

New Obstacles to Multiple Recurrence

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Abstract

We show that there is a set which is not a set of multiple recurrence despite being a set of recurrence for nil-Bohr sets. This answers Huang, Shao, and Ye’s “higher-order” version of Katznelson’s Question on Bohr recurrence and topological recurrence in the negative. Equivalently, we construct a set S so that there is a finite coloring of \mathbb{N} without three-term arithmetic progressions with common differences in S , but so that S lacks the usual polynomial obstacles to arithmetic progressions.

1 Introduction

A central and long-standing open problem in the field of topological dynamics is “Katznelson’s Question”, on whether a set of Bohr recurrence is a set of topological recurrence. This question was raised by a 1968 paper of Veech [21] and implicitly before that in a 1954 paper of Følner [5]. However, in a classic example of Stigler’s law, it is often referred to as “Katznelson’s question”, because Katznelson popularized the question and wrote a beautiful paper [16] connecting it to the chromatic number of abelian Cayley graphs. In this paper, via a Furstenberg correspondence principle type argument in symbolic dynamics, he shows that Bohr recurrence is equivalent to topological recurrence if and only if every abelian Cayley graph with finite chromatic number has a finite coloring via Bohr sets.

Let us recall the equivalent dynamical and combinatorial definitions of Bohr and topological recurrence, and some basics of topological dynamics. These appear widely in the literature (see for instance [16] [9] [6]). Recall that $\|x\|_{\mathbb{R}/\mathbb{Z}} = \min(\{x\}, 1 - \{x\})$ is the distance between x and the nearest integer.

Definition 1.1 (Bohr recurrence, combinatorial definition). *A set $S \subset \mathbb{Z}$ is a set of Bohr recurrence if and only if for all nonzero $\alpha_1, \dots, \alpha_k \in \mathbb{R}$ and all $\varepsilon > 0$ there exists some $s \in S$ so that for all $1 \leq i \leq k$, $\|\alpha_i s\|_{\mathbb{R}/\mathbb{Z}} < \varepsilon$.*

We will also formally define a Bohr set.

Definition 1.2 (Bohr set). *A Bohr set is a set S parameterized by a k -tuple of reals $(\alpha_1, \dots, \alpha_k)$ and ε , consisting of the s so that $\|\alpha_i s\|_{\mathbb{R}/\mathbb{Z}} < \varepsilon$ for all $1 \leq i \leq k$.*

The parameter k is called the *dimension* and ε is the *width*. Later we will define a complexity of a nil-Bohr set, which will be $\Theta(\max(k, 1/\varepsilon))$ for ordinary Bohr sets.

Equivalently a set is a set of Bohr recurrence if it is not contained in the complement of a Bohr set, or equivalently in a finite union of complements of one-dimensional Bohr sets. Sometimes

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this is referred to as a “*Bohr*₀ set” in the literature, with the term Bohr set referring to the more general class of translates of *Bohr*₀ sets. Here we will only deal with *Bohr*₀ sets, so we will refer to them simply as Bohr sets.

Definition 1.3 (Bohr recurrence, dynamical definition). *A set $S \subset \mathbb{Z}$ is a set of Bohr recurrence if and only if for all $k \in \mathbb{N}$ and all rotations $T(x_1, \dots, x_k) = (x_1 + \alpha_1, \dots, x_k + \alpha_k)$ on the torus $(\mathbb{R}/\mathbb{Z})^k$, there exists a point $x \in (\mathbb{R}/\mathbb{Z})^k$ and a subsequence S' of $n \in S$ so that $\lim_{n \in S'} T^n(x) = x$.*

The two definitions are readily seen to be equivalent. Since torus rotations are isometries, if $T^n x \rightarrow x$ is true for a single point then it is true for all points, so it would be equivalent to change the quantifier on x from “there exists” to “for all”.

We only consider S with $0 \notin S$, as if $0 \in S$ then all notions of recurrence are trivial. In general dealing with $S \subset \mathbb{N}$ is equivalent to dealing with $S \cup -S \subset \mathbb{Z}$, so we will assume for the remainder of this paper that $S \subset \mathbb{N}$. This is important, because a general topological system is not necessarily invertible, so T^{-1} may not make sense.

Recall that if S is a subset of a group G , the Cayley graph $\text{Cay}(G, S)$ on an abelian group G has all elements of G as vertices and a, b in G are adjacent if and only if $a - b \in (S \cup -S)$. Recall also that the chromatic number of a graph is the minimal number of colors needed to color the vertices so that no two vertices of the same color are adjacent.

Definition 1.4 (Topological recurrence, combinatorial definition). *A set $S \subset \mathbb{N}$ is a set of topological recurrence if and only if the Cayley graph $\text{Cay}(\mathbb{Z}, S)$ on \mathbb{Z} generated by S has infinite chromatic number.*

Topological recurrence is the most general form of recurrence, applying to all dynamical systems.

Definition 1.5 (Topological recurrence, dynamical definition). *A set $S \subset \mathbb{N}$ is a set of topological recurrence if for every compact dynamical system (X, T) , where X is a compact metric space and T is a continuous map on X , there exists a point $x \in X$ and a subsequence S' of S so that $\lim_{n \in S'} T^n x = x$.*

It was Katznelson [16] who proved the equivalence of these two definitions of topological recurrence, giving the eponymous conjecture its combinatorial form. One direction is relatively simple. If a set S is not a set of dynamical topological recurrence, then one can use the dynamical system to come up with a coloring of the Cayley graph. In particular there will be some ε so that $d(T^n x, x) > \varepsilon$ for all $n \in S$. Now cover the compact space by ε -balls for a suitably small ε , and color $m \in \mathbb{N}$ by which ε -ball $T^m(x)$ ends up in. The other direction, using a coloring to construct a dynamical system, follows from an argument of Katznelson that makes use of symbolic dynamics, and bears some resemblance to the Furstenberg correspondence principle.

A set of topological recurrence must be a set of Bohr recurrence. This is readily seen both from the topological definition, as a circle rotation is a special kind of dynamical system, and from the combinatorial version, as if S a set that is not a set of Bohr recurrence it is easy to construct a finite coloring by taking all of the $n\alpha_i$ modulo 1, up to an error of ε . Hence we can state Katznelson’s question explicitly as follows.

Conjecture 1.6 (Katznelson’s question). *If S is a set of Bohr recurrence, then is S necessarily a set of topological recurrence?*

In [6] it is stated that there is “no consensus among experts as to the expected answer”.

No discussion of the history of Katznelson’s question would be complete without also referencing the “Related Problem”. Bergelson asked whether there is a set which is a set of topological recurrence but not a set of measurable recurrence, or equivalently whether Cayley graph on the integers with a positive upper density independent set, but infinite chromatic number, and Kříž [17] resolved this question in the negative. His construction is based on the

boolean Cayley graph over $(\mathbb{Z}/2)^n$ generated by elements of Hamming weight $> n - 10\sqrt{n}$. One can check that elements of Hamming weight $\leq n/2 - 10\sqrt{n}$ form a positive density independent set in this graph. Furthermore, looking at the elements of Hamming weight $n/2 - 6\sqrt{n}$, this graph has a copy of the Kneser graph $KG(n, n/2 - 6\sqrt{n})$ and thus has chromatic number at least \sqrt{n} , and goes to infinity with n . Kříž projects this construction to the integers. See [11] for an exposition of his example by Griesmer and [10] for an extension.

In this paper, we will be concerned with the natural higher-order version of Katznelson's question. This was first explicitly asked by Huang, Shao, and Ye in 2011 ([14], [15]), and asked as Question 7 in [6]. Before stating it, we need to define topological multiple recurrence.

Definition 1.7 (Topological multiple recurrence, dynamical definition). *A set $S \subset \mathbb{N}$ is a set of topological d -recurrence if for every compact dynamical system (X, T) , where X is a compact metric space and T is a continuous map on X , there exists a point $x \in X$ and a subsequence S' of S so that for all $1 \leq d' \leq d$, $\lim_{n \in S'} T^{d'n} x = x$.*

In other words, $T^n x, T^{2n} x, \dots, T^{dn} x$ all go to x along this sequence S' of n . So we have $T^n x \rightarrow x, T^{2n} x \rightarrow x, \dots, T^{dn} x \rightarrow x$ simultaneously along this sequence S' . For $d = 1$, this is the ordinary notion of recurrence.

Definition 1.8 (Topological multiple recurrence, combinatorial definition). *A set $S \subset \mathbb{N}$ is a set of topological d -recurrence if for every finite coloring of \mathbb{N} , there is a monochromatic $(d+1)$ -term arithmetic progression with common difference in S .*

Equivalently, the $(d+1)$ -hypergraph with edges $(x, x+s, \dots, x+ds)$ for $s \in S$ has infinite chromatic number. In other words, restricted van der Waerden with differences in S is true for $(d+1)$ -term arithmetic progressions. For $d = 1$ this is readily seen to coincide with the Cayley graph definition. As before, this combinatorial definition is equivalent to the dynamical one, see for instance [13], Theorem 2.5. In [15], the notation $\mathcal{F}_{\text{Bir}_d}$ is used for the family of sets of topological d -recurrence.

For restricted van der Waerden, on top of the usual obstacles in Katznelson's question, there are higher degree obstacles. If s is far from a multiple of an irrational number like π , then $x, x+s, x+2s$ cannot all be too close modulo π . If $2s^2$ is far from a multiple of π , then because $x^2 - 2(x+s)^2 + (x+2s)^2 = 2s^2$, the squares of $x, x+s, x+2s$ cannot all be very close to each other modulo π as well, which yields a finite coloring where x is colored by x^2 modulo π up to a small error. In general there will be polynomial obstacles up to degree d . So the natural question here is whether there are any other obstacles, with the caveat that one has to also worry about "generalized polynomials" or "bracket polynomials" like $s[s]$. These generalized polynomials are formally defined in [15], but we will not define them yet.

In the dynamics literature, these correspond to nilmanifolds. Multiple recurrence for Bohr sets is equivalent to single recurrence, but there is a higher-order notion of Bohr recurrence called "nil-Bohr recurrence". It was originally defined dynamically in [12] and is the main object of study in [15]. We state it here for completeness. For a formal definition of a $(d\text{-step})$ nilsystem, one can for instance consult [15], but again it is not so crucial for the present paper.

Definition 1.9 (nil-Bohr multiple recurrence, dynamical definition). *A set $S \subset \mathbb{N}$ is a set of Nil_d -Bohr recurrence if for every d -step nil-system (X, T) there is a point $x \in X$ and a subsequence $S' \subset S$ so that for all $1 \leq d' \leq d$, $x = \lim_{n \in S'} T^{d'n} x$.*

As usual, there is also a combinatorial definition. Again, we state it here for completeness.

Definition 1.10 (nil-Bohr multiple recurrence, combinatorial definition). *A set $S \subset \mathbb{N}$ is a set of Nil_d -Bohr recurrence if for every finite family of generalized polynomials P_1, \dots, P_k with degree at most d and every ε , there exist $s \in S$ so that $\|P_i(s)\|_{\mathbb{R}/\mathbb{Z}} < \varepsilon$ for all $1 \leq i \leq k$.*

The set of n so that these generalized polynomials have small \mathbb{R}/\mathbb{Z} is called a "Nil-Bohr" set, or a " Nil_d -Bohr" set when we want to emphasize the parameter of d . An ordinary Bohr set is just a Nil_1 Bohr set.

The equivalence of the definitions is Theorem B of [15], where the notation $\mathcal{F}_{d,0}$ is used for the family of Nil_d -Bohr sets and $\mathcal{F}_{d,0}^*$ for the family of sets of Nil_d -Bohr recurrence. Again, we will not formally define “generalized polynomial”, but it is essentially just a polynomial with floors and ceilings, often also referred to as a “bracket polynomial” in the literature (see for instance [15], [19]). The “degree” of a generalized polynomial is just the degree in s ignoring the floors and ceilings, so for instance $s[s]$ will have degree 2.

Huang, Shao, and Ye also prove that one can also equivalently ask for the generalized polynomials P_i to be “special” generalized polynomials of an even simpler form. We will define these. First, following Section 4 of [15] we use the notation $[x] := \lfloor x + 1/2 \rfloor$ for the nearest integer function.

Definition 1.11 (Special Generalized Polynomials, after [15]). *We define L so that $L(x) = x$ and $L(x_1, \dots, x_\ell) = x_1[L(x_2, \dots, x_\ell)]$.*

Then the special generalized polynomials of degree at most d are the expressions

$$L(n^{j_1}a_1, \dots, n^{j_\ell}a_\ell)$$

such that $j_i \in \mathbb{N}$ and $a_i \in \mathbb{R}$ for $1 \leq i \leq \ell$, and $\sum_{i=1}^\ell j_i \leq d$.

In particular, we must have $\ell \leq d$.

Definition 1.12 (nil-Bohr multiple recurrence, simple combinatorial definition). *A set $S \subset \mathbb{N}$ is a set of Nil_d -Bohr recurrence if for every finite family of special generalized polynomials P_1, \dots, P_k with degree at most d and every ε , there exist $s \in S$ so that $\|P_i(s)\|_{\mathbb{R}/\mathbb{Z}} < \varepsilon$ for all $1 \leq i \leq k$.*

We have for instance that the set of n so that $1/4 < \{n^2\alpha\} < 3/4$ and the set of n so that $1/4 < \{\alpha n[n]\}$ are not sets of Nil_2 -Bohr recurrence, even though they are sets of Bohr recurrence.

The equivalence of the simple and ordinary combinatorial definitions of nil-Bohr multiple recurrence is Theorem 4.2.11 of [15]. In our proof, we will instead use a somewhat different form of “special generalized polynomial” from [19]. Leibman proves in his paper that these have the property that they jointly equi-distribute.

The natural higher-order analogue of Katznelson’s question is whether topological multiple recurrence is equivalent to nil-Bohr multiple recurrence, and specifically whether topological d -recurrence is equivalent to Nil_d -Bohr recurrence. This is Question B, the main question asked in [15], which in formulation B-III asks whether $\mathcal{F}_{Bir_d} = \mathcal{F}_{d,0}^*$. In this paper, we answer this question in the negative.

Theorem 1.13. *There is a set $S \subset \mathbb{N}$ which is not a set of topological 2-recurrence, but is a set of Nil_2 -Bohr recurrence.*

We really only need that it is a set of Nil_2 -Bohr recurrence, but it turns out to be no more difficult to also get this for $d > 2$. The same proof works.

Theorem 1.14. *There is a set $S \subset \mathbb{N}$ which is not a set of topological 2-recurrence, but is a set of Nil_d -Bohr recurrence for every d .*

Equivalently, we find a set S so that restricted van der Waerden with common difference $s \in S$ is false even for three-term progressions, but S intersects all Nil_d -Bohr sets, and so it does not have quadratic obstacles like $s : \|s^2\alpha\|_{\mathbb{R}/\mathbb{Z}} > \varepsilon$ (or higher degree obstacles like $s : \|s^3\alpha\|_{\mathbb{R}/\mathbb{Z}} > \varepsilon$) that would give an “obvious obstruction”.

2 Proof of Theorem 1.14

To prove Theorem 1.14 we must ultimately construct a set $S \subset \mathbb{N}$. However, we will first construct a set S in a different setting, and consider an appropriate projection.

Recall that $\ell^1(\mathbb{N}; \mathbb{R}/\mathbb{Z})$ consists of sequences (a_1, a_2, a_3, \dots) of elements \mathbb{R}/\mathbb{Z} so that the ℓ^1 norm $\sum_{i=1}^{\infty} \|a_i\|_{\mathbb{R}/\mathbb{Z}}$ is less than ∞ . This is an abelian group endowed with the usual addition operation.

The key fact about this group $\ell^1(\mathbb{N}; \mathbb{R}/\mathbb{Z})$ that we will exploit is that it is not a Hilbert space and so has no true “quadratic structure” (and similarly, no true cubic structure or quartic structure and so on). While one can define the square of the ℓ^2 norm to be $\sum_{i=1}^{\infty} \|a_i\|_{\mathbb{R}/\mathbb{Z}}^2$ which will be bounded by the usual ℓ^1 norm $\sum_{i=1}^{\infty} \|a_i\|_{\mathbb{R}/\mathbb{Z}}$ (and similarly one can define other ℓ^p norms), one can check that the parallelogram law $\|x+d\|_{\ell^2}^2 - 2\|x\|_{\ell^2}^2 + \|x-d\|_{\ell^2}^2 = 2\|d\|_{\ell^2}^2$ does not hold. For instance one can set $x = (0, 1/3, 2/3, 0, \dots)$ and $d = (1/3, 1/3, 1/3, 0, \dots)$. Then $\|d\|_2^2 = 1/3$ while $\|x-d\|_2 = \|x\|_2 = \|x+d\|_2$. However, this “fake quadratic structure” from this ℓ^2 norm and from the curvature of the circle will enable us to come up with a coloring that prevents three-term monochromatic progressions. Over the integers, the ℓ^2 norm will end up corresponding to something like an infinite sum of squares of fractional parts $\sum \{na_i\}^2$ which is an infinite-dimensional generalized polynomial and does not simplify to a low-dimensional generalized polynomial like $\{n^2\alpha\}$ where $\alpha = \sum a_i^2$. We now describe the construction.

2.1 Building the ℓ^1 example

Let δ_1 and δ_2 be small positive numbers so that δ_1/δ_2 is an integer and δ_2 is a suitably small function of δ_1 , so for instance $\delta_2 < \delta_1^3$. For instance one can pick $\delta_1 = 10^{-2}$ and $\delta_2 = 10^{-10}$. However, for ease of notation we will write δ_1 and δ_2 .

Definition 2.1. Let $S = S_{\infty}$ be the subset of sequences (a_1, a_2, \dots) from $\ell^1(\mathbb{N}; \mathbb{R}/\mathbb{Z})$ so that the following hold:

1. There is some $i \in \mathbb{N}$ such that $a_i \in (-\delta_1 - 2\delta_2, -\delta_1 + 2\delta_2) \subset \mathbb{R}/\mathbb{Z}$.
2. For $j \neq i$ we have $a_j \in (-2\delta_2, 2\delta_2) \subset \mathbb{R}/\mathbb{Z}$.
3. $\delta_1 - 2\delta_2 < \sum_{j \neq i} \|a_j\|_{\mathbb{R}/\mathbb{Z}} < \delta_1 + 2\delta_2$.

In other words, some a_i is in a small interval around $-\delta_1$ in \mathbb{R}/\mathbb{Z} and the other a_j 's are all very close to an integer so that their \mathbb{R}/\mathbb{Z} norms add up to about δ_1 .

We also define a finitary version S_m of S supported only on the first m coordinates. We use the notation $\ell^1(\mathbb{N}_{\leq m}; \mathbb{R}/\mathbb{Z})$ for the space $(\mathbb{R}/\mathbb{Z})^m$ equipped with the usual norm $\|(a_1, \dots, a_m)\| = \sum_{i=1}^m \|a_i\|_{\mathbb{R}/\mathbb{Z}}$. We will need this later for compactness purposes.

Definition 2.2. Let $\eta_m = 2^{-100m}$ and let S_m be the subset of sequences $(a_1, a_2, \dots, a_m) \in \ell^1(\mathbb{N}_{\leq m}; \mathbb{R}/\mathbb{Z})$ so that the following hold:

1. There is some $i \in \mathbb{N}$ such that $a_i \in (-\delta_1 - (2 - \eta_m)\delta_2, -\delta_1 + (2 - \eta_m)\delta_2) \subset \mathbb{R}/\mathbb{Z}$.
2. For $j \neq i$ we have $a_j \in (-(2 - \eta_m)\delta_2, (2 - \eta_m)\delta_2) \subset \mathbb{R}/\mathbb{Z}$.
3. $\delta_1 - (2 - \eta_m)\delta_2 < \sum_{j \neq i} \|a_j\|_{\mathbb{R}/\mathbb{Z}} < \delta_1 + (2 - \eta_m)\delta_2$.

Note that this the same definition except with the interval bounds $2\delta_2$ replaced by $(2 - \eta_m)\delta_2$ for some small constants $\eta_m = 2^{-100m} \rightarrow 0$. We must consider such S_m with positive and decreasing η_m for compactness purposes. We may set $\eta_{\infty} = 0$ so that $S_{\infty} = S$.

We first check that S meets every Bohr set in this setting. This will be important later, when we project S down to \mathbb{Z} . Also for $x \in (\mathbb{R}/\mathbb{Z})^k$ we define $|f(x)|$ to be the maximum of the \mathbb{R}/\mathbb{Z} norms of the coordinates.

Definition 2.3 (Bohr set, $\ell^1(\mathbb{N}_{\leq m}; \mathbb{R}/\mathbb{Z})$). Let $m \in \mathbb{N} \cup \infty$. We say that $B \subset \ell^1(\mathbb{N}_{\leq m}; \mathbb{R}/\mathbb{Z})$ is a Bohr set if there exists k and ε and a linear function $f : \ell^1(\mathbb{N}_{\leq m}; \mathbb{R}/\mathbb{Z}) \rightarrow (\mathbb{R}/\mathbb{Z})^k$ so that for B is the set of b so that $|f(b)| < \varepsilon$.

In other words, we must check that for all linear function $f : \ell^1(\mathbb{N}; \mathbb{R}/\mathbb{Z}) \rightarrow (\mathbb{R}/\mathbb{Z})^k$, there are points of S that are mapped arbitrarily close to the origin. What we actually need is a finitary version of this.

Theorem 2.4. Let B be a Bohr set with dimension k and width ε . Then there exists a function g so that if $m \in \mathbb{N} \cup \infty$ and $m > g(k, \varepsilon)$, $S_m \cap B \neq \emptyset$.

Proof. Let $\delta_2 e_j$ be the vector that is δ_2 in the j th coordinate and 0 in all other coordinates. For all ε , if m is sufficiently large as a function of k and ε we can find a set T of $1 + \delta_1/\delta_2$ natural numbers so that the following property holds. Call one of these numbers i . Then for any of the other numbers j , the vector $f_j := f(\delta_2 e_j)$ is within $\delta_2 \varepsilon$ of the vector $f_i := f(\delta_2 e_i)$. In other words $|f_j - f_i| \leq \delta_2 \varepsilon$, where the norm is taken in $(\mathbb{R}/\mathbb{Z})^k$.

The vector v with coordinate $-\delta_1$ in the i th coordinate, δ_2 in the j th coordinate for $T \ni j \neq i$, and 0 in all other coordinates is in S_m by definition. Since v is of the form $\sum_{T \ni j \neq i} (\delta_2 e_j - \delta_2 e_i)$, by the triangle inequality the norm of $f(v)$ in $(\mathbb{R}/\mathbb{Z})^k$ is bounded by $\sum_{T \ni j \neq i} \|f_j - f_i\| \leq (\delta_1/\delta_2) \delta_2 \varepsilon = \delta_1 \varepsilon < \varepsilon$. The same argument shows that S_m meets a Bohr set as long as m is a large enough function of the dimension k and the width $1/\varepsilon$ of the Bohr set. \square

In fact we only used here that the function is additive, since $\delta_1/\delta_2 \in \mathbb{N}$. The astute reader may notice that the linear functions $f : \ell^1(\mathbb{N}; \mathbb{R}/\mathbb{Z}) \rightarrow \mathbb{R}/\mathbb{Z}$ comprise the dual group $\ell^\infty(\mathbb{N}; \mathbb{R}\mathbb{Z})$, are given by a sequence of integers (b_1, b_2, \dots) with some C so that $|b_i| \leq C$ for all i , and send (a_1, a_2, \dots) to $\sum_{i \in \mathbb{N}} a_i b_i$. In particular, a function to $(\mathbb{R}/\mathbb{Z})^k$ will consist of k such functions, one for each coordinate.

2.2 The coloring

We now describe the coloring on $\ell^1(\mathbb{N}_{\leq m}; \mathbb{R}/\mathbb{Z})$ for $m \in \mathbb{N} \cup \{\infty\}$. Let $e(x) = e^{2\pi i x}$ be the exponential function from \mathbb{R}/\mathbb{Z} to the unit circle in \mathbb{C} . Note by the mean value theorem that $|e(x) - 1| \leq 2\pi \|x\|_{\mathbb{R}/\mathbb{Z}}$ where the norm on the left-hand side is taken in the complex numbers.

We will color the sequence $x = (a_1, \dots, a_m) \in \ell^1(\mathbb{N}_{\leq m}; \mathbb{R}/\mathbb{Z})$ by the value of $f(x) = \sum_{i=1}^m (e(a_i) - 1)$ or in \mathbb{C} , taken mod the Gaussian integers $\mathbb{Z}[i]$ up to a small error. Since $e(x) - 1$ is bounded by a constant multiple of the \mathbb{R}/\mathbb{Z} norm, for any $x \in \ell^1(\mathbb{N}; \mathbb{R}/\mathbb{Z})$, $f(x)$ will be well-defined even when $m = \infty$.

Definition 2.5 (Our Coloring). Partition $\mathbb{C}/\mathbb{Z}[i] \cong (\mathbb{R}/\mathbb{Z})^2$ into $O(1/\delta_2^2)$ sets of diameter at most δ_2 . Color $(a_1, \dots, a_m) \in \ell^1(\mathbb{N}_{\leq m}; \mathbb{R}/\mathbb{Z})$ by the set that contains $\sum_{i=1}^m (e(a_i) - 1)$ taken modulo $\mathbb{Z}[i]$.

We now show that our coloring lacks three-term monochromatic progressions with common difference in S .

Theorem 2.6. Let $x, s \in \ell^1(\mathbb{N}_{\leq m}; \mathbb{R}/\mathbb{Z})$ with $s \in S_m$. Then if $\ell^1(\mathbb{N}_{\leq m}; \mathbb{R}/\mathbb{Z})$ is colored as in Theorem 2.5, then x , $x + s$, and $x + 2s$ will not all receive the same color.

Before proving this, we recall a basic fact from geometry and trigonometry. If we consider three equally spaced points on the unit circle, if they cut out an arc which is about a δ proportion of the circle, then the middle point will have distance about δ^2 from the midpoint of the arc, by the standard Taylor approximation $1 - \cos(\delta) = \delta^2/2 + O(\delta^3)$. This distance is called the “sagitta”. So the contribution to $f(x) - 2f(x + s) + f(x + 2s)$ from the big i term so that $a_i \approx \delta_1$ will be about δ_1^2 , whereas the other contributions from the j terms will in total at most $\frac{\delta_1}{\delta_2} \delta_2^2 \approx \delta_1 \delta_2$ which is smaller.

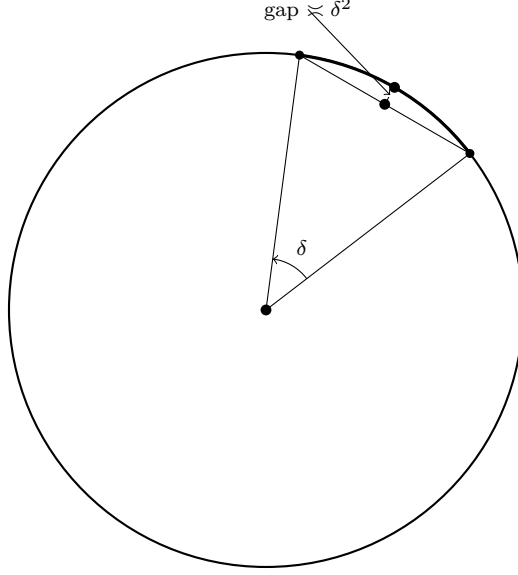


Figure 1: Curvature on the unit circle: an arc of proportion δ (central angle $2\pi\delta$) has sagitta $1 - \cos(\pi\delta) \asymp \delta^2$ between its arc-midpoint and chord-midpoint.

Proof. Let $s = (s_1, \dots, s_n) \in S \subset \ell^1(\mathbb{N}; \mathbb{R}/\mathbb{Z})$ and pick δ_1 to be a sufficiently small constant so that $1/\delta_1$ is an integer, and let $\delta_2 = \delta_1^5$ be an even smaller constant. In a slight abuse notation, we will use Θ and O for quantities that stay constant as $\delta_1 \rightarrow 0$, but we will end up making δ_1 a small enough constant itself. For any $x = (x_1, \dots, x_n, \dots) \in \ell^1(\mathbb{N}; \mathbb{R}/\mathbb{Z})$, we have that:

$$f(x) - 2f(x+s) + f(x+2s) = e(x_i) - 2e(x_i + s_i) + e(x_i + 2s_i) + \sum_{j \neq i} (e(x_j) - 2e(x_j + s_j) + e(x_j + 2s_j)).$$

The first term $e(x_i) - 2e(x_i + s_i) + e(x_i + 2s_i)$ has ℓ^2 norm $\Theta(\delta_1^2)$ in \mathbb{C} while the other terms have a total ℓ^2 norm in \mathbb{C} of

$$O\left(\sum_{j \neq i} \|s_j\|_{\mathbb{R}/\mathbb{Z}}^2\right) = O\left(\left(\max_{j \neq i} \|s_j\|_{\mathbb{R}/\mathbb{Z}}\right) \left(\sum_{j \neq i} \|s_j\|_{\mathbb{R}/\mathbb{Z}}\right)\right) = O(\delta_1 \delta_2) = O(\delta_1^6).$$

So if we choose δ_1 to be a small enough constant, $f(x) - 2f(x+s) + f(x+2s)$ has ℓ^2 norm $\Theta(\delta_1^2) > 2\delta_2$ in \mathbb{C} . This is less than $1/100$, and is far greater than $2\delta_2$ as well. This means that $f(x), f(x+s), f(x+2s)$ cannot all be within distance δ_2 of each other in $\mathbb{C}/\mathbb{Z}[i]$, as if they were then $f(x) - 2f(x+s) + f(x+2s) = (f(x) - f(x+s)) - (f(x+s) - f(x+2s))$ would be within $2\delta_2$ of a Gaussian integer by the triangle inequality. Hence $x, x+s, x+2s$ cannot all receive the same color under our coloring, and there is no monochromatic 3-term arithmetic progression with difference $s \in S$. \square

2.3 Projecting to \mathbb{Z}

We now need to project the example $S = S_\infty$ from the previous subsection onto a set $S = S_\mathbb{N}$ in \mathbb{Z} . Note that in order to avoid a notational clash for S , we refer to the sets from the previous section as S_m for $m \in \mathbb{N} \cup \{\infty\}$ and we refer to the image of the projection of S_∞ as $S_\mathbb{N}$.

We will pick a sufficiently quickly shrinking sequence of rationally independent irrational numbers $(\alpha_1, \alpha_2, \dots)$ and consider the projection map P sending $n \rightarrow (\{\alpha_1 n\}, \{\alpha_2 n\}, \dots) \in \ell^1(\mathbb{N}; \mathbb{R}/\mathbb{Z})$. We will pick our set of integers $S = S_{\mathbb{N}}$ to be the pre-image of the set $S_{\infty} \subset \ell^1(\mathbb{N}; \mathbb{R}/\mathbb{Z})$ from the last section under this map. Since P is a homomorphism, it is immediate to verify that there is a finite coloring of \mathbb{Z} so that there are no monochromatic 3-APs in \mathbb{Z} with common difference in $P(S) = S_{\mathbb{N}}$. Simply color $n \in \mathbb{Z}$ by the color of $P(n) \in \ell^1(\mathbb{N}; \mathbb{R}/\mathbb{Z})$. We also mention here that since α_i shrink sufficiently quickly, it is immediate that the image will be dense in $\ell^1(\mathbb{N}; \mathbb{R}/\mathbb{Z})$ with the usual ℓ^1 metric.

It remains to also check that $S_{\mathbb{N}}$ intersects any finite number of Bohr sets and nil-Bohr sets. We dealt with the Bohr sets in the last section, and to show that higher-order nil-Bohr sets are not an issue we will use standard results about the independence of higher degree generalized polynomials up to easily handled obstructions. Since S is constructed only from linear Bohr sets $n\alpha$, it is going to generally be independent of $n^2\alpha$ and intersect the set of n so that $\|n^2\alpha\|_{\mathbb{R}/\mathbb{Z}} < \varepsilon$. Of course $n^2\alpha$ and $(n^2 + n)\alpha$ will not be linearly independent of $n\alpha$, and so we may get constantly many more linear Bohr obstructions that can arise in this way.

Fortunately generalized polynomials can be generated by “basic generalized polynomials” that do equi-distribute unless there is a linear obstruction, so the usual Bohr set obstacles are all we need to check. In particular by the main result of [19] we may write our generalized polynomials as piecewise polynomial functions of these “basic generalized polynomials” and so any nil-Bohr set contains a nil-Bohr set generated by these basic generalized polynomials. We remark also that the “basic generalized polynomials” of [19] are slightly different than the “special generalized polynomials” of [15], and that Leibman uses the abbreviation “gen-polynomials” for generalized polynomials.

We now proceed to a rigorous proof. Let $\varepsilon_i \rightarrow 0$ be a sequence of positive numbers that goes to 0 sufficiently quickly, and pick $\alpha_1, \alpha_2, \dots$ going to 0 sufficiently quickly depending on ε_i . Let $S_{\mathbb{N}}$ be the inverse image of S_{∞} under the projection P . Let B be a nil-Bohr set. By Theorem 0.2 of [19], B contains a nil-Bohr set generated by basic generalized polynomials B_1, \dots, B_k and by Theorem 0.1 of [19] they jointly equi-distribute. Pick $m = m(B)$ sufficiently large depending on B , and consider the map (P_m, B) sending n to $(\{n\alpha_1\}, \dots, \{n\alpha_m\}, B_1(n), \dots, B_k(n))$ where $k = k(B)$. So we define $P_m(n) = (\{n\alpha_1\}, \dots, \{n\alpha_m\})$ and $B(n) = (B_1(n), \dots, B_k(n))$.

A *subtorus* of $(\mathbb{R}/\mathbb{Z})^{m+k}$ is defined to be the set of points where some set of linear forms with integer coefficients vanish, and the codimension is the number of independent linear forms needed to generate this vanishing set. For our purposes we need a slight modification of Leibman’s result.

Theorem 2.7 (Leibman). *Let $B = \{B_1, \dots, B_k\}$ be a family of equi-distributed basic generalized polynomials and let $\varepsilon > 0$. If $\alpha_1, \dots, \alpha_m$ are rationally independent irrational numbers, then the image of the map $n \rightarrow (\{n\alpha_1\}, \dots, \{n\alpha_m\}, B_1(n), \dots, B_k(n))$ which we denote as $(P_m(n), B(n))$ is ε -dense in some $O_k(1)$ codimension sub-torus of $(\mathbb{R}/\mathbb{Z})^{m+k}$.*

Proof. By Theorem 0.1 of [19], the last k coordinates will equidistribute over the torus $(\mathbb{R}/\mathbb{Z})^k$. Now reveal the α_i one by one. At each step, the new α_i will either be rationally independent from all of our existing gen-polynomials $\{n\beta\}$ (i.e. the already revealed $\{n\alpha_i\}$ and the $\{n\beta\}$ from B), in which case $\{n\alpha_i\}$ will jointly equidistribute, or it will be a rational linear combination of the β . Because the α_i are rationally independent, the latter case will occur only $O_k(1)$ times. \square

In Theorem 2.7, it is equivalent to restrict n to be bounded by some function F of $(\alpha_1, \dots, \alpha_m)$, ε , and B . For our purposes we will need an infinitary corollary of Theorem 2.7. We first record a slightly refined version.

First there exists a “complexity” function $c(B, \varepsilon)$ on nil-Bohr neighborhoods with some useful properties. We say that (B, ε) has complexity at most C if it can be expressed as the pointwise limit of nil-Bohr neighborhoods with dimension at most C , degree at most C , and width at least $1/C$. So C is a bound on the parameters of the nil-Bohr set. In order to compute the exact

complexity of (B, ε) , we take the infimum of these upper bounds. In particular, this complexity function goes to infinity with the inverse width $1/\varepsilon$, the degree d , and the dimension k , and a pointwise limit of (B, ε) with complexity at most C also has complexity at most C .

Theorem 2.8 (Leibman, refined). *There exists an explicit F' so that for any nil-Bohr set (B, ε) , it holds that $F((\alpha_1, \dots, \alpha_m), \varepsilon, B) \leq F'((\alpha_1, \dots, \alpha_m), c(B, \varepsilon))$.*

Proof. We prove this via compactness. Assume there exists $(\alpha_1, \dots, \alpha_m)$ and a sequence of (B, ε) with $c(B, \varepsilon) \leq C$ so that $F((\alpha_1, \dots, \alpha_m), \varepsilon, B)$ tends to infinity. By compactness, a subsequence of the (B, ε) has some nil-Bohr set $(B_\ell, \varepsilon_\ell)$ also of complexity bounded by C as a pointwise limit. But then $F(\alpha_1, \dots, \alpha_m, B_\ell, \varepsilon_\ell) = \infty$, a contradiction. \square

We need an infinitary corollary of this refinement for our purposes.

Theorem 2.9 (Leibman, infinitary corollary). *There exists a sufficiently quickly shrinking sequence α_i so that the following holds. Let B_1, \dots, B_k be equi-distributed basic generalized polynomials. If $m = m(\varepsilon, B_1, \dots, B_k)$ is sufficiently large, then the image of the map $n \mapsto (\{n\alpha_1\}, \dots, \{n\alpha_m\}, B_1(n), \dots, B_k(n)) = (P_m(n), B(n))$ for n ranging and sufficiently smaller than $1/\alpha_{m+1}$ is ε -dense in some $O_k(1)$ codimension sub-torus of $(\mathbb{R}/\mathbb{Z})^{m+k}$.*

Proof. By Theorem 2.8 we may pick a sequence of α_i that goes to zero quickly enough so that for any B and ε , for sufficiently large m we have $F(\alpha_1, \dots, \alpha_m, B, \varepsilon)$ is sufficiently smaller than $1/\alpha_{m+1}$. For instance if we guarantee that $1/\alpha_{m+1}$ is much larger than $F'(\alpha_1, \dots, \alpha_m, m) \geq F(\alpha_1, \dots, \alpha_m, B, \varepsilon)$ for $c(B, \varepsilon) \leq m$, then for any fixed (B, ε) we will have that $F(\alpha_1, \dots, \alpha_m, B, \varepsilon)$ will be sufficiently smaller than $1/\alpha_{m+1}$ for m which is a large enough function of $c = c(B, \varepsilon)$. \square

Theorem 2.10. *If we take a sequence of α_i satisfying the conditions of Theorem 2.9, then for any nil-Bohr set B , $P(S) \cap B \neq \emptyset$.*

Proof. Let ε_i be a sequence that goes to 0 sufficiently quickly. When n ranges but stays sufficiently small compared to $1/\alpha_{k+1}$, we have that for sufficiently large m $(P_m(n), B(n))$ intersects $S_m \times (-\varepsilon_k, \varepsilon_k)^k$, because m was chosen to be sufficiently large so that S_m meets all ordinary Bohr sets of complexity $O_k(1)$. So we can find some n so that $P_m(n) \in S_m$ and n is sufficiently smaller than $1/\alpha_{k+1}$. By the size bound on n we have $P(n) \in S_\infty$ and $n \in S_\mathbb{N}$. Since the coordinates of $B(n)$ have \mathbb{R}/\mathbb{Z} norm less than ε_k , n is in the nil-Bohr set B . This concludes the proof. \square

A remark about nilmanifolds is in order. We do not use them here, but our results could be phrased equivalently in those terms. Section 2 of [19] describes the correspondence between equi-distribution of generalized polynomials and the equi-distribution over nilmanifolds from [18]. The correspondence between generalized polynomials and nilmanifolds was first proved in [2] and is cited as Lemma 2.3 of [19]. Theorem 2.9 of [7] is the quantitative version of Theorem 1.9 of [7], the main theorem of [18]. As written, our construction is not explicit because of the compactness arguments. Using the quantitative results of Green-Tao [7], one can remedy this and make all of our compactness arguments effective.

3 Concluding Remarks and Open Questions

The most obvious open question here is Katznelson's question. In view of our disproof of the higher-order version, we conjecture that the answer should be negative.

Conjecture 3.1. *There is a set $S \subset \mathbb{N}$ which is a set of Bohr recurrence but not a set of topological recurrence.*

As usual we also state the combinatorial form of this conjecture.

Conjecture 3.2. *There is $S \subset \mathbb{N}$ so that the Cayley graph on \mathbb{Z} generated by S has a finite chromatic number, but does not admit a finite coloring via Bohr sets.*

One reason to believe Katznelson’s question has a positive answer is that a “zero density exception” version is known due to [5]. In particular, if A is a dense subset of \mathbb{Z} , then $S - S$ contains a $1 - o(1)$ proportion of a Bohr set. Katznelson’s question is equivalent to the same claim except with A syndetic and without the $o(1)$ proportion, see for instance [6]. A similar “zero density exception” of the higher-order Katznelson was proved in [15], but we have proved here higher-order Katznelson has a negative answer.

We generally believe that if Katznelson’s question has a positive answer, the proof somehow relies on topology. Indeed Glasscock, Koutsogiannis, and Richter [6] prove some special cases by using the intermediate value theorem, and [17] relies on the topological lower bound for the chromatic number of the Kneser graph.

We mention one important obstacle to extending something like our construction to settle the full Katznelson’s question. Let G be an Abelian group. Then an element of G^m in the generating set with sum of coordinates 0 can be written in the form $(a_1 - a_2, a_2 - a_3, \dots, a_{m-1} - a_m, a_m - a_1)$, so the cyclic shifts $(a_1, a_2, a_3, \dots, a_n)$ and $(a_2, a_3, \dots, a_n, a_1)$. Hence, we cannot simply take a function of each coordinate and add it up the way we considered $(e(x) - 1)$ here. Unfortunately, this kind of “cyclic shift” obstacle destroys many potential counterexamples to Katznelson’s question, and any kind of high-dimensional projection must break this sort of symmetry.

Another interesting direction is to consider the finite field setting. One can also consider analogues of Katznelson’s question over positive characteristic. This was asked by Ben Green over characteristic 2 ([8], Problems 52 and 53) and also by John Griesmer [9] over more general characteristic p . We can similarly ask the higher-order version of Katznelson’s question in characteristic p for $p > d$. We state two such versions for 3-term progressions.

Question 3.3. *Let $p_1, p_2, \dots, p_n, \dots \geq 3$ be an increasing sequence of primes, and say we have a sequence G_i of Cayley graphs on \mathbb{Z}/p_i with uniformly bounded chromatic number C . Is there a uniform constant K so that the generating set of G_i is contained in the union of K sets $\|P_j(x)/p\|_{\mathbb{R}/\mathbb{Z}} > 1/K$, where P_j are polynomials on \mathbb{Z}/p_i of degree at most 2?*

In other words, can we always find a bounded number K of quadratics $P_j(x)$ so that any x in the generating set is mapped by at least one of the quadratics to a point that is not in the interval $(-p/K, p/K)$?

Question 3.4. *Let $p \geq 3$ be a fixed prime, and say we have a sequence of Cayley graphs G_i on $(\mathbb{Z}/p)^n$ for a sequence of $n = n_i$ with uniformly bounded chromatic number C . Is there a uniform constant K so that the generating set of G_i is contained in the union of K sets $P_j(x) \neq 0$, where P_j are quadratic forms on $(\mathbb{Z}/p)^n$ of degree at most 2?*

A few comments are in order about why our example does not immediately resolve the above questions. Our example in some sense relies on the infinite bracket polynomial $\sum_i \{na_i\}^2$. In modulo 1, x and x^2 are independent. However, in modulo p the value of n determines the value of n^2 , and $\sum_i (na_i \bmod p)^2 = n^2 (\sum a_i^2 \bmod p)$. Our example also relies heavily on the existence of a norm, which is less straightforward in positive characteristic, though one can consider the p -adic norm or $\|x/p\|_{\mathbb{R}/\mathbb{Z}}$. In general, it is also true that \mathbb{Z}/p and \mathbb{R}/\mathbb{Z} “look different” from the standpoint of arithmetic progressions of length 3 or greater. See [4] for discussion about this.

We do not make any attempt at all to optimize the number of colors used in our construction. With our ideas it seems difficult to beat the bounds from the unit distance in the plane. Let us also mention that while it is simple to check that at least 3 colors are necessary for any example answering Katznelson’s Question in the negative [6], in this setting of 3-term and longer progressions we do not even know if we can have a counterexample using only 2 colors.

Concerning counterexamples with only 2 colors, we also mention here the 2-large versus large conjecture of Brown, Graham, and Landman [3]. A set S is *large* if it is a set of d -topological recurrence for all d , so that restricted van der Waerden with restricted common difference $s \in S$ is true for arbitrarily long arithmetic progressions. A set S is *2-large* if any 2-coloring of \mathbb{N} has arbitrarily large arithmetic progressions with common difference in S . Brown, Graham, and Landman asked if 2-large sets are large. We have constructed the first example of a set that is not large but meets every nil-Bohr set. We conjecture that it is necessary to do so to refute the 2-large versus large conjecture.

Conjecture 3.5. *If S is not a set of d -nilBohr recurrence for some d , then S is not 2-large.*

We also ask whether these sets, or some similarly constructed modifications, are 2-large.

Question 3.6. *Are the sets S of the kind constructed here 2-large?*

Another potential question we can ask is about the polynomial version of Katznelson’s question. What if we consider the Bergelson-Leibman polynomial van der Waerden Theorem [1] with restricted differences? We pose one such question.

Question 3.7. *For what sets S does a finite coloring of \mathbb{N} contain a monochromatic $\{x, x + d, x + d^2\}$ with $d \in S$?*

Progress on any of these questions would be interesting and would contribute to our general understanding of recurrence.

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