

Uniform inference for kernel instrumental variable regression

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Abstract

Instrumental variable regression is a foundational tool for causal analysis across the social and biomedical sciences. Recent advances use kernel methods to estimate nonparametric causal relationships, with general data types, while retaining a simple closed-form expression. Empirical researchers ultimately need reliable inference on causal estimates; however, uniform confidence sets for the method remain unavailable. To fill this gap, we develop valid and sharp confidence sets for kernel instrumental variable regression, allowing general nonlinearities and data types. Computationally, our bootstrap procedure requires only a single run of the kernel instrumental variable regression estimator. Theoretically, it relies on the same key assumptions. Overall, we provide a practical procedure for inference that substantially increases the value of kernel methods for causal analysis.

Keywords: Gaussian approximation, ill posed inverse problem, nonparametric regression, reproducing kernel Hilbert space.

1 Introduction and related work

Nonparametric instrumental variable regression is a leading framework for causal analysis from observation data (Newey and Powell, 2003; Ai and Chen, 2003; Hall and Horowitz, 2005; Blundell et al., 2007; Darolles et al., 2011). A recent literature advocates for kernel methods as natural extensions from linear models to nonlinear models (Singh et al., 2019; Dikkala et al., 2020). Similar to a linear method, a kernel method has a simple closed-form solution (Kimeldorf and Wahba, 1971). Unlike a linear method, a kernel method allows for rich nonlinearity in the causal relationship as well as general data types, such as preferences, sequences, and graphs, which often arise in economics and epidemiology.

This literature proposes nonparametric estimators and proves uniform consistency, yet uniform inference guarantees are unavailable; these causal estimators lack uniform confidence bands. Without confidence bands, social and biomedical scientists are reluctant to fully rely on these new estimators in causal analysis. More generally, uniform confidence bands appear to be absent from the recent, burgeoning literature on machine learning estimation of nonparametric instrumental variable regression. Our research question is how to construct them.

Our primary contribution is to develop a uniform confidence band for a kernel estimator of the nonparametric instrumental variable regression function. Our inferential procedure retains the practicality of kernel methods; computationally, it is a bootstrap that involves running kernel instrumental variable regression exactly once and sampling many anti-symmetric Gaussian multipliers. The anti-symmetry is effective at canceling out the complex bias of the estimator.

Our secondary contribution is to prove that the uniform confidence band is valid and sharp: it obtains coverage of at least, and not much more than, the nominal level. Formally, we derive nonasymptotic Gaussian and bootstrap couplings, which overcome the challenge of the ill-posed inverse problem inherent in nonparametric instrumental variable regression.

Nonasymptotic analysis is necessary because a stable Gaussian limit does not exist.

We show that well-known assumptions (Smale and Zhou, 2007; Caponnetto and De Vito, 2007; Mendelson and Neeman, 2010; Fischer and Steinwart, 2020) imply not only estimation but also inference guarantees. Our key assumptions are: (i) the data have a low effective dimension when expressed in the basis of the kernel; (ii) the true nonparametric instrumental variable regression function is smooth in terms of the basis of the kernel; and (iii) the expectation of the nonparametric instrumental variable regression function, conditional upon the instrumental variable, is a smooth function as well. These assumptions are called decay, source, and link conditions, respectively.

By studying inference, we complement several works on estimation and consistency of kernel methods for nonparametric instrumental variable regression. Previous work provides rates in mean square error after projection upon the instrument (Singh et al., 2019; Dikkala et al., 2020), in mean square error (Liao et al., 2020; Bennett et al., 2023a,b), in sup norm (Singh, 2020), and in interpolation norms (Meunier et al., 2024). Such rates can be used to verify conditions for inference on certain well-behaved functionals (Kallus et al., 2021; Ghassami et al., 2022; Chernozhukov et al., 2023). However, none of these works provide uniform confidence bands, which are the focus of the present work.

A recent paper provides uniform inference for kernel ridge regression (Singh and Vijaykumar, 2023). Kernel ridge regression is an easier estimation problem, which does not require ill-posed inversion of a conditional expectation operator. Previous results for kernel ridge regression do not apply to our setting. Still, we build on the broad structure of their argument. Specifically, we analyze Gaussian couplings (Zaitsev, 1987; Buzun et al., 2022) and bootstrap couplings (Freedman, 1981; Chernozhukov et al., 2014, 2016) in settings where the limit distribution may be degenerate (Andrews and Shi, 2013). In doing so, we develop new techniques

that may be used to provide uniform statistical inference in other ill-posed inverse problems.

Finally, our inferential procedure for a kernel instrumental variable estimator complements existing results for series estimators of nonparametric instrumental variable regression (Chen, 2007; Carrasco et al., 2007). The series procedures are designed for and theoretically justified in a setting with low- to moderate-dimensional Euclidean data (Belloni et al., 2015; Chen and Christensen, 2018; Chen et al., 2024). By contrast, our kernel procedure permits complex and nonstandard data, as long as the data have a low effective dimension relative to the kernel.

Section 2 recaps the kernel instrumental variable regression estimator and interprets our main assumptions: low effective dimension and high smoothness. Section 3 presents our main contribution: valid and sharp confidence sets for a kernel instrumental variable regression estimator. Section 4 concludes by discussing consequences for the uptake of kernel methods in causal analysis.

2 Model and assumptions

We begin by introducing some notation. We denote the L^2 norm by $\|f\|_2 = (\mathbb{E}[f(Z)^2])^{1/2}$, with empirical counterpart $\|f\|_{2,n} = \left(\frac{1}{n} \sum_{i=1}^n f(Z_i)^2\right)^{1/2}$. For any operator A mapping between Hilbert spaces, i.e. $A: \mathcal{H} \rightarrow \mathcal{H}'$, let $\|A\|_{\text{HS}}$ and $\|A\|_{\text{op}}$ denote the Hilbert-Schmidt and operator norm, respectively. We denote the eigendecomposition of a compact and self-adjoint operator A with eigenvalues $\{\nu_1(A), \nu_2(A), \dots\}$ and eigenfunctions $\{e_1(A), e_2(A), \dots\}$.

We use C , potentially with subscripts, to denote a positive constant that may only depend on the subscript parameter. For example, C_σ is a positive constant depending only on the parameter σ . We use \lesssim (or \lesssim_{C_σ}) to denote an inequality that holds up to some positive multiplicative constant (or function of σ). Equations and inequalities containing the parameter η are understood to hold with probability at least $1 - \eta$, where $\eta \in (0, 1)$. Throughout the

paper, we abbreviate $l(\eta) = \log(2/\eta)$.

2.1 Previous work: Closed form estimation of KIV

Our goal is to learn and conduct inference on the nonparametric instrumental variable regression function h_0 , which is defined as the solution to the following operator equation:

$$Y = h_0(X) + \varepsilon, \quad \mathbb{E}(\varepsilon|Z) = 0 \iff \mathbb{E}(Y|Z) = \mathbb{E}\{h_0(X)|Z\}. \quad (1)$$

We refer to Y as the outcome, X as the covariate, and Z as the instrument. The former formulation is clearly a generalized regression problem; when $X = Z$, $h_0(X) = \mathbb{E}(Y|X)$. However, in our setting, $X \neq Z$. The latter formulation has the “reduced form” function $\mathbb{E}(Y|Z)$ on the left-hand side, and the composition of a “first stage” conditional expectation operator $\mathbb{E}\{\dots|Z\}$ and a “second stage” function of interest $h_0(X)$ on the right-hand side. Isolating h_0 involves inverting the conditional expectation operator, which is an ill-posed task that makes this statistical problem challenging.

To estimate h_0 , we use a pair of approximating function spaces \mathcal{H}_x and \mathcal{H}_z that possess a specific structure: both are reproducing kernel Hilbert spaces (RKHSs) defined by the kernels $k_x : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$ and $k_z : \mathcal{Z} \times \mathcal{Z} \rightarrow \mathbb{R}$, respectively. We denote the associated feature maps by $\psi : \mathcal{X} \rightarrow \mathcal{H}_x$ and $\phi : \mathcal{Z} \rightarrow \mathcal{H}_z$, giving rise to the inner products $k_x(x, x') = \langle \psi(x), \psi(x') \rangle_{\mathcal{H}_x}$ and $k_z(z, z') = \langle \phi(z), \phi(z') \rangle_{\mathcal{H}_z}$. These spaces possess the reproducing property: $h(x) = \langle h, \psi(x) \rangle_{\mathcal{H}_x}$ for any $h \in \mathcal{H}_x$, and $f(z) = \langle f, \phi(z) \rangle_{\mathcal{H}_z}$ for any $f \in \mathcal{H}_z$. In other words, $\psi(x)$ is the dictionary of basis functions for \mathcal{H}_x , and likewise $\phi(z)$ for \mathcal{H}_z . The implied norms $\|h\|_{\mathcal{H}_x} = \langle h, h \rangle_{\mathcal{H}_x}^{1/2}$ and $\|f\|_{\mathcal{H}_z} = \langle f, f \rangle_{\mathcal{H}_z}^{1/2}$ quantify regularity. Concretely, the RKHS norm quantifies not only magnitude but also smoothness, generalizing the Sobolev norm.

Throughout the paper, we maintain a few regularity conditions to simplify the exposi-

tion. We assume that ψ and ϕ are measurable and bounded, i.e., $\sup_{x \in \mathcal{X}} \|\psi(x)\|_{\mathcal{H}_x} \leq \kappa_x$ and $\sup_{z \in \mathcal{Z}} \|\phi(z)\|_{\mathcal{H}_z} \leq \kappa_z$, which is satisfied by all kernels commonly used in practice. Additionally, we maintain that the residual $\varepsilon := Y - h_0(X)$ satisfies $|\varepsilon| \leq \bar{\sigma}$.

We now define covariance operators, which are central to our analysis. Let the symbol \otimes mean outer product. The covariance operator $S_x = \mathbb{E}\{\psi(X) \otimes \psi(X)^*\}$ satisfies $\langle u, S_x v \rangle_{\mathcal{H}_x} = \mathbb{E}[u(X)v(X)]$. The covariance operator $S_z = \mathbb{E}\{\phi(Z) \otimes \phi(Z)^*\}$ satisfies $\langle u, S_z v \rangle_{\mathcal{H}_z} = \mathbb{E}[u(Z)v(Z)]$. Finally, we define the cross covariance operator $S = \mathbb{E}\{\phi(Z) \otimes \psi(X)^*\}$, with adjoint $S^* = \mathbb{E}\{\psi(X) \otimes \phi(Z)^*\}$, satisfying $\langle f, Sh \rangle_{\mathcal{H}_z} = \langle S^* f, h \rangle_{\mathcal{H}_x} = \mathbb{E}[f(Z)h(X)]$. Together, these operators give rise to the modified covariance operator $T = S^* S_z^{-1} S$, which can be shown to satisfy $\langle f, Tf \rangle_{\mathcal{H}_x} = \mathbb{E}[\mathbb{E}\{f(X)|Z\}^2]$.

Kernel instrument variable regression (KIV) is a nonlinear extension of the standard two-stage least-squares (2SLS) method for linear estimation. In fact, 2SLS is a special case of KIV when k_x and k_z are linear kernels and with regularization set to zero. In other words, KIV generalizes from unregularized linear estimation to regularized nonlinear estimation. In the KIV model, the conditional expectation operator in (1) is given by $T^{\frac{1}{2}}$.

We consider two equivalent formulations of the estimator's objective. One formulation resembles regression, following Singh et al. (2019):

$$h_{\mu, \lambda} = \underset{h \in \mathcal{H}_x}{\operatorname{argmin}} \|(S_z + \mu)^{-1/2} S(h_0 - h)\|_{\mathcal{H}_z}^2 + \lambda \|h\|_{\mathcal{H}_x}^2.$$

The first term is a projected mean square error, with “first stage” regularization $\mu > 0$. The second term is a ridge penalty, with “second stage” regularization $\lambda > 0$. With $X = Z$ and $\mu = 0$, this objective reduces to the kernel ridge regression objective.

A second formulation is based on the conditional moment restriction $\mathbb{E}(Y - h_0(X)|Z) = 0$.

The adversarial formulation, following Dikkala et al. (2020) is

$$h_{\mu,\lambda} = \operatorname{argmin}_{h \in \mathcal{H}_x} \max_{f \in \mathcal{H}_z} 2\mathbb{E}[\{Y - h(X)\}f(Z)] - \|f\|_2^2 - \mu\|f\|_{\mathcal{H}_z}^2 + \lambda\|h\|_{\mathcal{H}_x}^2. \quad (2)$$

Intuitively, the adversary f maximizes the violation of the conditional moment. The estimator h minimizes the violation of the conditional moment, anticipating this adversary.

Regardless of the formulation, the estimator has a convenient closed-form solution due to the kernel trick. Specifically, the empirical analogues of both objectives are minimized by the following algorithm.

Algorithm 1 (Kernel instrumental variable regression). Given a sample $D = \{(Z_i, X_i, Y_i)\}_{i=1}^n$, kernels k_x and k_z , and regularization parameters $\lambda, \mu > 0$:

1. Compute the kernel matrices $K_{XX}, K_{ZZ} \in \mathbb{R}^{n \times n}$ with (i,j) th entries $k_x(X_i, X_j)$ and $k_z(Z_i, Z_j)$, respectively.
2. Compute the kernel vector $K_{xX} \in \mathbb{R}^{1 \times n}$ with i th entry $k_x(x, X_i)$
3. Estimate KIV as $\hat{h}(x) = K_{xX} \{K_{ZZ}(K_{ZZ} + n\mu I)^{-1} K_{XX} + n\lambda I\}^{-1} K_{ZZ}(K_{ZZ} + n\mu I)^{-1} Y$.

Example 1 (Linear kernel). Let $\mathcal{X} = \mathbb{R}^p$ and $\mathcal{Z} = \mathbb{R}^q$, so that the covariates and instruments are finite-dimensional vectors, and consider the linear kernels $k_x(x, x') = x^\top x'$ and $k_z(z, z') = z^\top z'$. Then, \mathcal{H}_x consists of linear functions of the form $h_\gamma(x) = \gamma^\top x$ for $\gamma \in \mathbb{R}^p$. Let $\mathbf{X} \in \mathbb{R}^{n \times p}$ and $\mathbf{Z} \in \mathbb{R}^{n \times q}$ be the design matrices. Then, the kernel objects become $K_{xX} = x\mathbf{X}^T$, $K_{XX} = \mathbf{X}\mathbf{X}^T$ and $K_{ZZ} = \mathbf{Z}\mathbf{Z}^T$, leading to

$$\hat{h}(x) = x\mathbf{X}^T \left[\mathbf{Z}\mathbf{Z}^T (\mathbf{Z}\mathbf{Z}^T + n\mu I_n)^{-1} \mathbf{X}\mathbf{X}^T + n\lambda I_n \right]^{-1} \mathbf{Z}\mathbf{Z}^T (\mathbf{Z}\mathbf{Z}^T + n\mu I_n)^{-1} Y = x^\top \hat{\gamma}$$

where $\hat{\gamma} = \left[\mathbf{X}^T \mathbf{Z} (\mathbf{Z}^T \mathbf{Z} + n\mu I_q)^{-1} \mathbf{Z}^T \mathbf{X} + n\lambda I_p \right]^{-1} \mathbf{X}^T \mathbf{Z} (\mathbf{Z}^T \mathbf{Z} + n\mu I_q)^{-1} \mathbf{Z}^T Y$ is regularized 2SLS.

Example 2 (Polynomial kernel). Let $\mathcal{X} = \mathbb{R}^p$ and $\mathcal{Z} = \mathbb{R}^q$ as before, but now choose the d -degree polynomial kernels with offsets $c_x, c_z \geq 0$: $k_x(x, x') = (x^\top x' + c_x)^d$ and $k_z(z, z') = (z^\top z' + c_z)^d$. Then there exist finite-dimensional feature maps $\psi: \mathbb{R}^p \rightarrow \mathbb{R}^{M_x}$ and $\phi: \mathbb{R}^q \rightarrow \mathbb{R}^{M_z}$ with $M_x = \binom{p+d}{d}$, and $M_z = \binom{q+d}{d}$, such that $k_x(x, x') = \langle \psi(x), \psi(x') \rangle_{\mathcal{H}_x}$ and $k_z(z, z') = \langle \phi(z), \phi(z') \rangle_{\mathcal{H}_z}$. The RKHS \mathcal{H}_x consists of polynomials of degree at most d , of the form $h(x) = w^\top \psi(x)$, with $w \in \mathbb{R}^{M_x}$.

Let $\Psi_X = [\psi(X_1), \dots, \psi(X_n)]^\top \in \mathbb{R}^{n \times M_x}$ and $\Phi_Z = [\phi(Z_1), \dots, \phi(Z_n)]^\top \in \mathbb{R}^{n \times M_z}$ be the design matrices. Then, the kernel objects are $K_{xX} = \psi(x)^\top \Psi_X^\top$, $K_{XX} = \Psi_X \Psi_X^\top$, $K_{ZZ} = \Phi_Z \Phi_Z^\top$,¹ and

$$\hat{h}(x) = \psi(x)^\top \Psi_X^\top \left[\Phi_Z \Phi_Z^\top (\Phi_Z \Phi_Z^\top + n\mu I_n)^{-1} \Psi_X \Psi_X^\top + n\lambda I_n \right]^{-1} \Phi_Z \Phi_Z^\top (\Phi_Z \Phi_Z^\top + n\mu I_n)^{-1} Y = \psi(x)^\top \hat{w},$$

where $\hat{w} = \left[\Psi_X^\top \Phi_Z (\Phi_Z^\top \Phi_Z + n\mu I_{M_z})^{-1} \Phi_Z^\top \Psi_X + n\lambda I_{M_x} \right]^{-1} \Psi_X^\top \Phi_Z (\Phi_Z^\top \Phi_Z + n\mu I_{M_z})^{-1} \Phi_Z^\top Y$ is regularized 2SLS with d -order polynomial expansions.

Example 3 (Preference kernel). A key advantage of kernel methods is that kernels can be chosen by the researcher to handle non-standard data. As a leading example, Singh and Vijaykumar (2023) consider student preferences data over 25 Boston schools. The space of preferences \mathcal{X} has dimension $25!$. It is not feasible to model these preferences with $25!$ indicators. However, it is feasible to model these preferences with kernels. A natural preference kernel is $k_x(x, x') = \exp\{-N(x, x')\}$, where $N(x, x')$ counts the number of pairwise disagreements between rankings x and x' . This kernel induces an RKHS \mathcal{H}_x over preferences.

Unobserved confounding is an important concern when studying school choice. If families receive a “nudge” of a randomly assigned default preference in an online system, this randomly assigned preference may be viewed as an instrument. It would be natural to use the preference kernel for the instrument as well: $k_z(z, z') = \exp\{-N(z, z')\}$, inducing an RKHS \mathcal{H}_z .

¹Equivalently, the (i, j) th entries of K_{XX} and K_{ZZ} are $(X_i^\top X_j + c_x)^d$ and $(Z_i^\top Z_j + c_z)^d$, respectively, while the i th entry of K_{xX} is $(x^\top X_i + c_x)^d$.

2.2 Goal: Valid and sharp confidence sets

In this paper, our goal is to construct confidence sets \hat{C}_n for the estimator in Algorithm 1, applicable to various data types. We would like these confidence sets to be computationally efficient: they should not require additional kernel evaluations or matrix inversions beyond Algorithm 1. Theoretically, we would like these confidence sets to be valid and sharp: they should contain the true nonparametric instrumental variable regression function h_0 with at least, but not much more than, nominal coverage. In what follows, we carefully define validity and sharpness.

Definition 1 (Validity). \hat{C}_n is τ -valid at level χ if $\mathbb{P}(h_0 \in \hat{C}_n) \geq 1 - \chi - \tau$.

Definition 2 (Sharpness). \hat{C}_n is (δ, τ) -sharp at level χ if $\mathbb{P}\left\{h_0 \in (1-\delta)\hat{C}_n + \delta\hat{h}\right\} \leq 1 - \chi + \tau$.

A valid confidence set contains h_0 with at least nominal coverage, up to a tolerance level τ . A sharp confidence set is not too conservative: if we slightly contract the set \hat{C}_n towards its center \hat{h} , i.e. if we examine $(1-\delta)\hat{C}_n + \delta\hat{h}$, then this contracted set should contain h_0 at most at the nominal level, up to a tolerance level τ . Intuitively, if $\tau = 0$ and $\delta = 0$, then validity and sharpness give exact coverage $\mathbb{P}(h_0 \in \hat{C}_n) = 1 - \chi$. We will show $\tau = \mathcal{O}(n^{-1})$ and $\delta = \log(n)^{-1}$ in our nonasymptotic analysis.

The following bias-variance decomposition illuminates the structure of our argument:

$$n^{1/2}(\hat{h} - h_0) = \underbrace{n^{1/2}\{(\hat{h} - h_{\mu, \lambda}) - \mathbb{E}_n(U)\}}_{\text{residual}} + \underbrace{n^{1/2}\mathbb{E}_n(U)}_{\text{pre-Gaussian}} + \underbrace{n^{1/2}(h_{\mu, \lambda} - h_0)}_{\text{bias}},$$

where $U_i \in \mathcal{H}_x$ is a mean-zero random function explicitly defined below. We prove that the residual term is asymptotically negligible in a strong sense. For the pre-Gaussian term, we provide Gaussian and bootstrap couplings, under a decay condition. Lastly, we show that the bias of the estimator vanishes under source and link conditions. These conditions are standard assumptions in the NPIV literature, though we take care while interpreting them in our setting.

2.3 Main assumption: Low effective dimension

Our main assumption is that the data have low effective dimensions relative to the bases of the kernels. Specifically, we assume that the covariance operators S_x , S_z , and T have eigenvalues that decay. Recall that $S_x = \mathbb{E}\{\psi(X) \otimes \psi(X)^*\}$, so decaying eigenvalues mean that relatively few dimensions of the features $\psi(X)$ can convey most of the information in the distribution of covariates X . We quantify the rate of spectral decay via the local width.

Definition 3 (Local width). Given $m > 0$, the local width of operator A is given by $\sigma^2(A, m) = \sum_{s>m} \nu_s(A)$, where $\{\nu_1(A), \nu_2(A), \dots\}$ are decreasing eigenvalues.

The local width is the tail sum of the eigenvalues. It quantifies how much information we lose when only considering the initial m dimensions.

In this paper, we prove that the pre-Gaussian term of KIV is of the form $n^{1/2}\mathbb{E}_n(U)$, where each summand is given by the expression

$$U_i = T_{\mu, \lambda}^{-1} \{M_i + M_i^* + N_i\} (h_0 - h_{\mu, \lambda}) + T_{\mu, \lambda}^{-1} S^* (S_z + \mu)^{-1} \phi(Z_i) \varepsilon_i.$$

In this compact notation, $T_{\mu, \lambda} = S^* (S_z + \mu)^{-1} S + \lambda$. Moreover, $M_i = S^* (S_z + \mu)^{-1} \{\phi(Z_i) \otimes \psi(X_i)^* - S\}$ and $N_i = S^* (S_z + \mu)^{-1} \{S_z - \phi(Z_i) \otimes \phi(Z_i)^*\} (S_z + \mu)^{-1} S$.

The complexity of the pre-Gaussian term, and therefore the challenge of Gaussian approximation, is quantified by the local width of $\Sigma := \mathbb{E}(U \otimes U^*)$. In our analysis, we show that the local width of Σ is bounded by the local width of $T = S^* S_z^{-1} S$ in the sense that $\sigma^2(\Sigma, m) \leq \frac{8\sigma^2}{\lambda^2} \sigma^2(T, m)$. Therefore, the fundamental condition for Gaussian approximation is $\sigma^2(T, m) \downarrow 0$ as $m \uparrow \infty$. This fundamental condition is also necessary for consistency of the KIV estimator, as shown in previous work. Appendix E provides concrete bounds on $\sigma^2(T, m)$ under low-level conditions. A simple sufficient condition is that the eigenvalues of S_x and S_z decay polynomially.

In this paper, we focus on the case of polynomial decay for simplicity. Our results naturally extend to exponential decay, appealing to the corresponding bounds on local width in Singh and Vijaykumar (2023).

2.4 Smoothness assumptions: Source and link conditions

Next, we assume that the target function h_0 is smooth, and that its conditional expectation is smooth. These are called source and link conditions in the nonparametric instrumental variable regression literature (Chen and Reiss, 2007; Caponnetto and De Vito, 2007). Such conditions are necessary for consistent estimation. Naturally, then, we will also use them to derive valid inference.

Assumption 1 (Source condition). *The target h_0 is smooth: there exists $\alpha \in [0,1]$ and $w_0 \in \mathcal{H}_x$ such that $h_0 = T^\alpha w_0$.*

Assumption 1 with $\alpha=0$ simply means that h_0 is correctly specified by \mathcal{H}_x . For $\alpha>0$, it means that h_0 is a particularly smooth element of \mathcal{H}_x .

Assumption 2 (Link condition). *Conditional expectations of smooth functions are smooth functions: the operator $S_z^{-(\frac{1}{2}+\beta)} S : \mathcal{H}_x \rightarrow \mathcal{H}_z$ is bounded, i.e. $\|S_z^{-(\frac{1}{2}+\beta)} S\|_{\text{op}} \leq r$ for some $\beta \in [1/2,1]$ and some $r < \infty$.*

Assumption 2 with $\beta=0$ simply means that the conditional expectation operator, when viewed as a mapping from \mathcal{H}_x to $L^2(Z)$, is bounded. For $\beta=0$, the link condition automatically holds.² With $\beta>0$, it means that the conditional expectation operator, when viewed as a mapping from \mathcal{H}_x to $\mathcal{H}_z^{\frac{1}{2}+\beta}$, is bounded. In other words, for any smooth function $h \in \mathcal{H}_x$, its conditional expectation $g(Z) = \mathbb{E}[h(X)|Z]$ is also smooth in the sense that $g \in \mathcal{H}_z^{\frac{1}{2}+\beta}$. Here,

²To see why, notice that \mathcal{H}_x embeds continuously in $L^2(X)$, and appeal to the law of total variance.

$\beta = \frac{1}{2}$, corresponds to the natural assumption that $g \in \mathcal{H}_z$, implying that the reduced form function $\mathbb{E}(Y|Z)$ is well-specified by \mathcal{H}_z .

Sobolev spaces are special cases of RKHSs. In Sobolev spaces, the source and link conditions amount to assumptions that the number of square integrable derivatives is high enough relative to the dimension of the data.

2.5 Technical assumption: Strong instrument

Finally, we require that the instrumental variable Z is strong enough, in the sense that it carries enough information about the covariate X . Such an assumption is standard in the instrumental variable literature, where it is often stated as a certain rank being large enough. In our setting, it becomes a condition that there are enough directions in \mathcal{H}_x that are well explained by \mathcal{H}_z . Let $\tilde{m}(\lambda, \mu) := \text{tr} T_{\mu, \lambda}^{-1} S^* (S_z + \mu)^{-1} S_z (S_z + \mu)^{-1} S T_{\mu, \lambda}^{-1}$ be the effective dimension of the nonparametric instrumental variable regression problem, generalizing the standard effective dimension (Caponnetto and De Vito, 2007; Fischer and Steinwart, 2020).

Assumption 3 (Strong instrument). *The instrument is strong enough: $\tilde{m}(\lambda, \mu) \gtrsim_{\rho_x, \omega_x} \lambda^{-\rho_x}$, where $\rho_x \in [1, 2]$ is the polynomial rate of decay for S_x .*

Formally, we assume that a specific effective dimension does not vanish too quickly as the regularization parameter λ vanishes. Later, we will prove that $\tilde{m}(\lambda, \mu) \lesssim_{\rho_x, \omega_x} \lambda^{-\rho_x}$. Assumption 3 assumes the matching lower bound $\tilde{m}(\lambda, \mu) \gtrsim \lambda^{-\rho_x}$, which rules out the possibility that the instrument is too weak.

Throughout the paper, we maintain that $\mu \leq \lambda \leq 1$, which means that we are not regularizing the first stage more than the second stage. This restriction on the regularization parameters is required for our bias argument.

3 Confidence bands with nonstandard data

Inference for KIV poses several challenges. First, recovering h_0 from the operator equation (1) is hard because it requires inverting a conditional expectation operator. Formally, the conditional expectation operator $S_z^{-1}S : h(\cdot) \rightarrow \mathbb{E}\{h(X)|Z=(\cdot)\}$ is an infinite-dimensional quantity that must be estimated from data and then inverted. Second, analyzing \hat{h} in Algorithm 1 is hard because it involves two regularization parameters. Compared to standard regression, additional regularization is unavoidable in order to non-parametrically approximate S_z^{-1} in the “first stage”. However, this additional regularization introduces complex bias, and it is unclear whether the regularization parameters can vanish quickly enough to control bias while also vanishing slowly enough to permit Gaussian approximation. Third, constructing a confidence set centered at \hat{h} becomes harder if we impose practical constraints: the confidence set should not require more computation than the estimator, and the confidence set should remain tractable whenever the estimator is tractable, e.g., with high-dimensional or nonstandard data.

To overcome these challenges, we propose an anti-symmetric Gaussian multiplier bootstrap that yields valid and sharp confidence sets for the KIV estimator. A virtue of our procedure (Algorithm 2) is that the costliest step of KIV estimation—invocation of the two kernel matrices—is performed only once. Theoretically, it yields confidence sets that are valid and sharp in \mathcal{H}_x -norm (Theorem 1), allowing the obvious choice of regularization $\lambda = \mu$. These sharp \mathcal{H}_x -norm confidence sets imply uniform confidence bands because $\|h - h_0\|_\infty \leq \kappa_x \|h - h_0\|_{\mathcal{H}_x}$ by the Cauchy-Schwarz inequality. Overall, we characterize a range of regimes with low effective dimension and high smoothness in which our method works.

3.1 This work: Bootstrap for KIV

We preview our inference procedure at a high level before filling in the details. First, we compute the KIV estimator, saving the kernel matrices and their regularized inverses, which will be reused in inference. This ensures the same $\mathcal{O}(n^3)$ computational complexity of estimation alone (Algorithm 1). Second, for each bootstrap iteration, we draw Gaussian multipliers and compute the bootstrap function \mathfrak{B} . Third, across bootstrap iterations, we calculate the quantile \hat{t}_χ of $\|\mathfrak{B}\|_{\mathcal{H}_x}$. Our confidence set \hat{C}_χ is the point estimate \hat{h} plus $n^{-1/2}\hat{t}_\chi$ inflated by an incremental factor $\{1+1/\log(n)\}$.

Importantly, $1/\log(n)$ is not a tuning parameter. We use this device to guarantee valid inference in many settings, inspired by Andrews and Shi (2013). Similar to Singh and Vijaykumar (2023), it is possible to replace $1/\log(n)$ with zero by placing stronger assumptions on the effective dimension of the data and then employing techniques of Chernozhukov et al. (2014) and Götze et al. (2019).

Our method provides \mathcal{H}_x -norm valued confidence sets. We can translate these confidence sets into uniform confidence bands, because a bounded kernel implies that for all $h \in \mathcal{H}_x$,

$$\sup_{x \in \mathcal{X}} |h(x)| = \sup_{x \in \mathcal{X}} |\langle h, \psi(x) \rangle_{\mathcal{H}_x}| \leq \|h\|_{\mathcal{H}_x} \sup_{x \in \mathcal{X}} \|\psi(x)\|_{\mathcal{H}_x} \leq \kappa_x \|h\|_{\mathcal{H}_x}.$$

Algorithm 2 (Confidence set for kernel instrumental variable regression). Given a sample $D = \{(Z_i, X_i, Y_i)\}_{i=1}^n$, kernels k_x and k_z , and regularization parameters $\lambda, \mu > 0$:

1. Compute the kernel matrices $K_{XX}, K_{ZZ} \in \mathbb{R}^{n \times n}$ as before.
2. Compute matrices $K = K_{ZZ}(K_{ZZ} + n\mu I)^{-1}$, $A = (KK_{XX} + n\lambda I)^{-1}$, and $C = 2K - K^2$.
3. Compute the KIV residuals $\hat{\varepsilon} \in \mathbb{R}^n$ by $\hat{\varepsilon} = Y - K_{XX}AKY$.
4. For each bootstrap iteration,
 - (a) draw multipliers $q \in \mathbb{R}^n$ from $\mathcal{N}(0, I - 11^\top/n)$, where $1 \in \mathbb{R}^n$ has entries equal to one;

- (b) compute the vector $\hat{\gamma} = n^{1/2} AC \text{diag}(\hat{\varepsilon})q$;
 - (c) compute the scalar $M = (\hat{\gamma}^\top K \hat{\gamma})^{1/2}$.
5. Across bootstrap iterations, compute the χ -quantile, \hat{t}_χ , of M .
6. Calculate the \mathcal{H}_x confidence set: $\hat{C}_\chi = \left\{ \hat{h} \pm \hat{t}_\chi n^{-1/2} h : \|h\|_{\mathcal{H}_x} \leq 1 + 1/\log(n) \right\}$.
7. Calculate the uniform confidence band: $\hat{C}_\chi(x) = \left[\hat{h}(x) \pm \hat{t}_\chi n^{-1/2} \kappa_x \{1 + 1/\log(n)\} \right]$.

Within Algorithm 2, we implicitly calculate $M = \|\mathfrak{B}\|_{\mathcal{H}_x}$ for the \mathcal{H}_x valued bootstrap function $\mathfrak{B} = \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n \frac{1}{\sqrt{2}} (\hat{V}_i - \hat{V}_j) h_{ij}$. Here, h_{ij} are independent and identically distributed standard Gaussians. Each summand \hat{V}_i is the empirical analogue of the pre-Gaussian summand U_i . Intuitively, by taking the difference $\hat{V}_i - \hat{V}_j$, we cancel the complex bias of KIV. In this way, we leverage symmetry. See Appendix A for details on the implicit estimator \hat{V}_i of U_i .

3.2 Main result: Valid and sharp inference

We present our main result: Theorem 1 proves that the confidence set in Algorithm 2 is valid and sharp in \mathcal{H}_x -norm. Moreover, Corollary 1 verifies that the confidence band in Algorithm 2 is valid in sup norm; it is a valid uniform confidence band for kernel instrumental variable regression, filling a crucial gap in the literature.

Our proof technique is largely agnostic about how low the effective dimensions of the data are. Fundamentally, we require that the local width in Definition 3 vanishes for the pre-Gaussian term, i.e. that $\sigma^2(\Sigma, m) \downarrow 0$ for $\Sigma = \mathbb{E}(U \otimes U^*)$ and U stated below Definition 3.

For exposition, we impose further structure on the problem, which implies the high level condition. In particular, we assume that the eigenvalues of the covariance operators S_x and S_z decay polynomially. This is a standard regime for RKHS analysis (Caponnetto and De Vito, 2007; Fischer and Steinwart, 2020), which generalizes the Sobolev setting. Formally, to state

Theorem 1, we impose that $\nu_s(S_x) \asymp \omega_x s^{-1/(\rho_x-1)}$ and $\nu_s(S_z) \asymp \omega_z s^{-1/(\rho_z-1)}$, where $\rho_x, \rho_z \in (1, 2]$ quantify the rates of polynomial decay, and where ω_x and ω_z are constants. These assumptions imply that the key local widths vanish: $\sigma^2(T, m) \downarrow 0$, and therefore $\sigma^2(\Sigma, m) \downarrow 0$, as required.

A weak regularity condition throughout our paper is that the kernels k_x and k_z are bounded. This condition is satisfied for kernels commonly used in practice. In the regularization regimes we consider, this in turn implies that the summands in the pre-Gaussian term satisfy $\|U_i\|_{\mathcal{H}_x} = \mathcal{O}(1/\sqrt{\lambda\mu})$ almost surely.³ Stronger assumptions could be imposed here, which we defer to future work.

Theorem 1 (Valid and sharp confidence sets). *For $\chi \in (0, 1)$, define \hat{t}_χ by $\mathbb{P}(\|\mathfrak{B}\| > \hat{t}_\chi | D) = \chi$. Suppose the data have low effective dimensions, i.e. $\nu_s(S_x) \asymp \omega_x s^{-1/(\rho_x-1)}$ and $\nu_s(S_z) \asymp \omega_z s^{-1/(\rho_z-1)}$. Suppose smoothness and strong instrument conditions hold, i.e., Assumptions 1, 2 and 3. Set (λ, μ) satisfying $\lambda \geq \mu$ and according to Table 2, e.g. $\lambda = \mu$.⁴ Suppose the effective dimensions are low enough and the smoothness is high enough according to Table 3. Then the \mathcal{H}_x confidence set in Algorithm 2 is $\mathcal{O}(1/n)$ -valid and $\{2/\log(n), \mathcal{O}(1/n)\}$ -sharp.*

Corollary 1 (Uniform confidence sets). *Under the assumptions of Theorem 1, the uniform confidence band in Algorithm 2 is $\mathcal{O}(1/n)$ -valid.*

3.3 Key intermediate results

Our main result, Theorem 1, ties together four intermediate results, which we present below: (i) a bias upper bound (Proposition 1), (ii) a Gaussian coupling (Theorem 2), (iii) a bootstrap coupling (Theorem 3), and (iv) a variance lower bound (Proposition 2). We summarize these four intermediate results in the leading case of polynomial decay for S_x and S_z (Table 1). To validate

³See Lemmas D.6 and D.7. We work with (λ, μ) regimes that guarantee $\|U_i\|_{\mathcal{H}_x} = \mathcal{O}(1/\sqrt{\lambda\mu})$, by imposing that λ and μ scale similarly. For example, when $\lambda = \mu^\iota$, we require that $\iota \in (0, 1]$ must be large enough.

⁴See Assumption D.1 for details.

Table 1: Intermediate results under polynomial decay of S_x and S_z

Bias upper bound: B	$n^{1/2}\lambda^\alpha$
Gaussian coupling: Q_\bullet	$\frac{1}{\lambda} \left(\frac{n\mu}{\lambda} \right)^{\frac{\rho_x-2}{2(3\rho_x-2)}}$
Gaussian coupling: Q_{res}	$\frac{\lambda^{\alpha-1}}{n\mu^{1+\rho_z}} + \frac{1}{n^{3/2}\lambda^{3/2}\mu^{\frac{1}{2}+\frac{3}{2}\rho_z}} + \frac{1}{n\lambda\mu^{1+\frac{1}{2}\rho_z}}$
Bootstrap coupling: R_\bullet	$\lambda^{-\rho_x/2}(n\mu)^{-\frac{\rho_x-2}{2(\rho_x-3)}}$
Bootstrap coupling: R_{res}	$\frac{\lambda^{\alpha-1}}{\mu^2 n^{1/2}} + \frac{1}{\lambda\mu^{3/2}n^{1/2}} + \frac{\lambda^{\alpha-3/2}}{n\mu^{2+\rho_z}} + \frac{1}{n^{3/2}\lambda^2\mu^{\frac{3}{2}+\frac{3}{2}\rho_z}} + \frac{1}{n\lambda^{3/2}\mu^{2+\frac{1}{2}\rho_z}}$
Variance lower bound: L	$\lambda^{-\rho_x/2}$

The first row gives the bias upper bound B . The second and third rows give the Gaussian coupling bound $Q=Q_\bullet+Q_{\text{res}}$. The fourth and fifth rows give the bootstrap coupling bound $R=R_\bullet+R_{\text{res}}$. The final row is the variance lower bound. Throughout, we suppress logarithmic factors and constants, and we impose $\lambda=\mu^\iota$ with $\iota\leq 1$.

inference, we must show that the errors arising from (i), (ii), and (iii) are dominated by (iv).

This leads to restrictions on the regularization parameters (λ,μ) (Table 2), and requirements that the effective dimension is low enough and smoothness is high enough (Table 3).

To begin, we characterize the bias of kernel instrumental variable regression under our smoothness assumptions. Under suitable regularity assumptions, our bias bound matches the well known bias bound of kernel ridge regression (Smale and Zhou, 2005; Caponnetto and De Vito, 2007; Fischer and Steinwart, 2020). We place two smoothness assumptions: a source condition ensuring that h_0 is smooth, and a link condition ensuring that the conditional expectation operator maps smooth functions to smooth functions. Both assumptions are in line with previous applications of RKHS methods to ill posed inverse problems (Nashed and Wahba, 1974; Singh et al., 2019; Meunier et al., 2024).

Proposition 1 (Bias upper bound). *Suppose that Assumptions 1 and 2 hold, and the regularization satisfies $\mu^\beta r < \lambda^{1/2}$. Then $n^{1/2}\|h_{\mu,\lambda} - h_0\|_{\mathcal{H}_x} \leq B = n^{1/2} \frac{\|h_{0,\lambda} - h_0\|_{\mathcal{H}_x}}{1 - C_\beta r \mu^\beta / \lambda^{1/2}} \leq n^{1/2} \frac{C_\alpha \lambda^\alpha \|T^{-\alpha} h_0\|_{\mathcal{H}_x}}{1 - C_\beta r \mu^\beta / \lambda^{1/2}}$.*

Under weak regularity conditions, the bias simplifies to $B = n^{1/2}\lambda^\alpha$, which is easy to interpret: for a smoother target function h_0 , the smoothness parameter α is larger, and the

bias vanishes more quickly. Overall, more smoothness translates into an easier estimation problem with better convergence rates. A simple and convenient regularity condition is that the conditional expectation operator is a bounded map from \mathcal{H}_x to \mathcal{H}_z , i.e. $\beta \geq \frac{1}{2}$. Under this convenient regularity condition, it suffices to place a mild restriction on the regularization: $\mu \leq r^{-2}\lambda$, which ensures that the denominator in Proposition 1 is bounded away from zero. In light of the adversarial formulation in (2), this mild restriction on regularization ensures that the adversary's strategy space is not too constrained, so that the adversary may adequately detect correlation between the instrument Z and the endogenous error ε . The mild restriction holds under natural choices of regularization: either $\lambda = \mu^\iota$ with $\iota \leq 1$, or $\mu = \lambda/C$ with $C > (C_\beta r)^{-2}$.

Equipped with this bias bound, we present two key results that underpin Theorem 1: a Gaussian coupling (Theorem 2), and a bootstrap coupling (Theorem 3).

Theorem 2 (Gaussian approximation). *Suppose the conditions of Proposition 1 hold. Next, suppose the data have low effective dimension, i.e. $\nu_s(S_x) \asymp \omega_x s^{-1/(\rho_x-1)}$ and $\nu_s(S_z) \asymp \omega_z s^{-1/(\rho_z-1)}$. Set the regularization (λ, μ) so that $\|U_i\|_{\mathcal{H}_x} \lesssim \frac{1}{\sqrt{\lambda\mu}}$.⁵ Finally, assume that n is sufficiently large.⁶ Then, there exists Gaussian $Z \in \mathcal{H}_x$, with covariance Σ , such that with probability $1 - \eta$, $\left\| n^{1/2}(\hat{h} - h_{\mu, \lambda}) - Z \right\|_{\mathcal{H}_x} \lesssim Q_\bullet \widetilde{M} \log(36/\eta) + Q_{\text{res}}$. In this compact notation, \widetilde{M} is an absolute constant, and the key quantities are*

$$Q_\bullet = \inf_{m \geq 1} \left\{ \frac{\sigma(T, m)}{\lambda} + \frac{m^2 \log(m^2)}{\sqrt{n\mu\lambda}} \right\}, \quad \mathbf{n}_z(\mu) = \text{tr}(S_z + \mu)^{-2} S_z,$$

$$Q_{\text{res}} = \left(\frac{l(\eta/6)^2 \mathbf{n}_z(\mu)}{n\lambda\mu} \|h_{\mu, \lambda} - h_0\|_{\mathcal{H}_x} + \frac{l(\eta/6)^3 \mathbf{n}_z(\mu)^{3/2}}{n^{3/2} \lambda^{3/2} \mu^{1/2}} + \frac{l(\eta/6)^2 \sqrt{\mathbf{n}_z(\mu)}}{n\lambda\mu} \right).$$

The quality of Gaussian approximation Z ultimately depends on a handful of quantities: spectral decay of T and S_z ; regularization parameters (λ, μ) ; and the bias $n^{1/2} \|h_{\mu, \lambda} - h_0\|_{\mathcal{H}_x}$.

⁵This mild condition requires $\lambda = \mu^{\iota/(2\beta)}$ for sufficiently large $\iota \in (0, 1]$, see e.g. Lemmas D.6 and D.7.

⁶See Assumption D.2 for a precise statement.

Complexity of the joint distribution of the covariate and the instrument is reflected by the local width $\sigma(T,m)$, and complexity of the marginal distribution of the instrument is reflected by $\mathbf{n}_z(\mu)$. If the effective dimension is higher, then these quantities are larger, and the quality of Gaussian approximation is worse. Gaussian approximation also degrades when the regularization parameters (λ,μ) are too small; intuitively, with less regularization, both the estimator and its Gaussian approximation are less stable. Finally, a large bias also complicates the Gaussian approximation.

Proposition 1 and Theorem 2 preview an important tension. Smaller (λ,μ) help estimation by limiting the bias, but possibly hurt inference by driving up complexity of the estimator. We use nonasymptotic analysis to thread this needle, achieving Gaussian approximation when (λ,μ) approach zero, even though no stable Gaussian limit exists. To sample from the sequence of approximating Gaussians Z , this paper proposes a new bootstrap procedure \mathfrak{B} . The following result proves the validity of the bootstrap.

Theorem 3 (Bootstrap approximation). *Suppose that the conditions of Theorem 2 hold. Further assume that n is sufficiently large.⁷ Then, there exists a Gaussian $Z' \in \mathcal{H}_x$ whose conditional distribution given U has covariance Σ , such that with probability $1-\eta$, we have $\mathbb{P}\left[\|Z' - \mathfrak{B}\| \lesssim \widetilde{M} \log(6/\eta)^{3/2} R_\bullet + R_{\text{res}} | U\right] \geq 1-\eta$. In this compact notation, \widetilde{M} is an absolute constant, and the key quantities are*

$$R_\bullet = \inf_{m \geq 1} \left[m^{1/4} \left\{ \frac{\tilde{\mathbf{m}}(\lambda,\mu)}{\mu \lambda n} + \frac{1}{n^2 \mu^2 \lambda^2} \right\}^{1/4} + \frac{\sigma(T,m)}{\lambda} \right], \quad \tilde{\mathbf{m}}(\lambda,\mu) = \text{tr} T_{\mu,\lambda}^{-1} S^* (S_z + \mu)^{-1} S_z (S_z + \mu)^{-1} S T_{\mu,\lambda}^{-1},$$

$$R_{\text{res}} = \sqrt{2l\left(\frac{2\eta}{15}\right)} \left[\left\{ \frac{l(\frac{\eta}{15})}{\mu^2 n^{1/2} \lambda} + \frac{l(\frac{\eta}{15})^2 \mathbf{n}_z(\mu)}{n \lambda^{3/2} \mu^2} \right\} \|h_{\mu,\lambda} - h_0\|_{\mathcal{H}_x} + \frac{l(\frac{\eta}{15})}{\lambda \mu^{3/2} n^{1/2}} + \frac{l(\frac{\eta}{15})^3 \mathbf{n}_z(\mu)^{3/2}}{n^{3/2} \lambda^2 \mu^{\frac{3}{2}}} + \frac{l(\frac{\eta}{15})^2 \sqrt{\mathbf{n}_z(\mu)}}{n \lambda^{3/2} \mu^2} \right].$$

Once again, a handful of quantities determine the quality of bootstrap approximation \mathfrak{B} :

⁷See Assumption H.1 for a precise statement.

spectral decay of T and S_z ; regularization parameters (λ, μ) ; and the bias $n^{1/2} \|h_{\mu, \lambda} - h_0\|_{\mathcal{H}_x}$.

Now, the spectral quantities are the local width $\sigma(T, m)$, the first stage effective dimension parameter $\mathbf{n}_z(\mu)$, and a new effective dimension parameter $\tilde{\mathbf{m}}(\lambda, \mu)$. As before, if the effective dimensions are higher, then these quantities are larger, and the quality of the bootstrap approximation degrades. As before, bootstrap approximation also degrades if (λ, μ) are too small, reinforcing the trade-off between estimation and inference.

The procedure $\mathfrak{B} = \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n \frac{1}{\sqrt{2}} (\hat{V}_i - \hat{V}_j) h_{ij}$ that we analyze in Theorem 3 is the empirical counterpart of an infeasible bootstrap $Z_{\mathfrak{B}} = \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n \left(\frac{V_i - V_j}{\sqrt{2}} \right) h_{ij}$. By briefly discussing $Z_{\mathfrak{B}}$, we clarify the role of anti-symmetry in our analysis. At a high level, \hat{V}_i approximates V_i , but V_i is biased for the desired U_i in the pre-Gaussian term: $V_i = U_i + \theta$, where $\mathbb{E}(U_i) = 0$ by construction and where $\theta \neq 0$ is a complex bias.⁸ Our key observation is that we can cancel out the bias θ by taking differences: $V_i - V_j = U_i - U_j$. This differencing is equivalent to generating the matrix of Gaussian multipliers $h \in \mathbb{R}^{n \times n}$, then constructing an anti-symmetric matrix of multipliers $h - h^\top$. Our technique expands a proposal of Freedman (1981), who addresses bias arising from non-orthogonality of errors in homoscedastic, fixed design regression.

The results in Proposition 1, Theorem 2, and Theorem 3 are all upper bounds that pertain to the fundamental decomposition

$$n^{1/2}(\hat{h} - h_0) = \underbrace{n^{1/2}\{(\hat{h} - h_{\mu, \lambda}) - \mathbb{E}_n(U)\}}_{\text{residual}} + \underbrace{n^{1/2}\mathbb{E}_n(U)}_{\text{pre-Gaussian}} + \underbrace{n^{1/2}(h_{\mu, \lambda} - h_0)}_{\text{bias}}.$$

Proposition 1 controls the nonrandom bias term. Theorem 2 constructs a Gaussian coupling Z for the pre-Gaussian term, and controls the residual. Theorem 3 allows us to sample from the approximating Gaussian Z via a feasible bootstrap \mathfrak{B} .

Our final result, which completes the paper, is a lower bound of the variance of the

⁸See Appendix F for details.

approximating Gaussian Z . Such a lower bound allows us to construct valid confidence sets. Specifically, we require a lower bound on the variance which is still larger than the upper bounds on the bias, Gaussian coupling error, and bootstrap coupling error.

Proposition 2 (Variance lower bound). *Let $Z \in \mathcal{H}_x$ be a Gaussian with covariance Σ . Suppose $\mathbb{E}(\varepsilon_i^2 | Z_i) \geq \underline{\sigma}^2$ almost surely. Finally, set (λ, μ) according to weak regularity conditions.⁹ Then, with probability $1 - \eta$, $\|Z\| \geq \sqrt{\frac{1}{4}\underline{\sigma}^2 \tilde{m}(\lambda, \mu)} - \left\{ 2 + \sqrt{2\ln(1/\eta)} \right\} \sqrt{\frac{2\underline{\sigma}^2}{\lambda}}$.*

Intuitively, for this lower bound to be meaningful, we require a strong instrument condition (Assumption 3). The strong instrument assumption prevents the effective dimension $\tilde{m}(\lambda, \mu)$ from collapsing as $\lambda, \mu \downarrow 0$. In other words, it imposes that a sufficiently rich set of directions in \mathcal{H}_x is well explained by \mathcal{H}_z . Viewed through this lens, our strong instrument assumption is an infinite-dimensional analogue of the familiar rank condition in classical instrumental variable analysis.

With our intermediate results in hand, as summarized by Table 1, we prove Theorem 1. First, we demonstrate that the bias is dominated by the variance, i.e. $B \ll L$. Then, we demonstrate that the Gaussian and bootstrap coupling errors are dominated by the variance, i.e. $Q \ll L$ and $R \ll L$ where $Q = Q_{\bullet} + Q_{\text{res}}$ and $R = R_{\bullet} + R_{\text{res}}$. For these orderings to hold, we require sensible restrictions on the regularization (λ, μ) , low enough effective dimension, and high enough smoothness, as summarized by Tables 2 and 3. Recall that as (λ, μ) vanish, B converges yet (Q, R) diverge. Our nonasymptotic analysis carefully navigates this tension between estimation and inference, in order to arrive at both uniform estimation and uniform inference for kernel instrumental variable regression.

⁹See Assumption D.1 for details.

4 Discussion: Inference for causal functions

Our main contribution is to develop an inference procedure for kernel instrumental regression. The procedure retains the simple closed-form structure of kernel estimators while delivering strong statistical guarantees, even with complex or nonstandard data types. By complementing flexible nonparametric estimation with reliable nonparametric inference, we aim to broaden the use of kernel methods for causal analysis, in the social and biomedical sciences. Looking ahead, our techniques may extend beyond instrumental variables to a wider class of causal functions (Singh et al., 2024, 2025).

References

- Ai, C. and Chen, X. (2003). Efficient estimation of models with conditional moment restrictions containing unknown functions. *Econometrica*, 71(6):1795–1843.
- Andrews, D. W. and Shi, X. (2013). Inference based on conditional moment inequalities. *Econometrica*, 81(2):609–666.
- Belloni, A., Chernozhukov, V., Chetverikov, D., and Kato, K. (2015). Some new asymptotic theory for least squares series: Pointwise and uniform results. *Journal of Econometrics*, 186(2):345–366.
- Bennett, A., Kallus, N., Mao, X., Newey, W. K., Syrgkanis, V., and Uehara, M. (2023a). Minimax instrumental variable regression and l_2 convergence guarantees without identification or closedness. In *Conference on Learning Theory*, pages 2291–2318. PMLR.
- Bennett, A., Kallus, N., Mao, X., Newey, W. K., Syrgkanis, V., and Uehara, M. (2023b). Source condition double robust inference on functionals of inverse problems. *arXiv:2307.13793*.
- Blundell, R., Chen, X., and Kristensen, D. (2007). Semi-nonparametric IV estimation of shape-invariant Engel curves. *Econometrica*, 75(6):1613–1669.
- Buzun, N., Shvetsov, N., and Dylov, D. V. (2022). Strong Gaussian approximation for the sum of random vectors. In *Conference on Learning Theory*, pages 1693–1715. PMLR.
- Caponnetto, A. and De Vito, E. (2007). Optimal rates for the regularized least-squares algorithm. *Foundations of Computational Mathematics*, 7:331–368.
- Carrasco, M., Florens, J.-P., and Renault, E. (2007). Linear inverse problems in structural econometrics estimation based on spectral decomposition and regularization. *Handbook of Econometrics*, 6:5633–5751.

- Chen, X. (2007). Large sample sieve estimation of semi-nonparametric models. *Handbook of Econometrics*, 6:5549–5632.
- Chen, X., Christensen, T., and Kankanala, S. (2024). Adaptive estimation and uniform confidence bands for nonparametric structural functions and elasticities. *Review of Economic Studies*, page rdae025.
- Chen, X. and Christensen, T. M. (2018). Optimal sup-norm rates and uniform inference on nonlinear functionals of nonparametric iv regression. *Quantitative Economics*, 9(1):39–84.
- Chen, X. and Reiss, M. (2007). On rate optimality for ill-posed inverse problems in econometrics. arXiv:0709.2003.
- Chernozhukov, V., Chetverikov, D., and Kato, K. (2014). Anti-concentration and honest, adaptive confidence bands. *The Annals of Statistics*, 42(5):1787–1818.
- Chernozhukov, V., Chetverikov, D., and Kato, K. (2016). Empirical and multiplier bootstraps for suprema of empirical processes of increasing complexity, and related Gaussian couplings. *Stochastic Processes and their Applications*, 126(12):3632–3651.
- Chernozhukov, V., Newey, W. K., and Singh, R. (2023). A simple and general debiased machine learning theorem with finite-sample guarantees. *Biometrika*, 110(1):257–264.
- Darolles, S., Fan, Y., Florens, J.-P., and Renault, E. (2011). Nonparametric instrumental regression. *Econometrica*, 79(5):1541–1565.
- De Vito, E., Rosasco, L., Caponnetto, A., De Giovannini, U., Odone, F., and Bartlett, P. (2005). Learning from examples as an inverse problem. *Journal of Machine Learning Research*, 6(5).
- Dikkala, N., Lewis, G., Mackey, L., and Syrgkanis, V. (2020). Minimax estimation of conditional moment models. *Advances in Neural Information Processing Systems*, 33:12248–12262.
- Fischer, S. and Steinwart, I. (2020). Sobolev norm learning rates for regularized least-squares algorithms. *The Journal of Machine Learning Research*, 21(1):8464–8501.
- Freedman, D. A. (1981). Bootstrapping regression models. *The Annals of Statistics*, 9(6):1218–1228.
- Ghassami, A., Ying, A., Shpitser, I., and Tchetgen, E. T. (2022). Minimax kernel machine learning for a class of doubly robust functionals with application to proximal causal inference. In *International Conference on Artificial Intelligence and Statistics*, pages 7210–7239. PMLR.
- Giné, E. and Nickl, R. (2021). *Mathematical foundations of infinite-dimensional statistical models*. Cambridge University Press.
- Götze, F., Naumov, A., Spokoiny, V., and Ulyanov, V. (2019). Large ball probabilities, Gaussian comparison and anti-concentration. *Bernoulli*, 25(4A):2538–2563.
- Hall, P. and Horowitz, J. L. (2005). Nonparametric methods for inference in the presence of instrumental variables. *The Annals of Statistics*, 33(6):2904–2929.

- Kallus, N., Mao, X., and Uehara, M. (2021). Causal inference under unmeasured confounding with negative controls: A minimax learning approach. *arXiv:2103.14029*.
- Kimeldorf, G. and Wahba, G. (1971). Some results on Tchebycheffian spline functions. *Journal of Mathematical Analysis and Applications*, 33(1):82–95.
- Liao, L., Chen, Y.-L., Yang, Z., Dai, B., Kolar, M., and Wang, Z. (2020). Provably efficient neural estimation of structural equation models: An adversarial approach. *Advances in Neural Information Processing Systems*, 33:8947–8958.
- Mendelson, S. and Neeman, J. (2010). Regularization in kernel learning. *The Annals of Statistics*, 38(1):526–565.
- Meunier, D., Li, Z., Christensen, T., and Gretton, A. (2024). Nonparametric instrumental regression via kernel methods is minimax optimal. *arXiv:2411.19653*.
- Nashed, M. Z. and Wahba, G. (1974). Regularization and approximation of linear operator equations in reproducing kernel spaces. *Bulletin of the American Mathematical Society*.
- Newey, W. K. and Powell, J. L. (2003). Instrumental variable estimation of nonparametric models. *Econometrica*, 71(5):1565–1578.
- Singh, R. (2020). Kernel methods for unobserved confounding: Negative controls, proxies, and instruments. *arXiv:2012.10315*.
- Singh, R., Sahani, M., and Gretton, A. (2019). Kernel instrumental variable regression. In *Advances in Neural Information Processing Systems*, pages 4593–4605.
- Singh, R. and Vijaykumar, S. (2023). Kernel ridge regression inference. *arXiv:2302.06578*.
- Singh, R., Xu, L., and Gretton, A. (2024). Kernel methods for causal functions: Dose, heterogeneous, and incremental response curves. *Biometrika*, 111(2):497–516.
- Singh, R., Xu, L., and Gretton, A. (2025). Sequential kernel embedding for mediated and time-varying dose response curves. *Bernoulli*, 31(4):3013–3033.
- Smale, S. and Zhou, D.-X. (2005). Shannon sampling II: Connections to learning theory. *Applied and Computational Harmonic Analysis*, 19(3):285–302.
- Smale, S. and Zhou, D.-X. (2007). Learning theory estimates via integral operators and their approximations. *Constructive Approximation*, 26(2):153–172.
- Steinwart, I. and Christmann, A. (2008). *Support vector machines*. Springer Science & Business Media.
- Zaitsev, A. Y. (1987). Estimates of the Lévy–Prokhorov distance in the multivariate central limit theorem for random variables with finite exponential moments. *Theory of Probability and Its Applications*, 31(2):203–220.

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A Algorithm derivation

We derive the closed form expressions of $\hat{h}(x)$ and $\mathfrak{B}(x)$, justifying Algorithms 1 and 2.

A.1 Equivalent objectives

Lemma A.1 (Dual characterization of the regularized KIV estimator). *The solution to the adversarial objective is equivalent to the solution of the regression objective*

$$h_{\mu,\lambda} = \underset{h}{\operatorname{argmin}} \| (S_z + \mu)^{-1/2} S(h_0 - h) \|_{\mathcal{H}_z}^2 + \lambda \| h \|_{\mathcal{H}_x}^2. \quad (3)$$

Proof. Observe that the adversary f_h is the solution to

$$\begin{aligned} f_h &= \underset{f}{\operatorname{argmax}} 2\mathbb{E}[\{Y - h(X)\}f(Z)] - \|f\|_2^2 - \mu \|f\|_{\mathcal{H}_z}^2 \\ &= \underset{f}{\operatorname{argmax}} 2\mathbb{E}[\{Y - h(X)\}\langle f, \phi(Z) \rangle_{\mathcal{H}_z}] - \langle f, (S_z + \mu)f \rangle_{\mathcal{H}_z}, \end{aligned}$$

where we used the definition of S_z and L^2 -norm. Boundedness of the feature map and its measurability imply Bochner integrability of the feature map (Steinwart and Christmann, 2008), which allows us to write the L^2 -norm as

$$\|f\|_2^2 = \mathbb{E}[f(Z)^2] = \mathbb{E}[\langle f, \phi(Z) \rangle_{\mathcal{H}_z}^2] = \mathbb{E}[\langle f, \phi(Z) \rangle_{\mathcal{H}_z} \langle f, \phi(Z) \rangle_{\mathcal{H}_z}] = \langle f, \mathbb{E}[\langle f, \phi(Z) \rangle_{\mathcal{H}_z} \phi(Z)] \rangle_{\mathcal{H}_z} = \langle f, S_z f \rangle_{\mathcal{H}_z}.$$

The first order condition implies that f_h has to satisfy

$$\begin{aligned} 2\mathbb{E}[\{Y-h(X)\}\phi(Z)]-2(S_z+\mu)f_h=0 &\iff (S_z+\mu)f_h=\mathbb{E}[\{Y-h(X)\}\phi(Z)]=S(h_0-h) \\ &\implies f_h=(S_z+\mu)^{-1}S(h_0-h). \end{aligned}$$

Here, we used that

$$\begin{aligned} \mathbb{E}[\{Y-h(X)\}\phi(Z)] &= \mathbb{E}[\{h_0(X)+\varepsilon-h(X)\}\phi(Z)] \\ &= \mathbb{E}[\{h_0(X)-h(X)\}\phi(Z)] \\ &= \mathbb{E}[\phi(Z)\langle\psi(X),h_0-h\rangle] \\ &= \mathbb{E}[\phi(Z)\otimes\psi(X)^*\{h_0-h\}]=S(h_0-h). \end{aligned}$$

Proceeding in similar fashion and plugging in the first order condition $(S_z+\mu)f_h=S(h_0-h)$, we can rewrite the adversarial objective as

$$\begin{aligned} &2\mathbb{E}[\{Y-h(X)\}f_h(Z)]-\|f_h\|_2^2-\mu\|f_h\|^2+\lambda\|h\|^2 \\ &= 2\langle S(h_0-h),f_h \rangle_{\mathcal{H}_z}-\langle f_h,(S_z+\mu)f_h \rangle_{\mathcal{H}_z}+\lambda\langle h,h \rangle_{\mathcal{H}_x} \\ &= 2\langle S(h_0-h),f_h \rangle_{\mathcal{H}_z}-\langle f_h,S(h_0-h) \rangle_{\mathcal{H}_z}+\lambda\langle h,h \rangle_{\mathcal{H}_x} \\ &= \langle S(h_0-h),f_h \rangle_{\mathcal{H}_z}+\lambda\langle h,h \rangle_{\mathcal{H}_x} \\ &= \langle S(h_0-h),(S_z+\mu)^{-1}S(h_0-h) \rangle_{\mathcal{H}_z}+\lambda\langle h,h \rangle_{\mathcal{H}_x} \\ &= \langle (S_z+\mu)^{-1/2}S(h_0-h),(S_z+\mu)^{-1/2}S(h_0-h) \rangle_{\mathcal{H}_z}+\lambda\langle h,h \rangle_{\mathcal{H}_x} \\ &= \|(S_z+\mu)^{-1/2}S(h_0-h)\|_{\mathcal{H}_z}^2+\lambda\|h\|_{\mathcal{H}_x}^2. \quad \square \end{aligned}$$

A.2 Closed form estimation

Consider the vector notation

$$\Psi_X:=\begin{bmatrix} \psi(X_1)^\top \\ \vdots \\ \psi(X_n)^\top \end{bmatrix}, \quad \Phi_Z:=\begin{bmatrix} \phi(Z_1)^\top \\ \vdots \\ \phi(Z_n)^\top \end{bmatrix}$$

and Gram matrices,

$$K_{XX} := \Psi_X \Psi_X^\top \in \mathbb{R}^{n \times n}, \quad (K_{XX})_{ij} = \psi(X_i)^\top \psi(X_j), \quad K_{xX} := \psi(x)^\top \Psi_X^\top,$$

$$K_{ZZ} := \Phi_Z \Phi_Z^\top \in \mathbb{R}^{n \times n}, \quad (K_{ZZ})_{ij} = \phi(Z_i)^\top \phi(Z_j).$$

Lemma A.2 (Point estimate). *Given $\lambda, \mu > 0$ the point estimator to (3) is given by*

$$\hat{h}(x) = K_{xX} \{ K_{ZZ} (K_{ZZ} + n\mu)^{-1} K_{XX} + n\lambda \}^{-1} K_{ZZ} (K_{ZZ} + n\mu)^{-1} Y.$$

Observe that for $K_{ZZ} (K_{ZZ} + n\mu)^{-1} = I$, this becomes the KRR solution (Kimeldorf and Wahba, 1971).

Proof. Recall that $\mathbb{E}[\{Y - h(X)\}\phi(Z)] = S(h_0 - h)$. With $\varepsilon_h = Y - h(X)$, the empirical objective of (3) becomes

$$\|(\hat{S}_z + \mu)^{-1/2} \mathbb{E}_n \{\varepsilon_h \phi(Z)\}\|_{\mathcal{H}_z}^2 + \lambda \|h\|_{\mathcal{H}_x}^2.$$

With $\mathbb{E}_n \{\varepsilon_h \phi(Z)\} = n^{-1} \Phi_Z^* \varepsilon_h$, the objective becomes

$$\begin{aligned} & \langle (\hat{S}_z + \mu)^{-1/2} n^{-1} \Phi_Z^* \varepsilon_h, (\hat{S}_z + \mu)^{-1/2} n^{-1} \Phi_Z^* \varepsilon_h \rangle_{\mathcal{H}_z} + \lambda \langle h, h \rangle_{\mathcal{H}_x} \\ &= n^{-2} \varepsilon_h^\top \Phi_Z (\hat{S}_z + \mu)^{-1} \Phi_Z^* \varepsilon_h + \lambda \langle h, h \rangle_{\mathcal{H}_x} \\ &= n^{-1} \varepsilon_h^\top \Phi_Z (\Phi_Z^* \Phi_Z + n\mu)^{-1} \Phi_Z^* \varepsilon_h + \lambda \langle h, h \rangle_{\mathcal{H}_x} \\ &= n^{-1} \varepsilon_h^\top K_{ZZ} (K_{ZZ} + n\mu)^{-1} \varepsilon_h + \lambda \langle h, h \rangle_{\mathcal{H}_x} \\ &= n^{-1} \varepsilon_h^\top K \varepsilon_h + \lambda \langle h, h \rangle_{\mathcal{H}_x}. \end{aligned}$$

By the representer theorem, we can write

$$\varepsilon_h = Y - \Psi_X h = Y - \Psi_X \Psi_X^* \alpha = Y - K_{XX} \alpha,$$

giving us $h = \Psi_X^* \alpha$. Thus, the objective becomes

$$\frac{1}{n} (Y - K_{XX} \alpha)^\top K (Y - K_{XX} \alpha) + \lambda \alpha^\top K_{XX} \alpha.$$

Setting the derivative with respect to α to zero yields the condition

$$0 = -\frac{2}{n} K_{XX} K (Y - K_{XX} \hat{\alpha}) + 2\lambda K_{XX} \hat{\alpha} = \frac{2}{n} K_{XX} \{-K(Y - K_{XX} \hat{\alpha}) + n\lambda \hat{\alpha}\}.$$

Setting the inner expression equal to zero gives

$$K(Y - K_{XX}\hat{\alpha}) = n\lambda\hat{\alpha} \iff KY = (KK_{XX} + n\lambda)\hat{\alpha} \iff \hat{\alpha} = (KK_{XX} + n\lambda)^{-1}KY.$$

Finally, we can write

$$\begin{aligned}\hat{h}(x) &= \psi(x)^*\Psi_X^*\hat{\alpha} \\ &= K_{xX}(KK_{XX} + n\lambda)^{-1}KY \\ &= K_{xX}\{K_{ZZ}(K_{ZZ} + n\mu)^{-1}K_{XX} + n\lambda\}^{-1}K_{ZZ}(K_{ZZ} + n\mu)^{-1}Y. \quad \square\end{aligned}$$

A.3 Closed form bootstrap

Proposition A.1 (Kernel bootstrap). *Evaluated at any point $x \in \mathcal{X}$, the bootstrap process admits the following finite-sample form*

$$\mathfrak{B}(x) = K_{xX}(KK_{XX} + n\lambda)^{-1}(2K - K^2)\beta,$$

with

$$\beta = \text{diag}(\hat{\varepsilon}) \frac{1}{\sqrt{2}}(h - h^\top)1, \quad K = K_{ZZ}(K_{ZZ} + n\mu)^{-1},$$

where $1 \in \mathbb{R}^n$ is a vector of ones and h is a matrix of i.i.d standard Gaussians h_{ij} .

Proof. Consider the decomposition

$$\begin{aligned}\frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n \left(\frac{\hat{V}_i - \hat{V}_j}{\sqrt{2}} \right) h_{ij} &= \underbrace{\frac{1}{n\sqrt{2}} \sum_{i=1}^n \sum_{j=1}^n \left(\hat{T}_{\mu,\lambda}^{-1} \hat{S}^*(\hat{S}_z + \mu)^{-1} \{ \phi(Z_i)\hat{\epsilon}_i - \phi(Z_j)\hat{\epsilon}_j \} \right) h_{ij}}_A \\ &\quad + \underbrace{\frac{1}{n\sqrt{2}} \sum_{i=1}^n \sum_{j=1}^n \left(\hat{T}_{\mu,\lambda}^{-1} \{ S_i^* - S_j^* \} (\hat{S}_z + \mu)^{-1} \mathbb{E}_n[\hat{\varepsilon}\phi(Z)] \right) h_{ij}}_B \\ &\quad - \underbrace{\frac{1}{n\sqrt{2}} \sum_{i=1}^n \sum_{j=1}^n \left(\hat{T}_{\mu,\lambda}^{-1} \hat{S}^*(\hat{S}_z + \mu)^{-1} (S_{z,i} - S_{z,j})(\hat{S}_z + \mu)^{-1} \mathbb{E}_n[\hat{\varepsilon}\phi(Z)] \right) h_{ij}}_C.\end{aligned}$$

We can utilize sum manipulation for all three expressions since generally

$$\begin{aligned}
\sum_{i=1}^n \sum_{j=1}^n (A_i - A_j) h_{ij} &= \sum_{i=1}^n \sum_{j=1}^n A_i h_{ij} - \sum_{i=1}^n \sum_{j=1}^n A_j h_{ij} \\
&= \sum_{i=1}^n A_i \sum_{j=1}^n h_{ij} - \sum_{j=1}^n A_j \sum_{i=1}^n h_{ij} \\
&= \sum_{i=1}^n A_i \sum_{j=1}^n h_{ij} - \sum_{i=1}^n A_i \sum_{j=1}^n h_{ji} \\
&= \sum_{i=1}^n A_i \sum_{j=1}^n (h_{ij} - h_{ji}).
\end{aligned}$$

Now, for A this means that

$$\frac{1}{\sqrt{2}} \sum_{i=1}^n \sum_{j=1}^n \{\phi(Z_i) \hat{\epsilon}_i - \phi(Z_j) \hat{\epsilon}_j\} h_{ij} = \sum_{i=1}^n \phi(Z_i) \hat{\epsilon}_i \sum_{j=1}^n \frac{(h_{ij} - h_{ji})}{\sqrt{2}} = \Phi_Z^* \beta.$$

Substituting this into A ,

$$\begin{aligned}
A &= \{\hat{S}^*(\hat{S}_z + \mu)^{-1} \hat{S} + \lambda\}^{-1} \hat{S}^*(\hat{S}_z + \mu)^{-1} \left(\frac{1}{n} \Phi_Z^* \beta \right) \\
&= \left\{ \frac{1}{n} \Psi_X^* \Phi_Z \left(\frac{1}{n} \Phi_Z^* \Phi_Z + \mu \right)^{-1} \frac{1}{n} \Phi_Z^* \Psi_X + \lambda \right\}^{-1} \frac{1}{n} \Psi_X^* \Phi_Z \left(\frac{1}{n} \Phi_Z^* \Phi_Z + \mu \right)^{-1} \left(\frac{1}{n} \Phi_Z^* \beta \right) \\
&= \{\Psi_X^* \Phi_Z (\Phi_Z^* \Phi_Z + n\mu)^{-1} \Phi_Z^* \Psi_X + n\lambda\}^{-1} \Psi_X^* \Phi_Z (\Phi_Z^* \Phi_Z + n\mu)^{-1} \Phi_Z^* \beta \\
&= (\Psi_X^* K \Psi_X + n\lambda)^{-1} \Psi_X^* K \beta \\
&= \Psi_X^* (K \Psi_X \Psi_X^* + n\lambda)^{-1} K \beta \\
&= \Psi_X^* (K K_{XX} + n\lambda)^{-1} K \beta.
\end{aligned}$$

For B , we have

$$\begin{aligned}
& \frac{1}{n\sqrt{2}} \sum_{i=1}^n \sum_{j=1}^n \left(\hat{T}_{\mu,\lambda}^{-1} \{S_i^* - S_j^*\} (\hat{S}_z + \mu)^{-1} \mathbb{E}_n[\hat{\varepsilon}\phi(Z)] \right) h_{ij} \\
&= \frac{1}{n} \sum_{i=1}^n \hat{T}_{\mu,\lambda}^{-1} S_i^* (\hat{S}_z + \mu)^{-1} \mathbb{E}_n[\hat{\varepsilon}\phi(Z)] \sum_{j=1}^n \left(\frac{h_{ij} - h_{ji}}{\sqrt{2}} \right) \\
&= \frac{1}{n^2} \sum_{i=1}^n \hat{T}_{\mu,\lambda}^{-1} S_i^* (\hat{S}_z + \mu)^{-1} \Phi_Z^* \beta \\
&= (\Psi_X^* K \Psi_X + n\lambda)^{-1} \frac{1}{n} \Psi_X^* \Phi_Z \left(\frac{1}{n} \Phi_Z^* \Phi_Z + \mu \right)^{-1} \Phi_Z^* \beta \\
&= (\Psi_X^* K \Psi_X + n\lambda)^{-1} \Psi_X^* \Phi_Z (\Phi_Z^* \Phi_Z + n\mu)^{-1} \Phi_Z^* \beta \\
&= \Psi_X^* (K K_{XX} + n\lambda)^{-1} K \beta.
\end{aligned}$$

Lastly, for C , we have

$$\begin{aligned}
& \frac{1}{n\sqrt{2}} \sum_{i=1}^n \sum_{j=1}^n \left(\hat{T}_{\mu,\lambda}^{-1} \hat{S}^* (\hat{S}_z + \mu)^{-1} (S_{z,i} - S_{z,j}) (\hat{S}_z + \mu)^{-1} \mathbb{E}_n[\hat{\varepsilon}\phi(Z)] \right) h_{ij} \\
&= \frac{1}{n} \sum_{i=1}^n \hat{T}_{\mu,\lambda}^{-1} \hat{S}^* (\hat{S}_z + \mu)^{-1} S_{z,i} (\hat{S}_z + \mu)^{-1} \mathbb{E}_n[\hat{\varepsilon}\phi(Z)] \sum_{j=1}^n \left(\frac{h_{ij} - h_{ji}}{\sqrt{2}} \right) \\
&= \frac{1}{n^2} \sum_{i=1}^n \hat{T}_{\mu,\lambda}^{-1} \hat{S}^* (\hat{S}_z + \mu)^{-1} S_{z,i} (\hat{S}_z + \mu)^{-1} \Phi_Z^* \beta \\
&= (\Psi_X^* K \Psi_X + n\lambda)^{-1} \hat{S}^* (\hat{S}_z + \mu)^{-1} \hat{S}_z (\hat{S}_z + \mu)^{-1} \Phi_Z^* \beta \\
&= (\Psi_X^* K \Psi_X + n\lambda)^{-1} \frac{1}{n} \Psi_X^* \Phi_Z \left(\frac{1}{n} \Phi_Z^* \Phi_Z + \mu \right)^{-1} \frac{1}{n} \Phi_Z^* \Phi_Z \left(\frac{1}{n} \Phi_Z^* \Phi_Z + \mu \right)^{-1} \Phi_Z^* \beta \\
&= (\Psi_X^* K \Psi_X + n\lambda)^{-1} \hat{S}^* (\hat{S}_z + \mu)^{-1} \hat{S}_z (\hat{S}_z + \mu)^{-1} \Phi_Z^* \beta \\
&= (\Psi_X^* K \Psi_X + n\lambda)^{-1} \Psi_X^* \Phi_Z (\Phi_Z^* \Phi_Z + n\mu)^{-1} \Phi_Z^* \Phi_Z (\Phi_Z^* \Phi_Z + n\mu)^{-1} \Phi_Z^* \beta \\
&= \Psi_X^* (K K_{XX} + n\lambda)^{-1} K^2 \beta.
\end{aligned}$$

To conclude,

$$\begin{aligned}
\mathfrak{B} &= 2\Psi_X^* (K K_{XX} + n\lambda)^{-1} K \beta - \Psi_X^* (K K_{XX} + n\lambda)^{-1} K^2 \beta, \\
&= \Psi_X^* (K K_{XX} + n\lambda)^{-1} (2K - K^2) \beta
\end{aligned}$$

and

$$\mathfrak{B}(x) = K_{xx} (KK_{XX} + n\lambda)^{-1} (2K - K^2)\beta.$$

One can see that for $K = I$ this exactly equals the closed-form expression of Singh and Vijaykumar (2023). \square

B Technical lemmas

B.1 Analysis

Lemma B.1 (Higher-order resolvent, cf. Singh and Vijaykumar, 2023, Lemma E.10). *Let V be a vector space and $A, B : V \rightarrow V$ be invertible linear operators. Then, for all $\ell \geq 1$, it holds*

$$A^{-1} - B^{-1} = A^{-1} \{(B - A)B^{-1}\}^\ell + \sum_{r=1}^{\ell-1} B^{-1} \{(B - A)B^{-1}\}^r.$$

When $\ell = 1$, this reduces to the familiar “resolvent identity”

$$A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1} \iff A^{-1} = A^{-1}(B - A)B^{-1} + B^{-1}.$$

Lemma B.2 (Non-commutative product rule). *Let define $\Delta A = \hat{A} - A$, $\Delta B = \hat{B} - B$, and $\Delta C = \hat{C} - C$. Then,*

$$\hat{A}\hat{B}\hat{C} = ABC + AB\Delta C + A\Delta BC + A\Delta B\Delta C + \Delta ABC + \Delta AB\Delta C + \Delta A\Delta BC + \Delta A\Delta B\Delta C.$$

Proof.

$$\begin{aligned} \hat{A}\hat{B}\hat{C} &= (A + \Delta A)(B + \Delta B)(C + \Delta C) \\ &= ABC + AB\Delta C + A\Delta BC + A\Delta B\Delta C \\ &\quad + \Delta ABC + \Delta AB\Delta C + \Delta A\Delta BC + \Delta A\Delta B\Delta C. \end{aligned} \quad \square$$

Lemma B.3 (Polar decomposition bound (c.f. De Vito et al., 2005)). *Let A be a bounded*

linear operator, then denote $|A| = \sqrt{A^* A}$ then,

$$\|A(A^* A + \lambda)^{-1}\|_{op} = \||A|(|A|^2 + \lambda)^{-1}\|_{op} = \sup_{t \in \text{spec}(\|A\|)} \frac{t}{t^2 + \lambda} = \frac{1}{2\lambda^{1/2}}.$$

The same holds for $\|(A^* A + \lambda)^{-1} A^*\|_{op}$.

Proof. The first equality comes from writing A with its polar decomposition $A = U|A|$. The second equality comes from

$$\|(A^* A + \lambda I)^{-1} A^*\|_{op} = \|(A(A^* A + \lambda I)^{-1})^*\|_{op} = \|A(A^* A + \lambda I)^{-1}\|_{op}. \quad \square$$

Lemma B.4 (Generalized parallelogram law). *Let A, B be bounded linear operators on a Hilbert space. The mixed term is controlled by the “squares,” i.e. $(A+B)(A+B)^* \leq 2AA^* + 2BB^*$.*

Proof. It suffices to show that

$$\begin{aligned} AA^* + AB^* + BA^* + BB^* &\leq 2AA^* + 2BB^* \\ \iff AB^* + BA^* &\leq AA^* + BB^* \\ \iff 0 &\leq (A-B)(A-B)^*. \quad \square \end{aligned}$$

Lemma B.5 (Young’s inequality for scalars). *Let $\bar{a}, \bar{b} \geq 0$ and $t > 0$. Then,*

$$2\bar{a}\bar{b} \leq \frac{\bar{a}^2}{t} + t\bar{b}^2.$$

In particular, if $\bar{b}^2 \leq C\bar{b}^2$ and $\bar{a}^2 \leq \frac{1}{8C^2}\bar{b}^2$, then, $(a+b)^2 \geq \frac{1}{4C}\bar{b}^2$.

Proof. For the first claim, observe that $(\sqrt{t}\bar{b} - \bar{a}/\sqrt{t})^2 \geq 0$, i.e.

$$t\bar{b}^2 + \frac{\bar{a}^2}{t} - 2\bar{a}\bar{b} \geq 0 \Rightarrow 2\bar{a}\bar{b} \leq \frac{\bar{a}^2}{t} + t\bar{b}^2.$$

For the “in particular” part, start with

$$(a+b)^2 = a^2 + b^2 + 2ab \geq a^2 + b^2 - 2|a||b| \geq \bar{b}^2 - 2\bar{a}\bar{b},$$

where we used $|a| \leq \bar{a}$ and $|b| \geq \underline{b}$ (and $a^2 \geq 0$). By Young's inequality (applied to \bar{a}, \bar{b}),

$$\underline{b}^2 - 2\bar{a}\bar{b} \geq \underline{b}^2 - \frac{\bar{a}^2}{t} - t\bar{b}^2.$$

Using $\underline{b}^2 \geq \bar{b}^2/C$,

$$\underline{b}^2 - \frac{\bar{a}^2}{t} - t\bar{b}^2 \geq \left(\frac{1}{C} - t\right)\bar{b}^2 - \frac{\bar{a}^2}{t}.$$

Choose $t = \frac{1}{2C}$ to get

$$\left(\frac{1}{C} - t\right)\bar{b}^2 - \frac{\bar{a}^2}{t} = \frac{1}{2C}\bar{b}^2 - 2C\bar{a}^2.$$

Finally, $\bar{a}^2 \leq \frac{1}{8C^2}\bar{b}^2$ implies $2C\bar{a}^2 \leq \frac{1}{4C}\bar{b}^2$, hence

$$(a+b)^2 \geq \frac{1}{2C}\bar{b}^2 - \frac{1}{4C}\bar{b}^2 = \frac{1}{4C}\bar{b}^2. \quad \square$$

Lemma B.6 (Young's inequality for operators). *Let A, B be bounded linear operators on a Hilbert space. Suppose the following conditions hold:*

$$1. \ 0 = \underline{\Sigma}_A \preceq AA^* \preceq \bar{\Sigma}_A;$$

$$2. \ \underline{\Sigma}_B \preceq BB^* \preceq \bar{\Sigma}_B \preceq C\underline{\Sigma}_B;$$

$$3. \ \bar{\Sigma}_A \preceq \frac{1}{8C^2}\bar{\Sigma}_B.$$

Then, $(A+B)(A+B)^* \succeq \frac{1}{4C}\bar{\Sigma}_B$.

Proof. Fix x in the Hilbert space. Then,

$$\begin{aligned} \|(A+B)^*x\|^2 &= \|A^*x\|^2 + \|B^*x\|^2 + 2\langle A^*x, B^*x \rangle \\ &\geq \|A^*x\|^2 + \|B^*x\|^2 - 2\|A^*x\|\|B^*x\| \\ &\geq \|B^*x\|^2 - 2\|A^*x\|\|B^*x\|. \end{aligned}$$

Define the scalars

$$\bar{a}^2 := \langle \bar{\Sigma}_A x, x \rangle, \quad \underline{b}^2 := \langle \underline{\Sigma}_B x, x \rangle, \quad \bar{b}^2 := \langle \bar{\Sigma}_B x, x \rangle.$$

By assumptions (1) and (2),

$$\|A^*x\|^2 = \langle AA^*x, x \rangle \leq \bar{a}^2, \quad \underline{b}^2 \leq \|B^*x\|^2 = \langle BB^*x, x \rangle \leq \bar{b}^2.$$

Hence

$$\|(A+B)^*x\|^2 \geq b^2 - 2\bar{a}\bar{b}.$$

Assumption (2) gives $\bar{b}^2 \leq C\underline{b}^2$, and (3) gives $\bar{a}^2 \leq \frac{1}{8C^2}\bar{b}^2$. Thus Lemma B.5 applies to $\bar{a}, \underline{b}, \bar{b}$ and yields

$$\|(A+B)^*x\|^2 \geq \frac{1}{4C}\bar{b}^2 = \frac{1}{4C}\langle \bar{\Sigma}_B x, x \rangle.$$

Equivalently,

$$\langle [(A+B)(A+B)^* - \frac{1}{4C}\bar{\Sigma}_B]x, x \rangle \geq 0 \quad \text{for all } x,$$

which is precisely $(A+B)(A+B)^* \succeq \frac{1}{4C}\bar{\Sigma}_B$. \square

Lemma B.7 (Pairwise centered identity with a linear map). *Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $a_1, \dots, a_n \in \mathcal{H}$, and let $L : \mathcal{H} \rightarrow \mathcal{H}$ be a bounded linear operator. Define the sample mean $\bar{a} := \frac{1}{n} \sum_{k=1}^n a_k$ and the centered vectors $\tilde{a}_i := a_i - \bar{a}$. Then,*

$$\frac{1}{n^2} \sum_{i,j=1}^n \left\| L \frac{a_i - a_j}{\sqrt{2}} \right\|^2 = \frac{1}{n} \sum_{i=1}^n \|L\tilde{a}_i\|^2.$$

Proof. Set $x_i := La_i \in \mathcal{H}$ and let $\bar{x} := \frac{1}{n} \sum_i x_i = L\bar{a}$ (by linearity of L). We use the standard identity

$$\sum_{i,j=1}^n \|x_i - x_j\|^2 = 2n \sum_{i=1}^n \|x_i - \bar{x}\|^2. \tag{4}$$

This holds because

$$\begin{aligned} \sum_{i,j} \|x_i - x_j\|^2 &= \sum_{i,j} (\|x_i\|^2 + \|x_j\|^2 - 2\langle x_i, x_j \rangle) = 2n \sum_i \|x_i\|^2 - 2 \left\| \sum_i x_i \right\|^2, \\ \sum_i \|x_i - \bar{x}\|^2 &= \sum_i \|x_i\|^2 - n\|\bar{x}\|^2 = \sum_i \|x_i\|^2 - \frac{1}{n} \left\| \sum_i x_i \right\|^2 \end{aligned}$$

and multiplying the second line by $2n$ yields (4). Now plug $x_i = La_i$ into (4) and divide by

$2n^2$, yielding

$$\frac{1}{2n^2} \sum_{i,j} \|La_i - La_j\|^2 = \frac{1}{n} \sum_i \|La_i - \overline{La}\|^2 = \frac{1}{n} \sum_i \|L(a_i - \bar{a})\|^2 = \frac{1}{n} \sum_i \|L\tilde{a}_i\|^2.$$

Finally, note that

$$\frac{1}{n^2} \sum_{i,j} \left\| L \frac{a_i - a_j}{\sqrt{2}} \right\|^2 = \frac{1}{2n^2} \sum_{i,j} \|La_i - La_j\|^2,$$

which matches the left-hand side above. \square

B.2 Probability Bounds

Lemma B.8 (Bernstein inequality (Proposition 2 of Caponnetto and De Vito, 2007)). *Suppose that ξ_i are i.i.d. random elements of a Hilbert space, which satisfy, for all $\ell \geq 2$*

$$\mathbb{E}\|\xi_i - \mathbb{E}(\xi_i)\|^\ell \leq \frac{1}{2} \ell! B^2 \left(\frac{A}{2} \right)^{\ell-2}.$$

Then for any $0 < \eta < 1$ it holds with probability at least $1 - \eta$ that

$$\left\| \frac{1}{n} \sum_{i=1}^n \xi_i - \mathbb{E}(\xi_i) \right\| \leq 2 \left\{ \sqrt{\frac{B^2 \log(2/\eta)}{n}} + \frac{A \log(2/\eta)}{n} \right\} \leq 2 \log(2/\eta) \left(\frac{A}{n} + \sqrt{\frac{B^2}{n}} \right).$$

In particular, this holds if $\mathbb{E}(\|\xi_i\|^2) \leq B^2$ and $\|\xi_i\| \leq A/2$ almost surely.

Lemma B.9 (Borell's inequality, Theorem 2.5.8 of Giné and Nickl, 2021). *Let G_t be a centered Gaussian process, a.s. bounded on T . Then for $u > 0$,*

$$\mathbb{P}\left(\sup_{t \in T} G_t - \mathbb{E} \sup_{t \in T} G_t > u \right) \vee \mathbb{P}\left(\sup_{t \in T} G_t - \mathbb{E} \sup_{t \in T} G_t < -u \right) \leq \exp\left(-\frac{u^2}{2\sigma_T^2}\right),$$

where $\sigma_T^2 = \sup_{t \in T} \mathbb{E} G_t^2$.

Lemma B.10 (Gaussian norm bound, cf. Singh and Vijaykumar, 2023). *Let Z be a Gaussian random element in a Hilbert space H such that $\mathbb{E}\|Z\|^2 < \infty$. Then, with probability $1 - \eta$,*

$$\|Z\| \leq \left\{ 1 + \sqrt{2 \log(1/\eta)} \right\} \sqrt{\mathbb{E}\|Z\|^2}.$$

In particular, if $A: H \rightarrow H$ is a trace-class operator, then, with probability $1 - \eta$ with respect to g

$$\|Ag\| \leq \left\{ 1 + \sqrt{2\log(1/\eta)} \right\} \|A\|_{\text{HS}}.$$

C Bias upper bound

To lighten notation, let $h_\lambda = h_{0,\lambda}$ and $T_\lambda = T_{0,\lambda}$.

C.1 Regression bias

Lemma C.1 (Regression bias bound). *Suppose Assumption 1 holds. Then,*

$$\|h_\lambda - h_0\| \leq \lambda^\alpha (\kappa_x^2)^{-\alpha} \|h_0\|.$$

Proof. First, we show that

$$\|h_\lambda - h_0\|^2 \leq \lambda^{2\alpha} (\kappa_x^2)^{-2\alpha} \|h_0\|^2.$$

Let (ν_j, e_j) be the eigendecomposition of $S^* S_z^{-1} S$ in the descending order of the eigenvalues.

Then,

$$w_0 = (S^* S_z^{-1} S)^{-\alpha} h_0 = \sum_j \nu_j^{-\alpha} e_j \langle e_j, h_0 \rangle.$$

Hence,

$$\|w_0\|^2 = \sum_j \nu_j^{-2\alpha} \langle e_j, h_0 \rangle^2 \leq \nu_1^{-2\alpha} \sum_j \langle e_j, h_0 \rangle^2 = \nu_1^{-2\alpha} \|h_0\|^2.$$

Note that

$$\begin{aligned} h_\lambda - h_0 &= [\{S^* S_z^{-1} S + \lambda\}^{-1} S^* S_z^{-1} S - I] h_0 \\ &= \sum_j \left(\frac{\nu_j}{\nu_j + \lambda} - 1 \right) e_j \langle e_j, h_0 \rangle \\ &= \sum_j \left(-\frac{\lambda}{\nu_j + \lambda} \right) e_j \langle e_j, h_0 \rangle. \end{aligned}$$

Therefore,

$$\begin{aligned}
\|h_\lambda - h_0\|^2 &= \sum_j \left(\frac{\lambda}{\nu_j + \lambda} \right)^2 \langle e_j, h_0 \rangle^2 \\
&= \sum_j \left(\frac{\lambda}{\nu_j + \lambda} \right)^2 \langle e_j, h_0 \rangle^2 \left(\frac{\lambda \nu_j \nu_j + \lambda}{\lambda \nu_j \nu_j + \lambda} \right)^{2\alpha} \\
&= \lambda^{2\alpha} \sum_j \nu_j^{-2\alpha} \langle e_j, h_0 \rangle^2 \left(\frac{\lambda}{\nu_j + \lambda} \right)^{2-2\alpha} \left(\frac{\nu_j}{\nu_j + \lambda} \right)^{2\alpha} \\
&\leq \lambda^{2\alpha} \sum_j \nu_j^{-2\alpha} \langle e_j, h_0 \rangle^2 \\
&= \lambda^{2\alpha} \|w_0\|^2 \\
&\leq \lambda^{2\alpha} (\kappa_x^2)^{-2\alpha} \|h_0\|^2.
\end{aligned}$$

The last inequality follows from Lemma E.1. \square

C.2 Instrumental variable regression bias

Lemma C.2 (Bias upper bound). *Suppose that Assumptions 1 and 2 hold for (α, β, r) , and $r\mu^\beta/\lambda^{1/2} < 1$. Then,*

$$\|T_{\mu,\lambda}^{-1} T_\mu h_0 - h_0\|_{\mathcal{H}_x} := b_{\mu,\lambda} \leq \frac{\|h_\lambda - h_0\|_{\mathcal{H}_x}}{1 - C_\beta r \mu^\beta / \lambda^{\frac{1}{2}}} \leq \frac{C_\alpha \lambda^\alpha \|T^{-\alpha} h_0\|_{\mathcal{H}_x}}{1 - C_\beta r \mu^\beta / \lambda^{\frac{1}{2}}}.$$

Proof. We begin by noting that

$$T^{-1/2} S^* S_z^{-1} S T^{-1/2} = T^{-1/2} T T^{-1/2} = I,$$

so $T^{-1/2} S^* S_z^{-1/2}$ is a unitary operator with norm one. By Lemma C.3 below, we may substitute

$$\begin{aligned}
\|T_\lambda^{-1} (T_\mu - T)x\| &= \|T_\lambda^{-1} S^* [\mu S_z^{-1} (S_z + \mu)^{-1}] Sx\| \\
&= \|[T_\lambda^{-1} T^{1/2}] [T^{-1/2} S^* S_z^{-1/2}] [\mu S_z^\beta (S_z + \mu)^{-1}] [S_z^{-1/2-\beta} S] x\| \\
&\leq \|T_\lambda^{-1} T^{1/2}\| \|T^{-1/2} S^* S_z^{-1/2}\| \|\mu S_z^\beta (S_z + \mu)^{-1}\| \|S_z^{-1/2-\beta} S\| \|x\|.
\end{aligned}$$

The first norm is bounded by $\lambda^{-1/2}$ by construction. The second norm is bounded by 1

from our earlier observation. The third is bounded by $C_\beta \mu^\beta$ by the second-stage bias bound (Lemma C.4). The third is bounded by r by Assumption 2. Thus, we obtain

$$\|T_\lambda^{-1}(T_\mu - T)\| \leq \frac{r\mu^\beta}{\sqrt{\lambda}}.$$

Thus,

$$\|[T_\lambda^{-1}(T_\mu - T)]^l b_\lambda\| \leq \left(\frac{r\mu^\beta}{\sqrt{\lambda}}\right)^l \|b_\lambda\|$$

and, using $\|T_\mu - T\| \leq \|T\| \leq \kappa_x^2$ and $\|T_{\mu,\lambda}^{-1}\| \leq \lambda^{-1}$, we also have that

$$\|T_{\mu,\lambda}^{-1}(T_\mu - T)[T_\lambda^{-1}(T_\mu - T)]^l b_\lambda\| \leq \frac{\kappa_x^2}{\lambda} \left(\frac{r\mu^\beta}{\sqrt{\lambda}}\right)^l \|b_\lambda\|.$$

Plugging these into our decomposition of Lemma C.3, we have for any $k \geq 1$ that

$$\|h_0 - T_{\mu,\lambda}^{-1}T_\mu h_0\| \leq \left[\frac{\kappa_x^2}{\lambda} \delta^{k-1} + \sum_{l=0}^{k-1} \delta^l \right] \|b_\lambda\|; \quad \delta = \left(\frac{r\mu^\beta}{\sqrt{\lambda}}\right).$$

Since $\delta < 1$, we recover the bound

$$\|h_0 - T_{\mu,\lambda}^{-1}T_\mu h_0\| \leq \lim_{k \uparrow \infty} \left[\frac{\kappa_x^2}{\lambda} \delta^{k-1} + \sum_{l=0}^{k-1} \delta^l \right] \|b_\lambda\| = \left(\frac{1}{1-\delta}\right) \|b_\lambda\|.$$

Finally, we use the standard regression bias bound (Lemma C.1) to bound $\|b_\lambda\|$.

Lemma C.3 (Decomposition). *Let $b_\lambda = h_0 - h_\lambda$. For any $k \geq 1$,*

$$h_0 - T_{\mu,\lambda}^{-1}T_\mu h_0 = T_{\lambda,\mu}^{-1}(T_\mu - T)[T_\lambda^{-1}(T_\mu - T)]^{k-1} b_\lambda + \sum_{l=0}^{k-1} [T_\lambda^{-1}(T_\mu - T)]^l b_\lambda.$$

Furthermore,

$$T - T_\mu = S^*(S_z^{-1} - (S_z + \mu)^{-1})S = \mu S^* S_z^{-1} (S_z + \mu)^{-1} S.$$

Proof. We begin by writing

$$h_0 - T_{\mu,\lambda}^{-1}T_\mu h_0 = (T_{\mu,\lambda}^{-1}T_{\mu,\lambda} - T_{\mu,\lambda}^{-1}T_\mu)h_0 = -\lambda T_{\mu,\lambda}^{-1}h_0.$$

By the resolvent identity $A^{-1} = B^{-1} + A^{-1}(B - A)B^{-1}$ with $A = T_{\mu,\lambda}$ and $B = T_\lambda$, we further obtain

$$\begin{aligned}-\lambda T_{\mu,\lambda}^{-1} h_0 &= -\lambda T_\lambda^{-1} h_0 + T_{\mu,\lambda}^{-1}(T_{\mu,\lambda} - T_\lambda)T_\lambda^{-1}(-\lambda h_0) \\ &= b_\lambda + T_{\mu,\lambda}^{-1}(T_\mu - T)T_\lambda^{-1}b_\lambda\end{aligned}$$

using $-\lambda T_\lambda h_0 = b_\lambda$ and $T_{\mu,\lambda} - T_\lambda = T_\mu - T$. This proves our claim with $k = 1$. To complete the proof by induction, note that the claim for $k+1$ follows from the claim for k by applying the resolvent identity to substitute

$$T_{\mu,\lambda}^{-1} = T_\lambda^{-1} + T_{\mu,\lambda}^{-1}(T_\mu - T)T_\lambda^{-1}. \quad \square$$

Lemma C.4 (Operator norm bound). *Let S_z be a positive semidefinite, self-adjoint operator. By spectral calculus, bounds for $\mu S_z^\beta (S_z + \mu I)^{-1}$ reduce to a scalar. In particular, for $\beta \in [0,1]$*

$$\|\mu S_z^\beta (S_z + \mu I)^{-1}\|_{\text{op}} \leq C_\beta \mu^\beta,$$

where $C_\beta = (1 - \beta)^{1-\beta} \beta^\beta$ is a constant dependent on β .

Proof. By the definition of the operator norm and the spectral theorem,

$$\|\mu S_z^\beta (S_z + \mu I)^{-1}\|_{\text{op}} = \left\{ \sup_t \frac{\mu t^\beta}{t + \mu} \right\} = (1 - \beta)^{1-\beta} \beta^\beta \mu^\beta \leq C_\beta \mu^\beta. \quad \square$$

Lemma C.5 (Projected bias). *Suppose that Assumptions 1 and 2 hold for (α, β, r) , and $r\mu^\beta / \lambda^{1/2} < 1$. Then,*

$$\|(S_z + \mu)^{-1/2} S(h_0 - T_{\mu,\lambda}^{-1} T_\mu h_0)\| \lesssim \frac{1}{1-\delta} C_\alpha \lambda^{(\alpha+1/2)\wedge 1},$$

where $\delta = \frac{r\mu^\beta}{\sqrt{\lambda}}$.

Proof. Define the bias element $b_\lambda := h_\lambda - h_0 = -\lambda T_\lambda^{-1} h_0$. Introduce $A := (S_z + \mu I)^{-1/2} S$. Note that

$$\|AT_\lambda^{-1/2}\| \leq \|S_z^{-1/2} ST_\lambda^{-1/2}\| \leq 1, \quad \|AT_{\mu,\lambda}^{-1}\| \leq \frac{1}{2\sqrt{\lambda}}.$$

For any integer $l \geq 0$,

$$[T_\lambda^{-1}(T_\mu - T)]^l = T_\lambda^{-1/2} \left(T_\lambda^{-1/2}(T_\mu - T) T_\lambda^{-1/2} \right)^l T_\lambda^{1/2}.$$

Hence, for every $l \geq 0$,

$$\begin{aligned} \|A[T_\lambda^{-1}(T_\mu - T)]^l b_\lambda\| &\leq \underbrace{\|AT_\lambda^{-1/2}\|}_{\leq 1} \underbrace{\|T_\lambda^{-1/2}(T_\mu - T) T_\lambda^{-1/2}\|}_{=\delta^l}^l \|T_\lambda^{1/2} b_\lambda\| \\ &= \delta^l \|T_\lambda^{1/2} b_\lambda\|, \end{aligned}$$

where we set

$$\begin{aligned} \delta &= \|T_\lambda^{-1/2}(T_\mu - T) T_\lambda^{-1/2}\| \\ &= \|T_\lambda^{-1/2} S^* S_z^{-1} \mu(S_z + \mu)^{-1} S T_\lambda^{-1/2}\| \leq \frac{1}{\sqrt{\lambda}} \|T_\lambda^{-1/2} S^* S_z^{-1/2} \mu S_z^\beta (S_z + \mu)^{-1} S_z^{-1/2-\beta} S\| \\ &\leq \frac{r \mu^\beta}{\sqrt{\lambda}} \end{aligned}$$

which is smaller than one by hypothesis. Lastly, for $\alpha \in [0, 1/2]$

$$\|T_\lambda^{1/2} b_\lambda\| = \lambda \|T_\lambda^{-1/2} h_0\| = \lambda \|T_\lambda^{-1/2} T^\alpha w_0\| \lesssim \xi_\alpha \lambda^{\alpha+1/2} \|T^{-\alpha} h_0\|,$$

while for $\alpha \in (1/2, 1]$,

$$\|T_\lambda^{1/2} b_\lambda\| = \lambda \|T_\lambda^{-1/2+\alpha} h_0\| \lesssim \xi_\alpha \lambda \|h_0\|$$

since $-1/2 + \alpha > 0$. Now, using Lemma C.3 for any integer $k \geq 1$,

$$A(h_0 - T_{\mu, \lambda}^{-1} T_\mu h_0) = AT_{\mu, \lambda}^{-1}(T_\mu - T)[T_\lambda^{-1}(T_\mu - T)]^{k-1} b_\lambda + \sum_{l=0}^{k-1} A[T_\lambda^{-1}(T_\mu - T)]^l b_\lambda.$$

Furthermore, for any integer $k \geq 1$,

$$\|A(h_0 - T_{\mu, \lambda}^{-1} T_\mu h_0)\| \leq \frac{1}{2\sqrt{\lambda}} \delta^{k-1} + \sum_{l=0}^{k-1} \delta^l \|T_\lambda^{1/2} b_{\mu, \lambda}\|.$$

In particular, for $k \rightarrow \infty$

$$\|A(h_0 - T_{\mu,\lambda}^{-1}T_\mu h_0)\| \leq \frac{1}{1-\delta} C_\alpha \lambda^{(\alpha+1/2) \wedge 1}. \quad \square$$

D Matching symbols

D.1 Covariance upper bound

We introduce some notation. Let $u = h_0 - h_{\mu,\lambda}$ be a function and $u(X_i) = \psi(X_i)^* u$ be a scalar, so that $S_i(h_0 - h_{\mu,\lambda}) = \{h_0(X_i) - h_{\mu,\lambda}(X_i)\}\phi(Z_i) = u(X_i)\phi(Z_i)$. A useful fact is that since $h_0 - h_{\mu,\lambda} = (I - T_{\mu,\lambda}^{-1}T_\mu)h_0$ and $(I - T_{\mu,\lambda}^{-1}T_\mu) \preceq I$, we have that $|u(X_i)| = |h_0(X_i) - h_{\mu,\lambda}(X_i)| \leq \kappa_x \|h_0 - h_{\mu,\lambda}\| = \bar{u}$, where \bar{u} is explicitly studied in the bias upper bound (Lemma C.2) argument. In addition, $v = (S_z + \mu)^{-1/2}S(h_0 - h_{\mu,\lambda})$ and $w = (S_z + \mu)^{-1/2}v$. Both live in \mathcal{H}_z and can be evaluated with $\phi(Z_i)$. Note that, for example, $S_i^*(S_z + \mu)^{-1}S(h_0 - h_{\mu,\lambda}) = \psi(X_i)w(Z_i)$ and $S_{z,i}(S_z + \mu)^{-1}S(h_0 - h_{\mu,\lambda}) = \phi(Z_i)w(Z_i)$, where $\|v\| \leq \bar{v}$ is studied in the projected bias argument (Lemma C.5). Also, $|w(Z_i)| \leq \kappa_z \|w\| = \bar{w}$. It holds that $\|w\| \leq \|v\| \frac{1}{\sqrt{\mu}}$.

Lemma D.1 (Covariance upper bound). *The covariance $\Sigma = \mathbb{E}(U_i \otimes U_i^*)$ is upper bounded by*

$$\Sigma \preceq (6\bar{w}^2 + 6\bar{u}^2 + 2\bar{\sigma}^2)T_{\mu,\lambda}^{-1}S^*(S_z + \mu)^{-1}S_z(S_z + \mu)^{-1}ST_{\mu,\lambda}^{-1} + 6\bar{v}^2\kappa_x^2T_{\mu,\lambda}^{-2}.$$

Proof. Write $U_i = U_{i1} + U_{i2}$, where

$$\begin{aligned} U_{i1} &= T_{\mu,\lambda}^{-1}\{S^*(S_z + \mu)^{-1}(S_i - S) + (S_i - S)^*(S_z + \mu)^{-1}S + S^*(S_z + \mu)^{-1}(S_z - S_{z,i})(S_z + \mu)^{-1}S\}(h_0 - h_{\mu,\lambda}) \\ U_{i2} &= T_{\mu,\lambda}^{-1}S^*(S_z + \mu)^{-1}\phi(Z_i)\varepsilon_i. \end{aligned}$$

Then, $\Sigma = \Sigma_{11} + \Sigma_{12} + \Sigma_{21} + \Sigma_{22}$ where $\Sigma_{\ell m} = \mathbb{E}(U_{i\ell} \otimes U_{im}^*)$. In particular by Lemma B.4, $\Sigma \preceq 2(\Sigma_{11} + \Sigma_{22})$. Consider the parts individually.

1. Σ_{11} : Define

$$\begin{aligned} U_{i11} &= \mathbb{E}\left\{T_{\mu,\lambda}^{-1}S^*(S_z + \mu)^{-1}(S_i - S)(h_0 - h_{\mu,\lambda})\right\}, \\ U_{i12} &= \mathbb{E}\left\{T_{\mu,\lambda}^{-1}(S_i - S)^*(S_z + \mu)^{-1}S(h_0 - h_{\mu,\lambda})\right\}, \\ U_{i13} &= \mathbb{E}\left\{T_{\mu,\lambda}^{-1}S^*(S_z + \mu)^{-1}(S_z - S_{z,i})(S_z + \mu)^{-1}S(h_0 - h_{\mu,\lambda})\right\}. \end{aligned}$$

Now, note again how

$$\Sigma_{11} = \sum_{i=1}^3 \sum_{j=1}^3 \Sigma_{1ij}$$

where $\Sigma_{1ij} = \mathbb{E}(U_{1ij} \otimes U_{1ij}^*)$ and

$$\begin{aligned} \Sigma_{111} &= \mathbb{E}\left(\{T_{\mu,\lambda}^{-1}S^*(S_z + \mu)^{-1}(S_i - S)(h_0 - h_{\mu,\lambda})\}\right) \otimes \left(\{T_{\mu,\lambda}^{-1}S^*(S_z + \mu)^{-1}(S_i - S)(h_0 - h_{\mu,\lambda})\}\right)^* \\ \Sigma_{112} &= \mathbb{E}\left(\{T_{\mu,\lambda}^{-1}S^*(S_z + \mu)^{-1}(S_i - S)(h_0 - h_{\mu,\lambda})\}\right) \otimes \left(\{T_{\mu,\lambda}^{-1}(S_i - S)^*(S_z + \mu)^{-1}S(h_0 - h_{\mu,\lambda})\}\right)^* \\ \Sigma_{113} &= \mathbb{E}\left(\{T_{\mu,\lambda}^{-1}S^*(S_z + \mu)^{-1}(S_i - S)(h_0 - h_{\mu,\lambda})\}\right) \otimes \left(\{T_{\mu,\lambda}^{-1}S^*(S_z + \mu)^{-1}(S_z - S_{z,i})(S_z + \mu)^{-1}S(h_0 - h_{\mu,\lambda})\}\right)^* \\ \Sigma_{121} &= \mathbb{E}\left(\{T_{\mu,\lambda}^{-1}(S_i - S)^*(S_z + \mu)^{-1}S(h_0 - h_{\mu,\lambda})\}\right) \otimes \left(\{T_{\mu,\lambda}^{-1}S^*(S_z + \mu)^{-1}(S_i - S)(h_0 - h_{\mu,\lambda})\}\right)^* \\ \Sigma_{122} &= \mathbb{E}\left(\{T_{\mu,\lambda}^{-1}(S_i - S)^*(S_z + \mu)^{-1}S(h_0 - h_{\mu,\lambda})\}\right) \otimes \left(\{T_{\mu,\lambda}^{-1}(S_i - S)^*(S_z + \mu)^{-1}S(h_0 - h_{\mu,\lambda})\}\right)^* \\ \Sigma_{123} &= \mathbb{E}\left(\{T_{\mu,\lambda}^{-1}(S_i - S)^*(S_z + \mu)^{-1}S(h_0 - h_{\mu,\lambda})\}\right) \otimes \left(\{T_{\mu,\lambda}^{-1}S^*(S_z + \mu)^{-1}(S_z - S_{z,i})(S_z + \mu)^{-1}S(h_0 - h_{\mu,\lambda})\}\right)^* \\ \Sigma_{131} &= \mathbb{E}\left(\{T_{\mu,\lambda}^{-1}S^*(S_z + \mu)^{-1}(S_z - S_{z,i})(S_z + \mu)^{-1}S(h_0 - h_{\mu,\lambda})\}\right) \otimes \left(\{T_{\mu,\lambda}^{-1}S^*(S_z + \mu)^{-1}(S_i - S)(h_0 - h_{\mu,\lambda})\}\right)^* \\ \Sigma_{132} &= \mathbb{E}\left(\{T_{\mu,\lambda}^{-1}S^*(S_z + \mu)^{-1}(S_z - S_{z,i})(S_z + \mu)^{-1}S(h_0 - h_{\mu,\lambda})\}\right) \otimes \left(\{T_{\mu,\lambda}^{-1}(S_i - S)^*(S_z + \mu)^{-1}S(h_0 - h_{\mu,\lambda})\}\right)^* \\ \Sigma_{133} &= \mathbb{E}\left(\{T_{\mu,\lambda}^{-1}S^*(S_z + \mu)^{-1}(S_z - S_{z,i})(S_z + \mu)^{-1}S(h_0 - h_{\mu,\lambda})\}\right) \\ &\quad \otimes \left(\{T_{\mu,\lambda}^{-1}S^*(S_z + \mu)^{-1}(S_z - S_{z,i})(S_z + \mu)^{-1}S(h_0 - h_{\mu,\lambda})\}\right)^*. \end{aligned}$$

We have that

$$\Sigma_{11} \preceq 3(\Sigma_{111} + \Sigma_{122} + \Sigma_{133}).$$

Thus, looking at the individual terms,

$$\begin{aligned}
\Sigma_{111} &\preceq \mathbb{E}\left(\{T_{\mu,\lambda}^{-1}S^*(S_z+\mu)^{-1}S_i(h_0-h_{\mu,\lambda})\}\right) \otimes \left(\{T_{\mu,\lambda}^{-1}S^*(S_z+\mu)^{-1}S_i(h_0-h_{\mu,\lambda})\}\right)^* \\
&= \mathbb{E}\left(T_{\mu,\lambda}^{-1}S^*(S_z+\mu)^{-1}u(X_i)\phi(Z_i)\phi(Z_i)^*u(X_i)(S_z+\mu)^{-1}ST_{\mu,\lambda}^{-1}\right) \\
&\preceq \bar{u}^2 T_{\mu,\lambda}^{-1} S^*(S_z+\mu)^{-1} S_z (S_z+\mu)^{-1} S T_{\mu,\lambda}^{-1}.
\end{aligned}$$

The other two diagonal terms require more care for Σ_{122}

$$\begin{aligned}
\Sigma_{122} &\preceq \mathbb{E}\left(\{T_{\mu,\lambda}^{-1}S_i^*(S_z+\mu)^{-1}S(h_0-h_{\mu,\lambda})\}\right) \otimes \left(\{T_{\mu,\lambda}^{-1}S_i^*(S_z+\mu)^{-1}S(h_0-h_{\mu,\lambda})\}\right)^* \\
&= \mathbb{E}\left(T_{\mu,\lambda}^{-1}\psi(X_i)w(Z_i)^2\psi(X_i)^*T_{\mu,\lambda}^{-1}\right).
\end{aligned}$$

Recall that, for general u, v

$$\langle u, S_z v \rangle_{\mathcal{H}_z} = \mathbb{E}_Z[u(Z)v(Z)].$$

Then,

$$\begin{aligned}
&\mathbb{E}\left(T_{\mu,\lambda}^{-1}\psi(X_i)w(Z_i)^2\psi(X_i)^*T_{\mu,\lambda}^{-1}\right) \\
&= \mathbb{E}_Z(w(Z_i)^2\mathbb{E}\{T_{\mu,\lambda}^{-1}\psi(X_i)\otimes\psi(X_i)^*T_{\mu,\lambda}^{-1}|Z_i\}) \\
&\preceq \mathbb{E}_Z(\langle\phi(Z_i), w\rangle_{\mathcal{H}_z}^2)T_{\mu,\lambda}^{-2}\kappa_x^2 \\
&= \langle(S_z+\mu)^{-1}S(h_0-h_{\mu,\lambda}), S_z(S_z+\mu)^{-1}S(h_0-h_{\mu,\lambda})\rangle_{\mathcal{H}_z} T_{\mu,\lambda}^{-2}\kappa_x^2 \\
&= \langle(S_z+\mu)^{-1/2}S(h_0-h_{\mu,\lambda}), (S_z+\mu)^{-1/2}S_z(S_z+\mu)^{-1/2}(S_z+\mu)^{-1/2}S(h_0-h_{\mu,\lambda})\rangle_{\mathcal{H}_z} T_{\mu,\lambda}^{-2}\kappa_x^2 \\
&\preceq \langle(S_z+\mu)^{-1/2}S(h_0-h_{\mu,\lambda}), (S_z+\mu)^{-1/2}S(h_0-h_{\mu,\lambda})\rangle_{\mathcal{H}_z} T_{\mu,\lambda}^{-2}\kappa_x^2 \\
&= \|(S_z+\mu)^{-1/2}S(h_0-h_{\mu,\lambda})\|_{\mathcal{H}_z}^2 T_{\mu,\lambda}^{-2}\kappa_x^2 \\
&\preceq \bar{v}^2 T_{\mu,\lambda}^{-2}\kappa_x^2.
\end{aligned}$$

Thus, $\Sigma_{122} \preceq \bar{v}^2 \kappa_x^2 T_{\mu,\lambda}^{-2}$. Lastly, for Σ_{133} we can write

$$\begin{aligned}
\Sigma_{133} &\preceq \mathbb{E} \left(\{T_{\mu,\lambda}^{-1} S^*(S_z + \mu)^{-1} S_{z,i} (S_z + \mu)^{-1} S(h_0 - h_{\mu,\lambda})\} \right) \otimes \left(\{T_{\mu,\lambda}^{-1} S^*(S_z + \mu)^{-1} S_{z,i} (S_z + \mu)^{-1} S(h_0 - h_{\mu,\lambda})\} \right)^* \\
&= \mathbb{E} \left(\{T_{\mu,\lambda}^{-1} S^*(S_z + \mu)^{-1} S_{z,i} w\} \right) \otimes \left(\{T_{\mu,\lambda}^{-1} S^*(S_z + \mu)^{-1} S_{z,i} w\} \right)^* \\
&= \mathbb{E} \left(\{T_{\mu,\lambda}^{-1} S^*(S_z + \mu)^{-1} \phi(Z_i) w(Z_i)\} \right) \otimes \left(\{T_{\mu,\lambda}^{-1} S^*(S_z + \mu)^{-1} \phi(Z_i) w(Z_i)\} \right)^* \\
&\preceq \bar{w}^2 T_{\mu,\lambda}^{-1} S^*(S_z + \mu)^{-1} S_z (S_z + \mu)^{-1} S T_{\mu,\lambda}^{-1}.
\end{aligned}$$

We can now combine the three diagonal terms, yielding

$$\begin{aligned}
\Sigma_{11} &\preceq 3\bar{u}^2 T_{\mu,\lambda}^{-1} S^*(S_z + \mu)^{-1} S_z (S_z + \mu)^{-1} S T_{\mu,\lambda}^{-1} + 3\bar{w}^2 T_{\mu,\lambda}^{-1} S^*(S_z + \mu)^{-1} S_z (S_z + \mu)^{-1} S T_{\mu,\lambda}^{-1} + 3\bar{v}^2 \kappa_x^2 T_{\mu,\lambda}^{-2} \\
&\preceq 3(\bar{u}^2 + \bar{w}^2) T_{\mu,\lambda}^{-1} S^*(S_z + \mu)^{-1} S_z (S_z + \mu)^{-1} S T_{\mu,\lambda}^{-1} + 3\bar{v}^2 \kappa_x^2 T_{\mu,\lambda}^{-2}.
\end{aligned}$$

2. For Σ_{22} , we use that $\varepsilon_i \leq \bar{\sigma}$. Then,

$$\begin{aligned}
0 &\preceq \Sigma_{22} \preceq \bar{\sigma}^2 \mathbb{E} \{T_{\mu,\lambda}^{-1} S^*(S_z + \mu)^{-1} \phi(Z_i)\} \otimes \{T_{\mu,\lambda}^{-1} S^*(S_z + \mu)^{-1} \phi(Z_i)\}^* \\
&= \bar{\sigma}^2 \mathbb{E} \{T_{\mu,\lambda}^{-1} S^*(S_z + \mu)^{-1} \phi(Z_i) \otimes \phi(Z_i)^* (S_z + \mu)^{-1} S T_{\mu,\lambda}^{-1}\} \\
&= \bar{\sigma}^2 T_{\mu,\lambda}^{-1} S^*(S_z + \mu)^{-1} S_z (S_z + \mu)^{-1} S T_{\mu,\lambda}^{-1}.
\end{aligned}$$

We collect results as

$$\begin{aligned}
\Sigma &\preceq 2(\Sigma_{11} + \Sigma_{22}) \\
&\preceq 6(\bar{w}^2 + \bar{u}^2) T_{\mu,\lambda}^{-1} S^*(S_z + \mu)^{-1} S_z (S_z + \mu)^{-1} S T_{\mu,\lambda}^{-1} + 6\bar{v}^2 \kappa_x^2 T_{\mu,\lambda}^{-2} + 2\bar{\sigma}^2 T_{\mu,\lambda}^{-1} S^*(S_z + \mu)^{-1} S_z (S_z + \mu)^{-1} S T_{\mu,\lambda}^{-1} \\
&= (6\bar{w}^2 + 6\bar{u}^2 + 2\bar{\sigma}^2) T_{\mu,\lambda}^{-1} S^*(S_z + \mu)^{-1} S_z (S_z + \mu)^{-1} S T_{\mu,\lambda}^{-1} + 6\bar{v}^2 \kappa_x^2 T_{\mu,\lambda}^{-2}. \quad \square
\end{aligned}$$

D.2 Covariance lower bound

We derive the variance lower bound of Z , where Z is a Gaussian element in \mathcal{H}_x with covariance Σ .

Assumption D.1 (Local width bound assumptions). *For our analysis we require that μ and λ adhere to the following relationship*

1. $\mu, \lambda < 1$,
2. $\lambda^{(2\alpha+1)\wedge 2} \lesssim_{C_{\sigma,K}} \mu \lesssim_{C_K} \lambda$,

where $C_{\sigma,K}$ and C_K are constants depending on σ and kernel constants, respectively.

Lemma D.2 (Covariance lower bound). *Suppose Assumption D.1 holds. Let Z be a Gaussian random element of \mathcal{H}_x with covariance Σ , and suppose $\mathbb{E}(\varepsilon_i^2 | Z_i) \geq \underline{\sigma}^2$ almost surely. Then with probability $1 - \eta$,*

$$\|Z\| \geq \sqrt{\frac{1}{4}\underline{\sigma}^2 \tilde{\mathfrak{m}}(\lambda, \mu)} - \left\{ 2 + \sqrt{2\ln(1/\eta)} \right\} \sqrt{\frac{2\bar{\sigma}^2}{\lambda}},$$

where $\tilde{\mathfrak{m}}(\lambda, \mu) = \text{tr} T_{\mu, \lambda}^{-1} S^* (S_z + \mu)^{-1} S_z (S_z + \mu)^{-1} S T_{\mu, \lambda}^{-1}$.

Proof. We lower bound $\mathbb{E}\|Z\|$ via the identity $\{\mathbb{E}(\|Z\|)\}^2 = \mathbb{E}(\|Z\|^2) - \mathbb{E}\{\|Z\| - \mathbb{E}(\|Z\|)\}^2$ then appeal to Lemma B.10. Let $B_{\mathcal{H}_x}$ be the unit ball in \mathcal{H}_x .

1. To upper bound $\mathbb{E}(\|Z\| - \mathbb{E}\|Z\|)^2$, we express $\|Z\|$ as the supremum of a Gaussian process: $\|Z\| = \sup_{t \in B_{\mathcal{H}_x}} \langle Z, t \rangle = \sup_{t \in B_{\mathcal{H}_x}} G_t$. By the variance upper bound

$$\sigma_T^2 = \sup_{t \in B_{\mathcal{H}_x}} \mathbb{E} \langle Z, t \rangle^2 = \|\Sigma\|_{op} \leq \frac{2\bar{\sigma}^2}{\lambda},$$

where the bound follows from

$$\Sigma \preceq (6\bar{w}^2 + 6\bar{u}^2 + 2\bar{\sigma}^2) T_{\mu, \lambda}^{-1} S^* (S_z + \mu)^{-1} S_z (S_z + \mu)^{-1} S T_{\mu, \lambda}^{-1} + 6\bar{v}^2 \kappa_x^2 T_{\mu, \lambda}^{-2}.$$

Using Assumption D.1 we can again focus on the \bar{w} term. Further it still holds that $\bar{w}^2 \leq \frac{\sigma^4}{72\bar{\sigma}^2}$. The latter implies that $12\bar{w}^2 \leq 2\bar{\sigma}^2$ since $\frac{\sigma^4}{72\bar{\sigma}^2} \leq \frac{\bar{\sigma}^2}{6} \iff \frac{\sigma^2}{\bar{\sigma}^2} \leq 12$ and $12\bar{w}^2 \leq 2\bar{\sigma}^2 \iff \bar{w}^2 \leq \frac{\bar{\sigma}^2}{6}$.

Using this,

$$\begin{aligned} \Sigma &\preceq 2(6\bar{w}^2 + 6\bar{u}^2 + 2\bar{\sigma}^2) T_{\mu, \lambda}^{-1} S^* (S_z + \mu)^{-1} S_z (S_z + \mu)^{-1} S T_{\mu, \lambda}^{-1} \\ &\preceq 2(12\bar{w}^2 + 2\bar{\sigma}^2) T_{\mu, \lambda}^{-1} S^* (S_z + \mu)^{-1} S_z (S_z + \mu)^{-1} S T_{\mu, \lambda}^{-1} \\ &\preceq 8\bar{\sigma}^2 T_{\mu, \lambda}^{-1} S^* (S_z + \mu)^{-1} S_z (S_z + \mu)^{-1} S T_{\mu, \lambda}^{-1}. \end{aligned}$$

Thus,

$$\begin{aligned}\|\Sigma\|_{op} &\leq \left\| 8\bar{\sigma}^2 T_{\mu,\lambda}^{-1} S^* (S_z + \mu)^{-1} S_z (S_z + \mu)^{-1} S T_{\mu,\lambda}^{-1} \right\|_{op} \\ &\leq \frac{8\bar{\sigma}^2}{4\lambda} \|(S_z + \mu)^{-1} S_z\|_{op} \\ &\leq \frac{8\bar{\sigma}^2}{4\lambda} = \frac{2\bar{\sigma}^2}{\lambda}.\end{aligned}$$

For $\mathbb{E}\|Z\|^2 = \text{tr}\Sigma$ we can use the same derivations and find that

$$\text{tr}\Sigma \leq \text{tr} 8\bar{\sigma}^2 T_{\mu,\lambda}^{-1} S^* (S_z + \mu)^{-1} S_z (S_z + \mu)^{-1} S T_{\mu,\lambda}^{-1} = 8\bar{\sigma}^2 \tilde{m}(\lambda, \mu),$$

where is defined as $\tilde{m}(\lambda, \mu) = \text{tr} T_{\mu,\lambda}^{-1} S^* (S_z + \mu)^{-1} S_z (S_z + \mu)^{-1} S T_{\mu,\lambda}^{-1}$. Thus, $\langle Z, t \rangle$ is almost surely bounded on $B_{\mathcal{H}_x}$ and by combining the two inequalities of Borell's inequality with a union bound, we have

$$\mathbb{P}\{(\|Z\| - \mathbb{E}\|Z\|)^2 \geq u\} = \mathbb{P}(|\|Z\| - \mathbb{E}\|Z\|| \geq \sqrt{u}) \leq 2\exp\left(-\frac{u}{2\frac{2\bar{\sigma}^2}{\lambda}}\right) = 2\exp\left(-\frac{u}{\frac{4\bar{\sigma}^2}{\lambda}}\right).$$

By integrating the tail,

$$\mathbb{E}(\|Z\| - \mathbb{E}\|Z\|)^2 = \int_0^\infty \mathbb{P}\{(\|Z\| - \mathbb{E}\|Z\|)^2 \geq u\} du \leq \int_0^\infty 2\exp\left(-\frac{u}{\frac{4\bar{\sigma}^2}{\lambda}}\right) du = 8\frac{\bar{\sigma}^2}{\lambda}.$$

2. We lower bound $\mathbb{E}(\|Z\|^2)$ by the variance lower bound (under the previously introduced regularity conditions), meaning that

$$\begin{aligned}\mathbb{E}\|Z\|^2 &= \text{tr}\Sigma \\ &\geq \text{tr} \frac{1}{4} \underline{\sigma}^2 T_{\mu,\lambda}^{-1} S^* (S_z + \mu)^{-1} S_z (S_z + \mu)^{-1} S T_{\mu,\lambda}^{-1} \\ &= \frac{1}{4} \underline{\sigma}^2 \tilde{m}(\lambda, \mu).\end{aligned}$$

3. Combining the bounds, we get that

$$\mathbb{E}\|Z\| \geq \sqrt{\frac{1}{4} \underline{\sigma}^2 \tilde{m}(\lambda, \mu) - 8\frac{\bar{\sigma}^2}{\lambda}} \geq \sqrt{\frac{1}{4} \underline{\sigma}^2 \tilde{m}(\lambda, \mu)} - \sqrt{8\frac{\bar{\sigma}^2}{\lambda}}.$$

In particular, choosing $u = \sqrt{2\|\Sigma\|_{op}\ln(1/\eta)}$, we conclude that with probability at least $1-\eta$,

$$\|Z\| \geq \mathbb{E}\|Z\| - \sqrt{2\|\Sigma\|_{op}\ln(1/\eta)}.$$

Therefore,

$$\begin{aligned} \|Z\| &\geq \sqrt{\frac{1}{4}\underline{\sigma}^2\tilde{\mathfrak{m}}(\lambda,\mu)} - \sqrt{\frac{8\bar{\sigma}^2}{\lambda}} - \sqrt{2\|\Sigma\|_{op}\ln(1/\eta)} \\ &\geq \sqrt{\frac{1}{4}\underline{\sigma}^2\tilde{\mathfrak{m}}(\lambda,\mu)} - \sqrt{8\frac{\bar{\sigma}^2}{\lambda}} - \sqrt{4\frac{\bar{\sigma}^2}{\lambda}\ln(1/\eta)} \\ &= \sqrt{\frac{1}{4}\underline{\sigma}^2\tilde{\mathfrak{m}}(\lambda,\mu)} - \left\{2 + \sqrt{2\ln(1/\eta)}\right\} \sqrt{\frac{2\bar{\sigma}^2}{\lambda}}. \quad \square \end{aligned}$$

D.3 Local width upper bound

Lemma D.3 (Local width bound). *Suppose Assumption D.1 holds. We can bound the local width of Σ with the local width of the covariance operator T by $\sigma(\Sigma,m) \leq \frac{\sqrt{8\bar{\sigma}}}{\lambda}\sigma(T,m)$ and $\sigma(\Sigma,0) \leq \bar{\sigma}\sqrt{8\tilde{\mathfrak{m}}(\lambda,\mu)}$, where $\tilde{\mathfrak{m}}(\lambda,\mu) = \text{tr}T_{\mu,\lambda}^{-1}S^*(S_z+\mu)^{-1}S_z(S_z+\mu)^{-1}ST_{\mu,\lambda}^{-1}$.*

Proof. Recall that under Assumption D.1, we showed in Lemma D.2 that

$$\Sigma \preceq 8\bar{\sigma}^2T_{\mu,\lambda}^{-1}S^*(S_z+\mu)^{-1}S_z(S_z+\mu)^{-1}ST_{\mu,\lambda}^{-1}$$

and

$$\sigma(\Sigma,0)^2 = \text{tr}\Sigma \leq \text{tr}8\bar{\sigma}^2T_{\mu,\lambda}^{-1}S^*(S_z+\mu)^{-1}S_z(S_z+\mu)^{-1}ST_{\mu,\lambda}^{-1} = 8\bar{\sigma}^2\tilde{\mathfrak{m}}(\lambda,\mu).$$

Now, for general m , since $S^*(S_z+\mu)^{-1}S_z(S_z+\mu)^{-1}S \preceq S^*S_z^{-1}S = T$ we have that

$$\Sigma \preceq 8\bar{\sigma}^2T_{\mu,\lambda}^{-1}TT_{\mu,\lambda}^{-1} \preceq \frac{8\bar{\sigma}^2}{\lambda^2}T$$

and subsequently

$$\sigma(\Sigma,m)^2 \leq \frac{8\bar{\sigma}^2}{\lambda^2}\sigma(T,m)^2. \quad \square$$

Lemma D.4 (Rate condition). *If $n \geq \max\{n_1, n_2\} := N_\delta$ then, $\delta \leq 1/2$ where*

$$n_1 = 16 \max\{\kappa_z^2 \mathbf{n}_z(\mu)^{-1} \mu^{-2}, \kappa \kappa_x^{1/2} \mathbf{m}(\lambda, \mu)^{-1} \lambda^{-2} \mu^{-1/2}\},$$

$$n_2 = 144 l(\eta)^2 \kappa \max\{\kappa_x^{1/2} \mathbf{m}(\lambda, \mu) \mu^{-1/2}, \frac{\kappa}{\mu \lambda} (2\kappa_z \sqrt{\mathbf{n}_z(\mu)} + 1)^2, \frac{\kappa_z^2 \mathbf{n}_z(\mu)}{\sqrt{\lambda \mu}}\}.$$

The effective dimensions are $\mathbf{m}(\lambda, \mu) = \text{tr}(T_{\mu, \lambda}^{-2} T)$ and $\mathbf{n}_z(\mu) = \text{tr}((S_z + \mu)^{-2} S_z)$.

Proof. We study how the high probability bounds of δ scale with n , μ , and λ . We have

$$\delta = \underbrace{\frac{\kappa \delta_z \tilde{\gamma}_1 + \tilde{\gamma}_1^2}{\lambda \mu}}_A + \underbrace{\frac{\kappa \delta_z (1 + 2\delta_z) + \tilde{\gamma}_1}{2\sqrt{\lambda \mu}}}_B + \underbrace{\tilde{\delta}(\mu, \lambda, \eta)}_C$$

$$\tilde{\gamma}_1 = \ln(2/\eta) \frac{\kappa}{n^{1/2}}$$

$$\delta_z = 2\kappa_z \ln(2/\eta) \left\{ \sqrt{\frac{\mathbf{n}_z(\mu)}{n}} \vee \frac{4\kappa_z}{n\mu} \right\}$$

$$\tilde{\delta}(\mu, \lambda, \eta) = 2\ln(2/\eta) \left(\frac{4\kappa \kappa_x^{1/2}}{n \lambda \mu^{1/2}} \vee \sqrt{\frac{\kappa \kappa_x^{1/2}}{n \mu^{1/2}} \mathbf{m}(\lambda, \mu)} \right).$$

For $n > 16 \{ \kappa_z^2 \mathbf{n}_z(\mu)^{-1} \mu^{-2} \vee \kappa \kappa_x^{1/2} \mathbf{m}(\lambda, \mu)^{-1} \lambda^{-2} \mu^{-1/2} \}$ the square root parts of the maxima dominate. Thus,

$$\tilde{\gamma}_1 = \ln(2/\eta) \frac{\kappa}{n^{1/2}}$$

$$\delta_z = 2\kappa_z \ln(2/\eta) \sqrt{\frac{\mathbf{n}_z(\mu)}{n}}$$

$$\tilde{\delta}(\mu, \lambda, \eta) = 2\ln(2/\eta) \sqrt{\frac{\kappa \kappa_x^{1/2}}{n \mu^{1/2}} \mathbf{m}(\lambda, \mu)}.$$

For A ,

$$\frac{l(\eta)^2 \kappa^2}{n \lambda \mu} (2\kappa_z \sqrt{\mathbf{n}_z(\mu)} + 1) \leq 1/6 \iff n_A \geq \frac{6l(\eta)^2 \kappa^2}{\lambda \mu} (2\kappa_z \sqrt{\mathbf{n}_z(\mu)} + 1).$$

For B ,

$$\frac{\kappa \delta_z + \tilde{\gamma}_1}{2\sqrt{\lambda \mu}} + \frac{\kappa 2\delta_z^2}{2\sqrt{\lambda \mu}} = \frac{l(\eta) \kappa (2\kappa_z \sqrt{\mathbf{n}_z(\mu)} + 1)}{2\sqrt{\lambda \mu}} \frac{1}{\sqrt{n}} + \frac{\kappa \cdot (2\kappa_z l(\eta) \sqrt{\mathbf{n}_z(\mu)})^2}{\sqrt{\lambda \mu}} \frac{1}{n}.$$

It suffices to make each summand at most $\frac{1}{12}$. Therefore, $B \leq \frac{1}{6}$ holds if

$$n_{B1} = \frac{36l(\eta)^2\kappa^2(2\kappa_z\sqrt{\mathbf{n}_z(\mu)}+1)^2}{\lambda\mu}, \quad \text{and} \quad n_{B2} = \frac{48l(\eta)^2\kappa\kappa_z^2\mathbf{n}_z(\mu)}{\sqrt{\lambda\mu}}.$$

Lastly, $C \leq 1/6 \iff n_C \geq 144l(\eta)^2\kappa\kappa_x^{1/2}\mathbf{m}(\lambda,\mu)\mu^{-1/2}$. One can clearly see that $n_{B1} > n_A$. Thus, $\delta \leq 1/2$ if, in addition to the n_1 used above

$$n \geq 144l(\eta)^2\kappa\max\{\kappa_x^{1/2}\mathbf{m}(\lambda,\mu)\mu^{-1/2}, \frac{\kappa}{\mu\lambda}(2\kappa_z\sqrt{\mathbf{n}_z(\mu)}+1)^2, \frac{\kappa_z^2\mathbf{n}_z(\mu)}{\sqrt{\lambda\mu}}\}. \quad \square$$

Assumption D.2 (Rate condition). *Assume n satisfies $n \geq \max\{n_1, n_2, N_{\delta_z}\}$ where*

$$\begin{aligned} n_1 &= 16\max\{\kappa_z^2\mathbf{n}_z(\mu)^{-1}\mu^{-2}, \kappa\kappa_x^{1/2}\mathbf{m}(\lambda,\mu)^{-1}\lambda^{-2}\mu^{-1/2}\}, \\ n_2 &= 144l(\eta)^2\kappa\max\{\kappa_x^{1/2}\mathbf{m}(\lambda,\mu)\mu^{-1/2}, \frac{\kappa}{\mu\lambda}(2\kappa_z\sqrt{\mathbf{n}_z(\mu)}+1)^2, \frac{\kappa_z^2\mathbf{n}_z(\mu)}{\sqrt{\lambda\mu}}\}, \\ N_{\delta_z} &= 16\kappa_z^2\ln(2/\eta)\{\ln(2/\eta)\mathbf{n}_z(\mu) \vee \mu^{-1}\}. \end{aligned}$$

D.4 Bounded summand

Lemma D.5 (Bounded U_i). *U_i is a-bounded almost surely, meaning*

$$\|U_i\| \leq a = \frac{2\kappa b_{\mu,\lambda} + 4\kappa\lambda^{-1/2}v_{\mu,\lambda} + 2\kappa_z\mu^{-1/2}v_{\mu,\lambda} + \bar{\sigma}\kappa_z}{2\sqrt{\lambda\mu}}.$$

Proof. We write

$$\begin{aligned} \|U_i\| &= \|T_{\mu,\lambda}^{-1}S^*(S_z + \mu)^{-1}(S_i - S)(h_0 - h_{\mu,\lambda})\| \\ &\quad + \|T_{\mu,\lambda}^{-1}(S_i - S)^*(S_z + \mu)^{-1}S(h_0 - h_{\mu,\lambda})\| \\ &\quad + \|T_{\mu,\lambda}^{-1}S^*(S_z + \mu)^{-1}(S_z - S_{z,i})(S_z + \mu)^{-1}S(h_0 - h_{\mu,\lambda})\| \\ &\quad + \|T_{\mu,\lambda}^{-1}S^*(S_z + \mu)^{-1}\phi(Z_i)\varepsilon_i\| \\ &\leq \frac{\kappa}{\sqrt{\lambda\mu}}b_{\mu,\lambda} + \frac{2\kappa}{\lambda}\sqrt{\frac{1}{\mu}}v_{\mu,\lambda} + \frac{1}{\sqrt{\lambda\mu}}\kappa_z\sqrt{\frac{1}{\mu}}v_{\mu,\lambda} + \frac{1}{2\sqrt{\lambda\mu}}\bar{\sigma}\kappa_z \\ &\leq \frac{2\kappa b_{\mu,\lambda} + 4\kappa\lambda^{-1/2}v_{\mu,\lambda} + 2\kappa_z\mu^{-1/2}v_{\mu,\lambda} + \bar{\sigma}\kappa_z}{2\sqrt{\lambda\mu}}. \end{aligned}$$

The first inequality follows from the triangle inequality in combination with Lemmas E.2 and B.3. \square

Lemma D.6 (Dominating rate in a under $\lambda = \mu^\iota$). *Suppose $v_{\mu,\lambda} \in \mathcal{O}(\lambda^{(\alpha+\frac{1}{2}) \wedge 1})$ and we choose the regime $\lambda = \mu^\iota$ with*

$$\iota \geq \begin{cases} \frac{1}{2\alpha+1}, & \alpha \in [0, \frac{1}{2}] \\ \frac{1}{2}, & \alpha \in (\frac{1}{2}, 1]. \end{cases}$$

Then, $\|U_i\| \leq a = \mathcal{O}\left(\frac{1}{\sqrt{\mu\lambda}}\right)$.

Proof. Since $\mu \leq \lambda$, the upper bound is controlled by

$$B_1 = \frac{v_{\mu,\lambda}}{\sqrt{\lambda}\mu} \quad \text{and} \quad B_2 = \frac{1}{\sqrt{\mu\lambda}}.$$

Consider the projected bias rate

$$v_{\mu,\lambda} \in \mathcal{O}(\lambda^{(\alpha+\frac{1}{2}) \wedge 1}).$$

Let $p := (\alpha + \frac{1}{2}) \wedge 1 \in [\frac{1}{2}, 1]$. Then $B_1 = \mathcal{O}(\lambda^{p-\frac{1}{2}}\mu^{-1})$, so

$$\frac{B_1}{B_2} = \mathcal{O}\left(\lambda^{p-\frac{1}{2}}\mu^{-1}\sqrt{\mu\lambda}\right) = \mathcal{O}(\lambda^p\mu^{-1/2}).$$

With the schedule $\lambda = \mu^\iota$, this becomes

$$\frac{B_1}{B_2} = \mathcal{O}(\mu^{\iota p - \frac{1}{2}}).$$

Hence $B_1 = \mathcal{O}(B_2)$ provided $\iota p - \frac{1}{2} \geq 0$, i.e.

$$\iota \geq \frac{1}{2p} = \frac{1}{2((\alpha + \frac{1}{2}) \wedge 1)} \iff \begin{cases} \iota \geq \frac{1}{2\alpha+1}, & \alpha \in [0, \frac{1}{2}] \\ \iota \geq \frac{1}{2}, & \alpha \in (\frac{1}{2}, 1]. \end{cases}$$

Under these conditions, $B_1 \leq \text{constant} \cdot B_2$, so the bound is dominated by B_2 , yielding

$$\|U_i\| \leq a = \mathcal{O}\left(\frac{1}{\sqrt{\mu\lambda}}\right). \quad \square$$

Lemma D.7 (Dominating rate in a by the variance scale under $\mu = \lambda/C$). *If $\mu = \lambda/C$ with $C > 1$ and $v_{\mu,\lambda} \in \mathcal{O}(\lambda^{(\alpha+\frac{1}{2}) \wedge 1})$ for $\alpha \in [0,1]$, then*

$$\|U_i\| \leq a = \mathcal{O}\left(\frac{1}{\sqrt{\mu\lambda}}\right).$$

Proof. By the projected-bias decomposition, the upper bound is controlled by

$$B_1 = \frac{v_{\mu,\lambda}}{\sqrt{\lambda}\mu} \quad \text{and} \quad B_2 = \frac{1}{\sqrt{\mu\lambda}}.$$

With $\mu = \lambda/C$ we have

$$B_1 = \frac{Cv_{\mu,\lambda}}{\lambda^{3/2}}, \quad B_2 = \frac{\sqrt{C}}{\lambda}, \quad \frac{B_1}{B_2} = \sqrt{C} \frac{v_{\mu,\lambda}}{\sqrt{\lambda}}.$$

Let $p := (\alpha + \frac{1}{2}) \wedge 1$. By hypothesis, $v_{\mu,\lambda} = \mathcal{O}(\lambda^p)$, hence $\frac{B_1}{B_2} = \sqrt{C} \mathcal{O}(\lambda^{p-\frac{1}{2}})$. Since $\alpha \in [0,1]$ implies $p \in [\frac{1}{2}, 1]$, we have $p - \frac{1}{2} \geq 0$. Consequently, $\|U_i\| \leq a = \mathcal{O}(1/\sqrt{\mu\lambda})$. \square

Corollary D.1 (Constant bound on a). *Depending on the regime $\lambda = \mu^\ell$ or $\mu = \lambda/C$, if the requirement on ι is satisfy, we can bound $\|U_i\| \leq a$ by*

$$a \leq \frac{\widetilde{M}}{\sqrt{\mu\lambda}}.$$

Here \widetilde{M} is a constant depending on the regime. For $\lambda = \mu^\ell$

$$\widetilde{M} := \frac{1}{2} \left(\frac{2\kappa C_\alpha}{1-c_\delta} \|T^{-\alpha} h_0\| + \frac{4\kappa C_\alpha}{1-c_\delta} + \frac{2\kappa_z C_\alpha}{1-c_\delta} + \bar{\sigma} \kappa_z \right)$$

and for $\lambda/C = \mu$

$$\widetilde{M} := \frac{1}{2} \left(\frac{2\kappa C_\alpha}{1-c_\delta} \|T^{-\alpha} h_0\| + \frac{4\kappa C_\alpha}{1-c_\delta} + \frac{2\kappa_z C_\alpha}{1-c_\delta} C^{1/2} + \bar{\sigma} \kappa_z \right).$$

Here, $1 - c_\delta$ gives a lower bound on the denominator introduced by the bias argument. It depends on the specific regime. For example, for $\mu = \lambda/C$,

$$c_\delta = \frac{r\sqrt{\mu}}{\sqrt{\lambda}} = \frac{r}{\sqrt{C}}.$$

E Bahadur representation

We derive a Bahadur representation, as a step towards deriving Q . In particular, we show the difference between $n^{1/2}(\hat{h} - h_{\mu,\lambda})$ and $n^{1/2}\mathbb{E}_n(U)$ is small, i.e. the residual vanishes. Here,

$$\begin{aligned} U_i &= T_{\mu,\lambda}^{-1} \{ S^*(S_z + \mu)^{-1}(S_i - S) + (S_i - S)^*(S_z + \mu)^{-1}S \\ &\quad + S^*(S_z + \mu)^{-1}(S_z - S_{z,i})(S_z + \mu)^{-1}S \} (h_0 - h_{\mu,\lambda}) + T_{\mu,\lambda}^{-1} S^*(S_z + \mu)^{-1} \phi(Z_i) \varepsilon_i. \end{aligned}$$

E.1 Helpful orderings

Lemma E.1 (Covariance operator bound). *If S_z is invertible, then we have the decomposition*

$$\begin{pmatrix} S_x & S^* \\ S & S_z \end{pmatrix} = \begin{pmatrix} I & S^* \\ 0 & S_z \end{pmatrix} \begin{pmatrix} S_x - S^* S_z^{-1} S & 0 \\ S_z^{-1} S & I \end{pmatrix}.$$

Moreover, $S_x \succeq S^* S_z^{-1} S \succeq 0$.

Proof. The decomposition can be checked by matrix multiplication under our maintained assumptions. Next, the lower bound $S^* S_z^{-1} S \succeq 0$ holds as $S^* S_z^{-1} S$ is the symmetric square of $S^* S_z^{-1/2}$. For the upper bound, note that $S_x - S^* S_z^{-1} S$ is symmetric, so it suffices to check non-negativity of eigenvalues. Suppose, to the contrary, that it has an eigenvector u_ν with negative eigenvalue $\nu < 0$. A straightforward computation using the decomposition from above then shows

$$\begin{pmatrix} u_\nu \\ -S_z^{-1} S u_\nu \end{pmatrix}^* \begin{pmatrix} S_x & S^* \\ S & S_z \end{pmatrix} \begin{pmatrix} u_\nu \\ -S_z^{-1} S u_\nu \end{pmatrix} = \begin{pmatrix} u_\nu \\ 0 \end{pmatrix}^* \begin{pmatrix} \{S_x - S^* S_z^{-1} S\} u_\nu \\ 0 \end{pmatrix} = \nu u_\nu^* u_\nu < 0.$$

This contradicts the hypothesized positive-definiteness. \square

Lemma E.2 (Bound on the regularized cross-covariance). *The operator norm of the conditional expectation operator is bounded: $\|(S_z + \mu I)^{-1} S\|_{op} \leq \sqrt{\frac{\kappa_x}{\mu}}$.*

Proof. Note that $S_z \preceq S_z + I\mu \implies S_z^{-1} \succeq (S_z + I\mu)^{-1}$. Thus, $S^*(S_z + \mu)^{-1} S \preceq S^* S_z^{-1} S \preceq S_x$. Where the last \preceq is followed by Lemma E.1. Now,

$$S^*(S_z + \mu)^{-1} S = \{(S_z + \mu)^{-1/2} S\}^* \{(S_z + \mu)^{-1/2} S\} \preceq S_x \implies \|(S_z + \mu)^{-1/2} S\| \leq \sqrt{\kappa_x}.$$

This means that

$$\|(S_z + \mu I)^{-1} S\| \leq \|(S_z + \mu I)^{-1/2}\| \cdot \|(S_z + \mu I)^{-1/2} S\| \leq \mu^{-1/2} \sqrt{\kappa_x}. \quad \square$$

E.2 High probability events

Lemma E.3 (Hilbert Schmidt bounds for primitive events). *$\|\hat{S} - S\|_{HS}$ and $\|\mathbb{E}_n(\phi_{Z_i} \varepsilon_i)\|_{HS}$ are bounded each with probability $1 - \eta$ for $n \geq 4$ by*

$$\|\hat{S} - S\|_{HS} \leq 2\ln(2/\eta) \left(\frac{4\kappa}{n} \vee \frac{2\kappa}{n^{1/2}} \right) \leq 8\ln(2/\eta) \frac{\kappa}{n^{1/2}} := \tilde{\gamma}_1$$

and

$$\|\mathbb{E}_n(\phi_{Z_i} \varepsilon_i)\|_{HS} \leq 2\ln(2/\eta) \left(\frac{2\kappa_z \bar{\sigma}}{n} + \frac{\kappa_z \bar{\sigma}}{n^{1/2}} \right) \leq 4\ln(2/\eta) \frac{\kappa_z \bar{\sigma}}{n^{1/2}} := \tilde{\gamma}_2.$$

Proof. For both bounds, we utilize Lemma B.8. For $\|\hat{S} - S\|_{HS}$, note that $\mathbb{E}(S_i - S) = 0$ and by the boundedness of the kernel $\|S_i - S\|_{HS} \leq 2\kappa$ and $\mathbb{E}\|S_i - S\|_{HS}^2 \leq (2\kappa)^2$. Let $A = 4\kappa$ and $B = 2\kappa$; the bound follows from concentration. For $\|\mathbb{E}_n(\phi_{Z_i} \varepsilon_i)\|_{HS}$ note that $\mathbb{E}(\phi_{Z_i} \varepsilon_i) = 0$. Also, $\|\phi_{Z_i} \varepsilon_i\|_{HS} \leq \kappa_z \bar{\sigma}$ and $\mathbb{E}(\|\phi_{Z_i} \varepsilon_i\|_{HS}^2) \leq \kappa_z^2 \bar{\sigma}^2$. Let $A = 2\kappa_z \bar{\sigma}$ and $B = \kappa_z \bar{\sigma}$; the bound follows from concentration. \square

Lemma E.4 (High-probability bound for preconditioned Z -covariances). *The Hilbert–Schmidt norm of the projected estimation error of S_z is bounded with high probability: $\|(S_z + \mu)^{-1} (\hat{S}_z - S_z)\|_{HS} \leq \delta_z$ with probability $1 - \eta$, where*

$$\delta_z = 2\kappa_z \ln(2/\eta) \left\{ \sqrt{\frac{\mathfrak{n}_z(\mu)}{n}} \vee \frac{4\kappa_z}{n\mu} \right\}.$$

A sufficient condition for ensuring $\delta_z \leq \frac{1}{2}$ is

$$n \geq 16\kappa_z^2 \ln(2/\eta) \left\{ \ln(2/\eta) \mathbf{n}_z(\mu) \vee \mu^{-1} \right\} := N_{\delta_z}.$$

Proof. The proof follows directly from matching symbols to Lemma F.2 of Singh and Vijayku-
mar (2023). The rate condition is simply the result of solving for n , in $\delta_z \leq 1/2$. \square

Lemma E.5 (High-probability bound on the projected feature–noise). *The projected empirical mean of the feature-noise is bounded with high probability: $\|(S_z + \mu)^{-1} \mathbb{E}_n(\phi_{Z_i} \varepsilon_i)\| \leq \tilde{\gamma}_3$ with probability $1 - \eta$, where*

$$\tilde{\gamma}_3 = 2 \ln(2/\eta) \left\{ \frac{2\bar{\sigma}\kappa_z}{n\mu} \vee \sqrt{\frac{\bar{\sigma}^2 \mathbf{n}_z(\mu)}{n}} \right\}.$$

Proof. Note that $\mathbb{E}\{(S_z + \mu)^{-1} \phi_{Z_i} \varepsilon_i\} = 0$ and also $\|(S_z + \mu)^{-1} \mathbb{E}_n(\phi_{Z_i} \varepsilon_i)\| \leq \frac{\bar{\sigma}\kappa_z}{\mu} \implies A = \frac{2\bar{\sigma}\kappa_z}{\mu}$. For the second moment, let (ν_j^z, e_j^z) be the eigendecomposition. Then

$$\mathbb{E} \|(S_z + \mu)^{-1} \phi_{Z_i} \varepsilon_i\|^2 \leq \bar{\sigma}^2 \sum_{j=1}^{\infty} \frac{\mathbb{E} \langle \phi_{Z_i}, e_j^z \rangle^2}{(\nu_j^z + \mu)^2} = \bar{\sigma}^2 \sum_{j=1}^{\infty} \frac{\nu_j^z}{(\nu_j^z + \mu)^2} = \bar{\sigma}^2 \mathbf{n}_z(\mu) \implies B = \bar{\sigma} \sqrt{\mathbf{n}_z(\mu)}.$$

Again, following Lemma B.8 with probability $1 - \eta$

$$\|(S_z + \mu)^{-1} \mathbb{E}_n(\phi_{Z_i} \varepsilon_i)\| \leq 2 \ln(2/\eta) \left\{ \frac{2\bar{\sigma}\kappa_z}{n\mu} \vee \sqrt{\frac{\bar{\sigma}^2 \mathbf{n}_z(\mu)}{n}} \right\} = \tilde{\gamma}_3. \quad \square$$

Lemma E.6 (Compounded operator bound). *With probability $1 - 2\eta$ the projected com-
pounded feature noise $T_{\mu, \lambda}^{-1} (\hat{S} - S)^* (\hat{S}_z + \mu)^{-1} \mathbb{E}_n(\phi_Z \varepsilon)$ is bounded:*

$$\left\| T_{\mu, \lambda}^{-1} (\hat{S} - S)^* (\hat{S}_z + \mu)^{-1} \mathbb{E}_n(\phi_Z \varepsilon) \right\| \leq \gamma_1 = \frac{\tilde{\gamma}_1 \tilde{\gamma}_2}{\lambda \mu}.$$

Proof. Note that,

$$\left\| T_{\mu, \lambda}^{-1} (\hat{S} - S)^* (\hat{S}_z + \mu)^{-1} \mathbb{E}_n(\phi_Z \varepsilon) \right\| \leq \frac{1}{\lambda \mu} \| (\hat{S} - S)^* \| \| \mathbb{E}_n(\phi_Z \varepsilon) \|.$$

Invoking Lemma E.3 plus an union bound gives the result. \square

Lemma E.7 (Inverse estimation error). Suppose, $\|(S_z + \mu)^{-1}(\hat{S}_z - S_z)\| = \delta_z \leq \frac{1}{2}$, then,

$$\|(\hat{S}_z + \mu)^{-1} - (S_z + \mu)^{-1}\| \leq \frac{\delta_z(1+2\delta_z)}{\mu}.$$

Proof. By the resolvent identity, $\forall l \geq 1$,

$$(\hat{S}_z + \mu)^{-1} - (S_z + \mu)^{-1} = (\hat{S}_z + \mu)^{-1} \{(S_z - \hat{S}_z)(S_z + \mu)^{-1}\}^l + \sum_{r=1}^{l-1} (S_z + \mu)^{-1} \{(S_z - \hat{S}_z)(S_z + \mu)^{-1}\}^r.$$

Now, let $l \rightarrow \infty$. Since $\|(S_z + \mu)^{-1}(\hat{S}_z - S_z)\| = \delta_z \leq \frac{1}{2}$ we have $\|S^*[(\hat{S}_z + \mu)^{-1} \{(S_z - \hat{S}_z)(S_z + \mu)^{-1}\}^l]\| \rightarrow 0$. Thus,

$$\underbrace{(S_z + \mu)^{-1}(S_z - \hat{S}_z)(S_z + \mu)^{-1}}_A + \underbrace{\sum_{r=2}^{\infty} (S_z + \mu)^{-1} \{(S_z - \hat{S}_z)(S_z + \mu)^{-1}\}^r}_B.$$

By hypothesis, $\|A\| \leq \frac{\delta_z}{\mu}$ and $\|B\| \leq \frac{\delta_z^2}{\mu(1-\delta_z)} \leq \frac{2\delta_z^2}{\mu}$. Thus, $\|A+B\| \leq \frac{\delta_z(1+2\delta_z)}{\mu}$. \square

Lemma E.8 (Projected inverse estimation error). Suppose $\|(S_z + \mu)^{-1}(\hat{S}_z - S_z)\| = \delta_z \leq \frac{1}{2}$. Then, $\left\| T_{\mu,\lambda}^{-1} S^* \left\{ (\hat{S}_z + \mu)^{-1} - (S_z + \mu)^{-1} \right\} \right\| \leq \frac{\delta_z(1+2\delta_z)}{2\sqrt{\lambda\mu}}$.

Proof. By the resolvent identity, $\forall l \geq 1$,

$$(\hat{S}_z + \mu)^{-1} - (S_z + \mu)^{-1} = (\hat{S}_z + \mu)^{-1} \{(S_z - \hat{S}_z)(S_z + \mu)^{-1}\}^l + \sum_{r=1}^{l-1} (S_z + \mu)^{-1} \{(S_z - \hat{S}_z)(S_z + \mu)^{-1}\}^r.$$

Now, let $l \rightarrow \infty$. Since $\|(S_z + \mu)^{-1}(\hat{S}_z - S_z)\| = \delta_z \leq \frac{1}{2}$ we have $\|S^*[(\hat{S}_z + \mu)^{-1} \{(S_z - \hat{S}_z)(S_z + \mu)^{-1}\}^l]\| \rightarrow 0$. Thus,

$$\begin{aligned} \left\| T_{\mu,\lambda}^{-1} S^* \left\{ (\hat{S}_z + \mu)^{-1} - (S_z + \mu)^{-1} \right\} \right\| &\leq \|T_{\mu,\lambda}^{-1} S^*(S_z + \mu)^{-1}(S_z - \hat{S}_z)(S_z + \mu)^{-1}\| \\ &\quad + \|T_{\mu,\lambda}^{-1} S^*(S_z + \mu)^{-1} \sum_{r=2}^{\infty} \{(S_z - \hat{S}_z)(S_z + \mu)^{-1}\}^r\| \\ &\leq \frac{\delta_z}{2\sqrt{\mu\lambda}} + \frac{1}{2\sqrt{\mu\lambda}} \frac{\delta_z^2}{1-\delta_z} \leq \frac{\delta_z}{2\sqrt{\mu\lambda}} + \frac{2\delta_z^2}{2\sqrt{\mu\lambda}} = \frac{\delta_z(1+2\delta_z)}{2\sqrt{\mu\lambda}}, \end{aligned}$$

where the first inequality in the last line follows from Lemma B.3 with $A^* = S^*(S_z + \mu)^{1/2}$ and by the hypothesis. \square

Lemma E.9 (Sandwich error bound). *The projected error $T_{\mu,\lambda}^{-1}(\hat{S}-S)^*(\hat{S}_z+\mu)^{-1}(\hat{S}-S)$ is bounded with high-probability: with probability $1-\eta$,*

$$\left\| T_{\mu,\lambda}^{-1}(\hat{S}-S)^*(\hat{S}_z+\mu)^{-1}(\hat{S}-S) \right\| \leq \delta_1 = \frac{\tilde{\gamma}_1^2}{\lambda\mu}.$$

Proof. Note that,

$$\left\| T_{\mu,\lambda}^{-1}(\hat{S}-S)^*(\hat{S}_z+\mu)^{-1}(\hat{S}-S) \right\| \leq \frac{1}{\lambda\mu} \|(\hat{S}-S)^*\|^2.$$

Invoking Lemma E.3 gives the result. \square

Lemma E.10 (High-probability bound for the operator deviation). *The scaled estimation error of the regularized covariance operator T_μ is bound with high probability: with probability $1-3\eta$,*

$$\left\| T_{\mu,\lambda}^{-1}(\hat{T}_\mu - T_\mu) \right\|_{op} \leq \delta := \frac{\kappa\delta_z\tilde{\gamma}_1 + \tilde{\gamma}_1^2}{\lambda\mu} + \frac{\kappa\delta_z(1+2\delta_z) + \tilde{\gamma}_1}{2\sqrt{\lambda\mu}} + \tilde{\delta}(\mu, \lambda, \eta),$$

where $\tilde{\delta}(\mu, \lambda, n) = 2\ln(2/\eta) \left(\frac{4\kappa\kappa_x^{1/2}}{n\lambda\mu^{1/2}} \vee \sqrt{\frac{\kappa\kappa_x^{1/2}}{n\mu^{1/2}} \mathfrak{m}(\lambda, \mu)} \right)$.

Proof. We can write

$$\begin{aligned} T_{\mu,\lambda}^{-1}(\hat{T}_\mu - T_\mu) &= T_{\mu,\lambda}^{-1}\{\hat{S}^*(\hat{S}_z + \mu)^{-1}\hat{S} - S^*(S_z + \mu)^{-1}S\} \\ &= \underbrace{T_{\mu,\lambda}^{-1}\{\hat{S}^*(\hat{S}_z + \mu)^{-1}\hat{S} - S^*(\hat{S}_z + \mu)^{-1}\hat{S}\}}_{(I)} \\ &\quad + \underbrace{T_{\mu,\lambda}^{-1}\{S^*(\hat{S}_z + \mu)^{-1}\hat{S} - S^*(S_z + \mu)^{-1}\hat{S}\}}_{(II)} \\ &\quad + \underbrace{T_{\mu,\lambda}^{-1}\{S^*(S_z + \mu)^{-1}\hat{S} - S^*(S_z + \mu)^{-1}S\}}_{(III)}. \end{aligned}$$

We aim to bound the following terms:

$$\begin{aligned}
(I) \quad & \left\| T_{\mu,\lambda}^{-1} (\hat{S} - S)^* (\hat{S}_z + \mu)^{-1} \hat{S} \right\| = \left\| T_{\mu,\lambda}^{-1} (\hat{S} - S)^* (\hat{S}_z + \mu)^{-1} (S_z - \hat{S}_z) (S_z + \mu)^{-1} \hat{S} \right. \\
& \quad \left. + T_{\mu,\lambda}^{-1} (\hat{S} - S)^* (S_z + \mu)^{-1} \hat{S} \pm T_{\mu,\lambda}^{-1} (\hat{S} - S)^* (S_z + \mu)^{-1} S \right\| \\
& = \left\| T_{\mu,\lambda}^{-1} (\hat{S} - S)^* (\hat{S}_z + \mu)^{-1} (S_z - \hat{S}_z) (S_z + \mu)^{-1} \hat{S} \right. \\
& \quad \left. + T_{\mu,\lambda}^{-1} (\hat{S} - S)^* (S_z + \mu)^{-1} (\hat{S} - S) + T_{\mu,\lambda}^{-1} (\hat{S} - S)^* (S_z + \mu)^{-1} S \right\| \\
& \leq \underbrace{\left\| T_{\mu,\lambda}^{-1} (\hat{S} - S)^* (\hat{S}_z + \mu)^{-1} (S_z - \hat{S}_z) (S_z + \mu)^{-1} \hat{S} \right\|}_{(I.1)} \\
& \quad + \underbrace{\left\| T_{\mu,\lambda}^{-1} (\hat{S} - S)^* (S_z + \mu)^{-1} (\hat{S} - S) \right\|}_{(I.2)} \\
& \quad + \underbrace{\left\| T_{\mu,\lambda}^{-1} (\hat{S} - S)^* (S_z + \mu)^{-1} S \right\|}_{(I.3)}.
\end{aligned}$$

We continue by again analyzing the individual terms

$$(I.1) \quad \left\| T_{\mu,\lambda}^{-1} (\hat{S} - S)^* (\hat{S}_z + \mu)^{-1} (S_z - \hat{S}_z) (S_z + \mu)^{-1} \hat{S} \right\| \leq \frac{\kappa \delta_z}{\lambda \mu} \|(\hat{S} - S)^*\| \leq \frac{\kappa \delta_z \tilde{\gamma}_1}{\lambda \mu} \text{ with probability } 1 - \eta$$

using the hypothesis and Lemma E.3.

$$(I.2) \quad \left\| T_{\mu,\lambda}^{-1} (\hat{S} - S)^* (S_z + \mu)^{-1} (\hat{S} - S) \right\| \leq \frac{\tilde{\gamma}_1^2}{\lambda \mu} \text{ with probability } 1 - \eta \text{ using Lemma E.3 again.}$$

(I.3) This becomes more involved. Note that $\mathbb{E}\{T_{\mu,\lambda}^{-1} (S_i - S)^* (S_z + \mu)^{-1} S\} = 0$. Also,

$$\|T_{\mu,\lambda}^{-1} (S_i - S)^* (S_z + \mu)^{-1} S\| \leq \frac{2\kappa}{\lambda} \|(S_z + \mu)^{-1} S\| \leq \frac{2\kappa \kappa_x^{1/2}}{\lambda \mu^{1/2}}$$

by Lemma E.2, hence $A := \frac{4\kappa\kappa_x^{1/2}}{\lambda\mu^{1/2}}$. The second moment is given by

$$\begin{aligned}
\mathbb{E}\{\|T_{\mu,\lambda}^{-1}(S_i - S)^*(S_z + \mu)^{-1}S\|^2\} &= \mathbb{E}\{\text{tr}(T_{\mu,\lambda}^{-1}(S_i - S)^*(S_z + \mu)^{-1}S)^*(T_{\mu,\lambda}^{-1}(S_i - S)^*(S_z + \mu)^{-1}S)\} \\
&= \mathbb{E}\{\text{tr}S^*(S_z + \mu)^{-1}(S_i - S)T_{\mu,\lambda}^{-2}(S_i - S)^*(S_z + \mu)^{-1}S\} \\
&\leq \mathbb{E}\{\text{tr}S^*(S_z + \mu)^{-1}S_i T_{\mu,\lambda}^{-2} S_i^* (S_z + \mu)^{-1}S\} \\
&\leq \frac{\kappa\kappa_x^{1/2}}{\sqrt{\mu}} \mathbb{E}\{\text{tr}T_{\mu,\lambda}^{-2} S_i^* (S_z + \mu)^{-1}S\} \\
&= \frac{\kappa\kappa_x^{1/2}}{\sqrt{\mu}} \text{tr}T_{\mu,\lambda}^{-1} S^* (S_z + \mu)^{-1}S T_{\mu,\lambda}^{-1} \\
&\leq \frac{\kappa\kappa_x^{1/2}}{\sqrt{\mu}} \text{tr}T_{\mu,\lambda}^{-1} S^* S_z^{-1} S T_{\mu,\lambda}^{-1} = \frac{\kappa\kappa_x^{1/2}}{\sqrt{\mu}} \mathbf{m}(\lambda, \mu).
\end{aligned}$$

Thus, $B^2 = \frac{\kappa\kappa_x^{1/2}}{\sqrt{\mu}} \mathbf{m}(\lambda, \mu)$ and with probability $1 - \eta$, we can bound (I.3) with

$$\left\| T_{\mu,\lambda}^{-1}(\hat{S} - S)^*(S_z + \mu)^{-1}S \right\| \leq 2\ln(2/\eta) \left(\frac{4\kappa\kappa_x^{1/2}}{n\lambda\mu^{1/2}} \vee \sqrt{\frac{\kappa\kappa_x^{1/2}}{n\mu^{1/2}} \mathbf{m}(\lambda, \mu)} \right) := \tilde{\delta}(\mu, \lambda, \eta).$$

Now combining this with (I.1) and (I.2) with probability $1 - 2\eta$

$$\left\| T_{\mu,\lambda}^{-1}(\hat{S} - S)^*(\hat{S}_z + \mu)^{-1}\hat{S} \right\| \leq \frac{\kappa\delta_z\tilde{\gamma}_1}{\lambda\mu} + \frac{\tilde{\gamma}_1^2}{\lambda\mu} + \tilde{\delta}(\mu, \lambda, \eta).$$

We turn to the remaining terms.

(II) $\left\| T_{\mu,\lambda}^{-1}S^*\left\{(\hat{S}_z + \mu)^{-1} - (S_z + \mu)^{-1}\right\}\hat{S} \right\| \leq \frac{\kappa\delta_z(1+2\delta_z)}{2\sqrt{\lambda\mu}}$. This follows directly by applying Lemma E.8.

(III) Note that $\left\| T_{\mu,\lambda}^{-1}S^*(S_z + \mu)^{-1}(\hat{S} - S) \right\| \leq \|T_{\mu,\lambda}^{-1}S^*(S_z + \mu)^{-1}\| \|(\hat{S} - S)\| \leq \frac{\tilde{\gamma}_1}{2\sqrt{\lambda\mu}}$. This holds on the event from Lemma E.3 and using Lemma B.3 with $A^* = S^*(S_z + \mu)^{1/2}$. Combining the three bounds with a union for the distinct high probability events involved, we have with probability $1 - 3\eta$

$$\left\| T_{\mu,\lambda}^{-1}(\hat{T}_\mu - T_\mu) \right\|_{op} \leq \delta = \frac{\kappa\delta_z\tilde{\gamma}_1 + \tilde{\gamma}_1^2}{\lambda\mu} + \frac{\kappa\delta_z(1+2\delta_z) + \tilde{\gamma}_1}{2\sqrt{\lambda\mu}} + \tilde{\delta}(\mu, \lambda, \eta). \quad \square$$

Lemma E.11 (Linearization). Suppose $\|T_{\mu,\lambda}^{-1}(\hat{T}_\mu - T_\mu)\|_{\text{HS}} \leq \delta < 1$. Then, for all $k \geq 1$

$$(\hat{T}_{\mu,\lambda}^{-1} - T_{\mu,\lambda}^{-1})u = A_1 u + A_2 T_{\mu,\lambda}^{-1}u + A_3 T_{\mu,\lambda}^{-1}u, \quad \|A_1\|_{\text{HS}} \leq \frac{\delta^k}{\lambda}, \quad \|A_2\|_{\text{HS}} \leq \delta, \quad \|A_3\|_{\text{HS}} \leq \frac{\delta^2}{1-\delta}.$$

Furthermore, when $\delta \leq 1/2$ then, $\|(\hat{T}_{\mu,\lambda}^{-1} - T_{\mu,\lambda}^{-1})u\| \leq 2\delta\|T_{\mu,\lambda}^{-1}u\|$.

Proof. By the iterated resolvent identity Lemma B.1,

$$\begin{aligned} \hat{T}_{\mu,\lambda}^{-1} - T_{\mu,\lambda}^{-1} &= \hat{T}_{\mu,\lambda}^{-1}\{(T_\mu - \hat{T}_\mu)T_{\mu,\lambda}^{-1}\}^k + \sum_{r=1}^{k-1} T_{\mu,\lambda}^{-1}\{(T_\mu - \hat{T}_\mu)T_{\mu,\lambda}^{-1}\}^r \\ &= \hat{T}_{\mu,\lambda}^{-1}\{(T_\mu - \hat{T}_\mu)T_{\mu,\lambda}^{-1}\}^k + T_{\mu,\lambda}^{-1}(T_\mu - \hat{T}_\mu)T_{\mu,\lambda}^{-1} + \sum_{r=2}^{k-1} T_{\mu,\lambda}^{-1}\{(T_\mu - \hat{T}_\mu)T_{\mu,\lambda}^{-1}\}^r \\ &= A_1 + A_2 T_{\mu,\lambda}^{-1} + A_3 T_{\mu,\lambda}^{-1}. \end{aligned}$$

Now, using $\|T_{\mu,\lambda}^{-1}(\hat{T}_\mu - T_\mu)\|_{\text{HS}} \leq \delta < 1$,

$$\begin{aligned} \|A_1\|_{\text{HS}} &= \|\hat{T}_{\mu,\lambda}^{-1}\{(T_\mu - \hat{T}_\mu)T_{\mu,\lambda}^{-1}\}^k\|_{\text{HS}} \leq \frac{\delta^k}{\lambda} \\ \|A_2\|_{\text{HS}} &= \|T_{\mu,\lambda}^{-1}(T_\mu - \hat{T}_\mu)\|_{\text{HS}} \leq \delta \\ \|A_3\|_{\text{HS}} &= \left\| \sum_{r=2}^{k-1} \{T_{\mu,\lambda}^{-1}\{(T_\mu - \hat{T}_\mu)\}^r\} \right\|_{\text{HS}} \leq \sum_{r=2}^{k-1} \delta^r \leq \sum_{r=2}^{\infty} \delta^r \leq \frac{\delta^2}{1-\delta}. \end{aligned}$$

If in addition we assume $\delta \leq 1/2$, then

$$\begin{aligned} \|(\hat{T}_{\mu,\lambda}^{-1} - T_{\mu,\lambda}^{-1})u\| &\leq \|A_1\|_{\text{HS}}\|u\| + \|A_2\|_{\text{HS}}\|T_{\mu,\lambda}^{-1}u\| + \|A_3\|_{\text{HS}}\|T_{\mu,\lambda}^{-1}u\| \\ &\leq \frac{\delta^k}{\lambda}\|u\| + (\delta + \frac{\delta^2}{1-\delta})\|T_{\mu,\lambda}^{-1}u\| \\ &= \frac{\delta^k}{\lambda}\|u\| + \frac{\delta}{1-\delta}\|T_{\mu,\lambda}^{-1}u\| \underset{k \rightarrow \infty, \delta \leq 1/2}{\leq} 2\delta\|T_{\mu,\lambda}^{-1}u\|. \quad \square \end{aligned}$$

Lemma E.12 (Projected inverse estimation error with noise bound). Suppose $\|(S_z + \mu)^{-1}(\hat{S}_z - S_z)\|_{\text{HS}} \leq \delta_z \leq \frac{1}{2}$. Then, with probability $1 - \eta$, $\|T_{\mu,\lambda}^{-1}S^*\{(\hat{S}_z + \mu)^{-1} - (S_z + \mu)^{-1}\}\mathbb{E}_n(\phi_Z \varepsilon)\| \leq \xi_1 = \frac{\delta_z \tilde{\gamma}_2}{\sqrt{\lambda \mu}}$.

Proof. Using the resolvent identity, we can write for all $l \geq 1$

$$(\hat{S}_z + \mu)^{-1} - (S_z + \mu)^{-1} = (\hat{S}_z + \mu)^{-1} \{ (\hat{S}_z - S_z)(S_z + \mu)^{-1} \}^l + \sum_{r=1}^{l-1} (S_z + \mu)^{-1} \{ (\hat{S}_z - S_z)(S_z + \mu)^{-1} \}^r.$$

For $l \rightarrow \infty$ using the same argument as in Lemma E.11,

$$\begin{aligned} \|T_{\mu,\lambda}^{-1} S^* \{ (\hat{S}_z + \mu)^{-1} - (S_z + \mu)^{-1} \} \mathbb{E}_n(\phi_Z \varepsilon) \| &= \|T_{\mu,\lambda}^{-1} S^* (S_z + \mu)^{-1} \sum_{r=1}^{\infty} \{ (\hat{S}_z - S_z)(S_z + \mu)^{-1} \}^r \mathbb{E}_n(\phi_Z \varepsilon) \| \\ &\leq \|T_{\mu,\lambda}^{-1} S^* (S_z + \mu)^{-1}\| \left\| \sum_{r=1}^{\infty} \{ (\hat{S}_z - S_z)(S_z + \mu)^{-1} \}^r \mathbb{E}_n(\phi_Z \varepsilon) \right\| \\ &\leq \frac{1}{2\sqrt{\lambda\mu}} 2\delta_z \tilde{\gamma}_2 = \frac{\delta_z \tilde{\gamma}_2}{\sqrt{\lambda\mu}}. \quad \square \end{aligned}$$

Lemma E.13 (Empirical projected noise bound). *Suppose $\|(S_z + \mu)^{-1}(\hat{S}_z - S_z)\|_{HS} \leq \delta_z \leq \frac{1}{2}$. Then, with probability at least $1 - 3\eta$ it holds that*

$$\left\| T_{\mu,\lambda}^{-1} \hat{S}^* (\hat{S}_z + \mu)^{-1} \mathbb{E}_n\{\phi(Z) \varepsilon\} \right\| \leq \gamma = \frac{\kappa 2\delta_z \tilde{\gamma}_3}{\lambda} + \frac{\tilde{\gamma}_1 \tilde{\gamma}_3}{\lambda} + \frac{\tilde{\gamma}_2}{2\sqrt{\lambda\mu}}.$$

Proof. Note that,

$$\begin{aligned} &T_{\mu,\lambda}^{-1} \hat{S}^* (\hat{S}_z + \mu)^{-1} \mathbb{E}_n\{\phi_Z \varepsilon\} \pm T_{\mu,\lambda}^{-1} \hat{S}^* (S_z + \mu)^{-1} \mathbb{E}_n\{\phi_Z \varepsilon\} \\ &= T_{\mu,\lambda}^{-1} \hat{S}^* \{ (\hat{S}_z + \mu)^{-1} - (S_z + \mu)^{-1} \} \mathbb{E}_n(\phi_Z \varepsilon) \\ &\quad + T_{\mu,\lambda}^{-1} \hat{S}^* (S_z + \mu)^{-1} \mathbb{E}_n(\phi_Z \varepsilon) \pm T_{\mu,\lambda}^{-1} S^* (S_z + \mu)^{-1} \mathbb{E}_n(\phi_Z \varepsilon) \\ &= \underbrace{T_{\mu,\lambda}^{-1} \hat{S}^* \{ (\hat{S}_z + \mu)^{-1} - (S_z + \mu)^{-1} \} \mathbb{E}_n(\phi_Z \varepsilon)}_{\text{I}} \\ &\quad + \underbrace{T_{\mu,\lambda}^{-1} (\hat{S} - S)^* (S_z + \mu)^{-1} \mathbb{E}_n(\phi_Z \varepsilon)}_{\text{II}} \\ &\quad + \underbrace{T_{\mu,\lambda}^{-1} S^* (S_z + \mu)^{-1} \mathbb{E}_n(\phi_Z \varepsilon)}_{\text{III}}. \end{aligned}$$

(I) In the first term,

$$\begin{aligned}\|T_{\mu,\lambda}^{-1}\hat{S}^*\{(\hat{S}_z+\mu)^{-1}-(S_z+\mu)^{-1}\}\mathbb{E}_n(\phi_Z\varepsilon)\| &\leq \frac{\kappa}{\lambda}\|\{(\hat{S}_z+\mu)^{-1}-(S_z+\mu)^{-1}\}\mathbb{E}_n(\phi_Z\varepsilon)\| \\ &\leq \frac{2\kappa\delta_z\tilde{\gamma}_3}{\lambda}.\end{aligned}$$

This follows by the event from Lemma E.5, using the assumption and the same application of the resolvent identity as in Lemma E.12.

(II) In the second term,

$$\|T_{\mu,\lambda}^{-1}(\hat{S}-S)^*(S_z+\mu)^{-1}\mathbb{E}_n(\phi_z\varepsilon)\| \leq \frac{1}{\lambda}\|(\hat{S}-S)^*\|\|(S_z+\mu)^{-1}\mathbb{E}_n(\phi_z\varepsilon)\| \leq \frac{1}{\lambda}\tilde{\gamma}_1\tilde{\gamma}_3$$

on the events from Lemmas E.3 and E.5.

(III) In the third term,

$$\|T_{\mu,\lambda}^{-1}S^*(S_z+\mu)^{-1}\mathbb{E}_n(\phi_z\varepsilon)\| \leq \|T_{\mu,\lambda}^{-1}S^*(S_z+\mu)^{-1}\|\|\mathbb{E}_n(\phi_z\varepsilon)\| \leq \frac{\tilde{\gamma}_2}{2\sqrt{\lambda\mu}},$$

on the event from Lemma E.3, using Lemma B.3 with $A^*=S^*(S_z+\mu)^{1/2}$.

Combining all three bounds, we get with probability at least $1-3\eta$

$$\gamma = \frac{\kappa 2\delta_z\tilde{\gamma}_3}{\lambda} + \frac{\tilde{\gamma}_1\tilde{\gamma}_3}{\lambda} + \frac{\tilde{\gamma}_2}{2\sqrt{\lambda\mu}}. \quad \square$$

E.3 Main result

Lemma E.14 (Abstract Bahadur representation). *Suppose*

$$\left\|T_{\mu,\lambda}^{-1}\hat{S}^*(\hat{S}_z+\mu)^{-1}\mathbb{E}_n\{\phi(Z)\varepsilon\}\right\| \leq \gamma, \quad \|(S_z+\mu)^{-1}(\hat{S}_z-S_z)\| \leq \delta_z \leq \frac{1}{2}, \quad \left\|T_{\mu,\lambda}^{-1}(\hat{T}_\mu-T_\mu)\right\|_{op} \leq \delta \leq \frac{1}{2},$$

$$\left\|T_{\mu,\lambda}^{-1}(\hat{S}-S)^*(\hat{S}_z+\mu)^{-1}\mathbb{E}_n(\phi_Z\varepsilon)\right\| \leq \gamma_1, \quad \|\hat{S}-S\| \leq \tilde{\gamma}_1,$$

and

$$\|T_{\mu,\lambda}^{-1}S^*\{(\hat{S}_z+\mu)^{-1}-(S_z+\mu)^{-1}\}\mathbb{E}_n(\phi_Z\varepsilon)\| \leq \xi_1.$$

Then,

$$\hat{h} - h_{\mu,\lambda} = \mathbb{E}_n(U_i) + u, \quad \|u\| \leq 2\delta\gamma + \gamma_1 + \xi_1 + (2\delta^2 + R)\|h_0 - h_{\mu,\lambda}\|$$

with

$$R := \frac{\delta_z}{\sqrt{\lambda\mu}} \{ \delta_z(\kappa + \tilde{\gamma}_1) + \tilde{\gamma}_1/2 \} + \frac{\tilde{\gamma}_1}{\lambda\mu} \{ \tilde{\gamma}_1 + \delta_z(1 + 2\delta_z)(\kappa + \tilde{\gamma}_1) \}$$

and

$$\begin{aligned} U_i &= T_{\mu,\lambda}^{-1} \{ S^*(S_z + \mu)^{-1}(S_i - S) + (S_i - S)^*(S_z + \mu)^{-1}S \\ &\quad + S^*(S_z + \mu)^{-1}(S_z - S_{z,i})(S_z + \mu)^{-1}S \} (h_0 - h_{\mu,\lambda}) + T_{\mu,\lambda}^{-1} S^*(S_z + \mu)^{-1} \phi(Z_i) \varepsilon_i. \end{aligned}$$

Subsequently,

$$\begin{aligned} \mathbb{E}_n(U) &= T_{\mu,\lambda}^{-1} \{ S^*(S_z + \mu)^{-1}(\hat{S} - S) + (\hat{S} - S)^*(S_z + \mu)^{-1}S \} (h_0 - h_{\mu,\lambda}) \\ &\quad + T_{\mu,\lambda}^{-1} S^*(S_z + \mu)^{-1} \mathbb{E}_n(\phi_Z \varepsilon) \\ &\quad + T_{\mu,\lambda}^{-1} \{ S^*(S_z + \mu)^{-1}(S_z - \hat{S}_z)(S_z + \mu)^{-1}S \} (h_0 - h_{\mu,\lambda}). \end{aligned}$$

Proof. We proceed in steps

1. First, we decompose

$$\hat{h} = \hat{T}_{\mu,\lambda} \hat{S}^*(\hat{S}_z + \mu)^{-1} \mathbb{E}_n\{Y \phi(Z)\}, \quad h_{\mu,\lambda} = T_{\mu,\lambda}^{-1} T_\mu h_0.$$

Thus,

$$\begin{aligned} \hat{h} - h_{\mu,\lambda} &= \hat{T}_{\mu,\lambda}^{-1} \hat{S}^*(\hat{S}_z + \mu)^{-1} \mathbb{E}_n\{Y \phi(Z)\} - T_{\mu,\lambda}^{-1} T_\mu h_0 \\ &= \hat{T}_{\mu,\lambda}^{-1} \hat{S}^*(\hat{S}_z + \mu)^{-1} \mathbb{E}_n[\{h_0(X) + \varepsilon\} \phi(Z)] - T_{\mu,\lambda}^{-1} T_\mu h_0 \\ &= \hat{T}_{\mu,\lambda}^{-1} \hat{S}^*(\hat{S}_z + \mu)^{-1} \mathbb{E}_n(\phi_Z \varepsilon) + \hat{T}_{\mu,\lambda}^{-1} \hat{T}_\mu h_0 - T_{\mu,\lambda}^{-1} T_\mu h_0 \\ &= (i) + (ii). \end{aligned}$$

Here, $(i) = \hat{T}_{\mu,\lambda}^{-1} \hat{S}^*(\hat{S}_z + \mu)^{-1} \mathbb{E}_n(\phi_Z \varepsilon)$ and $(ii) = \hat{T}_{\mu,\lambda}^{-1} \hat{T}_\mu h_0 - T_{\mu,\lambda}^{-1} T_\mu h_0$.

2. Observe that $(i) = (\hat{T}_{\mu,\lambda}^{-1} - T_{\mu,\lambda}^{-1}) P_n + T_{\mu,\lambda}^{-1} P_n$ where $P_n = \hat{S}^*(\hat{S}_z + \mu)^{-1} \mathbb{E}_n(\phi_Z \varepsilon)$. This

follows by writing

$$\hat{T}_{\mu,\lambda}^{-1} = (\hat{T}_{\mu,\lambda}^{-1} - T_{\mu,\lambda}^{-1}) + T_{\mu,\lambda}^{-1}.$$

Invoking Lemma E.11 and the assumed high probability events, the first term can be bounded by

$$\|(\hat{T}_{\mu,\lambda}^{-1} - T_{\mu,\lambda}^{-1})P_n\| \leq 2\delta \|T_{\mu,\lambda}^{-1}P_n\| \leq 2\delta\gamma.$$

Developing the remaining term,

$$\begin{aligned} T_{\mu,\lambda}^{-1}P_n &= T_{\mu,\lambda}^{-1}(\hat{S} - S)^*(\hat{S}_z + \mu)^{-1}\mathbb{E}_n(\phi_Z\varepsilon) + T_{\mu,\lambda}^{-1}S^*(\hat{S}_z + \mu)^{-1}\mathbb{E}_n(\phi_Z\varepsilon) \\ &= T_{\mu,\lambda}^{-1}(\hat{S} - S)^*(\hat{S}_z + \mu)^{-1}\mathbb{E}_n(\phi_Z\varepsilon) + T_{\mu,\lambda}^{-1}S^*(\hat{S}_z + \mu)^{-1}\mathbb{E}_n(\phi_Z\varepsilon) \pm T_{\mu,\lambda}^{-1}S^*(S_z + \mu)^{-1}\mathbb{E}_n(\phi_Z\varepsilon) \\ &= T_{\mu,\lambda}^{-1}(\hat{S} - S)^*(\hat{S}_z + \mu)^{-1}\mathbb{E}_n(\phi_Z\varepsilon) + T_{\mu,\lambda}^{-1}S^*\{(\hat{S}_z + \mu)^{-1} - (S_z + \mu)^{-1}\}\mathbb{E}_n(\phi_Z\varepsilon) \\ &\quad + T_{\mu,\lambda}^{-1}S^*(S_z + \mu)^{-1}\mathbb{E}_n(\phi_Z\varepsilon), \end{aligned}$$

where by hypothesis $\|T_{\mu,\lambda}^{-1}(\hat{S} - S)^*(\hat{S}_z + \mu)^{-1}\mathbb{E}_n(\phi_Z\varepsilon)\| \leq \gamma_1$ and $\|T_{\mu,\lambda}^{-1}S^*\{(\hat{S}_z + \mu)^{-1} - (S_z + \mu)^{-1}\}\mathbb{E}_n(\phi_Z\varepsilon)\| \leq \xi_1$.

3. Next, we show that $(ii) = (\hat{T}_{\mu,\lambda}^{-1} - T_{\mu,\lambda}^{-1})Q_n + T_{\mu,\lambda}^{-1}Q_n$ for $Q_n = (\hat{T}_\mu - T_\mu)(h_0 - h_{\mu,\lambda})$. This argument is more involved. Note that

$$\begin{aligned} \hat{T}_{\mu,\lambda}^{-1}\hat{T}_\mu h_0 - T_{\mu,\lambda}^{-1}T_\mu h_0 &= (\hat{T}_{\mu,\lambda}^{-1} - T_{\mu,\lambda}^{-1})\hat{T}_\mu h_0 + T_{\mu,\lambda}^{-1}\hat{T}_\mu h_0 - T_{\mu,\lambda}^{-1}T_\mu h_0 \\ &= (\hat{T}_{\mu,\lambda}^{-1} - T_{\mu,\lambda}^{-1})(\hat{T}_\mu - T_\mu)h_0 + (\hat{T}_{\mu,\lambda}^{-1} - T_{\mu,\lambda}^{-1})T_\mu h_0 + T_{\mu,\lambda}^{-1}\hat{T}_\mu h_0 - T_{\mu,\lambda}^{-1}T_\mu h_0 \\ &= (\hat{T}_{\mu,\lambda}^{-1} - T_{\mu,\lambda}^{-1})(\hat{T}_\mu - T_\mu)h_0 - \hat{T}_{\mu,\lambda}^{-1}(\hat{T}_\mu - T_\mu)h_{\mu,\lambda} + T_{\mu,\lambda}^{-1}(\hat{T}_\mu - T_\mu)h_0, \end{aligned}$$

where in the last line we use the resolvent identity to write

$$(\hat{T}_{\mu,\lambda}^{-1} - T_{\mu,\lambda}^{-1})T_\mu h_0 = \hat{T}_{\mu,\lambda}^{-1}(T_\mu - \hat{T}_\mu)T_{\mu,\lambda}^{-1}T_\mu h_0 = \hat{T}_{\mu,\lambda}^{-1}(T_\mu - \hat{T}_\mu)h_{\mu,\lambda} = -\hat{T}_{\mu,\lambda}^{-1}(\hat{T}_\mu - T_\mu)h_{\mu,\lambda}.$$

To conclude the argument, we note that

$$\begin{aligned}
& -\hat{T}_{\mu,\lambda}^{-1}(\hat{T}_\mu - T_\mu)h_{\mu,\lambda} + T_{\mu,\lambda}^{-1}(\hat{T}_\mu - T_\mu)h_0 \pm T_{\mu,\lambda}^{-1}(\hat{T}_\mu - T_\mu)h_{\mu,\lambda} \\
& = (T_{\mu,\lambda}^{-1} - \hat{T}_{\mu,\lambda}^{-1})(\hat{T}_\mu - T_\mu)h_{\mu,\lambda} + T_{\mu,\lambda}^{-1}(\hat{T}_\mu - T_\mu)(h_0 - h_{\mu,\lambda}) \\
& = (\hat{T}_{\mu,\lambda}^{-1} - T_{\mu,\lambda}^{-1})(\hat{T}_\mu - T_\mu)(-h_{\mu,\lambda}) + T_{\mu,\lambda}^{-1}(\hat{T}_\mu - T_\mu)(h_0 - h_{\mu,\lambda}) \\
& = (\hat{T}_{\mu,\lambda}^{-1} - T_{\mu,\lambda}^{-1})(\hat{T}_\mu - T_\mu)(-h_{\mu,\lambda}) + T_{\mu,\lambda}^{-1}Q_n.
\end{aligned}$$

By linearization and the high probability events,

$$\|(\hat{T}_{\mu,\lambda}^{-1} - T_{\mu,\lambda}^{-1})Q_n\| \leq 2\delta \|T_{\mu,\lambda}^{-1}Q_n\| \leq 2\delta^2 \|h_0 - h_{\mu,\lambda}\|.$$

Developing the remaining term,

$$\begin{aligned}
T_{\mu,\lambda}^{-1}Q_n &= T_{\mu,\lambda}^{-1}(\hat{T}_\mu - T_\mu)(h_0 - h_{\mu,\lambda}) \\
&= T_{\mu,\lambda}^{-1}\{\hat{S}^*(\hat{S}_z + \mu)^{-1}\hat{S} - S^*(S_z + \mu)^{-1}S\}(h_0 - h_{\mu,\lambda}).
\end{aligned}$$

We apply Lemma B.2 to the expression giving us

$$\hat{S}^*(\hat{S}_z + \mu)^{-1}\hat{S} - S^*(S_z + \mu)^{-1}S.$$

Now we utilize Lemma B.2 with

$$\hat{A} = \hat{S}^*, \quad A = S^*, \quad \hat{B} = (\hat{S}_z + \mu)^{-1}, \quad B = (S_z + \mu)^{-1}, \quad \hat{C} = \hat{S}, \quad C = S,$$

and differences

$$\Delta A = \hat{S}^* - S^* := \Delta S^*, \quad \Delta B = (\hat{S}_z + \mu)^{-1} - (S_z + \mu)^{-1}, \quad \Delta C = \hat{S} - S := \Delta S.$$

Then, Lemma B.2 allows us to write

$$\begin{aligned}
\hat{S}^*(\hat{S}_z + \mu)^{-1}\hat{S} &= S^*(S_z + \mu)^{-1}S + S^*(S_z + \mu)^{-1}\Delta C + S^*\Delta BS + S^*\Delta B\Delta C \\
&\quad + \Delta A(S_z + \mu)^{-1}S + \Delta A(S_z + \mu)^{-1}\Delta C + \Delta A\Delta BS + \Delta A\Delta B\Delta C.
\end{aligned}$$

Thus, the difference is

$$\begin{aligned}\hat{S}^*(\hat{S}_z + \mu)^{-1}\hat{S} - S^*(S_z + \mu)^{-1}S &= S^*(S_z + \mu)^{-1}\Delta S + S^*\Delta BS + S^*\Delta B\Delta S \\ &\quad + \Delta S^*(S_z + \mu)^{-1}S + \Delta S^*(S_z + \mu)^{-1}\Delta S \\ &\quad + \Delta S^*\Delta BS + \Delta S^*\Delta B\Delta S.\end{aligned}$$

Let

$$\Delta B = \underbrace{(S_z + \mu)^{-1}(S_z - \hat{S}_z)(S_z + \mu)^{-1}}_{\Delta_1 B} + \underbrace{\sum_{r=2}^{\infty} (S_z + \mu)^{-1}\{(S_z - \hat{S}_z)(S_z + \mu)^{-1}\}^r}_{\Delta_2 B}.$$

Note the application of Lemma E.7. Thus, $\|\Delta B\| \leq \frac{\delta_z(1+2\delta_z)}{\mu}$. Since $S^*(S_z + \mu)^{-1}\Delta S$, $\Delta S^*(S_z + \mu)^{-1}S$, and $S^*\Delta_1 BS$ are part of the Bahadur representation, we control the following terms with the assumed events and Lemma B.3 taking $A^* = S^*(S_z + \mu)^{1/2}$.

1. $T_{\mu,\lambda}^{-1}S^*\Delta_2 BS$:

$$\|T_{\mu,\lambda}^{-1}S^*\Delta_2 BS\| \leq \kappa \frac{\delta_z^2}{2\sqrt{\lambda\mu}(1-\delta_z)} \leq \kappa \frac{\delta_z^2}{\sqrt{\lambda\mu}}.$$

2. $T_{\mu,\lambda}^{-1}S^*\Delta B\Delta S$:

$$\|T_{\mu,\lambda}^{-1}S^*\Delta B\Delta S\| \leq \frac{\delta_z(1+2\delta_z)}{2\sqrt{\lambda\mu}} \tilde{\gamma}_1.$$

3. $T_{\mu,\lambda}^{-1}\Delta S^*(S_z + \mu)^{-1}\Delta S$:

$$\|T_{\mu,\lambda}^{-1}\Delta S^*(S_z + \mu)^{-1}\Delta S\| \leq \frac{\tilde{\gamma}_1^2}{\lambda\mu}.$$

4. $T_{\mu,\lambda}^{-1}\Delta S^*\Delta BS$:

$$\|T_{\mu,\lambda}^{-1}\Delta S^*\Delta BS\| \leq \frac{\tilde{\gamma}_1\delta_z(1+2\delta_z)\kappa}{\lambda\mu}.$$

5. $T_{\mu,\lambda}^{-1}\Delta S^*\Delta B\Delta S$:

$$\|T_{\mu,\lambda}^{-1}\Delta S^*\Delta B\Delta S\| \leq \frac{\delta_z(1+2\delta_z)}{\lambda\mu} \tilde{\gamma}_1^2.$$

Together, the difference $\hat{S}^*(\hat{S}_z + \mu)^{-1}\hat{S} - S^*(S_z + \mu)^{-1}S$, without the terms accounted for in

the Bahadur, can be bounded by the sum of the five bounds

$$\begin{aligned} & \kappa \frac{\delta_z^2}{\sqrt{\lambda\mu}} + \frac{\delta_z(1+2\delta_z)}{2\sqrt{\lambda\mu}} \tilde{\gamma}_1 + \frac{\tilde{\gamma}_1^2}{\lambda\mu} + \frac{\tilde{\gamma}_1\delta_z(1+2\delta_z)\kappa}{\lambda\mu} + \frac{\delta_z(1+2\delta_z)}{\lambda\mu} \tilde{\gamma}_1^2 \\ &= \frac{\delta_z}{\sqrt{\lambda\mu}} \left\{ \delta_z(\kappa + \tilde{\gamma}_1) + \frac{\tilde{\gamma}_1}{2} \right\} + \frac{\tilde{\gamma}_1}{\lambda\mu} \left\{ \tilde{\gamma}_1 + \delta_z(1+2\delta_z)(\kappa + \tilde{\gamma}_1) \right\}. \end{aligned}$$

Together with the first part of the proof,

$$\|u\| \leq 2\delta\gamma + \gamma_1 + \xi_1 + \left[2\delta^2 + \frac{\delta_z^2}{\sqrt{\lambda\mu}} \left\{ \delta_z(\kappa + \tilde{\gamma}_1) + \frac{\tilde{\gamma}_1}{2} \right\} + \frac{\tilde{\gamma}_1}{\lambda\mu} \left\{ \tilde{\gamma}_1 + \delta_z(1+2\delta_z)(\kappa + \tilde{\gamma}_1) \right\} \right] \|h_0 - h_{\mu,\lambda}\|. \quad \square$$

Theorem E.1 (Bahadur representation). *Suppose $n \geq \max\{N_{\delta_z}, N_\delta\}$, then, with probability $1 - 5\eta$, $\hat{h} - h_{\mu,\lambda} = \mathbb{E}_n(U) + u$ for some u with $\|u\| \lesssim V(n, \mu, \lambda, \eta) + B(n, \mu, \lambda, \eta, h_0) := \Delta_U$, where*

$$\begin{aligned} l(\eta) &= \ln(2/\eta), \\ \mathfrak{n}_z(\mu) &= \text{tr}((S_z + \mu)^{-2} S_z), \\ V(n, \mu, \lambda, \eta) &= \left(\frac{\mathfrak{n}_z(\mu)^{3/2} l(\eta)^3}{n^{3/2} \lambda^{3/2} \mu^{1/2}} \vee \frac{\mathfrak{n}_z(\mu) \tilde{\delta}(\mu, \lambda, \eta) l(\eta)^2}{n\lambda} \vee \frac{\mathfrak{n}_z(\mu)^{1/2} l(\eta)^2}{n\lambda\mu} \vee \frac{\tilde{\delta}(\mu, \lambda, \eta) l(\eta)}{\sqrt{n\lambda\mu}} \right), \\ B(n, \mu, \lambda, \eta, h_0) &= \left(\frac{l(\eta)^2 \mathfrak{n}_z(\mu)}{n\lambda\mu} \vee \tilde{\delta}(\mu, \lambda, \eta)^2 \right) \|h_0 - h_{\mu,\lambda}\|, \\ \tilde{\delta}(\mu, \lambda, \eta) &= 2\ln(2/\eta) \left(\frac{4\kappa\kappa_x^{1/2}}{n\lambda\mu^{1/2}} \vee \sqrt{\frac{\kappa\kappa_x^{1/2}}{n\mu^{1/2}}} \mathfrak{m}(\lambda, \mu) \right), \\ \mathfrak{m}(\lambda, \mu) &= \text{tr} T_{\mu, \lambda}^{-1} S^* (S_z + \mu)^{-1} S T_{\mu, \lambda}^{-1} = \text{tr} \{(T_\mu + \lambda)^{-2} T_\mu\}, \quad T_\mu = S^* (S_z + \mu)^{-1} S. \end{aligned}$$

Proof. Recall the following bounds from the high probability events in Lemma E.14:

$$\begin{aligned} \gamma_1 &= \frac{\tilde{\gamma}_1 \tilde{\gamma}_2}{\lambda\mu}, \\ \gamma &= \frac{\kappa 2\delta_z \tilde{\gamma}_3}{\lambda} + \frac{\tilde{\gamma}_1 \tilde{\gamma}_3}{\lambda} + \frac{\tilde{\gamma}_2}{2\sqrt{\lambda\mu}}, \\ \xi_1 &= \frac{\delta_z \tilde{\gamma}_2}{\sqrt{\lambda\mu}}. \end{aligned}$$

Note that for these bounds to hold, each of the primitive events needs to hold. Those are, $\tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_3$ each with probability $1 - \eta$. The δ_z bound holds by Lemma E.4 with probability

$1-\eta$ and δ holds on the events $\tilde{\gamma}_1, \delta_z$ and a Bernstein inequality with probability $1-\eta$ again. Thus, using a union bound with probability $1-5\eta$, all the bounds hold and

$$\|u\| \leq 2\delta\gamma + \gamma_1 + \xi_1 + \left[2\delta^2 + \frac{\delta_z}{\sqrt{\lambda\mu}} \left\{ \delta_z(\kappa + \tilde{\gamma}_1) + \frac{\tilde{\gamma}_1}{2} \right\} + \frac{\tilde{\gamma}_1}{\lambda\mu} \left\{ \tilde{\gamma}_1 + \delta_z(1+2\delta_z)(\kappa + \tilde{\gamma}_1) \right\} \right] \|h_0 - h_{\mu,\lambda}\|.$$

Now, let $l(\eta) = \ln(2/\eta)$ collecting terms while suppressing constants,

$$\begin{aligned} & \delta\gamma + \gamma_1 + \xi_1 \\ &= \mathcal{O} \left(\frac{\mathbf{n}_z(\mu)^{3/2} l(\eta)^3}{n^{3/2} \lambda^{3/2} \mu^{1/2}} \vee \frac{\mathbf{n}_z(\mu) \tilde{\delta}(\mu, \lambda, \eta) l(\eta)^2}{n\lambda} \vee \frac{\mathbf{n}_z(\mu)^{1/2} l(\eta)^2}{n\lambda\mu} \vee \frac{\tilde{\delta}(\mu, \lambda, \eta) l(\eta)}{\sqrt{n\lambda\mu}} \vee \frac{l(\eta)^2}{n\lambda\mu} \vee \frac{\mathbf{n}_z(\mu)^{1/2} l(\eta)^2}{n\sqrt{\lambda\mu}} \right) \\ &= \mathcal{O} \left(\frac{\mathbf{n}_z(\mu)^{3/2} l(\eta)^3}{n^{3/2} \lambda^{3/2} \mu^{1/2}} \vee \frac{\mathbf{n}_z(\mu) \tilde{\delta}(\mu, \lambda, \eta) l(\eta)^2}{n\lambda} \vee \frac{\mathbf{n}_z(\mu)^{1/2} l(\eta)^2}{n\lambda\mu} \vee \frac{\tilde{\delta}(\mu, \lambda, \eta) l(\eta)}{\sqrt{n\lambda\mu}} \right) \\ &\implies \delta\gamma + \gamma_1 + \xi_1 \lesssim \left(\frac{\mathbf{n}_z(\mu)^{3/2} l(\eta)^3}{n^{3/2} \lambda^{3/2} \mu^{1/2}} \vee \frac{\mathbf{n}_z(\mu) \tilde{\delta}(\mu, \lambda, \eta) l(\eta)^2}{n\lambda} \vee \frac{\mathbf{n}_z(\mu)^{1/2} l(\eta)^2}{n\lambda\mu} \vee \frac{\tilde{\delta}(\mu, \lambda, \eta) l(\eta)}{\sqrt{n\lambda\mu}} \right) := V(n, \mu, \lambda, \eta). \end{aligned}$$

For the terms involving $\|h_0 - h_{\mu,\lambda}\|$ we can write

$$\begin{aligned} & \left[2\delta^2 + \frac{\delta_z}{\sqrt{\lambda\mu}} \left\{ \delta_z(\kappa + \tilde{\gamma}_1) + \frac{\tilde{\gamma}_1}{2} \right\} + \frac{\tilde{\gamma}_1}{\lambda\mu} \left\{ \tilde{\gamma}_1 + \delta_z(1+2\delta_z)(\kappa + \tilde{\gamma}_1) \right\} \right] \|h_0 - h_{\mu,\lambda}\| \\ &= \mathcal{O} \left(\left\{ \frac{l(\eta)^2 \mathbf{n}_z(\mu)}{n\lambda\mu} \vee \tilde{\delta}(\mu, \lambda, \eta)^2 \vee \frac{l(\eta)^2 \mathbf{n}_z(\mu)}{n\sqrt{\lambda\mu}} \vee \frac{l(\eta)^2 \mathbf{n}_z(\mu)^{1/2}}{n\lambda\mu} \right\} \|h_0 - h_{\mu,\lambda}\| \right) \\ &= \mathcal{O} \left(\left\{ \frac{l(\eta)^2 \mathbf{n}_z(\mu)}{n\lambda\mu} \vee \tilde{\delta}(\mu, \lambda, \eta)^2 \right\} \|h_0 - h_{\mu,\lambda}\| \right) \\ &\implies \left[2\delta^2 + \frac{\delta_z}{\sqrt{\lambda\mu}} \left\{ \delta_z(\kappa + \tilde{\gamma}_1) + \frac{\tilde{\gamma}_1}{2} \right\} + \frac{\tilde{\gamma}_1}{\lambda\mu} \left\{ \tilde{\gamma}_1 + \delta_z(1+2\delta_z)(\kappa + \tilde{\gamma}_1) \right\} \right] \|h_0 - h_{\mu,\lambda}\| \\ &\lesssim \left\{ \frac{l(\eta)^2 \mathbf{n}_z(\mu)}{n\lambda\mu} \vee \tilde{\delta}(\mu, \lambda, \eta)^2 \right\} \|h_0 - h_{\mu,\lambda}\| := B(n, \mu, \lambda, \eta, h_0). \end{aligned}$$

Combining this $\|u\| \lesssim V(n, \mu, \lambda, \eta) + B(n, \mu, \lambda, \eta, h_0) := \Delta_U$. \square

F Feasible bootstrap

We derive a feasible bootstrap in order to derive R . Specifically, we aim to bound $\|Z_{\mathfrak{B}} - \mathfrak{B}\|$.

Recall that

$$Z_{\mathfrak{B}} = \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n \left(\frac{V_i - V_j}{\sqrt{2}} \right) h_{ij}, \quad \mathfrak{B} = \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n \left(\frac{\hat{V}_i - \hat{V}_j}{\sqrt{2}} \right) h_{ij}$$

with

$$\begin{aligned} V_i &= T_{\mu, \lambda}^{-1} \{ S^*(S_z + \mu)^{-1} \phi(Z_i) (Y_i - h_{\mu, \lambda}(X_i)) + S_i^*(S_z + \mu)^{-1} S(h_0 - h_{\mu, \lambda}) \\ &\quad - S^*(S_z + \mu)^{-1} S_{z,i} (S_z + \mu)^{-1} S(h_0 - h_{\mu, \lambda}) \}. \end{aligned}$$

The feasible version contains

$$\begin{aligned} \hat{V}_i &= \hat{T}_{\mu, \lambda}^{-1} \{ \hat{S}^*(\hat{S}_z + \mu)^{-1} \phi(Z_i) \hat{\epsilon}_i + S_i^*(\hat{S}_z + \mu)^{-1} \mathbb{E}_n[\hat{\varepsilon} \phi(Z)] \\ &\quad - \hat{S}^*(\hat{S}_z + \mu)^{-1} S_{z,i} (\hat{S}_z + \mu)^{-1} \mathbb{E}_n[\hat{\varepsilon} \phi(Z)] \}. \end{aligned}$$

This has already been derived in Appendix A.2. To lighten notation, let

$$\begin{aligned} w_i &= S^*(S_z + \mu)^{-1} \phi(Z_i) (Y_i - h_{\mu, \lambda}(X_i)) + S_i^*(S_z + \mu)^{-1} S(h_0 - h_{\mu, \lambda}) - S^*(S_z + \mu)^{-1} S_{z,i} (S_z + \mu)^{-1} S(h_0 - h_{\mu, \lambda}), \\ \hat{w}_i &= \hat{S}^*(\hat{S}_z + \mu)^{-1} \phi(Z_i) \hat{\epsilon}_i + S_i^*(\hat{S}_z + \mu)^{-1} \mathbb{E}_n[\hat{\varepsilon} \phi(Z)] - \hat{S}^*(\hat{S}_z + \mu)^{-1} S_{z,i} (\hat{S}_z + \mu)^{-1} \mathbb{E}_n[\hat{\varepsilon} \phi(Z)], \\ w_{ij} &= \frac{w_i - w_j}{\sqrt{2}}, \\ \hat{w}_{ij} &= \frac{\hat{w}_i - \hat{w}_j}{\sqrt{2}}. \end{aligned}$$

F.1 Helpful orderings

Lemma F.1 (Regularized empirical block decomposition). *Let $\hat{S}_x, \hat{S}_z \succeq 0$. Then, for any $\mu > 0$, we have the decomposition*

$$\begin{pmatrix} \hat{S}_x & \hat{S}^* \\ \hat{S} & \hat{S}_z + \mu \end{pmatrix} = \begin{pmatrix} I & \hat{S}^* \\ 0 & \hat{S}_z + \mu \end{pmatrix} \begin{pmatrix} \hat{S}_x - \hat{S}^*(\hat{S}_z + \mu)^{-1} \hat{S} & 0 \\ (\hat{S}_z + \mu)^{-1} \hat{S} & I \end{pmatrix}.$$

Moreover, the following inequalities hold $\hat{S}_x \succeq \hat{S}^*(\hat{S}_z + \mu I)^{-1}\hat{S} \succeq 0$.

Proof. The left-hand side is

$$\begin{pmatrix} \hat{S}_x & \hat{S}^* \\ \hat{S} & \hat{S}_z + \mu I \end{pmatrix} = \begin{pmatrix} \hat{S}_x & \hat{S}^* \\ \hat{S} & \hat{S}_z \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \mu \end{pmatrix}.$$

Thus, the left-hand side is positive semidefinite. The operator $\hat{S}_x - \hat{S}^*(\hat{S}_z + \mu I)^{-1}\hat{S}$ is self-adjoint. Assume, towards a contradiction, that it has an eigenvector $u_\nu \neq 0$ with eigenvalue $\nu < 0$. Using the factorization above, a direct computation gives

$$\begin{aligned} & \begin{pmatrix} u_\nu \\ -(\hat{S}_z + \mu I)^{-1}\hat{S}u_\nu \end{pmatrix}^* \begin{pmatrix} \hat{S}_x & \hat{S}^* \\ \hat{S} & \hat{S}_z + \mu I \end{pmatrix} \begin{pmatrix} u_\nu \\ -(\hat{S}_z + \mu I)^{-1}\hat{S}u_\nu \end{pmatrix} \\ &= \begin{pmatrix} u_\nu \\ 0 \end{pmatrix}^* \begin{pmatrix} \{\hat{S}_x - \hat{S}^*(\hat{S}_z + \mu I)^{-1}\hat{S}\}u_\nu \\ 0 \end{pmatrix} = \nu \|u_\nu\|^2 < 0. \end{aligned}$$

But $\begin{pmatrix} \hat{S}_x & \hat{S}^* \\ \hat{S} & \hat{S}_z + \mu I \end{pmatrix}$ is positive semidefinite, so its quadratic form cannot be negative. Thus,

$$\hat{S}_x - \hat{S}^*(\hat{S}_z + \mu I)^{-1}\hat{S} \succeq 0.$$

□

Lemma F.2 (Bound on the regularized empirical cross-covariance). *The preconditioned empirical cross-covariance is bounded in operator norm; specifically,*

$$\|(\hat{S}_z + \mu I)^{-1/2}\hat{S}\| \leq \sqrt{\kappa_x}.$$

Proof. By Lemma F.1, we have the operator inequality

$$\hat{S}^*(\hat{S}_z + \mu I)^{-1}\hat{S} \preceq \hat{S}_x.$$

Taking operator norms of both sides, this yields

$$\|(\hat{S}_z + \mu I)^{-1/2}\hat{S}\|^2 = \|\hat{S}^*(\hat{S}_z + \mu I)^{-1}\hat{S}\| \leq \|\hat{S}_x\| \leq \kappa_x.$$

Finally, taking square roots gives the claimed bound. \square

Lemma F.3 (High-probability bound for the empirical operator deviation). *Assume $\delta \leq 1/2$. Then,*

$$\|\hat{T}_{\mu,\lambda}^{-1}(\hat{T}_\mu - T_\mu)\|_{op} \leq \delta_B = 3\delta.$$

Proof. We have that

$$\begin{aligned} \|\hat{T}_{\mu,\lambda}^{-1}(\hat{T}_\mu - T_\mu)\|_{op} &= \|\hat{T}_{\mu,\lambda}^{-1}(\hat{T}_\mu - T_\mu) \pm T_{\mu,\lambda}^{-1}(\hat{T}_\mu - T_\mu)\|_{op} \\ &\leq \|(\hat{T}_{\mu,\lambda}^{-1} - T_{\mu,\lambda}^{-1})(\hat{T}_\mu - T_\mu)\|_{op} + \|T_{\mu,\lambda}^{-1}(\hat{T}_\mu - T_\mu)\|_{op} \\ &\leq 2\delta \|T_{\mu,\lambda}^{-1}(\hat{T}_\mu - T_\mu)\|_{op} + \delta \leq 2\delta^2 + \delta \leq 3\delta. \end{aligned} \quad \square$$

Lemma F.4 (Empirical linearization). *We show an empirical version of Lemma E.11. Suppose $\|\hat{T}_{\mu,\lambda}^{-1}(\hat{T}_\mu - T_\mu)\|_{op} \leq \delta_B < 1$. Then, for all $k \geq 1$,*

$$(T_{\mu,\lambda}^{-1} - \hat{T}_{\mu,\lambda}^{-1})u = A_1u + A_2\hat{T}_{\mu,\lambda}^{-1}u + A_3\hat{T}_{\mu,\lambda}^{-1}u, \quad \|A_1\|_{HS} \leq \frac{\delta_B^k}{\lambda}, \quad \|A_2\|_{HS} \leq \delta_B, \quad \|A_3\|_{HS} \leq \frac{\delta_B^2}{1-\delta_B}.$$

Furthermore, when $\delta_B \leq 1/2$ then $\|(\hat{T}_{\mu,\lambda}^{-1} - T_{\mu,\lambda}^{-1})u\| \leq 2\delta_B \|\hat{T}_{\mu,\lambda}^{-1}u\|$.

Proof. By the iterated resolvent identity Lemma B.1,

$$\begin{aligned} T_{\mu,\lambda}^{-1} - \hat{T}_{\mu,\lambda}^{-1} &= T_{\mu,\lambda}^{-1}\{(\hat{T}_\mu - T_\mu)\hat{T}_{\mu,\lambda}^{-1}\}^k + \sum_{r=1}^{k-1} \hat{T}_{\mu,\lambda}^{-1}\{(\hat{T}_\mu - T_\mu)\hat{T}_{\mu,\lambda}^{-1}\}^r \\ &= T_{\mu,\lambda}^{-1}\{(\hat{T}_\mu - T_\mu)\hat{T}_{\mu,\lambda}^{-1}\}^k + \hat{T}_{\mu,\lambda}^{-1}\{(\hat{T}_\mu - T_\mu)\hat{T}_{\mu,\lambda}^{-1}\} + \sum_{r=2}^{k-1} \hat{T}_{\mu,\lambda}^{-1}\{(\hat{T}_\mu - T_\mu)\hat{T}_{\mu,\lambda}^{-1}\}^r \\ &= A_1 + A_2\hat{T}_{\mu,\lambda}^{-1} + A_3\hat{T}_{\mu,\lambda}^{-1}. \end{aligned}$$

Now, using $\|\hat{T}_{\mu,\lambda}^{-1}(\hat{T}_\mu - T_\mu)\|_{HS} \leq \delta_B < 1$,

$$\begin{aligned} \|A_1\|_{HS} &= \|T_{\mu,\lambda}^{-1}\{(\hat{T}_\mu - T_\mu)\hat{T}_{\mu,\lambda}^{-1}\}^k\|_{HS} \leq \frac{\delta_B^k}{\lambda} \\ \|A_2\|_{HS} &= \|(\hat{T}_\mu - T_\mu)\hat{T}_{\mu,\lambda}^{-1}\|_{HS} \leq \delta_B \\ \|A_3\|_{HS} &= \left\| \sum_{r=2}^{k-1} \{(\hat{T}_\mu - T_\mu)\hat{T}_{\mu,\lambda}^{-1}\}^r \right\|_{HS} \leq \sum_{r=2}^{k-1} \delta_B^r \leq \sum_{r=2}^{\infty} \delta_B^r \leq \frac{\delta_B^2}{1-\delta_B}. \end{aligned}$$

If in addition we assume $\delta_B \leq 1/2$, then

$$\begin{aligned} \|(\hat{T}_{\mu,\lambda}^{-1} - T_{\mu,\lambda}^{-1})u\| &\leq \|A_1\|_{HS}\|u\| + \|A_2\|_{HS}\|\hat{T}_{\mu,\lambda}^{-1}u\| + \|A_3\|_{HS}\|\hat{T}_{\mu,\lambda}^{-1}u\| \\ &\leq \frac{\delta_B^k}{\lambda}\|u\| + (\delta_B + \frac{\delta_B^2}{1-\delta_B})\|\hat{T}_{\mu,\lambda}^{-1}u\| \\ &= \frac{\delta_B^k}{\lambda}\|u\| + \frac{\delta_B}{1-\delta_B}\|\hat{T}_{\mu,\lambda}^{-1}u\| \underset{k \rightarrow \infty, \delta_B \leq 1/2}{\leq} 2\delta_B\|\hat{T}_{\mu,\lambda}^{-1}u\|. \quad \square \end{aligned}$$

F.2 High probability events

Lemma F.5 (Estimation error bound of the Z covariance operator). $\|\hat{S}_z - S_z\|_{HS}$ is bounded with probability $1 - \eta$ for $n \geq 4$ by

$$\|\hat{S}_z - S_z\|_{HS} \leq \tilde{\gamma}_z = 2\ln(2/\eta) \left(\frac{4\kappa_z}{n} + \frac{2\kappa_z}{n^{1/2}} \right) \leq 8\ln(2/\eta) \frac{\kappa_z}{n^{1/2}}.$$

Proof. The proof follows from the same arguments as in Lemma E.3. \square

Lemma F.6 (High-probability bound on the empirical feature–noise). The empirical second moment of the feature–noise product is bounded: with probability $1 - \eta$

$$\mathbb{E}_n \|\phi(Z_i)\epsilon_i\|^2 \leq \gamma',$$

where

$$\gamma' := 2\bar{\sigma}^2\kappa_z^2 + 2\ln\left(\frac{2}{\eta}\right) \left(\frac{2\bar{\sigma}^2\kappa_z^2}{n} + \sqrt{\frac{\bar{\sigma}^4\kappa_z^4}{n}} \right).$$

Proof. Let $\xi_i = \|\phi(Z_i)\epsilon_i\|^2$. By boundedness, $0 \leq \xi_i \leq \bar{\sigma}^2\kappa_z^2$, and

$$\mathbb{E}\xi_i = \sigma^2 \mathbb{E}\|\phi(Z_i)\|^2 \leq \bar{\sigma}^2\kappa_z^2.$$

Therefore, $\mathbb{E}\xi_i^2 \leq \bar{\sigma}^4\kappa_z^4$. Applying Lemma B.8 to $\{\xi_i\}_{i=1}^n$ yields

$$\mathbb{E}_n \xi_i \leq 2\bar{\sigma}^2\kappa_z^2 + 2\ln\left(\frac{2}{\eta}\right) \left(\frac{2\bar{\sigma}^2\kappa_z^2}{n} + \sqrt{\frac{\bar{\sigma}^4\kappa_z^4}{n}} \right)$$

with probability at least $1 - \eta$. \square

Lemma F.7 (High-probability bounds for preconditioned Z -covariances). *With probability at least $1-\eta$, the deviation of the empirical covariance concentrates in the Hilbert–Schmidt norm:*

$$\|(\hat{S}_z - S_z)(S_z + \mu)^{-1/2}\|_{\text{HS}} \leq \delta'_z,$$

where

$$\delta'_z := 2\ln\left(\frac{2}{\eta}\right) \left(\frac{4\kappa_z^2}{\sqrt{\mu n}} + \sqrt{\frac{\kappa_z^2 \text{tr}(S_z(S_z + \mu)^{-1})}{n}} \right).$$

Moreover, with probability at least $1-\eta$,

$$\mathbb{E}_n \| (S_z + \mu)^{-1/2} S_{z,i} \|^2_{\text{HS}} \leq \delta''''_\mu.$$

Proof. Let, $\xi_i = (S_{z,i} - S_z)(S_z + \mu)^{-1/2}$. First, $\mathbb{E}\xi_i = 0$. Next, $\|\xi_i\|_{\text{HS}} \leq \frac{2\kappa_z^2}{\sqrt{\mu}}$ and

$$\mathbb{E}\| (S_{z,i} - S_z)(S_z + \mu)^{-1/2} \|^2_{\text{HS}} \leq \kappa_z^2 \text{tr} S_z (S_z + \mu)^{-1}.$$

Using Lemma B.8, with probability $1-\eta$

$$\|(\hat{S}_z - S_z)(S_z + \mu)^{-1/2}\|_{\text{HS}} \leq 2\ln(2/\eta) \left(\frac{4\kappa_z^2}{\sqrt{\mu n}} + \sqrt{\frac{\kappa_z^2 \text{tr} S_z (S_z + \mu)^{-1}}{n}} \right) := \delta'_z.$$

For the second statement, let $\xi_i = \| (S_z + \mu)^{-1/2} S_{z,i} \|^2$. Clearly, $\xi_i \leq \frac{\kappa_z^2}{\mu}$. Furthermore, $\mathbb{E}\xi_i \leq \mathbb{E}\| (S_z + \mu)^{-1/2} S_{z,i} \|^2 = \mathbb{E}\text{tr}(S_z + \mu)^{-1} S_{z,i}^2 \leq \kappa_z^2 \mathbb{E}\text{tr}(S_z + \mu)^{-1} S_{z,i} = \kappa_z^2 \text{tr}(S_z + \mu)^{-1} S_z$. Also, $E\xi_i^2 \leq \frac{\kappa_z^4 \text{tr}(S_z + \mu)^{-1} S_z}{\mu}$. By Lemma B.8, with probability $1-\eta$,

$$\mathbb{E}_n \| (S_z + \mu)^{-1/2} S_{z,i} \|^2 \leq \kappa_z^2 \text{tr}(S_z + \mu)^{-1} S_z + 2\ln(2/\eta) \left\{ \frac{2\kappa_z^2}{\mu n} + \sqrt{\frac{\kappa_z^4 \text{tr}(S_z + \mu)^{-1} S_z}{\mu n}} \right\} := \delta''''_\mu.$$

□

Lemma F.8 (Empirical resolvent-weighted operator bounds). *Assume $\| (S_z + \mu)^{-1} (\hat{S}_z - S_z) \| = \delta_z \leq \frac{1}{2}$. Then, with probability at least $1-\eta$,*

$$\mathbb{E}_n \| S_{z,i} (\hat{S}_z + \mu)^{-1} \|^2 \leq 2 \left\{ \kappa_z^2 \frac{\delta_z (1+2\delta_z)}{\mu} \right\}^2 + 4\ln\frac{2}{\eta} \left\{ \frac{2\kappa_z^4}{\mu^2 n} + \sqrt{\frac{\kappa_z^6 \mathbf{n}_z(\mu)}{\mu^2 n}} \right\} + 2\kappa_z^2 \mathbf{n}_z(\mu) =: \delta'_\mu,$$

where $\mathbf{n}_z(\mu) := \text{tr}((S_z + \mu)^{-2} S_z)$. In addition,

$$\mathbb{E}_n \|S_i^*(\hat{S}_z + \mu)^{-1}\|^2 \leq 2 \left\{ \kappa \frac{\delta_z(1+2\delta_z)}{\mu} \right\}^2 + 4 \ln \frac{2}{\eta} \left\{ \frac{2\kappa^2}{\mu^2 n} + \sqrt{\frac{\kappa^2 \kappa_x^2 \mathbf{n}_z(\mu)}{\mu^2 n}} \right\} + 2\kappa_x^2 \mathbf{n}_z(\mu) =: \delta''_\mu.$$

Consequently, with probability at least $1 - \eta$,

$$\mathbb{E}_n \|S_i^*(S_z + \mu)^{-1}\|^2 \leq 4 \ln \frac{2}{\eta} \left\{ \frac{2\kappa^2}{\mu^2 n} + \sqrt{\frac{\kappa^2 \kappa_x^2 \mathbf{n}_z(\mu)}{\mu^2 n}} \right\} + 2\kappa_x^2 \mathbf{n}_z(\mu) =: \delta'''_\mu.$$

Proof. For both parts we use the basic decomposition

$$\mathbb{E}_n \|T(\hat{S}_z + \mu)^{-1}\|^2 \leq 2\mathbb{E}_n \|T[(\hat{S}_z + \mu)^{-1} - (S_z + \mu)^{-1}]\|^2 + 2\mathbb{E}_n \|T(S_z + \mu)^{-1}\|^2,$$

with the appropriate choice of T .

(i) Take $T = S_{z,i}$. By Lemma E.7 and $\delta_z \leq \frac{1}{2}$,

$$\|S_{z,i}^*[(\hat{S}_z + \mu)^{-1} - (S_z + \mu)^{-1}]\| \leq \kappa_z^2 \frac{\delta_z(1+2\delta_z)}{\mu}.$$

For the second term, let $\xi_i := \|S_{z,i}^*(S_z + \mu)^{-1}\|^2$. Then $\xi_i \leq \kappa_z^4 / \mu^2$ and

$$\mathbb{E} \xi_i = \mathbb{E} \text{tr}(S_{z,i}^*(S_z + \mu)^{-2} S_{z,i}) \leq \kappa_z^2 \text{tr}((S_z + \mu)^{-2} S_z) = \kappa_z^2 \mathbf{n}_z(\mu),$$

which implies $\mathbb{E} \xi_i^2 \leq (\kappa_z^6 / \mu^2) \mathbf{n}_z(\mu)$. Applying Lemma B.8 yields the claimed bound.

(ii) Take $T = S_i^*$. By Lemma E.7 and $\delta_z \leq \frac{1}{2}$,

$$\|S_i^*[(\hat{S}_z + \mu)^{-1} - (S_z + \mu)^{-1}]\| \leq \kappa \frac{\delta_z(1+2\delta_z)}{\mu}.$$

For the second term, with $\xi_i := \|S_i^*(S_z + \mu)^{-1}\|^2 \leq \kappa^2 / \mu^2$,

$$\mathbb{E} \xi_i = \mathbb{E} \|S_i^*(S_z + \mu)^{-1}\|^2 \leq \mathbb{E} \text{tr}(S_i^*(S_z + \mu)^{-2} S_i) = \mathbb{E} [\|\psi(X_i)\|^2 \text{tr}((S_z + \mu)^{-2} S_{z,i})] \leq \kappa_x^2 \mathbf{n}_z(\mu),$$

hence $\mathbb{E} \xi_i^2 \leq (\kappa^2 \kappa_x^2 / \mu^2) \mathbf{n}_z(\mu)$. Lemma B.8 gives the desired result for $\mathbb{E}_n \|S_i^*(\hat{S}_z + \mu)^{-1}\|^2$, and the subsequent bound for $\mathbb{E}_n \|S_i^*(S_z + \mu)^{-1}\|^2$ follows by the same Bernstein step applied

directly to ξ_i . □

F.3 Combining events

Lemma F.9 (Consistency). *Assume the same events hold as in the abstract Bahadur representation (Lemma E.14). In addition, assume that,*

$$\begin{aligned} \|(\hat{S}_z - S_z)(S_z + \mu)^{-1/2}\| &\leq \delta'_z, \\ \|(\hat{S} - S)(S_z + \mu)^{-1/2}\| &\leq \delta''_z, \\ \|S - \hat{S}\| &\leq \tilde{\gamma}_1, \\ \|\mathbb{E}_n[\phi(Z_i)\epsilon_i]\| &\leq \tilde{\gamma}_2. \end{aligned}$$

Denote the bias and projected bias as $\|h_0 - h_{\mu,\lambda}\| \leq b_{\mu,\lambda}$ and $\|(S_z + \mu)^{-1/2}S(h_0 - h_{\mu,\lambda})\| \leq v_{\mu,\lambda}$, respectively. Then,

$$\|h_{\mu,\lambda} - \hat{h}\| \leq \Delta_{h_{\mu,\lambda}} := \frac{1}{2\sqrt{\lambda\mu}}\tilde{\gamma}_1 b_{\mu,\lambda} + \frac{1}{\lambda}\delta''_z v_{\mu,\lambda} + \frac{1}{2\sqrt{\lambda\mu}}\tilde{\gamma}_2 + \frac{1}{2\sqrt{\lambda\mu}}\delta'_z b_{\mu,\lambda} + \Delta_U.$$

Here, Δ_U is the error of the Bahadur representation explicitly derived in Lemma E.14. Also

$$\|h_0 - \hat{h}\| \leq b_{\mu,\lambda} + \Delta_{h_{\mu,\lambda}} := \Delta_h.$$

Proof. By Lemma E.14,

$$\hat{h} - h_{\mu,\lambda} = \mathbb{E}_n(U_i) + \Delta_U,$$

where

$$\begin{aligned} \mathbb{E}_n(U) &= T_{\mu,\lambda}^{-1}\{S^*(S_z + \mu)^{-1}(\hat{S} - S) + (\hat{S} - S)^*(S_z + \mu)^{-1}S\}(h_0 - h_{\mu,\lambda}) \\ &\quad + T_{\mu,\lambda}^{-1}S^*(S_z + \mu)^{-1}\mathbb{E}_n(\phi_Z \varepsilon) \\ &\quad + T_{\mu,\lambda}^{-1}\{S^*(S_z + \mu)^{-1}(S_z - \hat{S}_z)(S_z + \mu)^{-1}S\}(h_0 - h_{\mu,\lambda}). \end{aligned}$$

Using that $\|\hat{h} - h_{\mu,\lambda}\| \leq \|\mathbb{E}_n(U_i)\| + \Delta_U$, we can focus on the individual terms using the bias

$(b_{\mu,\lambda})$, projected bias $(v_{\mu,\lambda})$, and Lemma F.7:

$$\|\mathbb{E}_n(U_i)\| \leq \frac{1}{2\sqrt{\lambda\mu}}\tilde{\gamma}_1 b_{\mu,\lambda} + \frac{1}{\lambda}\delta_z'' v_{\mu,\lambda} + \frac{1}{2\sqrt{\lambda\mu}}\tilde{\gamma}_2 + \frac{1}{2\sqrt{\lambda\mu}}\delta_z' b_{\mu,\lambda}.$$

Combining this with Lemma E.14 gives the result. For $\Delta_h = \|h_0 - \hat{h}\|$, we can write $\Delta_h \leq \|h_0 - h_{\mu,\lambda}\| + \|h_{\mu,\lambda} - \hat{h}\| \leq b_{\mu,\lambda} + \Delta_{h_{\mu,\lambda}}$. \square

Lemma F.10 (High-probability bound for the operator deviation). *With probability at least $1 - 4\eta$,*

$$\left\| T_{\mu,\lambda}^{-1}(\hat{T}_\mu - T_\mu) \right\|_{op} \leq \delta,$$

where

$$\delta = \frac{\kappa\delta_z\tilde{\gamma}_1 + \tilde{\gamma}_1^2}{\lambda\mu} + \frac{\tilde{\gamma}_z\sqrt{\kappa_x}}{2\sqrt{\lambda\mu}} + \frac{\tilde{\gamma}_1}{2\sqrt{\lambda\mu}} + \tilde{\delta}(\mu, \lambda, n),$$

and

$$\tilde{\delta}(\mu, \lambda, n) = 2\ln\left(\frac{2}{\eta}\right) \left(\frac{4\kappa\kappa_x^{1/2}}{n\lambda\mu^{1/2}} \vee \sqrt{\frac{\kappa\kappa_x^{1/2}}{n\mu^{1/2}} \mathfrak{m}(\lambda, \mu)} \right).$$

Proof. We can write

$$\begin{aligned} T_{\mu,\lambda}^{-1}(\hat{T}_\mu - T_\mu) &= T_{\mu,\lambda}^{-1}\{\hat{S}^*(\hat{S}_z + \mu)^{-1}\hat{S} - S^*(S_z + \mu)^{-1}S\} \\ &= \underbrace{T_{\mu,\lambda}^{-1}\{\hat{S}^*(\hat{S}_z + \mu)^{-1}\hat{S} - S^*(\hat{S}_z + \mu)^{-1}\hat{S}\}}_{(I)} \\ &\quad + \underbrace{T_{\mu,\lambda}^{-1}\{S^*(\hat{S}_z + \mu)^{-1}\hat{S} - S^*(S_z + \mu)^{-1}\hat{S}\}}_{(II)} \\ &\quad + \underbrace{T_{\mu,\lambda}^{-1}\{S^*(S_z + \mu)^{-1}\hat{S} - S^*(S_z + \mu)^{-1}S\}}_{(III)}. \end{aligned}$$

We aim to bound each of the three terms. We consider them individually.

$$\begin{aligned}
(I) \quad & \left\| T_{\mu,\lambda}^{-1} (\hat{S} - S)^* (\hat{S}_z + \mu)^{-1} \hat{S} \right\| = \left\| T_{\mu,\lambda}^{-1} (\hat{S} - S)^* (\hat{S}_z + \mu)^{-1} (S_z - \hat{S}_z) (S_z + \mu)^{-1} \hat{S} \right. \\
& \quad \left. + T_{\mu,\lambda}^{-1} (\hat{S} - S)^* (S_z + \mu)^{-1} \hat{S} \pm T_{\mu,\lambda}^{-1} (\hat{S} - S)^* (S_z + \mu)^{-1} S \right\| \\
& = \left\| T_{\mu,\lambda}^{-1} (\hat{S} - S)^* (\hat{S}_z + \mu)^{-1} (S_z - \hat{S}_z) (S_z + \mu)^{-1} \hat{S} \right. \\
& \quad \left. + T_{\mu,\lambda}^{-1} (\hat{S} - S)^* (S_z + \mu)^{-1} (\hat{S} - S) + T_{\mu,\lambda}^{-1} (\hat{S} - S)^* (S_z + \mu)^{-1} S \right\| \\
& \leq \underbrace{\left\| T_{\mu,\lambda}^{-1} (\hat{S} - S)^* (\hat{S}_z + \mu)^{-1} (S_z - \hat{S}_z) (S_z + \mu)^{-1} \hat{S} \right\|}_{(I.1)} \\
& \quad + \underbrace{\left\| T_{\mu,\lambda}^{-1} (\hat{S} - S)^* (S_z + \mu)^{-1} (\hat{S} - S) \right\|}_{(I.2)} \\
& \quad + \underbrace{\left\| T_{\mu,\lambda}^{-1} (\hat{S} - S)^* (S_z + \mu)^{-1} S \right\|}_{(I.3)}.
\end{aligned}$$

We continue by again analyzing the individual terms

$$(I.1) \quad \left\| T_{\mu,\lambda}^{-1} (\hat{S} - S)^* (\hat{S}_z + \mu)^{-1} (S_z - \hat{S}_z) (S_z + \mu)^{-1} \hat{S} \right\| \leq \frac{\kappa \delta_z}{\lambda \mu} \|(\hat{S} - S)^*\| \leq \frac{\kappa \delta_z \tilde{\gamma}_1}{\lambda \mu} \text{ with probability } 1 - \eta$$

using the hypothesis and Lemma E.3.

$$(I.2) \quad \left\| T_{\mu,\lambda}^{-1} (\hat{S} - S)^* (S_z + \mu)^{-1} (\hat{S} - S) \right\| \leq \frac{\tilde{\gamma}_1^2}{\lambda \mu} \text{ with probability } 1 - \eta \text{ using Lemma E.3 again.}$$

(I.3) This becomes more involved. Note that $\mathbb{E}\{T_{\mu,\lambda}^{-1} (S_i - S)^* (S_z + \mu)^{-1} S\} = 0$. Also,

$$\|T_{\mu,\lambda}^{-1} (S_i - S)^* (S_z + \mu)^{-1} S\| \leq \frac{2\kappa}{\lambda} \|(S_z + \mu)^{-1} S\| \leq \frac{2\kappa \kappa_x^{1/2}}{\lambda \mu^{1/2}}$$

by Lemma E.2, hence $A := \frac{4\kappa\kappa_x^{1/2}}{\lambda\mu^{1/2}}$. The second moment is given by

$$\begin{aligned}
\mathbb{E}\{\|T_{\mu,\lambda}^{-1}(S_i - S)^*(S_z + \mu)^{-1}S\|^2\} &= \mathbb{E}\{\text{tr}(T_{\mu,\lambda}^{-1}(S_i - S)^*(S_z + \mu)^{-1}S)^*(T_{\mu,\lambda}^{-1}(S_i - S)^*(S_z + \mu)^{-1}S)\} \\
&= \mathbb{E}\{\text{tr}S^*(S_z + \mu)^{-1}(S_i - S)T_{\mu,\lambda}^{-2}(S_i - S)^*(S_z + \mu)^{-1}S\} \\
&\leq \mathbb{E}\{\text{tr}S^*(S_z + \mu)^{-1}S_i T_{\mu,\lambda}^{-2}S_i^*(S_z + \mu)^{-1}S\} \\
&\leq \frac{\kappa\kappa_x^{1/2}}{\sqrt{\mu}} \mathbb{E}\{\text{tr}T_{\mu,\lambda}^{-2}S_i^*(S_z + \mu)^{-1}S\} \\
&= \frac{\kappa\kappa_x^{1/2}}{\sqrt{\mu}} \text{tr}T_{\mu,\lambda}^{-1}S^*(S_z + \mu)^{-1}S T_{\mu,\lambda}^{-1} \\
&\leq \frac{\kappa\kappa_x^{1/2}}{\sqrt{\mu}} \text{tr}T_{\mu,\lambda}^{-1}S^*S_z^{-1}S T_{\mu,\lambda}^{-1} = \frac{\kappa\kappa_x^{1/2}}{\sqrt{\mu}} \mathfrak{m}(\lambda, \mu).
\end{aligned}$$

Thus, $B^2 = \frac{\kappa\kappa_x^{1/2}}{\sqrt{\mu}} \mathfrak{m}(\lambda, \mu)$ and with probability $1 - \eta$ we can bound (I.3) with

$$\left\| T_{\mu,\lambda}^{-1}(\hat{S} - S)^*(S_z + \mu)^{-1}S \right\| \leq 2\ln(2/\eta) \left(\frac{4\kappa\kappa_x^{1/2}}{n\lambda\mu^{1/2}} \vee \sqrt{\frac{\kappa\kappa_x^{1/2}}{n\mu^{1/2}} \mathfrak{m}(\lambda, \mu)} \right) := \tilde{\delta}(\mu, \lambda, \eta).$$

Now combining this with (I.1) and (I.2), with probability $1 - 3\eta$,

$$\left\| T_{\mu,\lambda}^{-1}(\hat{S} - S)^*(\hat{S}_z + \mu)^{-1}\hat{S} \right\| \leq \frac{\kappa\delta_z\tilde{\gamma}_1}{\lambda\mu} + \frac{\tilde{\gamma}_1^2}{\lambda\mu} + \tilde{\delta}(\mu, \lambda, \eta).$$

For (II) we use the resolvent identity, Lemma F.5, and F.2. With probability $1 - \eta$,

$$\left\| T_{\mu,\lambda}^{-1}S^*(S_z + \mu)^{-1}(\hat{S}_z - S_z)(\hat{S}_z + \mu)^{-1}\hat{S} \right\| \leq \frac{\tilde{\gamma}_z\sqrt{\kappa_x}}{2\sqrt{\lambda\mu}}.$$

Lastly, for we use that (III) $\left\| T_{\mu,\lambda}^{-1}S^*(S_z + \mu)^{-1}(\hat{S} - S) \right\| \leq \|T_{\mu,\lambda}^{-1}S^*(S_z + \mu)^{-1}\| \|(\hat{S} - S)\| \leq \frac{\tilde{\gamma}_1}{2\sqrt{\lambda\mu}}$. This holds on the event from Lemma E.3 and using Lemma B.3 with $A^* = S^*(S_z + \mu)^{1/2}$. Combining the three bounds with a union for the distinct high probability events involved, we have with probability $1 - 4\eta$,

$$\left\| T_{\mu,\lambda}^{-1}(\hat{T}_\mu - T_\mu) \right\|_{op} \leq \delta = \frac{\kappa\delta_z\tilde{\gamma}_1 + \tilde{\gamma}_1^2}{\lambda\mu} + \frac{\tilde{\gamma}_z\sqrt{\kappa_x}}{2\sqrt{\lambda\mu}} + \frac{\tilde{\gamma}_1}{2\sqrt{\lambda\mu}} + \tilde{\delta}(\mu, \lambda, \eta). \quad \square$$

Lemma F.11 (Δ_1 -bound). Suppose the following inequalities hold

$$\|(S_z + \mu)^{-1}(\hat{S}_z - S_z)\| = \delta_z \leq \frac{1}{2},$$

$$\|\mathbb{E}_n[\phi(Z_i)\epsilon_i]\| \leq \tilde{\gamma}_2,$$

$$\mathbb{E}_n\|\phi(Z_i)\epsilon_i\|^2 \leq \gamma',$$

$$\mathbb{E}_n\|S_{z,i}^*(\hat{S}_z + \mu)^{-1}\|^2 \leq \delta'_\mu,$$

$$\mathbb{E}_n\|S_i^*(\hat{S}_z + \mu)^{-1}\|^2 \leq \delta''_\mu.$$

Then, conditional on data, with probability $1 - \eta$,

$$\begin{aligned} \|\hat{T}_{\mu,\lambda}^{-1}u\| &\leq \left\{1 + \sqrt{2\log(1/\eta)}\right\} \sqrt{\mathbb{E}\|\hat{T}_{\mu,\lambda}^{-1}u\|^2} \\ &\leq \left\{1 + \sqrt{2\log(1/\eta)}\right\} \sqrt{14\left\{\frac{\kappa^2\Delta_h^2}{\lambda\mu} + \frac{4\kappa^4\kappa_x\Delta_h^2}{\lambda^2\mu} + \frac{\kappa_z^2\kappa_x\Delta_h^2}{4\lambda\mu^2} + 4\Delta_h + \frac{\gamma'}{4\mu\lambda} + \frac{\tilde{\gamma}_2^2\delta''_\mu}{\lambda^2} + \frac{\tilde{\gamma}_2^2\delta'_\mu}{4\lambda\mu}\right\}}. \end{aligned}$$

If additionally $\|\hat{T}_{\mu,\lambda}^{-1}(\hat{T}_\mu - T_\mu)\|_{op} \leq \delta_B < 1/2$. Then, with probability $1 - \eta$,

$$\|\Delta_1\| \leq 2\delta_B \left\{1 + \sqrt{2\log(1/\eta)}\right\} \sqrt{14\left\{\frac{\kappa^2\Delta_h^2}{\lambda\mu} + \frac{4\kappa^4\kappa_x\Delta_h^2}{\lambda^2\mu} + \frac{\kappa_z^2\kappa_x\Delta_h^2}{4\lambda\mu^2} + 4\Delta_h + \frac{\gamma'}{4\mu\lambda} + \frac{\tilde{\gamma}_2^2\delta''_\mu}{\lambda^2} + \frac{\tilde{\gamma}_2^2\delta'_\mu}{4\lambda\mu}\right\}},$$

where $\Delta_h = \|h_0 - \hat{h}\|$.

Proof. Let $u = \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n \hat{w}_{ij} h_{ij}$. Conditional on the data, $\hat{T}_{\mu,\lambda}^{-1}u$ is Gaussian. Using Lemma B.10, with probability $1 - \eta$,

$$\|\hat{T}_{\mu,\lambda}^{-1}u\| \leq \left\{1 + \sqrt{2\log(1/\eta)}\right\} \sqrt{\mathbb{E}\|\hat{T}_{\mu,\lambda}^{-1}u\|^2}.$$

Note that $\mathbb{E}_h\|\hat{T}_{\mu,\lambda}^{-1}u\|^2 = \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n \|\hat{T}_{\mu,\lambda}^{-1}\hat{w}_{ij}\|^2$. Within each term, we have

$$\|\hat{T}_{\mu,\lambda}^{-1}\hat{w}_{ij}\|^2 = \frac{1}{2} \|\hat{T}_{\mu,\lambda}^{-1}\{\hat{w}_i - \hat{w}_j\}\|^2 \leq \|\hat{T}_{\mu,\lambda}^{-1}\hat{w}_i\|^2 + \|\hat{T}_{\mu,\lambda}^{-1}\hat{w}_j\|^2.$$

We can rewrite the following quantities

$$\begin{aligned}\phi(Z_i)\hat{\epsilon}_i &= \phi(Z_i)(Y_i - \hat{h}(X_i)) = \phi(Z_i)(h_0(X_i) - \hat{h}(X_i) + \epsilon_i) = S_i(h_0 - \hat{h}) + \phi(Z_i)\epsilon_i \\ \mathbb{E}_n[\hat{\epsilon}\phi(Z)] &= \mathbb{E}_n[(Y - \hat{h}(X))\phi(Z)] = \hat{S}(h_0 - \hat{h}) + \mathbb{E}_n[\epsilon\phi(Z)].\end{aligned}$$

Using this, \hat{w}_i becomes

$$\begin{aligned}\hat{w}_i &= \hat{S}^*(\hat{S}_z + \mu)^{-1}(S_i(h_0 - \hat{h}) + \phi(Z_i)\epsilon_i) \\ &\quad + S_i^*(\hat{S}_z + \mu)^{-1}(\hat{S}(h_0 - \hat{h}) + \mathbb{E}_n[\epsilon\phi(Z)]) \\ &\quad - \hat{S}^*(\hat{S}_z + \mu)^{-1}S_{z,i}(\hat{S}_z + \mu)^{-1}(\hat{S}(h_0 - \hat{h}) + \mathbb{E}_n[\epsilon\phi(Z)]).\end{aligned}$$

Combining the terms gives

$$\begin{aligned}\hat{w}_i &= \underbrace{\left\{ \hat{S}^*(\hat{S}_z + \mu)^{-1}S_i + S_i^*(\hat{S}_z + \mu)^{-1}\hat{S} - \hat{S}^*(\hat{S}_z + \mu)^{-1}S_{z,i}(\hat{S}_z + \mu)^{-1}\hat{S} \right\}}_A(h_0 - \hat{h}) \\ &\quad + \underbrace{\hat{S}^*(\hat{S}_z + \mu)^{-1}\phi(Z_i)\epsilon_i}_B \\ &\quad + \underbrace{\left\{ S_i^*(\hat{S}_z + \mu)^{-1} - \hat{S}^*(\hat{S}_z + \mu)^{-1}S_{z,i}(\hat{S}_z + \mu)^{-1} \right\}}_C\mathbb{E}_n[\epsilon\phi(Z)].\end{aligned}$$

We can use the parallelogram law to write

$$\frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n \|\hat{T}_{\mu,\lambda}^{-1} \hat{w}_{ij}\|^2 \leq \frac{2}{n} \sum_{i=1}^n \|\hat{T}_{\mu,\lambda}^{-1} \hat{w}_i\|^2.$$

Now, focus on the individual components of this expression and note that by Lemma B.3

$$\|\hat{T}_{\mu,\lambda}^{-1} S^*(\hat{S}_z + \mu)^{-1}\|_{op} \leq \frac{1}{2\sqrt{\lambda\mu}}.$$

(A) We add and subtract the estimated $\hat{T}_{\mu,\lambda}^{-1}\hat{T}_\mu$, resulting in

$$\begin{aligned}
& \|\hat{T}_{\mu,\lambda}^{-1}\left\{\hat{S}^*(\hat{S}_z+\mu)^{-1}S_i+S_i^*(\hat{S}_z+\mu)^{-1}\hat{S}-\hat{S}^*(\hat{S}_z+\mu)^{-1}S_{z,i}(\hat{S}_z+\mu)^{-1}\hat{S}\right\}(h_0-\hat{h})\pm 2\hat{T}_{\mu,\lambda}^{-1}\left\{\hat{S}^*(\hat{S}_z+\mu)^{-1}\hat{S}\right\}\Delta_h\| \\
& \leq \|\hat{T}_{\mu,\lambda}^{-1}\hat{S}^*(\hat{S}_z+\mu)^{-1}(S_i-\hat{S})\|\Delta_h + \|\hat{T}_{\mu,\lambda}^{-1}(S_i-\hat{S})^*(\hat{S}_z+\mu)^{-1}\hat{S}\|\Delta_h + \|\hat{T}_{\mu,\lambda}^{-1}\hat{S}^*(\hat{S}_z+\mu)^{-1}S_{z,i}(\hat{S}_z+\mu)^{-1}\hat{S}\|\Delta_h \\
& + \|2\hat{T}_{\mu,\lambda}^{-1}\left\{\hat{S}^*(\hat{S}_z+\mu)^{-1}\hat{S}\right\}\|\Delta_h \\
& \leq \frac{1}{2\sqrt{\lambda\mu}}\|S_i-\hat{S}\|\Delta_h + \frac{\kappa\sqrt{\kappa_x}}{\lambda\sqrt{\mu}}\|S_i-\hat{S}\|\Delta_h + \frac{\kappa_z\sqrt{\kappa_x}}{2\sqrt{\lambda\mu}}\Delta_h + 2\Delta_h.
\end{aligned}$$

(B) Next,

$$\|\hat{T}_{\mu,\lambda}^{-1}\hat{S}^*(\hat{S}_z+\mu)^{-1}\phi(Z_i)\epsilon_i\| \leq \frac{1}{2\sqrt{\mu\lambda}}\|\phi(Z_i)\epsilon_i\|.$$

(C) Finally,

$$\begin{aligned}
& \|\hat{T}_{\mu,\lambda}^{-1}\left\{S_i^*(\hat{S}_z+\mu)^{-1}-\hat{S}^*(\hat{S}_z+\mu)^{-1}S_{z,i}(\hat{S}_z+\mu)^{-1}\right\}\mathbb{E}_n[\epsilon\phi(Z)]\| \\
& \leq \frac{1}{\lambda}\|S_i^*(\hat{S}_z+\mu)^{-1}\|\|\mathbb{E}_n[\epsilon\phi(Z)]\| + \frac{1}{2\sqrt{\lambda\mu}}\|S_{z,i}(\hat{S}_z+\mu)^{-1}\|\|\mathbb{E}_n[\epsilon\phi(Z)]\|
\end{aligned}$$

Recall we have split $\hat{T}_{\mu,\lambda}^{-1}\hat{w}_i$ into seven terms. Combining them,

$$\begin{aligned}
2\frac{1}{n}\sum_{i=1}^n\|\hat{T}_{\mu,\lambda}^{-1}\hat{w}_i\|^2 & \leq 2\mathbb{E}_n\|\hat{T}_{\mu,\lambda}^{-1}\hat{w}_i\|^2 \\
& \leq 14\left\{\frac{\kappa^2\Delta_h^2}{\lambda\mu} + \frac{4\kappa^4\kappa_x\Delta_h^2}{\lambda^2\mu} + \frac{\kappa_z^2\kappa_x\Delta_h^2}{4\lambda\mu^2} + 4\Delta_h\right. \\
& \quad + \frac{1}{4\mu\lambda}\mathbb{E}_n\|\phi(Z_i)\epsilon_i\|^2 \\
& \quad \left. + \frac{\|\mathbb{E}_n[\epsilon\phi(Z)]\|^2}{\lambda^2}\mathbb{E}_n\|S_i^*(\hat{S}_z+\mu)^{-1}\|^2 + \frac{\|\mathbb{E}_n[\epsilon\phi(Z)]\|^2}{4\lambda\mu}\mathbb{E}_n\|S_{z,i}(\hat{S}_z+\mu)^{-1}\|^2\right\}.
\end{aligned}$$

Now with the events from Lemma E.3, F.6, F.8, and F.8,

$$2\frac{1}{n}\sum_{i=1}^n\|\hat{T}_{\mu,\lambda}^{-1}\hat{w}_i\|^2 \leq 14\left\{\frac{\kappa^2\Delta_h^2}{\lambda\mu} + \frac{4\kappa^4\kappa_x\Delta_h^2}{\lambda^2\mu} + \frac{\kappa_z^2\kappa_x\Delta_h^2}{4\lambda\mu^2} + 4\Delta_h + \frac{\gamma'}{4\mu\lambda} + \frac{\tilde{\gamma}_2^2\delta_\mu''}{\lambda^2} + \frac{\tilde{\gamma}_2^2\delta_\mu'}{4\lambda\mu}\right\}.$$

Furthermore, by Lemma F.4, if $\delta_B < 1/2$, then with probability $1 - \eta$,

$$\begin{aligned} \|\Delta_1\| &= \|(\hat{T}_{\mu,\lambda}^{-1} - T_{\mu,\lambda}^{-1})u\| = \|(T_{\mu,\lambda}^{-1} - \hat{T}_{\mu,\lambda}^{-1})u\| \leq 2\delta_B \|\hat{T}_{\mu,\lambda}^{-1}u\| \\ &\leq 2\delta_B \left\{ 1 + \sqrt{2\log(1/\eta)} \right\} \sqrt{14 \left\{ \frac{\kappa^2 \Delta_h^2}{\lambda\mu} + \frac{4\kappa^4 \kappa_x \Delta_h^2}{\lambda^2 \mu} + \frac{\kappa_z^2 \kappa_x \Delta_h^2}{4\lambda\mu^2} + 4\Delta_h + \frac{\gamma'}{4\mu\lambda} + \frac{\tilde{\gamma}_2^2 \delta_\mu''}{\lambda^2} + \frac{\tilde{\gamma}_2^2 \delta_\mu'}{4\lambda\mu} \right\}}. \end{aligned}$$

□

Lemma F.12 (Δ_2 -bound). Let, $\Delta_2 = T_{\mu,\lambda}^{-1} \left\{ \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n (\hat{w}_{ij} - w_{ij}) h_{ij} \right\}$. Suppose the following bounds hold:

$$\begin{aligned} \|(S_z + \mu)^{-1}(\hat{S}_z - S_z)\| &= \delta_z \leq \frac{1}{2} \\ \|(\hat{S}_z - S_z)(S_z + \mu)^{-1/2}\| &\leq \delta'_z \\ \|\hat{T}_{\mu,\lambda}^{-1}(\hat{T}_\mu - T_\mu)\| &\leq \delta_B \leq \frac{1}{2} \\ \|S - \hat{S}\| &\leq \tilde{\gamma}_1 \\ \|\mathbb{E}_n[\phi(Z_i)\epsilon_i]\| &\leq \tilde{\gamma}_2 \\ \mathbb{E}_n \|\phi(Z_i)\epsilon_i\|^2 &\leq \gamma' \\ \mathbb{E}_n \|S_{z,i}^*(\hat{S}_z + \mu)^{-1}\|^2 &\leq \delta'_\mu \\ \mathbb{E}_n \|S_i^*(\hat{S}_z + \mu)^{-1}\|^2 &\leq \delta''_\mu \\ \mathbb{E}_n \|S_i^*(S_z + \mu)^{-1}\|^2 &\leq \delta'''_\mu \\ \mathbb{E}_n \|(S_z + \mu)^{-1/2} S_{z,i}\|^2 &\leq \delta''''_\mu. \end{aligned}$$

Then, conditional on data, with probability $1 - \eta$,

$$\|\Delta_2\| \leq \left\{ 1 + \sqrt{2\log(1/\eta)} \right\} \sqrt{\bar{\Delta}_2},$$

where

$$\begin{aligned} \bar{\Delta}_2 &= 30(1 + \delta_B) \left\{ \frac{\tilde{\gamma}_1^2}{\lambda^2} (\delta'_\mu b_{\mu,\lambda}^2 + \gamma') + \frac{\delta_z^2}{\lambda\mu} (\kappa^2 b_{\mu,\lambda}^2 + \gamma') + \frac{\kappa^2}{4\lambda\mu} \Delta_{h_{\mu,\lambda}}^2 + \frac{1}{\lambda^2} \delta''_\mu (\delta'_z)^2 v_{\mu,\lambda}^2 + \frac{\tilde{\gamma}_1^2}{\lambda^2} \delta''_\mu b_{\mu,\lambda}^2 + \frac{\kappa^2}{\lambda^2 \mu} \Delta_{h_{\mu,\lambda}}^2 + \frac{\tilde{\gamma}_2^2}{\lambda^2} \delta''_\mu \right. \\ &\quad \left. + \frac{\tilde{\gamma}_1^2}{\lambda^2 \mu} \delta'''_\mu v_{\mu,\lambda}^2 + \frac{\delta_z^2}{\lambda\mu} \delta''''_\mu v_{\mu,\lambda}^2 + \frac{1}{4\lambda\mu} \delta'_\mu (\delta'_z)^2 v_{\mu,\lambda}^2 + \frac{\tilde{\gamma}_1^2 b_{\mu,\lambda}^2}{4\lambda\mu} \delta'_\mu + \frac{\kappa_z^4 \kappa^2}{4\lambda\mu^2} \Delta_{h_{\mu,\lambda}}^2 + \frac{\tilde{\gamma}_2^2}{4\lambda\mu} \delta'_\mu \right\} \end{aligned}$$

and $\Delta_{h_{\mu,\lambda}} = \|h_{\mu,\lambda} - \hat{h}\|$ according to Lemma F.9.

Proof. Let $u = \frac{1}{n} \sum_{i=1}^n \sum_{j=1}^n (\hat{w}_{ij} - w_{ij}) h_{ij}$. Conditional on the data, with probability $1-\eta$,

$$\|T_{\mu,\lambda}^{-1} u\| \leq \left\{ 1 + \sqrt{2 \log(1/\eta)} \right\} \sqrt{\mathbb{E} \|T_{\mu,\lambda}^{-1} u\|^2}.$$

Furthermore, $\mathbb{E}_h \|T_{\mu,\lambda}^{-1} u\|^2 = \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n \|T_{\mu,\lambda}^{-1} (\hat{w}_{ij} - w_{ij})\|^2$. Note that

$$\hat{w}_{ij} - w_{ij} = \frac{\hat{w}_i - w_i}{\sqrt{2}} - \frac{\hat{w}_j - w_j}{\sqrt{2}}.$$

Considering each $\hat{w}_i - w_i$ we can write

$$\begin{aligned} & \hat{w}_i - w_i \\ &= S^*(S_z + \mu)^{-1} \phi(Z_i) (Y_i - h_{\mu,\lambda}(X_i)) + S_i^*(S_z + \mu)^{-1} S(h_0 - h_{\mu,\lambda}) - S^*(S_z + \mu)^{-1} S_{z,i} (S_z + \mu)^{-1} S(h_0 - h_{\mu,\lambda}) \\ &\quad - \{\hat{S}^*(\hat{S}_z + \mu)^{-1} \phi(Z_i) \hat{\epsilon}_i + S_i^*(\hat{S}_z + \mu)^{-1} \mathbb{E}_n[\hat{\varepsilon} \phi(Z)] - \hat{S}^*(\hat{S}_z + \mu)^{-1} S_{z,i} (\hat{S}_z + \mu)^{-1} \mathbb{E}_n[\hat{\varepsilon} \phi(Z)]\} \\ &= S^*(S_z + \mu)^{-1} \phi(Z_i) (Y_i - h_{\mu,\lambda}(X_i)) + S_i^*(S_z + \mu)^{-1} S(h_0 - h_{\mu,\lambda}) - S^*(S_z + \mu)^{-1} S_{z,i} (S_z + \mu)^{-1} S(h_0 - h_{\mu,\lambda}) \\ &\quad - \{\hat{S}^*(\hat{S}_z + \mu)^{-1} \phi(Z_i) (Y_i - \hat{h}(X_i)) + S_i^*(\hat{S}_z + \mu)^{-1} \mathbb{E}_n[(Y - \hat{h}(X)) \phi(Z)] \\ &\quad - \hat{S}^*(\hat{S}_z + \mu)^{-1} S_{z,i} (\hat{S}_z + \mu)^{-1} \mathbb{E}_n[(Y_i - \hat{h}(X)) \phi(Z)]\}. \end{aligned}$$

Within each term of $\|T_{\mu,\lambda}^{-1} (\hat{w}_{ij} - w_{ij})\|^2$,

$$\|T_{\mu,\lambda}^{-1} (\hat{w}_{ij} - w_{ij})\|^2 = \frac{1}{2} \|T_{\mu,\lambda}^{-1} \{(\hat{w}_i - w_i) - (\hat{w}_j - w_j)\}\|^2 \leq \|T_{\mu,\lambda}^{-1} \Delta w_i\|^2 + \|T_{\mu,\lambda}^{-1} \Delta w_j\|^2.$$

Using the parallelogram law again,

$$\begin{aligned} \mathbb{E}_h \|T_{\mu,\lambda}^{-1} u\|^2 &= \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n \|T_{\mu,\lambda}^{-1} (\hat{w}_{ij} - w_{ij})\|^2 \leq \frac{2}{n} \sum_{i=1}^n \|T_{\mu,\lambda}^{-1} \Delta w_i\|^2 \leq \frac{2}{n} \sum_{i=1}^n \|(T_{\mu,\lambda}^{-1} - \hat{T}_{\mu,\lambda}^{-1}) \Delta w_i\|^2 + \|\hat{T}_{\mu,\lambda}^{-1} \Delta w_i\|^2 \\ &\leq \frac{2}{n} \sum_{i=1}^n (1 + \delta_B) \|\hat{T}_{\mu,\lambda}^{-1} \Delta w_i\|^2. \end{aligned}$$

Now, we decompose $\hat{T}_{\mu,\lambda}^{-1} \Delta w_i$ further:

$$\begin{aligned}
\hat{T}_{\mu,\lambda}^{-1} \Delta w_i = & \hat{T}_{\mu,\lambda}^{-1} \left\{ \underbrace{\left[S^*(S_z + \mu)^{-1} - \hat{S}^*(\hat{S}_z + \mu)^{-1} \right] \phi(Z_i) (Y_i - h_{\mu,\lambda}(X_i)) }_A \right. \\
& + \underbrace{\hat{S}^*(\hat{S}_z + \mu)^{-1} \phi(Z_i) (\hat{h}(X_i) - h_{\mu,\lambda}(X_i)) }_B \\
& + \underbrace{S_i^* \left[(S_z + \mu)^{-1} - (\hat{S}_z + \mu)^{-1} \right] S(h_0 - h_{\mu,\lambda}) }_C \\
& + \underbrace{S_i^* (\hat{S}_z + \mu)^{-1} \left(S(h_0 - h_{\mu,\lambda}) - E_n[(Y_i - \hat{h}(X_i)) \phi(Z_i)] \right) }_D \\
& - \underbrace{\left[S^*(S_z + \mu)^{-1} - \hat{S}^*(\hat{S}_z + \mu)^{-1} \right] S_{z,i} (S_z + \mu)^{-1} S(h_0 - h_{\mu,\lambda}) }_E \\
& - \underbrace{\hat{S}^*(\hat{S}_z + \mu)^{-1} S_{z,i} \left[(S_z + \mu)^{-1} - (\hat{S}_z + \mu)^{-1} \right] S(h_0 - h_{\mu,\lambda}) }_F \\
& \left. - \hat{S}^*(\hat{S}_z + \mu)^{-1} S_{z,i} (\hat{S}_z + \mu)^{-1} \left(S(h_0 - h_{\mu,\lambda}) - \mathbb{E}_n[(Y_i - \hat{h}(X_i)) \phi(Z_i)] \right) \right\}_G.
\end{aligned}$$

(A) By Lemma F.13,

$$\begin{aligned}
\|\hat{T}_{\mu,\lambda}^{-1} \left[S^*(S_z + \mu)^{-1} - \hat{S}^*(\hat{S}_z + \mu)^{-1} \right] \phi(Z_i) (Y_i - h_{\mu,\lambda}(X_i))\| \leq & \frac{\tilde{\gamma}_1}{\lambda} \| (S_z + \mu)^{-1} \phi(Z_i) (Y_i - h_{\mu,\lambda}(X_i)) \| \\
& + \frac{\delta_z}{\sqrt{\lambda \mu}} \| \phi(Z_i) (Y_i - h_{\mu,\lambda}(X_i)) \|.
\end{aligned}$$

Generally, note that $\phi(Z_i) (Y_i - h_{\mu,\lambda}(X_i)) = S_i (h_0 - h_{\mu,\lambda}) + \phi(Z_i) \epsilon_i$. For the first term,

$$\frac{\tilde{\gamma}_1}{\lambda} \| (S_z + \mu)^{-1} \phi(Z_i) (Y_i - h_{\mu,\lambda}(X_i)) \| \leq \frac{\tilde{\gamma}_1}{\lambda} \left\{ \| (S_z + \mu)^{-1} S_i \| b_{\mu,\lambda} + \| \phi(Z_i) \epsilon_i \| \right\}.$$

For the second term,

$$\frac{\delta_z}{\sqrt{\lambda \mu}} \| \phi(Z_i) (Y_i - h_{\mu,\lambda}(X_i)) \| \leq \frac{\delta_z}{\sqrt{\lambda \mu}} \left\{ \kappa b_{\mu,\lambda} + \| \phi(Z_i) \epsilon_i \| \right\}.$$

Together, this yields

$$\begin{aligned} & \|\hat{T}_{\mu,\lambda}^{-1} \left[S^*(S_z + \mu)^{-1} - \hat{S}^*(\hat{S}_z + \mu)^{-1} \right] \phi(Z_i) (Y_i - h_{\mu,\lambda}(X_i)) \| \\ & \leq \frac{\tilde{\gamma}_1}{\lambda} \left\{ \|(S_z + \mu)^{-1} S_i\| b_{\mu,\lambda} + \|\phi(Z_i) \epsilon_i\| \right\} + \frac{\delta_z}{\sqrt{\lambda \mu}} \left\{ \kappa b_{\mu,\lambda} + \|\phi(Z_i) \epsilon_i\| \right\}. \end{aligned}$$

(B) Next,

$$\|\hat{T}_{\mu,\lambda}^{-1} \hat{S}^*(\hat{S}_z + \mu)^{-1} \phi(Z_i) (\hat{h}(X_i) - h_{\mu,\lambda}(X_i))\| = \|\hat{T}_{\mu,\lambda}^{-1} \hat{S}^*(\hat{S}_z + \mu)^{-1} S_i (\hat{h} - h_{\mu,\lambda})\| \leq \frac{\kappa}{2\sqrt{\lambda \mu}} \Delta_{h_{\mu,\lambda}}.$$

(C) Next,

$$\begin{aligned} \|\hat{T}_{\mu,\lambda}^{-1} S_i^* \left[(S_z + \mu)^{-1} - (\hat{S}_z + \mu)^{-1} \right] S (h_0 - h_{\mu,\lambda})\| &= \|\hat{T}_{\mu,\lambda}^{-1} S_i^* (\hat{S}_z + \mu)^{-1} (S_z - \hat{S}_z) (S_z + \mu)^{-1} S (h_0 - h_{\mu,\lambda})\| \\ &\leq \frac{1}{\lambda} \|S_i^* (\hat{S}_z + \mu)^{-1}\| \delta'_z v_{\mu,\lambda}. \end{aligned}$$

(D) Next,

$$\begin{aligned} & \|\hat{T}_{\mu,\lambda}^{-1} S_i^* (\hat{S}_z + \mu)^{-1} \left(S(h_0 - h_{\mu,\lambda}) - E_n[(Y_i - \hat{h}(X_i)) \phi(Z_i)] \right)\| \\ &= \|\hat{T}_{\mu,\lambda}^{-1} S_i^* (\hat{S}_z + \mu)^{-1} \left\{ (S - \hat{S})(h_0 - h_{\mu,\lambda}) - \hat{S}(h_{\mu,\lambda} - \hat{h}) - \mathbb{E}_n(\epsilon \phi(Z)) \right\}\| \\ &\leq \frac{\tilde{\gamma}_1}{\lambda} \|S_i^* (\hat{S}_z + \mu)^{-1}\| b_{\mu,\lambda} + \frac{\kappa}{\lambda \sqrt{\mu}} \Delta_{h_{\mu,\lambda}} + \frac{\tilde{\gamma}_2}{\lambda} \|S_i^* (\hat{S}_z + \mu)^{-1}\|. \end{aligned}$$

(E) By Lemma F.13,

$$\begin{aligned} & \|\hat{T}_{\mu,\lambda}^{-1} \left[S^*(S_z + \mu)^{-1} - \hat{S}^*(\hat{S}_z + \mu)^{-1} \right] S_{z,i} (S_z + \mu)^{-1} S (h_0 - h_{\mu,\lambda})\| \\ &\leq \frac{\tilde{\gamma}_1}{\lambda} \|(S_z + \mu)^{-1} S_{z,i} (S_z + \mu)^{-1} S (h_0 - h_{\mu,\lambda})\| + \frac{\delta_z}{\sqrt{\lambda \mu}} \|S_{z,i} (S_z + \mu)^{-1} S (h_0 - h_{\mu,\lambda})\| \\ &\leq \frac{\tilde{\gamma}_1}{\lambda \sqrt{\mu}} \|(S_z + \mu)^{-1} S_{z,i}\| v_{\mu,\lambda} + \frac{\delta_z}{\sqrt{\lambda \mu}} \|(S_z + \mu)^{-1/2} S_{z,i}\| v_{\mu,\lambda}. \end{aligned}$$

(F) Next,

$$\|\hat{T}_{\mu,\lambda}^{-1} \hat{S}^*(\hat{S}_z + \mu)^{-1} S_{z,i} \left[(S_z + \mu)^{-1} - (\hat{S}_z + \mu)^{-1} \right] S (h_0 - h_{\mu,\lambda})\| \leq \frac{1}{2\sqrt{\lambda \mu}} \|S_{z,i} (\hat{S}_z + \mu)^{-1}\| \delta'_z v_{\mu,\lambda}.$$

(G) Finally,

$$\begin{aligned}
& \|\hat{T}_{\mu,\lambda}^{-1}\hat{S}^*(\hat{S}_z + \mu)^{-1}S_{z,i}(\hat{S}_z + \mu)^{-1}\left(S(h_0 - h_{\mu,\lambda}) - \mathbb{E}_n[(Y_i - \hat{h}(X_i))\phi(Z_i)]\right)\| \\
& \leq \frac{1}{2\sqrt{\lambda\mu}} \left\{ \|S_{z,i}(\hat{S}_z + \mu)^{-1}(S - \hat{S})(h_0 - h_{\mu,\lambda})\| + \|S_{z,i}(\hat{S}_z + \mu)^{-1}\hat{S}(h_{\mu,\lambda} - \hat{h})\| + \|S_{z,i}(\hat{S}_z + \mu)^{-1}\mathbb{E}_n(\epsilon\phi(Z))\| \right\} \\
& \leq \frac{\tilde{\gamma}_1 b_{\mu,\lambda}}{2\sqrt{\lambda\mu}} \|S_{z,i}(\hat{S}_z + \mu)^{-1}\| + \frac{\kappa_z^2 \kappa}{2\sqrt{\lambda\mu}} \Delta_{h_{\mu,\lambda}} + \frac{\tilde{\gamma}_2}{2\sqrt{\mu\lambda}} \|S_{z,i}(\hat{S}_z + \mu)^{-1}\|.
\end{aligned}$$

We combine these bounds for $\frac{2}{n} \sum_{i=1}^n (1 + \delta_B) \|\hat{T}_{\mu,\lambda}^{-1} \Delta w_i\|^2$. Note that we split $\hat{T}_{\mu,\lambda}^{-1} \Delta w_i$ into seven terms (A-G). However, some of them were split into more terms. Overall, we obtained 15 terms, meaning we can bound the norm $\|\hat{T}_{\mu,\lambda}^{-1} \Delta w_i\|^2$ by the sum of the individual norms times 15 (i.e. $\|\hat{T}_{\mu,\lambda}^{-1} \Delta w_i\|^2 \leq 15 \sum \|\cdot\|^2$). As such,

$$\begin{aligned}
& \frac{2}{n} \sum_{i=1}^n (1 + \delta_B) \|\hat{T}_{\mu,\lambda}^{-1} \Delta w_i\|^2 \\
& \leq 30(1 + \delta_B) \left\{ \frac{\tilde{\gamma}_1^2}{\lambda^2} (\delta'_\mu b_{\mu,\lambda}^2 + \gamma') + \frac{\delta_z^2}{\lambda\mu} (\kappa^2 b_{\mu,\lambda}^2 + \gamma') + \frac{\kappa^2}{4\lambda\mu} \Delta_{h_{\mu,\lambda}}^2 + \frac{1}{\lambda^2} \delta''_\mu (\delta'_z)^2 v_{\mu,\lambda}^2 + \frac{\tilde{\gamma}_1^2}{\lambda^2} \delta''_\mu b_{\mu,\lambda}^2 + \frac{\kappa^2}{\lambda^2\mu} \Delta_{h_{\mu,\lambda}}^2 + \frac{\tilde{\gamma}_2^2}{\lambda^2} \delta''_\mu \right. \\
& \quad \left. + \frac{\tilde{\gamma}_1^2}{\lambda^2\mu} \delta'''_\mu v_{\mu,\lambda}^2 + \frac{\delta_z^2}{\lambda\mu} \delta'''_\mu v_{\mu,\lambda}^2 + \frac{1}{4\lambda\mu} \delta'_\mu (\delta'_z)^2 v_{\mu,\lambda}^2 + \frac{\tilde{\gamma}_1^2 b_{\mu,\lambda}^2}{4\lambda\mu} \delta'_\mu + \frac{\kappa_z^4 \kappa^2}{4\lambda\mu^2} \Delta_{h_{\mu,\lambda}}^2 + \frac{\tilde{\gamma}_2^2}{4\lambda\mu} \delta'_\mu \right\} := \bar{\Delta}_2.
\end{aligned}$$

Therefore, conditional on the data and the assumed high probability events, with probability $1 - \eta$,

$$\|\Delta_2\| \leq \left\{ 1 + \sqrt{2\log(1/\eta)} \right\} \sqrt{\bar{\Delta}_2}.$$

□

Lemma F.13 (Inverse estimation error bound). *If $\delta_z \leq 1/2$ and $\|\hat{S} - S\|_{op} \leq \tilde{\gamma}_1$ then, for any $u \in \mathcal{H}_z$,*

$$\|\hat{T}_{\mu,\lambda}^{-1} \{S^*(S_z + \mu)^{-1} - \hat{S}^*(\hat{S}_z + \mu)^{-1}\} u\| \leq \frac{\tilde{\gamma}_1}{\lambda} \|(S_z + \mu)^{-1} u\| + \frac{1}{\sqrt{\lambda\mu}} \delta_z \|u\|.$$

Proof. Using the resolvent identity, we can write

$$\begin{aligned}
& \|\hat{T}_{\mu,\lambda}^{-1}\{S^*(S_z+\mu)^{-1} - \hat{S}^*(\hat{S}_z+\mu)^{-1}\}u\| \\
&= \|\hat{T}_{\mu,\lambda}^{-1}\{S^*(S_z+\mu)^{-1} - \hat{S}^*(\hat{S}_z+\mu)^{-1} \pm \hat{S}^*(S_z+\mu)^{-1}\}u\| \\
&= \|\hat{T}_{\mu,\lambda}^{-1}\{(S-\hat{S})^*(S_z+\mu)^{-1} - \hat{S}^*\{(\hat{S}_z+\mu)^{-1} - (S_z+\mu)^{-1}\}u\}\| \\
&\leq \frac{\tilde{\gamma}_1}{\lambda} \|(S_z+\mu)^{-1}u\| + \|\hat{T}_{\mu,\lambda}^{-1}\hat{S}^*(\hat{S}_z+\mu)^{-1}\sum_{r=1}^{\infty}\{(\hat{S}_z-S_z)(S_z+\mu)^{-1}\}^r u\| \\
&\leq \frac{\tilde{\gamma}_1}{\lambda} \|(S_z+\mu)^{-1}u\| + \frac{1}{2\sqrt{\lambda\mu}} 2\delta_z \|u\|.
\end{aligned}$$

□

F.4 Main result

Lemma F.14 (Decomposition of the bootstrap error). $\mathfrak{B} - Z_{\mathfrak{B}} = \Delta_1 + \Delta_2$, where $\Delta_1 = (\hat{T}_{\mu,\lambda}^{-1} - T_{\mu,\lambda}^{-1})(\frac{1}{n}\sum_{i=1}^n\sum_{j=1}^n\hat{w}_{ij}h_{ij})$ and $\Delta_2 = T_{\mu,\lambda}^{-1}\left\{\frac{1}{n}\sum_{i=1}^n\sum_{j=1}^n(\hat{w}_{ij} - w_{ij})h_{ij}\right\}$.

Proof. The proof follows by matching symbols with Lemma G.3 in Singh and Vijaykumar (2023). □

Theorem F.1 (Feasible bootstrap). Suppose $n \geq \max\{N_{\delta_z}, 3N_{\delta}\}$ such that $\delta_B = 3\delta \leq 1/2$ and $\delta_z \leq 1/2$. Then with probability $1 - 15\eta$,

$$\|\mathfrak{B} - Z_{\mathfrak{B}}\| \leq C_{\eta} \left[\frac{1}{\lambda} A + \frac{1}{\lambda\sqrt{\mu}} B + \frac{1}{\sqrt{\lambda\mu}} C + \frac{1}{\sqrt{\lambda\mu}} D \right],$$

where

$$\begin{aligned}
A &= 2\delta_B \sqrt{14} \left(\tilde{\gamma}_2 \sqrt{\delta''_\mu} + \sqrt{30(1+\delta_B)} \left(\tilde{\gamma}_1 (b_{\mu,\lambda} \sqrt{\delta'_\mu} + \sqrt{\gamma'}) + \delta'_z v_{\mu,\lambda} \sqrt{\delta''_\mu} + \tilde{\gamma}_1 b_{\mu,\lambda} \sqrt{\delta''_\mu} + \tilde{\gamma}_2 \sqrt{\delta''_\mu} \right) \right), \\
B &= 2\delta_B \sqrt{14} \left(2\kappa^2 \sqrt{\kappa_x} \Delta_h \right) + \sqrt{30(1+\delta_B)} \left(\kappa \Delta_{h_{\mu,\lambda}} + \tilde{\gamma}_1 v_{\mu,\lambda} \sqrt{\delta'''_\mu} \right), \\
C &= 2\delta_B \sqrt{14} \left(\kappa \Delta_h + \frac{1}{2} \sqrt{\gamma'} + \frac{1}{2} \tilde{\gamma}_2 \sqrt{\delta'_\mu} + 2\sqrt{\Delta_h \lambda \mu} \right) \\
&\quad + \sqrt{30(1+\delta_B)} \left(\delta_z (\kappa b_{\mu,\lambda} + \sqrt{\gamma'}) + \frac{1}{2} \kappa \Delta_{h_{\mu,\lambda}} + \delta_z v_{\mu,\lambda} \sqrt{\delta'''_\mu} + \frac{1}{2} \delta'_z v_{\mu,\lambda} \sqrt{\delta'_\mu} + \frac{1}{2} \tilde{\gamma}_1 b_{\mu,\lambda} \sqrt{\delta'_\mu} + \frac{1}{2} \tilde{\gamma}_2 \sqrt{\delta'_\mu} \right), \\
D &= 2\delta_B \sqrt{14} \left(\frac{1}{2} \kappa_z \sqrt{\kappa_x} \Delta_h \right) + \sqrt{30(1+\delta_B)} \left(\frac{1}{2} \kappa \kappa_z^2 \Delta_{h_{\mu,\lambda}} \right).
\end{aligned}$$

Proof. Let $C_\eta = \left\{ 1 + \sqrt{2\log(1/\eta)} \right\}$. We first collect the events and then determine the probability. Using Lemma F.11 and Lemma F.12, with probability $1 - 2\eta$,

$$\begin{aligned}
\|\mathfrak{B} - Z_{\mathfrak{B}}\| &\leq \|\Delta_1\| + \|\Delta_2\| \\
&\leq 2\delta_B C_\eta \sqrt{14} \left(\frac{\kappa^2 \Delta_h^2}{\lambda \mu} + \frac{4\kappa^4 \kappa_x \Delta_h^2}{\lambda^2 \mu} + \frac{\kappa_z^2 \kappa_x \Delta_h^2}{4\lambda \mu^2} + 4\Delta_h + \frac{\gamma'}{4\lambda \mu} + \frac{\tilde{\gamma}_2^2 \delta''_\mu}{\lambda^2} + \frac{\tilde{\gamma}_2^2 \delta'_\mu}{4\lambda \mu} \right) \\
&\quad + C_\eta \left[30(1+\delta_B) \left\{ \frac{\tilde{\gamma}_1^2}{\lambda^2} (\delta'_\mu b_{\mu,\lambda}^2 + \gamma') + \frac{\delta_z^2}{\lambda \mu} (\kappa^2 b_{\mu,\lambda}^2 + \gamma') + \frac{\kappa^2}{4\lambda \mu} \Delta_{h_{\mu,\lambda}}^2 + \frac{1}{\lambda^2} \delta''_\mu (\delta'_z)^2 v_{\mu,\lambda}^2 \right. \right. \\
&\quad \left. \left. + \frac{\tilde{\gamma}_1^2}{\lambda^2} \delta''_\mu b_{\mu,\lambda}^2 + \frac{\kappa^2}{\lambda^2 \mu} \Delta_{h_{\mu,\lambda}}^2 + \frac{\tilde{\gamma}_2^2}{\lambda^2} \delta''_\mu + \frac{\tilde{\gamma}_1^2}{\lambda^2 \mu} \delta'''_\mu v_{\mu,\lambda}^2 + \frac{\delta_z^2}{\lambda \mu} \delta'''_\mu v_{\mu,\lambda}^2 \right. \right. \\
&\quad \left. \left. + \frac{1}{4\lambda \mu} \delta'_\mu (\delta'_z)^2 v_{\mu,\lambda}^2 + \frac{\tilde{\gamma}_1^2 b_{\mu,\lambda}^2}{4\lambda \mu} \delta'_\mu + \frac{\kappa_z^4 \kappa^2}{4\lambda \mu^2} \Delta_{h_{\mu,\lambda}}^2 + \frac{\tilde{\gamma}_2^2}{4\lambda \mu} \delta'_\mu \right\} \right]^{1/2} \\
&\leq C_\eta \left[2\delta_B \sqrt{14} \left(\frac{\kappa}{\sqrt{\lambda \mu}} \Delta_h + \frac{2\kappa^2 \sqrt{\kappa_x}}{\lambda \sqrt{\mu}} \Delta_h + \frac{\kappa_z \sqrt{\kappa_x}}{2\sqrt{\lambda \mu}} \Delta_h + \frac{2\sqrt{\Delta_h \lambda \mu}}{\sqrt{\lambda \mu}} + \frac{\sqrt{\gamma'}}{2\sqrt{\lambda \mu}} + \frac{\tilde{\gamma}_2}{\lambda} \sqrt{\delta''_\mu} + \frac{\tilde{\gamma}_2}{2\sqrt{\lambda \mu}} \sqrt{\delta'_\mu} \right) \right. \\
&\quad \left. + \sqrt{30(1+\delta_B)} \left(\frac{\tilde{\gamma}_1}{\lambda} (b_{\mu,\lambda} \sqrt{\delta'_\mu} + \sqrt{\gamma'}) + \frac{\delta_z}{\sqrt{\lambda \mu}} (\kappa b_{\mu,\lambda} + \sqrt{\gamma'}) + \frac{\kappa}{2\sqrt{\lambda \mu}} \Delta_{h_{\mu,\lambda}} \right. \right. \\
&\quad \left. \left. + \frac{1}{\lambda} \delta'_z v_{\mu,\lambda} \sqrt{\delta''_\mu} + \frac{\tilde{\gamma}_1}{\lambda} b_{\mu,\lambda} \sqrt{\delta''_\mu} + \frac{\kappa}{\lambda \sqrt{\mu}} \Delta_{h_{\mu,\lambda}} + \frac{\tilde{\gamma}_2}{\lambda} \sqrt{\delta''_\mu} + \frac{\tilde{\gamma}_1}{\lambda \sqrt{\mu}} v_{\mu,\lambda} \sqrt{\delta'''_\mu} + \frac{\delta_z}{\sqrt{\lambda \mu}} v_{\mu,\lambda} \sqrt{\delta''_\mu} \right. \right. \\
&\quad \left. \left. + \frac{\delta'_z v_{\mu,\lambda}}{2\sqrt{\lambda \mu}} \sqrt{\delta'_\mu} + \frac{\tilde{\gamma}_1 b_{\mu,\lambda}}{2\sqrt{\lambda \mu}} \sqrt{\delta'_\mu} + \frac{\kappa \kappa_z^2}{2\sqrt{\lambda \mu}} \Delta_{h_{\mu,\lambda}} + \frac{\tilde{\gamma}_2}{2\sqrt{\lambda \mu}} \sqrt{\delta'_\mu} \right) \right] \\
&= C_\eta \left[\frac{1}{\lambda} A + \frac{1}{\lambda \sqrt{\mu}} B + \frac{1}{\sqrt{\lambda \mu}} C + \frac{1}{\sqrt{\lambda \mu}} D \right].
\end{aligned}$$

Here,

$$\begin{aligned}
A &= 2\delta_B \sqrt{14} \left(\tilde{\gamma}_2 \sqrt{\delta''_\mu} + \sqrt{30(1+\delta_B)} \left(\tilde{\gamma}_1 (b_{\mu,\lambda} \sqrt{\delta'_\mu} + \sqrt{\gamma'}) + \delta'_z v_{\mu,\lambda} \sqrt{\delta''_\mu} + \tilde{\gamma}_1 b_{\mu,\lambda} \sqrt{\delta''_\mu} + \tilde{\gamma}_2 \sqrt{\delta''_\mu} \right) \right), \\
B &= 2\delta_B \sqrt{14} \left(2\kappa^2 \sqrt{\kappa_x} \Delta_h \right) + \sqrt{30(1+\delta_B)} \left(\kappa \Delta_{h_{\mu,\lambda}} + \tilde{\gamma}_1 v_{\mu,\lambda} \sqrt{\delta'''_\mu} \right), \\
C &= 2\delta_B \sqrt{14} \left(\kappa \Delta_h + \frac{1}{2} \sqrt{\gamma'} + \frac{1}{2} \tilde{\gamma}_2 \sqrt{\delta'_\mu} + 2\sqrt{\Delta_h \lambda \mu} \right) \\
&\quad + \sqrt{30(1+\delta_B)} \left(\delta_z (\kappa b_{\mu,\lambda} + \sqrt{\gamma'}) + \frac{1}{2} \kappa \Delta_{h_{\mu,\lambda}} + \delta_z v_{\mu,\lambda} \sqrt{\delta'''_\mu} + \frac{1}{2} \delta'_z v_{\mu,\lambda} \sqrt{\delta'_\mu} + \frac{1}{2} \tilde{\gamma}_1 b_{\mu,\lambda} \sqrt{\delta'_\mu} + \frac{1}{2} \tilde{\gamma}_2 \sqrt{\delta'_\mu} \right), \\
D &= 2\delta_B \sqrt{14} \left(\frac{1}{2} \kappa_z \sqrt{\kappa_x} \Delta_h \right) + \sqrt{30(1+\delta_B)} \left(\frac{1}{2} \kappa \kappa_z^2 \Delta_{h_{\mu,\lambda}} \right).
\end{aligned}$$

Now, the events with bounds $\delta'_\mu, \delta''_\mu, \delta'''_\mu, \delta''''_\mu, \tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}$ each hold with probability $1-\eta$. Moreover, δ_z, δ'_z also hold with probability $1-\eta$. Notice that δ_B depends only on δ (Lemma F.10) which holds with $1-2\eta$ since we have already accounted for $\tilde{\gamma}_1$ and δ_z . When using the consistency bounds from Lemma F.9, the bound holds with probability $1-13\eta$ when taking into account the 2η from before. For $\Delta_{h_{\mu,\lambda}}$, we need to consider the unconsidered events from our Bahadur representation. Note $\|\hat{h} - h_{\mu,\lambda}\| \leq \|\mathbb{E}_n(U_i)\| + \Delta_U$ with

$$\|\mathbb{E}_n(U_i)\| \leq \frac{1}{2\sqrt{\lambda\mu}} \tilde{\gamma}_1 b_{\mu,\lambda} + \frac{1}{\lambda} \delta''_z v_{\mu,\lambda} + \frac{1}{2\sqrt{\lambda\mu}} \tilde{\gamma}_2 + \frac{1}{2\sqrt{\lambda\mu}} \delta'_z b_{\mu,\lambda}.$$

Here, only δ''_z is new. In Δ_U (see Theorem E.1) we observe that only $\tilde{\gamma}_3$ (through γ) is unaccounted for. Thus, we can use the derived bound for Δ_U with an additional probability of $1-\eta$ and

$$\Delta_U \leq 2\delta\gamma + \gamma_1 + \xi_1 + \left[2\delta^2 + \frac{\delta_z}{\sqrt{\lambda\mu}} \{ \delta_z (\kappa + \tilde{\gamma}_1) + \frac{\tilde{\gamma}_1}{2} \} + \frac{\tilde{\gamma}_1}{\lambda\mu} \{ \tilde{\gamma}_1 + \delta_z (1+2\delta_z) (\kappa + \tilde{\gamma}_1) \} \right] \|h_0 - h_{\mu,\lambda}\|.$$

Given that Δ_h 's probabilistic part is only, $\Delta_{h_{\mu,\lambda}}$ we can bound the overall $\|\mathfrak{B} - Z\|$ with the above events and probability $1-15\eta$. \square

G Gaussian coupling

G.1 Polynomial decay

Assumption G.1 (Spectrum of S_x). S_x is an operator with eigenvalues $\nu_s(S_x)$ arranged in nonincreasing order. The spectrum of S_z decays polynomially, i.e.

$$\nu_s(S_x) \asymp \omega_x s^{-1/(\rho_x - 1)},$$

with $\rho_x \in [1, 2]$.

Assumption G.2 (Spectrum of S_z). S_z is a positive, self-adjoint operator with eigenvalues $\nu_s(S_z)$ arranged in nonincreasing order. The spectrum of S_z decays polynomially, i.e.

$$\nu_s(S_z) \asymp \omega_z s^{-1/(\rho_z - 1)},$$

with $\rho_z \in [1, 2]$.

Lemma G.1 (Effective dimension bound). Suppose Assumptions G.1 and G.2 hold. Then, the effective dimensions are bounded by

$$\mathfrak{n}_z(\mu) \lesssim_{\rho_z, \omega_z} \mu^{-\rho_x},$$

$$\mathfrak{m}(\lambda, \mu) \lesssim_{\rho_x, \omega_x} \lambda^{-\rho_x},$$

$$\tilde{\mathfrak{m}}(\lambda, \mu) \lesssim_{\rho_x, \omega_x} \lambda^{-\rho_x},$$

with $\rho_x, \rho_z \in [1, 2]$.

Proof. The bound for $\mathfrak{n}_z(\mu)$ follows directly from Proposition K.2 of Singh and Vijaykumar (2023) by taking $\rho_z = 1 + 1/\beta$, where this β represents the notation used in their paper. For $\mathfrak{m}(\lambda, \mu) = \text{tr}\{(T_\mu + \lambda)^{-2} T_\mu\}$ note that $T_\mu \preceq T \preceq S_x$. Following Singh and Vijaykumar (2023), define a generalized effective dimension of S_x as

$$\psi(m, c) = \sum_{s=m+1}^{\infty} \frac{\nu_s}{(\nu_s + \lambda)^c}.$$

Here,

$$\mathfrak{m}(\lambda, \mu) \leq \frac{1}{\lambda} \text{tr}\{(T_\mu + \lambda)^{-1} T_\mu\} \leq \frac{1}{\lambda} \text{tr}\{(T + \lambda)^{-1} T\} \leq \frac{1}{\lambda} \text{tr}\{(S_x + \lambda)^{-1} S_x\} = \frac{1}{\lambda} \psi(0, 1) \lesssim_{\rho_x, \omega_x} \frac{1}{\lambda} \frac{1}{\lambda^{\rho_x - 1}} = \lambda^{-\rho_x}.$$

The last inequality follows again by applying their Proposition K.2. Lastly,

$$\tilde{\mathfrak{m}}(\lambda, \mu) = \text{tr} T_{\mu, \lambda}^{-1} S^* (S_z + \mu)^{-1} S_z (S_z + \mu)^{-1} S T_{\mu, \lambda}^{-1} \leq \text{tr} T_{\mu, \lambda}^{-1} S^* (S_z + \mu)^{-1} S T_{\mu, \lambda}^{-1} = \mathfrak{m}(\lambda, \mu) \lesssim_{\rho_x, \omega_x} \lambda^{-\rho_x}.$$

□

G.2 Main result

Remark G.1 (Rate of $\Delta_{h_{\mu, \lambda}}$). We analyze the rate of $\Delta_{h_{\mu, \lambda}}$. For this, note that when considering $\mu \leq \lambda$,

$$\begin{aligned} \Delta_{h_{\mu, \lambda}} &= \mathcal{O}\left(\frac{1}{2\sqrt{\lambda\mu}}\tilde{\gamma}_1 b_{\mu, \lambda} + \frac{1}{\lambda}\delta''_z v_{\mu, \lambda} + \frac{1}{2\sqrt{\lambda\mu}}\tilde{\gamma}_2 + \frac{1}{2\sqrt{\lambda\mu}}\delta'_z b_{\mu, \lambda} + \Delta_U\right) \\ &= \mathcal{O}\left(\frac{1}{\sqrt{n\lambda\mu}}b_{\mu, \lambda} + \frac{1}{\lambda}\sqrt{\frac{\mu\mathfrak{n}_z(\mu)}{n}}v_{\mu, \lambda} + \frac{1}{\sqrt{n\lambda\mu}} + \frac{1}{\sqrt{\lambda\mu}}\sqrt{\frac{\mu\mathfrak{n}_z(\mu)}{n}}b_{\mu, \lambda} + \Delta_U\right) \\ &= \mathcal{O}\left(\frac{1}{\lambda}\sqrt{\frac{\mu\mathfrak{n}_z(\mu)}{n}}v_{\mu, \lambda} + \frac{1}{\sqrt{n\lambda\mu}} + \frac{1}{\sqrt{\lambda\mu}}\sqrt{\frac{\mu\mathfrak{n}_z(\mu)}{n}}b_{\mu, \lambda} + \Delta_U\right) \\ &= \mathcal{O}\left(\frac{1}{\sqrt{n\lambda\mu}} + \frac{1}{\sqrt{\lambda\mu}}\sqrt{\frac{\mu\mathfrak{n}_z(\mu)}{n}}b_{\mu, \lambda} + \Delta_U\right). \end{aligned}$$

Theorem G.1 (Gaussian approximation). *Suppose n satisfies the rate Assumption D.2 i.e. $n \geq \max\{N_{\delta_z}, N_\delta\}$, $\|U_i\| \leq a \lesssim \frac{\widetilde{M}}{\sqrt{\lambda\mu}}$ and that Assumptions G.1 and G.2 hold. In addition, suppose the assumptions for the bias upper bound hold (Lemma C.2). Then, there exists a sequence $(Z_i)_{1 \leq i \leq n}$ of Gaussians in \mathcal{H}_x , with covariance Σ such that with probability $1 - \eta$,*

$$\|\sqrt{n}(\hat{h} - h_{\mu, \lambda}) - \frac{1}{\sqrt{n}} \sum_{i=1}^n Z_i\| \lesssim Q_\bullet(T, n, \lambda, \mu) \widetilde{M} \log(36/\eta) + Q_{\text{res}},$$

where

$$Q_\bullet = \inf_{m \geq 1} \left\{ \frac{\sigma(T, m)}{\lambda} + \frac{m^2 \log(m^2)}{\sqrt{n\mu\lambda}} \right\}$$

and

$$Q_{\text{res}} = \left(\frac{\log(12/\eta)^2 \mathbf{n}_z(\mu)}{n\lambda\mu} b_{\mu,\lambda} + \frac{\log(12/\eta)^3 \mathbf{n}_z(\mu)^{3/2}}{n^{3/2}\lambda^{3/2}\mu^{\frac{1}{2}}} + \frac{\log(12/\eta)^2 \sqrt{\mathbf{n}_z(\mu)}}{n\lambda\mu} \right).$$

Proof. Recall that by Theorem E.1, with probability $1 - 5\eta$,

$$\|\sqrt{n}(\hat{h} - h_{\mu,\lambda}) - \frac{1}{\sqrt{n}} \sum_{i=1}^n U_i\| \lesssim B(n, \mu, \lambda, \eta, h_0) + V(n, \mu, \lambda, \eta).$$

Now, we study the behavior V and B by plugging in the tractable behavior of the effective dimensions, while noting that $\mu \leq \lambda$ by the bias argument. Let $l(\eta) = \ln(2/\eta)$ and recall that

$$\begin{aligned} \tilde{\delta}(\mu, \lambda, \eta) &= 2l(\eta) \left(\frac{4\kappa\kappa_x^{1/2}}{n\lambda\mu^{1/2}} \vee \sqrt{\frac{\kappa\kappa_x^{1/2}}{n\mu^{1/2}\lambda^{\rho_x}}} \right) = \mathcal{O}\left(l(\eta)\sqrt{\frac{1}{n\mu^{1/2}\lambda^{\rho_x}}}\right), \\ V(n, \mu, \lambda, \eta) &= \mathcal{O}\left(\frac{l(\eta)^3}{n^{3/2}\lambda^{3/2}\mu^{1/2+(3/2)\rho_z}} \vee \frac{l(\eta)^3}{n^{3/2}\lambda^{1+(1/2)\rho_x}\mu^{1/2+\rho_z}} \vee \frac{l(\eta)^2}{n\lambda\mu^{1+(1/2)\rho_z}} \vee \frac{l(\eta)^2}{n\lambda^{1/2+(1/2)\rho_x}\mu^{3/4}}\right), \\ B(n, \mu, \lambda, \eta, h_0) &= \mathcal{O}\left(\frac{l(\eta)^2}{n\lambda\mu^{1+\rho_z}} \vee \tilde{\delta}(\mu, \lambda, \eta)^2\right) b_{\mu,\lambda} = \mathcal{O}\left(\frac{l(\eta)^2}{n\lambda\mu^{1+\rho_z}} b_{\mu,\lambda}\right). \end{aligned}$$

Now, we proceed to simplify using $\mu \leq \lambda$:

$$\begin{aligned} B + V &= \mathcal{O}\left(\frac{l(\eta)^2}{n\lambda^2\mu^{1+\rho_z}} b_{\mu,\lambda} \vee \frac{l(\eta)^3}{n^{3/2}\lambda^{3/2}\mu^{1/2+(3/2)\rho_z}} \vee \frac{l(\eta)^3}{n^{3/2}\lambda^{1+(1/2)\rho_x}\mu^{1/2+\rho_z}} \vee \frac{l(\eta)^2}{n\lambda\mu^{1+(1/2)\rho_z}} \vee \frac{l(\eta)^2}{n\lambda^{1/2+(1/2)\rho_x}\mu^{3/4}}\right) \\ &= \mathcal{O}\left(\frac{l(\eta)^2}{n\lambda\mu^{1+\rho_z}} b_{\mu,\lambda} \vee \frac{l(\eta)^3}{n^{3/2}\lambda^{3/2}\mu^{\frac{1}{2}+\frac{3}{2}\rho_z}} \vee \frac{l(\eta)^2}{n\lambda\mu^{1+\frac{1}{2}\rho_z}}\right). \end{aligned}$$

We arrive at the final form for polynomial decay:

$$\|\sqrt{n}(\hat{h} - h_{\mu,\lambda}) - \frac{1}{\sqrt{n}} \sum_{i=1}^n U_i\| \lesssim \left(\frac{l(\eta)^2}{n\lambda\mu^{1+\rho_z}} b_{\mu,\lambda} + \frac{l(\eta)^3}{n^{3/2}\lambda^{3/2}\mu^{\frac{1}{2}+\frac{3}{2}\rho_z}} + \frac{l(\eta)^2}{n\lambda\mu^{1+\frac{1}{2}\rho_z}} \right).$$

In general form,

$$\|\sqrt{n}(\hat{h} - h_{\mu,\lambda}) - \frac{1}{\sqrt{n}} \sum_{i=1}^n U_i\| \lesssim \left(\frac{l(\eta)^2 \mathbf{n}_z(\mu)}{n\lambda\mu} b_{\mu,\lambda} + \frac{l(\eta)^3 \mathbf{n}_z(\mu)^{3/2}}{n^{3/2}\lambda^{3/2}\mu^{\frac{1}{2}}} + \frac{l(\eta)^2 \sqrt{\mathbf{n}_z(\mu)}}{n\lambda\mu} \right).$$

Recall the upper bound for the $\|U_i\| \lesssim \frac{\widetilde{M}}{\sqrt{\lambda\mu}}$ where \widetilde{M} encompasses all constants in the upper bound; see Corollary D.1 for details. Now, using Theorem A.1 from Singh and Vijaykumar

(2023) together with Lemma D.3, with probability $1-\eta$,

$$\begin{aligned} \left\| \frac{1}{\sqrt{n}} \sum_{i=1}^n (U_i - Z_i) \right\| &\lesssim \inf_{m \geq 1} \left\{ \sqrt{\log(6/\eta)} \sigma(\Sigma, m) + \frac{am^2 \log(m^2/\eta)}{\sqrt{n}} \right\} \\ &\lesssim \inf_{m \geq 1} \left\{ \sqrt{\log(6/\eta)} \frac{\sqrt{8}\bar{\sigma}}{\lambda} \sigma(T, m) + \frac{am^2 \log(m^2/\eta)}{\sqrt{n}} \right\} \\ &\lesssim \inf_{m \geq 1} \left\{ \sqrt{\log(6/\eta)} \frac{\widetilde{M}}{\lambda} \sigma(T, m) + \frac{\widetilde{M}m^2 \log(m^2/\eta)}{\sqrt{n}\lambda\mu} \right\}. \end{aligned}$$

Using a union bound, the Gaussian approximation holds with $1-6\eta$. Rescaling $\eta/6$ gives the result. \square

H Bootstrap coupling

Assumption H.1 (Combined rate condition). *Assume a universal n that satisfies all rate conditions imposed with*

$$n \geq \max\{3N_\delta, N_{\delta_z}, \mu^{-2}\lambda^{-1}\tilde{\mathfrak{m}}(\lambda\mu)^{-1}\}.$$

Remark H.1 (Dominant terms in δ). We study the dominant terms in δ from Lemma F.10. Note that this is a *different* δ than the one we used for the Gaussian coupling:

$$\begin{aligned} \delta &= \mathcal{O}\left(\frac{\delta_z \tilde{\gamma}_1 + \tilde{\gamma}_1^2}{\lambda\mu} + \frac{\tilde{\gamma}_z}{\sqrt{\lambda\mu}} + \frac{\tilde{\gamma}_1}{\sqrt{\lambda\mu}} + \tilde{\delta}(\mu, \lambda, \eta)\right) \\ &= \mathcal{O}\left(\frac{\sqrt{\mathfrak{n}_z(\mu)}}{n\mu\lambda} \vee \frac{1}{n^{3/2}\lambda\mu^2} \vee \frac{1}{\sqrt{n\lambda}\mu} \vee \frac{1}{n\sqrt{\lambda}\mu^{3/2}} \vee \sqrt{\frac{\mathfrak{m}(\lambda, \mu)}{n\mu^{1/2}}}\right) \\ &= \mathcal{O}\left(\frac{1}{\sqrt{n\lambda}\mu} \vee \frac{1}{n\sqrt{\lambda}\mu^{3/2}} \vee \sqrt{\frac{1}{n\mu^{1/2}\lambda^{\rho_x}}}\right) \\ &= \mathcal{O}\left(\frac{1}{\sqrt{n\lambda}\mu} \vee \frac{1}{n\sqrt{\lambda}\mu^{3/2}}\right) = \frac{1}{\sqrt{n\lambda}\mu} \mathcal{O}\left(1 \vee \frac{1}{\sqrt{n\mu}}\right) = \mathcal{O}\left(\frac{1}{\sqrt{n\lambda}\mu}\right). \end{aligned}$$

All terms converge to zero because of the rate condition, taking $\lambda=\mu^\nu$. This rate also holds for δ_B .

Lemma H.1 (Feasible bootstrap rate). *Suppose n satisfies the combined rate condition Assumption H.1 and that Assumptions G.1 and G.2 hold. In addition, suppose that the*

assumptions for the bias upper bound hold (Lemma C.2). Then, with probability $1 - 15\eta$,

$$\|\mathfrak{B} - Z_{\mathfrak{B}}\| \lesssim C_\eta \left[\frac{l(\eta)}{\mu^2 \sqrt{n} \lambda} b_{\mu, \lambda} + \frac{l(\eta)}{\lambda \mu^{3/2} \sqrt{n}} + \frac{l(\eta)^2 \mathfrak{n}_z(\mu)}{n \lambda^{3/2} \mu^2} b_{\mu, \lambda} + \frac{l(\eta)^3 \mathfrak{n}_z(\mu)^{3/2}}{n^{3/2} \lambda^2 \mu^{\frac{3}{2}}} + \frac{l(\eta)^2 \sqrt{\mathfrak{n}_z(\mu)}}{n \lambda^{3/2} \mu^2} \right].$$

Proof. Recall the elements of the feasible bootstrap

$$\begin{aligned} A &= \mathcal{O}(2\delta_B \sqrt{14} \left(\tilde{\gamma}_2 \sqrt{\delta''_\mu} + \sqrt{30(1+\delta_B)} \left(\tilde{\gamma}_1 (b_{\mu, \lambda} \sqrt{\delta'_\mu} + \sqrt{\gamma'}) + \delta'_z v_{\mu, \lambda} \sqrt{\delta''_\mu} + \tilde{\gamma}_1 b_{\mu, \lambda} \sqrt{\delta''_\mu} + \tilde{\gamma}_2 \sqrt{\delta''_\mu} \right) \right) \\ &= \mathcal{O} \left(\delta_B \left(\tilde{\gamma}_2 \sqrt{\delta''_\mu} \right) + \tilde{\gamma}_1 + \delta'_z v_{\mu, \lambda} \sqrt{\delta''_\mu} + \tilde{\gamma}_2 \sqrt{\delta''_\mu} \right) \\ &= \mathcal{O} \left(\delta_B \sqrt{\delta''_\mu} (n^{-1/2}) \vee \sqrt{\frac{\mu \mathfrak{n}_z(\mu)}{n}} v_{\mu, \lambda} \sqrt{\delta''_\mu} \vee \frac{1}{\sqrt{n}} \sqrt{\delta''_\mu} \right) \\ &= \mathcal{O} \left(\sqrt{\delta''_\mu} \left\{ \delta_B n^{-1/2} \vee \sqrt{\frac{\mu \mathfrak{n}_z(\mu)}{n}} v_{\mu, \lambda} \vee \frac{1}{\sqrt{n}} \right\} \right) \\ &= \mathcal{O}(\sqrt{\delta''_\mu} n^{-1/2}), \\ B &= \mathcal{O}(2\delta_B \sqrt{14} \left(2\kappa^2 \sqrt{\kappa_x} \Delta_h \right) + \sqrt{30(1+\delta_B)} \left(\kappa \Delta_{h_{\mu, \lambda}} + \tilde{\gamma}_1 v_{\mu, \lambda} \sqrt{\delta'''_\mu} \right)) \\ &= \mathcal{O}(\delta_B (b_{\mu, \lambda} + \Delta_{h_{\mu, \lambda}}) + \Delta_{h_{\mu, \lambda}}) \\ &= \mathcal{O}(\delta_B b_{\mu, \lambda} + \Delta_{h_{\mu, \lambda}}), \\ C &= \mathcal{O}(2\delta_B \sqrt{14} \left(\kappa \Delta_h + \frac{1}{2} \sqrt{\gamma'} + \frac{1}{2} \tilde{\gamma}_2 \sqrt{\delta'_\mu} + 2 \sqrt{\Delta_h \lambda \mu} \right) \\ &\quad + \sqrt{30(1+\delta_B)} \left(\delta_z (\kappa b_{\mu, \lambda} + \sqrt{\gamma'}) + \frac{1}{2} \kappa \Delta_{h_{\mu, \lambda}} + \delta_z v_{\mu, \lambda} \sqrt{\delta'''_\mu} + \frac{1}{2} \delta'_z v_{\mu, \lambda} \sqrt{\delta'_\mu} + \frac{1}{2} \tilde{\gamma}_1 b_{\mu, \lambda} \sqrt{\delta'_\mu} + \frac{1}{2} \tilde{\gamma}_2 \sqrt{\delta'_\mu} \right)) \\ &= \mathcal{O} \left((\delta_B \left(\Delta_h + \tilde{\gamma}_2 \sqrt{\delta'_\mu} \right) + \left(\delta_z b_{\mu, \lambda} + \Delta_{h_{\mu, \lambda}} + \delta_z v_{\mu, \lambda} \sqrt{\delta'''_\mu} + \delta'_z v_{\mu, \lambda} \sqrt{\delta'_\mu} + \tilde{\gamma}_2 \sqrt{\delta'_\mu} \right)) \right) \\ &= \mathcal{O} \left(\delta_B \left(b_{\mu, \lambda} + \tilde{\gamma}_2 \sqrt{\delta'_\mu} \right) + \left(\delta_z b_{\mu, \lambda} + \Delta_{h_{\mu, \lambda}} + \delta_z v_{\mu, \lambda} \sqrt{\delta'_\mu} + \tilde{\gamma}_2 \sqrt{\delta'_\mu} \right) \right) \\ &= \mathcal{O}(\sqrt{\delta'_\mu} (\delta_B \tilde{\gamma}_2 + \delta_z v_{\mu, \lambda} + \tilde{\gamma}_2) + \delta_z b_{\mu, \lambda} + \Delta_{h_{\mu, \lambda}}) \\ &= \mathcal{O}(\sqrt{\delta'_\mu} (\delta_z v_{\mu, \lambda} + \tilde{\gamma}_2) + \delta_z b_{\mu, \lambda} + \Delta_{h_{\mu, \lambda}}), \\ D &= 2\delta_B \sqrt{14} \left(\frac{1}{2} \kappa_z \sqrt{\kappa_x} \Delta_h \right) + \sqrt{30(1+\delta_B)} \left(\frac{1}{2} \kappa \kappa_z^2 \Delta_{h_{\mu, \lambda}} \right) \\ &= \mathcal{O}(\delta_B b_{\mu, \lambda} + \Delta_{h_{\mu, \lambda}}). \end{aligned}$$

We can now study the error bound of the bootstrap in detail:

$$\|\mathfrak{B} - Z_{\mathfrak{B}}\| \leq C_\eta \left[\frac{1}{\lambda} A + \frac{1}{\lambda\sqrt{\mu}} B + \frac{1}{\sqrt{\lambda\mu}} C + \frac{1}{\sqrt{\lambda}\mu} D \right].$$

Since $\mu \leq \lambda$, $c_{det} = \frac{1}{\lambda\sqrt{\mu}} + \frac{1}{\sqrt{\lambda\mu}} \lesssim \frac{1}{\sqrt{\lambda\mu}}$. Thus,

$$\begin{aligned} \|\mathfrak{B} - Z_{\mathfrak{B}}\| &\lesssim C_\eta \left[\frac{1}{\lambda} \sqrt{\delta''_\mu} n^{-1/2} + c_{det} \delta_B b_{\mu,\lambda} + (c_{det} + \frac{1}{\sqrt{\lambda\mu}}) \Delta_{h_{\mu,\lambda}} + \frac{1}{\sqrt{\lambda\mu}} (\sqrt{\delta'_\mu} (\delta_z v_{\mu,\lambda} + \tilde{\gamma}_2) + \delta_z b_{\mu,\lambda}) \right] \\ &\lesssim C_\eta \left[\frac{1}{\lambda} \sqrt{\delta''_\mu} n^{-1/2} + \frac{1}{\sqrt{\lambda\mu}} \delta_B b_{\mu,\lambda} + \frac{1}{\sqrt{\lambda\mu}} \Delta_{h_{\mu,\lambda}} + \frac{1}{\sqrt{\lambda\mu}} (\sqrt{\delta'_\mu} (\delta_z v_{\mu,\lambda} + n^{-1/2}) + \delta_z b_{\mu,\lambda}) \right] \\ &\lesssim C_\eta \left[\frac{1}{\sqrt{\lambda\mu}} \delta_B b_{\mu,\lambda} + \frac{1}{\sqrt{\lambda\mu}} \left(\frac{1}{\sqrt{n\lambda\mu}} + \frac{1}{\sqrt{\lambda\mu}} \sqrt{\frac{\mu \mathbf{n}_z(\mu)}{n}} b_{\mu,\lambda} + Q_{\text{res}} \right) + \frac{1}{\sqrt{\lambda\mu}} (\sqrt{\delta'_\mu} (\delta_z v_{\mu,\lambda} + n^{-1/2}) + \delta_z b_{\mu,\lambda}) \right] \\ &\lesssim C_\eta \left[\frac{1}{\sqrt{\lambda\mu}} \delta_B b_{\mu,\lambda} + \frac{1}{\sqrt{\lambda\mu}} \left(\frac{1}{\sqrt{n\lambda\mu}} + \frac{1}{\sqrt{\lambda\mu}} \sqrt{\frac{\mu \mathbf{n}_z(\mu)}{n}} b_{\mu,\lambda} + Q_{\text{res}} \right) \right] \\ &\lesssim C_\eta \left[\frac{1}{\sqrt{\lambda\mu}} \delta_B b_{\mu,\lambda} + \frac{1}{\sqrt{\lambda\mu}} \left(\frac{1}{\sqrt{n\lambda\mu}} + \frac{1}{\sqrt{\lambda\mu}} \sqrt{\frac{\mu \mathbf{n}_z(\mu)}{n}} b_{\mu,\lambda} + Q_{\text{res}} \right) \right] \\ &\lesssim C_\eta \left[\frac{1}{\sqrt{\lambda\mu}} \frac{1}{\sqrt{n\lambda\mu}} \lambda^\alpha + \frac{1}{\sqrt{\lambda\mu}} \left(\frac{1}{\sqrt{n\lambda\mu}} + \frac{1}{\sqrt{n\lambda\mu^{\rho_z}}} \lambda^\alpha + Q_{\text{res}} \right) \right] \\ &\lesssim C_\eta \left[\frac{1}{\sqrt{\lambda\mu}} \frac{1}{\sqrt{n\lambda\mu}} \lambda^\alpha + \frac{1}{\sqrt{\lambda\mu}} \left(\frac{1}{\sqrt{n\lambda\mu}} + Q_{\text{res}} \right) \right]. \end{aligned}$$

For Q_{res} we already derived

$$Q_{\text{res}} \lesssim \left(\frac{1}{n\lambda\mu^{1+\rho_z}} \lambda^\alpha + \frac{1}{n^{3/2}\lambda^{3/2}\mu^{\frac{1}{2}+\frac{3}{2}\rho_z}} + \frac{1}{n\lambda\mu^{1+\frac{1}{2}\rho_z}} \right).$$

Plugging this in gives, for polynomial decay,

$$\|\mathfrak{B} - Z_{\mathfrak{B}}\| \lesssim C_\eta \left[\frac{\lambda^{\alpha-1}}{\mu^2\sqrt{n}} + \frac{1}{\lambda\mu^{3/2}\sqrt{n}} + \frac{\lambda^\alpha}{n\lambda^{3/2}\mu^{2+\rho_z}} + \frac{1}{n^{3/2}\lambda^2\mu^{\frac{3}{2}+\frac{3}{2}\rho_z}} + \frac{1}{n\lambda^{3/2}\mu^{2+\frac{1}{2}\rho_z}} \right].$$

In general form,

$$\|\mathfrak{B} - Z_{\mathfrak{B}}\| \lesssim C_\eta \left[\frac{l(\eta)}{\mu^2\sqrt{n\lambda}} b_{\mu,\lambda} + \frac{l(\eta)}{\lambda\mu^{3/2}\sqrt{n}} + \frac{l(\eta)^2 \mathbf{n}_z(\mu)}{n\lambda^{3/2}\mu^2} b_{\mu,\lambda} + \frac{l(\eta)^3 \mathbf{n}_z(\mu)^{3/2}}{n^{3/2}\lambda^2\mu^{\frac{3}{2}}} + \frac{l(\eta)^2 \sqrt{\mathbf{n}_z(\mu)}}{n\lambda^{3/2}\mu^2} \right],$$

which holds with probability $1 - 15\eta$. \square

Theorem H.1 (Bootstrap approximation). *Suppose n satisfies the combined rate condition of*

Assumption H.1, and that *Assumptions G.1* and *G.2* hold. In addition, suppose that the assumptions for the bias upper bound hold (*Lemma C.2*). Then, there exists a random variable Z whose conditional distribution given U is Gaussian with covariance Σ , such that with probability $1-\eta$,

$$\mathbb{P}\left[\|\mathfrak{B} - Z'\|_{\mathcal{H}_x} \lesssim \widetilde{M} \log(6/\eta)^{3/2} R_\bullet(n, \lambda, \mu) + R_{\text{res}} | U\right] \geq 1 - \eta,$$

where

$$R_\bullet = \inf_{m \geq 1} \left[m^{1/4} \left\{ \frac{\tilde{\mathfrak{m}}(\lambda, \mu)}{\mu \lambda n} + \frac{1}{n^2 \mu^2 \lambda^2} \right\}^{1/4} + \frac{\sigma(T, m)}{\lambda} \right],$$

$$R_{\text{res}} = \sqrt{2l\left(\frac{2\eta}{15}\right)} \left[\left\{ \frac{l(\frac{\eta}{15})}{\mu^2 n^{1/2} \lambda} + \frac{l(\frac{\eta}{15})^2 \mathfrak{n}_z(\mu)}{n \lambda^{3/2} \mu^2} \right\} \|h_{\mu, \lambda} - h_0\|_{\mathcal{H}_x} + \frac{l(\frac{\eta}{15})}{\lambda \mu^{3/2} n^{1/2}} + \frac{l(\frac{\eta}{15})^3 \mathfrak{n}_z(\mu)^{3/2}}{n^{3/2} \lambda^2 \mu^{\frac{3}{2}}} + \frac{l(\frac{\eta}{15})^2 \sqrt{\mathfrak{n}_z(\mu)}}{n \lambda^{3/2} \mu^2} \right].$$

Proof. Let, $l(\eta) = \log(2/\eta)$ and $\|h_{\mu, \lambda} - h_0\|_{\mathcal{H}_x} := b_{\mu, \lambda}$, Recall from *Lemma H.1*, with probability $1-\eta$,

$$\begin{aligned} \|\mathfrak{B} - Z_{\mathfrak{B}}\|_{\mathcal{H}_x} &\lesssim \sqrt{2 \log(15/\eta)} \left[\frac{\log(30/\eta)}{\mu^2 \sqrt{n} \lambda} b_{\mu, \lambda} + \frac{\log(30/\eta)}{\lambda \mu^{3/2} \sqrt{n}} + \frac{\log(30/\eta)^2 \mathfrak{n}_z(\mu)}{n \lambda^{3/2} \mu^2} b_{\mu, \lambda} \right. \\ &\quad \left. + \frac{\log(30/\eta)^3 \mathfrak{n}_z(\mu)^{3/2}}{n^{3/2} \lambda^2 \mu^{\frac{3}{2}}} + \frac{\log(30/\eta)^2 \sqrt{\mathfrak{n}_z(\mu)}}{n \lambda^{3/2} \mu^2} \right]. \end{aligned}$$

This defines R_{res} . Now, apply *Corollary B.1* from *Singh and Vijaykumar (2023)* taking $W = Z_{\mathfrak{B}}$ and $W' = \mathfrak{B}$. Then, there must exist Z' with the desired conditional distribution, such that with probability $1-\eta$, conditional upon $\sigma(U)$,

$$\|Z' - \mathfrak{B}\|_{\mathcal{H}_x} \lesssim C' \log(6/\eta)^{3/2} \inf_{m \geq 1} \left[m^{1/4} \left\{ \frac{a^2 \sigma^2(\Sigma, 0)}{n} + \frac{a^4}{n^2} \right\}^{1/4} + \sigma(\Sigma, m) \right] + R_{\text{res}}.$$

Similar for the Gaussian coupling we use that $a \leq \frac{\widetilde{M}}{\sqrt{\lambda \mu}}$, due to *Corollary D.1* and *Lemma D.3*:

$$\begin{aligned} \|Z - \mathfrak{B}\|_{\mathcal{H}_x} &\lesssim C' \log(6/\eta)^{3/2} \inf_{m \geq 1} \left[m^{1/4} \left\{ \frac{a^2 \sigma^2(\Sigma, 0)}{n} + \frac{a^4}{n^2} \right\}^{1/4} + \sigma(\Sigma, m) \right] + R_{\text{res}} \\ &\lesssim \widetilde{M} \log(6/\eta)^{3/2} \inf_{m \geq 1} \left[m^{1/4} \left\{ \frac{\tilde{\mathfrak{m}}(\lambda, \mu)}{\mu \lambda n} + \frac{1}{n^2 \mu^2 \lambda^2} \right\}^{1/4} + \frac{\sigma(T, m)}{\lambda} \right] + R_{\text{res}}. \quad \square \end{aligned}$$

I Uniform confidence band

I.1 High level summary

1. $Q(n, \lambda, \mu, \eta) = Q_{\bullet}(T, n, \lambda, \mu) \widetilde{M} \log(12/\eta) + Q_{\text{res}};$
2. $R(n, \lambda, \mu, \eta) = \widetilde{M} \log(6/\eta)^{3/2} R_{\bullet}(n, \lambda, \mu) + R_{\text{res}};$
3. $L(\lambda, \mu, \eta) = \sqrt{\frac{1}{4} \sigma^2 \tilde{m}(\lambda, \mu)} - \left\{ 2 + \sqrt{2 \ln(1/\eta)} \right\} \sqrt{\frac{2\bar{\sigma}^2}{\lambda}};$
4. $B(\lambda, \mu) = \sqrt{n} \frac{C_{\alpha} \lambda^{\alpha} \|T^{-\alpha} h_0\|_{\mathcal{H}_x}}{1 - C_{\beta} r \sqrt{\mu/\lambda}}.$

We now explicitly provide upper bounds for (Q, R, L, B) under polynomial decay.

B and L: By the bias upper bound

$$B(\lambda, \mu) = \sqrt{n} \frac{C_{\alpha} \lambda^{\alpha} \|T^{-\alpha} h_0\|_{\mathcal{H}_x}}{1 - C_{\beta} r \sqrt{\mu/\lambda}}.$$

Lemma D.2 implies that

$$\|Z\| \gtrsim L(\lambda, \mu, \eta) = \sqrt{\frac{1}{4} \sigma^2 \tilde{m}(\lambda, \mu)} - \left\{ 2 + \sqrt{2 \ln(1/\eta)} \right\} \sqrt{\frac{2\bar{\sigma}^2}{\lambda}}.$$

Q: Note that $Q = Q_{\bullet} + Q_{\text{res}}$. The latter is already derived in Theorem G.1. For Q_{\bullet} , following Proposition K.1 of Singh and Vijaykumar (2023), $\sigma(T, m) \lesssim_{\rho_x, \omega} m^{1/2 - 1/(2\rho_x - 2)}$. Recall that

$$Q_{\bullet} = \inf_{m \geq 1} \left\{ \frac{\sigma(T, m)}{\lambda} + \frac{m^2 \log(m^2)}{\sqrt{n \mu \lambda}} \right\}.$$

Equating main terms and solving for m ,

$$\begin{aligned} \frac{m^{\frac{1}{2} - \frac{1}{2\rho_x - 2}}}{\lambda} &= \frac{m^2}{\sqrt{n \lambda \mu}} \\ m^{\frac{1}{2} - \frac{1}{2\rho_x - 2} - 2} &= \left(\frac{\lambda}{n \mu} \right)^{1/2} \\ m^{-\frac{3}{2} - \frac{1}{2\rho_x - 2}} &= \left(\frac{\lambda}{n \mu} \right)^{1/2} \\ m &= \left(\frac{n \mu}{\lambda} \right)^{\frac{\rho_x - 1}{3\rho_x - 2}}. \end{aligned}$$

Finally, we can derive an upper bound for Q_\bullet by plugging this m in

$$Q_\bullet \lesssim \frac{\sigma(T,m)}{\lambda} \lesssim_{\rho_x,\omega} \frac{m^{\frac{1}{2}-\frac{1}{2\rho_x-2}}}{\lambda} = \frac{1}{\lambda} \left(\frac{n\mu}{\lambda} \right)^{\frac{\rho_x-2}{2(3\rho_x-2)}}.$$

R: $R = R_\bullet + R_{\text{res}}$. The latter is derived in Theorem H.1. For R_\bullet , recall that

$$R_\bullet(n,\lambda,\mu) = \inf_{m \geq 1} \left[m^{1/4} \left\{ \frac{\tilde{m}(\lambda,\mu)}{\mu\lambda n} + \frac{1}{n^2\mu^2\lambda^2} \right\}^{1/4} + \frac{\sigma(T,m)}{\lambda} \right].$$

Now in order for, $\tilde{m}(\lambda,\mu)/(n\lambda\mu) > \frac{1}{n^2\lambda^2\mu^2}$ we need $n > (\tilde{m}(\lambda,\mu)\mu\lambda)^{-1}$. This directly follows from the combined rate Assumption H.1 when taking $\lambda = \mu$. Equating the main terms and solving for m yields

$$\begin{aligned} m^{1/4} \left(\frac{\lambda^{-\rho_x}}{n\lambda\mu} \right)^{1/4} &= \frac{m^{\frac{1}{2}-\frac{1}{2\rho_x-2}}}{\lambda} \\ m^{1/4} \left(\frac{\lambda^{-\rho_x}}{n\lambda\mu} \right)^{1/4} &= \frac{m^{\frac{1}{2}-\frac{1}{2\rho_x-2}}}{\lambda} \\ m^{-\frac{1}{4}+\frac{1}{2\rho_x-2}} &= (n\mu)^{1/4} \lambda^{(\rho_x-3)/4} \\ m &= ((n\mu)^{1/4} \lambda^{(\rho_x-3)/4})^{-\frac{4(\rho_x-1)}{\rho_x-3}} \\ m &= (n\mu)^{-\frac{\rho_x-1}{\rho_x-3}} \lambda^{-(\rho_x-1)}. \end{aligned}$$

Plugging this in gives us the upper bound for R_\bullet :

$$R_\bullet \lesssim \frac{\sigma(T,m)}{\lambda} \lesssim_{\rho_x,\omega_x} \frac{m^{\frac{1}{2}-\frac{1}{2\rho_x-2}}}{\lambda} \lesssim_{\rho_x,\omega_x} \lambda^{-\rho_x/2} (n\mu)^{-\frac{\rho_x-2}{2(\rho_x-3)}}.$$

I.2 Restrictions for valid inference

We derive restrictions on possible data-generating processes that allow us to use the guarantees of Singh and Vijaykumar (2023).

I.2.1 $B \ll L$

We require

$$n^{1/2} \frac{\lambda^\alpha}{1 - \sqrt{\mu/\lambda}} \ll \lambda^{-\rho_x/2}.$$

Rearranging yields

$$n^{1/2} \ll (1 - \sqrt{\mu/\lambda}) \lambda^{-(\rho_x/2 + \alpha)} \implies n \ll (1 - \sqrt{\mu/\lambda})^2 \lambda^{-(\rho_x + 2\alpha)} \lesssim \lambda^{-(\rho_x + 2\alpha)}.$$

I.2.2 $Q+R \ll L$

We now determine the sample size n required to ensure that the approximation errors Q_\bullet and R_\bullet are dominated by the variance $L = \lambda^{-\rho_x/2}$.

Condition for Q_\bullet : Using the Q_\bullet derivation from before, the condition $Q_\bullet \ll L$ requires

$$\begin{aligned} \frac{1}{\lambda} \left(\frac{n\mu}{\lambda} \right)^{\frac{\rho_x-2}{2(3\rho_x-2)}} &\ll \lambda^{-\rho_x/2} \\ \left(\frac{n\mu}{\lambda} \right)^{\frac{\rho_x-2}{2(3\rho_x-2)}} &\ll \lambda^{1-\rho_x/2} \\ \frac{n\mu}{\lambda} &\gg \left(\lambda^{\frac{2-\rho_x}{2}} \right)^{\frac{2(3\rho_x-2)}{\rho_x-2}} \\ n &\gg \mu^{-1} \lambda^{-(3\rho_x-3)}. \end{aligned}$$

Condition for R_\bullet : Using the R_\bullet derivation from before, the condition $R_\bullet \ll L$ requires that

$$\lambda^{-\rho_x/2} (n\mu)^{-\frac{\rho_x-2}{2(\rho_x-3)}} \ll \lambda^{-\rho_x/2}.$$

Canceling the $\lambda^{-\rho_x/2}$ term, this simplifies to

$$(n\mu)^{-\frac{\rho_x-2}{2(\rho_x-3)}} \ll 1 \iff n \gg \mu^{-1}.$$

Conditions for Q_{res} : Using the definition of Q_{res} from Theorem G.1 and plugging in the bias and effective dimension upper bound, we can derive

$$\begin{aligned} \frac{\lambda^{\alpha-1}}{n\mu^{1+\rho_z}} &\leq \lambda^{-\rho_x/2} \iff n \geq \lambda^{\alpha-1+\frac{\rho_x}{2}} \mu^{-(1+\rho_z)} \\ \frac{1}{n^{3/2} \lambda^{3/2} \mu^{\frac{1}{2} + \frac{3}{2}\rho_z}} &\leq \lambda^{-\frac{\rho_x}{2}} \iff n \geq \lambda^{-1+\frac{\rho_x}{3}} \mu^{-\frac{1}{3}-\rho_z} \\ \frac{1}{n\lambda\mu^{1+\frac{1}{2}\rho_z}} &\leq \lambda^{-\frac{\rho_x}{2}} \iff n \geq \lambda^{-1+\frac{\rho_x}{2}} \mu^{-(1+\frac{1}{2}\rho_z)}. \end{aligned}$$

Conditions for R_{res} : Using the definition of R_{res} from Theorem G.1 and plugging in the bias and effective dimension upper bound, we can derive

$$\begin{aligned}
 (1) \quad & \frac{\lambda^{\alpha-1}}{\mu^2 \sqrt{n}} \leq \lambda^{-\rho_x/2} \iff n \geq \mu^{-4} \lambda^{2\alpha-2+\rho_x} \\
 (2) \quad & \frac{1}{\lambda \mu^{3/2} \sqrt{n}} \leq \lambda^{-\rho_x/2} \iff n \geq \mu^{-3} \lambda^{-2+\rho_x} \\
 (3) \quad & \frac{\lambda^{\alpha-3/2}}{n \mu^{2+\rho_z}} \leq \lambda^{-\rho_x/2} \iff n \geq \mu^{-(2+\rho_z)} \lambda^{\alpha-\frac{3}{2}+\frac{\rho_x}{2}} \\
 (4) \quad & \frac{1}{n^{3/2} \lambda^2 \mu^{\frac{3}{2}+\frac{3}{2}\rho_z}} \leq \lambda^{-\rho_x/2} \iff n \geq \mu^{-(1+\rho_z)} \lambda^{-\frac{4}{3}+\frac{\rho_x}{3}} \\
 (5) \quad & \frac{1}{n \lambda^{3/2} \mu^{2+\frac{1}{2}\rho_z}} \leq \lambda^{-\rho_x/2} \iff n \geq \mu^{-(2+\frac{1}{2}\rho_z)} \lambda^{-\frac{3}{2}+\frac{\rho_x}{2}}.
 \end{aligned}$$

Combining restrictions gives the conditions summarized by Table 2.

Table 2: Final restrictions on λ, μ .

Component	Sufficient restriction on λ, μ
$B \ll L$	$n \ll \lambda^{-(\rho_x+2\alpha)}$
$Q_\bullet \ll L$	$n \gg \mu^{-1} \lambda^{-(3\rho_x-3)}$
$R_\bullet \ll L$	$n \gg \mu^{-1}$
$Q_{\text{res}}^{(1)} \ll L$	$n \gg \lambda^{\alpha-1+\frac{\rho_x}{2}} \mu^{-(1+\rho_z)}$
$Q_{\text{res}}^{(2)} \ll L$	$n \gg \lambda^{-1+\frac{\rho_x}{3}} \mu^{-\frac{1}{3}-\rho_z}$
$Q_{\text{res}}^{(3)} \ll L$	$n \gg \lambda^{-1+\frac{\rho_x}{2}} \mu^{-(1+\frac{1}{2}\rho_z)}$
$R_{\text{res}}^{(1)} \ll L$	$n \gg \mu^{-4} \lambda^{2\alpha-2+\rho_x}$
$R_{\text{res}}^{(2)} \ll L$	$n \gg \mu^{-3} \lambda^{-2+\rho_x}$
$R_{\text{res}}^{(3)} \ll L$	$n \gg \mu^{-(2+\rho_z)} \lambda^{\alpha-\frac{3}{2}+\frac{\rho_x}{2}}$
$R_{\text{res}}^{(4)} \ll L$	$n \gg \mu^{-(1+\rho_z)} \lambda^{-\frac{4}{3}+\frac{\rho_x}{3}}$
$R_{\text{res}}^{(5)} \ll L$	$n \gg \mu^{-(2+\frac{1}{2}\rho_z)} \lambda^{-\frac{3}{2}+\frac{\rho_x}{2}}$

I.2.3 Parameter restrictions from combined bounds

We now derive the conditions on parameters α, ρ_x, ρ_z , and ι required to ensure that the set of data generating processes defined by $Q+R \ll L$ (lower bound) and $B \ll L$ (upper bound) is non-empty. Recall the upper bound is $n \lesssim \lambda^{-(\rho_x+2\alpha)}$. We substitute $\lambda = \mu^\iota$ and analyze the

regime $\mu \rightarrow 0$. For Q_\bullet , the condition $\mu^{-1} \lambda^{-(3\rho_x - 3)} \ll \lambda^{-(\rho_x + 2\alpha)}$ becomes

$$\mu^{-(1+\iota(2\rho_x - 3 - 2\alpha))} \ll 1 \iff 1 + \iota(2\rho_x - 3 - 2\alpha) < 0 \iff 2\rho_x < 3 + 2\alpha - \frac{1}{\iota}.$$

For R_\bullet , the condition $\mu^{-1} \ll \lambda^{-(\rho_x + 2\alpha)}$ implies $\mu^{-1} \ll \mu^{-\iota(\rho_x + 2\alpha)}$. This holds for $\mu \rightarrow 0$ if and only if

$$1 < \iota(\rho_x + 2\alpha) \iff \rho_x + 2\alpha > \frac{1}{\iota}.$$

Next, we analyze the restrictions imposed by Q_{res} . We require the lower bound exponent to be strictly larger than the upper bound exponent (in terms of μ^{-k} decay):

$$(1) \quad \lambda^{-(\rho_x + 2\alpha)} \gtrsim \lambda^{\alpha - 1 + \rho_x/2} \mu^{-(1 + \rho_z)} \implies \rho_z + 1 < \iota \left(3\alpha + \frac{3}{2}\rho_x - 1 \right)$$

$$(2) \quad \lambda^{-(\rho_x + 2\alpha)} \gtrsim \lambda^{-1 + \rho_x/3} \mu^{-(\frac{1}{3} + \rho_z)} \implies \rho_z + \frac{1}{3} < \iota \left(2\alpha + \frac{4}{3}\rho_x - 1 \right)$$

$$(3) \quad \lambda^{-(\rho_x + 2\alpha)} \gtrsim \lambda^{-1 + \rho_x/2} \mu^{-(1 + \frac{1}{2}\rho_z)} \implies \rho_z + 2 < \iota(4\alpha + 3\rho_x - 2).$$

Similarly, the conditions from R_{res} yield

$$(1) \quad \mu^{-4} \lambda^{2\alpha - 2 + \rho_x} \ll \lambda^{-(\rho_x + 2\alpha)} \iff \iota(4\alpha + 2\rho_x - 2) > 4$$

$$(2) \quad \mu^{-3} \lambda^{-2 + \rho_x} \ll \lambda^{-(\rho_x + 2\alpha)} \iff \iota(2\alpha + 2\rho_x - 2) > 3$$

$$(3) \quad \mu^{-(2 + \rho_z)} \lambda^{\alpha - \frac{3}{2} + \frac{\rho_x}{2}} \ll \lambda^{-(\rho_x + 2\alpha)} \iff 2\rho_z + 4 < \iota(6\alpha + 3\rho_x - 3)$$

$$(4) \quad \mu^{-(1 + \rho_z)} \lambda^{-\frac{4}{3} + \frac{\rho_x}{3}} \ll \lambda^{-(\rho_x + 2\alpha)} \iff 3\rho_z + 3 < \iota(6\alpha + 4\rho_x - 4)$$

$$(5) \quad \mu^{-(2 + \frac{1}{2}\rho_z)} \lambda^{-\frac{3}{2} + \frac{\rho_x}{2}} \ll \lambda^{-(\rho_x + 2\alpha)} \iff \rho_z + 4 < \iota(4\alpha + 3\rho_x - 3).$$

These combined parameter restrictions are summarized in Table 3.

Table 3: Parameter restrictions ensuring $Q+R \ll L$ and $B \ll L$ simultaneously

Component	Restriction
Q_\bullet	$2\rho_x < 3 + 2\alpha - \frac{1}{\iota}$
R_\bullet	$\rho_x + 2\alpha > \frac{1}{\iota}$
$Q_{\text{res}}^{(1)}$	$\rho_z + 1 < \iota \left(3\alpha + \frac{3}{2}\rho_x - 1 \right)$
$Q_{\text{res}}^{(2)}$	$\rho_z + \frac{1}{3} < \iota \left(2\alpha + \frac{4}{3}\rho_x - 1 \right)$
$Q_{\text{res}}^{(3)}$	$\rho_z + 2 < \iota(4\alpha + 3\rho_x - 2)$
$R_{\text{res}}^{(1)}$	$\iota(4\alpha + 2\rho_x - 2) > 4$
$R_{\text{res}}^{(2)}$	$\iota(2\alpha + 2\rho_x - 2) > 3$
$R_{\text{res}}^{(3)}$	$2\rho_z + 4 < \iota(6\alpha + 3\rho_x - 3)$
$R_{\text{res}}^{(4)}$	$3\rho_z + 3 < \iota(6\alpha + 4\rho_x - 4)$
$R_{\text{res}}^{(5)}$	$\rho_z + 4 < \iota(4\alpha + 3\rho_x - 3)$

I.3 Valid and sharp inference

Proof of Theorem 1. We appeal to Proposition 1 of Singh and Vijaykumar (2023). To establish the claimed validity and sharpness, it suffices to show that there exist choices of η and δ satisfying:

$$\frac{\Delta(n, \lambda, \mu, \eta) + B(\lambda, \mu)}{L(\lambda, \mu, 1 - \alpha - 2\eta) - \Delta(n, \lambda, \mu, \eta)} \leq \delta \leq \frac{1}{2},$$

where $\Delta := Q + R$ represents the total approximation error. We select $\eta = 1/n$ and $\delta = 1/\log(n)$. The parameter restrictions summarized in Tables 2 and 3 ensure that the bias B and approximation error Δ decay strictly faster than the variance lower bound L . Specifically, our bounds imply

$$\frac{B}{L} = \mathcal{O}(n^{-\varepsilon_1}) \quad \text{and} \quad \frac{\Delta}{L} = \mathcal{O}(n^{-\varepsilon_2})$$

for some $\varepsilon_1, \varepsilon_2 > 0$. Consequently, for sufficiently large n , we have $\Delta \leq L/2$, allowing us to bound the left hand side

$$\frac{\Delta + B}{L - \Delta} \leq \frac{\Delta + B}{L/2} = 2 \left(\frac{\Delta}{L} + \frac{B}{L} \right) = \mathcal{O}(n^{-(\varepsilon_1 \wedge \varepsilon_2)}).$$

Since a polynomial decay dominates a logarithmic decay, we have $\mathcal{O}(n^{-\varepsilon}) = o(1/\log(n))$. Therefore, for sufficiently large n ,

$$\frac{\Delta+B}{L-\Delta} \leq \frac{1}{\log(n)} = \delta.$$

Finally, since n is large, $\delta = 1/\log(n) \leq 1/2$. Thus, the condition on the incremental factor is satisfied. By Proposition 1 of Singh and Vijaykumar (2023), this choice of δ guarantees that the confidence sets are valid with tolerance $\tau = \mathcal{O}(\eta) = \mathcal{O}(1/n)$ and satisfy sharpness with slack $2\delta = 2/\log(n)$. \square