eta_t T) a_t^{-71/128}(1) + (\eta_t \kappa)^{89/128} (\eta_t T)^{1/4} \log(\eta_t T) a_t^{-3/2}(1) \\ & \quad \left. + (\eta_t T)^{1/2} \log^2(\eta_t T) \delta^{-2} a_t^{-2}(1) (\eta_t \kappa)^{41/64}, \delta + (\eta_t T)^{3/8} \log^{3/2}(\eta_t T) \delta^{-3/2} a_t^{-245/256}(1) \right\}.\end{aligned}$$

(2) For any $t \in [T]$ such that $a_t(0) \geq 5b_1^{-4}b_2^{-1}$, there holds

$$\begin{aligned}& \mathbb{E} \left[\left| \frac{1}{p_{t+1}} - \frac{1}{\bar{p}_{t+1}} \right| \right] \\ & \leq \tilde{C} \max \left\{ (\eta_t T)^{1/4} \log(\eta_t T) a_t^{-71/128}(0) + (\eta_t T)^{3/8} \log^{3/2}(\eta_t T) \delta^{-3/2} a_t^{-245/256}(0), \right. \\ & \quad (\eta_t T)^{1/4} \log(\eta_t T) a_t^{-71/128}(0) + (\eta_t \kappa)^{89/128} (\eta_t T)^{1/4} \log(\eta_t T) a_t^{-3/2}(0) \\ & \quad \left. + (\eta_t T)^{1/2} \log^2(\eta_t T) \delta^{-2} a_t^{-2}(0) (\eta_t \kappa)^{41/64}, \delta + (\eta_t T)^{3/8} \log^{3/2}(\eta_t T) \delta^{-3/2} a_t^{-245/256}(0) \right\}.\end{aligned}$$

Proof. Without loss of generality, we only prove the first part. For any $t \in [T]$ such that $a_t(1) \geq 5b_1^{-4}b_2^{-1}$, it is easy to see that $2 + b_1(b_2/6)^{1/4} a_t^{1/4}(1) \leq 2b_1 b_2^{1/4} a_t^{1/4}(1)$. We then consider the following three cases:

(1) $a_t(0) \leq (1 - \delta)a_t(1)$.

By Chebyshev's inequality and Lemma C.3, we have

$$\Pr(\hat{a}_t(1) \leq \hat{a}_t(0))$$

$$\begin{aligned}
&\leq \Pr\left(\frac{|\hat{a}_t(1) - a_t(1)|}{a_t(1)} + \frac{|\hat{a}_t(0) - a_t(0)|}{a_t(1)} \geq \delta\right) \quad (\text{Chebyshev's inequality}) \\
&\leq \Pr\left(\frac{|\hat{a}_t(1) - a_t(1)|}{a_t(1)} \geq \frac{\delta}{2}\right) + \Pr\left(\frac{|\hat{a}_t(0) - a_t(0)|}{a_t(1)} \geq \frac{\delta}{2}\right) \\
&\leq \frac{4 \operatorname{Var}(\hat{a}_t(1))}{\delta^2 a_t^2(1)} + \frac{4 \operatorname{Var}(\hat{a}_t(0))}{\delta^2 a_t^2(1)} \\
&\leq \frac{4C(\eta_t T)^{1/2} \log^2(\eta_t T) \left(2 + b_1(b_2/6)^{1/4} a_t^{1/4}(0)\right)^{25/16}}{\delta^2 a_t^2(1)} \\
&\quad + \frac{4C(\eta_t T)^{1/2} \log^2(\eta_t T) \left(2 + b_1(b_2/6)^{1/4} a_t^{1/4}(1)\right)^{25/16}}{\delta^2 a_t^2(1)} \quad (\text{Lemma C.3}) \\
&\leq \frac{8C(\eta_t T)^{1/2} \log^2(\eta_t T) \left(2 + b_1(b_2/6)^{1/4} a_t^{1/4}(1)\right)^{25/16}}{\delta^2 a_t^2(1)} \\
&\leq \frac{8 \cdot 2^{25/16} C b_1^{25/16} b_2^{25/64} (\eta_t T)^{1/2} \log^2(\eta_t T) a_t^{25/64}(1)}{\delta^2 a_t^2(1)} \quad (\text{since } a_t(0) \leq a_t(1)) \\
&\leq \frac{2^{73/16} C b_1^{25/16} b_2^{25/64} (\eta_t T)^{1/2} \log^2(\eta_t T)}{\delta^2 a_t^{103/64}(1)}. \tag{68}
\end{aligned}$$

Under event $\{\hat{a}_t(1) \geq \hat{a}_t(0)\}$, we have $p_{t+1}, \bar{p}_{t+1} \geq 1/2$. By definition, p_{t+1} and \bar{p}_{t+1} satisfy:

$$\begin{aligned}
&-\frac{a_t(1) \cdot \frac{\eta_{t+1}}{\eta_t}}{\bar{p}_{t+1}^2} + \frac{a_t(0) \cdot \frac{\eta_{t+1}}{\eta_t}}{(1 - \bar{p}_{t+1})^2} + \Psi'(\bar{p}_{t+1}) = 0, \\
&-\frac{\hat{a}_t(1) \cdot \frac{\eta_{t+1}}{\eta_t}}{p_{t+1}^2} + \frac{\hat{a}_t(0) \cdot \frac{\eta_{t+1}}{\eta_t}}{(1 - p_{t+1})^2} + \Psi'(p_{t+1}) = 0.
\end{aligned}$$

By Lemma 4.14 and Lemma C.1, we have

$$\begin{aligned}
\left| \frac{1}{p_{t+1}} - \frac{1}{\bar{p}_{t+1}} \right| &\leq \frac{b_2^2}{2b_3} \frac{|a_t(1) - \hat{a}_t(1)| \cdot \frac{\eta_{t+1}}{\eta_t}}{a_t(1) \cdot \frac{\eta_{t+1}}{\eta_t}} + \frac{b_1 b_2}{b_3} \left(1 + \frac{2}{b_3}\right) \frac{\bar{p}_{t+1}}{1 - \bar{p}_{t+1}} \cdot \frac{|a_t(0) - \hat{a}_t(0)| \cdot \frac{\eta_{t+1}}{\eta_t}}{a_t(1) \cdot \frac{\eta_{t+1}}{\eta_t}} \quad (\text{Lemma C.1}) \\
&\leq \frac{b_2^2}{2b_3} \frac{|a_t(1) - \hat{a}_t(1)| \cdot \frac{\eta_{t+1}}{\eta_t}}{a_t(1) \cdot \frac{\eta_{t+1}}{\eta_t}} \\
&\quad + \frac{b_1 b_2}{b_3} \left(1 + \frac{2}{b_3}\right) \frac{\left(1 + b_1(b_2/6)^{1/4} a_t^{1/4}(1) \cdot \left(\frac{\eta_{t+1}}{\eta_t}\right)^{1/4}\right) |a_t(0) - \hat{a}_t(0)|}{a_t(1)} \quad (\text{Lemma 4.14}) \\
&\leq \frac{b_2^2}{2b_3} \frac{|a_t(1) - \hat{a}_t(1)|}{a_t(1)} + \frac{2b_1^2 b_2^{5/4}}{b_3} \left(1 + \frac{2}{b_3}\right) \frac{|a_t(0) - \hat{a}_t(0)|}{a_t^{3/4}(1)} \quad (\text{since } \eta_{t+1} \leq \eta_t). \tag{69}
\end{aligned}$$

By calculation, we have

$$\begin{aligned}
\mathbb{E}\left[\left|\frac{1}{p_{t+1}} - \frac{1}{\bar{p}_{t+1}}\right|\right] &= \mathbb{E}\left[\left|\frac{1}{p_{t+1}} - \frac{1}{\bar{p}_{t+1}}\right| \mathbf{1}[\hat{a}_t(1) \geq \hat{a}_t(0)]\right] + \mathbb{E}\left[\left|\frac{1}{p_{t+1}} - \frac{1}{\bar{p}_{t+1}}\right| \mathbf{1}[\hat{a}_t(1) \leq \hat{a}_t(0)]\right] \\
&\triangleq D_1 + D_2. \tag{70}
\end{aligned}$$

For term D_1 , by Lemma C.3, (69) and Cauchy-Schwarz inequality, we have

$$D_1 \leq \mathbb{E}\left[\frac{b_2^2}{2b_3} \frac{|a_t(1) - \hat{a}_t(1)|}{a_t(1)} + \frac{2b_1^2 b_2^{5/4}}{b_3} \left(1 + \frac{2}{b_3}\right) \frac{|a_t(0) - \hat{a}_t(0)|}{a_t^{3/4}(1)}\right]$$

$$\begin{aligned}
&\leq \frac{b_2^2}{2b_3} a_t^{-1}(1) \operatorname{Var}^{1/2}(\hat{a}_t(1)) + \frac{2b_1^2 b_2^{5/4}}{b_3} \left(1 + \frac{2}{b_3}\right) a_t^{-3/4}(1) \operatorname{Var}^{1/2}(\hat{a}_t(0)) \quad (\text{Cauchy-Schwarz inequality}) \\
&\leq \frac{C^{1/2} b_2^2}{2b_3} a_t^{-1}(1) (\eta_t T)^{1/4} \log(\eta_t T) \left(2 + b_1(b_2/6)^{1/4} a_t^{1/4}(0)\right)^{25/32} \\
&\quad + \frac{2C^{1/2} b_1^2 b_2^{5/4}}{b_3} \left(1 + \frac{2}{b_3}\right) a_t^{-3/4}(1) (\eta_t T)^{1/4} \log(\eta_t T) \left(2 + b_1(b_2/6)^{1/4} a_t^{1/4}(1)\right)^{25/32} \quad (\text{Lemma C.3}) \\
&\leq \frac{2^{-7/32} C^{1/2} b_1^{25/32} b_2^{281/128}}{b_3} a_t^{-1}(1) (\eta_t T)^{1/4} \log(\eta_t T) a_t^{25/128}(1) \\
&\quad + \frac{2^{57/32} C^{1/2} b_1^{89/32} b_2^{185/128}}{b_3} \left(1 + \frac{2}{b_3}\right) a_t^{-3/4}(1) (\eta_t T)^{1/4} \log(\eta_t T) a_t^{25/128}(1) \\
&\leq \frac{C^{1/2} b_1^{25/32} b_2^{281/128}}{b_3} (\eta_t T)^{1/4} \log(\eta_t T) a_t^{-103/128}(1) \\
&\quad + \frac{2^{57/32} C^{1/2} b_1^{89/32} b_2^{185/128}}{b_3} \left(1 + \frac{2}{b_3}\right) (\eta_t T)^{1/4} \log(\eta_t T) a_t^{-71/128}(1). \tag{71}
\end{aligned}$$

Under event $\{\hat{a}_t(1) \leq \hat{a}_t(0)\}$, we have $p_{t+1} \leq 1/2$ and $\bar{p}_{t+1} \geq 1/2$. By (68), Corollary 4.15 and Hölder's inequality, we have

$$\begin{aligned}
D_2 &= \operatorname{E} \left[\left| \frac{1}{p_{t+1}} - \frac{1}{\bar{p}_{t+1}} \right| \mathbf{1} [\hat{a}_t(1) \leq \hat{a}_t(0)] \right] \\
&\leq \operatorname{E} \left[\frac{1}{p_{t+1}} \mathbf{1} [\hat{a}_t(1) \leq \hat{a}_t(0)] \right] \\
&\leq \left(\operatorname{E} \left[\frac{1}{p_{t+1}^4} \right] \right)^{1/4} (\Pr(\hat{a}_t(1) \leq \hat{a}_t(0)))^{3/4} \quad (\text{Hölder's inequality}) \\
&\leq \left(2 + b_1(b_2/6)^{1/4} a_t^{1/4}(0) \right) \left(\frac{2^{73/16} C b_1^{25/16} b_2^{25/64} (\eta_t T)^{1/2} \log^2(\eta_t T)}{\delta^2 a_t^{103/64}(1)} \right)^{3/4} \quad (\text{by (68) and Corollary 4.15}) \\
&\leq 2b_1 b_2^{1/4} a_t^{1/4}(1) \left(\frac{2^{73/16} C b_1^{25/16} b_2^{25/64} (\eta_t T)^{1/2} \log^2(\eta_t T)}{\delta^2 a_t^{103/64}(1)} \right)^{3/4} \\
&\leq 2^{283/64} C^{3/4} b_1^{139/64} b_2^{139/256} (\eta_t T)^{3/8} \log^{3/2}(\eta_t T) \delta^{-3/2} a_t^{-245/256}(1). \tag{72}
\end{aligned}$$

Hence by (70), (71) and (72), we have

$$\begin{aligned}
&\operatorname{E} \left[\left| \frac{1}{p_{t+1}} - \frac{1}{\bar{p}_{t+1}} \right| \right] \\
&\leq \frac{C^{1/2} b_1^{25/32} b_2^{281/128}}{b_3} (\eta_t T)^{1/4} \log(\eta_t T) a_t^{-103/128}(1) \\
&\quad + \frac{2^{57/32} C^{1/2} b_1^{89/32} b_2^{185/128}}{b_3} \left(1 + \frac{2}{b_3}\right) (\eta_t T)^{1/4} \log(\eta_t T) a_t^{-71/128}(1) \\
&\quad + 2^{283/64} C^{3/4} b_1^{139/64} b_2^{139/256} (\eta_t T)^{3/8} \log^{3/2}(\eta_t T) \delta^{-3/2} a_t^{-245/256}(1) \\
&\leq \left[\frac{C^{1/2} b_1^{25/32} b_2^{281/128}}{b_3} + \frac{2^{57/32} C^{1/2} b_1^{89/32} b_2^{185/128}}{b_3} \left(1 + \frac{2}{b_3}\right) \right] (\eta_t T)^{1/4} \log(\eta_t T) a_t^{-71/128}(1) \\
&\quad + 2^{283/64} C^{3/4} b_1^{139/64} b_2^{139/256} (\eta_t T)^{3/8} \log^{3/2}(\eta_t T) \delta^{-3/2} a_t^{-245/256}(1). \tag{73}
\end{aligned}$$

(2) $a_t(1) \leq (1 - \delta) a_t(0)$.

By Chebyshev's inequality and Lemma C.3, we have

$$\begin{aligned}
\Pr \left(\frac{|\hat{a}_t(0) - a_t(0)|}{a_t(0)} \geq \frac{\delta}{2} \right) &\leq \frac{4 \operatorname{Var}(\hat{a}_t(0))}{\delta^2 a_t^2(0)} \quad (\text{Chebyshev's inequality}) \\
&\leq \frac{4C(\eta_t T)^{1/2} \log^2(\eta_t T) \left(2 + b_1(b_2/6)^{1/4} a_t^{1/4}(1)\right)^{25/16}}{\delta^2 a_t^2(0)} \quad (\text{Lemma C.3}) \\
&\leq \frac{4 \cdot 2^{25/16} C b_1^{25/16} b_2^{25/64} (\eta_t T)^{1/2} \log^2(\eta_t T) a_t^{25/64}(0)}{\delta^2 a_t^2(0)} \\
&\leq \frac{2^{57/16} C b_1^{25/16} b_2^{25/64} (\eta_t T)^{1/2} \log^2(\eta_t T)}{\delta^2 a_t^{103/64}(0)} \\
&\leq \frac{2^{57/16} C b_1^{25/16} b_2^{25/64} (\eta_t T)^{1/2} \log^2(\eta_t T)}{\delta^2 a_t^{103/64}(1)} \quad (\text{since } a_t(0) \geq a_t(1)) . \quad (74)
\end{aligned}$$

Since $a_t(0) = \eta_t E \hat{A}_t(0) \leq \eta_t \kappa$ by Corollary B.17, we also have

$$\begin{aligned}
\Pr \left(\frac{|\hat{a}_t(1) - a_t(1)|}{a_t(1)} \geq \frac{\delta}{2} \right) &\leq \frac{4 \operatorname{Var}(\hat{a}_t(1))}{\delta^2 a_t^2(1)} \quad (\text{Chebyshev's inequality}) \\
&\leq \frac{4C(\eta_t T)^{1/2} \log^2(\eta_t T) \left(2 + b_1(b_2/6)^{1/4} a_t^{1/4}(0)\right)^{25/16}}{\delta^2 a_t^2(1)} \quad (\text{Lemma C.3}) \\
&\leq \frac{4 \cdot 2^{25/16} C b_1^{25/16} b_2^{25/64} (\eta_t T)^{1/2} \log^2(\eta_t T) (\eta_t \kappa)^{25/64}}{\delta^2 a_t^2(1)} \\
&= \frac{2^{57/16} C b_1^{25/16} b_2^{25/64} (\eta_t T)^{1/2} \log^2(\eta_t T) (\eta_t \kappa)^{25/64}}{\delta^2 a_t^2(1)} . \quad (75)
\end{aligned}$$

Under event $\{|\hat{a}_t(0) - a_t(0)| \leq \frac{\delta}{2} a_t(0)\}$ and event $\{|\hat{a}_t(1) - a_t(1)| \leq \frac{\delta}{2} a_t(1)\}$, we have

$$\begin{aligned}
\hat{a}_t(0) - \hat{a}_t(1) &\geq (a_t(0) - a_t(1)) - |\hat{a}_t(0) - a_t(0)| - |\hat{a}_t(1) - a_t(1)| \\
&\geq (1 - (1 - \delta)) a_t(0) - \frac{\delta}{2} a_t(0) - \frac{\delta}{2} a_t(1) \\
&\geq \delta a_t(0) - \frac{\delta}{2} a_t(0) - \frac{\delta}{2} a_t(0) \quad (\text{since } a_t(0) \geq a_t(1)) \\
&= 0 .
\end{aligned}$$

This implies that $\hat{a}_t(0) \geq \hat{a}_t(1)$. Hence by definition, p_{t+1} and \bar{p}_{t+1} satisfy:

$$\begin{aligned}
-\frac{a_t(1) \cdot \frac{\eta_{t+1}}{\eta_t}}{\bar{p}_{t+1}^2} + \frac{a_t(0) \cdot \frac{\eta_{t+1}}{\eta_t}}{(1 - \bar{p}_{t+1})^2} + \Psi'(\bar{p}_{t+1}) &= 0 , \\
-\frac{\hat{a}_t(1) \cdot \frac{\eta_{t+1}}{\eta_t}}{p_{t+1}^2} + \frac{\hat{a}_t(0) \cdot \frac{\eta_{t+1}}{\eta_t}}{(1 - p_{t+1})^2} + \Psi'(p_{t+1}) &= 0 .
\end{aligned}$$

Since $\delta < 1/4$, by Lemma C.1 and the proof in Lemma 4.14, we have

$$\begin{aligned}
&\left| \frac{1}{p_{t+1}} - \frac{1}{\bar{p}_{t+1}} \right| \\
&\leq \frac{b_1 b_2}{b_3} \left(1 + \frac{2}{b_3}\right) \frac{1 - \bar{p}_{t+1}}{\bar{p}_{t+1}} \frac{|\hat{a}_t(0) - a_t(0)| \cdot \frac{\eta_{t+1}}{\eta_t}}{a_t(0) \cdot \frac{\eta_{t+1}}{\eta_t}} \\
&\quad + \frac{2b_1^2 b_2}{b_3^2} \left(1 + \frac{2}{b_3}\right) \frac{1 - p_{t+1}}{p_{t+1}} \frac{|\hat{a}_t(1) - a_t(1)| \cdot \frac{\eta_{t+1}}{\eta_t}}{a_t(1) \cdot \frac{\eta_{t+1}}{\eta_t}} \quad (\text{Lemma C.1})
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{b_1 b_2}{b_3} \left(1 + \frac{2}{b_3}\right) \frac{\left(1 + b_1(b_2/6)^{1/4} a_t^{1/4}(0) \cdot \left(\frac{\eta_{t+1}}{\eta_t}\right)^{1/4}\right) |\hat{a}_t(0) - a_t(0)|}{a_t(0)} \\
&\quad + \frac{2b_1^2 b_2}{b_3^2} \left(1 + \frac{2}{b_3}\right) \cdot b_1(b_2/b_3)^{1/2} \left(\frac{\hat{a}_t(0) \cdot \frac{\eta_{t+1}}{\eta_t}}{\hat{a}_t(1) \cdot \frac{\eta_{t+1}}{\eta_t}}\right)^{1/2} \frac{|\hat{a}_t(1) - a_t(1)|}{a_t(1)} \quad (\text{Lemma 4.14}) \\
&\leq \frac{2b_1^2 b_2^{5/4}}{b_3} \left(1 + \frac{2}{b_3}\right) \frac{a_t^{1/4}(0) |\hat{a}_t(0) - a_t(0)|}{a_t(0)} \\
&\quad + \frac{2b_1^3 b_2^{3/2}}{b_3^{5/2}} \left(1 + \frac{2}{b_3}\right) \left(\frac{a_t(0) + \frac{\delta}{2} a_t(0)}{a_t(1) - \frac{\delta}{2} a_t(1)}\right)^{1/2} \frac{|\hat{a}_t(1) - a_t(1)|}{a_t(1)} \quad (\text{since } \eta_{t+1} \leq \eta_t) \\
&\leq \frac{2b_1^2 b_2^{5/4}}{b_3} \left(1 + \frac{2}{b_3}\right) \frac{|\hat{a}_t(0) - a_t(0)|}{a_t^{3/4}(0)} + \frac{2b_1^3 b_2^{3/2}}{b_3^{5/2}} \left(1 + \frac{2}{b_3}\right) \left(\frac{1 + \frac{\delta}{2}}{1 - \frac{\delta}{2}}\right)^{1/2} \frac{(\eta_t \kappa)^{1/2} |\hat{a}_t(1) - a_t(1)|}{a_t^{3/2}(1)} \\
&\leq \frac{2b_1^2 b_2^{5/4}}{b_3} \left(1 + \frac{2}{b_3}\right) \frac{|\hat{a}_t(0) - a_t(0)|}{a_t^{3/4}(0)} + \frac{2b_1^3 b_2^{3/2}}{b_3^{5/2}} \left(1 + \frac{2}{b_3}\right) \left(\frac{1 + \frac{1}{2} \cdot \frac{1}{4}}{1 - \frac{1}{2} \cdot \frac{1}{4}}\right)^{1/2} \frac{(\eta_t \kappa)^{1/2} |\hat{a}_t(1) - a_t(1)|}{a_t^{3/2}(1)} \\
&\leq \frac{2b_1^2 b_2^{5/4}}{b_3} \left(1 + \frac{2}{b_3}\right) \frac{|\hat{a}_t(0) - a_t(0)|}{a_t^{3/4}(0)} + \frac{4b_1^3 b_2^{3/2}}{b_3^{5/2}} \left(1 + \frac{2}{b_3}\right) \frac{(\eta_t \kappa)^{1/2} |\hat{a}_t(1) - a_t(1)|}{a_t^{3/2}(1)}. \tag{76}
\end{aligned}$$

By calculation, we have

$$\begin{aligned}
\mathbb{E} \left[\left| \frac{1}{p_{t+1}} - \frac{1}{\bar{p}_{t+1}} \right| \right] &= \mathbb{E} \left[\left| \frac{1}{p_{t+1}} - \frac{1}{\bar{p}_{t+1}} \right| \mathbf{1} [\hat{a}_t(0) \geq \hat{a}_t(1)] \right] + \mathbb{E} \left[\left| \frac{1}{p_{t+1}} - \frac{1}{\bar{p}_{t+1}} \right| \mathbf{1} [\hat{a}_t(0) \leq \hat{a}_t(1)] \right] \\
&\triangleq D_1 + D_2. \tag{77}
\end{aligned}$$

For term D_1 , by Lemma C.3, (76) and Cauchy-Schwarz inequality, we have

$$\begin{aligned}
D_1 &\leq \mathbb{E} \left[\frac{2b_1^2 b_2^{5/4}}{b_3} \left(1 + \frac{2}{b_3}\right) \frac{|\hat{a}_t(0) - a_t(0)|}{a_t^{3/4}(0)} + \frac{4b_1^3 b_2^{3/2}}{b_3^{5/2}} \left(1 + \frac{2}{b_3}\right) \frac{(\eta_t \kappa)^{1/2} |\hat{a}_t(1) - a_t(1)|}{a_t^{3/2}(1)} \right] \\
&\leq \frac{2b_1^2 b_2^{5/4}}{b_3} \left(1 + \frac{2}{b_3}\right) a_t^{-3/4}(1) \text{Var}^{1/2}(\hat{a}_t(0)) + \frac{4b_1^3 b_2^{3/2}}{b_3^{5/2}} \left(1 + \frac{2}{b_3}\right) (\eta_t \kappa)^{1/2} a_t^{-3/2}(1) \text{Var}^{1/2}(\hat{a}_t(1)) \quad (\text{Cauchy}) \\
&\leq \frac{2b_1^2 b_2^{5/4}}{b_3} \left(1 + \frac{2}{b_3}\right) a_t^{-3/4}(1) C^{1/2}(\eta_t T)^{1/4} \log(\eta_t T) \left(2 + b_1(b_2/6)^{1/4} a_t^{1/4}(1)\right)^{25/32} \\
&\quad + \frac{4b_1^3 b_2^{3/2}}{b_3^{5/2}} \left(1 + \frac{2}{b_3}\right) (\eta_t \kappa)^{1/2} a_t^{-3/2}(1) C^{1/2}(\eta_t T)^{1/4} \log(\eta_t T) \left(2 + b_1(b_2/6)^{1/4} a_t^{1/4}(0)\right)^{25/32} \quad (\text{Lemma C.3}) \\
&\leq \frac{2^{57/32} C^{1/2} b_1^{89/32} b_2^{185/128}}{b_3} \left(1 + \frac{2}{b_3}\right) a_t^{-3/4}(1) (\eta_t T)^{1/4} \log(\eta_t T) a_t^{25/128}(1) \\
&\quad + \frac{2^{89/32} C^{1/2} b_1^{121/32} b_2^{217/128}}{b_3^{5/2}} \left(1 + \frac{2}{b_3}\right) (\eta_t \kappa)^{1/2} a_t^{-3/2}(1) (\eta_t T)^{1/4} \log(\eta_t T) a_t^{25/128}(0) \\
&\leq \frac{2^{57/32} C^{1/2} b_1^{89/32} b_2^{185/128}}{b_3} \left(1 + \frac{2}{b_3}\right) (\eta_t T)^{1/4} \log(\eta_t T) a_t^{-71/128}(1) \\
&\quad + \frac{2^{89/32} C^{1/2} b_1^{121/32} b_2^{217/128}}{b_3^{5/2}} \left(1 + \frac{2}{b_3}\right) (\eta_t \kappa)^{89/128} (\eta_t T)^{1/4} \log(\eta_t T) a_t^{-3/2}(1). \tag{78}
\end{aligned}$$

Under event $\{\hat{a}_t(1) \geq \hat{a}_t(0)\}$, we have $p_{t+1} \geq 1/2$ and $\bar{p}_{t+1} \leq 1/2$. Then by (74), (75), Corollary B.17 and Corollary 4.15, we have

$$D_2 = \mathbb{E} \left[\left| \frac{1}{p_{t+1}} - \frac{1}{\bar{p}_{t+1}} \right| \mathbf{1} [\hat{a}_t(1) \geq \hat{a}_t(0)] \right]$$

$$\begin{aligned}
&\leq \mathbb{E} \left[\frac{1}{\bar{p}_{t+1}} \mathbf{1} [\hat{a}_t(1) \geq \hat{a}_t(0)] \right] \\
&= \frac{1}{\bar{p}_{t+1}} \Pr(\hat{a}_t(1) \geq \hat{a}_t(0)) \\
&\leq \left(2 + b_1(b_2/6)^{1/4} a_t^{1/4}(0) \right) \left(\Pr \left(\frac{|\hat{a}_t(1) - a_t(1)|}{a_t(1)} \geq \frac{\delta}{2} \right) + \Pr \left(\frac{|\hat{a}_t(0) - a_t(0)|}{a_t(0)} \geq \frac{\delta}{2} \right) \right) \quad (\text{Corollary 4.15}) \\
&\leq 2b_1 b_2^{1/4} a_t^{1/4}(0) \left(\frac{2^{57/16} C b_1^{25/16} b_2^{25/64} (\eta_t T)^{1/2} \log^2(\eta_t T)}{\delta^2 a_t^{103/64}(1)} \right. \\
&\quad \left. + \frac{2^{57/16} C b_1^{25/16} b_2^{25/64} (\eta_t T)^{1/2} \log^2(\eta_t T) (\eta_t \kappa)^{25/64}}{\delta^2 a_t^2(1)} \right) \quad (\text{by (74) and (75)}) \\
&\leq 2^{73/16} C b_1^{41/16} b_2^{41/64} (\eta_t \kappa)^{1/4} (\eta_t T)^{1/2} \log^2(\eta_t T) \delta^{-2} \left(a_t^{-103/64}(1) + a_t^{-2}(1) (\eta_t \kappa)^{25/64} \right) \\
&\leq 2^{89/16} C b_1^{41/16} b_2^{41/64} (\eta_t T)^{1/2} \log^2(\eta_t T) \delta^{-2} a_t^{-2}(1) (\eta_t \kappa)^{41/64} \quad (\text{Corollary B.17}) . \tag{79}
\end{aligned}$$

Hence by (77), (78) and (79), we have

$$\begin{aligned}
&\mathbb{E} \left[\left| \frac{1}{p_{t+1}} - \frac{1}{\bar{p}_{t+1}} \right| \right] \\
&\leq \frac{2^{57/32} C^{1/2} b_1^{89/32} b_2^{185/128}}{b_3} \left(1 + \frac{2}{b_3} \right) (\eta_t T)^{1/4} \log(\eta_t T) a_t^{-71/128}(1) \\
&\quad + \frac{2^{89/32} C^{1/2} b_1^{121/32} b_2^{217/128}}{b_3^{5/2}} \left(1 + \frac{2}{b_3} \right) (\eta_t \kappa)^{89/128} (\eta_t T)^{1/4} \log(\eta_t T) a_t^{-3/2}(1) \\
&\quad + 2^{89/16} C b_1^{41/16} b_2^{41/64} (\eta_t T)^{1/2} \log^2(\eta_t T) \delta^{-2} a_t^{-2}(1) (\eta_t \kappa)^{41/64} . \tag{80}
\end{aligned}$$

(3) $a_t(0) \geq (1 - \delta)a_t(1)$ and $a_t(1) \geq (1 - \delta)a_t(0)$.

By Chebyshev's inequality and Lemma C.3, we have

$$\begin{aligned}
\Pr \left(\frac{|\hat{a}_t(0) - a_t(0)|}{a_t(1)} \geq \delta \right) &\leq \frac{\text{Var}(\hat{a}_t(0))}{\delta^2 a_t^2(1)} \quad (\text{Chebyshev's inequality}) \\
&\leq \frac{C(\eta_t T)^{1/2} \log^2(\eta_t T) \left(2 + b_1(b_2/6)^{1/4} a_t^{1/4}(1) \right)^{25/16}}{\delta^2 a_t^2(1)} \quad (\text{Lemma C.3}) \\
&\leq \frac{2^{25/16} C b_1^{25/16} b_2^{25/64} (\eta_t T)^{1/2} \log^2(\eta_t T) a_t^{25/64}(1)}{\delta^2 a_t^2(1)} \\
&= \frac{2^{25/16} C b_1^{25/16} b_2^{25/64} (\eta_t T)^{1/2} \log^2(\eta_t T)}{\delta^2 a_t^{103/64}(1)} . \tag{81}
\end{aligned}$$

Similarly, since $\delta < 1/4$, we can prove that

$$\begin{aligned}
\Pr \left(\frac{|\hat{a}_t(1) - a_t(1)|}{a_t(0)} \geq \delta \right) &\leq \frac{2^{25/16} C b_1^{25/16} b_2^{25/64} (\eta_t T)^{1/2} \log^2(\eta_t T)}{\delta^2 a_t^{103/64}(0)} \\
&\leq \frac{2^{25/16} C b_1^{25/16} b_2^{25/64} (\eta_t T)^{1/2} \log^2(\eta_t T)}{\delta^2 (1 - \delta)^{103/64} a_t^{103/64}(1)} \\
&\leq \frac{2^{41/16} C b_1^{25/16} b_2^{25/64} (\eta_t T)^{1/2} \log^2(\eta_t T)}{\delta^2 a_t^{103/64}(1)} . \tag{82}
\end{aligned}$$

Note that under event $\{|\hat{a}_t(1) - a_t(1)| \leq \delta a_t(0)\}$ and event $\{|\hat{a}_t(0) - a_t(0)| \leq \delta a_t(1)\}$, we have

$$\frac{\hat{a}_t(0)}{\hat{a}_t(1)} \leq \frac{a_t(0) + \delta a_t(1)}{a_t(1) - \delta a_t(0)} \leq \frac{\left(1 + \frac{\delta}{1-\delta} \right) a_t(0)}{(1 - 2\delta) a_t(0)} = \frac{1}{(1 - \delta)(1 - 2\delta)} ,$$

$$\frac{\hat{a}_t(1)}{\hat{a}_t(0)} \leq \frac{a_t(1) + \delta a_t(0)}{a_t(0) - \delta a_t(1)} \leq \frac{\left(1 + \frac{\delta}{1-\delta}\right) a_t(1)}{(1-2\delta) a_t(1)} = \frac{1}{(1-\delta)(1-2\delta)} .$$

Then by Lemma C.2, under event $\{a_t(1) \leq a_t(0)\}$ and event $\{\hat{a}_t(1) \leq \hat{a}_t(0)\}$, we have

$$\begin{aligned} \left| \frac{1}{p_{t+1}} - \frac{1}{\bar{p}_{t+1}} \right| &\leq \left| \frac{1}{p_{t+1}} - 2 \right| + \left| \frac{1}{\bar{p}_{t+1}} - 2 \right| \\ &\leq \frac{b_1 b_2}{b_3} \cdot \frac{|a_t(1) - a_t(0)|}{a_t(1)} + \frac{b_1 b_2}{b_3} \cdot \frac{|\hat{a}_t(1) - \hat{a}_t(0)|}{\hat{a}_t(1)} \\ &\leq \frac{b_1 b_2}{b_3} \left| \frac{a_t(0)}{a_t(1)} - 1 \right| + \frac{b_1 b_2}{b_3} \left| \frac{\hat{a}_t(0)}{\hat{a}_t(1)} - 1 \right| \\ &\leq \frac{b_1 b_2}{b_3} \left(\frac{1}{1-\delta} - 1 \right) + \frac{b_1 b_2}{b_3} \left(\frac{1}{(1-\delta)(1-2\delta)} - 1 \right) \\ &= \frac{4b_1 b_2}{b_3} \frac{\delta}{1-2\delta} \\ &\leq \frac{8b_1 b_2}{b_3} \delta . \end{aligned}$$

Under event $\{a_t(1) \geq a_t(0)\}$ and event $\{\hat{a}_t(1) \geq \hat{a}_t(0)\}$, there holds $p_{t+1}, \bar{p}_{t+1} \geq 1/2$. Hence we have

$$\begin{aligned} \left| \frac{1}{p_{t+1}} - \frac{1}{\bar{p}_{t+1}} \right| &\leq \left| \frac{1}{p_{t+1}} - 2 \right| + \left| \frac{1}{\bar{p}_{t+1}} - 2 \right| \\ &\leq \left| \frac{\frac{1}{2} - p_{t+1}}{\frac{1}{2} p_{t+1}} \right| + \left| \frac{\frac{1}{2} - \bar{p}_{t+1}}{\frac{1}{2} \bar{p}_{t+1}} \right| \\ &\leq \left| \frac{\frac{1}{2} - p_{t+1}}{\left(1 - \frac{1}{2}\right) (1 - p_{t+1})} \right| + \left| \frac{\frac{1}{2} - \bar{p}_{t+1}}{\left(1 - \frac{1}{2}\right) (1 - \bar{p}_{t+1})} \right| \\ &\leq \left| \frac{1}{1 - p_{t+1}} - 2 \right| + \left| \frac{1}{1 - \bar{p}_{t+1}} - 2 \right| \\ &\leq \frac{8b_1 b_2}{b_3} \delta . \end{aligned}$$

Similarly we can prove that under event $\{a_t(1) \geq a_t(0)\} \cap \{\hat{a}_t(1) \leq \hat{a}_t(0)\}$ or event $\{a_t(1) \leq a_t(0)\} \cap \{\hat{a}_t(1) \geq \hat{a}_t(0)\}$, we have

$$\left| \frac{1}{p_{t+1}} - \frac{1}{\bar{p}_{t+1}} \right| \leq \frac{8b_1 b_2}{b_3} \delta . \quad (83)$$

Since $\delta < 1/4$, by (81), (82), (83), Corollary (4.15) and Hölder's inequality, we have

$$\begin{aligned} &\mathbb{E} \left[\left| \frac{1}{p_{t+1}} - \frac{1}{\bar{p}_{t+1}} \right| \right] \\ &\leq \mathbb{E} \left[\left| \frac{1}{p_{t+1}} - \frac{1}{\bar{p}_{t+1}} \right| \mathbf{1} [|\hat{a}_t(1) - a_t(1)| \leq \delta a_t(0)] \cdot \mathbf{1} [|\hat{a}_t(0) - a_t(0)| \leq \delta a_t(1)] \right] \\ &\quad + \mathbb{E} \left[\left(\frac{1}{p_{t+1}} + \frac{1}{\bar{p}_{t+1}} \right) \mathbf{1} [|\hat{a}_t(1) - a_t(1)| \geq \delta a_t(0)] \right] + \mathbb{E} \left[\left(\frac{1}{p_{t+1}} + \frac{1}{\bar{p}_{t+1}} \right) \mathbf{1} [|\hat{a}_t(0) - a_t(0)| \geq \delta a_t(1)] \right] \\ &\leq \frac{8b_1 b_2}{b_3} \delta + \left(\mathbb{E}^{1/4} \left(\frac{1}{p_{t+1}^4} \right) + \mathbb{E}^{1/4} \left(\frac{1}{\bar{p}_{t+1}^4} \right) \right) (\Pr(|\hat{a}_t(1) - a_t(1)| \geq \delta a_t(0)))^{3/4} \\ &\quad + \left(\mathbb{E}^{1/4} \left(\frac{1}{p_{t+1}^4} \right) + \mathbb{E}^{1/4} \left(\frac{1}{\bar{p}_{t+1}^4} \right) \right) (\Pr(|\hat{a}_t(0) - a_t(0)| \geq \delta a_t(1)))^{3/4} \quad (\text{by (83) and Hölder's inequality}) \\ &\leq \left[\left(\frac{2^{25/16} C b_1^{25/16} b_2^{25/64} (\eta_t T)^{1/2} \log^2(\eta_t T)}{\delta^2 a_t^{103/64}(1)} \right)^{3/4} + \left(\frac{2^{41/16} C b_1^{25/16} b_2^{25/64} (\eta_t T)^{1/2} \log^2(\eta_t T)}{\delta^2 a_t^{103/64}(1)} \right)^{3/4} \right] \end{aligned}$$

$$\begin{aligned}
& \times \left(2 \cdot \left(2 + b_1(b_2/6)^{1/4} a_t^{1/4}(0) \right) \right) + \frac{8b_1 b_2}{b_3} \delta \quad (\text{by (81), (82) and Corollary (4.15)}) \\
& \leq \frac{8b_1 b_2}{b_3} \delta + 8b_1 b_2^{1/4} a_t^{1/4}(0) \cdot \frac{2^{123/64} C^{3/4} b_1^{75/64} b_2^{75/256} (\eta_t T)^{3/8} \log^{3/2}(\eta_t T)}{\delta^{3/2} a_t^{309/256}(1)} \\
& \leq \frac{8b_1 b_2}{b_3} \delta + 2^{315/64} a_t^{1/4}(0) C^{3/4} b_1^{139/64} b_2^{139/256} (\eta_t T)^{3/8} \log^{3/2}(\eta_t T) \delta^{-3/2} a_t^{-309/256}(1) \\
& \leq \frac{8b_1 b_2}{b_3} \delta + 2^{315/64} (1 - \delta)^{-1/4} a_t^{1/4}(1) C^{3/4} b_1^{139/64} b_2^{139/256} (\eta_t T)^{3/8} \log^{3/2}(\eta_t T) \delta^{-3/2} a_t^{-309/256}(1) \\
& \leq \frac{8b_1 b_2}{b_3} \delta + 33 C^{3/4} b_1^{139/64} b_2^{139/256} (\eta_t T)^{3/8} \log^{3/2}(\eta_t T) \delta^{-3/2} a_t^{-245/256}(1) . \tag{84}
\end{aligned}$$

Hence by (73), (80) and (84), we have

$$\begin{aligned}
& \mathbb{E} \left[\left| \frac{1}{p_{t+1}} - \frac{1}{\bar{p}_{t+1}} \right| \right] \\
& \leq \tilde{C} \max \left\{ (\eta_t T)^{1/4} \log(\eta_t T) a_t^{-71/128}(1) + (\eta_t T)^{3/8} \log^{3/2}(\eta_t T) \delta^{-3/2} a_t^{-245/256}(1), \right. \\
& \quad (\eta_t T)^{1/4} \log(\eta_t T) a_t^{-71/128}(1) + (\eta_t \kappa)^{89/128} (\eta_t T)^{1/4} \log(\eta_t T) a_t^{-3/2}(1) \\
& \quad \left. + (\eta_t T)^{1/2} \log^2(\eta_t T) \delta^{-2} a_t^{-2}(1) (\eta_t \kappa)^{41/64}, \delta + (\eta_t T)^{3/8} \log^{3/2}(\eta_t T) \delta^{-3/2} a_t^{-245/256}(1) \right\} ,
\end{aligned}$$

where

$$\begin{aligned}
\tilde{C} = \max \left\{ & \left[\frac{C^{1/2} b_1^{25/32} b_2^{281/128}}{b_3} + \frac{2^{57/32} C^{1/2} b_1^{89/32} b_2^{185/128}}{b_3} \left(1 + \frac{2}{b_3} \right) \right], \\
& 2^{283/64} C^{3/4} b_1^{139/64} b_2^{139/256}, \frac{2^{57/32} C^{1/2} b_1^{89/32} b_2^{185/128}}{b_3} \left(1 + \frac{2}{b_3} \right), \\
& \frac{2^{89/32} C^{1/2} b_1^{121/32} b_2^{217/128}}{b_3^{5/2}} \left(1 + \frac{2}{b_3} \right), 2^{89/16} C b_1^{41/16} b_2^{41/64}, \\
& \left. \frac{8b_1 b_2}{b_3}, 33 C^{3/4} b_1^{139/64} b_2^{139/256} \right\} .
\end{aligned}$$

□

By selecting an appropriate constant $\delta > 0$ and a suitable threshold in Lemma C.4, we obtain the following simplified result stated in the corollary.

Corollary C.5. *Suppose T is large enough. Under Assumptions 1-3 and Condition 1, there exist a constant $K_2 > 0$ such that:*

(1) *For any $t \in [T]$ such that $a_t(1) \geq (\eta_t T)^{7/10}$, there holds:*

$$\mathbb{E} \left[\left| \frac{1}{p_{t+1}} - \frac{1}{\bar{p}_{t+1}} \right| \right] \leq K_2 (\eta_t T)^{-2/25} .$$

(2) *For any $t \in [T]$ such that $a_t(0) \geq (\eta_t T)^{7/10}$, there holds:*

$$\mathbb{E} \left[\left| \frac{1}{1 - p_{t+1}} - \frac{1}{1 - \bar{p}_{t+1}} \right| \right] \leq K_2 (\eta_t T)^{-2/25} .$$

Proof. We only prove the first part. It is easy to see that when T is large enough, $(\eta_t T)^{7/10} \geq (\eta_T T)^{7/10} \geq 5b_1^{-4}b_2^{-1}$. For $0 < \delta < 1/4$, by Lemma C.4, we have

$$\begin{aligned} & \mathbb{E} \left[\left| \frac{1}{p_{t+1}} - \frac{1}{\bar{p}_{t+1}} \right| \right] \\ & \leq \tilde{C} \max \left\{ (\eta_t T)^{1/4} \log(\eta_t T) a_t^{-71/128}(1) + (\eta_t T)^{3/8} \log^{3/2}(\eta_t T) \delta^{-3/2} a_t^{-245/256}(1), \right. \\ & \quad (\eta_t T)^{1/4} \log(\eta_t T) a_t^{-71/128}(1) + (\eta_t \kappa)^{89/128} (\eta_t T)^{1/4} \log(\eta_t T) a_t^{-3/2}(1) \\ & \quad \left. + (\eta_t T)^{1/2} \log^2(\eta_t T) \delta^{-2} a_t^{-2}(1) (\eta_t \kappa)^{41/64}, \delta + (\eta_t T)^{3/8} \log^{3/2}(\eta_t T) \delta^{-3/2} a_t^{-245/256}(1) \right\} \\ & \triangleq \tilde{C} \max \{B_1, B_2, B_3\}. \end{aligned}$$

By Corollary B.17, we have $\eta_t \kappa \lesssim \eta_t T$. Then by choosing $\delta = (\eta_t T)^{-2/25}$, we have

$$\begin{aligned} B_1 & \lesssim \log(\eta_t T) (\eta_t T)^{\frac{1}{4} - \frac{71}{128} \cdot \frac{7}{10}} + \log^{3/2}(\eta_t T) (\eta_t T)^{\frac{3}{2} \cdot \frac{2}{25} + \frac{3}{8} - \frac{245}{256} \cdot \frac{7}{10}} \\ & \lesssim (\eta_t T)^{-\frac{2}{25}}, \\ B_2 & \lesssim \log(\eta_t T) (\eta_t T)^{\frac{1}{4} - \frac{71}{128} \cdot \frac{7}{10}} + \log(\eta_t T) (\eta_t T)^{\frac{89}{128} + \frac{1}{4} - \frac{3}{2} \cdot \frac{7}{10}} + \log^2(\eta_t T) (\eta_t T)^{2 \cdot \frac{2}{25} + \frac{1}{2} - 2 \cdot \frac{7}{10} + \frac{41}{64}} \\ & \lesssim (\eta_t T)^{-\frac{2}{25}}, \\ B_3 & \lesssim (\eta_t T)^{-\frac{2}{25}} + \log^{3/2}(\eta_t T) (\eta_t T)^{\frac{3}{2} \cdot \frac{2}{25} + \frac{3}{8} - \frac{245}{256} \cdot \frac{7}{10}} \\ & \lesssim (\eta_t T)^{-\frac{2}{25}}. \end{aligned}$$

Hence there exists constant $K_2 > 0$ (independent of t and T) such that

$$\mathbb{E} \left[\left| \frac{1}{p_{t+1}} - \frac{1}{\bar{p}_{t+1}} \right| \right] \leq \max \{B_1, B_2, B_3\} \leq K_2 (\eta_t T)^{-2/25}.$$

□

Based on Corollary C.5, the following lemma verifies the first convergence result in (44).

Lemma C.6. *Under Assumptions 1-3 and Condition 1, there holds:*

$$\mathbb{E} \left[\frac{1}{T} \sum_{t=1}^T (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^2 \cdot \left| \frac{1}{p_t} - \frac{1}{\bar{p}_t} \right| \right] \rightarrow 0.$$

Proof. Let $\mathcal{G} = \{2 \leq t \leq T : a_{t-1}(1) \geq (\eta_{t-1} T)^{7/10}\}$. Then we can obtain the following decomposition:

$$\begin{aligned} & \mathbb{E} \left(\frac{1}{T} \sum_{t=1}^T (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^2 \cdot \left| \frac{1}{p_t} - \frac{1}{\bar{p}_t} \right| \right) \\ & = \frac{1}{T} \sum_{t \in \bar{\mathcal{G}}} (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^2 \cdot \mathbb{E} \left[\left| \frac{1}{p_t} - \frac{1}{\bar{p}_t} \right| \right] + \frac{1}{T} \sum_{t \in \mathcal{G}} (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^2 \cdot \mathbb{E} \left[\left| \frac{1}{p_t} - \frac{1}{\bar{p}_t} \right| \right] \\ & \triangleq B_1 + B_2. \end{aligned} \tag{85}$$

Define $t_0 = 1$ and suppose $\bar{\mathcal{G}} = \{t_1, \dots, t_N\}$, where $t_1 < \dots < t_N$. For any $t \in [T]$, by Cauchy-Schwarz inequality, (55), Corollary B.4, Corollary 4.15, Corollary B.17, Assumption 1 and Assumption 3, we have

$$\mathbb{E}[\hat{A}_t(1)] - \mathbb{E}[\hat{A}_{t-1}(1)]$$

$$\begin{aligned}
&= \mathbb{E} \left[\frac{\mathbf{1}[Z_t = 1]}{p_t} \cdot (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle)^2 \right] \\
&= \mathbb{E} [(y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle)^2] \\
&= (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^2 + \sum_{s=1}^{t-1} \Pi_{t,s}^2 y_s^2(1) \mathbb{E} \left[\frac{1}{p_s} - 1 \right] \\
&\lesssim y_t^2(1) + \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle^2 + R_t((t-1) \vee \eta_t^{-1})^{-1} \sum_{s=1}^{t-1} |\Pi_{t,s}| R_s y_s^2(1) \eta_s^{1/4} T^{1/4} \quad (\text{Corollary B.4, 4.15 and B.17}) \\
&\lesssim T^{1/2} + R_t T^{1/4} + R_t \eta_t T^{1/8} \sum_{s=1}^{t-1} |\Pi_{t,s}| R_s \cdot R_s^{-1/2} y_s^2(1) \quad (\text{Assumption 1 and (55)}) \\
&\lesssim T^{1/2} + T^{1/2} (T R_t^{-4})^{-1/4} + T^{1/8-1/2} R_t^{1-2+1-1/2} \left(\sum_{s=1}^{t-1} \Pi_{t,s}^2 \right)^{1/2} \left(\sum_{s=1}^{t-1} y_s^4(1) \right)^{1/2} \quad (\text{Cauchy-Schwarz}) \\
&\lesssim T^{1/2} + T^{1/8-1/2+1/2} R_t^{-1/2} \Pi_{t,t}^{1/2} \quad (\text{Assumption 1 and Assumption 3}) \\
&\lesssim T^{1/2} + T^{1/8} R_t^{-1/2+1} \eta_t^{1/2} \quad (\text{Corollary B.4}) \\
&= T^{1/2} + T^{1/8-1/4} R_t^{-1/2+1-1} \\
&\lesssim T^{1/2} + T^{-1/8} R_t^{-1/2} \\
&\lesssim T^{1/2}.
\end{aligned}$$

Hence there exists constant $K_3 > 0$, such that $\mathbb{E}[\hat{A}_t(1)] - \mathbb{E}[\hat{A}_{t-1}(1)] \leq K_3 T^{1/2}$ for any $t \in [T]$. Moreover, the proof also establishes that $(y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^2 \leq \mathbb{E}[\hat{A}_t(1)] - \mathbb{E}[\hat{A}_{t-1}(1)]$. Hence by the definition of $\bar{\mathcal{G}}$, Corollary 4.15, Corollary B.17 and Assumption 3, we have

$$\begin{aligned}
B_1 &\lesssim T^{-1} \sum_{t \in \bar{\mathcal{G}}} (\mathbb{E}[\hat{A}_t(1)] - \mathbb{E}[\hat{A}_{t-1}(1)]) (\eta_t T)^{1/4} \quad (\text{Corollary 4.15 and Corollary B.17}) \\
&= T^{-1} \sum_{k=1}^N (\mathbb{E}[\hat{A}_{t_k}(1)] - \mathbb{E}[\hat{A}_{t_{k-1}}(1)]) (\eta_{t_k} T)^{1/4} \\
&= T^{-1} \sum_{k=1}^{N-1} \mathbb{E}[\hat{A}_{t_k}(1)] \left((\eta_{t_k} T)^{1/4} - (\eta_{t_{k+1}} T)^{1/4} \right) + T^{-1} \mathbb{E}[\hat{A}_{t_N}(1)] (\eta_{t_N} T)^{1/4} - T^{-1} \mathbb{E}[\hat{A}_{t_0}(1)] (\eta_{t_1} T)^{1/4} \\
&\leq T^{-1} \sum_{k=1}^{N-1} \left(\eta_{t_k-1}^{-1} \cdot \eta_{t_k-1} \mathbb{E}[\hat{A}_{t_k-1}(1)] + K_3 T^{1/2} \right) \left((\eta_{t_k} T)^{1/4} - (\eta_{t_{k+1}} T)^{1/4} \right) \\
&\quad + T^{-1} \left(\eta_{t_N-1}^{-1} \cdot \eta_{t_N-1} \mathbb{E}[\hat{A}_{t_N-1}(1)] + K_3 T^{1/2} \right) (\eta_{t_N} T)^{1/4} \\
&\leq T^{-1} \sum_{k=1}^{N-1} \eta_{t_k-1}^{-1} (\eta_{t_k-1} T)^{7/10} \left((\eta_{t_k} T)^{1/4} - (\eta_{t_{k+1}} T)^{1/4} \right) + T^{-1} \eta_{t_N-1}^{-1} (\eta_{t_N-1} T)^{7/10} (\eta_{t_N} T)^{1/4} \\
&\quad + K_3 T^{-1/2} \left[\sum_{k=1}^{N-1} \left((\eta_{t_k} T)^{1/4} - (\eta_{t_{k+1}} T)^{1/4} \right) + (\eta_{t_N} T)^{1/4} \right] \quad (\text{by the definition of } \bar{\mathcal{G}}) \\
&\leq K_3 T^{-1/2} (\eta_{t_1} T)^{1/4} + T^{-1} \sum_{k=1}^{N-1} \eta_{t_k}^{-1} (\eta_{t_k} T)^{7/10} \left((\eta_{t_k} T)^{1/4} - (\eta_{t_{k+1}} T)^{1/4} \right) + T^{-1} \eta_{t_N}^{-1} (\eta_{t_N} T)^{7/10} (\eta_{t_N} T)^{1/4} \\
&\leq K_3 T^{-1/2+1/8} + (\eta_{t_N} T)^{-1+7/10+1/4} + \sum_{k=1}^{N-1} (\eta_{t_k} T)^{-1+7/10} \left((\eta_{t_k} T)^{1/4} - (\eta_{t_{k+1}} T)^{1/4} \right) \\
&\leq o(1) + (\eta_T T)^{-1/20} + \sum_{k=1}^{N-1} \left((\eta_{t_k} T)^{1/4} \right)^{-6/5} \left((\eta_{t_k} T)^{1/4} - (\eta_{t_{k+1}} T)^{1/4} \right) \quad (\text{Assumption 3})
\end{aligned}$$

$$\begin{aligned}
&\leq o(1) + \sum_{k=1}^{N-1} \int_{(\eta_{t_{k+1}} T)^{1/4}}^{(\eta_{t_k} T)^{1/4}} x^{-6/5} dx \\
&\leq o(1) + \int_{(\eta_{t_N} T)^{1/4}}^{(\eta_{t_1} T)^{1/4}} x^{-6/5} dx \\
&\lesssim o(1) + (\eta_{t_N} T)^{-\frac{1}{4} \cdot \frac{1}{5}} \\
&\leq o(1) + (\eta_T T)^{-1/20} \\
&\rightarrow 0 \quad (\text{Assumption 3}) .
\end{aligned} \tag{86}$$

By the definition of set \mathcal{G} and Corollary C.5, we have

$$\max_{t \in \mathcal{G}} \mathbb{E} \left[\left| \frac{1}{p_t} - \frac{1}{\bar{p}_t} \right| \right] \lesssim \max_{t \in \mathcal{G}} (\eta_{t-1} T)^{-2/25} \leq (\eta_T T)^{-2/25} .$$

Hence by Lemma B.11 and Assumption 3, we can bound B_2 by:

$$\begin{aligned}
B_2 &= \frac{1}{T} \sum_{t \in \mathcal{G}} (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^2 \cdot \mathbb{E} \left[\left| \frac{1}{p_t} - \frac{1}{\bar{p}_t} \right| \right] \\
&\leq T^{-1} \sum_{t \in \mathcal{G}} (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^2 (\eta_T T)^{-2/25} \\
&\lesssim T^{-1} \cdot T (\eta_T T)^{-2/25} \quad (\text{Lemma B.11}) \\
&\leq (\eta_T T)^{-2/25} \\
&\rightarrow 0 \quad (\text{Assumption 3}) .
\end{aligned} \tag{87}$$

Hence by (85), (86), (87), we have

$$\mathbb{E} \left(\frac{1}{T} \sum_{t=1}^T (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^2 \cdot \left| \frac{1}{p_t} - \frac{1}{\bar{p}_t} \right| \right) = B_1 + B_2 \rightarrow 0 .$$

□

Before verifying the remaining convergence results in (44), we first establish the explicit form of the tracking error terms.

Lemma C.7. *The expected prediction tracking terms can be computed as*

$$\begin{aligned}
\mathbb{E} [\langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) - \boldsymbol{\beta}_t(1) \rangle^2] &= \sum_{s=1}^{t-1} \Pi_{t,s}^2 y_s^2(1) \mathbb{E} \left[\frac{1}{p_s} - 1 \right] , \\
\mathbb{E} [\langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(0) - \boldsymbol{\beta}_t(0) \rangle^2] &= \sum_{s=1}^{t-1} \Pi_{t,s}^2 y_s^2(0) \mathbb{E} \left[\frac{1}{1-p_s} - 1 \right] .
\end{aligned}$$

The remaining convergence results in (44) are easier to establish using the previous lemmas.

Lemma C.8. *Under Assumptions 1-3 and Condition 1, there holds:*

$$\mathbb{E} \left[\frac{1}{T} \sum_{t=1}^T \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) - \boldsymbol{\beta}_t^*(1) \rangle^2 \cdot \left| \frac{1}{p_t} - \frac{1}{\bar{p}_t} \right| \right] \rightarrow 0 .$$

Proof. By Lemma B.15, Corollary 4.15, Corollary B.17, Proposition 5.4 and Lemma C.7, We have

$$\mathbb{E} \left[\frac{1}{T} \sum_{t=1}^T \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) - \boldsymbol{\beta}_t^*(1) \rangle^2 \cdot \left| \frac{1}{p_t} - \frac{1}{\bar{p}_t} \right| \right]$$

$$\begin{aligned}
&\lesssim T^{-1+7/26} \mathbb{E} \left[\sum_{t=1}^T R_t^{-4/11} \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) - \boldsymbol{\beta}_t^*(1) \rangle^2 \right] \quad (\text{Proposition 5.4}) \\
&= T^{-1+7/26} \sum_{t=1}^T R_t^{-4/11} \sum_{s=1}^{t-1} \Pi_{t,s}^2 y_s^2(1) \mathbb{E} \left[\frac{1}{p_s} - 1 \right] \quad (\text{Lemma C.7}) \\
&\lesssim T^{-1+7/26+1/4} \sum_{t=1}^T R_t^{-4/11} \sum_{s=1}^{t-1} \Pi_{t,s}^2 y_s^2(1) \eta_s^{1/4} \quad (\text{Corollary 4.15 and Corollary B.17}) \\
&\lesssim T^{-1+7/26+1/4-1/8} \sum_{t=1}^T R_t^{-4/11} \sum_{s=1}^{t-1} R_s^{-1/2} \Pi_{t,s}^2 y_s^2(1) \\
&\lesssim T^{-1+7/26+1/4-1/8} \cdot T^{3/4-(4/11+1/2)/4} \quad (\text{Lemma B.15}) \\
&= T^{-41/572} \\
&\rightarrow 0 .
\end{aligned}$$

□

Lemma C.9. *Under Assumptions 1-3 and Condition 1, there holds:*

$$\mathbb{E} \left[\frac{1}{T} \sum_{t=1}^T \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) - \boldsymbol{\beta}_t^*(1) \rangle^2 \cdot \left| \frac{1}{p_t} - 1 \right| \right] \rightarrow 0 .$$

Proof. Note that the almost sure bound in Proposition 5.4 also serves as an upper bound for $1/\bar{p}_t$. Then by the same proof in Lemma C.8, the result is proved. □

Lemma C.10. *Under Assumptions 1-3 and Condition 1, there holds:*

$$\mathbb{E} \left[\frac{1}{T} \sum_{t=1}^T (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle) \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) - \boldsymbol{\beta}_t^*(1) \rangle \cdot \left| \frac{1}{p_t} - \frac{1}{\bar{p}_t} \right| \right] \rightarrow 0 .$$

Proof. By Lemma C.6, Lemma C.8 and Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\mathbb{E} \left[\frac{1}{T} \sum_{t=1}^T (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle) \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) - \boldsymbol{\beta}_t^*(1) \rangle \cdot \left| \frac{1}{p_t} - \frac{1}{\bar{p}_t} \right| \right] \\
&\leq \left(\mathbb{E} \left[\frac{1}{T} \sum_{t=1}^T (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^2 \cdot \left| \frac{1}{p_t} - \frac{1}{\bar{p}_t} \right| \right] \right)^{1/2} \left(\mathbb{E} \left[\frac{1}{T} \sum_{t=1}^T \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) - \boldsymbol{\beta}_t^*(1) \rangle^2 \cdot \left| \frac{1}{p_t} - \frac{1}{\bar{p}_t} \right| \right] \right)^{1/2} \\
&\rightarrow 0 .
\end{aligned}$$

□

Lemma C.11. *Under Assumptions 1-3 and Condition 1, there holds*

$$\mathbb{E} \left[\frac{1}{T} \sum_{t=1}^T (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle) \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) - \boldsymbol{\beta}_t^*(1) \rangle \cdot \left| \frac{1}{\bar{p}_t} - 1 \right| \right] \rightarrow 0 .$$

Proof. By Cauchy-Schwarz inequality, Lemma B.11, Lemma B.15, Lemma C.7, Lemma 4.14 and Corollary B.17, we have

$$\begin{aligned}
&\mathbb{E} \left[\frac{1}{T} \sum_{t=1}^T |y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle| |\langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) - \boldsymbol{\beta}_t^*(1) \rangle| \cdot \left| \frac{1}{\bar{p}_t} - 1 \right| \right] \\
&\lesssim \mathbb{E} \left[\frac{1}{T} \sum_{t=1}^T |y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle| |\langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) - \boldsymbol{\beta}_t^*(1) \rangle| \cdot \eta_t^{1/4} T^{1/4} \right] \quad (\text{Lemma 4.14 and Corollary B.17})
\end{aligned}$$

$$\begin{aligned}
&\lesssim T^{1/8} \mathbb{E} \left[\frac{1}{T} \sum_{t=1}^T R_t^{-1/2} |y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle| |\langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) - \boldsymbol{\beta}_t^*(1) \rangle| \right] \\
&\lesssim T^{-1+1/8} \left(\sum_{t=1}^T (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^2 \right)^{1/2} \left(\mathbb{E} \left[\sum_{t=1}^T R_t^{-1} \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) - \boldsymbol{\beta}_t^*(1) \rangle^2 \right] \right)^{1/2} \quad (\text{Cauchy-Schwarz}) \\
&\lesssim T^{-1+1/8+1/2} \left(\sum_{t=1}^T R_t^{-1} \sum_{s=1}^{t-1} \Pi_{t,s}^2 y_s^2(1) \mathbb{E} \left[\frac{1}{p_s} - 1 \right] \right)^{1/2} \quad (\text{Lemma B.11 and Lemma C.7}) \\
&\lesssim T^{-1+1/8+1/2+1/16} \left(\sum_{t=1}^T R_t^{-1} \sum_{s=1}^{t-1} R_s^{-1/2} \Pi_{t,s}^2 y_s^2(1) \right)^{1/2} \quad (\text{Lemma 4.14 and Corollary B.17}) \\
&\lesssim T^{-1+1/8+1/2+1/16} \cdot \left(T^{3/4-(1+1/2)/4} \right)^{1/2} \quad (\text{Lemma B.15}) \\
&= T^{-1/8} \\
&\rightarrow 0 .
\end{aligned}$$

□

Based on Lemma C.6-C.11, we now verify the first convergence result in (42) in the following lemma.

Lemma C.12. *Under Assumptions 1-3 and Condition 1, there holds:*

$$\begin{aligned}
&\frac{1}{T} \left[\sum_{t=1}^T \{y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle\}^2 \cdot \left(\frac{1}{p_t} - 1 \right) - \mathbb{E} \left[\sum_{t=1}^T \{y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle\}^2 \cdot \left(\frac{1}{p_t} - 1 \right) \right] \right] \xrightarrow{p} 0 , \\
&\frac{1}{T} \left[\sum_{t=1}^T \{y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle\}^2 \cdot \left(\frac{1}{1-p_t} - 1 \right) - \mathbb{E} \left[\sum_{t=1}^T \{y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle\}^2 \cdot \left(\frac{1}{1-p_t} - 1 \right) \right] \right] \xrightarrow{p} 0 .
\end{aligned}$$

Proof. By Markov's inequality and Lemma C.6, we have

$$\left| \frac{1}{T} \sum_{t=1}^T (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^2 \cdot \left(\frac{1}{p_t} - \frac{1}{\bar{p}_t} \right) \right| \leq \frac{1}{T} \sum_{t=1}^T (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^2 \cdot \left| \frac{1}{p_t} - \frac{1}{\bar{p}_t} \right| \xrightarrow{p} 0 .$$

On the other hand, we also have

$$\left| \mathbb{E} \left[\frac{1}{T} \sum_{t=1}^T (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^2 \cdot \left(\frac{1}{p_t} - \frac{1}{\bar{p}_t} \right) \right] \right| \leq \mathbb{E} \left[\frac{1}{T} \sum_{t=1}^T (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^2 \cdot \left| \frac{1}{p_t} - \frac{1}{\bar{p}_t} \right| \right] \rightarrow 0 .$$

These two convergence results imply that

$$\frac{1}{T} \sum_{t=1}^T (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^2 \cdot \left(\frac{1}{p_t} - \frac{1}{\bar{p}_t} \right) - \mathbb{E} \left[\frac{1}{T} \sum_{t=1}^T (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^2 \cdot \left(\frac{1}{p_t} - \frac{1}{\bar{p}_t} \right) \right] \xrightarrow{p} 0 .$$

By Lemma C.8, Lemma C.9, Lemma C.10 and Lemma C.11, we can similarly prove the other four convergence results, which verifies the following:

$$\frac{1}{T} \left[\sum_{t=1}^T \{y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle\}^2 \cdot \left(\frac{1}{p_t} - 1 \right) - \mathbb{E} \left[\sum_{t=1}^T \{y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle\}^2 \cdot \left(\frac{1}{p_t} - 1 \right) \right] \right] \xrightarrow{p} 0 .$$

The second convergence is proved due to the symmetry. □

We can similarly verify the final convergence result in (42) in the following lemma.

Lemma C.13. Under Assumptions 1-3 and Condition 1, there holds:

$$\frac{1}{T} \left[\sum_{t=1}^T \{y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle\} \{y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle\} - \mathbb{E} \left[\sum_{t=1}^T \{y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle\} \{y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle\} \right] \right] \xrightarrow{P} 0 .$$

Proof. We proof the result by similar method as in Lemma C.12. First, we decompose the random term:

$$\begin{aligned} & \frac{1}{T} \sum_{t=1}^T (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle) (y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle) \\ &= \frac{1}{T} \sum_{t=1}^T (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle) (y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(0) \rangle) + \frac{1}{T} \sum_{t=1}^T (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle) \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(0) - \boldsymbol{\beta}_t(0) \rangle \\ &= \frac{1}{T} \sum_{t=1}^T (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle) (y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(0) \rangle) + \frac{1}{T} \sum_{t=1}^T (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle) \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(0) - \boldsymbol{\beta}_t(0) \rangle \\ &\quad + \frac{1}{T} \sum_{t=1}^T (y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(0) \rangle) \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) - \boldsymbol{\beta}_t(1) \rangle . \end{aligned}$$

Note that the first term is nonrandom. Hence by Markov's inequality, it suffices to show that

$$\begin{aligned} & \mathbb{E} \left[\frac{1}{T} \sum_{t=1}^T |y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle| |\langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(0) - \boldsymbol{\beta}_t(0) \rangle| \right] \rightarrow 0 , \\ & \mathbb{E} \left[\frac{1}{T} \sum_{t=1}^T |y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(0) \rangle| |\langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) - \boldsymbol{\beta}_t(1) \rangle| \right] \rightarrow 0 . \end{aligned}$$

By Lemma C.7, Lemma B.15, Lemma 4.14 and Corollary B.17, we have

$$\begin{aligned} \mathbb{E} \left[\sum_{t=1}^T \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) - \boldsymbol{\beta}_t(1) \rangle^2 \right] &= \sum_{t=1}^T \sum_{s=1}^{t-1} \Pi_{t,s}^2 y_s^2(1) \mathbb{E} \left[\frac{1}{p_s} - 1 \right] \\ &\lesssim T^{1/8} \sum_{t=1}^T \sum_{s=1}^{t-1} R_s^{-1/2} \Pi_{t,s}^2 y_s^2(1) \\ &\lesssim T^{1/8} \cdot T^{3/4-1/2/4} \\ &= T^{3/4} . \end{aligned} \tag{88}$$

A similar result can be derived for $\mathbb{E} \left[\sum_{t=1}^T \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(0) - \boldsymbol{\beta}_t(0) \rangle^2 \right]$. Hence by (88), Cauchy-Schwarz inequality, Lemma B.11 and Assumption 3, we have

$$\begin{aligned} & \mathbb{E} \left[\frac{1}{T} \sum_{t=1}^T |y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle| |\langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(0) - \boldsymbol{\beta}_t(0) \rangle| \right] \\ &\leq \frac{1}{T} \sum_{t=1}^T \left(\mathbb{E} (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle)^2 \right)^{1/2} \left(\mathbb{E} \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(0) - \boldsymbol{\beta}_t(0) \rangle^2 \right)^{1/2} \quad (\text{Cauchy-Schwarz inequality}) \\ &\leq \frac{1}{T} \left(\mathbb{E} \left[\sum_{t=1}^T (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle)^2 \right] \right)^{1/2} \left(\sum_{t=1}^T \mathbb{E} \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(0) - \boldsymbol{\beta}_t(0) \rangle^2 \right)^{1/2} \quad (\text{Cauchy-Schwarz inequality}) \\ &\lesssim T^{-1+3/8} \cdot \left(\mathbb{E} \left[\sum_{t=1}^T (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle + \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) - \boldsymbol{\beta}_t(1) \rangle)^2 \right] \right)^{1/2} \quad (\text{by (88)}) \\ &= T^{-5/8} \cdot \left(\sum_{t=1}^T (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^2 + 2 \sum_{t=1}^T (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle) \mathbb{E} [\langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) - \boldsymbol{\beta}_t(1) \rangle] \right) \end{aligned}$$

$$\begin{aligned}
& + \mathbb{E} \left[\sum_{t=1}^T \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) - \boldsymbol{\beta}_t(1) \rangle^2 \right]^{1/2} \\
& = T^{-5/8} \cdot \left(\sum_{t=1}^T (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^2 + \mathbb{E} \left[\sum_{t=1}^T \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) - \boldsymbol{\beta}_t(1) \rangle^2 \right] \right)^{1/2} \\
& \lesssim T^{-5/8+1/2} \quad (\text{Lemma B.11 and (88)}) \\
& = T^{-1/8} \\
& \rightarrow 0 .
\end{aligned}$$

Hence the result is proved by similar argument as in Lemma C.12. \square

By Lemma C.12 and Lemma C.13, we can verify the stable variance condition in the following lemma:

Lemma C.14. *Under Assumptions 1-4 and Condition 1, there holds: $V_T^2 \xrightarrow{P} 1$.*

Proof. By Lemma C.12, Lemma C.13 and the simplified form of V_T^2 , the result is proved. \square

C.2.2 Conditional Lyapunov Condition

We first derive a simplified form of the conditional Lyapunov condition. Setting $\delta = 2$, we obtain the following result by applying the Cauchy-Schwarz inequality:

$$\begin{aligned}
\sum_{t=1}^T \mathbb{E}[X_{t,T}^4 | \mathcal{F}_{t-1,T}] & \lesssim \frac{1}{T^4 (\text{Var}(\hat{\tau}))^2} \sum_{t=1}^T \mathbb{E} \left[(y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle)^4 \cdot \left(\frac{\mathbf{1}[Z_t = 1]}{p_t} - 1 \right)^4 \middle| \mathcal{F}_{t-1,T} \right] \\
& \quad + \frac{1}{T^4 (\text{Var}(\hat{\tau}))^2} \sum_{t=1}^T \mathbb{E} \left[(y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle)^4 \cdot \left(\frac{\mathbf{1}[Z_t = 0]}{1-p_t} - 1 \right)^4 \middle| \mathcal{F}_{t-1,T} \right] \quad (\text{Cauchy-Schwarz}) \\
& \lesssim \frac{1}{T^4 (\text{Var}(\hat{\tau}))^2} \sum_{t=1}^T \left[(y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle)^4 \cdot \frac{1}{p_t^3} + (y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle)^4 \cdot \frac{1}{(1-p_t)^3} \right] .
\end{aligned}$$

By Markov's inequality and non-superefficiency condition (Corollary 5.2), it suffices to prove that

$$\begin{aligned}
& \frac{1}{T^2} \sum_{t=1}^T \mathbb{E} \left[(y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle)^4 \cdot \frac{1}{p_t^3} \right] \rightarrow 0 , \\
& \frac{1}{T^2} \sum_{t=1}^T \mathbb{E} \left[(y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle)^4 \cdot \frac{1}{(1-p_t)^3} \right] \rightarrow 0 .
\end{aligned}$$

Owing to the symmetry between the treated and control groups, we only need to prove the first convergence. Using the inequality $(a+b)^4 \leq 8a^4 + 8b^4$, it suffices to show that

$$\begin{aligned}
& \frac{1}{T^2} \sum_{t=1}^T \mathbb{E} \left[(y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^4 \cdot \frac{1}{p_t^3} \right] \rightarrow 0 , \\
& \frac{1}{T^2} \sum_{t=1}^T \mathbb{E} \left[\langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) - \boldsymbol{\beta}_t^*(1) \rangle^4 \cdot \frac{1}{p_t^3} \right] \rightarrow 0 . \tag{89}
\end{aligned}$$

The two convergence results in (89) are proved separately in Lemma C.15 and Lemma C.18.

Lemma C.15. *Under Assumptions 1-3 and Condition 1, there holds:*

$$\frac{1}{T^2} \sum_{t=1}^T \mathbb{E} \left[(y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^4 \cdot \frac{1}{p_t^3} \right] \rightarrow 0 .$$

Proof. By Lemma B.7, Corollary 4.15, Corollary B.17 and Assumption 3, we have

$$\begin{aligned}
& \frac{1}{T^2} \sum_{t=1}^T \mathbb{E} \left[(y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^4 \cdot \frac{1}{p_t^3} \right] \\
&= \frac{1}{T^2} \sum_{t=1}^T (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^4 \mathbb{E} \left[\frac{1}{p_t^3} \right] \\
&\lesssim T^{-2} \sum_{t=1}^T (\eta_t T)^{3/4} (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^4 \quad (\text{Corollary 4.15 and Corollary B.17}) \\
&\leq T^{-2+3/4} \sum_{t=1}^T \eta_t^{-1/4} \cdot \eta_t (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^4 \\
&\leq T^{-5/4} \eta_T^{-1/4} \cdot R_T^2 T^{1/2} \quad (\text{by Lemma B.7 and } \eta_T \leq \eta_t) \\
&= (T R_T^{-4})^{-5/8} \\
&\rightarrow 0 \quad (\text{Assumption 3}) .
\end{aligned}$$

□

Before verifying the second convergence result in (89), we first establish the following result, which is a direct corollary of Lemma C.3.

Corollary C.16. *Under Assumptions 1-3 and Condition 1, for any $t \in [T]$, there holds:*

$$\max \left\{ \mathbb{E} \hat{A}_t^2(1), \mathbb{E} \hat{A}_t^2(0) \right\} \leq \tilde{\kappa} ,$$

where $\tilde{\kappa} = (4c_1^4 + o(1))T^2$ is a constant that does not depend on t .

Proof. We only prove the result for $\mathbb{E} \hat{A}_t^2(1)$. By Lemma C.3 and Assumption 3, we have

$$\begin{aligned}
\text{Var}(\hat{A}_t(1)) &\lesssim \eta_t^{-2} (\eta_t T)^{1/2} \log^2(\eta_t T) \left(2 + b_1(b_2/6)^{1/4} a_t^{1/4}(0) \right)^{25/16} \quad (\text{Lemma C.3}) \\
&\lesssim T^2 \cdot (\eta_t T)^{-2} (\eta_t T)^{1/2} \log^2(\eta_t T) (\eta_t T)^{\frac{1}{4} \cdot \frac{25}{16}} \\
&= T^2 \cdot (\eta_t T)^{-71/64} \log^2(\eta_t T) \\
&= o(T^2) \quad (\text{Assumption 3}) .
\end{aligned}$$

Hence by Corollary B.17 we have

$$\mathbb{E} \hat{A}_t^2(1) = (\mathbb{E} \hat{A}_t(1))^2 + \text{Var}(\hat{A}_t(1)) \leq (2c_1^2 + o(1))^2 T^2 + o(T^2) = (4c_1^4 + o(1))T^2 .$$

□

Based on Corollary C.16, we refine the result of Lemma B.18 in the following lemma.

Lemma C.17. *Under Assumptions 1-3 and Condition 1, there holds:*

$$\max_{1 \leq s_3 \neq s_2 \neq s_1 \leq T} \left| \mathbb{E} \left[\left(\frac{\mathbf{1}[Z_{s_1} = 1]}{p_{s_1}} - 1 \right)^2 \left(\frac{\mathbf{1}[Z_{s_2} = 1]}{p_{s_2}} - 1 \right) \left(\frac{\mathbf{1}[Z_{s_3} = 1]}{p_{s_3}} - 1 \right) \right] \right| = \mathcal{O}(T^{81/512}) .$$

Proof. By Corollary C.16 and similar method in Corollary 4.15, we can prove that for any $0 \leq k \leq 8$ and any $t \in [T]$, there holds:

$$\max \left\{ \mathbb{E} \left[\frac{1}{p_t^k} \right], \mathbb{E} \left[\frac{1}{(1-p_t)^k} \right] \right\} \leq (\eta_t T)^{k/4} .$$

By this result, we can sharpen the uniform upper bound derived in Lemma B.18. By similar method as in the proof of Lemma B.18, we can show

$$\begin{aligned}
& \max_{1 \leq s_3 \neq s_2 \neq s_1 \leq T} \left| E \left[\left(\frac{\mathbf{1}[Z_{s_1} = 1]}{p_{s_1}} - 1 \right)^2 \left(\frac{\mathbf{1}[Z_{s_2} = 1]}{p_{s_2}} - 1 \right) \left(\frac{\mathbf{1}[Z_{s_3} = 1]}{p_{s_3}} - 1 \right) \right] \right| \\
& \lesssim \max_{1 \leq s_3 < s_2 < s_1 \leq T} E \left[\frac{1}{p_{s_1}} \frac{\mathbf{1}[Z_{s_2} = 1]}{p_{s_2}} \frac{\mathbf{1}[Z_{s_3} = 1]}{p_{s_3}} \right] \\
& \leq \max_{1 \leq s_3 < s_2 < s_1 \leq T} \left(E \left[\frac{1}{p_{s_1}^8} \right] \right)^{1/8} \left(E \left[\frac{\mathbf{1}[Z_{s_2} = 1]}{p_{s_2}^{8/7}} \frac{\mathbf{1}[Z_{s_3} = 1]}{p_{s_3}^{8/7}} \right] \right)^{7/8} \\
& \leq \max_{1 \leq s_3 < s_2 < s_1 \leq T} \left(E \left[\frac{1}{p_{s_1}^8} \right] \right)^{1/8} \left(E \left[\frac{1}{p_{s_2}^{1/7}} \frac{\mathbf{1}[Z_{s_3} = 1]}{p_{s_3}^{8/7}} \right] \right)^{7/8} \\
& \leq \max_{1 \leq s_3 < s_2 < s_1 \leq T} \left(E \left[\frac{1}{p_{s_1}^8} \right] \right)^{1/8} \left(E \left[\frac{1}{p_{s_2}^8} \right] \right)^{\frac{7}{8} \cdot \frac{1}{56}} \left(E \left[\frac{\mathbf{1}[Z_{s_3} = 1]}{p_{s_3}^{\frac{8}{7} \cdot \frac{56}{55}}} \right] \right)^{\frac{7}{8} \cdot \frac{55}{56}} \\
& = \max_{1 \leq s_3 < s_2 < s_1 \leq T} \left(E \left[\frac{1}{p_{s_1}^8} \right] \right)^{1/8} \left(E \left[\frac{1}{p_{s_2}^8} \right] \right)^{\frac{7}{8} \cdot \frac{1}{56}} \left(E \left[\frac{1}{p_{s_3}^{\frac{9}{55}}} \right] \right)^{\frac{7}{8} \cdot \frac{55}{56}} \\
& \lesssim (\eta_1 T)^{(8 \cdot \frac{1}{8} + 8 \cdot \frac{7}{8} \cdot \frac{1}{56} + \frac{9}{55} \cdot \frac{7}{8} \cdot \frac{55}{56})/4} \\
& = (\eta_1 T)^{81/256} .
\end{aligned}$$

Hence we have

$$\max_{1 \leq s_3 \neq s_2 \neq s_1 \leq T} \left| E \left[\left(\frac{\mathbf{1}[Z_{s_1} = 1]}{p_{s_1}} - 1 \right)^2 \left(\frac{\mathbf{1}[Z_{s_2} = 1]}{p_{s_2}} - 1 \right) \left(\frac{\mathbf{1}[Z_{s_3} = 1]}{p_{s_3}} - 1 \right) \right] \right| = \mathcal{O}(T^{81/512}) .$$

□

We now verify the second convergence result in (89) with the help of Lemma C.17.

Lemma C.18. *Under Assumptions 1-3 and Condition 1, there holds:*

$$\frac{1}{T^2} \sum_{t=1}^T E \left[\langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) - \boldsymbol{\beta}_t^*(1) \rangle^4 \cdot \frac{1}{p_t^3} \right] \rightarrow 0 .$$

Proof. By Assumption 3, Proposition 5.4 and similar method as in the proof of Corollary B.19 using the refined bound in Lemma C.17, we have

$$\begin{aligned}
& \frac{1}{T^2} \sum_{t=1}^T E \left[\frac{1}{p_t^3} \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) - \boldsymbol{\beta}_t^*(1) \rangle^4 \right] \\
& \lesssim T^{-2+21/26} \sum_{t=1}^T E \left[R_t^{-12/11} \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) - \boldsymbol{\beta}_t^*(1) \rangle^4 \right] \quad (\text{Proposition 5.4}) \\
& \lesssim T^{-31/26} \sum_{t=1}^T E \left[R_t^{-1} \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) - \boldsymbol{\beta}_t^*(1) \rangle^4 \right] \\
& \lesssim T^{-31/26} \left(R_T^8 \eta_T^3 T \cdot T^{3/8} + R_T^{13/2} \eta_T^{7/4} T^{7/8} \cdot T^{5/16} + R_T^6 \eta_T^2 T \cdot T^{9/32} \right. \\
& \quad \left. + R_T^4 \eta_T T \cdot T^{81/512} \right) \quad (\text{using similar proof as in Corollary B.19 and Lemma C.17}) \\
& = (TR_T^{-4})^{-137/104} \cdot R_T^{-85/26} + (TR_T^{-4})^{-183/208} \cdot R_T^{-27/52} + (TR_T^{-4})^{-379/416} \cdot R_T^{-171/104}
\end{aligned}$$

$$+ (TR_T^{-4})^{-3555/6656} \cdot R_T^{-227/1664} \\ = o(1) \quad (\text{Assumption 3}) .$$

□

Based on Lemma C.15 and Lemma C.18. We now verify conditional Lyapunov condition.

Lemma C.19. *Under Assumptions 1-4 and Condition 1, there holds $\sum_{t=1}^T \mathbb{E}[X_{t,T}^4 | \mathcal{F}_{t-1,T}] \xrightarrow{p} 0$.*

Proof. By Corollary 5.2, Cauchy-Schwarz inequality and AM-GM inequality, we have

$$\begin{aligned} & \mathbb{E}\left[\sum_{t=1}^T \mathbb{E}[X_{t,T}^4 | \mathcal{F}_{t-1,T}]\right] \\ & \lesssim \frac{1}{T^2} \sum_{t=1}^T \mathbb{E}\left[\left(y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle\right)^4 \cdot \frac{1}{p_t^3}\right] + \frac{1}{T^2} \sum_{t=1}^T \mathbb{E}\left[\left(\langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) - \boldsymbol{\beta}_t^*(1) \rangle\right)^4 \cdot \frac{1}{p_t^3}\right] \\ & \quad + \frac{1}{T^2} \sum_{t=1}^T \mathbb{E}\left[\left(y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(0) \rangle\right)^4 \cdot \frac{1}{(1-p_t)^3}\right] + \frac{1}{T^2} \sum_{t=1}^T \mathbb{E}\left[\left(\langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) - \boldsymbol{\beta}_t^*(0) \rangle\right)^4 \cdot \frac{1}{(1-p_t)^3}\right], \end{aligned}$$

which should converge to 0 by Lemma C.15, Lemma C.18 and symmetric results regarding the controlled group ($k = 0$). This implies that the conditional Lyapunov condition should hold by Markov's inequality. □

Finally, under Lemma C.14 and Lemma C.19, we can verify the central limit theorem:

Theorem C.20. *Under Assumptions 1-4 and Condition 1, the standardized adaptive Horvitz-Thompson estimator is asymptotically standard normal:*

$$\frac{\hat{\tau} - \tau}{\sqrt{\text{Var}(\hat{\tau})}} \xrightarrow{d} \mathcal{N}(0, 1) .$$

C.3 Non-superefficiency

In this section, we verify the non-superefficiency condition in Corollary 5.2. We first introduce the following notations:

$$\begin{aligned} \tilde{A}_T(1) &= \sum_{t=1}^T \frac{\mathbf{1}[Z_t = 1]}{p_t} \cdot (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^2 , \\ \tilde{A}_T(0) &= \sum_{t=1}^T \frac{\mathbf{1}[Z_t = 0]}{1-p_t} \cdot (y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(0) \rangle)^2 , \\ \tilde{p} &= \operatorname{argmin}_{p \in (0,1)} \sum_{t=1}^T \left(\frac{\mathbf{1}[Z_t = 1]}{p_t} \cdot (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^2 \cdot \frac{1}{p} \right. \\ &\quad \left. + \frac{\mathbf{1}[Z_t = 0]}{1-p_t} \cdot (y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(0) \rangle)^2 \cdot \frac{1}{1-p} + \eta_T^{-1} \Psi(p) \right) , \\ \check{p} &= \operatorname{argmin}_{p \in (0,1)} \sum_{t=1}^T \left((y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^2 \cdot \frac{1}{p} + (y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(0) \rangle)^2 \cdot \frac{1}{1-p} + \eta_T^{-1} \Psi(p) \right) . \end{aligned}$$

In Proposition 4.11, we have shown that the upper bound of the prediction regret is sublinear. The following lemma further establishes that the lower bound of the prediction regret is also sublinear.

Lemma C.21. Under Assumptions 1-3 and Condition 1, for $k \in \{0, 1\}$, there holds: $\mathbb{E}[\mathcal{R}_T^{\text{pred}}] = o(T)$.

Proof. We only prove the result for $k = 1$. By Lemma 4.9 and Assumption 3, we have already proved that the positive part of $\mathbb{E}[\mathcal{R}_T^{\text{pred}}]$ is of order $o(T)$. It suffice to prove that the negative part is also of order $o(T)$. By direct calculation, we have

$$\begin{aligned} & \ell_t(\beta_t(1), \beta_t(0)) \\ &= \frac{\mathcal{E}(0)}{\mathcal{E}(1)} (y_t(1) - \langle \mathbf{x}_t, \beta_t(1) \rangle)^2 + \frac{\mathcal{E}(1)}{\mathcal{E}(0)} (y_t(0) - \langle \mathbf{x}_t, \beta_t(0) \rangle)^2 + 2(y_t(1) - \langle \mathbf{x}_t, \beta_t(1) \rangle) \cdot (y_t(0) - \langle \mathbf{x}_t, \beta_t(0) \rangle) \\ &= \frac{\mathcal{E}(0)}{\mathcal{E}(1)} (y_t(1) - \langle \mathbf{x}_t, \beta_t^*(1) \rangle)^2 + \frac{2\mathcal{E}(0)}{\mathcal{E}(1)} (y_t(1) - \langle \mathbf{x}_t, \beta_t^*(1) \rangle) \langle \mathbf{x}_t, \beta_t^*(1) - \beta_t(1) \rangle + \frac{\mathcal{E}(0)}{\mathcal{E}(1)} \langle \mathbf{x}_t, \beta_t^*(1) - \beta_t(1) \rangle^2 \\ &\quad + \frac{\mathcal{E}(1)}{\mathcal{E}(0)} (y_t(0) - \langle \mathbf{x}_t, \beta_t^*(0) \rangle)^2 + \frac{2\mathcal{E}(1)}{\mathcal{E}(0)} (y_t(0) - \langle \mathbf{x}_t, \beta_t^*(0) \rangle) \langle \mathbf{x}_t, \beta_t^*(0) - \beta_t(0) \rangle + \frac{\mathcal{E}(1)}{\mathcal{E}(0)} \langle \mathbf{x}_t, \beta_t^*(0) - \beta_t(0) \rangle^2 \\ &\quad + 2(y_t(1) - \langle \mathbf{x}_t, \beta_t^*(1) \rangle) (y_t(0) - \langle \mathbf{x}_t, \beta_t^*(0) \rangle) + 2(y_t(1) - \langle \mathbf{x}_t, \beta_t^*(1) \rangle) \langle \mathbf{x}_t, \beta_t^*(0) - \beta_t(0) \rangle \\ &\quad + 2(y_t(0) - \langle \mathbf{x}_t, \beta_t^*(0) \rangle) \langle \mathbf{x}_t, \beta_t^*(1) - \beta_t(1) \rangle + \langle \mathbf{x}_t, \beta_t^*(1) - \beta_t(1) \rangle \langle \mathbf{x}_t, \beta_t^*(0) - \beta_t(0) \rangle . \end{aligned} \quad (90)$$

Since $\mathbb{E}[\beta_t(k)] = \beta_t^*(k)$ for any $k \in \{0, 1\}$ and $t \in [T]$, terms 2, 5, 8 and 9 in (90) have expectation as 0. We denote

$$\bar{L}_t(\beta(1), \beta(0)) = \sum_{s=1}^{t-1} \ell_s(\beta(1), \beta(0)) + \eta_{t-1}^{-1} m(\beta(1), \beta(0))$$

and let $(\tilde{\beta}_t^*(1), \tilde{\beta}_t^*(0))$ denote its minimizer. Then by (88) and similar calculation as in Lemma 4.9, we have

$$\begin{aligned} & \mathbb{E}[\mathcal{R}_T^{\text{pred}}] \\ &= \mathbb{E} \left[\sum_{t=1}^T \ell_t(\beta_t(1), \beta_t(0)) - \sum_{t=1}^T \ell_t(\beta^*(1), \beta^*(0)) \right] \\ &= \sum_{t=1}^T \ell_t(\beta_t^*(1), \beta_t^*(0)) - \sum_{t=1}^T \ell_t(\beta^*(1), \beta^*(0)) \\ &\quad + \frac{\mathcal{E}(1)}{\mathcal{E}(0)} \mathbb{E} \left[\sum_{t=1}^T \langle \mathbf{x}_t, \beta_t^*(1) - \beta_t(1) \rangle^2 \right] + \frac{\mathcal{E}(0)}{\mathcal{E}(1)} \mathbb{E} \left[\sum_{t=1}^T \langle \mathbf{x}_t, \beta_t^*(0) - \beta_t(0) \rangle^2 \right] \\ &\quad + 2 \mathbb{E} \left[\sum_{t=1}^T \langle \mathbf{x}_t, \beta_t^*(1) - \beta_t(1) \rangle \langle \mathbf{x}_t, \beta_t^*(0) - \beta_t(0) \rangle \right] \\ &= o(T) + \sum_{t=1}^T \ell_t(\beta_t^*(1), \beta_t^*(0)) - \sum_{t=1}^T \ell_t(\beta^*(1), \beta^*(0)) \quad (\text{by (88)}) \\ &= o(T) + \eta_{T+1}^{-1} m(\beta^*(1), \beta^*(0)) + \sum_{t=1}^T \ell_t(\beta_t^*(1), \beta_t^*(0)) + L_1(\beta_1^*(1), \beta_1^*(0)) - L_{T+1}(\beta_{T+1}^*(1), \beta_{T+1}^*(0)) \\ &\quad + L_{T+1}(\beta_{T+1}^*(1), \beta_{T+1}^*(0)) - L_{T+1}(\beta^*(1), \beta^*(0)) \\ &= o(T) + \eta_{T+1}^{-1} m(\beta^*(1), \beta^*(0)) + L_{T+1}(\beta_{T+1}^*(1), \beta_{T+1}^*(0)) - L_{T+1}(\beta^*(1), \beta^*(0)) \\ &\quad + \sum_{t=1}^T \ell_t(\beta_t^*(1), \beta_t^*(0)) + \sum_{t=1}^T (L_t(\beta_t^*(1), \beta_t^*(0)) - L_{t+1}(\beta_{t+1}^*(1), \beta_{t+1}^*(0))) \\ &= o(T) + \eta_{T+1}^{-1} m(\beta^*(1), \beta^*(0)) + L_{T+1}(\beta_{T+1}^*(1), \beta_{T+1}^*(0)) - L_{T+1}(\beta^*(1), \beta^*(0)) \end{aligned}$$

$$\begin{aligned}
& + \sum_{t=1}^T (\bar{L}_{t+1}(\beta_t^*(1), \beta_t^*(0)) - \bar{L}_{t+1}(\beta_{t+1}^*(1), \beta_{t+1}^*(0))) \\
& + \sum_{t=1}^T (\bar{L}_{t+1}(\beta_{t+1}^*(1), \beta_{t+1}^*(0)) - L_{t+1}(\beta_{t+1}^*(1), \beta_{t+1}^*(0))) \\
& \geq o(T) + (\eta_{T+1}^{-1} m(\beta^*(1), \beta^*(0)) + L_{T+1}(\beta_{T+1}^*(1), \beta_{T+1}^*(0)) - L_{T+1}(\beta^*(1), \beta^*(0))) \\
& + \sum_{t=1}^T (\eta_t^{-1} - \eta_{t+1}^{-1}) m(\beta_{t+1}^*(1), \beta_{t+1}^*(0)) \\
& + \sum_{t=1}^T (\bar{L}_{t+1}(\tilde{\beta}_{t+1}^*(1), \tilde{\beta}_{t+1}^*(0)) - \bar{L}_{t+1}(\beta_{t+1}^*(1), \beta_{t+1}^*(0))) \quad (\text{optimality of } (\tilde{\beta}_{t+1}^*(1), \tilde{\beta}_{t+1}^*(0))) \\
& \triangleq o(T) + S_1 + S_2 + S_3 . \tag{91}
\end{aligned}$$

For simplicity, we denote $\mathbf{Y}_t(1) = (y_1(1), \dots, y_t(1))^{\top}$ and $\mathbf{Y}_t(0) = (y_1(0), \dots, y_t(0))^{\top}$ for $t \in [T]$. By direct calculation, we have

$$\begin{aligned}
S_1 &= \eta_{T+1}^{-1} m(\beta^*(1), \beta^*(0)) + L_{T+1}(\beta_{T+1}^*(1), \beta_{T+1}^*(0)) - L_{T+1}(\beta^*(1), \beta^*(0)) \\
&= -\frac{\mathcal{E}(0)}{\mathcal{E}(1)} \left[\left\| \left(\mathbf{I}_d - \mathbf{X}_T (\mathbf{X}_T^{\top} \mathbf{X}_T)^{-1} \mathbf{X}_T^{\top} \right) \mathbf{Y}_T(1) \right\|_2^2 - \left\| \left(\mathbf{I}_d - \mathbf{X}_T (\mathbf{X}_T^{\top} \mathbf{X}_T + \eta_T^{-1} \mathbf{I}_d)^{-1} \mathbf{X}_T^{\top} \right) \mathbf{Y}_T(1) \right\|_2^2 \right. \\
&\quad \left. - \eta_T^{-1} \mathbf{Y}_T^{\top}(1) \mathbf{X}_T (\mathbf{X}_T^{\top} \mathbf{X}_T + \eta_T^{-1} \mathbf{I}_d)^{-2} \mathbf{X}_T^{\top} \mathbf{Y}_T(1) \right] \\
&\quad - \frac{\mathcal{E}(1)}{\mathcal{E}(0)} \left[\left\| \left(\mathbf{I}_d - \mathbf{X}_T (\mathbf{X}_T^{\top} \mathbf{X}_T)^{-1} \mathbf{X}_T^{\top} \right) \mathbf{Y}_T(0) \right\|_2^2 - \left\| \left(\mathbf{I}_d - \mathbf{X}_T (\mathbf{X}_T^{\top} \mathbf{X}_T + \eta_T^{-1} \mathbf{I}_d)^{-1} \mathbf{X}_T^{\top} \right) \mathbf{Y}_T(0) \right\|_2^2 \right. \\
&\quad \left. - \eta_T^{-1} \mathbf{Y}_T^{\top}(0) \mathbf{X}_T (\mathbf{X}_T^{\top} \mathbf{X}_T + \eta_T^{-1} \mathbf{I}_d)^{-2} \mathbf{X}_T^{\top} \mathbf{Y}_T(0) \right] \\
&\quad - 2 \left[\langle \left(\mathbf{I}_d - \mathbf{X}_T (\mathbf{X}_T^{\top} \mathbf{X}_T)^{-1} \mathbf{X}_T^{\top} \right) \mathbf{Y}_T(1), \left(\mathbf{I}_d - \mathbf{X}_T (\mathbf{X}_T^{\top} \mathbf{X}_T)^{-1} \mathbf{X}_T^{\top} \right) \mathbf{Y}_T(0) \rangle \right. \\
&\quad \left. - \langle \left(\mathbf{I}_d - \mathbf{X}_T (\mathbf{X}_T^{\top} \mathbf{X}_T + \eta_T^{-1} \mathbf{I}_d)^{-1} \mathbf{X}_T^{\top} \right) \mathbf{Y}_T(1), \left(\mathbf{I}_d - \mathbf{X}_T (\mathbf{X}_T^{\top} \mathbf{X}_T + \eta_T^{-1} \mathbf{I}_d)^{-1} \mathbf{X}_T^{\top} \right) \mathbf{Y}_T(0) \rangle \right. \\
&\quad \left. - \eta_T^{-1} \langle \mathbf{X}_T (\mathbf{X}_T^{\top} \mathbf{X}_T + \eta_T^{-1} \mathbf{I}_d)^{-1} \mathbf{X}_T^{\top} \mathbf{Y}_T(1), \mathbf{X}_T (\mathbf{X}_T^{\top} \mathbf{X}_T + \eta_T^{-1} \mathbf{I}_d)^{-1} \mathbf{X}_T^{\top} \mathbf{Y}_T(0) \rangle \right] \\
&= -\frac{\mathcal{E}(0)}{\mathcal{E}(1)} \left[-\mathbf{Y}_T^{\top}(1) \left(\mathbf{X}_T (\mathbf{X}_T^{\top} \mathbf{X}_T)^{-1} \mathbf{X}_T^{\top} \right) \mathbf{Y}_T(1) + \mathbf{Y}_T^{\top}(1) \left(\mathbf{X}_T (\mathbf{X}_T^{\top} \mathbf{X}_T + \eta_T^{-1} \mathbf{I}_d)^{-1} \mathbf{X}_T^{\top} \right) \mathbf{Y}_T(1) \right] \\
&\quad - \frac{\mathcal{E}(1)}{\mathcal{E}(0)} \left[-\mathbf{Y}_T^{\top}(0) \left(\mathbf{X}_T (\mathbf{X}_T^{\top} \mathbf{X}_T)^{-1} \mathbf{X}_T^{\top} \right) \mathbf{Y}_T(0) + \mathbf{Y}_T^{\top}(0) \left(\mathbf{X}_T (\mathbf{X}_T^{\top} \mathbf{X}_T + \eta_T^{-1} \mathbf{I}_d)^{-1} \mathbf{X}_T^{\top} \right) \mathbf{Y}_T(0) \right] \\
&\quad - 2 \left[-\mathbf{Y}_T^{\top}(1) \left(\mathbf{X}_T (\mathbf{X}_T^{\top} \mathbf{X}_T)^{-1} \mathbf{X}_T^{\top} \right) \mathbf{Y}_T(0) + \mathbf{Y}_T^{\top}(1) \left(\mathbf{X}_T (\mathbf{X}_T^{\top} \mathbf{X}_T + \eta_T^{-1} \mathbf{I}_d)^{-1} \mathbf{X}_T^{\top} \right) \mathbf{Y}_T(0) \right] \\
&= \left(\left(\frac{\mathcal{E}(0)}{\mathcal{E}(1)} \right)^{1/2} \mathbf{Y}_T(1) + \left(\frac{\mathcal{E}(1)}{\mathcal{E}(0)} \mathbf{Y}_T(0) \right)^{1/2} \right)^{\top} \left[\left(\mathbf{X}_T (\mathbf{X}_T^{\top} \mathbf{X}_T)^{-1} \mathbf{X}_T^{\top} \right) - \left(\mathbf{X}_T (\mathbf{X}_T^{\top} \mathbf{X}_T + \eta_T^{-1} \mathbf{I}_d)^{-1} \mathbf{X}_T^{\top} \right) \right] \\
&\quad \times \left(\left(\frac{\mathcal{E}(0)}{\mathcal{E}(1)} \right)^{1/2} \mathbf{Y}_T(1) + \left(\frac{\mathcal{E}(1)}{\mathcal{E}(0)} \right)^{1/2} \mathbf{Y}_T(0) \right) \\
&\geq \left(\left(\frac{\mathcal{E}(0)}{\mathcal{E}(1)} \right)^{1/2} \mathbf{Y}_T(1) + \left(\frac{\mathcal{E}(1)}{\mathcal{E}(0)} \right)^{1/2} \mathbf{Y}_T(0) \right)^{\top} \left[\mathbf{X}_T (\mathbf{X}_T^{\top} \mathbf{X}_T)^{-1} \mathbf{X}_T^{\top} + \mathbf{X}_T (\mathbf{X}_T^{\top} \mathbf{X}_T + \eta_T^{-1} \mathbf{I}_d)^{-1} \mathbf{X}_T^{\top} \right] \\
&\quad \times \left(\left(\frac{\mathcal{E}(0)}{\mathcal{E}(1)} \right)^{1/2} \mathbf{Y}_T(1) + \left(\frac{\mathcal{E}(1)}{\mathcal{E}(0)} \right)^{1/2} \mathbf{Y}_T(0) \right) \\
&= 0 . \tag{92}
\end{aligned}$$

For any $t \in [T]$, by Cauchy-Schwarz inequality, Lemma B.3 and Assumption 1, we have

$$\|\beta_{t+1}^*(1)\|_2^2 = \mathbf{Y}_t^{\top}(1) \mathbf{X}_t (\mathbf{X}_t^{\top} \mathbf{X}_t + \eta_{t+1}^{-1} \mathbf{I}_d)^{-2} \mathbf{X}_t^{\top} \mathbf{Y}_t(1)$$

$$\begin{aligned}
&\leq \left(\sum_{s=1}^t y_s^2(1) \right) \left\| \mathbf{X}_t (\mathbf{X}_t^\top \mathbf{X}_t + \eta_{t+1}^{-1} \mathbf{I}_d)^{-2} \mathbf{X}_t^\top \right\|_2 \\
&\leq \left(\sum_{s=1}^t y_s^2(1) \right) \left\| (\mathbf{X}_t^\top \mathbf{X}_t + \eta_{t+1}^{-1} \mathbf{I}_d)^{-1} \right\|_2 \\
&\leq \left(\sum_{s=1}^t y_s^4(1) \right)^{1/2} t^{1/2} (\gamma_0 \vee c_2 \vee 1) (t \vee \eta_{t+1}^{-1})^{-1} \quad (\text{Cauchy-Schwarz inequality and Lemma B.3}) \\
&\lesssim T^{1/2} (t \vee \eta_{t+1}^{-1})^{-1/2} \quad (\text{Assumption 1}) \\
&\lesssim T^{1/2} (\eta_{t+1}^{-1})^{-1/2} .
\end{aligned}$$

Similar result can be obtained for $\beta_{t+1}^*(0)$. Hence we can bound $-S_2$ by:

$$\begin{aligned}
-S_2 &= \sum_{t=1}^T (\eta_{t+1}^{-1} - \eta_t^{-1}) \left(\frac{\mathcal{E}(0)}{\mathcal{E}(1)} \|\beta_{t+1}^*(1)\|_2^2 + \frac{\mathcal{E}(1)}{\mathcal{E}(0)} \|\beta_{t+1}^*(0)\|_2^2 + 2\langle \beta_{t+1}^*(1), \beta_{t+1}^*(0) \rangle \right) \\
&\lesssim \sum_{t=1}^T (\eta_{t+1}^{-1} - \eta_t^{-1}) \left(\|\beta_{t+1}^*(1)\|_2^2 + \frac{\mathcal{E}(1)}{\mathcal{E}(0)} \|\beta_{t+1}^*(0)\|_2^2 \right) \\
&\lesssim T^{1/2} \sum_{t=1}^T (\eta_{t+1}^{-1} - \eta_t^{-1}) (\eta_{t+1}^{-1})^{-1/2} \\
&\leq T^{1/2} \int_0^{\eta_{T+1}^{-1}} x^{-1/2} dx \quad (\eta_t \text{ is decreasing}) \\
&\lesssim T^{1/2} \eta_{T+1}^{-1/2} \\
&= T^{1/2} \eta_T^{-1/2} . \tag{93}
\end{aligned}$$

By the explicit form of $\beta_{t+1}^*(1)$ and $\tilde{\beta}_{t+1}^*(1)$, we have

$$\begin{aligned}
&\|\mathbf{Y}_t(1) - \mathbf{X}_t \beta_{t+1}^*(1)\|_2^2 + \eta_t^{-1} \|\beta_{t+1}^*(1)\|_2^2 - \left(\|\mathbf{Y}_t(1) - \mathbf{X}_t \tilde{\beta}_{t+1}^*(1)\|_2^2 + \eta_t^{-1} \|\tilde{\beta}_{t+1}^*(1)\|_2^2 \right) \\
&= (\mathbf{Y}_t^\top(1) \mathbf{Y}_t(1) - 2\mathbf{Y}_t^\top(1) \mathbf{X}_t \beta_{t+1}^*(1) + (\beta_{t+1}^*(1))^\top (\mathbf{X}_t^\top \mathbf{X}_t + \eta_t^{-1} \mathbf{I}_d) \beta_{t+1}^*(1)) \\
&\quad - \left(\mathbf{Y}_t^\top(1) \mathbf{Y}_t(1) - 2\mathbf{Y}_t^\top(1) \mathbf{X}_t \tilde{\beta}_{t+1}^*(1) + (\tilde{\beta}_{t+1}^*(1))^\top (\mathbf{X}_t^\top \mathbf{X}_t + \eta_t^{-1} \mathbf{I}_d) \tilde{\beta}_{t+1}^*(1) \right) \\
&= -2\mathbf{Y}_t^\top(1) \mathbf{X}_t (\mathbf{X}_t^\top \mathbf{X}_t + \eta_{t+1}^{-1} \mathbf{I}_d)^{-1} \mathbf{X}_t^\top \mathbf{Y}_t(1) + 2\mathbf{Y}_t^\top(1) \mathbf{X}_t (\mathbf{X}_t^\top \mathbf{X}_t + \eta_t^{-1} \mathbf{I}_d)^{-1} \mathbf{X}_t^\top \mathbf{Y}_t(1) \\
&\quad + \mathbf{Y}_t^\top(1) \mathbf{X}_t (\mathbf{X}_t^\top \mathbf{X}_t + \eta_{t+1}^{-1} \mathbf{I}_d)^{-1} (\mathbf{X}_t^\top \mathbf{X}_t + \eta_t^{-1} \mathbf{I}_d) (\mathbf{X}_t^\top \mathbf{X}_t + \eta_{t+1}^{-1} \mathbf{I}_d)^{-1} \mathbf{X}_t^\top \mathbf{Y}_t(1) \\
&\quad - \mathbf{Y}_t^\top(1) \mathbf{X}_t (\mathbf{X}_t^\top \mathbf{X}_t + \eta_t^{-1} \mathbf{I}_d)^{-1} (\mathbf{X}_t^\top \mathbf{X}_t + \eta_t^{-1} \mathbf{I}_d) (\mathbf{X}_t^\top \mathbf{X}_t + \eta_t^{-1} \mathbf{I}_d)^{-1} \mathbf{X}_t^\top \mathbf{Y}_t(1) \\
&= \mathbf{Y}_t^\top(1) \left(-2\mathbf{X}_t (\mathbf{X}_t^\top \mathbf{X}_t + \eta_{t+1}^{-1} \mathbf{I}_d)^{-1} \mathbf{X}_t^\top + 2\mathbf{X}_t (\mathbf{X}_t^\top \mathbf{X}_t + \eta_t^{-1} \mathbf{I}_d)^{-1} \mathbf{X}_t^\top \right. \\
&\quad \left. + \mathbf{X}_t (\mathbf{X}_t^\top \mathbf{X}_t + \eta_{t+1}^{-1} \mathbf{I}_d)^{-1} (\mathbf{X}_t^\top \mathbf{X}_t + \eta_t^{-1} \mathbf{I}_d) (\mathbf{X}_t^\top \mathbf{X}_t + \eta_{t+1}^{-1} \mathbf{I}_d)^{-1} \mathbf{X}_t^\top - \mathbf{X}_t (\mathbf{X}_t^\top \mathbf{X}_t + \eta_t^{-1} \mathbf{I}_d)^{-1} \mathbf{X}_t^\top \right) \mathbf{Y}_t(1) .
\end{aligned}$$

By the same method, we also have

$$\begin{aligned}
&\|\mathbf{Y}_t(0) - \mathbf{X}_t \beta_{t+1}^*(0)\|_2^2 + \eta_t^{-1} \|\beta_{t+1}^*(0)\|_2^2 - \left(\|\mathbf{Y}_t(0) - \mathbf{X}_t \tilde{\beta}_{t+1}^*(0)\|_2^2 + \eta_t^{-1} \|\tilde{\beta}_{t+1}^*(0)\|_2^2 \right) \\
&= \mathbf{Y}_t^\top(0) \left(-2\mathbf{X}_t (\mathbf{X}_t^\top \mathbf{X}_t + \eta_{t+1}^{-1} \mathbf{I}_d)^{-1} \mathbf{X}_t^\top + 2\mathbf{X}_t (\mathbf{X}_t^\top \mathbf{X}_t + \eta_t^{-1} \mathbf{I}_d)^{-1} \mathbf{X}_t^\top \right. \\
&\quad \left. + \mathbf{X}_t (\mathbf{X}_t^\top \mathbf{X}_t + \eta_{t+1}^{-1} \mathbf{I}_d)^{-1} (\mathbf{X}_t^\top \mathbf{X}_t + \eta_t^{-1} \mathbf{I}_d) (\mathbf{X}_t^\top \mathbf{X}_t + \eta_{t+1}^{-1} \mathbf{I}_d)^{-1} \mathbf{X}_t^\top - \mathbf{X}_t (\mathbf{X}_t^\top \mathbf{X}_t + \eta_t^{-1} \mathbf{I}_d)^{-1} \mathbf{X}_t^\top \right) \mathbf{Y}_t(0) , \\
&\langle \mathbf{Y}_t(1) - \mathbf{X}_t \beta_{t+1}^*(1), \mathbf{Y}_t(0) - \mathbf{X}_t \beta_{t+1}^*(0) \rangle + \eta_t^{-1} \langle \beta_{t+1}^*(1), \beta_{t+1}^*(0) \rangle \\
&\quad - (\langle \mathbf{Y}_t(1) - \mathbf{X}_t \beta_{t+1}^*(1), \mathbf{Y}_t(0) - \mathbf{X}_t \beta_{t+1}^*(0) \rangle + \eta_t^{-1} \langle \beta_{t+1}^*(1), \beta_{t+1}^*(0) \rangle)
\end{aligned}$$

$$\begin{aligned}
&= \mathbf{Y}_t^\top(1) \left(-2\mathbf{X}_t (\mathbf{X}_t^\top \mathbf{X}_t + \eta_{t+1}^{-1} \mathbf{I}_d)^{-1} \mathbf{X}_t^\top + 2\mathbf{X}_t (\mathbf{X}_t^\top \mathbf{X}_t + \eta_t^{-1} \mathbf{I}_d)^{-1} \mathbf{X}_t^\top \right. \\
&\quad \left. + \mathbf{X}_t (\mathbf{X}_t^\top \mathbf{X}_t + \eta_{t+1}^{-1} \mathbf{I}_d)^{-1} (\mathbf{X}_t^\top \mathbf{X}_t + \eta_t^{-1} \mathbf{I}_d) (\mathbf{X}_t^\top \mathbf{X}_t + \eta_{t+1}^{-1} \mathbf{I}_d)^{-1} \mathbf{X}_t^\top - \mathbf{X}_t (\mathbf{X}_t^\top \mathbf{X}_t + \eta_t^{-1} \mathbf{I}_d)^{-1} \mathbf{X}_t^\top \right) \mathbf{Y}_t(0) .
\end{aligned}$$

Hence

$$\begin{aligned}
&\bar{L}_{t+1}(\beta_{t+1}^*(1), \beta_{t+1}^*(0)) - \bar{L}_{t+1}(\tilde{\beta}_{t+1}^*(1), \tilde{\beta}_{t+1}^*(0)) \\
&= \left[\left(\frac{\mathcal{E}(0)}{\mathcal{E}(1)} \right)^{1/2} \mathbf{Y}_t(1) + \left(\frac{\mathcal{E}(1)}{\mathcal{E}(0)} \right)^{1/2} \mathbf{Y}_t(0) \right]^\top \left(-2\mathbf{X}_t (\mathbf{X}_t^\top \mathbf{X}_t + \eta_{t+1}^{-1} \mathbf{I}_d)^{-1} \mathbf{X}_t^\top + 2\mathbf{X}_t (\mathbf{X}_t^\top \mathbf{X}_t + \eta_t^{-1} \mathbf{I}_d)^{-1} \mathbf{X}_t^\top \right. \\
&\quad \left. + \mathbf{X}_t (\mathbf{X}_t^\top \mathbf{X}_t + \eta_{t+1}^{-1} \mathbf{I}_d)^{-1} (\mathbf{X}_t^\top \mathbf{X}_t + \eta_t^{-1} \mathbf{I}_d) (\mathbf{X}_t^\top \mathbf{X}_t + \eta_{t+1}^{-1} \mathbf{I}_d)^{-1} \mathbf{X}_t^\top - \mathbf{X}_t (\mathbf{X}_t^\top \mathbf{X}_t + \eta_t^{-1} \mathbf{I}_d)^{-1} \mathbf{X}_t^\top \right) \\
&\quad \times \left[\left(\frac{\mathcal{E}(0)}{\mathcal{E}(1)} \right)^{1/2} \mathbf{Y}_t(1) + \left(\frac{\mathcal{E}(1)}{\mathcal{E}(0)} \right)^{1/2} \mathbf{Y}_t(0) \right] \\
&\lesssim \left(\sum_{s=1}^t y_s^2(1) + \sum_{s=1}^t y_s^2(0) \right) \left\| \mathbf{X}_t \left(-2(\mathbf{X}_t^\top \mathbf{X}_t + \eta_{t+1}^{-1} \mathbf{I}_d)^{-1} + 2(\mathbf{X}_t^\top \mathbf{X}_t + \eta_t^{-1} \mathbf{I}_d)^{-1} - (\mathbf{X}_t^\top \mathbf{X}_t + \eta_t^{-1} \mathbf{I}_d)^{-1} \right. \right. \\
&\quad \left. \left. + (\mathbf{X}_t^\top \mathbf{X}_t + \eta_{t+1}^{-1} \mathbf{I}_d)^{-1} (\mathbf{X}_t^\top \mathbf{X}_t + \eta_t^{-1} \mathbf{I}_d) (\mathbf{X}_t^\top \mathbf{X}_t + \eta_{t+1}^{-1} \mathbf{I}_d)^{-1} \right) \mathbf{X}_t^\top \right\|_2 \\
&= \left(\sum_{s=1}^t y_s^2(1) + \sum_{s=1}^t y_s^2(0) \right) \left\| \mathbf{X}_t^\top \mathbf{X}_t \left(-2(\mathbf{X}_t^\top \mathbf{X}_t + \eta_{t+1}^{-1} \mathbf{I}_d)^{-1} + 2(\mathbf{X}_t^\top \mathbf{X}_t + \eta_t^{-1} \mathbf{I}_d)^{-1} - (\mathbf{X}_t^\top \mathbf{X}_t + \eta_t^{-1} \mathbf{I}_d)^{-1} \right. \right. \\
&\quad \left. \left. + (\mathbf{X}_t^\top \mathbf{X}_t + \eta_{t+1}^{-1} \mathbf{I}_d)^{-1} (\mathbf{X}_t^\top \mathbf{X}_t + \eta_t^{-1} \mathbf{I}_d) (\mathbf{X}_t^\top \mathbf{X}_t + \eta_{t+1}^{-1} \mathbf{I}_d)^{-1} \right) \right\|_2 \\
&\triangleq \left(\sum_{s=1}^t y_s^2(1) + \sum_{s=1}^t y_s^2(0) \right) \|\mathbf{V}\|_2 .
\end{aligned}$$

Let $\lambda_1, \dots, \lambda_d$ be all eigenvalues of $\mathbf{X}_t^\top \mathbf{X}_t$. Since $\mathbf{X}_t^\top \mathbf{X}_t$, $\mathbf{X}_t^\top \mathbf{X}_t + \eta_t^{-1} \mathbf{I}_d$ and $\mathbf{X}_t^\top \mathbf{X}_t + \eta_{t+1}^{-1} \mathbf{I}_d$ can be simultaneously orthogonally diagonalized, if λ is the eigenvalue of $\mathbf{X}_t^\top \mathbf{X}_t$, then the corresponding eigenvalue of \mathbf{V} is as:

$$\begin{aligned}
&\lambda \left(-\frac{2}{\lambda + \eta_{t+1}^{-1}} + \frac{2}{\lambda + \eta_t^{-1}} - \frac{1}{\lambda + \eta_t^{-1}} + \frac{\lambda + \eta_t^{-1}}{(\lambda + \eta_{t+1}^{-1})^2} \right) \\
&= \lambda \cdot \frac{-2(\lambda + \eta_{t+1}^{-1})(\lambda + \eta_t^{-1}) + (\lambda + \eta_{t+1}^{-1})^2 + (\lambda + \eta_t^{-1})^2}{(\lambda + \eta_{t+1}^{-1})^2(\lambda + \eta_t^{-1})} \\
&= \lambda \cdot \frac{(\lambda + \eta_{t+1}^{-1} - \lambda - \eta_t^{-1})^2}{(\lambda + \eta_{t+1}^{-1})^2(\lambda + \eta_t^{-1})} \\
&\leq (\eta_{t+1}^{-1} - \eta_t^{-1})^2 \frac{1}{(\lambda + \eta_{t+1}^{-1})^2} .
\end{aligned}$$

By similar proof as in Lemma B.3, $(\lambda + \eta_{t+1}^{-1})^{-1} \leq (\gamma_0 \vee c_2 \vee 1)(t \vee \eta_{t+1}^{-1})^{-1}$, then we have $\|\mathbf{V}\|_2 \leq (\gamma_0 \vee c_2 \vee 1)^2(\eta_{t+1}^{-1} - \eta_t^{-1})^2(t \vee \eta_{t+1}^{-1})^{-2}$. Hence by Cauchy-Schwarz inequality, we can bound S_3 as:

$$\begin{aligned}
-S_3 &= \sum_{t=1}^T \left(\bar{L}_{t+1}(\beta_{t+1}^*(1), \beta_{t+1}^*(0)) - \bar{L}_{t+1}(\tilde{\beta}_{t+1}^*(1), \tilde{\beta}_{t+1}^*(0)) \right) \\
&\lesssim \sum_{t=1}^T \left(\sum_{s=1}^t y_s^2(1) + \sum_{s=1}^t y_s^2(0) \right) \|\mathbf{V}\|_2 \\
&\lesssim \sum_{t=1}^T \left[\left(\sum_{s=1}^t y_s^4(1) \right)^{1/2} + \left(\sum_{s=1}^t y_s^4(0) \right)^{1/2} \right] t^{1/2} (\eta_{t+1}^{-1} - \eta_t^{-1})^2 (t \vee \eta_{t+1}^{-1})^{-2} \quad (\text{Cauchy-Schwarz inequality})
\end{aligned}$$

$$\begin{aligned}
&\lesssim T^{1/2} \sum_{t=1}^T (\eta_{t+1}^{-1} - \eta_t^{-1})^2 (t \vee \eta_{t+1}^{-1})^{-3/2} \\
&\leq T^{1/2} \sum_{t=1}^T (\eta_{t+1}^{-1} - \eta_t^{-1})(\eta_{t+1}^{-1})^{-1/2} \\
&\leq T^{1/2} \int_0^{\eta_{T+1}^{-1}} x^{-1/2} dx \quad (\eta_t \text{ is decreasing}) \\
&\lesssim T^{1/2} \eta_{T+1}^{-1/2} \\
&= T^{1/2} \eta_T^{-1/2}. \tag{94}
\end{aligned}$$

Hence by (91), (92), (93) and (94), we proved that the negative part of $E[\mathcal{R}_T^{\text{pred}}]$ is of order $o(T) + O(T^{1/2} \eta_T^{-1/2}) = o(T)$ by Assumption 3. Hence the result is proved. \square

The following lemma upper bounds the variability of the inverse probability for the last subject T . Its proof follows a similar line of reasoning as that of Lemma C.4. We recommend that readers consult the proof of Lemma C.4 first. For simplicity, some intermediate steps are omitted in the proof of Lemma C.22.

Lemma C.22. *Under Assumptions 1-3 and Condition 1, there holds:*

$$\max \left\{ E \left[\left| \frac{1}{\tilde{p}} - \frac{1}{\check{p}} \right|^{5/4} \right], E \left[\left| \frac{1}{1-\tilde{p}} - \frac{1}{1-\check{p}} \right|^{5/4} \right] \right\} = \mathcal{O}(T^{-5/24} R_T^{1/6}),$$

Proof. We first bound the variance of $\tilde{A}_T(1)$ and $\tilde{A}_T(0)$. For $\tilde{A}_T(1)$, by Corollary 4.15, Corollary B.17 and (55) in the proof of Lemma C.3, we have

$$\begin{aligned}
&\text{Var}(\tilde{A}_T(1)) \\
&= \text{Var} \left(\sum_{t=1}^T \frac{\mathbf{1}[Z_t = 1]}{p_t} \cdot (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^2 \right) \\
&= \text{Var} \left(\sum_{t=1}^T \left(\frac{\mathbf{1}[Z_t = 1]}{p_t} - 1 \right) \cdot (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^2 \right) \\
&= \sum_{t=1}^T \text{Var} \left(\left(\frac{\mathbf{1}[Z_t = 1]}{p_t} - 1 \right) \cdot (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^2 \right) \quad (\text{variance of sum of MDS}) \\
&= \sum_{t=1}^T (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^4 E \left[\frac{1-p_t}{p_t} \right] \\
&\leq 2 \sum_{t=1}^T y_t^4(1) E \left[\frac{1}{p_t} \right] + 2 \sum_{t=1}^T \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle^4 E \left[\frac{1-p_t}{p_t} \right] \\
&\lesssim \mathcal{O}(T^{9/8}) + T^{5/4} \max_{t \in [T]} R_t a_t^{1/4}(0) \quad (\text{by (55), Assumption 1, Corollary 4.15}) \\
&\lesssim T^{5/4} \cdot \max_{t \in [T]} R_t (\eta_t T)^{1/4} \quad (\text{Corollary B.17}) \\
&= T^{11/8} \cdot \max_{t \in [T]} R_t^{1/2} \\
&\leq T^{11/8} R_T^{1/2}.
\end{aligned}$$

Hence there exists constant $C > 0$ such that $\text{Var}(\tilde{A}_T(1)) \leq CT^{11/8} R_T^{1/2}$. Similarly, we can prove that $\text{Var}(\tilde{A}_T(0)) \leq CT^{11/8} R_T^{1/2}$. Suppose $0 < \delta < 1/4$ is a fixed constant. Then we consider the following three cases:

$$(1) \quad A_T^*(0) \leq (1 - \delta) A_T^*(1).$$

For $k \in \{0, 1\}$, denote events $\mathcal{G}_k = \left\{ |\tilde{A}_T(k) - A_T^*(k)| < \frac{\delta}{2}(A_T^*(1) \vee A_T^*(0)) \right\}$ and $\tilde{\mathcal{G}}_k = \left\{ |\tilde{A}_T(k) - A_T^*(k)| < \frac{1}{2}(A_T^*(1) \vee A_T^*(0)) \right\}$. By Lemma 4.9 and Lemma C.21, it is easy to see that $A_T^*(1) = \Theta(T)$ and $A_T^*(0) = \Theta(T)$. Hence by Chebyshev's inequality, we have

$$\begin{aligned} \Pr(\mathcal{G}_1^c) &\lesssim \delta^{-2} T^{-2} \text{Var}(\tilde{A}_T(1)) \lesssim \delta^{-2} T^{-2} T^{11/8} R_T^{1/2} \lesssim \delta^{-2} T^{-5/8} R_T^{1/2}, \\ \Pr(\mathcal{G}_0^c) &\lesssim \delta^{-2} T^{-2} \text{Var}(\tilde{A}_T(0)) \lesssim \delta^{-2} T^{-5/8} R_T^{1/2}, \\ \Pr(\tilde{\mathcal{G}}_1^c) &\lesssim T^{-2} \text{Var}(\tilde{A}_T(1)) \lesssim T^{-5/8} R_T^{1/2}, \\ \Pr(\tilde{\mathcal{G}}_0^c) &\lesssim T^{-2} \text{Var}(\tilde{A}_T(0)) \lesssim T^{-5/8} R_T^{1/2}. \end{aligned} \quad (95)$$

By Assumption 1, under event $\tilde{\mathcal{G}}_1$ and $\tilde{\mathcal{G}}_0$, we have

$$\frac{\tilde{A}_T(0)}{\tilde{A}_T(1)} \leq \frac{A_T^*(0) + \frac{1}{2}A_T^*(0)}{A_T^*(1) - \frac{1}{2}A_T^*(1)} \leq \frac{3c_1}{c_0},$$

which implies that $1/\tilde{p} \leq 2 + 3^{1/2} b_1 (2b_2/b_3)^{1/2} (c_1/c_0)^{1/2}$ by Lemma 4.14. Moreover we have $1/\check{p} \leq 2 + b_1 (2b_2/b_3)^{1/2} (c_1/c_0)^{1/2}$ by Lemma 4.14 and Assumption 1. Under event \mathcal{G}_1 ($\mathcal{G}_1 \subseteq \tilde{\mathcal{G}}_1$) and \mathcal{G}_0 ($\mathcal{G}_0 \subseteq \tilde{\mathcal{G}}_0$), we have $\tilde{A}_T(1) \geq \tilde{A}_T(0)$. Note that, by definition \tilde{p} and \check{p} should satisfy the following first-order equations:

$$\begin{aligned} -\frac{\tilde{A}_T(1)}{\tilde{p}^2} + \frac{\tilde{A}_T(0)}{(1-\tilde{p})^2} + \eta_T^{-1} \Psi'(\tilde{p}) &= 0, \\ -\frac{A_T^*(1)}{\check{p}^2} + \frac{A_T^*(0)}{(1-\check{p})^2} + \eta_T^{-1} \Psi'(\check{p}) &= 0. \end{aligned}$$

Hence by Lemma C.1, we can show that

$$\left| \frac{1}{\tilde{p}} - \frac{1}{\check{p}} \right| \lesssim \left| \frac{A_T^*(1) - \tilde{A}_T(1)}{A_T^*(1)} \right| + \left| \frac{A_T^*(0) - \tilde{A}_T(0)}{A_T^*(0)} \right|,$$

which indicates that

$$\left| \frac{1}{\tilde{p}} - \frac{1}{\check{p}} \right|^{5/4} \lesssim \left(\frac{A_T^*(1) - \tilde{A}_T(1)}{A_T^*(1)} \right)^{5/4} + \left(\frac{A_T^*(0) - \tilde{A}_T(0)}{A_T^*(0)} \right)^{5/4}. \quad (96)$$

Hence by Corollary 4.15, Corollary B.17, (95), (96) and Hölder's inequality, we have

$$\begin{aligned} &\mathbb{E} \left[\left| \frac{1}{\tilde{p}} - \frac{1}{\check{p}} \right|^{5/4} \right] \\ &= \mathbb{E} \left[\left| \frac{1}{\tilde{p}} - \frac{1}{\check{p}} \right|^{5/4} \mathbf{1} [\mathcal{G}_1 \cap \mathcal{G}_0] \right] + \mathbb{E} \left[\left| \frac{1}{\tilde{p}} - \frac{1}{\check{p}} \right|^{5/4} \mathbf{1} [\mathcal{G}_1^c \cup \mathcal{G}_0^c] \right] \\ &\leq \mathbb{E} \left[\left| \frac{1}{\tilde{p}} - \frac{1}{\check{p}} \right|^{5/4} \mathbf{1} [\mathcal{G}_1 \cap \mathcal{G}_0] \right] + \mathbb{E} \left[\left| \frac{1}{\tilde{p}} - \frac{1}{\check{p}} \right|^{5/4} \mathbf{1} [(\mathcal{G}_1^c \cup \mathcal{G}_0^c) \cap \tilde{\mathcal{G}}_1 \cap \tilde{\mathcal{G}}_0] \right] + \mathbb{E} \left[\left| \frac{1}{\tilde{p}} - \frac{1}{\check{p}} \right|^{5/4} \mathbf{1} [\tilde{\mathcal{G}}_1^c \cup \tilde{\mathcal{G}}_0^c] \right] \\ &\lesssim \mathbb{E} \left[\left(\frac{A_T^*(1) - \tilde{A}_T(1)}{A_T^*(1)} \right)^{5/4} + \left(\frac{A_T^*(0) - \tilde{A}_T(0)}{A_T^*(0)} \right)^{5/4} \right] + \mathbb{E} \left[\mathbf{1} [(\mathcal{G}_1^c \cup \mathcal{G}_0^c) \cap \tilde{\mathcal{G}}_1 \cap \tilde{\mathcal{G}}_0] \right] \\ &\quad + \mathbb{E} \left[\left(\frac{1}{\tilde{p}} - \frac{1}{\check{p}} \right)^{5/4} \mathbf{1} [\tilde{\mathcal{G}}_1^c \cup \tilde{\mathcal{G}}_0^c] \right] \end{aligned}$$

$$\begin{aligned}
&\lesssim T^{-5/4} \text{Var}^{5/8}(\tilde{A}_T(1)) + T^{-5/4} \text{Var}^{5/8}(\tilde{A}_T(0)) + \Pr(\mathcal{G}_1^c \cup \mathcal{G}_0^c) + E \left[\left| \frac{1}{\tilde{p}^4} \right| + \left| \frac{1}{\check{p}^4} \right| \right]^{5/16} \Pr^{11/16}(\tilde{\mathcal{G}}_1^c \cup \tilde{\mathcal{G}}_0^c) \\
&\lesssim T^{-25/64} R_T^{5/16} + \delta^{-2} T^{-5/8} R_T^{1/2} + (\eta_T T)^{5/16} \cdot (T^{-5/8} R_T^{1/2})^{11/16} \\
&\lesssim T^{-25/64} R_T^{5/16} + \delta^{-2} T^{-5/8} R_T^{1/2} + T^{-35/128} R_T^{-9/32}.
\end{aligned} \tag{97}$$

(2) $A_T(1) \leq (1 - \delta) A_T(0)$.

By similar method as in case (1), we can prove that

$$E \left[\left| \frac{1}{\tilde{p}} - \frac{1}{\check{p}} \right|^{5/4} \right] \lesssim T^{-25/64} R_T^{5/16} + \delta^{-2} T^{-5/8} R_T^{1/2} + T^{-35/128} R_T^{-9/32}. \tag{98}$$

(3) $A_T(0) \geq (1 - \delta) A_T(1)$ and $A_T(1) \geq (1 - \delta) A_T(0)$.

By similar proof as in Lemma C.4 and in case (1), we can prove that

$$E \left[\left| \frac{1}{\tilde{p}} - \frac{1}{\check{p}} \right|^{5/4} \right] \lesssim \delta + \delta^{-2} T^{-5/8} R_T^{1/2} + T^{-35/128} R_T^{-9/32}. \tag{99}$$

By choosing $\delta = T^{-5/24} R_T^{1/6}$, we can verify that

$$\begin{aligned}
T^{-25/64} R_T^{5/16} &= T^{-5/24} R_T^{1/6} \cdot (T R_T^{-4})^{-35/192} \cdot R_T^{-7/12} = o(\delta), \\
\delta^{-2} T^{-5/8} R_T^{1/2} &= \delta, \\
T^{-35/128} R_T^{-9/32} &= T^{-5/24} R_T^{1/6} \cdot T^{-25/384} R_T^{-43/96} = o(\delta).
\end{aligned}$$

Hence the result is proved by (97), (98) and (99). \square

The following lemma is proved by a similar argument as in Lemma C.1. It characterizes the difference in the inverse probabilities when different step sizes are used.

Lemma C.23. *Let $A, B, \eta \geq \tilde{\eta}$ be positive constants. Suppose p, \tilde{p} satisfy*

$$\begin{aligned}
-\frac{A}{p^2} + \frac{B}{(1-p)^2} + \eta^{-1} \Psi'(p) &= 0, \\
-\frac{A}{\tilde{p}^2} + \frac{B}{(1-\tilde{p})^2} + \tilde{\eta}^{-1} \Psi'(\tilde{p}) &= 0.
\end{aligned}$$

(1) If $A \geq B$, then under Condition 1 we can attain the following upper bound:

$$\left| \frac{1}{1-p} - \frac{1}{1-\tilde{p}} \right| \leq \frac{6b_1}{b_3} \left(1 + \frac{2}{b_3} \right)^5 \frac{|\eta^{-1} - \tilde{\eta}^{-1}|}{A} \left(\frac{p}{1-p} \right) \left(\frac{\tilde{p}}{1-\tilde{p}} \right)^4.$$

(2) If $A \leq B$, then under Condition 1 we can attain the following upper bound:

$$\left| \frac{1}{p} - \frac{1}{\tilde{p}} \right| \leq \frac{6b_1}{b_3} \left(1 + \frac{2}{b_3} \right)^5 \frac{|\eta^{-1} - \tilde{\eta}^{-1}|}{B} \left(\frac{1-p}{p} \right) \left(\frac{1-\tilde{p}}{\tilde{p}} \right)^4.$$

Proof. We only prove the first part. Since $A \geq B$, we have $p, \tilde{p} \geq 1/2$. For $u = \phi^{-1}(p) \geq 0$ and $\tilde{u} = \phi^{-1}(\tilde{p}) \geq 0$, it is easy to see that u and \tilde{u} satisfy

$$A \left(\frac{1}{\phi(u)} \right)' + B \left(\frac{1}{1-\phi(u)} \right)' + \eta^{-1}(u + 3u^2) = 0,$$

$$A \left(\frac{1}{\phi(\tilde{u})} \right)' + B \left(\frac{1}{1 - \phi(\tilde{u})} \right)' + \tilde{\eta}^{-1}(\tilde{u} + 3\tilde{u}^2) = 0 . \quad (100)$$

By subtracting the first equation in (100) from the second equation, we have

$$\begin{aligned} & A \left[\left(\frac{1}{\phi(u)} \right)' - \left(\frac{1}{\phi(\tilde{u})} \right)' \right] + B \left[\left(\frac{1}{1 - \phi(u)} \right)' - \left(\frac{1}{1 - \phi(\tilde{u})} \right)' \right] \\ & + \eta^{-1}(u - \tilde{u})(1 + 3u + 3\tilde{u}) + (\eta^{-1} - \tilde{\eta}^{-1})(\tilde{u} + 3\tilde{u}^2) = 0 . \end{aligned} \quad (101)$$

The convexity assumption in Condition 1 implies that $A \left[\left(\frac{1}{\phi(u)} \right)' - \left(\frac{1}{\phi(\tilde{u})} \right)' \right], B \left[\left(\frac{1}{1 - \phi(u)} \right)' - \left(\frac{1}{1 - \phi(\tilde{u})} \right)' \right]$ and $\eta^{-1}(u - \tilde{u})(1 + 3u + 3\tilde{u})$ always have the same sign. Hence by Lemma B.16, we have

$$\begin{aligned} & \left| A \left[\left(\frac{1}{\phi(u)} \right)' - \left(\frac{1}{\phi(\tilde{u})} \right)' \right] + B \left[\left(\frac{1}{1 - \phi(u)} \right)' - \left(\frac{1}{1 - \phi(\tilde{u})} \right)' \right] + \eta^{-1}(u - \tilde{u})(1 + 3u + 3\tilde{u}) \right| \\ & \geq A \left| \left(\frac{1}{\phi(u)} \right)' - \left(\frac{1}{\phi(\tilde{u})} \right)' \right| \\ & \geq \frac{b_3}{2} \frac{A}{(1 + \tilde{u})(1 + u)(1 + \tilde{u} \wedge u)} \cdot |u - \tilde{u}| . \end{aligned} \quad (102)$$

Then by (101), (102) and Lemma B.16, we have

$$\begin{aligned} |u - \tilde{u}| & \leq \left(\frac{b_3}{2} \frac{A}{(1 + \tilde{u})(1 + u)(1 + \tilde{u} \wedge u)} \right)^{-1} |\eta^{-1} - \tilde{\eta}^{-1}|(\tilde{u} + 3\tilde{u}^2) \quad (\text{by (101) and (102)}) \\ & \leq \frac{2}{b_3} \frac{|\eta^{-1} - \tilde{\eta}^{-1}|}{A} (1 + u)(1 + \tilde{u})^2 (\tilde{u} + 3\tilde{u}^2) \\ & \leq \frac{6}{b_3} \frac{|\eta^{-1} - \tilde{\eta}^{-1}|}{A} (1 + u)(1 + \tilde{u})^4 \\ & \leq \frac{6}{b_3} \frac{|\eta^{-1} - \tilde{\eta}^{-1}|}{A} \left[\frac{2}{b_3} \left(\frac{1}{1-p} - \frac{1}{1-1/2} \right) + 1 \right] \left[\frac{2}{b_3} \left(\frac{1}{1-\tilde{p}} - \frac{1}{1-1/2} \right) + 1 \right]^4 \quad (\text{Lemma B.16}) \\ & \leq \frac{6}{b_3} \left(1 + \frac{2}{b_3} \right)^5 \frac{|\eta^{-1} - \tilde{\eta}^{-1}|}{A} \left(\frac{p}{1-p} \right) \left(\frac{\tilde{p}}{1-\tilde{p}} \right)^4 . \end{aligned}$$

Hence we can prove by Lemma B.16 that

$$\left| \frac{1}{1-p} - \frac{1}{1-\tilde{p}} \right| \leq b_1 |u - \tilde{u}| \leq \frac{6b_1}{b_3} \left(1 + \frac{2}{b_3} \right)^5 \frac{|\eta^{-1} - \tilde{\eta}^{-1}|}{A} \left(\frac{p}{1-p} \right) \left(\frac{\tilde{p}}{1-\tilde{p}} \right)^4 .$$

□

With the help of Lemma C.21, Lemma C.22 and Lemma C.23, now we can derive the exact form for the asymptotic variance of $\hat{\tau}$ in the following theorem.

Theorem 5.1. *Under Assumptions 1-3 and Condition 1, the asymptotic variance of the adaptive AIPW estimator under SIGMOID-FTRL is the oracle variance:*

$$T \cdot \text{Var}(\hat{\tau}) = 2(1 + \rho)\mathcal{E}(1)\mathcal{E}(0) + o(1) .$$

Proof. By Lemma 3.3, we have

$$T \cdot \text{Var}(\hat{\tau}) = T \cdot V^* + \frac{1}{T} E[\mathcal{R}_T^{\text{prob}}] + \frac{1}{T} E[\mathcal{R}_T^{\text{pred}}] .$$

By Lemma 4.9 and Lemma C.21, we have shown that $\frac{1}{T} \mathbb{E}[\mathcal{R}_T^{\text{pred}}] = o(1)$. Hence it suffices to prove that $\frac{1}{T} \mathbb{E}[\mathcal{R}_T^{\text{prob}}] = o(1)$. We introduce the following notations:

$$\begin{aligned}\hat{f}_t(p) &= \frac{\mathbf{1}[Z_t = 1]}{p_t} \cdot (y_t(1) - \langle x_t, \beta_t(1) \rangle)^2 \cdot \frac{1}{p} + \frac{\mathbf{1}[Z_t = 0]}{1-p_t} \cdot (y_t(0) - \langle x_t, \beta_t(0) \rangle)^2 \cdot \frac{1}{1-p} , \\ \hat{F}_t(p) &= \sum_{s=1}^{t-1} \hat{f}_s(p) + \frac{1}{\eta_t} \Psi(p) , \\ \tilde{F}_t(p) &= \sum_{s=1}^{t-1} \hat{f}_s(p) + \frac{1}{\eta_{t-1}} \Psi(p) , \\ \hat{h}_t(u) &= \frac{\mathbf{1}[Z_t = 1]}{p_t} \cdot (y_t(1) - \langle x_t, \beta_t(1) \rangle)^2 \cdot \frac{1}{\phi(u)} + \frac{\mathbf{1}[Z_t = 0]}{1-p_t} \cdot (y_t(0) - \langle x_t, \beta_t(0) \rangle)^2 \cdot \frac{1}{1-\phi(u)} , \\ \hat{H}_t(u) &= \sum_{s=1}^{t-1} \hat{h}_s(u) + \frac{1}{\eta_t} \psi(u) , \\ \tilde{H}_t(u) &= \sum_{s=1}^{t-1} \hat{h}_s(u) + \frac{1}{\eta_{t-1}} \psi(u) .\end{aligned}$$

Let $\tilde{p}_t, u_t, \tilde{u}_t$ be the minimizer of function \tilde{F}_t, \hat{H}_t and \tilde{H}_t , respectively. By similar proof as in Lemma 4.9, we have

$$\begin{aligned}& \frac{1}{T} \mathbb{E}[\mathcal{R}_T^{\text{prob}}] \\&= \frac{1}{T} \mathbb{E} \left[\sum_{t=1}^T f_t(p_t) - \sum_{t=1}^T f_t(p^*) \right] \\&= \frac{1}{T} \mathbb{E} \left[\sum_{t=1}^T \hat{f}_t(p_t) - \sum_{t=1}^T \hat{f}_t(p^*) \right] \\&= \frac{1}{T} \mathbb{E} \left[\frac{1}{\eta_{T+1}} \Psi(p^*) + \sum_{t=1}^T \hat{f}_t(p_t) + \sum_{t=1}^T (\hat{F}_t(p_t) - \hat{F}_{t+1}(p_{t+1})) + \hat{F}_{T+1}(p_{T+1}) - \hat{F}_{T+1}(p^*) \right] \\&= \frac{1}{T} \mathbb{E} \left[\frac{1}{\eta_{T+1}} \Psi(p^*) + \sum_{t=1}^T (\tilde{F}_{t+1}(p_t) - \hat{F}_{t+1}(p_{t+1})) + \hat{F}_{T+1}(p_{T+1}) - \hat{F}_{T+1}(p^*) \right] \\&= \frac{1}{T} \frac{1}{\eta_T} \Psi(p^*) + \frac{1}{T} \sum_{t=1}^T \mathbb{E} [\tilde{F}_{t+1}(p_t) - \tilde{F}_{t+1}(\tilde{p}_{t+1})] - \frac{1}{T} \sum_{t=1}^T \mathbb{E} [\tilde{F}_{t+1}(p_{t+1}) - \tilde{F}_{t+1}(\tilde{p}_{t+1})] \\&\quad + \frac{1}{T} \sum_{t=1}^T \mathbb{E} [\tilde{F}_{t+1}(p_{t+1}) - \hat{F}_{t+1}(p_{t+1})] + \frac{1}{T} \mathbb{E} [\hat{F}_{T+1}(p_{T+1}) - \hat{F}_{T+1}(p^*)] \\&= o(1) - \frac{1}{T} \sum_{t=1}^T \mathbb{E} [\tilde{F}_{t+1}(p_{t+1}) - \tilde{F}_{t+1}(\tilde{p}_{t+1})] - \frac{1}{T} \sum_{t=1}^T (\eta_{t+1}^{-1} - \eta_t^{-1}) \mathbb{E} [\Psi(p_{t+1})] \\&\quad + \frac{1}{T} \mathbb{E} [\hat{F}_{T+1}(p_{T+1}) - \hat{F}_{T+1}(\tilde{p})] + \frac{1}{T} \mathbb{E} [\hat{F}_{T+1}(\tilde{p}) - \hat{F}_{T+1}(\check{p})] \\&\quad + \frac{1}{T} \mathbb{E} [\hat{F}_{T+1}(\check{p}) - \hat{F}_{T+1}(p^*)] \quad (\text{Proposition 4.7}) \\&\triangleq o(1) - S_1 - S_2 + S_3 + S_4 + S_5 .\end{aligned} \tag{103}$$

For S_1 , by the definition of \tilde{F} and \tilde{H} , we have

$$0 \leq \frac{1}{T} \mathbb{E} \left[\sum_{t=1}^T (\tilde{F}_{t+1}(p_{t+1}) - \tilde{F}_{t+1}(\tilde{p}_{t+1})) \right] \quad (\text{optimality of } \tilde{p}_{t+1})$$

$$\begin{aligned}
&= \frac{1}{T} \sum_{t=1}^T \mathbb{E} \left[\left(\tilde{H}_{t+1}(u_{t+1}) - \tilde{H}_{t+1}(\tilde{u}_{t+1}) \right) \right] \\
&= \frac{1}{T} \mathbb{E} \left[\sum_{t=1}^T \left[\langle \nabla \tilde{H}_{t+1}(\tilde{u}_{t+1}), u_{t+1} - \tilde{u}_{t+1} \rangle + \mathcal{B}_{\tilde{H}_{t+1}}(u_{t+1} | \tilde{u}_{t+1}) \right] \right] \\
&= \frac{1}{T} \mathbb{E} \left[\sum_{t=1}^T \mathcal{B}_{\tilde{H}_{t+1}}(u_{t+1} | \tilde{u}_{t+1}) \right] \quad (\text{optimality of } \tilde{u}_{t+1}) .
\end{aligned}$$

For any $t \in [T]$, if $\hat{A}_t(1) \geq \hat{A}_t(0)$, then $u_{t+1}, \tilde{u}_{t+1} \geq 0$. Since $\eta_{t+1} \leq \eta_t$, it is easy to show that $\tilde{u}_{t+1} \geq u_{t+1}$ and $\tilde{p}_{t+1} \geq p_{t+1}$ by similar method as in Lemma 4.14. Hence for $f_1(u) = 1/\phi(u)$ and $f_2(u) = 1/(1-\phi(u))$, by Lemma B.16 we have

$$\begin{aligned}
\mathcal{B}_{f_1}(u_{t+1} | \tilde{u}_{t+1}) &= f_1(u_{t+1}) - f_1(\tilde{u}_{t+1}) - f'_1(\tilde{u}_{t+1})(u_{t+1} - \tilde{u}_{t+1}) \\
&= \frac{1}{\phi(u_{t+1})} - \frac{1}{\phi(\tilde{u}_{t+1})} - \left(\frac{1}{\phi(\tilde{u}_{t+1})} \right)' (u_{t+1} - \tilde{u}_{t+1}) \\
&\leq b_2 \cdot \frac{\tilde{u}_{t+1} - u_{t+1}}{(1 + u_{t+1})(1 + \tilde{u}_{t+1})} , \\
\mathcal{B}_{f_2}(u_{t+1} | \tilde{u}_{t+1}) &= f_2(u_{t+1}) - f_2(\tilde{u}_{t+1}) - f'_2(\tilde{u}_{t+1})(u_{t+1} - \tilde{u}_{t+1}) \\
&= \frac{1}{1 - \phi(u_{t+1})} - \frac{1}{1 - \phi(\tilde{u}_{t+1})} - \left(\frac{1}{1 - \phi(\tilde{u}_{t+1})} \right)' (u_{t+1} - \tilde{u}_{t+1}) \\
&\leq b_2 \cdot \frac{\tilde{u}_{t+1} - u_{t+1}}{(1 + u_{t+1})(1 + \tilde{u}_{t+1})} , \\
\mathcal{B}_\psi(u_{t+1} | \tilde{u}_{t+1}) &= \psi(u_{t+1}) - \psi(\tilde{u}_{t+1}) - \psi'(\tilde{u}_{t+1})(u_{t+1} - \tilde{u}_{t+1}) \\
&= \frac{1}{2} u_{t+1}^2 + u_{t+1}^3 - \frac{1}{2} \tilde{u}_{t+1}^2 - \tilde{u}_{t+1}^3 - (\tilde{u}_{t+1} + 3\tilde{u}_{t+1}^2)(u_{t+1} - \tilde{u}_{t+1}) \\
&= \frac{1}{2} (u_{t+1} - \tilde{u}_{t+1})^2 (1 + 2u_{t+1} + 4\tilde{u}_{t+1}) .
\end{aligned}$$

On the other side, by the definition of p_{t+1} and \tilde{p}_{t+1} , there holds:

$$\begin{aligned}
&-\frac{\hat{A}_t(1)}{\tilde{p}_{t+1}^2} + \frac{\hat{A}_t(0)}{(1 - \tilde{p}_{t+1})^2} + \eta_t^{-1} \Psi'(\tilde{p}_{t+1}) = 0 , \\
&-\frac{\hat{A}_t(1)}{p_{t+1}^2} + \frac{\hat{A}_t(0)}{(1 - p_{t+1})^2} + \eta_{t+1}^{-1} \Psi'(p_{t+1}) = 0 .
\end{aligned}$$

Then by Lemma B.16, Lemma C.1 and Lemma C.23, we have

$$\begin{aligned}
&\mathcal{B}_{\tilde{H}_{t+1}}(u_{t+1} | \tilde{u}_{t+1}) \\
&= \hat{A}_t(1) \cdot \mathcal{B}_{f_1}(u_{t+1} | \tilde{u}_{t+1}) + \hat{A}_t(0) \cdot \mathcal{B}_{f_2}(u_{t+1} | \tilde{u}_{t+1}) + \eta_t^{-1} \mathcal{B}_\psi(u_{t+1} | \tilde{u}_{t+1}) \\
&\lesssim \hat{A}_t(1) \cdot \frac{\tilde{u}_{t+1} - u_{t+1}}{(1 + u_{t+1})(1 + \tilde{u}_{t+1})} + \hat{A}_t(0) \cdot \frac{\tilde{u}_{t+1} - u_{t+1}}{(1 + u_{t+1})(1 + \tilde{u}_{t+1})} + \eta_t^{-1} (\tilde{u}_{t+1} - u_{t+1})^2 (1 + \tilde{u}_{t+1} + u_{t+1}) \\
&\lesssim \hat{A}_t(1) \cdot \frac{\tilde{u}_{t+1} - u_{t+1}}{(1 + u_{t+1})(1 + \tilde{u}_{t+1})} + \eta_t^{-1} (\tilde{u}_{t+1} - u_{t+1})^2 (1 + \tilde{u}_{t+1}) \quad (\text{since } \hat{A}_t(1) \geq \hat{A}_t(0) \text{ and } \tilde{u}_{t+1} \geq u_{t+1}) \\
&\lesssim \hat{A}_t(1) \cdot \frac{\frac{1}{1 - \tilde{p}_{t+1}} - \frac{1}{1 - p_{t+1}}}{(1 + u_{t+1})(1 + \tilde{u}_{t+1})} + \eta_t^{-1} \left(\frac{1}{1 - \tilde{p}_{t+1}} - \frac{1}{1 - p_{t+1}} \right)^2 (1 + \tilde{u}_{t+1}) \quad (\text{Lemma B.16}) \\
&\lesssim \hat{A}_t(1) \cdot \left(\frac{\tilde{p}_{t+1}}{1 - \tilde{p}_{t+1}} \right)^{-1} \left(\frac{p_{t+1}}{1 - p_{t+1}} \right)^{-1} \cdot \frac{\eta_{t+1}^{-1} - \eta_t^{-1}}{\hat{A}_t(1)} \left(\frac{\tilde{p}_{t+1}}{1 - \tilde{p}_{t+1}} \right) \left(\frac{p_{t+1}}{1 - p_{t+1}} \right)^4 \quad (\text{Lemma C.23}) \\
&\quad + \eta_t^{-1} \left(\frac{\tilde{p}_{t+1}}{1 - \tilde{p}_{t+1}} \frac{\eta_t \hat{A}_t(1) - \eta_{t+1} \hat{A}_t(1)}{\eta_t \hat{A}_t(1)} + \frac{p_{t+1}}{1 - p_{t+1}} \frac{\eta_t \hat{A}_t(0) - \eta_{t+1} \hat{A}_t(0)}{\eta_t \hat{A}_t(0)} \right)^2 \left(\frac{\tilde{p}_{t+1}}{1 - \tilde{p}_{t+1}} \right) \quad (\text{Lemma C.1})
\end{aligned}$$

$$\lesssim (\eta_{t+1}^{-1} - \eta_t^{-1}) \left(\frac{p_{t+1}}{1-p_{t+1}} \right)^3 + \eta_t^{-1} \left(\frac{\eta_{t+1}^{-1} - \eta_t^{-1}}{\eta_{t+1}^{-1}} \right)^2 \left(\frac{\tilde{p}_{t+1}}{1-\tilde{p}_{t+1}} \right)^3 \quad (\text{since } \tilde{p}_{t+1} \geq p_{t+1}) .$$

If $\hat{A}_t(1) \leq \hat{A}_t(0)$, by symmetry we can similarly prove that

$$\mathcal{B}_{\tilde{H}_{t+1}}(u_{t+1}|\tilde{u}_{t+1}) \lesssim (\eta_{t+1}^{-1} - \eta_t^{-1}) \left(\frac{1-p_{t+1}}{p_{t+1}} \right)^3 + \eta_t^{-1} \left(\frac{\eta_{t+1}^{-1} - \eta_t^{-1}}{\eta_{t+1}^{-1}} \right)^2 \left(\frac{1-\tilde{p}_{t+1}}{\tilde{p}_{t+1}} \right)^3 .$$

Since $\eta_t \geq \eta_{t+1}$, by Corollary 4.15, Corollary B.17 and Assumption 3, we have

$$\begin{aligned} & \frac{1}{T} \mathbb{E} \left[\sum_{t=1}^T \mathcal{B}_{\tilde{H}_{t+1}}(u_{t+1}|\tilde{u}_{t+1}) \right] \\ & \lesssim \frac{1}{T} \sum_{t=1}^T (\eta_{t+1}^{-1} - \eta_t^{-1}) \mathbb{E} \left[\left(\frac{1}{1-p_{t+1}} \right)^3 + \left(\frac{1}{p_{t+1}} \right)^3 \right] + \frac{1}{T} \sum_{t=1}^T \eta_t^{-1} \left(\frac{\eta_{t+1}^{-1} - \eta_t^{-1}}{\eta_{t+1}^{-1}} \right)^2 \mathbb{E} \left[\left(\frac{1}{1-\tilde{p}_{t+1}} \right)^3 + \left(\frac{1}{\tilde{p}_{t+1}} \right)^3 \right] \\ & \lesssim \frac{1}{T} \sum_{t=1}^T (\eta_{t+1}^{-1} - \eta_t^{-1}) \eta_{t+1}^{3/4} T^{3/4} + \frac{1}{T} \sum_{t=1}^T \eta_t^{-1} \left(\frac{\eta_{t+1}^{-1} - \eta_t^{-1}}{\eta_{t+1}^{-1}} \right)^2 \eta_t^{3/4} T^{3/4} \quad (\text{Corollary 4.15 and Corollary B.17}) \\ & \leq T^{-1/4} \left[\sum_{t=1}^T (\eta_{t+1}^{-1} - \eta_t^{-1}) (\eta_{t+1}^{-1})^{-3/4} + \sum_{t=1}^T \eta_t^{-1/4} (\eta_{t+1}^{-1} - \eta_t^{-1}) \eta_{t+1} \right] \\ & \leq T^{-1/4} \left[\sum_{t=1}^T (\eta_{t+1}^{-1} - \eta_t^{-1}) (\eta_{t+1}^{-1})^{-3/4} + \sum_{t=1}^T (\eta_{t+1}^{-1} - \eta_t^{-1}) (\eta_{t+1}^{-1})^{-3/4} \right] \quad (\text{since } \eta_t^{-1/4} \leq \eta_{t+1}^{-1/4}) \\ & \leq 2T^{-1/4} \int_{\eta_1^{-1}}^{\eta_{T+1}^{-1}} x^{-3/4} dx \\ & \lesssim T^{-1/4} \eta_T^{-1/4} \\ & \rightarrow 0 \quad (\text{Assumption 3}) , \end{aligned}$$

which implies that

$$S_1 = \frac{1}{T} \mathbb{E} \left[\sum_{t=1}^T \mathcal{B}_{\tilde{H}_{t+1}}(u_{t+1}|\tilde{u}_{t+1}) \right] = o(1) . \quad (104)$$

For $p_{t+1} \geq 1/2$, by Lemma B.16, we have

$$\phi^{-1}(p_{t+1}) = u_{t+1} - 0 \leq \frac{2}{b_3} \left(\frac{1}{1-p_{t+1}} - \frac{1}{1-1/2} \right) \leq \frac{1}{1-p_{t+1}} .$$

Similarly, we can prove that for $p_{t+1} \leq 1/2$, we have $-\phi^{-1}(p_{t+1}) \lesssim \frac{1}{p_{t+1}}$. Hence by Corollary 4.15 and Corollary B.17, we have

$$\begin{aligned} S_2 &= \frac{1}{T} \sum_{t=1}^T (\eta_{t+1}^{-1} - \eta_t^{-1}) \mathbb{E} [\Psi(p_{t+1})] \\ &= \frac{1}{T} \sum_{t=1}^T (\eta_{t+1}^{-1} - \eta_t^{-1}) \mathbb{E} \left[\frac{1}{2} (\phi^{-1}(p_{t+1}))^2 + |\phi^{-1}(p_{t+1})|^3 \right] \\ &\leq \frac{1}{T} \sum_{t=1}^T (\eta_{t+1}^{-1} - \eta_t^{-1}) \eta_{t+1}^{3/4} T^{3/4} \quad (\text{Corollary 4.15 and Corollary B.17}) \\ &\leq T^{-1/4} \sum_{t=1}^T (\eta_{t+1}^{-1} - \eta_t^{-1}) (\eta_{t+1}^{-1})^{-3/4} \end{aligned}$$

$$\begin{aligned}
&\leq T^{-1/4} \int_{\eta_T^{-1}}^{\eta_{T+1}^{-1}} x^{-3/4} dx \\
&\lesssim T^{-1/4} \eta_T^{-1/4} \\
&\rightarrow 0 \quad (\text{Assumption 3}) .
\end{aligned} \tag{105}$$

Let $\tilde{u} = \phi^{-1}(\tilde{p})$. Then by Lemma 4.3 and Lemma B.20, we have

$$\begin{aligned}
0 &\leq \frac{1}{T} \mathbb{E} [\hat{F}_{T+1}(\tilde{p}) - \hat{F}_{T+1}(p_{T+1})] \quad (\text{optimality of } p_{T+1}) \\
&= \frac{1}{T} \mathbb{E} [\hat{H}_{T+1}(\tilde{u}) - \hat{H}_{T+1}(u_{T+1})] \\
&= \frac{1}{T} \mathbb{E} [\langle \nabla \hat{H}_{T+1}(u_{T+1}), \tilde{u} - u_{T+1} \rangle + \mathcal{B}_{\hat{H}_{T+1}}(\tilde{u}|u_{T+1})] \\
&= \frac{1}{T} \mathbb{E} [\mathcal{B}_{\hat{H}_{T+1}}(\tilde{u}|u_{T+1})] \quad (\text{optimality of } u_{T+1}) \\
&\leq \frac{1}{T} \mathbb{E} \left[\sup_{z \in \mathbb{R}} \left\{ \langle \nabla \hat{H}_{T+1}(u_{T+1}) - \nabla \hat{H}_{T+1}(\tilde{u}), z - u_t \rangle - \frac{1}{\eta_T} \mathcal{B}_\psi(z|\tilde{u}) \right\} \right] \quad (\text{Lemma B.20}) \\
&= \frac{1}{T} \mathbb{E} \left[\sup_{z \in \mathbb{R}} \left\{ \langle -\nabla \hat{H}_{T+1}(\tilde{u}), u_t - z \rangle - \frac{1}{\eta_T} \mathcal{B}_\psi(z|\tilde{u}) \right\} \right] \quad (\text{optimality of } u_{T+1}) \\
&\leq \frac{1}{T} \mathbb{E} \left[\sup_{z \in \mathbb{R}} \left\{ \langle -\nabla \hat{H}_{T+1}(\tilde{u}), u_t - z \rangle - \frac{1}{2\eta_T} (z - \tilde{u})^2 (1 + |\tilde{u}|) \right\} \right] \quad (\text{Lemma 4.3}) \\
&\leq \frac{\eta_T}{2T(1 + |\tilde{u}|)} \mathbb{E} [(\nabla \hat{H}_{T+1}(\tilde{u}))^2] \\
&\leq T^{-1} \eta_T \mathbb{E} [(\nabla \hat{H}_{T+1}(\tilde{u}))^2] .
\end{aligned} \tag{106}$$

By definition, $\tilde{u} = \phi^{-1}(\tilde{p})$ should satisfy the following first order equation:

$$\begin{aligned}
&\sum_{t=1}^T \left[\frac{\mathbf{1}[Z_t = 1]}{p_t} (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^2 \cdot \left(\frac{1}{\phi(\tilde{u})} \right)' \right. \\
&\quad \left. + \frac{\mathbf{1}[Z_t = 0]}{1-p_t} (y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(0) \rangle)^2 \cdot \left(\frac{1}{1-\phi(\tilde{u})} \right)' \right] + \eta_T^{-1} \psi'(\tilde{u}) = 0 .
\end{aligned}$$

Then by the definition of \hat{H}_{T+1} , Condition 1, Cauchy-Schwarz inequality and AM-GM inequality, we have

$$\begin{aligned}
&\mathbb{E} [(\nabla \hat{H}_{T+1}(\tilde{u}))^2] \\
&= \mathbb{E} \left[\left(\sum_{t=1}^T \nabla \hat{h}_t(\tilde{u}) + \eta_T^{-1} \psi(\tilde{u}) \right)^2 \right] \\
&= \mathbb{E} \left[\left(\sum_{t=1}^T \frac{\mathbf{1}[Z_t = 1]}{p_t} \cdot (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle)^2 \cdot \left(\frac{1}{\phi(\tilde{u})} \right)' \right. \right. \\
&\quad \left. \left. + \frac{\mathbf{1}[Z_t = 0]}{1-p_t} \cdot (y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle)^2 \cdot \left(\frac{1}{1-\phi(\tilde{u})} \right)' + \eta_T^{-1} \psi(\tilde{u}) \right)^2 \right] \\
&= \mathbb{E} \left[\left(\sum_{t=1}^T \frac{\mathbf{1}[Z_t = 1]}{p_t} \cdot [(y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle)^2 - (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^2] \cdot \left(\frac{1}{\phi(\tilde{u})} \right)' \right. \right. \\
&\quad \left. \left. + \frac{\mathbf{1}[Z_t = 0]}{1-p_t} \cdot [(y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle)^2 - (y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(0) \rangle)^2] \cdot \left(\frac{1}{\phi(\tilde{u})} \right)' \right)^2 \right] \quad (\text{subtract the first-order equation})
\end{aligned}$$

$$\begin{aligned}
&\lesssim \mathbb{E} \left[\left(\sum_{t=1}^T \frac{\mathbf{1}[Z_t = 1]}{p_t} \cdot [(y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle)^2 - (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^2] \cdot \left(\frac{1}{\phi(\tilde{u})} \right)' \right)^2 \right] \\
&\quad + \mathbb{E} \left[\left(\sum_{t=1}^T \frac{\mathbf{1}[Z_t = 0]}{1-p_t} \cdot [(y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle)^2 - (y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(0) \rangle)^2] \cdot \left(\frac{1}{1-\phi(\tilde{u})} \right)' \right)^2 \right] \\
&\lesssim \mathbb{E} \left[\left(\sum_{t=1}^T \left(\frac{\mathbf{1}[Z_t = 1]}{p_t} - 1 \right) [(y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle)^2 - (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^2] \right)^2 \right] \\
&\quad + \mathbb{E} \left[\left(\sum_{t=1}^T \left(\frac{\mathbf{1}[Z_t = 0]}{1-p_t} - 1 \right) [(y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle)^2 - (y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(0) \rangle)^2] \right)^2 \right] \\
&\quad + \mathbb{E} \left[\left(\sum_{t=1}^T [(y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle)^2 - (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^2] \right)^2 \right] \\
&\quad + \mathbb{E} \left[\left(\sum_{t=1}^T [(y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle)^2 - (y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(0) \rangle)^2] \right)^2 \right] \quad (\text{Cauchy-Schwarz and AM-GM}) \\
&\triangleq B_1 + B_2 + B_3 + B_4 . \tag{107}
\end{aligned}$$

By the proof in Lemma C.3 and Lemma C.22, Cauchy-Schwarz inequality and AM-GM inequality, we have

$$\begin{aligned}
B_1 &= \text{Var} \left(\sum_{t=1}^T \left(\frac{\mathbf{1}[Z_t = 1]}{p_t} - 1 \right) [(y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle)^2 - (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^2] \right) \\
&\lesssim \text{Var} \left(\sum_{t=1}^T \left(\frac{\mathbf{1}[Z_t = 1]}{p_t} - 1 \right) (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle)^2 \right) + \text{Var} \left(\sum_{t=1}^T \left(\frac{\mathbf{1}[Z_t = 1]}{p_t} - 1 \right) (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^2 \right) \\
&\lesssim \eta_T^{-2} (\eta_T T)^{1/2} \log^2(\eta_T T) (\eta_T T)^{25/64} + T^{5/4} R_T (\eta_T T)^{1/4} \quad (\text{by the proofs in Lemma C.3 and C.22}) \\
&\lesssim T^{3/2} R_T^2 \cdot (\eta_T T)^{-1+1/2+25/64} \log^2(\eta_T T) \\
&= o(T^{3/2} R_T^2) , \\
B_2 &= o(T^{3/2} R_T^2) , \\
B_3 &\lesssim \eta_T^{-2} (\eta_T T)^{1/2} \log^2(\eta_T T) (\eta_T T)^{25/64} \quad (\text{check the detailed proof in Lemma C.3}) \\
&= o(T^{3/2} R_T^2) , \\
B_4 &= o(T^{3/2} R_T^2) . \tag{108}
\end{aligned}$$

Hence by (107) and (108), we have $\mathbb{E}[(\nabla \hat{H}_{T+1}(\tilde{u}))^2] \lesssim B_1 + B_2 + B_3 + B_4 = o(T^{3/2} R_T^2)$. Then by (106), we have

$$|S_3| \lesssim T^{-1} \eta_T \mathbb{E}[(\nabla \hat{H}_{T+1}(\tilde{u}))^2] = T^{-1} T^{-1/2} R_T^{-2} \cdot o(T^{3/2} R_T^2) = o(1) . \tag{109}$$

For S_4 , we have the following decomposition:

$$\begin{aligned}
S_4 &= \frac{1}{T} \sum_{t=1}^T \mathbb{E} \left[\frac{\mathbf{1}[Z_t = 1]}{p_t} \cdot (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle)^2 \cdot \frac{1}{\tilde{p}} + \frac{\mathbf{1}[Z_t = 0]}{1-p_t} \cdot (y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle)^2 \cdot \frac{1}{1-\tilde{p}} \right] \\
&\quad - \frac{1}{T} \sum_{t=1}^T \mathbb{E} \left[\frac{\mathbf{1}[Z_t = 1]}{p_t} \cdot (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle)^2 \cdot \frac{1}{\check{p}} + \frac{\mathbf{1}[Z_t = 0]}{1-p_t} \cdot (y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle)^2 \cdot \frac{1}{1-\check{p}} \right] \\
&\quad + (\eta_T T)^{-1} \mathbb{E}[\Psi(\tilde{p}) - \Psi(\check{p})]
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{T} \sum_{t=1}^T \mathbb{E} \left[\frac{\mathbf{1}[Z_t = 1]}{p_t} \cdot (y_t(1) - \langle x_t, \beta_t(1) \rangle)^2 \cdot \left(\frac{1}{\tilde{p}} - \frac{1}{\check{p}} \right) \right] \\
&\quad + \frac{1}{T} \sum_{t=1}^T \mathbb{E} \left[\frac{\mathbf{1}[Z_t = 0]}{1-p_t} \cdot (y_t(0) - \langle x_t, \beta_t(0) \rangle)^2 \cdot \left(\frac{1}{1-\tilde{p}} - \frac{1}{1-\check{p}} \right) \right] + (\eta_T T)^{-1} \mathbb{E} [\Psi(\tilde{p}) - \Psi(\check{p})] \\
&\triangleq S_{4,1} + S_{4,2} + S_{4,3} .
\end{aligned} \tag{110}$$

By Cauchy-Schwarz inequality and AM-GM inequality, we have the following decomposition for $S_{4,1}$:

$$\begin{aligned}
|S_{4,1}| &= \left| \frac{1}{T} \sum_{t=1}^T \mathbb{E} \left[\frac{\mathbf{1}[Z_t = 1]}{p_t} (y_t(1) - \langle x_t, \beta_t(1) \rangle)^2 \cdot \left(\frac{1}{\tilde{p}} - \frac{1}{\check{p}} \right) \right] \right| \\
&\leq \frac{1}{T} \sum_{t=1}^T \mathbb{E} \left[\frac{\mathbf{1}[Z_t = 1]}{p_t} (y_t(1) - \langle x_t, \beta_t(1) \rangle)^2 \cdot \left| \frac{1}{\tilde{p}} - \frac{1}{\check{p}} \right| \right] \\
&= \frac{1}{T} \sum_{t=1}^T \mathbb{E} \left[\frac{\mathbf{1}[Z_t = 1]}{p_t} (y_t(1) - \langle x_t, \beta_t^*(1) \rangle + \langle x_t, \beta_t^*(1) - \beta_t(1) \rangle)^2 \cdot \left| \frac{1}{\tilde{p}} - \frac{1}{\check{p}} \right| \right] \\
&\lesssim \frac{1}{T} \sum_{t=1}^T \mathbb{E} \left[\frac{\mathbf{1}[Z_t = 1]}{p_t} (y_t(1) - \langle x_t, \beta_t^*(1) \rangle)^2 \cdot \left| \frac{1}{\tilde{p}} - \frac{1}{\check{p}} \right| \right] \\
&\quad + \frac{1}{T} \sum_{t=1}^T \mathbb{E} \left[\frac{\mathbf{1}[Z_t = 1]}{p_t} \langle x_t, \beta_t(1) - \beta_t^*(1) \rangle^2 \cdot \left| \frac{1}{\tilde{p}} - \frac{1}{\check{p}} \right| \right] \\
&\triangleq D_1 + D_2 .
\end{aligned} \tag{111}$$

By Corollary B.11, Corollary 4.15, Corollary B.17, Lemma C.22, Hölder's inequality and Assumption 3, we have

$$\begin{aligned}
D_1 &= \frac{1}{T} \sum_{t=1}^T \mathbb{E} \left[\frac{\mathbf{1}[Z_t = 1]}{p_t} \cdot (y_t(1) - \langle x_t, \beta_t^*(1) \rangle)^2 \cdot \left| \frac{1}{\tilde{p}} - \frac{1}{\check{p}} \right| \right] \\
&= \frac{1}{T} \sum_{t=1}^T (y_t(1) - \langle x_t, \beta_t^*(1) \rangle)^2 \cdot \mathbb{E} \left[\frac{\mathbf{1}[Z_t = 1]}{p_t} \cdot \left| \frac{1}{\tilde{p}} - \frac{1}{\check{p}} \right| \right] \\
&\leq \frac{1}{T} \sum_{t=1}^T (y_t(1) - \langle x_t, \beta_t^*(1) \rangle)^2 \cdot \left(\mathbb{E} \left[\frac{\mathbf{1}[Z_t = 1]}{p_t^5} \right] \right)^{1/5} \left(\mathbb{E} \left[\left| \frac{1}{\tilde{p}} - \frac{1}{\check{p}} \right|^{5/4} \right] \right)^{4/5} \quad (\text{Hölder's inequality}) \\
&\lesssim \frac{1}{T} \sum_{t=1}^T (y_t(1) - \langle x_t, \beta_t^*(1) \rangle)^2 \cdot \left(\mathbb{E} \left[\frac{1}{p_t^4} \right] \right)^{1/5} \left(T^{-5/24} R_T^{1/6} \right)^{4/5} \quad (\text{Lemma C.22}) \\
&\lesssim \left(\frac{1}{T} \sum_{t=1}^T (y_t(1) - \langle x_t, \beta_t^*(1) \rangle)^2 \right)^{1/5} \cdot T^{\frac{1}{8} \cdot \frac{4}{5}} \cdot T^{-1/6} R_T^{2/15} \quad (\text{Corollary 4.15 and Corollary B.17}) \\
&\lesssim (T R_T^{-4})^{-1/15} R_T^{-2/15} \quad (\text{Corollary B.11}) \\
&= o(1) \quad (\text{Assumption 3}) .
\end{aligned} \tag{112}$$

By the definition of $\tilde{A}_T(0)$, Corollary B.11, Lemma 4.14 and Proposition 5.4, we have

$$\begin{aligned}
\frac{1}{\tilde{p}} &\leq \eta_T^{1/4} \tilde{A}_T^{1/4}(0) \leq \eta_T^{1/4} \cdot (T^{7/26} R_T^{-4/11} \cdot A_T^*(0))^{1/4} \leq \eta_T^{1/4} \cdot (T^{33/26} R_T^{-4/11})^{1/4} = T^{5/26} R_T^{-13/22} , \\
\frac{1}{\check{p}} &\leq \eta_T^{1/4} (A_T^*(0))^{1/4} \lesssim \eta_T^{1/4} T^{1/4} \lesssim T^{5/26} R_T^{-13/22} .
\end{aligned}$$

Hence by law of iterated expectation and (88), we have

$$\begin{aligned}
D_2 &= \frac{2}{T} \sum_{t=1}^T \mathbb{E} \left[\frac{\mathbf{1}[Z_t = 1]}{p_t} \cdot \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) - \boldsymbol{\beta}_t^*(1) \rangle^2 \cdot \left| \frac{1}{\tilde{p}} - \frac{1}{\check{p}} \right| \right] \\
&\lesssim T^{-1+5/26} R_T^{-13/22} \sum_{t=1}^T \mathbb{E} \left[\frac{\mathbf{1}[Z_t = 1]}{p_t} \cdot \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) - \boldsymbol{\beta}_t^*(1) \rangle^2 \right] \\
&\leqslant T^{-21/26} R_T^{-13/22} \sum_{t=1}^T \mathbb{E} [\langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) - \boldsymbol{\beta}_t^*(1) \rangle^2] \quad (\text{by law of iterated expectation}) \\
&\lesssim T^{-21/26} R_T^{-13/22} \cdot T^{3/4} \quad (\text{by (88)}) \\
&= T^{-3/52} R_T^{-13/22} \\
&= o(1) . \tag{113}
\end{aligned}$$

By (111), (112) and (113), we have $S_{4,1} = o(1)$. Similarly, we can prove that $S_{4,2} = o(1)$. For $S_{4,3}$, by similar method as in bounding S_2 , Corollary B.11 and Corollary 4.15, we have

$$\begin{aligned}
|S_{4,3}| &\leqslant (\eta_T T)^{-1} \mathbb{E} [\Psi(\check{p})] + (\eta_T T)^{-1} \Psi(\check{p}) \\
&\lesssim (\eta_T T)^{-1} \mathbb{E} \left[\frac{1}{\check{p}^3} + \frac{1}{(1-\check{p})^3} \right] + (\eta_T T)^{-1} \mathbb{E} \left[\frac{1}{\check{p}^3} + \frac{1}{(1-\check{p})^3} \right] \\
&\lesssim (\eta_T T)^{-1} \left[\eta_T^{3/4} \mathbb{E}^{3/4} \tilde{A}_T(0) + \eta_T^{3/4} \mathbb{E}^{3/4} \tilde{A}_T(1) + \eta_T^{3/4} (A_T^*(0))^{3/4} + \eta_T^{3/4} \mathbb{E}^{3/4} (A_T^*(1))^{3/4} \right] \quad (\text{Corollary 4.15}) \\
&\lesssim (\eta_T T)^{-1} \left[\eta_T^{3/4} (A_T^*(0))^{3/4} + \eta_T^{3/4} (A_T^*(1))^{3/4} \right] \\
&\lesssim (\eta_T T)^{-1} \cdot (\eta_T T)^{3/4} \quad (\text{Corollary B.11}) \\
&= (\eta_T T)^{-1/4} \\
&= o(1) \quad (\text{Assumption 3}) .
\end{aligned}$$

Hence by (110), we have $S_4 = S_{4,1} + S_{4,2} + S_{4,3} = o(1)$. For S_5 , we have the following decomposition:

$$\begin{aligned}
S_5 &= \frac{1}{T} \sum_{t=1}^T \mathbb{E} \left[\frac{\mathbf{1}[Z_t = 1]}{p_t} \cdot (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle)^2 \cdot \frac{1}{\check{p}} + \frac{\mathbf{1}[Z_t = 0]}{1-p_t} \cdot (y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle)^2 \cdot \frac{1}{1-\check{p}} \right] \\
&\quad - \frac{1}{T} \sum_{t=1}^T \mathbb{E} \left[\frac{\mathbf{1}[Z_t = 1]}{p_t} \cdot (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle)^2 \cdot \frac{1}{p^*} + \frac{\mathbf{1}[Z_t = 0]}{1-p_t} \cdot (y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle)^2 \cdot \frac{1}{1-p^*} \right] \\
&\quad + (\eta_T T)^{-1} [\Psi(\check{p}) - \Psi(p^*)] \\
&= \frac{1}{T} \sum_{t=1}^T \mathbb{E} \left[(y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle)^2 \cdot \frac{1}{\check{p}} + (y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle)^2 \cdot \frac{1}{1-\check{p}} \right] \\
&\quad - \frac{1}{T} \sum_{t=1}^T \mathbb{E} \left[(y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle)^2 \cdot \frac{1}{p^*} + (y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle)^2 \cdot \frac{1}{1-p^*} \right] \quad (\text{since } \check{p} \text{ and } p^* \text{ are nonrandom}) \\
&\quad + (\eta_T T)^{-1} [\Psi(\check{p}) - \Psi(p^*)] \\
&= (\eta_T T)^{-1} [\Psi(\check{p}) - \Psi(p^*)] + \frac{1}{T} \sum_{t=1}^T \mathbb{E} \left[(y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle)^2 \right] \cdot \left(\frac{1}{\check{p}} - \frac{1}{p^*} \right) \\
&\quad + \frac{1}{T} \sum_{t=1}^T \mathbb{E} \left[(y_t(0) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(0) \rangle)^2 \right] \cdot \left(\frac{1}{1-\check{p}} - \frac{1}{1-p^*} \right) \\
&\triangleq S_{5,1} + S_{5,2} + S_{5,3} . \tag{114}
\end{aligned}$$

By similar method as in the previous step, we can prove that $S_{5,1} = o(1)$. For $S_{5,2}$, we consider the following four cases:

- (1) $\mathcal{E}_1 \geq \mathcal{E}_0$ and $A_T^*(1) \geq A_T^*(0)$.

In such case, by definition, \check{p} and p^* should satisfy:

$$\begin{aligned} -\frac{A_T^*(1)}{\check{p}^2} + \frac{A_T^*(0)}{(1-\check{p})^2} + \eta_T^{-1}\Psi(\check{p}) &= 0 , \\ -\frac{T \cdot \mathcal{E}^2(1)}{(p^*)^2} + \frac{T \cdot \mathcal{E}^2(0) + \Delta}{(1-p^*)^2} + \eta_T^{-1}\Psi(p^*) &= 0 , \end{aligned}$$

where $\Delta \triangleq -\eta_T^{-1}(1-p^*)^2\Psi(p^*)$. Since p^* is bounded away from 0 and 1 by Assumption 1, we have $\Delta = \mathcal{O}(\eta_T^{-1}) = o(T)$ by Assumption 3. By Assumption 1, Lemma B.11 and similar proof as in Lemma C.21, it is easy to see that $A_T^*(1) = \Theta(T)$ and $A_T^*(0) = \Theta(T)$, then we have $1/\check{p}, 1/(1-\check{p}) = \mathcal{O}(1)$ and $1/p^*, 1/(1-p^*) = \mathcal{O}(1)$ by Lemma 4.14. Then by Assumption 1, Lemma C.1, Lemma B.11 and similar proof as in Lemma C.21, we have

$$\begin{aligned} \left| \frac{1}{\check{p}} - \frac{1}{p^*} \right| &\lesssim \frac{|A_T^*(1) - T \cdot \mathcal{E}^2(1)|}{T \cdot \mathcal{E}^2(1)} + \frac{|A_T^*(0) - T \cdot \mathcal{E}^2(0) - \Delta|}{T \cdot \mathcal{E}^2(1)} \quad (\text{Lemma C.1}) \\ &\lesssim T^{-1} \cdot o(T) + T^{-1} \cdot o(T) \quad (\text{Assumption 1, Lemma B.11 and Lemma C.21}) \\ &= o(1) . \end{aligned}$$

- (2) $\mathcal{E}_1 \leq \mathcal{E}_0$ and $A_T^*(1) \leq A_T^*(0)$.

We can prove that $\left| \frac{1}{\check{p}} - \frac{1}{p^*} \right| = o(1)$ by the same method as in case (1).

- (3) $\mathcal{E}_1 \geq \mathcal{E}_0$ and $A_T^*(1) \leq A_T^*(0)$.

In such case, \check{p} and p^* satisfies:

$$\begin{aligned} -\frac{A_T^*(1)}{\check{p}^2} + \frac{A_T^*(0)}{(1-\check{p})^2} + \eta_T^{-1}\Psi(\check{p}) &= 0 , \\ -\frac{T \cdot \mathcal{E}^2(1)}{(p^*)^2} + \frac{T \cdot \mathcal{E}^2(0) + \Delta}{(1-p^*)^2} + \eta_T^{-1}\Psi(p^*) &= 0 , \end{aligned}$$

where $\Delta \triangleq -\eta_T^{-1}(1-p^*)^2\Psi(p^*) = o(T)$. Then by Lemma B.11 and similar proof as in Lemma C.21, we have

$$\begin{aligned} 0 &\leq A_T^*(0) - A_T^*(1) \leq T \cdot \mathcal{E}^2(0) - T \cdot \mathcal{E}^2(1) + |A_T^*(1) - T \cdot \mathcal{E}^2(1)| + |A_T^*(0) - T \cdot \mathcal{E}^2(0)| \\ &\leq |A_T^*(1) - T \cdot \mathcal{E}^2(1)| + |A_T^*(0) - T \cdot \mathcal{E}^2(0)| \\ &= o(T) . \end{aligned}$$

Then by Lemma C.2, we have

$$\left| \frac{1}{\check{p}} - 2 \right| = \left| \frac{\check{p} - (1/2)}{\check{p} \cdot (1/2)} \right| \leq \left| \frac{\check{p} - (1/2)}{(1-\check{p}) \cdot (1/2)} \right| = \left| \frac{1}{1-\check{p}} - 2 \right| \leq \frac{A_T^*(0) - A_T^*(1)}{2A_T^*(1)} = o(1) .$$

Similarly we can prove that $T \cdot \mathcal{E}^2(1) - T \cdot \mathcal{E}^2(0) = o(T)$. Then by Assumption 1 and Lemma C.2, we have

$$\left| \frac{1}{p^*} - 2 \right| \leq \frac{T \cdot \mathcal{E}^2(1) - T \cdot \mathcal{E}^2(0) - \Delta}{T \cdot \mathcal{E}^2(0) + \Delta} = o(1) .$$

Hence we have

$$\left| \frac{1}{\check{p}} - \frac{1}{p^*} \right| \leq \left| \frac{1}{\check{p}} - 2 \right| + \left| \frac{1}{p^*} - 2 \right| = o(1) .$$

(4) $\mathcal{E}_1 \leq \mathcal{E}_0$ and $A_T^*(1) \geq A_T^*(0)$.

We can prove that $\left| \frac{1}{\bar{p}} - \frac{1}{p^*} \right| = o(1)$ by the same method as in case (3).

By the discussions in these four cases, we proved that $\left| \frac{1}{\bar{p}} - \frac{1}{p^*} \right| = o(1)$. Then by Corollary B.17 and (88), we have

$$\begin{aligned} |S_{5,2}| &\leq \frac{1}{T} \sum_{t=1}^T \mathbb{E} \left[(y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) \rangle)^2 \right] \cdot \left| \frac{1}{\bar{p}} - \frac{1}{p^*} \right| \\ &= \frac{1}{T} \left(\sum_{t=1}^T (y_t(1) - \langle \mathbf{x}_t, \boldsymbol{\beta}_t^*(1) \rangle)^2 + \mathbb{E} \left[\sum_{t=1}^T \langle \mathbf{x}_t, \boldsymbol{\beta}_t(1) - \boldsymbol{\beta}^*(1) \rangle^2 \right] \right) \cdot \left| \frac{1}{\bar{p}} - \frac{1}{p^*} \right| \\ &\lesssim T^{-1} \cdot T \cdot o(1) \\ &= o(1). \end{aligned}$$

Similarly, we can show that $S_{5,3} = o(1)$. Hence $S_5 = o(1)$ by (114), which indicates that $\frac{1}{T} \mathbb{E}[\mathcal{R}_T^{\text{prob}}] = o(1) - S_1 - S_2 + S_3 + S_4 + S_5 = o(1)$ by (103), (104), (105) and (109). Since $T \cdot V^* = 2(1 + \rho)\mathcal{E}(1)\mathcal{E}(0)$, the result is proved. \square

The explicit form of the asymptotic variance in Theorem 5.1 establishes the equivalence between Assumption 4 and the non-superefficiency condition, which is stated in the Corollary 5.2.

Corollary 5.2 (Non-Superefficiency). *Under Assumptions 1-4 and Condition 1, $\liminf_{T \rightarrow \infty} T \cdot \text{Var}(\hat{\tau}) > 0$.*

Proof. It is easy to see that $-1 \leq \rho \leq 1$ by Cauchy-Schwarz inequality. Since $c_0 \leq \mathcal{E}(0), \mathcal{E}(1) \leq c_1$ by Assumption 1, the result is proved by Assumption 4 and Theorem 5.1. \square

C.4 Variance Estimator

Recall that the variance estimator is proposed as:

$$\begin{aligned} \hat{A}(1) &= \frac{1}{T} \sum_{t=1}^T Q_{t,t} Y_t^2 \frac{\mathbf{1}[Z_t = 1]}{p_t} + \frac{1}{T} \sum_{t=1}^T \sum_{s \neq t} Q_{t,s} Y_t Y_s \frac{\mathbf{1}[Z_t = 1, Z_s = 1]}{p_s p_t} \quad \text{and} \\ \hat{A}(0) &= \frac{1}{T} \sum_{t=1}^T Q_{t,t} Y_t^2 \frac{\mathbf{1}[Z_t = 0]}{1 - p_t} + \frac{1}{T} \sum_{t=1}^T \sum_{s \neq t} Q_{t,s} Y_t Y_s \frac{\mathbf{1}[Z_t = 0, Z_s = 0]}{(1 - p_s)(1 - p_t)}. \end{aligned}$$

In this section, we establish the consistency of the proposed variance estimator and characterize its convergence rate. The following lemma provides an upper bound for the deterministic terms involved in controlling the variance of the estimator.

Lemma C.24. *Under Assumptions 1-2, for $k \in \{0, 1\}$, there holds:*

$$(1) \sum_{t=1}^T Q_{t,t}^2 y_t^4(k) \leq c_1^4 T.$$

$$(2) \sum_{1 \leq t_1 \neq t_2 \leq T} Q_{t_1, t_2}^2 y_{t_1}^2(k) y_{t_2}^2(k) \left(\frac{R_{t_2}}{R_{t_1}} \right)^{1/2} \left(\frac{R_{t_1}}{R_{t_2}} \right)^{1/2} \leq c_1^4 c_2 R_T^2.$$

Proof. We only prove the result for $k = 1$.

(1) We have $\mathbf{Q} = (\mathbf{I}_T - \mathbf{X}_T (\mathbf{X}_T^\top \mathbf{X}_T)^{-1} \mathbf{X}_T^\top)$ by definition. Since $\mathbf{Q} \leq \mathbf{I}_T$, we have $Q_{t,t} \leq \|\mathbf{Q}\|_2 \leq 1$ for any $t \in [T]$. Hence by Assumption 1, we have

$$\sum_{t=1}^T Q_{t,t}^2 y_t^4(1) \leq c_1^4 T.$$

(2) Denote $\mathbf{H} = \mathbf{X}_T (\mathbf{X}_T^\top \mathbf{X}_T)^{-1} \mathbf{X}_T^\top$ and

$$\widetilde{\mathbf{H}} = \text{diag} \left\{ \left(\frac{R_T}{R_1} \right)^{1/2}, \dots, \left(\frac{R_T}{R_T} \right)^{1/2} \right\} \mathbf{H} \text{diag} \left\{ \left(\frac{R_T}{R_1} \right)^{1/2}, \dots, \left(\frac{R_T}{R_T} \right)^{1/2} \right\} .$$

It is easy to see that $\widetilde{\mathbf{H}}$ is positive definite. Denote $\mathbf{Y} = (y_1^2(1), \dots, y_T^2(1))^\top$. For any $1 \leq i \neq j \leq T$, it is easy to see that $H_{ij} = -Q_{ij}$. Hence we have

$$\begin{aligned} & \sum_{1 \leq t_1 \neq t_2 \leq T} Q_{t_1, t_2}^2 y_{t_1}^2(1) y_{t_2}^2(1) \left(\frac{R_T}{R_{t_1}} \right)^{1/2} \left(\frac{R_T}{R_{t_2}} \right)^{1/2} \\ &= \sum_{1 \leq t_1 \neq t_2 \leq T} H_{t_1, t_2}^2 y_{t_1}^2(1) y_{t_2}^2(1) \left(\frac{R_T}{R_{t_1}} \right)^{1/2} \left(\frac{R_T}{R_{t_2}} \right)^{1/2} \\ &\leq \sum_{1 \leq t_1, t_2 \leq T} H_{t_1, t_2}^2 y_{t_1}^2(1) y_{t_2}^2(1) \left(\frac{R_T}{R_{t_1}} \right)^{1/2} \left(\frac{R_T}{R_{t_2}} \right)^{1/2} \\ &= \mathbf{Y}^\top (\mathbf{H} \circ \widetilde{\mathbf{H}}) \mathbf{Y} . \end{aligned}$$

The t -th diagonal element of $\widetilde{\mathbf{H}}$ is bounded by:

$$\tilde{H}_{t,t} = \left(\frac{R_T}{R_t} \right) H_{t,t} \leq \left(\frac{R_T}{R_t} \right) \|\mathbf{x}_t\|_2^2 \|(\mathbf{X}_T^\top \mathbf{X}_T)^{-1}\|_2 \leq c_2 T^{-1} \left(\frac{R_T}{R_t} \right) R_t^2 \leq c_2 R_T^2 T^{-1} .$$

Then by Theorem 5.3.4 in Horn and Johnson, 2012, we have

$$\begin{aligned} \|\mathbf{H} \circ \widetilde{\mathbf{H}}\|_2 &\leq \|\mathbf{H}\|_2 \max_{t=1, \dots, T} \tilde{H}_{t,t} \\ &\leq \|\mathbf{X}_T (\mathbf{X}_T^\top \mathbf{X}_T)^{-1} \mathbf{X}_T^\top\|_2 \cdot c_2 R_T^2 T^{-1} \\ &\leq \|\mathbf{X}_T^\top \mathbf{X}_T (\mathbf{X}_T^\top \mathbf{X}_T)^{-1}\|_2 \cdot c_2 R_T^2 T^{-1} \\ &\leq c_2 R_T^2 T^{-1} . \end{aligned}$$

Hence by Assumption 1, we have

$$\begin{aligned} & \sum_{1 \leq t_1 \neq t_2 \leq T} Q_{t_1, t_2}^2 y_{t_1}^2(1) y_{t_2}^2(1) \left(\frac{R_T}{R_{t_1}} \right)^{1/2} \left(\frac{R_T}{R_{t_2}} \right)^{1/2} \\ &\leq \mathbf{Y}^\top (\mathbf{H} \circ \widetilde{\mathbf{H}}) \mathbf{Y} \\ &\leq \|\mathbf{H} \circ \widetilde{\mathbf{H}}\|_2 \sum_{t=1}^T y_t^4(1) \\ &\leq c_1^4 c_2 R_T^2 . \end{aligned}$$

□

We now bound the variance of the estimated squared OLS in the following theorem:

Theorem 5.6. *Under Assumptions 1-3 and Condition 1, for each treatment $k \in \{0, 1\}$, we have that the estimated squared OLS residuals satisfy*

$$\mathbb{E}[\hat{A}(k)] = A(k) \quad \text{and} \quad \text{Var}(\hat{A}(k)^2) = \mathcal{O}(\{T^{-5/12} R^{2/3}\}^2) .$$

Proof. We only prove the result for $k = 1$. We first verify the unbiasedness of $\hat{A}(1)$. By law of iterated expectation, we have

$$\begin{aligned}
& \mathbb{E}[\hat{A}(1)] \\
&= \frac{1}{T} \mathbb{E} \left[\sum_{t=1}^T Q_{t,t} y_t^2(1) \cdot \frac{\mathbf{1}[Z_t = 1]}{p_t} + \sum_{1 \leq t_1 \neq t_2 \leq T} Q_{t_1,t_2} y_{t_1}(1) y_{t_2}(1) \cdot \frac{\mathbf{1}[Z_{t_1} = 1]}{p_{t_1}} \frac{\mathbf{1}[Z_{t_2} = 1]}{p_{t_2}} \right] \\
&= \frac{1}{T} \sum_{t=1}^T \mathbb{E} \left[\sum_{t=1}^T Q_{t,t} y_t^2(1) \cdot \frac{\mathbf{1}[Z_t = 1]}{p_t} \right] + \frac{2}{T} \sum_{1 \leq t_1 < t_2 \leq T} \mathbb{E} \left[\mathbb{E} \left[Q_{t_1,t_2} y_{t_1}(1) y_{t_2}(1) \cdot \frac{\mathbf{1}[Z_{t_1} = 1]}{p_{t_1}} \frac{\mathbf{1}[Z_{t_2} = 1]}{p_{t_2}} \mid \mathcal{F}_{t_2-1} \right] \right] \\
&= \frac{1}{T} \sum_{t=1}^T Q_{t,t} y_t^2(1) + \frac{2}{T} \sum_{1 \leq t_1 < t_2 \leq T} \mathbb{E} \left[Q_{t_1,t_2} y_{t_1}(1) y_{t_2}(1) \cdot \frac{\mathbf{1}[Z_{t_1} = 1]}{p_{t_1}} \right] \\
&= \frac{1}{T} \sum_{t=1}^T Q_{t,t} y_t^2(1) + \frac{2}{T} \sum_{1 \leq t_1 < t_2 \leq T} Q_{t_1,t_2} y_{t_1}(1) y_{t_2}(1) \\
&= \frac{1}{T} \mathbf{y}(k)^\top \mathbf{Q} \mathbf{y}(k) \\
&= A(1).
\end{aligned}$$

Now we turn to bounding the variance of $\hat{A}(1)$. By Cauchy-Schwarz inequality and AM-GM inequality, we have:

$$\begin{aligned}
& \text{Var}(\hat{A}(1)) \\
&= \frac{1}{T^2} \text{Var} \left(\sum_{t=1}^T Q_{t,t} y_t^2(1) \cdot \frac{\mathbf{1}[Z_t = 1]}{p_t} + \sum_{1 \leq t_1 \neq t_2 \leq T} Q_{t_1,t_2} y_{t_1}(1) y_{t_2}(1) \cdot \frac{\mathbf{1}[Z_{t_1} = 1]}{p_{t_1}} \frac{\mathbf{1}[Z_{t_2} = 1]}{p_{t_2}} \right) \\
&\lesssim \frac{1}{T^2} \text{Var} \left(\sum_{t=1}^T Q_{t,t} y_t^2(1) \cdot \frac{\mathbf{1}[Z_t = 1]}{p_t} \right) + \frac{1}{T^2} \text{Var} \left(\sum_{1 \leq t_1 \neq t_2 \leq T} Q_{t_1,t_2} y_{t_1}(1) y_{t_2}(1) \cdot \frac{\mathbf{1}[Z_{t_1} = 1]}{p_{t_1}} \frac{\mathbf{1}[Z_{t_2} = 1]}{p_{t_2}} \right) \\
&\lesssim \frac{1}{T^2} \sum_{t=1}^T Q_{t,t}^2 y_t^4(1) \text{Var} \left(\frac{\mathbf{1}[Z_t = 1]}{p_t} - 1 \right) + \frac{1}{T^2} \sum_{1 \leq t_1 \neq t_2 \leq T} Q_{t_1,t_2}^2 y_{t_1}^2(1) y_{t_2}^2(1) \text{Var} \left(\frac{\mathbf{1}[Z_{t_1} = 1]}{p_{t_1}} \frac{\mathbf{1}[Z_{t_2} = 1]}{p_{t_2}} \right) \\
&\quad + \frac{1}{T^2} \sum_{1 \leq t_2 \neq t_3 < t_1 \leq T} Q_{t_1,t_2} Q_{t_1,t_3} y_{t_1}^2(1) y_{t_2}(1) y_{t_3}(1) \text{Cov} \left(\frac{\mathbf{1}[Z_{t_1} = 1]}{p_{t_1}} \frac{\mathbf{1}[Z_{t_2} = 1]}{p_{t_2}}, \frac{\mathbf{1}[Z_{t_1} = 1]}{p_{t_1}} \frac{\mathbf{1}[Z_{t_3} = 1]}{p_{t_3}} \right) \\
&\triangleq S_1 + S_2 + S_3.
\end{aligned}$$

By similar method as in the proof of Lemma B.18 and Lemma C.17, we can show that

$$\begin{aligned}
& \max_{1 \leq t \leq T} \text{Var} \left(\frac{\mathbf{1}[Z_t = 1]}{p_t} - 1 \right) = \mathcal{O}(T^{1/8}), \\
& \max_{1 \leq t_1 \neq t_2 \leq T} \text{Var} \left(\frac{\mathbf{1}[Z_{t_1} = 1]}{p_{t_1}} \frac{\mathbf{1}[Z_{t_2} = 1]}{p_{t_2}} \right) = \mathcal{O}(T^{9/32}), \\
& \max_{1 \leq t_2 \neq t_3 < t_1 \leq T} \left| \text{Cov} \left(\frac{\mathbf{1}[Z_{t_1} = 1]}{p_{t_1}} \frac{\mathbf{1}[Z_{t_2} = 1]}{p_{t_2}}, \frac{\mathbf{1}[Z_{t_1} = 1]}{p_{t_1}} \frac{\mathbf{1}[Z_{t_3} = 1]}{p_{t_3}} \right) \right| \\
&\lesssim \left(2 + b_1 (b_2/6)^{1/4} \eta_{t_1}^{1/4} \mathbb{E}^{1/4} \hat{A}_{t_1}(0) \right)^{81/64} \left(\frac{R_{t_1}}{R_{t_2}} \right)^{9/128} \left(\frac{R_{t_1}}{R_{t_3}} \right)^{9/128}.
\end{aligned}$$

Then by Lemma C.24, AM-GM inequality and Assumption 3, we have

$$|S_1| \lesssim T^{-1} \cdot T^{1/8} = T^{-7/8},$$

$$|S_2| \lesssim R_T^2 T^{-2} \cdot T^{9/32} = T^{-1} \cdot (T R_T^{-4})^{-23/32} \cdot R_T^{-7/8} = o(T^{-1}),$$

$$\begin{aligned}
|S_3| &\leq \frac{1}{T^2} \sum_{1 \leq t_2 \neq t_3 < t_1 \leq T} |Q_{t_1, t_2}| |Q_{t_1, t_3}| |y_{t_1}^2(1)| |y_{t_2}(1)| |y_{t_3}(1)| \left| \text{Cov} \left(\frac{\mathbf{1}[Z_{t_1} = 1]}{p_{t_1}}, \frac{\mathbf{1}[Z_{t_2} = 1]}{p_{t_2}}, \frac{\mathbf{1}[Z_{t_1} = 1]}{p_{t_1}}, \frac{\mathbf{1}[Z_{t_3} = 1]}{p_{t_3}} \right) \right| \\
&\lesssim \frac{1}{T^2} \sum_{1 \leq t_2 \neq t_3 < t_1 \leq T} |Q_{t_1, t_2}| |Q_{t_1, t_3}| |y_{t_1}^2(1)| |y_{t_2}(1)| |y_{t_3}(1)| (\eta_{t_1} T)^{81/256} \left(\frac{R_{t_1}}{R_{t_2}} \right)^{9/128} \left(\frac{R_{t_1}}{R_{t_3}} \right)^{9/128} \\
&\leq \frac{1}{T^2} \sum_{1 \leq t_2 \neq t_3 < t_1 \leq T} |Q_{t_1, t_2}| |Q_{t_1, t_3}| |y_{t_1}^2(1)| |y_{t_2}(1)| |y_{t_3}(1)| (\eta_T T)^{81/256} \left(\frac{R_T}{R_{t_1}} \right)^{81/128} \left(\frac{R_{t_1}}{R_{t_2}} \right)^{9/128} \left(\frac{R_{t_1}}{R_{t_3}} \right)^{9/128} \\
&= \frac{1}{T^2} \sum_{1 \leq t_2 \neq t_3 < t_1 \leq T} |Q_{t_1, t_2}| |Q_{t_1, t_3}| |y_{t_1}^2(1)| |y_{t_2}(1)| |y_{t_3}(1)| (\eta_T T)^{81/256} \left(\frac{R_T}{R_{t_1}} \right)^{63/128} \left(\frac{R_T}{R_{t_2}} \right)^{9/128} \left(\frac{R_T}{R_{t_3}} \right)^{9/128} \\
&\lesssim T^{-1} (\eta_T T)^{81/256} \sum_{1 \leq t_2 < t_1 \leq T} Q_{t_1, t_2}^2 y_{t_1}^2(1) y_{t_2}^2(1) \left(\frac{R_T}{R_{t_1}} \right)^{63/128} \left(\frac{R_T}{R_{t_2}} \right)^{9/64} \\
&\lesssim T^{-1} (\eta_T T)^{81/256} \sum_{1 \leq t_1 \neq t_2 \leq T} Q_{t_1, t_2}^2 y_{t_1}^2(1) y_{t_2}^2(1) \left(\frac{R_T}{R_{t_1}} \right)^{1/2} \left(\frac{R_T}{R_{t_2}} \right)^{1/2} \\
&\lesssim T^{-1} (\eta_T T)^{81/256} R_T^2 \\
&= T^{-1} R_T^2 (T R_T^{-4})^{81/512} \\
&\lesssim T^{-1} R_T^2 (T R_T^{-4})^{1/6}.
\end{aligned}$$

Hence by Lemma B.5, we have

$$\begin{aligned}
\text{Var}(\hat{A}(1)) &\lesssim S_1 + S_2 + S_3 \\
&\lesssim T^{-7/8} + T^{-1} + T^{-1} R_T^2 (T R_T^{-4})^{1/6} \\
&= T^{-1} R_T^2 (T R_T^{-4})^{1/6} \cdot \left(T^{-1/24} R_T^{-4/3} + T^{-1/6} R_T^{-4/3} + 1 \right) \\
&\lesssim T^{-1} R_T^2 (T R_T^{-4})^{1/6} \\
&= T^{-5/6} R_T^{4/3} \\
&\lesssim (T^{-5/12} R^{2/3})^2.
\end{aligned}$$

□

The following corollary is implied by Theorem 5.6, which implies the consistency of the variance estimator.

Corollary 5.7. *Under Assumptions 1-3 and Condition 1, $T \cdot \widehat{\text{VB}} - T \cdot \text{VB} = \mathcal{O}_p(T^{-5/12} R^{2/3})$.*

Proof. Recall that $T \cdot \text{VB} = 4\mathcal{E}(1)\mathcal{E}(0)$ and $T \cdot \widehat{\text{VB}} = 4\hat{\mathcal{E}}(1)\hat{\mathcal{E}}(0)$. By the definition of $\hat{\mathcal{E}}(k)$ and Theorem 5.6, we have

$$|\hat{\mathcal{E}}^2(k) - \mathcal{E}^2(k)| \leq |\hat{A}(k) - \mathcal{E}(k)| = |\hat{A}(k) - A(k)| = \mathcal{O}_p(T^{-5/12} R^{2/3}).$$

Since $\mathcal{E}(k) = \Theta(1)$ by Assumption 1, it is easy to show that $|\hat{\mathcal{E}}(k) - \mathcal{E}(k)| = \mathcal{O}_p(T^{-5/24} R^{1/3})$. Then by Assumption 1, we can directly show that

$$\hat{\mathcal{E}}(1)\hat{\mathcal{E}}(0) - \mathcal{E}(1)\mathcal{E}(0) = \mathcal{O}_p(T^{-5/12} R^{2/3}),$$

which implies that $T \cdot \widehat{\text{VB}} - T \cdot \text{VB} = \mathcal{O}_p(T^{-5/12} R^{2/3})$. □

C.5 Wald-type Confidence Intervals

Under the central limit theorem (Theorem C.20) and the consistency of the variance estimator (Corollary 5.7), it is easy to prove the following lemma regarding the asymptotic coverage of the Wald-type confidence intervals.

Corollary 5.8. *Under Assumptions 1-4 and Condition 1, the Wald-type intervals cover at the nominal level: $\liminf_{T \rightarrow \infty} \Pr(\tau \in \widehat{\text{CI}}_\alpha) \geq 1 - \alpha$.*