

Adaptive SGD with Line-Search and Polyak Stepsizes: Nonconvex Convergence and Accelerated Rates

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Abstract

We extend the convergence analysis of AdaSLS and AdaSPS in [Jiang and Stich, 2024] to the nonconvex setting, presenting a unified convergence analysis of stochastic gradient descent with adaptive Armijo line-search (AdaSLS) and Polyak stepsize (AdaSPS) for nonconvex optimization. Our contributions include: (1) an $\mathcal{O}(1/\sqrt{T})$ convergence rate for general nonconvex smooth functions, (2) an $\mathcal{O}(1/T)$ rate under quasar-convexity and interpolation, and (3) an $\mathcal{O}(1/T)$ rate under the strong growth condition for general nonconvex functions.

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1 Introduction

We consider the finite-sum smooth optimization problem:

$$\min_{x \in \mathbb{R}^d} \left\{ f(x) \triangleq \frac{1}{n} \sum_{i=1}^n f_i(x) \right\}, \quad (1)$$

This formulation is widely used in machine learning problems, particularly in empirical risk minimization, where the large number of data points n makes Stochastic Gradient Descent (SGD) [Robbins and Monro, 1951] and its variants [Bottou et al., 2018] the preferred optimization methods for training neural networks due to their computational efficiency.

The pioneering work of [Vaswani et al., 2019] introduced Armijo line-search to SGD for neural network training, demonstrating exceptional performance in interpolation regimes. Concurrently, Stochastic Polyak Stepsize (SPS) methods [Loizou et al., 2021, Berrada et al., 2020, Polyak, 1987] gained rapid interest due to their ability to utilize local curvature information while requiring only knowledge of individual optimal function values f_i^* . However, both approaches face significant limitations: they fail to converge in non-interpolated settings and cannot automatically adapt across different optimization regimes.

Recent advances by [Jiang and Stich, 2024] addressed these limitations by developing AdaSLS (Adaptive Stochastic Line-Search) and AdaSPS (Adaptive Stochastic Polyak Stepsize), which achieve unified convergence guarantees for convex functions in both interpolation and non-interpolation settings.

Nevertheless, most existing analyses assume that each component function f_i satisfies specific convexity conditions. In the case of strong convexity, there exists $\mu > 0$ such that for all $x, y \in \mathbb{R}^d$,

$$f_i(y) \geq f_i(x) + \langle \nabla f_i(x), y - x \rangle + \frac{\mu}{2} \|y - x\|^2. \quad (2)$$

For general convexity, the weaker condition holds for all $x, y \in \mathbb{R}^d$:

$$f_i(y) \geq f_i(x) + \langle \nabla f_i(x), y - x \rangle. \quad (3)$$

In nonconvex settings, where these convexity assumptions are removed, the L -Lipschitz smoothness of f_i (as defined in Definition 1) implies weak convexity. This arises from the quadratic upper bound property:

$$f_i(y) \leq f_i(x) + \langle \nabla f_i(x), y - x \rangle + \frac{L}{2} \|y - x\|^2, \quad (4)$$

which can be equivalently expressed as the function $f_i(x) + \frac{L}{2} \|x\|^2$ being convex, characterizing L -weak convexity.

The convergence behavior of these adaptive methods in nonconvex optimization remains an open theoretical question, a critical gap given the prevalence of nonconvex objectives in modern machine learning.

1.1 Our Contributions

This paper provides a unified convergence analysis for AdaSLS and AdaSPS in nonconvex optimization. Generalizing the results of [Jiang and Stich, 2024], we first establish that both methods achieve $\mathcal{O}(1/\sqrt{T})$ rate for general smooth nonconvex functions. Beyond this baseline guarantee, we prove that an accelerated $\mathcal{O}(1/T)$ rate is achievable under specific conditions: for both algorithms under quasar-convexity and interpolation, and for AdaSLS under the strong growth condition for general nonconvex functions. Our work thus extends the adaptive stepsize framework into the nonconvex domain, offering robust convergence guarantees that match or exceed classical results while adapting to the problem’s geometry.

The key differences between our work and [Jiang and Stich, 2024] can be seen in Table 1:

Stepsize	Interpolation/Strong Growth				Non-interpolation			
	str-cvx	cvx	non-cvx	input	str-cvx	cvx	non-cvx	input
SPS/SPS _{max} [Loizou et al., 2021]	$\mathcal{O}(\log(\frac{1}{\varepsilon}))$	$\mathcal{O}(\frac{1}{\varepsilon})$	$\mathcal{O}(\frac{1}{\varepsilon})$ [Gower et al., 2021]	$f_{i_t}^*$	$\varepsilon \geq \Omega(\sigma_f^2)$	$\varepsilon \geq \Omega(\sigma_f^2)$	—	$f_{i_t}^*$
SLS [Vaswani et al., 2019]	$\mathcal{O}(\log(\frac{1}{\varepsilon}))$	$\mathcal{O}(\frac{1}{\varepsilon})$	—	<i>None</i>	$\varepsilon \geq \Omega(\sigma_f^2)$	$\varepsilon \geq \Omega(\sigma_f^2)$	—	<i>None</i>
DecSPS [Orvieto et al., 2022]	$\mathcal{O}(\frac{1}{\varepsilon^2})$	$\mathcal{O}(\frac{1}{\varepsilon^2})$	—	$\ell_{i_t}^*$	$\mathcal{O}(\frac{1}{\varepsilon^2})$	$\mathcal{O}(\frac{1}{\varepsilon^2})$	—	$\ell_{i_t}^*$
AdaSLS [Jiang and Stich, 2024]	$\mathcal{O}(\log(\frac{1}{\varepsilon}))$	$\mathcal{O}(\frac{1}{\varepsilon})$	$\mathcal{O}(\frac{1}{\varepsilon})$ (this work)	<i>None</i>	$\mathcal{O}(\frac{1}{\varepsilon^2})$	$\mathcal{O}(\frac{1}{\varepsilon^2})$	$\mathcal{O}(\frac{1}{\varepsilon^2})$ (this work)	<i>None</i>
AdaSPS [Jiang and Stich, 2024]	$\mathcal{O}(\log(\frac{1}{\varepsilon}))$	$\mathcal{O}(\frac{1}{\varepsilon})$	$\mathcal{O}(\frac{1}{\varepsilon})$ (this work)	$f_{i_t}^*$	$\mathcal{O}(\frac{1}{\varepsilon^2})$	$\mathcal{O}(\frac{1}{\varepsilon^2})$	$\mathcal{O}(\frac{1}{\varepsilon^2})$ (this work)	$\ell_{i_t}^*$

Table 1: Summary of convergence behaviors of the considered adaptive stepsizes for smooth functions. The error metrics for strongly convex, convex, and nonconvex problems are $\|x_T - x_*\|^2$, $f(\bar{x}_T) - f(x_*)$, and $\|\nabla f(\bar{x}_T)\|^2$, respectively. The notation $\Omega(\cdot)$ indicates the size of the neighborhood that they can converge to.

2 Preliminaries

2.1 Algorithm Overview

We consider the standard stochastic gradient descent (SGD) framework for minimizing the finite-sum objective:

$$x_{t+1} = x_t - \eta_t \nabla f_{i_t}(f x_t),$$

where i_t is a randomly sampled index at iteration t , and $\eta_t > 0$ is the step-size. The performance of SGD heavily depends on the choice of η_t . In this work, we analyze two adaptive step-size strategies that automatically adjust η_t based on local curvature and historical information: **AdaSLS** and **AdaSPS**.

AdaSLS [Jiang and Stich, 2024] combines a stochastic Armijo line-search with a global adaptive step-size. At each iteration, it first performs a backtracking line-search to determine a local scaling

factor γ_t , then sets the effective step-size as:

$$\eta_t = \min \left\{ \frac{\gamma_t}{c_l \sqrt{\sum_{s=0}^t \gamma_s \|\nabla f_{i_s}(\mathbf{x}_s)\|^2}}, \eta_{t-1} \right\},$$

where $c_l > 0$ is a tuning parameter.

AdaSPS [Jiang and Stich, 2024] leverages a Polyak-type step-size normalized by cumulative function value gaps:

$$\eta_t = \min \left\{ \frac{f_{i_t}(\mathbf{x}_t) - \ell_{i_t}^*}{c_p \|\nabla f_{i_t}(\mathbf{x}_t)\|^2} \cdot \frac{1}{\sqrt{\sum_{s=0}^t (f_{i_s}(\mathbf{x}_s) - \ell_{i_s}^*)}}, \eta_{t-1} \right\},$$

where $\ell_{i_t}^* \leq f_{i_t}^*$ is a lower bound of the minimal function value for the mini-batch, and $c_p > 0$ is a constant.

Both methods are designed to be robust across interpolation and non-interpolation settings. Complete algorithmic descriptions are provided in Appendix A.

2.2 Assumptions and Definitions

Definition 1 (L-smoothness). *Each f_i is L -smooth, i.e., $\|\nabla f_i(x) - \nabla f_i(y)\| \leq L\|x - y\|$ for all $x, y \in \mathbb{R}^d$.*

Definition 2 (Strong Growth Condition). *There exists $\rho_s > 0$ such that*

$$\mathbb{E}_i \|\nabla f_i(x)\|^2 \leq \rho_s \|\nabla f(x)\|^2 \quad \forall x \in \mathbb{R}^d. \quad (5)$$

Definition 3 (Quasar-convexity). *Let x_* be a global minimizer of f . We say that f is quasar-convex with parameter θ if:*

$$\nabla f(x)^\top (x - x_*) \geq \theta \cdot [f(x) - f(x_*)] \quad \forall x \in \mathbb{R}^d. \quad (6)$$

Definition 4 (Interpolation). *The interpolation condition holds if $\sigma_f^2 = 0$, where*

$$\sigma_f^2 \triangleq f(x_*) - \mathbb{E}_{i_t} \left[\inf_x f_{i_t}(x) \right]. \quad (7)$$

Definition 5 (Estimation Error). *Denote the estimation error by*

$$\text{err}_f^2 \triangleq \mathbb{E}_{i_t} [\inf_x f_{i_t}(x) - \ell_{i_t}^*]. \quad (8)$$

3 Convergence Analysis

We begin by recalling key technical lemmas that form the foundation of our analysis.

Lemma 1 (Sequence bound [Jiang and Stich, 2024]). *For any non-negative sequence $\{a_t\}_{0 \leq t \leq T}$, it holds that:*

$$\sqrt{\sum_{t=0}^T a_t} \leq \sum_{t=0}^T \frac{a_t}{\sqrt{\sum_{i=0}^t a_i}} \leq 2 \sqrt{\sum_{t=0}^T a_t}. \quad (9)$$

Lemma 2 (Step-size bounds for AdaSLS [Jiang and Stich, 2024]). *Suppose each f_i is L -smooth, then the step-size of AdaSLS satisfies:*

$$\min \left\{ \frac{1-\rho}{L}, \gamma_{\max} \right\} \frac{1}{c_\ell \sqrt{\sum_{s=0}^t \gamma_s \|\nabla f_{i_s}(x_s)\|^2}} \leq \eta_t \leq \frac{\gamma_t}{c_\ell \sqrt{\sum_{s=0}^t \gamma_s \|\nabla f_{i_s}(x_s)\|^2}} \quad \forall t \geq 0. \quad (10)$$

Lemma 3 (Step-size bounds for AdaSPS [Jiang and Stich, 2024]). *Suppose each f_i is L -smooth, then the step-size of AdaSPS satisfies:*

$$\frac{1}{2c_p L} \frac{1}{\sqrt{\sum_{s=0}^t f_{is}(x_s) - \ell_{i_s}^*}} \leq \eta_t \leq \frac{f_{it}(x_t) - \ell_{i_t}^*}{c_p \|\nabla f_{i_t}(x_t)\|^2} \frac{1}{\sqrt{\sum_{s=0}^t f_{is}(x_s) - \ell_{i_s}^*}} \quad \forall t \geq 0. \quad (11)$$

3.1 General Nonconvex Convergence without Interpolation

Our first result establishes the convergence rate for general nonconvex smooth functions.

Theorem 1 (General nonconvex convergence for AdaSLS). *Suppose each f_i is L -smooth and the iterates $\{x_t\}$ satisfy $\|x_t - x_*\| \leq D$ for all t . Then, it holds that:*

$$\min_{0 \leq t \leq T-1} \mathbb{E} \|\nabla f(x_t)\|^2 \leq \left(\frac{c_\ell L D^2}{2 \min \left\{ \frac{1-\rho}{L}, \gamma_{\max} \right\}} + \frac{L}{2c_\ell} \right) \sqrt{\frac{L D^2 / 2 + \sigma_f^2}{\rho}} \cdot \frac{1}{\sqrt{T}} \quad \forall T \geq 1. \quad (12)$$

Proof. By the L -smoothness of f , we have the descent inequality:

$$\begin{aligned} f(x_{t+1}) - f(x_t) &\leq \nabla f(x_t)^\top (x_{t+1} - x_t) + \frac{L}{2} \|x_{t+1} - x_t\|^2 \\ &= -\eta_t \nabla f(x_t)^\top \nabla f_{i_t}(x_t) + \frac{L}{2} \eta_t^2 \|\nabla f_{i_t}(x_t)\|^2. \end{aligned}$$

Taking expectation gives:

$$\begin{aligned} \mathbb{E} \left[\frac{f(x_{t+1}) - f(x_t)}{\eta_t} \right] &\leq -\mathbb{E} \left[\mathbb{E}_{i_t} [\nabla f(x_t)^\top \nabla f_{i_t}(x_t)] \right] + \frac{L}{2} \mathbb{E} [\eta_t \|\nabla f_{i_t}(x_t)\|^2] \\ &= -\mathbb{E} \|\nabla f(x_t)\|^2 + \frac{L}{2} \mathbb{E} [\eta_t \|\nabla f_{i_t}(x_t)\|^2]. \end{aligned}$$

Rearranging terms and summing from $t = 0$ to $T - 1$:

$$\begin{aligned}
\sum_{t=0}^{T-1} \mathbb{E} \|\nabla f(x_t)\|^2 &\leq \sum_{t=0}^{T-1} \mathbb{E} \left[\frac{f(x_t) - f(x_{t+1})}{\eta_t} \right] + \frac{L}{2} \sum_{t=0}^{T-1} \mathbb{E} [\eta_t \|\nabla f_{i_t}(x_t)\|^2] \\
&= \mathbb{E} \left[\frac{f(x_0)}{\eta_0} - \frac{f(x_T)}{\eta_{T-1}} + \sum_{t=0}^{T-2} \left(\frac{1}{\eta_{t+1}} - \frac{1}{\eta_t} \right) f(x_{t+1}) \right] + \frac{L}{2} \sum_{t=0}^{T-1} \mathbb{E} [\eta_t \|\nabla f_{i_t}(x_t)\|^2] \\
&\leq \mathbb{E} \left[\frac{f(x_0)}{\eta_0} - \frac{f(x_T)}{\eta_{T-1}} + \sum_{t=0}^{T-2} \left(\frac{1}{\eta_{t+1}} - \frac{1}{\eta_t} \right) \max_{0 \leq t \leq T-1} f(x_t) \right] + \frac{L}{2} \sum_{t=0}^{T-1} \mathbb{E} [\eta_t \|\nabla f_{i_t}(x_t)\|^2] \\
&= \mathbb{E} \left[\frac{f(x_0)}{\eta_0} - \frac{f(x_T)}{\eta_{T-1}} + \left(\frac{1}{\eta_{T-1}} - \frac{1}{\eta_0} \right) \max_{0 \leq t \leq T-1} f(x_t) \right] + \frac{L}{2} \sum_{t=0}^{T-1} \mathbb{E} [\eta_t \|\nabla f_{i_t}(x_t)\|^2] \\
&\leq \underbrace{\mathbb{E} \left[\frac{\max_{0 \leq t \leq T-1} f(x_t) - f(x_T)}{\eta_{T-1}} \right]}_{\triangleq I} + \underbrace{\frac{L}{2} \sum_{t=0}^{T-1} \mathbb{E} [\eta_t \|\nabla f_{i_t}(x_t)\|^2]}_{\triangleq II}. \tag{13}
\end{aligned}$$

Let us denote the two terms as I and II . By L -smoothness of f :

$$f(x_t) - f(x_*) \leq \frac{L}{2} \|x_t - x_*\|^2 \leq \frac{L}{2} D^2 \quad \forall t. \tag{14}$$

It then follows from Lemma 2 that:

$$I \leq \mathbb{E} \left[\frac{LD^2}{2\eta_{T-1}} \right] \leq \frac{c_\ell LD^2}{2 \min \left\{ \frac{1-\rho}{L}, \gamma_{\max} \right\}} \mathbb{E} \left[\sqrt{\sum_{s=0}^{T-1} \gamma_s \|\nabla f_{i_s}(x_s)\|^2} \right].$$

For the second part:

$$\begin{aligned}
II &\leq \frac{L}{2} \mathbb{E} \left[\sum_{t=0}^{T-1} \frac{\gamma_t \|\nabla f_{i_t}(x_t)\|^2}{c_\ell \sqrt{\sum_{s=0}^t \gamma_s \|\nabla f_{i_s}(x_s)\|^2}} \right] \quad (\text{by Lemma 2}) \\
&\leq \frac{L}{2c_\ell} \mathbb{E} \left[\sqrt{\sum_{t=0}^{T-1} \gamma_t \|\nabla f_{i_t}(x_t)\|^2} \right] \quad (\text{by Lemma 1}).
\end{aligned}$$

Hence, combining I and II :

$$\begin{aligned}
\sum_{t=0}^{T-1} \mathbb{E} \|\nabla f(x_t)\|^2 &\leq I + II \\
&\leq \left(\frac{c_\ell LD^2}{2 \min \left\{ \frac{1-\rho}{L}, \gamma_{\max} \right\}} + \frac{L}{2c_\ell} \right) \mathbb{E} \left[\sqrt{\sum_{t=0}^{T-1} \gamma_t \|\nabla f_{i_t}(x_t)\|^2} \right] \\
&\leq \left(\frac{c_\ell LD^2}{2 \min \left\{ \frac{1-\rho}{L}, \gamma_{\max} \right\}} + \frac{L}{2c_\ell} \right) \sqrt{\mathbb{E} \left[\sum_{t=0}^{T-1} \gamma_t \|\nabla f_{i_t}(x_t)\|^2 \right]} \quad (\text{Jensen's inequality}).
\end{aligned}$$

Let $C_1 = \frac{c_\ell LD^2}{2 \min\{\frac{1-\rho}{L}, \gamma_{\max}\}} + \frac{L}{2c_\ell}$. By Armijo's line search:

$$\begin{aligned}
\sum_{t=0}^{T-1} \mathbb{E} \|\nabla f(x_t)\|^2 &\leq C_1 \sqrt{\mathbb{E} \left[\sum_{t=0}^{T-1} \frac{f_{i_t}(x_t) - f_{i_t}(x_{t+1})}{\rho} \right]} \\
&\leq C_1 \sqrt{\mathbb{E} \left[\sum_{t=0}^{T-1} \frac{f_{i_t}(x_t) - \inf_x f_{i_t}(x)}{\rho} \right]} \\
&= \frac{C_1}{\sqrt{\rho}} \sqrt{\sum_{t=0}^{T-1} \mathbb{E} \left[f_{i_t}(x_t) - f(x_*) + f(x_*) - \inf_x f_{i_t}(x) \right]} \\
&= \frac{C_1}{\sqrt{\rho}} \sqrt{\sum_{t=0}^{T-1} \mathbb{E} [f(x_t) - f(x_*)] + T \left(f(x_*) - \mathbb{E}_{i_t} \left[\inf_x f_{i_t}(x) \right] \right)} \\
&\leq \frac{C_1}{\sqrt{\rho}} \sqrt{\sum_{t=0}^{T-1} \frac{LD^2}{2} + T\sigma_f^2}.
\end{aligned}$$

Therefore, dividing both sides by T :

$$\min_{0 \leq t \leq T-1} \mathbb{E} \|\nabla f(x_t)\|^2 \leq \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E} \|\nabla f(x_t)\|^2 \leq C_1 \sqrt{\frac{LD^2/2 + \sigma_f^2}{\rho}} \cdot \frac{1}{\sqrt{T}}.$$

□

Theorem 1 shows that AdaSLS achieves the $\mathcal{O}(1/\epsilon^2)$ convergence rate for general nonconvex L-smooth functions, matching the classical result for SGD with diminishing step-size. We now establish a similar guarantee for AdaSPS.

Theorem 2 (General nonconvex convergence for AdaSPS). *Suppose each f_i is L -smooth and the iterates $\{x_t\}$ satisfy $\|x_t - x_*\| \leq D$ for all t . Then, it holds that:*

$$\min_{0 \leq t \leq T-1} \mathbb{E} \|\nabla f(x_t)\|^2 \leq \left(c_p L^2 D^2 + \frac{L}{2c_p} \right) \sqrt{LD^2/2 + \sigma_f^2 + \text{err}_f^2} \cdot \frac{1}{\sqrt{T}} \quad \forall T \geq 1. \quad (15)$$

Proof. Using the descent inequality derived from L-smoothness and by the lower bound on the step size of AdaSPS (Lemma 3):

$$I \leq \mathbb{E} \left[\frac{LD^2}{2\eta_{T-1}} \right] \leq c_p L^2 D^2 \mathbb{E} \left[\sqrt{\sum_{s=0}^{T-1} (f_{i_s}(x_s) - \ell_{i_s}^*)} \right]$$

For the second part, by the upper bound on the step size:

$$\begin{aligned}
II &\leq \frac{L}{2} \mathbb{E} \left[\sum_{t=0}^{T-1} \frac{f_{i_t}(x_t) - \ell_{i_t}^*}{c_p \sqrt{\sum_{s=0}^t (f_{i_s}(x_s) - \ell_{i_s}^*)}} \right] \\
&\leq \frac{L}{2c_p} \mathbb{E} \left[\sqrt{\sum_{t=0}^{T-1} (f_{i_t}(x_t) - \ell_{i_t}^*)} \right] \quad (\text{by Lemma 1})
\end{aligned}$$

Combining I and II :

$$\begin{aligned}
\sum_{t=0}^{T-1} \mathbb{E} \|\nabla f(x_t)\|^2 &\leq I + II \\
&\leq \left(c_p L^2 D^2 + \frac{L}{2c_p} \right) \mathbb{E} \left[\sqrt{\sum_{t=0}^{T-1} (f_{i_t}(x_t) - \ell_{i_t}^*)} \right] \\
&\leq \left(c_p L^2 D^2 + \frac{L}{2c_p} \right) \sqrt{\mathbb{E} \left[\sum_{t=0}^{T-1} (f_{i_t}(x_t) - \ell_{i_t}^*) \right]} \quad (\text{Jensen's inequality})
\end{aligned}$$

Let $C_2 = c_p L^2 D^2 + \frac{L}{2c_p}$, we have:

$$\begin{aligned}
\sum_{t=0}^{T-1} \mathbb{E} \|\nabla f(x_t)\|^2 &\leq C_2 \sqrt{\sum_{t=0}^{T-1} \mathbb{E} \left[f_{i_t}(x_t) - f(x_*) + f(x_*) - \inf_x f_{i_t}(x) + \inf_x f_{i_t}(x) - \ell_{i_t}^* \right]} \\
&= C_2 \sqrt{\sum_{t=0}^{T-1} \mathbb{E} [f(x_t) - f(x_*)] + T \left(f(x_*) - \mathbb{E}_{i_t} \left[\inf_x f_{i_t}(x) \right] \right) + T \left(\mathbb{E}_{i_t} \left[\inf_x f_{i_t}(x) - \ell_{i_t}^* \right] \right)} \\
&\leq C_2 \sqrt{\sum_{t=0}^{T-1} \frac{LD^2}{2} + T \left(\sigma_f^2 + \text{err}_f^2 \right)}
\end{aligned}$$

Therefore:

$$\min_{0 \leq t \leq T-1} \mathbb{E} \|\nabla f(x_t)\|^2 \leq \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E} \|\nabla f(x_t)\|^2 \leq C_2 \sqrt{LD^2/2 + \sigma_f^2 + \text{err}_f^2} \cdot \frac{1}{\sqrt{T}}$$

□

3.2 Accelerated Convergence

Although the $\mathcal{O}(1/\sqrt{T})$ rate holds for general nonconvex optimization, faster convergence can be established under additional structural assumptions. We now present accelerated rates under quasr-convexity and interpolation conditions.

Theorem 3 (Quasar-convex + interpolation for AdaSLS). *Suppose that (i) each f_i is L -smooth and quasar-convex with parameter $\theta \in (0, 1]$; (ii) bounded iterates $\|x_t - x_*\| \leq D$; (iii) $\sigma_f^2 = 0$. Then, with $c_\ell = \left(\theta \rho \sqrt{\gamma_0 \|\nabla f_{i_0}(x_0)\|^2}\right)^{-1}$, we have:*

$$\min_{0 \leq t \leq T-1} \mathbb{E} \|\nabla f(x_t)\|^2 \leq \left(\frac{c_\ell L D^2}{2 \min \left\{ \frac{1-\rho}{L}, \gamma_{\max} \right\}} + \frac{L}{2c_\ell} \right) \cdot \frac{c_\ell D^2}{\theta \cdot \min \left\{ \frac{1-\rho}{L}, \gamma_{\max} \right\}} \cdot \frac{1}{T} \quad \forall T \geq 1. \quad (16)$$

Proof. Following the proof of Theorem 1, we have:

$$\sum_{t=0}^{T-1} \mathbb{E} \|\nabla f(x_t)\|^2 \leq \left(\frac{c_\ell L D^2}{2 \min \left\{ \frac{1-\rho}{L}, \gamma_{\max} \right\}} + \frac{L}{2c_\ell} \right) \mathbb{E} \left[\sqrt{\sum_{t=0}^{T-1} \gamma_t \|\nabla f_{i_t}(x_t)\|^2} \right]. \quad (17)$$

Now we establish a tighter estimate under quasar-convexity and interpolation. By Lemma 1:

$$\begin{aligned} \sqrt{\sum_{t=0}^{T-1} \gamma_t \|\nabla f_{i_t}(x_t)\|^2} &\leq \sum_{t=0}^{T-1} \frac{\gamma_t \|\nabla f_{i_t}(x_t)\|^2}{\sqrt{\sum_{s=0}^t \gamma_s \|\nabla f_{i_s}(x_s)\|^2}} \\ &\leq \frac{c_\ell}{\min \left\{ \frac{1-\rho}{L}, \gamma_{\max} \right\}} \sum_{t=0}^{T-1} \gamma_t \|\nabla f_{i_t}(x_t)\|^2 \cdot \eta_t \quad (\text{by Lemma 2}) \\ &\leq \frac{c_\ell}{\rho \cdot \min \left\{ \frac{1-\rho}{L}, \gamma_{\max} \right\}} \sum_{t=0}^{T-1} \eta_t [f_{i_t}(x_t) - f_{i_t}(x_{t+1})] \quad (\text{by Armijo}) \\ &\leq \frac{c_\ell}{\rho \cdot \min \left\{ \frac{1-\rho}{L}, \gamma_{\max} \right\}} \sum_{t=0}^{T-1} \eta_t [f_{i_t}(x_t) - \inf_x f_{i_t}(x)] \\ &= \frac{c_\ell}{\rho \cdot \min \left\{ \frac{1-\rho}{L}, \gamma_{\max} \right\}} \sum_{t=0}^{T-1} \eta_t [f_{i_t}(x_t) - f_{i_t}(x_*)] \quad (\text{by interpolation}). \end{aligned}$$

On the other hand:

$$\begin{aligned} \|x_{t+1} - x_*\|^2 &= \|x_t - x_*\|^2 + \|x_{t+1} - x_t\|^2 + 2(x_t - x_*)^\top (x_{t+1} - x_t) \\ &= \|x_t - x_*\|^2 + \|\eta_t \nabla f_{i_t}(x_t)\|^2 - 2\eta_t \nabla f_{i_t}(x_t)^\top (x_t - x_*) \\ &\leq \|x_t - x_*\|^2 + \eta_t \frac{\gamma_t \|\nabla f_{i_t}(x_t)\|^2}{c_\ell \sqrt{\sum_{s=0}^t \gamma_s \|\nabla f_{i_s}(x_s)\|^2}} - 2\eta_t \nabla f_{i_t}(x_t)^\top (x_t - x_*) \quad (\text{Lemma 2}) \\ &\leq \|x_t - x_*\|^2 + \eta_t \frac{f_{i_t}(x_t) - f_{i_t}(x_{t+1})}{c_\ell \rho \sqrt{\sum_{s=0}^t \gamma_s \|\nabla f_{i_s}(x_s)\|^2}} - 2\eta_t \nabla f_{i_t}(x_t)^\top (x_t - x_*) \quad (\text{Armijo}). \end{aligned}$$

By quasar-convexity:

$$\begin{aligned} \left(2\theta - \frac{f_{i_t}(x_t) - f_{i_t}(x_{t+1})}{c_\ell \rho \sqrt{\sum_{s=0}^t \gamma_s \|\nabla f_{i_s}(x_s)\|^2}} \right) \eta_t [f_{i_t}(x_t) - f_{i_t}(x_*)] &\leq \theta \eta_t [f_{i_t}(x_t) - f_{i_t}(x_*)] \\ &\leq \|x_t - x_*\|^2 - \|x_{t+1} - x_*\|^2. \end{aligned}$$

Telescoping yields:

$$\sqrt{\sum_{t=0}^{T-1} \gamma_t \|\nabla f_{i_t}(x_t)\|^2} \leq \frac{c_\ell D^2}{\theta \cdot \min\left\{\frac{1-\rho}{L}, \gamma_{\max}\right\}}. \quad (18)$$

Combining with the earlier inequality:

$$\begin{aligned} \min_{0 \leq t \leq T-1} \mathbb{E} \|\nabla f(x_t)\|^2 &\leq \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E} \|\nabla f(x_t)\|^2 \\ &\leq \left(\frac{c_\ell L D^2}{2 \min\left\{\frac{1-\rho}{L}, \gamma_{\max}\right\}} + \frac{L}{2c_\ell} \right) \cdot \frac{c_\ell D^2}{\theta \cdot \min\left\{\frac{1-\rho}{L}, \gamma_{\max}\right\}} \cdot \frac{1}{T}. \end{aligned}$$

□

The quasar-convexity and interpolation conditions also enable accelerated convergence for AdaSPS, as shown in the following result.

Theorem 4 (Quasar-convex + interpolation for AdaSPS). *Suppose that (i) each f_i is L -smooth and quasar-convex with parameter $\theta \in (0, 1]$; (ii) bounded iterates $\|x_t - x_*\| \leq D$; (iii) $\sigma_f^2 = \text{err}_f^2 = 0$. Then, with $c_p = [\theta(f_{i_0}(x_0) - \ell_{i_0}^*)]^{-1}$, we have:*

$$\min_{0 \leq t \leq T-1} \mathbb{E} \|\nabla f(x_t)\|^2 \leq \left(c_p L^2 D^2 + \frac{L}{2c_p} \right) \cdot \frac{2c_p L D^2}{\theta} \cdot \frac{1}{T} \quad \forall T \geq 1. \quad (19)$$

Proof. From Theorem 2, we have:

$$\sum_{t=0}^{T-1} \mathbb{E} \|\nabla f(x_t)\|^2 \leq \left(c_p L^2 D^2 + \frac{L}{2c_p} \right) \mathbb{E} \left[\sqrt{\sum_{s=0}^{T-1} (f_{i_s}(x_s) - \ell_{i_s}^*)} \right]. \quad (20)$$

We now bound the cumulative function gaps. The iterate update gives:

$$\begin{aligned} \|x_{t+1} - x_*\|^2 &= \|x_t - x_*\|^2 + \|x_{t+1} - x_t\|^2 + 2(x_t - x_*)^\top (x_{t+1} - x_t) \\ &= \|x_t - x_*\|^2 + \|\eta_t \nabla f_{i_t}(x_t)\|^2 - 2\eta_t \nabla f_{i_t}(x_t)^\top (x_t - x_*) \\ &\leq \|x_t - x_*\|^2 + \eta_t \frac{f_{i_t}(x_t) - \ell_{i_t}^*}{c_p \sqrt{\sum_{s=0}^t f_{i_s}(x_s) - \ell_{i_s}^*}} - 2\eta_t \nabla f_{i_t}(x_t)^\top (x_t - x_*) \\ &= \|x_t - x_*\|^2 + \eta_t \frac{f_{i_t}(x_t) - f_{i_t}(x_*)}{c_p \sqrt{\sum_{s=0}^t f_{i_s}(x_s) - \ell_{i_s}^*}} - 2\eta_t \nabla f_{i_t}(x_t)^\top (x_t - x_*). \end{aligned}$$

By quasar-convexity:

$$\eta_t [f_{i_t}(x_t) - f_{i_t}(x_*)] \leq \frac{1}{\theta} (\|x_t - x_*\|^2 - \|x_{t+1} - x_*\|^2). \quad (21)$$

Therefore:

$$\sqrt{\sum_{s=0}^{T-1} (f_{i_s}(x_s) - \ell_{i_s}^*)} \leq \frac{2c_p L D^2}{\theta}.$$

Combining with the initial inequality:

$$\begin{aligned} \min_{0 \leq t \leq T-1} \mathbb{E} \|\nabla f(x_t)\|^2 &\leq \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E} \|\nabla f(x_t)\|^2 \\ &\leq \left(c_p L^2 D^2 + \frac{L}{2c_p} \right) \cdot \frac{2c_p L D^2}{\theta} \cdot \frac{1}{T}. \end{aligned}$$

□

Beyond interpolation settings, another important scenario that permits accelerated convergence is the strong growth condition, which bounds the relationship between stochastic and full gradients.

Theorem 5 (Strong growth condition for AdaSLS). *In addition to the assumptions of Theorem 1, if we further assume that (i) strong growth holds with constant ρ_s ; (ii) bounded iterates $\|x_t - x_*\| \leq D$. Then AdaSLS achieves:*

$$\min_{0 \leq t \leq T-1} \mathbb{E} \|\nabla f(x_t)\|^2 \leq \gamma_{\max} \rho_s \cdot \left(\frac{c L D^2}{2 \min \left\{ \frac{1-\rho}{L}, \gamma_{\max} \right\}} + \frac{L}{2c} \right)^2 \cdot \frac{1}{T} \quad \forall T \geq 1. \quad (22)$$

Proof. Note that $\{\gamma_t\}_{t \geq 0}$ is a sequence satisfying $\gamma_t \leq \gamma_{\max}$. We begin with the inequality:

$$\sum_{t=0}^{T-1} \gamma_t \|\nabla f_{i_t}(x_t)\|^2 \leq \gamma_{\max} \sum_{t=0}^{T-1} \|\nabla f_{i_t}(x_t)\|^2$$

Using the strong growth condition:

$$\mathbb{E}_{i_t} \|\nabla f_{i_t}(x_t)\|^2 \leq \rho_s \|\nabla f(x_t)\|^2 \quad \forall x \in \mathbb{R}^d$$

Taking full expectation gives:

$$\mathbb{E} \|\nabla f_{i_t}(x_t)\|^2 = \mathbb{E} [\mathbb{E}_{i_t} \|\nabla f_{i_t}(x_t)\|^2] \leq \rho_s \mathbb{E} \|\nabla f(x_t)\|^2$$

Summing over $t = 0$ to $T - 1$:

$$\begin{aligned} \mathbb{E} \left[\sum_{t=0}^{T-1} \|\nabla f_{i_t}(x_t)\|^2 \right] &= \sum_{t=0}^{T-1} \mathbb{E} \|\nabla f_{i_t}(x_t)\|^2 \\ &\leq \rho_s \sum_{t=0}^{T-1} \mathbb{E} \|\nabla f(x_t)\|^2 \end{aligned} \quad (23)$$

By Theorem 1, we have:

$$\begin{aligned}
\sum_{t=0}^{T-1} \mathbb{E} \|\nabla f(x_t)\|^2 &\leq C_1 \mathbb{E} \left[\sqrt{\sum_{t=0}^{T-1} \gamma_t \|\nabla f_{i_t}(x_t)\|^2} \right] \\
&\leq C_1 \sqrt{\mathbb{E} \left[\sum_{t=0}^{T-1} \gamma_t \|\nabla f_{i_t}(x_t)\|^2 \right]} \quad (\text{by Jensen's Inequality}) \\
&\leq C_1 \sqrt{\gamma_{\max}} \cdot \sqrt{\mathbb{E} \left[\sum_{t=0}^{T-1} \|\nabla f_{i_t}(x_t)\|^2 \right]} \tag{24}
\end{aligned}$$

where $C_1 = \frac{c_\ell L D^2}{2 \min\{\frac{1-\rho}{L}, \gamma_{\max}\}} + \frac{L}{2c_\ell}$.

Let $S_1 = \sum_{t=0}^{T-1} \mathbb{E} \|\nabla f(x_t)\|^2$ and $S_2 = \mathbb{E} \left[\sum_{t=0}^{T-1} \|\nabla f_{i_t}(x_t)\|^2 \right]$.

From (23) and (24), we have:

$$\begin{aligned}
S_2 &\leq \rho_s S_1, \\
S_1 &\leq C_1 \sqrt{\gamma_{\max}} \cdot \sqrt{S_2}.
\end{aligned}$$

Combining these:

$$S_1 \leq C_1^2 \gamma_{\max} \rho_s$$

Therefore:

$$\begin{aligned}
\min_{0 \leq t \leq T-1} \mathbb{E} \|\nabla f(x_t)\|^2 &\leq \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E} \|\nabla f(x_t)\|^2 \\
&\leq \frac{C_1^2 \gamma_{\max} \rho_s}{T}.
\end{aligned}$$

□

4 Conclusion

We have established comprehensive convergence guarantees for AdaSLS and AdaSPS in nonconvex optimization settings. Our analysis demonstrates that both adaptive methods achieve $\mathcal{O}(1/\sqrt{T})$ convergence for general nonconvex smooth functions, matching the classical SGD rate. Under more favorable conditions—quasar-convexity with interpolation and the strong growth condition—we prove accelerated $\mathcal{O}(1/T)$ rates for AdaSLS.

These results bridge an important theoretical gap, extending the adaptive step-size framework from convex to nonconvex optimization while maintaining strong convergence guarantees.

5 Open Questions and Future Work

While this work provides a comprehensive analysis of adaptive SGD methods in nonconvex settings, several important questions remain open:

5.1 AdaSPS under Strong Growth Condition

A notable gap in our analysis is the convergence rate of AdaSPS under the strong growth condition. Although we established $\mathcal{O}(1/T)$ convergence for AdaSLS in Theorem 5, extending this result to AdaSPS presents significant technical challenges. The main difficulty arises from the different step-size structures:

- **AdaSLS** uses gradient norms in the denominator: $\eta_t \propto \frac{\gamma_t}{\sqrt{\sum \gamma_s \|\nabla f_{i_s}(x_s)\|^2}}$
- **AdaSPS** uses function value gaps: $\eta_t \propto \frac{f_{i_t}(x_t) - \ell_{i_t}^*}{\sqrt{\sum (f_{i_s}(x_s) - \ell_{i_s}^*)^2}}$

Under the strong growth condition, the gradient norm structure of AdaSLS allows a direct connection to $\|\nabla f(x_t)\|^2$, enabling the accelerated rate. For AdaSPS, establishing a similar connection between function value gaps and gradient norms under strong growth remains an open problem.

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A Algorithm Details

A.1 AdaSLS Algorithm

Algorithm 1 AdaSLS

Require: $x_0 \in \mathbb{R}^d, T \in \mathbb{N}^+, c_t > 0$

- 1: set $\eta_{-1} = +\infty$
 - 2: **for** $t = 0$ to $T - 1$ **do**
 - 3: uniformly sample $i_t \subseteq [n]$
 - 4: obtain γ_t via backtracking line-search (Algorithm 2)
 - 5: set $\eta_t = \min \left\{ \frac{\gamma_t}{c_t \sqrt{\sum_{s=0}^t \gamma_s \|\nabla f_{i_s}(x_s)\|^2}}, \eta_{t-1} \right\}$
 - 6: $x_{t+1} = x_t - \eta_t \nabla f_{i_t}(x_t)$
 - 7: **end for**
 - 8: **return** $\bar{x}_T = \frac{1}{T} \sum_{t=0}^{T-1} x_t$
-

Algorithm 2 Backtracking line-search for AdaSLS

Require: $\beta \in [\frac{1}{2}, 1)$, $\rho \in (0, 1)$, $\gamma_{\max} > 0$

- 1: $\gamma = \gamma_{\max}$
 - 2: **while** $f_{i_t}(x_t - \gamma \nabla f_{i_t}(x_t)) > f_{i_t}(x_t) - \rho \gamma \|\nabla f_{i_t}(x_t)\|^2$ **do**
 - 3: $\gamma = \beta \gamma$
 - 4: **end while**
 - 5: **return** $\gamma_t = \gamma$
-

A.2 AdaSPS Algorithm

Algorithm 3 AdaSPS

Require: $x_0 \in \mathbb{R}^d, T \in \mathbb{N}^+, c_p > 0$

- 1: set $\eta_{-1} = +\infty$
 - 2: **for** $t = 0$ to $T - 1$ **do**
 - 3: uniformly sample $i_t \subseteq [n]$
 - 4: provide a lower bound $\ell_{i_t}^* \leq f_{i_t}^*$
 - 5: set $\eta_t = \min \left\{ \frac{f_{i_t}(x_t) - \ell_{i_t}^*}{c_p \|\nabla f_{i_t}(x_t)\|^2 \sqrt{\sum_{s=0}^t (f_{i_s}(x_s) - \ell_{i_s}^*)^2}}, \eta_{t-1} \right\}$
 - 6: $x_{t+1} = x_t - \eta_t \nabla f_{i_t}(x_t)$
 - 7: **end for**
 - 8: **return** $\bar{x}_T = \frac{1}{T} \sum_{t=0}^{T-1} x_t$
-