

Arctic Auctions, Linear Fisher Markets, and Rational Convex Programs

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November 27, 2025

Abstract

This paper unifies two foundational constructs from economics and algorithmic game theory—the Arctic Auction and the linear Fisher market—to address the efficient allocation of differentiated goods in complex markets. We focus on the *Arctic Auction* [Kle10, Kle18], a mechanism designed to address the challenge of allocating liquidity efficiently across banks pledging heterogeneous collateral of varying quality, and was used for this purpose by the Bank of England and IMF, among other institutions.

Our main contribution is to show that an equilibrium for the Arctic Auction is captured by a *Rational Convex Program* (RCP) [Vaz12a], thereby revealing deep structural regularities between market design and convex optimization. Building directly on primal–dual techniques for the linear Fisher market [DPSV08], we derive the first combinatorial polynomial-time algorithm for computing Arctic Auction equilibria. This result establishes that the Arctic Auction, long valued for its practical success, also enjoys strong algorithmic and theoretical foundations.

^{*}Supported in part by NSF grant CCF-2230414.

1 Introduction

This paper brings together two foundational notions from economics and algorithmic game theory (AGT), namely Arctic Auctions and linear Fisher markets, to address the challenge of efficiently allocating differentiated goods in complex markets.

We focus on Klemperer’s **Arctic Auction** [Kle18], a novel and powerful mechanism designed to solve resource allocation problems, notably those encountered in central banking. Our core technical contribution is to establish a fundamental link between this auction design and optimization theory: we show that the equilibrium of the Arctic Auction is captured by a **Rational Convex Program (RCP)** [Vaz12a]. Next, by building on primal-dual techniques developed by Devanur, Papadimitriou, Saberi, and Vazirani for the **linear Fisher market** [DPSV08], we are able to derive the **first combinatorial polynomial-time algorithm** for computing an equilibrium for the Arctic Auction. This work not only provides an exact computational solution for a mechanism of immediate practical importance but also solidifies the theoretical connection between modern market design and convex optimization.

The **Product-Mix Auction** of Klemperer [Kle10, Kle18, Kle08] is a general sealed-bid mechanism that computes competitive-equilibrium allocations across multiple substitutable goods by solving an optimization that maximizes total surplus, where bidders submit mutually exclusive bids across product categories and the mechanism determines market-clearing prices and allocations simultaneously. The design of the Product-Mix Auction was motivated by the 2007–2008 financial crisis, when the **Bank of England** faced the challenge of allocating liquidity efficiently across banks pledging heterogeneous collateral of varying quality. The Arctic Auction¹ is a specialized implementation of the product-mix auction designed for central bank liquidity operations, with the key difference being that it handles settings where bidders (banks) may choose not to spend all their money if prices exceed their valuation thresholds, and it remains computationally stable even when some asset markets “freeze”. Variants of the Arctic Auction have since been adopted in policy modeling contexts by the **IMF** and other institutions as a general tool for crisis-time resource allocation when assets are imperfect substitutes and traditional market-clearing mechanisms fail.

The linear Fisher market was the first market model for which an exact efficient algorithm was given [DPSV08]; henceforth we will call this the **DPSV algorithm**. Equilibrium allocation and prices for this model are captured by the celebrated Eisenberg-Gale convex program [EG59]. However, this program was not sufficient to derive an efficient algorithm for computing an **exact equilibrium** — that needed an algorithm for diophantine approximation which was given later in [Jai07]. The Arctic Auction can be viewed as a variant of the linear Fisher market in which buyers need not spend all their money and the utility of buyer i for good j gives an *upper bound on the price* at which i will be happy to buy good j , i.e., if the price of j equals this bound, i is indifferent between getting money back or buying this good and if it exceeds this bound, i will not buy this good [Kle18].

We note that incorporating such upper bounds in the DPSV algorithm is far from straightforward for the following basic reason: The crucial property which yields efficient computability for a

¹The name “Arctic” was given to such auctions since the issue of bidders not spending all their money first arose in Iceland’s planned auction [Kle18].

linear Fisher market is **weak gross substitutability**, i.e., if the demand of one good increases, the (equilibrium) price of another good cannot decrease. This property supports an algorithm which monotonically raises prices of goods, i.e., it never needs to reduce any price. Prices are dual variables of Eisenberg-Gale convex program, and it is well-known that primal-dual algorithms either raise or lower duals monotonically to achieve efficiency; schemes that raise *and* lower dual variables have not panned out in the past². For the Arctic Auction, when the price of some good j exceeds buyer i 's upper bound, j should be returned to i . But in doing so, we will be forced to lower the price of good j , since its demand has decreased. The full picture is even more complicated: at any point in the algorithm, different goods have different prices, possibly unrelated to buyer i 's upper bounds, and for the same good, the upper bounds of different buyers are different.

Notwithstanding these challenges, we proceeded to study a convex program appearing in Chen et al. [CYZ07]. On applying KKT conditions, we realized that not only it captures equilibrium allocations and prices for the Arctic Auction but it is also an RCP, Theorems 1 and 2, respectively. These are exceptionally strong structural properties, and there must be an underlying reason for their existence. That reason turned out to be the fact that in the definition of Arctic Auction, the **upper bound on the price** at which buyer i will be happy to buy a good j is **precisely her utility**, u_{ij} , for good j . The proof of Theorem 1 provided two more clues: the importance of the notion of *bang per buck*, Definition 2, and the importance of the event $\alpha_i = 1$, where α_i is the bang per buck of buyer i .

These facts dramatically changed the outlook—they brought hope despite the complicated picture painted above. Indeed, we believe that if the upper bound at which buyer i will be happy to buy a good j is not **exactly** u_{ij} , no convex program would capture an equilibrium of the resulting market. The fact that convex program (1) is an RCP encouraged us to seek a combinatorial polynomial-time algorithm for Arctic Auction. It was natural to extend the DPSV Algorithm. However, this algorithm uses the sophisticated machinery of **balanced flows**. Adapting these ideas to the Arctic Auction, i.e., arranging the return of money to buyers without violating any of the conditions, is quite non-trivial, see Theorem 3.

We note that this happy outcome is further testament to the exceptional skill that went into the design of Arctic Auction (and lucky breaks that yielded critical insights which eventually blended well with the DPSV algorithm).

This raises an interesting question: Is this happy outcome a one-off? I.e., can RCP and combinatorial polynomial-time algorithms go much further? We believe the answer is “yes”. As evidence, in Section 6 we present a linear Fisher market model with constant marginal costs for production. We show that an optimal solution to this model is captured by an RCP and we give a simple combinatorial polynomial-time algorithm for it.

2 Related Works

The notion of Product-Mix Auction was given by Klemperer [Kle10, Kle18]. The Arctic Auction was also introduced by Klemperer, as a specialized implementation of the product-mix auc-

²The only exception we are aware of is Edmonds' weighted matching algorithm [Edm65].

tion, designed for central bank liquidity operations [Kle18]. [FFW11] gave an early operational playbook showing how central banks price liquidity across collateral classes via Product-Mix Auctions. Fichtl [Fic22] gave a proof of correctness of an algorithm proposed by Paul Klemperer and DotEcon for finding equilibrium prices for the Arctic Auction if the number of goods is very small, e.g., three. The algorithm was explicitly designed for solving a real practical problem in this regime, even though it is exponential time in the worst case. It maximizes auctioneer surplus (a harder goal) as opposed to our algorithm which maximizes total surplus.

[BK19] analyzed the existence of equilibrium in Product-Mix Auctions when goods are indivisible. They introduce the concept of “demand types” to characterize solutions when standard assumptions break down. [BGKL24] developed polynomial-time algorithms to find competitive equilibrium prices and quantities for bidders in strong-substitutes product-mix auctions by using submodular minimization for price-finding and a novel constrained matching approach for supply allocation, directly utilizing the auction’s explicit bidding language. [BKL24] established that the Product-Mix Auction’s simple geometric bidding language uniquely represents all concave substitutes and strong-substitutes valuations, thereby providing a new characterization for these preference classes and ensuring the auction implements competitive equilibrium allocations.

[FGL23] study pricing in quasi-linear, budget-constrained multi-good markets and show that the elementwise-minimal competitive equilibrium prices are simultaneously seller revenue maximizing and constrained welfare maximizing, a structural result obtained without assuming gross substitutability or homogeneity. Their work is complementary to ours: while they characterize the geometry and optimality of equilibrium price vectors in markets with budgets, our paper develops an algorithmic framework for computing such equilibria for the Arctic Auction.

In the conference version, [DPSV02] proved that their algorithm for the linear Fisher market terminates with a rational equilibrium if all parameters in the given instance are rational numbers. A corollary of this result is that the Eisenberg-Gale convex program, which captures equilibrium allocations and prices for the linear Fisher market, always has rational optimal solutions. A direct proof of this fact was later given in [Vaz07].

Hence the Eisenberg-Gale convex program, a non-linear program, “behaves” like a linear program, an intriguing phenomenon indeed. Ye [Ye07] gave other convex programs which exhibit this behavior. Drawing analogy with a fundamental phenomenon in combinatorial optimization, namely the existence of LP-relaxations for some problems which always admit integral optimal solutions, e.g., for maximum matching and max-flow [LP86, Sch86] (i.e., these LPs “behave” like integer programs), [Vaz12a] formally defined the notion of a Rational Convex Program (RCP), see Definition 3. The importance of this notion is that the existence of such a program gives a strong indication that the problem admits a combinatorial polynomial-time algorithm.

RCPs have been found for several other market models, e.g., for the dichotomous-utilities case of the linear Arrow-Debreu Nash bargaining one-sided matching market [GTV24]; 2-player Nash and nonsymmetric bargaining games [Vaz12b]; and a perfect price discrimination market model with production [GV11]. [DGV16] gave an RCP for the linear Arrow–Debreu market. [CDG⁺17] gave new convex programs for natural generalizations of the linear Fisher market, in particular for the quasi-linear case; however, their program is different from the one studied in Section 3. Although they established rationality of equilibria for some markets, they did not explore the notion of rational convex programs.

Several market equilibrium works utilize the flow-based technique introduced in [DPSV08]. Building on [DPSV08], Orlin gave a strongly polynomial time algorithm for computing an equilibrium for the linear Fisher market [Orl10]. [DM15] gave a combinatorial polynomial algorithm for the linear Arrow-Debreu market and [DGM16] gave an algorithm with an improved running time. [VY25] used these ideas to give a strongly polynomial algorithm for computing the Hylland-Zeckhauser equilibrium [HZ79] for the case that each agent's utilities come from a bi-valued set. [GTV24] gave an ϵ -approximate equilibrium algorithm for the Arrow-Debreu extension of the classic Hylland-Zeckhauser mechanism [HZ79] for a one-sided matching market. [JV10] define the notion of Eisenberg–Gale markets and give efficient algorithms for several resource allocation markets in this class. [Vaz10] gave a polynomial time algorithm for computing an equilibrium for the spending constraint market model.

3 Rational Convex Program for Arctic Auction

Definition 1. (Klemperer [Kle18]) An *Arctic Auction* is a variant of the linear Fisher market in which buyers need not spend all their money and the utility of buyer i for good j gives an *upper bound on the price* at which i will be happy to buy good j . The market consists of a set B of n buyers, with buyer i having money $m_i \in \mathbb{Q}_+$, and a set G of m divisible goods, w.l.o.g. one unit of each. The buyers have linear utilities for goods; assume $u_{ij} \in \mathbb{Q}_+$ is the utility of buyer i for a unit of good j . If prices of all goods j (weakly) exceed u_{ij} then i may be returned a part or all of her money; the money returned is denoted by s_i . If x_{ij} is the amount of good j allocated to buyer i , then her total utility from this bundle is:

$$u_i(x, s_i) = \sum_{j \in G} u_{ij}x_{ij} + s_i = w_i(x_i) + s_i,$$

where x_i is the restriction of x to buyer i and $w_i(x_i) = \sum_{j \in G} u_{ij}x_{ij}$. If p_j is the price of one unit of good j then the worth of this bundle is $\sum_{j \in G} x_{ij}p_j + s_i$.

We will make the mild assumption that each buyer has positive utility for some good and each good is desired by some buyer. Under these conditions, *equilibrium allocations and prices exist*, satisfying:

1. Each buyer gets a *utility maximizing bundle*.
2. The *market clears*, i.e., all goods are sold and all money is either spent or returned. (In addition, all prices will be positive.)

Note 1. All parameters in all models defined in this paper, e.g., Definition 1, will be assumed to be non-negative rational numbers, i.e., belonging to \mathbb{Q}_+ .

Definition 2. W.r.t. prices p , define the *maximum bang per buck* of buyer i to be

$$\alpha_i = \max_{j \in G} \left\{ \frac{u_{ij}}{p_j} \right\}.$$

Furthermore, if the above equality holds for good j , then we will say that j is a *maximum bang per buck good for i* . For the sake of succinctness we will say that α_i is the *mbpb of buyer i* and j is an *mbpb good for i* .

Since the utility function of each buyer is linear, it is easy to show that her utility maximizing bundle must contain only mbpb goods. Lemma 1 follows easily from Definition 1.

Lemma 1. *Equilibrium allocations and prices for the Arctic Auction satisfy the following conditions for each buyer $i \in B$.*

1. If $\exists j \in G$ s.t. $x_{ij} > 0$ then j is an mbpb good for i . Furthermore, since $p_j \leq u_{ij}$ (because u_{ij} is an upper bound on the price at which i is happy to buy j), $\alpha_i \geq 1$.
2. If $s_i > 0$ and $\exists j \in G$ s.t. $x_{ij} > 0$ then $\alpha_i = 1$ and each good is bought by i at the upper bound of its price.
3. If $\forall j \in G$, $x_{ij} = 0$ then $\forall j \in G$, $p_j \geq u_{ij}$, i.e., the price of each good is at least its upper bound, and hence $\alpha_i \leq 1$.

We next study a variant of the Eisenberg-Gale convex program, given by Chen et al. [CYZ07]:

$$\begin{aligned} \max \quad & \left(\sum_{i \in B} m_i (\log(w_i(x_i) + s_i)) - s \right) \\ \text{s.t.} \quad & \sum_{i \in B} x_{ij} \leq 1 \quad \forall j \in G, \\ & \sum_{i \in B} s_i - s = 0, \\ & x_{ij} \geq 0 \quad \forall i \in B, j \in G, \\ & s_i \geq 0 \quad \forall i \in B \end{aligned} \tag{1}$$

Let p_j and λ be the dual variables corresponding to the first and second constraints of (1). Optimal solutions to the primal and dual variables must satisfy KKT conditions, in addition to the constraints of (1):

1. $\forall j \in G : p_j \geq 0$.
2. $\forall j \in G : p_j > 0 \implies \sum_{i \in B} x_{ij} = 1$.
3. $1 - \lambda \geq 0$.
4. $s > 0 \implies \lambda = 1$
5. $\forall i \in B, \forall j \in G : \frac{u_{ij}}{p_j} \leq \frac{w_i(x_i) + s_i}{m_i}$.
6. $\forall i \in B, \forall j \in G : x_{ij} > 0 \implies \frac{u_{ij}}{p_j} = \frac{w_i(x_i) + s_i}{m_i}$.
7. $\forall i \in B : \lambda \geq \frac{m_i}{w_i(x_i) + s_i}$.

8.

$$\forall i \in B : s_i > 0 \implies \lambda = \frac{m_i}{w_i(x_i) + s_i}.$$

Lemma 2. Optimal primal and dual solutions to convex program (1) satisfy the statements given in Lemma 1.

Proof. **Statement 1:** Assume $x_{ik} > 0$ for $k \in G$. Then by (6),

$$\frac{u_{ik}}{p_k} = \frac{w_i(x_i) + s_i}{m_i}.$$

By (5),

$$\forall j \in G : \frac{u_{ij}}{p_j} \leq \frac{w_i(x_i) + s_i}{m_i},$$

Therefore k is an mbpb good for i and

$$\frac{w_i(x_i) + s_i}{m_i} = \alpha_i.$$

By (7),

$$\frac{w_i(x_i) + s_i}{m_i} \geq \frac{1}{\lambda}.$$

By condition (3), $1/\lambda \geq 1$. Hence we get that

$$\alpha_i \geq \frac{1}{\lambda} \geq 1.$$

Therefore if $x_{ij} > 0$,

$$\frac{u_{ij}}{p_j} = \alpha_i \geq 1.$$

Hence $p_j \leq u_{ij}$, i.e., the price of j does not exceed u_{ij} .

Statement 2: Assume that $s_i > 0$ and $x_{ik} > 0$ for $k \in G$. Clearly $s > 0$, hence by condition (4), $\lambda = 1$.

By Statement (1) and condition (8),

$$\alpha_i = \frac{w_i(x_i) + s_i}{m_i} = \frac{1}{\lambda} = 1.$$

Now for each good j , if $x_{ij} > 0$ then $u_{ij}/p_j = 1$, i.e., j is bought at the upper bound of its price.

Statement 3: Assume that $\forall j \in G, x_{ij} = 0$. If so, $w_i(x_i) = 0$ and by (3) and (7),

$$\frac{s_i}{m_i} \geq \frac{1}{\lambda} \geq 1.$$

Therefore $s_i > 0$ and hence $s > 0$. Now by (4), $\lambda = 1$. Further, by (8), $s_i = m_i$. Now by (5), $\forall j \in G, p_j \geq u_{ij}$, i.e., the price of each good is at least its upper bound and hence $\alpha_i \leq 1$. \square

Theorem 1. Optimal primal and dual solutions to convex program (1) give equilibrium allocations and prices for the arctic auction.

Proof. By Definition 1, two facts need to be established.

1). The market clears: By the (mild) assumptions in Definition 1, each good j has an interested buyer, hence $p_j > 0$. Now, by KKT condition (2), it is fully sold. Next we prove that each buyer i spends all her money. There are three cases:

Case 1: $s_i = 0$. The money spent by i is

$$\sum_{j \in G} x_{ij} p_j = \sum_{j \in G} x_{ij} \frac{m_i u_{ij}}{w_i(x_i)} = m_i.$$

Case 2: $s_i > 0$ and $\exists j \in G$ s.t. $x_{ij} > 0$. By Statement (2), $\alpha_i = 1$ and $m_i = w_i(x_i) + s_i$. Furthermore, since each good is bought by i at the upper bound of its price,

$$w_i(x_i) = \sum_{j \in G} u_{ij} x_{ij} = \sum_{j \in G} p_j x_{ij},$$

i.e., the money spent on goods. Hence, s_i is precisely the unspent money.

Case 3: $\forall j \in G$, $x_{ij} = 0$. As shown in Statement (3), $s_i = m_i$.

2). Buyers get optimal bundles: By Statement (1) of Lemma 2, each good bought by i is an mbpb good for i . Furthermore, if money is returned, in Case 2 (Case 3), it provides exactly (at least) as much utility as mbpb goods. \square

Definition 3. (Vazirani [Vaz12a]) A nonlinear convex program is said to be a *rational convex program* (RCP) if for any setting of its parameters to rational numbers such that it has a finite optimal solution, it admits an optimal solution that is rational and can be written using polynomially many bits in the number of bits needed to write all the parameters. (As stated in Note 1, for all models studied in this paper, all parameters will be assumed to be non-negative rational numbers.)

Theorem 2. Convex program (1) is a rational convex program.

Proof. Let us assume that all parameters, i.e., the u_{ij} s and m_i s, are rational numbers. We will show that equilibrium allocations and prices form a solution to a linear system hence proving the theorem.

We first give the variables of this linear system. For each good j , define a new variable, q_j , which is meant to be $1/p_j$. The linear system will solve for the m q_j s. The p_j s will be obtained by taking reciprocals of q_j s. Among the x_{ij} s, guess the ones that are strictly positive. Assume there are k such variables and discard the rest. Lastly, guess the variables s_i that are strictly positive. Assume there are l of these and discard the rest.

Finally, we give the $m + k + l$ equations of the linear system over these $m + k + l$ variables. For each good j , include the equality given in condition (2). For each positive x_{ij} , include the equality given in condition (6). For each positive s_i , include the equality given in condition (8) with λ replaced by 1. Finally, if all s_i 's are zero and $s = 0$, λ plays a role only in condition (7) and this is easily satisfied by taking λ to be the largest allowed value, namely $\lambda = 1$. \square

4 An Efficient Primal-Dual Algorithm for Arctic Auction

The fact that program (1) is a rational convex program, Theorem 2, opens up the possibility of obtaining a polynomial time combinatorial algorithm for arctic auction.

4.1 Balanced Flow in Network $\mathcal{N}(p, r)$

Algorithm 1 builds on the polynomial time algorithm for a linear Fisher model given in [DPSV08]. In particular, it also critically uses the notion of *balanced flow* from [DPSV08]. Section 8 of the latter paper shows how to compute a balanced flow via n max-flow computations. Remark 1 explains the reason for using balanced flows and ℓ_2 norm of the surplus vector.

Definition 4. Define $\mathcal{N}(p, r)$ to be a directed network on vertices $B \cup G \cup \{s, t\}$, where s and t are the source and sink of $\mathcal{N}(p, r)$. The network has two vectors as parameters, p and r , where p_j is the current price of good j and r_i is the **money returned** to buyer i . The **left-over money** of i is defined to be $(m_i - r_i)$. The vectors p and r evolve as the algorithm proceeds, hence changing the network itself, as detailed in Algorithm 1. The edges of $\mathcal{N}(p, r)$ are:

1. $\forall j \in G$: edge (s, j) with capacity p_j .
2. $\forall i \in B$: edge (i, t) with capacity $m_i - r_i$, i.e., the left-over money of i .
3. $\forall j \in G$ and $\forall i \in B$: if j is an mbpb good for i , then the edge (j, i) with capacity infinity.

If f is a feasible flow in $\mathcal{N}(p, r)$, its *value* is the amount of flow going from s to t and is denoted by $|f|$.

Definition 5. W.r.t. flow f in network $\mathcal{N}(p, r)$, for each buyer i define $\text{surplus}(i)$ to be $(m_i - r_i) - f(i, t)$, i.e., the left-over capacity of edge (i, t) . Let $\gamma(f)$ denote the vector of surpluses of all buyers. Then f is said to be a *balanced flow* if it minimizes the ℓ_2 norm of the surplus vector, i.e., $\|f\|_2$, over all feasible flows f in $\mathcal{N}(p, r)$.

Clearly a balanced flow will be a max-flow in $\mathcal{N}(p, r)$. Moreover, all balanced flows have the same surplus vector (Corollary 8.7 in [DPSV08]) and a balanced flow will make the surpluses of the buyers as equal as possible. Let f be a balanced flow in $\mathcal{N}(p, r)$ and let $\mathcal{R}(f)$ be the corresponding residual network. The following property plays a key role in the DPSV algorithm as well as in Algorithm 1.

Property 1: Let $i, i' \in B$. If $\text{surplus}(i') < \text{surplus}(i)$ then there is no path from i' to i in $\mathcal{R}(f)$.

Theorem 8.4 in [DPSV08] shows that f is a balanced flow iff Property 1 holds. The proof of the main direction is straightforward: if there were a path from i' to i in $\mathcal{R}(p)$, we could find a circulation which increases $\text{surplus}(i')$ and decreases $\text{surplus}(i)$ thereby making the flow more balanced, leading to a contradiction.

4.2 Description of Algorithm 1

Similar to [DPSV08], Algorithm 1 is also a *primal-dual algorithm*. In our algorithm, the dual steps work on the dual variables, i.e., p_j s, by raising the price of a well-chosen set of goods. A max-flow in $\mathcal{N}(p, r)$ determines the primal variables, i.e., x_{ij} s. The task of the primal steps is:

1. Modify $\mathcal{N}(p, r)$ appropriately so the subnetwork induced by $B \cup G$ contains all and only mbpb edges.
2. Determine money returned, i.e., r .

Similar to [DPSV08], Algorithm 1 maintains:

Invariant: Algorithm 1 maintains the invariant that $(\{s\}, B \cup G \cup \{t\})$ is a minimum $s-t$ cut in $\mathcal{N}(p, r)$.

Because of this Invariant, the prices computed by the algorithm at any point are (weakly) bounded by equilibrium prices, i.e., the algorithm resorts to raising prices only, and not raising *and* lowering prices.

Let MIN denote the minimum money of a buyer, i.e., $\text{MIN} = \min_{i \in B} m_i$. Step 1 of Algorithm 1 initializes the price of each good to MIN/m , ensuring that each buyer can buy all m goods with her money. This ensures that $\alpha_i > 1$ and after Step 5, when $\mathcal{N}(p, r)$ contains all mbpb edges, it satisfies the Invariant. Step 2 initializes the mbpb, α_i , of each buyer i and sets r_i to 0. At the beginning of each phase, a balanced flow is computed in $\mathcal{N}(p, r)$ and the set $I \subseteq B$ of buyers having the largest surplus is identified (Step 4). $J \subseteq G$ consists of the set of goods desired by buyers in I .

By Property 1, there is no residual path from a buyer in $(B - I)$ to a buyer in I . Therefore there is no flow going from goods in J to buyers in $(B - I)$. In Step 6 of the New Phase, such edges are dropped. Let $Z \subseteq B - I$ be buyers all of whose mbpb edges are in I ; clearly, after Step 6, such buyers will have no edges to G . We will call them **zero-degree buyers** and will put them in the special set Z . For $i \in Z$, edge (i, t) carries no flow and $\text{surplus}(i) = m_i$. Importantly, the mbpb, α_i of such a buyer i decreases as prices of goods in J increase. In contrast, all mbpb goods of buyers in $(B - I) - Z$ are in $G - J$ and since the price of these goods remains unchanged, their mbpb remains unchanged. The algorithm keeps track of buyers in Z for the following two events:

1. If $\exists i \in Z$ s.t. $\alpha_i = 1$, Step 9(c)(iv) is triggered, which removes i from B and $\mathcal{N}(p, r)$. Since edge (i, t) carried no flow, the removal of i does not violate the Invariant. Following this, the algorithm continues raising the prices of goods in J . Observe that until i 's money is returned, we cannot raise prices of goods in J .
2. If $\exists i \in Z$ and $j \in (G - J)$ s.t. j becomes a mbpb good for i , Step 9(c)(v) is triggered, which adds (i, j) to $\mathcal{N}(p, r)$ with infinity capacity. i is not a zero-degree buyer anymore and is

Algorithm 1. Algorithm for Arctic Auction

Initialization

1. $\forall j \in G: p_j \leftarrow \text{MIN}/m.$
2. $\forall i \in B: \alpha_i \leftarrow \max_{j \in G} \left\{ \frac{u_{ij}}{p_j} \right\}$ and $r_i \leftarrow 0.$
3. Compute $\mathcal{N}(p, r).$
4. $\forall j \in G$ s.t. $\deg_{\mathcal{N}(p, r)}(j) = 0,$ $p_j \leftarrow \max_{i \in B} \left\{ \frac{u_{ij}}{\alpha_i} \right\}.$
5. Recompute $\mathcal{N}(p, r).$

New phase

1. $f \leftarrow$ balanced flow in $\mathcal{N}(p, r).$
2. $\forall i \in B:$ $\text{surplus}(i) \leftarrow (m_i - r_i) - f(i, t).$
3. $\delta \leftarrow \max_{i \in B} \{ \text{surplus}(i) \}.$
4. $I \leftarrow \arg \max_{i \in B} \{ \text{surplus}(i) \}.$
5. $J \leftarrow \Gamma(I).$
6. Remove edges going from $B - I$ to $J.$
7. Let $Z \subseteq (B - I)$ be buyers having no edges from G in $\mathcal{N}(p, r).$
8. Record initial prices of goods in $J:$ $\forall j \in J: \bar{p}_j \leftarrow p_j.$

9. New iteration

- (a) $\theta \leftarrow 1.$
- (b) $\forall j \in J: p_j \leftarrow \bar{p}_j \cdot \theta.$ Define capacity of edge (s, j) to be $p_j.$
- (c) Raise θ continuously **until**:
 - i. **New edge:** $\exists i \in I$ and $j \in (G - J)$ s.t. j is a mbpb good for $i.$
 - A. Add (i, j) with capacity infinity to $\mathcal{N}(p, r).$
 - B. Recompute balanced flow, $f,$ in $\mathcal{N}(p, r).$
 - C. $I' \leftarrow \{i \in (B - I) \mid \exists \text{path from } i \text{ to } I \text{ in } \mathcal{R}(f)\}.$
 - D. $I \leftarrow (I \cup I').$
 - E. $J \leftarrow \Gamma(I).$
 - F. Start new iteration.
 - ii. **A set goes tight:** $\exists S \subseteq J : \text{worth}(S) = \text{worth}(\Gamma(S)).$
 - A. If $S \neq G,$ start new phase.
 - B. If $S = G,$ Output current allocations and prices and **Halt.**
 - iii. **Money is returned to a buyer:** $\exists i \in I$ s.t. $\alpha_i = 1.$
 - A. Set i 's left-over money to 0. If $(s, G \cup B \cup t)$ is an $s-t$ min-cut, return all of i 's money, remove i from B and $\mathcal{N}(p, r),$ and go to next phase.
 - B. If $(s, G \cup B \cup t)$ is not an $s-t$ min-cut, find a maximal $s-t$ min-cut, say $(s \cup S \cup T, (G - S) \cup (B - T) \cup t).$
 - C. Return $m_i - (\text{worth}(S) - \text{worth}(T))$ money to $i.$
 - D. Go to next phase.
 - iv. **Remove a buyer from $Z:$** If $\exists i \in Z$ s.t. $\alpha_i = 1,$ return all of i 's money, remove i from B and $\mathcal{N}(p, r),$ and continue raising $\theta.$
 - v. **New edge in $Z:$** If $\exists i \in Z$ and $j \in (G - J)$ s.t. j is mbpb good for $i,$ add (j, i) to $\mathcal{N}(p, r),$ move i from Z to $(B - I) - Z,$ and continue raising $\theta.$

moved from Z to $(B - I) - Z$, and we continue raising θ .

The main changes from [DPSV08] are *primal steps* Step 9(c)(iv), Step 9(c)(v) and Step 9(c)(iii). The last step is triggered when α_i of a buyer $i \in I$ becomes 1. In this case, either i is removed after returning all her money or is retained after returning a part of her money.

Notation $\Gamma(S)$ will denote the *neighborhood* of set S in $\mathcal{N}(p, r)$ as follows: If $S \subseteq G$ then $\Gamma(S) \subseteq B$ and if $S \subseteq B$ then $\Gamma(S) \subseteq G$.

We will define the *worth of a set* as follows: If $S \subseteq G$ then $\text{worth}(S)$ is defined to be the sum of prices of goods in S and if $S \subseteq B$ then $\text{worth}(S)$ is defined to be the sum of leftover money of buyers in S . By the Invariant, $\forall S \subseteq G : \text{worth}(S) \leq \text{worth}(\Gamma(S))$. Note that while computing $\text{worth}(T)$ in Step 9(c)(iii)(C), we will assume that the money of i is zero.

The algorithm runs in *phases*. A phase is partitioned into *iterations*. During each iteration, the algorithm increases the prices of goods in set J . For $j \in J$, \bar{p}_j will denote the price of j at the beginning of the iteration. Its current price, p_j , is $\bar{p}_j \cdot \theta$, where θ is initialized to 1 and is raised continuously at rate 1. As mentioned above, at the beginning of an iteration, edges from J to $B - I$ will be dropped; by definition, there are no edges from $(G - J)$ to I . Hence the subnetwork induced on (I, J) is decoupled from $(G - I, B - J)$. An iteration ends when a new edge enters $\mathcal{N}(p, r)$, in Step 9(c)(i), or when the phase ends.

A phase ends when one of these events happens:

1. A set $S \subseteq G$ goes *tight*, i.e., $\text{worth}(S) = \text{worth}(\Gamma(S))$, in Step 9(c)(ii).
2. $\alpha_i = 1$ for some buyer i and her money is fully returned, in Step 9(c)(iii)(A).
3. $\alpha_i = 1$ for some buyer i and her money is partially returned, in Step 9(c)(iii)(D).

In the third possibility, in Step 9(c)(iii)(B), $(s, G \cup B \cup t)$ is not an $s-t$ min-cut in $\mathcal{N}(p, r)$, indicating that returning all money to i will violate the Invariant. Hence if $(s \cup S \cup T, (G - S) \cup (B - T) \cup t)$ is a maximal $s-t$ min-cut, then $\text{worth}(S) > \text{worth}(T)$. As mentioned above, while computing $\text{worth}(T)$ in Step 9(c)(iii)(C), we will assume that the money of i is zero.

In Step 9(c)(iii)(C), the algorithm will return $r_i = m_i - (\text{worth}(S) - \text{worth}(T))$ money to i . As a result, $(s, G \cup B \cup t)$ becomes an $s-t$ min-cut again $\mathcal{N}(p, r)$, and the Invariant is restored, as shown in Lemma 3. Moreover, S is now a tight set and therefore the phase ends.

Lemma 3. *In Step 9(c)(iii)(C), after returning $m_i - (\text{worth}(S) - \text{worth}(T))$ money to i , $(s, G \cup B \cup t)$ becomes an $s-t$ min-cut again, thereby restoring the Invariant. At this point, $(s \cup S \cup T, (G - S) \cup (B - T) \cup t)$ is also an $s-t$ min-cut in $\mathcal{N}(p, r)$, i.e., S is a tight set.*

Proof. Before Step 9(c)(iii)(A) is executed, the Invariant holds and therefore $(s, G \cup B \cup t)$ is an $s-t$ min-cut in $\mathcal{N}(p, r)$. After Step 9(c)(iii)(A) is executed and all of i 's money is returned, $(s, G \cup B \cup t)$ is not an $s-t$ min-cut but $(s \cup S \cup T, (G - S) \cup (B - T) \cup t)$ is. At this point, let us continuously increase the money of buyer i from zero until $(s, G \cup B \cup t)$ also becomes an $s-t$ min-cut. Let β be the money of i when this happens. Then clearly, $\text{worth}(S) = \text{worth}(T) + \beta$. Therefore on setting $r_i = m_i - (\text{worth}(S) - \text{worth}(T))$ the Invariant is restored and S is a newly tight set. \square

Observe that at this point, α_i is still 1. Now there are two possibilities: First, i enters I in a future phase. At the start of that phase, Step 9(c)(iii) will kick in again since $\alpha_i = 1$. Again i will either be returned all or part of her money. In the former case, the prices of i 's mbpb goods can be raised. After increasing prices, $\alpha_i < 1$, but this is fine since at these high prices, i prefers her money back, and has already been arranged. In the latter case, i is in a tight set again, and the whole process will repeat.

Second, the algorithm terminates without i entering the active set I again. One possibility is that i is removed in Step 9(c)(iv) with all her money returned. Otherwise at termination, in the final max-flow in Step 9(c)(ii)(B), i will be allocated her mbpb goods worth her remaining money. Since α_i is still 1, i is happy with a combination of money back and her mbpb goods. Note that the algorithm terminates when $(\{s\} \cup B \cup G, \{t\})$ becomes a min-cut.

If $\alpha_i > 1$ throughout the run of the algorithm, Step 9(c)(iii) will not kick in and no money will be returned to i . At termination, in Step 9(c)(ii)(B), i will be allocated her mbpb goods worth her total money. Algorithm 1 raises prices monotonically and therefore for each buyer i , α_i decreases monotonically. Combining with all the observations made above, including those for Steps 9(c)(iv) and 9(c)(v), we get Theorem 3, proving correctness of Algorithm 1 on the main new feature of the Arctic Auction over and above a linear Fisher market.

Theorem 3. *For each buyer i , Algorithm 1 allocates a bundle of goods and money satisfying:*

1. *If $\alpha_i > 1$, i will be allocated her mbpb goods worth her entire money.*
2. *If $\alpha_i = 1$, i may be returned money and any goods allocated will be her mbpb goods. The total worth of the bundle will be m_i .*
3. *If $\alpha_i < 1$, i will be returned all her money.*

5 Proof of Running Time

As in [DPSV08], our proof is based on the following potential function

$$\Phi = \sum_{i \in B} \gamma_i^2$$

We will show that in a phase, this potential decreases by at least a multiplicative factor of $1 - 1/n^3$. We will classify phases into three types based on how they terminate, and we quantify the progress made in each type of phase.

Type I: Phase ends in Step 9(c)(ii), when a set $S \subseteq J$ goes tight.

Type II: Phase ends in Step 9(c)(iii)(A), when a buyer i has all money returned and is removed.

Type III: Phase ends in Step 9(c)(iii)(D), when buyer i has partial money returned.

Remark 1. The main difficulty in establishing the running time of Algorithm 1, as well as the DPSV algorithm, comes from the use of balanced flows and ℓ_2 norm of the surplus vector. The

question arises: Can we not argue about the ℓ_1 norm of the surplus vector? It turns out there is an infinite family of instances for the DPSV algorithm for which the decrease in the ℓ_1 norm of the surplus vector is inverse exponential.

The next question is: what is the advantage of using ℓ_2 norm? To answer this, consider two 2-dimensional vectors $(1, 0)$ and $(1/2, 1/2)$. The ℓ_1 norm of both vectors is 1. On the other hand, the ℓ_2 norms are 1 and $1/\sqrt{2}$, respectively. In an iteration, we may go from the first to close to the second vector, resulting in almost no improvement in the ℓ_1 norm-based potential function. On the other hand, there is a substantial improvement in the ℓ_2 norm-based potential function. Indeed, certain iterations of the algorithm do not lead to much decrease in the total surplus, but they make the surplus vector more balanced, thereby increasing the potential for a substantial improvement in future iterations.

Lemma 4, which is based on Lemma 8.5 and Corollary 8.6 in [DPSV08], plays a critical role in [DPSV08] and our proof for showing that sufficient progress is made towards decreasing the ℓ_2^2 norm of the surplus vector in an iteration. For this reason, we have provided a detailed proof, which is more intuitive and complete than Lemma 8.5 from [DPSV08], and may be of independent interest.

Lemma 4. *Let p and p^* be the price vectors at the beginning and end of an iteration of the algorithm in which no money is returned (so r is fixed). Let f and f^* be balanced flows in $N(p, r)$ and $N(p^*, r)$, respectively. Suppose that for some buyer i and some $\sigma > 0$ we have*

$$\gamma_i(f^*) = \gamma_i(f) - \sigma.$$

Then

$$\|\gamma(f^*)\|_2^2 \leq \|\gamma(f)\|_2^2 - \sigma^2.$$

Proof. Since this iteration raises prices but never returns money, the capacities of edges from the source do not decrease, and the capacities into t remain fixed. If this iteration ended in Step 9(c)(i), $N(p^*, r)$ contains an additional edge, compared to $N(p, r)$. Hence f is feasible in $N(p^*, r)$. Since f^* is a balanced (and thus maximum) flow in $N(p^*, r)$, we have $|f^*| \geq |f|$.

Consider the difference $f^* - f$ in the residual network of flow f in $N(p^*, r)$. This difference decomposes into a set of $s \rightarrow t$ augmenting paths of total value $|f^*| - |f|$ together with a circulation of value $|f|$. Select from this decomposition exactly those paths/cycles that traverse the edge (i, t) , and *augment* f along these to obtain a feasible flow h in $N(p^*, r)$. By construction, the flow on (i, t) increases by exactly σ , and therefore

$$\gamma_i(h) = \gamma_i(f) - \sigma = \gamma_i(f^*).$$

Since f^* is the balanced flow for $N(p^*, r)$ and h is a feasible flow in that network, we have

$$\|\gamma(f^*)\|_2 \leq \|\gamma(h)\|_2.$$

Thus it suffices to prove

$$\|\gamma(h)\|_2^2 \leq \|\gamma(f)\|_2^2 - \sigma^2.$$

For establishing this inequality, we need to consider only buyers whose surplus increases in h (other than i). For each such buyer j , let $\sigma_j \geq 0$ denote its surplus increase. Flow conservation in the selected augmentation implies $\sum_j \sigma_j \leq \sigma$.

Furthermore, in f^* there is a residual path from i to each such j (because the portions of $f^* - f$ used to form h include these paths). Now Property 1 of balanced flows (no residual path from strictly smaller surplus to strictly larger surplus), implies

$$\gamma_i(f^*) \geq \gamma_j(f^*).$$

Since $\gamma_i(h) = \gamma_i(f^*)$ and $\gamma_j(h) \leq \gamma_j(f^*)$, we have for all affected j :

$$\gamma_i(h) \geq \gamma_j(h).$$

Finally, for concreteness, assume that there are k such buyers j namely i_1, i_2, \dots, i_k and positive values $\sigma_1, \dots, \sigma_k$ with $\sum_{l=1}^k \sigma_l \leq \sigma$ such that for $1 \leq l \leq k$:

$$\gamma_i(h) = \gamma_i(f) - \sigma \geq \gamma_{i_l}(f) + \sigma_l = \gamma_{i_l}(h)$$

Now using Lemma 5 (i.e., which is Lemma 8.3 from [DPSV08]), we get

$$\|\gamma(f)\|_2^2 - \|\gamma(h)\|_2^2 \geq \sigma^2.$$

Combining this with $\|\gamma(f^*)\|_2 \leq \|\gamma(h)\|_2$ yields

$$\|\gamma(f)\|_2^2 - \|\gamma(f^*)\|_2^2 \geq \sigma^2,$$

as required. □

Lemma 5. (Lemma 8.3 from [DPSV08]) Assume that $a \geq b_i \geq 0$ for $i = 1, 2, \dots, n$. Assume further that $\sum_{l=1}^k \sigma_l \leq \sigma$, where $\sigma, \sigma_l \geq 0$. Then

$$\|(a + \sigma, b_1 - \sigma_1, b_2 - \sigma_2, \dots, b_n - \sigma_n)\|^2 \leq \|(a, b_1, b_2, \dots, b_n)\|^2 - \sigma^2.$$

Next we quantify the decrease the in the potential Φ in a Type I phase.

Lemma 6. (Bounding Type I Phases) During any Type I phase, the potential

$$\Phi = \sum_{i \in B} \gamma_i^2$$

decreases by at least a multiplicative factor of $1 - 1/n^3$.

Proof. Let p_0 be the price vector at the start of the Type I phase and let f_0 be the balanced flow in $N(p_0, r)$, where r remains fixed throughout the phase. Let

$$\delta = \max_{i \in B} \gamma_i(f_0) \quad \text{and} \quad I_0 = \{ i \in B : \gamma_i(f_0) = \delta \}.$$

Let $J_0 = \Gamma(I_0)$ denote the set of goods connected to buyers in I_0 .

During a Type I phase the algorithm repeatedly increases the prices of all goods in J , and whenever a new maximum bang-per-buck edge enters the equality graph (Step 9(c)(i)), we update I and J and recompute a balanced flow, thereby starting a new *iteration* of the phase. Let these iterations be $0, 1, \dots, k$. Since each such iteration adds at least one new good to J and $|J| \leq n$, we have $k \leq n$.

For each iteration t , let p_t be the price vector and let I_t be the corresponding set of active buyers. Define

$$\delta_t = \min_{i \in I_t} \gamma_i(p_t).$$

At the start of the phase $\delta_0 = \delta$, and at the end of the phase (event (1) of Step 9(c)) we have $\delta_k = 0$ because some buyer in I_k has surplus 0.

Since $\delta_0 = \delta$ and $\delta_k = 0$ and there are at most $k \leq n$ iterations, there exists an iteration $t \in \{1, \dots, k\}$ such that

$$\delta_{t-1} - \delta_t \geq \frac{\delta}{n}. \quad (1)$$

We next show that this drop occurs in the surplus of a single buyer already in I_{t-1} . If $i' \in I_t \setminus I_{t-1}$ is a newly added buyer, then by Step 9(c)(i)(C) there is a path in the residual graph of the balanced flow (for prices p_t) from i' to some buyer $h \in I_{t-1}$. By Property 1 of balanced flows, which prohibits residual paths from lower surplus to higher surplus buyers, we obtain $\gamma_{i'}(p_t) \geq \gamma_h(p_t) \geq \delta_{t-1}$. Hence δ_t cannot be achieved by a newly added buyer, and therefore there exists $i \in I_{t-1}$ such that

$$\gamma_i(p_{t-1}) - \gamma_i(p_t) \geq \delta_{t-1} - \delta_t \geq \frac{\delta}{n}. \quad (2)$$

Let $\sigma = \gamma_i(p_{t-1}) - \gamma_i(p_t)$; then $\sigma \geq \delta/n$. Now we apply Lemma 4 to balanced flows f_{t-1} and f_t at prices p_{t-1} and p_t . If the surplus of some buyer decreases by σ , then

$$\|\gamma(f_{t-1})\|_2^2 - \|\gamma(f_t)\|_2^2 \geq \sigma^2. \quad (3)$$

Therefore, from iteration $t-1$ to iteration t we have

$$\Phi_{t-1} - \Phi_t \geq \sigma^2 \geq \frac{\delta^2}{n^2}. \quad (4)$$

In all other iterations of the phase, the potential Φ is nonincreasing: when prices rise, we recompute a balanced flow at the new prices, resulting in a surplus vector with ℓ_2 -norm no larger than before. Thus overall, from the start to the end of the Type I phase,

$$\Phi_{\text{after}} \leq \Phi_{\text{before}} - \frac{\delta^2}{n^2}. \quad (5)$$

Finally, at the start of the phase every buyer has surplus at most δ , so

$$\Phi_{\text{before}} = \sum_{i \in B} \gamma_i^2 \leq n \delta^2.$$

Combining this with (5) gives

$$\Phi_{\text{after}} \leq \Phi_{\text{before}} - \frac{\delta^2}{n^2} \leq \Phi_{\text{before}} \left(1 - \frac{1}{n^3}\right),$$

which completes the proof. \square

Lemma 7. (Bounding Type II Phases) *There are at most n Type II phases throughout the algorithm.*

Proof. Each Type II phase removes one buyer from B . Since we start with at most n buyers, there can be at most n such phases. \square

We next consider Type III phases which end in Step 9(c)(iii)(D), i.e., when buyer i has partial money returned.

Lemma 8. (Bounding Type III Phases) *During any Type III phase, the potential Φ decreases by at least a multiplicative factor of $1 - 1/(4n^3)$.*

Proof. We will use the notation set up in the proof of Lemma 6. Let δ be the maximum surplus of a buyer at the start of this phase. There are $k \leq n$ iterations, with iteration k ending in the return of money ρ_i to buyer i . Let r be the vector of money returned at the start of the phase and $r + \rho_i$ at the end. Now there are two cases:

Case 1: There is a buyer $j \in I_{k-1}$ whose surplus drops to $\leq \delta/2$.
If so, there exists an iteration $t \in \{1, \dots, k-1\}$ such that

$$\delta_{t-1} - \delta_t \geq \frac{\delta}{2n}. \quad (1)$$

As in Lemma 6, applying Lemma 4, the desired result follows.

Case 2: The minimum surplus of a buyer in I_{k-1} is $> \delta/2$.

Since i is in a tight set at the end of the phase, its surplus drops to zero. Let f^* be a balanced flow in $N(p^*, r)$ when α_i becomes 1 and before money is returned to i . Let the drop in the surplus of i up to this point in iteration k be δ_k (note that δ_k may be < 0). When money is returned to i , her surplus drops further by ρ_i , therefore,

$$\delta_k + \rho_i > \delta/2.$$

Hence one of δ_k and ρ_i is at least $\delta/4$. In the first case, the argument of Lemma 6 applies and gives the desired result. In the second case, on returning money, the surplus of i drops by at least $\delta/4$ and the surplus of other buyers remains unchanged, resulting in a drop in the ℓ_1 norm of the surplus vector by the same amount. Consequently, the ℓ_2^2 norm of the surplus vector drops by at least $(\delta^2/4)$ and it may decrease further on recomputing a balanced flow. Since Φ at the start of the phase is $\leq n\delta^2$, we get:

$$\Phi_{\text{after}} \leq \Phi_{\text{before}} - \left(\frac{\delta}{4}\right)^2 \leq \Phi_{\text{before}} \left(1 - \frac{1}{16n}\right) \leq \Phi_{\text{before}} \left(1 - \frac{1}{4n^3}\right),$$

which completes the proof. □

We now prove that Algorithm 1 runs in time polynomial in n , $\log U$, and the bit-size of the initial money vector (m_i) . Our analysis mirrors that of DPSV [DPSV08], but includes two differences specific to the Arctic Auction: (i) buyers may have money returned during the algorithm, and hence their effective budgets $(m_i - r_i)$ decrease over time; and (ii) the network $N(p, r)$ contains buyer-return arcs that must be accounted for in the bit-size analysis. Importantly, the return of money never increases any coordinate of the surplus vector, and therefore never increases the potential function Φ .

Theorem 4. (Running Time)

Let all numbers m_i and u_{ij} be nonnegative rationals whose numerators and denominators are written with at most L bits in total, and let $M = \sum_{i \in B} m_i$ and $U = \max_{i,j} u_{ij}$. Then Algorithm 1 terminates after at most

$$O(n^3(\log n + n \log U + \log M))$$

phases, and performs at most

$$O(n^5(\log n + n \log U + \log M))$$

max-flow computations.

Proof. At the start of the algorithm, $r_i = 0$ for all i , so $0 \leq \gamma_i \leq m_i$. Thus

$$\Phi_{\text{init}} = \sum_i \gamma_i^2 \leq \sum_i m_i^2 \leq M^2.$$

Whenever money is returned, $(m_i - r_i)$ only decreases. Therefore γ_i never increases, and hence

$$\Phi \leq \Phi_{\text{init}} \leq M^2 \quad \text{throughout the algorithm.}$$

Thus decreasing money does not worsen the upper bound for Arctic Auction, compared to [DPSV08].

Potential drop per phase: By Lemma 6, in every Type I phase,

$$\Phi_{\text{after}} \leq \left(1 - \frac{1}{n^3}\right) \Phi_{\text{before}}.$$

By Lemma 8 in every Type III phase,

$$\Phi_{\text{after}} \leq \left(1 - \frac{1}{4n^3}\right) \Phi_{\text{before}}.$$

By Lemma 7, each Type II phase removes one buyer permanently, and hence there are at most n Type II phases.

Thus for every Type I or Type III phase, we have the uniform multiplicative decrease

$$\Phi_{\text{after}} \leq \left(1 - \frac{1}{4n^3}\right) \Phi_{\text{before}}. \quad (2)$$

Lower bound on nonzero potential: All prices, flows, surpluses, and money-return values arise as solutions of linear systems of equations with coefficients taken from $\{m_i\}$, $\{u_{ij}\}$ and previous values of (p, r) , all of which have bit-size at most $O(\log n + n \log U + \log M)$ by the same argument used in [DPSV08]. Thus every rational number appearing in the algorithm has denominator of size at most $2^{O(\log n + n \log U + \log M)}$.

Hence if any $\gamma_i > 0$, then

$$\gamma_i \geq 2^{-O(\log n + n \log U + \log M)},$$

and therefore

$$\Phi \geq 2^{-O(\log n + n \log U + \log M)}. \quad (3)$$

This lower bound does *not* depend on the current total money $\sum_i (m_i - r_i)$ and therefore remains valid even as money is returned.

Bounding the total number of phases and max-flow computations: Let K denote the number of Type I plus Type III phases. By repeated application of (2), we obtain

$$\Phi_{\text{final}} \leq \left(1 - \frac{1}{4n^3}\right)^K \Phi_{\text{init}} \leq \left(1 - \frac{1}{4n^3}\right)^K M^2.$$

As long as the algorithm has not terminated, $\Phi > 0$, and hence by (3),

$$2^{-O(\log n + n \log U + \log M)} \leq \left(1 - \frac{1}{4n^3}\right)^K M^2.$$

Taking logarithms and using $\ln(1 - x) \leq -x$, we obtain

$$K = O\left(n^3(\log n + n \log U + \log M)\right).$$

Each phase consists of at most n iterations, since each iteration adds a new mbpb edge and no such edge can be added twice. Since the balanced-flow procedure of DPSV requires n max-flow computations (Section 8 of [DPSV08]), each iteration requires at most $O(n)$ max-flow computations. Therefore each phase uses $O(n^2)$ max-flow calls, giving a total of $O(n^5(\log n + n \log U + \log M))$ max-flows. \square

6 Linear Fisher Market with Constant Marginal Costs

Definition 6. The *Linear Fisher market with constant marginal costs for production* differs from Definition 1 in two respects: first, there is no upper bound on the amount of goods available for sale, and second, for each good j , the production cost incurred by the seller for good j is given by a function $c_j : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $c_j(y_j) = d_j \cdot y_j$, where $d_j > 0$ is a rational parameter. Because supply is unbounded, we are not seeking a *competitive equilibrium*³.

Allocations and prices must satisfy: each buyer gets a utility maximizing bundle of goods and the money of each buyer is either spent on goods or is returned. The seller's *revenue* is defined to be

$$\sum_{i \in B} m_i - s.$$

For each good j , let $y_j = \sum_{i \in B} x_{ij}$, i.e., the amount of good j sold. Then the *profit* of the seller is defined to be:

$$\sum_{i \in B} m_i - s - \sum_{j \in G} c_j(y_j).$$

We will show that the following variant of program (1) captures optimal allocations and prices for the market given in Definition 6.

$$\begin{aligned} \max \quad & \sum_{i \in B} m_i (\log(w_i(x_i) + s_i) - s - \sum_{j \in G} d_j \cdot y_j) \\ \text{s.t.} \quad & \sum_{i \in B} x_{ij} - y_j = 0 \quad \forall j \in G, \\ & \sum_{i \in B} s_i - s = 0, \\ & x_{ij} \geq 0 \quad \forall i \in B, j \in G, \\ & s_i \geq 0 \quad \forall i \in B \end{aligned} \tag{4}$$

Let p_j and λ be the dual variables corresponding to the first and second constraints of (4). Optimal solutions to the primal and dual variables must satisfy KKT conditions, in addition to the constraints of (4):

1. $\forall j \in G : d_j - p_j \geq 0$.
2. $\forall j \in G : y_j > 0 \implies d_j = p_j$.
3. $1 - \lambda \geq 0$.
4. $s > 0 \implies \lambda = 1$
5. $\forall i \in B, \forall j \in G : \frac{u_{ij}}{p_j} \leq \frac{w_i(x_i) + s_i}{m_i}$.

³This model is not used for central-bank liquidity matters and therefore "arctic auction" has been dropped from the name.

6.

$$\forall i \in B, \forall j \in G : x_{ij} > 0 \implies \frac{u_{ij}}{p_j} = \frac{w_i(x_i) + s_i}{m_i}.$$

7.

$$\forall i \in B : \lambda \geq \frac{m_i}{w_i(x_i) + s_i}.$$

8.

$$\forall i \in B : s_i > 0 \implies \lambda = \frac{m_i}{w_i(x_i) + s_i}.$$

It is easy to see that the statements given in Lemma 1 hold for this market as well and optimal primal and dual solutions to convex program (4) satisfy them via the same proof as given in Lemma 2.

Theorem 5. *There exist optimal primal and dual solutions to convex program (4) which form a solution to a Fisher market with constant marginal costs for production. At such a solution, the seller's revenue is maximized and his profit will be zero.*

Proof. Three facts need to be established.

1). Buyers get optimal bundles: By Lemma 1, each good bought by i is an mbpb good for i , and if money is returned, it provides at least as much utility as mbpb goods. Furthermore, for each buyer i , all her money is either spent on mbpb goods or is returned, i.e.,

$$m_i = \sum_{j \in G} u_{ij}x_{ij} + s_i.$$

2). The seller maximizes her revenue: We will consider three cases:

Case 1: $\alpha_i > 1$. If so, i 's mbpb goods give her more utility than money. Therefore in every optimal primal, $s_i = 0$, i.e., i spends all her money on goods and it becomes the seller's revenue.

Case 2: $\alpha_i = 1$. If so, i gets the same utility from her mbpb goods as from money. In this case, we will assume that i spends all her money on goods and it becomes the seller's revenue.

Case 3: $\alpha_i < 1$. If so, i derives more utility from money than her mbpb goods. Therefore in every optimal primal, $s_i = m_i$.

3). The seller's profit will be zero: By KKT condition (2), if $y_j > 0$, $p_j = d_j$. As a result, all the revenue made from selling good j , i.e., $p_j \cdot y_j$, will be used for paying cost, since $d_j \cdot y_j = p_j \cdot y_j$. Therefore seller's profit will be zero. \square

Theorem 6. *Convex program (4) is a rational convex program.*

Proof. The proof is easier than the one for Theorem 2 because for each good j that is sold to a non-zero extent, $p_j = d_j$, establishing its rationality. Corresponding to non-zero x_{ij} s and s_i s, we will use the linear equations given in Theorem 2 to obtain the required linear system. \square

6.1 Efficient Greedy Algorithm

The algorithm for obtaining a solution is straightforward. We set that the price p_j of each good j to d_j . If for a buyer i , $\alpha_i < 1$, i.e., the price of each good j is greater than her upper bound of u_{ij} , then we return money m_i to i and remove her from consideration.

For the remaining buyers, $\alpha_i \geq 1$. For each such buyer i , we find M_i , the set of her mbpb goods. Allocate goods $j \in M_i$ worth m_i money arbitrarily, i.e., $x_{ij} > 0$ only if $j \in M_i$ and $\sum_{j \in G} x_{ij} \cdot p_i = m_i$. These prices and allocations will constitute a solution.

7 Discussion

Following up on [DPSV08], Orlin gave a strongly polynomial time algorithm for computing an equilibrium for the linear Fisher market [Orl10]. Is such an extension possible for the Arctic Auction as well?

8 Acknowledgements

I wish to express my gratitude to Paul Klemperer and Edwin Lock for their generous help which got me started on this project. Paul introduced me to the elegant notion of Arctic Auction and Edwin informed me about several useful references.

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