POINT-LIKE NON-COMMUTATIVE FAMILIES OF BOUNDING COCHAINS

ELAD KOSLOFF AND JAKE P. SOLOMON

ABSTRACT. We define genus zero open Gromov-Witten invariants with boundary and interior constraints for a Lagrangian submanifold of arbitrary even dimension. The definition relies on constructing a canonical family of bounding cochains that satisfy the point-like condition of the second author and Tukachinsky. Since the Lagrangian is even dimensional, the parameter of the family is odd. Thus, to avoid the vanishing of invariants with more than one boundary constraint, the parameter must be non-commutative. The invariants are defined either when the Lagrangian is a rational cohomology sphere or when the Lagrangian is fixed by an anti-symplectic involution, has dimension 2 modulo 4, and its cohomology is that of a sphere aside from degree 1 modulo 4. In dimension 2, these invariants recover Welschinger's invariants.

We develop an obstruction theory for the existence and uniqueness of bounding cochains in a Fukaya A_∞ algebra with non-commutative coefficients. The obstruction classes belong to twisted cohomology groups of the Lagrangian instead of the de Rham cohomology of the commutative setting. A spectral sequence is constructed to compute the twisted cohomology groups. The extension of scalars of an A_∞ algebra by a non-commutative ring is treated in detail. A theory of pseudo-completeness is introduced to guarantee the convergence of the Maurer-Cartan equation, which defines bounding cochains, even though the non-commutative parameter is given zero filtration.

CONTENTS

1. Introduction

1.1. **Overview.** An open Gromov-Witten invariant is a count of J-holomorphic curves in a symplectic manifold X with boundary in a Lagrangian submanifold $L \subset X$ representing a homology class $d \in H_2(X, L)$. To obtain a finite count, the curves are constrained to pass through a collection of cycles on X and L, called interior and boundary constraints respectively. The count should not depend on the choice of tame almost complex structure J and it should depend only on the homology class of the cycles used as constraints. The main difficulty in defining open Gromov-Witten invariants is that as one varies J or the constraints in a one-parameter family, J-holomorphic curves can degenerate and disappear through the process of boundary bubbling. In some situations [5,17,30,33], boundary bubbles can be shown to cancel out using symmetries of the pair (X, L). However, in the presence of boundary constraints, when L has dimension greater than 3, the cancellation of bubbles based on symmetries breaks down. The second author and Tukachinsky [35–37] used the Fukaya-Oh-Ohta-Ono theory of bounding cochains [6, 8] to define open Gromov-Witten invariants without requiring any symmetry of (X, L), in arbitrary dimension, with boundary and interior constraints. To this end, they define the notion of a "point-like" family of bounding cochains and prove its existence and uniqueness up to gauge-equivalence under appropriate conditions. However, in even dimensions, the parity of the parameter of the family is odd, and since it is graded commutative, invariants with more than one boundary constraint vanish.

The present work develops the theory of point-like families of bounding cochains with non-commutative parameter and uses it to define open Gromov-Witten invariants with an arbitrary number of boundary constraints that need not vanish. As in [35–37], these invariants are defined without requiring any symmetry of (X, L), in arbitrary dimension, and interior constraints are allowed as well. In dimension 2 the open Gromov-Witten invariants of the present work are shown to recover Welschinger's enumerative invariants [41]. So, computations of Welschinger's invariants [4, 23–25] give examples where the open Gromov-Witten invariants of the present work with multiple boundary constraints are non-zero. A forthcoming article of the first author [28] extends the open WDVV equation of [37] to the invariants of the present work, which should allow extensive calculations in arbitrary dimension. While the present work assumes the Lagrangian submanifold is orientable, the techniques developed here applied to the Fukaya A_∞ algebras of non-orientable Lagrangians developed in [26, 27] should give open Gromov-Witten invariants of non-orientable Lagrangians.

The existence and uniqueness of non-commutative families of bounding cochains is controlled by obstruction classes that belong to a twisted version of the de Rham cohomology of the Lagrangian submanifold. The twisted cohomology is computed from the ordinary cohomology of the Lagrangian by means of a spectral sequence. The non-triviality of the differentials of this spectral sequence force the vanishing of certain obstructions that do not appear in the commutative setting. the obstruction classes to belong to the twisted cohomology, we assign zero filtration to the non-commutative parameter. Thus, the Maurer-Cartan equation, which a bounding cochain satisfies by definition, a priori need not converge. To prove convergence, we show that Fukaya A_∞ algebras satisfy a property called pseudo-completeness.

1.2. Statement of the results.

1.2.1. *A_∞ algebras.* To formulate our results, we recall relevant notation from [36]. For an element x of a graded abelian group we denote by $|x|$ the degree of x. Consider a symplectic manifold (X, ω) of even complex dimension $0 < n$, and a connected, Lagrangian submanifold L with relative spin structure $\mathfrak{s} = \mathfrak{s}_L$. Let J be an ω-tame almost complex structure on X. Denote by $\mu : H_2(X, L) \to \mathbb{Z}$ the Maslov index. Denote by $A^*(L)$ the ring of differential forms on L with coefficients in $\mathbb{R}$. Let $\Pi = \Pi_L$ be the quotient of $H_2(X, L; \mathbb{Z})$ by a possibly trivial subgroup S_L contained in the kernel of the homomorphism $\omega \oplus \mu : H_2(X, L; \mathbb{Z}) \to \mathbb{R} \oplus \mathbb{Z}$. Thus the homomorphisms ω, μ, descend to Π. Denote by β_0 the zero element of Π. We use a Novikov ring Λ which is a completion of a subring of the group ring of Π. The precise definition follows. Denote by T^β the element of the group ring corresponding to $\beta \in \Pi$, so $T^{\beta_1} T^{\beta_2} = T^{\beta_1 + \beta_2}$. Then,

$$\Lambda = \left\{ \sum_{i=1}^\infty a_i T^{\beta_i} \middle| a_i \in \mathbb{R}, \beta_i \in \Pi, \omega(\beta_i) \geq 0, \omega(\beta_i) = 0 \implies \beta_i = \beta_0, \lim_{i \to \infty} \omega(\beta_i) = \infty \right\}. \quad (1)$$

A grading on Λ is defined by declaring T^β to be of degree $\mu(\beta)$. Denote also

$$\Lambda^+ = \left\{ \sum_{i=1}^\infty a_i T^{\beta_i} \in \Lambda \middle| \omega(\beta_i) > 0 \quad \forall i \right\}.$$

We use a family of A_∞ structures on $A^*(L) \otimes \Lambda$ following [6, 8], based on the results of [36]. Let $\mathcal{M}_{k+1,l}(\beta)$ be the moduli space of genus zero J-holomorphic open stable maps $u : (\Sigma, \partial\Sigma) \to (X, L)$ of degree $[u_*([\Sigma, \partial\Sigma])] = \beta \in \Pi$ with one boundary component, $k + 1$

boundary marked points, and l interior marked points. The boundary points are labeled according to their cyclic order. The space $\mathcal{M}_{k+1,l}(\beta)$ carries evaluation maps associated to boundary marked points $evb_j^\beta : \mathcal{M}_{k+1,l}(\beta) \to L$, $j = 0, \ldots, k$, and evaluation maps associated to interior marked points $evi_j^\beta : \mathcal{M}_{k+1,l}(\beta) \to X$, $j = 1, \ldots, l$.

We assume that all J-holomorphic genus zero open stable maps with one boundary component are regular, the moduli spaces $\mathcal{M}_{k+1,l}(\beta; J)$ are smooth orbifolds with corners, and the evaluation maps evb_0^β are proper submersions. Examples include $(\mathbb{C}P^n, \mathbb{R}P^n)$ with the standard symplectic and complex structures or, more generally, flag varieties, Grassmannians, and products thereof. See [36, Example 1.4, Remark 1.5]. Throughout the paper we fix a connected component $\mathcal{J}$ of the space of ω-tame almost complex structures satisfying our assumptions. All almost complex structures are taken from $\mathcal{J}$. Throughout the paper we fix a connected component $\mathcal{J}$ of the space of ω-tame almost complex structures satisfying our assumptions. All almost complex structures are taken from $\mathcal{J}$. The definition of open Gromov-Witten invariants in the present paper extends to arbitrary targets (X, ω, L) and $\mathcal{J}$ the space of all ω-tame almost complex structures given the virtual fundamental class technique of Kuranishi structures [1, 2, 6–16, 18]. Alternatively, it should be possible to use the polyfold theory of [19–22, 29]. To make the present paper more accessible, we use the self-contained treatment of Fukaya A_∞ algebras for Lagrangian submanifolds of [36]. The treatment suffices for the calculations of [28]. The treatment is sufficiently general to cover an infinite family of Lagrangian submanifolds for which no open Gromov-Witten invariants were defined previously, as mentioned above. The techniques introduced in this paper are no simpler for the Lagrangians covered by the treatment of [36] than in the general case. The relative spin structure $\mathfrak{s}$ determines an orientation on the moduli spaces $\mathcal{M}_{k+1,l}(\beta)$ as in [8, Chapter 8]. For $m > 0$ denote by $A^m(X, L)$ differential m-forms on X that vanish on L, and denote by $A^0(X, L)$ functions on X that are constant on L. The exterior differential d makes $A^*(X, L)$ into a complex. Let

$$R := \Lambda[[t_0, \ldots, t_N]], \qquad Q := \mathbb{R}[t_0, \ldots, t_N],$$

be graded commutative power series and polynomial rings, and let

$$C := R \otimes A^*(L), \qquad \text{and} \qquad D := Q \otimes A^*(X, L).$$

Here and throughout the paper, unless otherwise mentioned, tensor products are implicitly completed. Let $S := \mathbb{R}\langle s \rangle$ denote the ring of polynomials in a non-commutative formal variable s of degree $|s| = 1 - n$, which is necessarily odd, and let

$$\hat{R} := S \otimes R, \qquad \hat{C} := S \otimes R \otimes A^*(L).$$

The grading on these complexes takes into account the degrees of s, t_j, T^β, and the degree of differential forms. For an $\mathbb{R}$-algebra Υ, write

$$\widetilde{H}^*(X, L; \Upsilon) := H^*(A^*(X, L) \otimes \Upsilon, d).$$

Observe that

$$\widetilde{H}^*(X, L; \Upsilon) \simeq \left(H^0(L; \mathbb{R}) \oplus H^{>0}(X, L; \mathbb{R}) \right) \otimes \Upsilon, \qquad \widetilde{H}^*(X, L; Q) = H^*(D).$$

Given a graded module M, we write M_j or $(M)_j$ for the degree j part. Let

$$R^+ := \hat{R}\Lambda^+ \triangleleft \hat{R}, \qquad \mathcal{I}_{\hat{R}} := \langle t_0, \ldots, t_N \rangle + R^+ \triangleleft \hat{R}, \qquad \mathcal{I}_Q := \langle t_0, \ldots, t_N \rangle \triangleleft Q,$$

be the ideals generated by the formal variables. Define $\hat{R}^*$ to be the units of the ring $\hat{R}$. For $x \in S$ we write
$$x = x^0 + x^1, \qquad x^0 \in \mathbb{R}\langle s^2 \rangle, \qquad x^1 \in s \cdot \mathbb{R}\langle s^2 \rangle.$$
We can extend this definition to elements of tensor products M of S such as $M = \hat{R}$ and $M = \hat{C}$. An $x \in M$ such that $x = x^0$ will be called even. Likewise an $x \in M$ such that $x = x^1$ will be called odd. The sub-modules of even and odd elements will be denoted by M^{even} and M^{odd} respectively.

Let $\gamma \in \mathcal{I}_Q D$ be a closed form with $|\gamma| = 2$. For example, given closed differential forms $\gamma_j \in A^*(X, L)$ for $j = 0, \ldots, N$, take t_j of degree $2 - |\gamma_j|$ and $\gamma := \sum_{j=0}^{N} t_j \gamma_j$. Define structure maps
$$\hat{\mathfrak{m}}_k^\gamma : \hat{C}^{\otimes k} \longrightarrow \hat{C}$$
by
$$\hat{\mathfrak{m}}_k^\gamma(\alpha_1, \ldots, \alpha_k) :=$$
$$= \delta_{k,1} \cdot d\alpha_1 + (-1)^{\sum_{j=1}^k j(|\alpha_j|+1)+1} \sum_{\substack{\beta \in \Pi \\ l \geq 0}} T^\beta \frac{1}{l!} evb_{0*}^\beta \left(\bigwedge_{j=1}^l (evi_j^\beta)^* \gamma \wedge \bigwedge_{j=1}^k (evb_j^\beta)^* \alpha_j \right).$$

The push-forward $(evb_0^\beta)_*$ is defined by integration over the fiber; it is well-defined because evb_0^β is a proper submersion. The condition $\gamma \in \mathcal{I}_Q D$ ensures that the infinite sum converges. Intuitively, γ should be thought of as interior constraints, while α_j are boundary constraints. Then the output is a cochain on L that is "Poincaré dual" to the image of the boundaries of disks that satisfy the given constraints.

Define the pairing $\langle\ ,\ \rangle : \hat{C} \otimes \hat{C} \to \hat{R}^{odd}$ by
$$\langle \xi, \eta \rangle := (-1)^{|(\eta)|} \left(\int_L \xi \wedge \eta \right)^1.$$

In Section 4 we show that $(\hat{C}, \{\hat{\mathfrak{m}}_k^\gamma\}_{k \geq 0}, \langle\ ,\ \rangle, 1)$ is an n-dimensional curved cyclic unital A_∞ algebra.

Definition 1.1. A **bounding pair** with respect to J is a pair (γ, b) where $\gamma \in \mathcal{I}_Q D$ is closed with $|\gamma| = 2$ and $b \in (\langle s \rangle + \mathcal{I}_{\hat{R}}) \hat{C}$ with $|b| = 1$, such that
$$\sum_{k \geq 0} \hat{\mathfrak{m}}_k^\gamma(b^{\otimes k}) = c \cdot 1, \qquad c \in \mathcal{I}_{\hat{R}} \cap Z\left(\hat{R}\right), \qquad |c| = 2. \tag{2}$$

In this situation, following [8], b is called a **bounding cochain** for $\hat{\mathfrak{m}}^\gamma$.

Remark 1.2. The condition that c belongs to the center of $\hat{R}$ is crucial for bounding cochains in the present context to behave analogously to bounding cochains of Fukaya A_∞ algebras over commutative rings.

Definition 1.3. A **unit bounding pair** (γ, b) is a bounding pair such that $\int_L b^1 \in s\hat{R}^*$.

The superpotential is defined by
$$\Omega(\gamma, b) := (-1)^n \left(\sum_{k \geq 0} \frac{1}{(k+1)} \langle \hat{\mathfrak{m}}_k^\gamma(b^{\otimes k}), b \rangle \right)$$

Remark 1.4. The superpotential in [35] includes an $\mathfrak{m}_{-1}$ term to account for disks without boundary constraints, and the enhanced superpotential of [37] adds a term to correct for disks degenerating into spheres. These terms cannot be included in the superpotential in the non-commutative setting of the present paper for the following reason. The pairing $\langle \cdot, \cdot \rangle$ is by definition the odd part of the Poincaré pairing. It is necessary to take the odd part in order for the pairing to be cyclically symmetric. See Lemma 3.31. Cyclic symmetry is used in the proof that the superpotential is invariant under gauge equivalence as asserted by Theorem 1 below. On the other hand, terms arising from disks without boundary marked points or spheres are of even parity. They are not invariant by themselves, but in the commutative setting they are augmented by contributions from other terms of the superpotential. In the non-commutative setting, the rest of the superpotential is odd, so it cannot contribute to the invariance of even terms.

Definition 4.20 gives a notion of gauge equivalence between a bounding pair (γ, b) with respect to J and a bounding pair (γ', b') with respect to another almost complex structure J'. Let $\sim$ denote the resulting equivalence relation.

Theorem 1 (Invariance of the super-potential). *If $(\gamma, b) \sim (\gamma', b')$, then $\Omega(\gamma, b) = \Omega(\gamma', b')$.*

To obtain invariants from Ω, we must understand the space of gauge equivalence classes of bounding pairs. Assume $n > 0$. Define a map

$$\varrho : \{\text{unit bounding pairs}\}/\sim \; \longrightarrow \; (\mathcal{I}_Q \widetilde{H}^*(X, L; Q))_2 \oplus (\hat{R}^*)_0^{even} \oplus (\mathcal{I}_{\hat{R}})_{1-n}^{even} \Big/ (\mathcal{I}_{\hat{R}}^{odd})^2$$

by

$$\varrho([\gamma, b]) := \left([\gamma], \frac{\int_L b^1}{s}, \int_L b^0 \right). \tag{3}$$

We prove in Lemma 4.23 that ϱ is well defined.

Theorem 2 (Classification of bounding pairs – rational cohomology spheres). *Assume $H^*(L; \mathbb{R}) = H^*(S^n; \mathbb{R})$. Then ϱ is bijective.*

In the presence of an anti-symplectic involution, the assumptions on L can be relaxed. A **real setting** is a quadruple (X, L, ω, ϕ) where $\phi : X \to X$ is an anti-symplectic involution such that $L \subset fix(\phi)$. Throughout the paper, whenever we discuss a real setting, we fix a connected subset $\mathcal{J}_\phi \subset \mathcal{J}$ consisting of $J \in \mathcal{J}$ such that $\phi^* J = -J$. All almost complex structures of a real setting are taken from $\mathcal{J}_\phi$. If we use virtual fundamental class techniques, we can treat any ω-tame almost complex structure J satisfying $\phi^* J = -J$. Whenever we discuss a real setting, we take $S_L \subset H_2(X, L; \mathbb{Z})$ with $\text{Im}(\text{Id} + \phi_*) \subset S_L$, so ϕ_* acts on $\Pi_L = H_2(X, L; Z)/S_L$ as $-\text{Id}$. Also, the formal variables t_i have even degree. We denote by $\widetilde{H}_\phi^{even}(X, L; \mathbb{R})$ (resp. $H_\phi^{even}(X; \mathbb{R})$) the direct sum over k of the $(-1)^k$-eigenspace of ϕ^* acting on $\widetilde{H}^{2k}(X, L; \mathbb{R})$ (resp. $H^{2k}(X; \mathbb{R})$). Note that the Poincaré duals of ϕ-invariant almost complex submanifolds of X disjoint from L belong to $\widetilde{H}_\phi^{even}(X, L; \mathbb{R})$. Extend the action of ϕ^* to $\Lambda, Q, R, \hat{R}, S, C, \hat{C}$ and D, by taking

$$\phi^* T^\beta = (-1)^{\mu(\beta)/2} T^\beta, \qquad \phi^* t_i = (-1)^{|t_i|/2} t_i, \qquad \phi^*(s) = (-1)^{\frac{|s|(|s|-1)}{2}} s. \tag{4}$$

Elements $a \in \Lambda, Q, R, C, D$, and pairs thereof are called **real** if

$$\phi^* a = -a. \tag{5}$$

For a group Z on which ϕ^* acts, let $Z^{-\phi^*} \subset Z$ denote the elements fixed by $-\phi^*$. Let

$$\varrho_\phi : \{\text{real bounding pairs}\}/\sim \; \longrightarrow \; (\mathcal{I}_Q \widetilde{H}^*(X,L;Q))_2^{-\phi^*} \oplus (\hat{R}^*)_0^{even}$$

be given by the first two coordinates of the formula for ϱ. Then, we obtain the following variant of Theorem 2.

Theorem 3 (Classification of bounding pairs – real spin case). *Suppose* (X,L,ω,ϕ) *is a real setting,* $\mathfrak{s}$ *is a spin structure, and* $n \equiv 2 \pmod 4$. *Moreover, assume* $H^i(L;\mathbb{R}) \simeq H^i(S^n;\mathbb{R})$ *for* $i \not\equiv 1 \pmod 4$. *Then* ϱ_ϕ *is bijective.*

1.2.2. *Open Gromov-Witten invariants.* When the hypothesis of either Theorem 2 or 3 is satisfied, we define open Gromov-Witten invariants as follows. In the case of Theorem 3, take S_L containing $\mathrm{Im}(\mathrm{Id} + \phi_*)$. In the case of Theorem 2 (resp. Theorem 3) let $W_L = \widetilde{H}^*(X,L;\mathbb{R})$ (resp. $W_L = \widetilde{H}_\phi^{even}(X,L;\mathbb{R})$). Fix $\Gamma_0, \ldots, \Gamma_N$, a basis of W_L, set $|t_j| = 2 - |\Gamma_j|$, and take

$$\Gamma := \sum_{j=0}^N t_j \Gamma_j \in (\mathcal{I}_Q \widetilde{H}^*(X,L;Q))_2.$$

By Theorem 2 (resp. Theorem 3), choose a unit bounding pair (γ, b) such that

$$\varrho([\gamma,b]) = (\Gamma,1,0) \qquad (\text{resp. } \varrho_\phi([\gamma,b]) = (\Gamma,1)). \tag{6}$$

By Theorem 1, the superpotential $\Omega = \Omega(\gamma,b)$ is independent of the choice of (γ,b).

Definition 1.5. The **open Gromov-Witten invariants** of (X,L),

$$\mathrm{OGW}_{\beta,k} = \mathrm{OGW}_{\beta,k}^L : W_L^{\otimes l} \to \mathbb{R},$$

are defined by setting

$$\mathrm{OGW}_{\beta,k}(\Gamma_{i_1}, \ldots, \Gamma_{i_l}) := \text{ the coefficient of } T^\beta \text{ in } \partial_{t_{i_1}} \cdots \partial_{t_{i_l}} \partial_s^k \Omega|_{s=0,t_j=0}$$

and extending linearly to general input.

Remark 1.6. Theorems 2 and 3 say that a bounding cochain is determined up to equivalence by the cohomology class of its part that has degree n in $A^*(L)$. In general, the degree n part of b must be "corrected" by non-closed forms of lower degrees in order to solve equation (2). Equation (6) says that the degree n part of the bounding cochain parameterizes multiples of the point as the formal variable s varies. Thus, following [35], such a bounding cochain is called *point-like*.

Remark 1.7. In the special case that $n = 2$, the cohomological assumption is always satisfied. This explains the significance of dimension 2 in Welschinger's work [39]. See Theorem 4 for a comparison of Welschinger's invariants with the invariants of the present work.

1.2.3. *Comparison with Welschinger invariants.* For a symplectic manifold (X,ω) of dimension $n = 2$ with an anti-symplectic involution $\phi : X \to X$, we proceed to compare the invariants defined above with Welschinger's invariants [39]. We begin with some relevant notation. Let $Y \subset \mathit{fix}(\phi)$ be closed and open. Define the doubling map

$$\chi_Y : H_2(X,Y;\mathbb{Z}) \to H_2(X;\mathbb{Z}) \tag{7}$$

as follows. For $\beta \in H_2(X,Y;\mathbb{Z})$ represent β by a singular chain $\sigma \in C_2(X,Y;\mathbb{Z})$. Then $\chi_Y(\beta) = [\sigma - \phi_\# \sigma]$. Observe that χ_Y descends to the quotient of $H_2(X,Y;\mathbb{Z})$ by any subgroup

contained in $\mathrm{Im}(\mathrm{Id} + \phi_*)$. We denote the map on the quotient by χ_Y as well. In particular, as long as $S_L \subset \mathrm{Im}(\mathrm{Id} + \phi_*)$, we have $\chi_L : \Pi_L \to H_2(X; \mathbb{Z})$. Thus, for the following theorem, we take $S_L \subset \mathrm{Im}(\mathrm{Id} + \phi_*)$. When we consider invariants arising from Theorem 3, the opposite inclusion holds as well, so $S_L = \mathrm{Im}(\mathrm{Id} + \phi_*)$.

For $d \in H_2(X; \mathbb{Z}), l \in \mathbb{Z}_{\geq 0}$, write

$$k_{d,l} = \frac{c_1(X)(d) - 2l - 1}{2}.$$

Let $\mathcal{W}_{d,l}$ denote Welschinger's invariant counting real rational J-holomorphic curves in (X, ϕ) of degree d with real locus in L passing through $k_{d,l}$ real points and l pairs of ϕ-conjugate points. Recall from Remark 1.7 that when $n = 2$, the hypothesis of Theorem 3 is always satisfied, so the invariants $\mathrm{OGW}_{\beta,k}$ are defined.

Theorem 4 (Comparison with Welschinger's invariants). *Suppose $n = 2$. Let $A \in \widetilde{H}_\phi^{2n}(X, L; \mathbb{R})$ be the Poincaré dual of a point. Let $d \in H_2(X; \mathbb{Z}), l \in \mathbb{Z}_{\geq 0}$, such that $k_{d,l} \geq 0$. Then,*

$$\sum_{\chi_L(\beta)=d} \mathrm{OGW}_{\beta, k_{d,l}}(A^{\otimes l}) = \pm 2^{1-l} \cdot \mathcal{W}_{d,l}.$$

1.3. **General notation.** We write $I := [0, 1]$ for the closed unit interval. Use i to denote the inclusion $i : L \hookrightarrow X$. By abuse of notation, we also use i for $\mathrm{Id} \times i : I \times L \to I \times X$. The meaning in each case should be clear from the context.

Denote by pt the map (from any space) to a point.

Define a filtration

$$\nu : R \longrightarrow \mathbb{R}_{\geq 0}$$

by

$$\nu\left(\sum_{j=0}^{\infty} a_j T^{\beta_j} \prod_{a=0}^{N} t_a^{l_{aj}}\right) = \inf_{a_j \neq 0}\left(\omega(\beta_j) + \sum_{a=0}^{N} l_{aj}\right).$$

Let Υ' be an $\mathbb{R}$ vector space and let $\Upsilon = \Upsilon' \otimes R$. Then, equipping Υ' with the trivial filtration, ν induces a filtration on Υ which we also denote by ν. Whenever a tensor product (resp. direct sum) of modules with filtration is written, we mean the completed tensor product (resp. direct sum). Whenever a tensor product is written, we mean the completed tensor product over $\mathbb{R}$.

Write $A^*(L; R)$ for $R \otimes A^*(L)$. Similarly, $A^*(X; R)$ and $A^*(X, L; R)$ stand for $R \otimes A^*(X)$ and $R \otimes A^*(X, L)$, respectively.

We implicitly assume that elements of graded rings and modules are homogeneous and write $|\cdot|$ for the degree. Given α, a homogeneous differential form with coefficients in R, denote by $|\alpha|_A$ the degree of the differential form, ignoring the grading of R.

For a possibly non-homogeneous α, denote by $(\alpha)_j$ the form that is the part of degree j in α. In particular, $|(\alpha)_j|_A = j$. For a graded module M, the notation M_j or $(M)_j$ stands for the degree j part of the module, which in the present context involves degrees of forms as well as degrees of variables.

Let Υ' be an $\mathbb{R}$-vector space, $\Upsilon'' = R, Q$, or Λ, and let $\Upsilon = \Upsilon' \otimes \Upsilon''$. For $x \in \Upsilon$ and $\lambda \in \Upsilon''$, denote by $[\lambda](x) \in \Upsilon'$ the coefficient of λ in x.

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2. A_∞ ALGEBRAS OVER AN ASSOCIATIVE ALGEBRA

2.1. **Graded algebras.** We recall that if V is a vector space and G an abelian group, then a G-grading of V is a decomposition of V into a direct sum of subspaces $V = \bigoplus_{\alpha \in G} V_\alpha$. If $v \in V_\alpha$, then we say that v is homogeneous of degree α and we write $|v| = \alpha$. A G-graded algebra is an algebra A with a G grading for which $A_\alpha A_\beta \subset A_{\alpha+\beta}$. Let A and B be associative graded algebras, graded by $\mathbb{Z}$. Their tensor product $A \otimes B$ is the graded algebra whose space is the tensor product of the spaces of A and B, with the induced $\mathbb{Z}$-grading and the product operation defined by

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{(|a_2|)(|b_1|)} a_1 a_2 \otimes b_1 b_2, \qquad a_i \in A, \qquad b_i \in B.$$

If M is a graded A bimodule and N is a graded B bimodule, then $M \otimes N$ is naturally a graded $A \otimes B$ bimodule. Explicitly, for $a \in A, b \in B, m \in M$ and $n \in N$, we have

$$(a \otimes b)(m \otimes n) = (-1)^{(|b|)(|m|)} am \otimes bn, \qquad (m \otimes n)(a \otimes b) = (-1)^{(|a|)(|n|)} ma \otimes nb.$$

The bracket $[\cdot, \cdot]$ in a graded associative algebra is defined by

$$[a, b] = ab - (-1)^{|a||b|} ba.$$

If A is an associative algebra with a bracket $[x, y]$, then the centralizer of a subset S of A is defined to be

$$Z(S) := \{x \in A \mid [x, s] = 0 \text{ for all } s \in S\}.$$

For a graded vector space V, let $V[k]$ be a copy of V with the shifted grading $V[k]_i = V_{k+i}$.

Definition 2.1. Let M, N, be differential graded associative algebras over a field $\Bbbk$. A differential graded $\Bbbk$ linear map $F : M \to N$ is said to be **graded cyclic** if for all $s_1, \ldots s_n \in M$, we have

$$F(s_1 \cdots s_n) = (-1)^{|s_n| \sum_{i=1}^{n-1} |s_i|} F(s_n \cdot s_1 \cdots s_{n-1}).$$

Example 2.2. The identity morphism of a graded commutative algebra is a graded cyclic morphism. $M_n(R)$ denote the algebra of $n \times n$ matrices with entries in a graded commutative $\Bbbk$ algebra R. Then the trace map $\text{tr} : M_n(R) \to R$ is a graded cyclic morphism.

Definition 2.3. A **filtered ring** is a ring $\mathcal{N}$ with a map $\zeta_\mathcal{N} : \mathcal{N} \to \mathbb{R}_{\geq 0}$ such that

$$\zeta_\mathcal{N}(x) = \infty \iff x = 0, \qquad \zeta_\mathcal{N}(xy) \geq \zeta_\mathcal{N}(x) + \zeta_\mathcal{N}(y), \qquad \zeta_\mathcal{N}(x+y) \geq \min(\zeta_\mathcal{N}(x), \zeta_\mathcal{N}(y)).$$

We write $F_{\zeta_\mathcal{N}}^a \mathcal{N} = \{x \in \mathcal{N} | \zeta_\mathcal{N}(x) \geq a\}$ and $F_{\zeta_\mathcal{N}}^{>a} \mathcal{N} = \{x \in \mathcal{N} | \zeta_\mathcal{N}(x) > a\}$. For $E \in \mathbb{R}$ abbreviate by $\mathcal{N}_E$ the rings $F_{\zeta_\mathcal{N}}^E \mathcal{N} / F_{\zeta_\mathcal{N}}^{>E} \mathcal{N}$. The **residue ring** is $\overline{\mathcal{N}} = \mathcal{N}_0$.

A **filtered dga** over a field $\Bbbk$ is a dga $\mathcal{N}$ over $\Bbbk$ equipped with a filtration $\zeta_\mathcal{N}$ that makes $\mathcal{N}$ a filtered ring such that the differential d satisfies $\zeta_\mathcal{N}(dx) \geq \zeta_\mathcal{N}(x)$ for $x \in \mathcal{N}$ and each graded part $\mathcal{N}_m$ is a closed subspace of $\mathcal{N}$.

A **filtered module** over $\mathcal{N}$ is an $\mathcal{N}$-module $\mathcal{C}$ along with a map $\zeta_\mathcal{C} : \mathcal{C} \to \mathbb{R}_{\geq 0}$ such that

$$\zeta_\mathcal{C}(x) = \infty \iff x = 0, \qquad \zeta_\mathcal{C}(xy) \geq \zeta_\mathcal{N}(x) + \zeta_\mathcal{C}(y), \qquad \zeta_\mathcal{C}(x+y) \geq \min(\zeta_\mathcal{C}(x), \zeta_\mathcal{C}(y)).$$

We write $F^{\geq a}_{\zeta_{\mathcal{C}}} \mathcal{C} = \{x \in \mathcal{C} | \zeta_{\mathcal{C}}(x) \geq a\}$ and $F^{>a}_{\zeta_{\mathcal{C}}} \mathcal{C} = \{x \in \mathcal{C} | \zeta_{\mathcal{C}}(x) > a\}$. For $E \in \mathbb{R}$ abbreviate by $\mathcal{C}_E$ the $\overline{\mathcal{N}}$ modules $F^E_{\zeta_{\mathcal{C}}} \mathcal{C} / F^{>E}_{\zeta_{\mathcal{C}}} \mathcal{C}$. The **residue module** is the $\overline{\mathcal{N}}$-module $\overline{\mathcal{C}} = \mathcal{C}_0$. For $n \in \mathcal{N}$ we write $\bar{n} \in \overline{\mathcal{N}}$ for the residue. For $f : \mathcal{C} \to \mathcal{D}$ a homomorphism of $\mathcal{N}$-modules, we write $\bar{f} : \overline{\mathcal{C}} \to \overline{\mathcal{D}}$ for the induced homomorphism of $\overline{\mathcal{N}}$-modules.

Definition 2.4. Let $\mathcal{N}$ be a filtered commutative unital ring. An $\overline{\mathcal{N}}$ **structure** on $\mathcal{N}$ is an $\overline{\mathcal{N}}$ algebra structure on $\mathcal{N}$ such that the residue map $\mathcal{N} \to \overline{\mathcal{N}}$ is an augmentation. Equivalently, it is a homomorphism $\overline{\mathcal{N}} \to \mathcal{N}$ such that the composition $\overline{\mathcal{N}} \to \mathcal{N} \to \overline{\mathcal{N}}$ is the identity. Assume now that $\mathcal{N}$ is provided with an $\overline{\mathcal{N}}$ structure, and let $\mathcal{C}$ be a filtered $\mathcal{N}$ module. An $\overline{\mathcal{N}}$ **structure** on $\mathcal{C}$ is a filtered $\mathcal{N}$-module isomorphism $\mathcal{N} \otimes_{\overline{\mathcal{N}}} \overline{\mathcal{C}} \xrightarrow{\sim} \mathcal{C}$ that induces the identity homomorphism

$$\overline{\mathcal{C}} \simeq \overline{\mathcal{N} \otimes_{\overline{\mathcal{N}}} \overline{\mathcal{C}}} \to \overline{\mathcal{C}}.$$

Lemma 2.5. *Let $\mathcal{C}$ be a filtered $\mathcal{N}$ module with an $\overline{\mathcal{N}}$ structure given by $g : \mathcal{N} \otimes_{\overline{\mathcal{N}}} \overline{\mathcal{C}} \to \mathcal{C}$. For all $b \in \overline{\mathcal{C}}$ we have*

$$\overline{g(1 \otimes b)} = b.$$

Proof. By the definition of an $\overline{\mathcal{N}}$ structure, the composition

$$\overline{\mathcal{C}} \to \overline{\mathcal{N} \otimes_{\overline{\mathcal{N}}} \overline{\mathcal{C}}} \xrightarrow{\bar{g}} \overline{\mathcal{C}}$$
$$b \mapsto \bar{1} \otimes b \mapsto \bar{g}(\bar{1} \otimes b)$$

is the identity. Hence $b = \bar{g}(\bar{1} \otimes b) = \overline{g(1 \otimes b)}$. $\qquad\qquad\square$

Lemma 2.6. *Let $\mathcal{C}$ be a filtered $\mathcal{N}$ module with an $\overline{\mathcal{N}}$ structure given by $g : \mathcal{N} \otimes_{\overline{\mathcal{N}}} \overline{\mathcal{C}} \to \mathcal{C}$, and let f be its inverse. Suppose $\alpha \in \mathcal{C}$. Then*

$$f(\alpha) \equiv 1 \otimes \bar{\alpha} \pmod{F^{>0}\mathcal{C}}.$$

Proof. It is enough to show that $\overline{f(\alpha)} = \bar{1} \otimes \bar{\alpha}$. Indeed by Lemma 2.5

$$\bar{g}(\bar{1} \otimes \bar{\alpha}) = \overline{g(1 \otimes \bar{\alpha})} = \bar{\alpha}.$$

This implies

$$\bar{1} \otimes \bar{\alpha} = \bar{f}(\bar{g}(\bar{1} \otimes \bar{\alpha})) = \bar{f}(\bar{\alpha}).$$
$$\square$$

Let $\mathcal{N}$ be a filtered commutative ring with an $\overline{\mathcal{N}}$ structure given by $h : \overline{\mathcal{N}} \to \mathcal{N}$. Let $\mathcal{C}$ be a filtered $\mathcal{N}$ module with a $\overline{\mathcal{N}}$ structure given by $g : \mathcal{N} \otimes_{\overline{\mathcal{N}}} \overline{\mathcal{C}} \xrightarrow{\sim} \mathcal{C}$. Let $A \subset \mathcal{N}$ be a filtered unital subring such that $h(\overline{\mathcal{N}}) \subset A$. Let $h_A : \overline{\mathcal{N}} \to A$ denote the corestriction.

Lemma 2.7. *We have $\bar{A} = \overline{\mathcal{N}}$. It follows that $h_A : \bar{A} = \overline{\mathcal{N}} \to A$ is an $\bar{A}$ structure.*

Proof. Let $\pi : \mathcal{N} \to \overline{\mathcal{N}}$ be the residue map. As $h(\overline{\mathcal{N}}) \subset A$ it follows that

$$\overline{\mathcal{N}} = \pi \circ h(\overline{\mathcal{N}}) \subset \bar{A} \subset \overline{\mathcal{N}}.$$

So, $\bar{A} = \overline{\mathcal{N}}$. Let $\pi_A = \pi|_A : A \to \bar{A}$ denote the residue map. Then $\pi_A \circ h_A = \pi \circ h = \mathrm{Id}$. So, $h_A : \bar{A} \to A$ is an $\bar{A}$ structure. $\qquad\square$

Let

$$\mathcal{C}_A := g(A \otimes_{\bar{A}} \overline{\mathcal{C}}).$$

Lemma 2.8. *We have $\overline{\mathcal{C}_A} = \overline{\mathcal{C}}$.*

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Proof. Keeping in mind Lemma 2.7, since g preserves filtration, we have

$$\overline{\mathcal{C}_A} = \overline{g(A \otimes_{\bar{A}} \overline{\mathcal{C}})} = \bar{g}(\overline{A} \otimes_{\bar{A}} \overline{\mathcal{C}}) = \bar{g}(\bar{\mathcal{N}} \otimes_{\bar{\mathcal{N}}} \overline{\mathcal{C}}) = \overline{\mathcal{C}}.$$

$\square$

Let

$$g_A : A \otimes_{\bar{A}} \overline{\mathcal{C}_A} \to \mathcal{C}_A$$

be given by $g_A = g|_{A \otimes_{\bar{A}} \overline{\mathcal{C}}}$.

Lemma 2.9. g_A *is an* $\bar{A}$ *structure on* $\mathcal{C}_A$.

Proof. In Lemma 2.8 we showed $\overline{\mathcal{C}} = \overline{\mathcal{C}_A}$, and in Lemma 2.7 we have shown that $\bar{A} = \bar{\mathcal{N}}$. So, we have $\bar{g} = \bar{g}_A$. Hence the fact that g is an $\mathcal{N}$ structure on $\mathcal{C}$ implies that g_A induces the identity

$$
\begin{array}{ccccc}
\overline{\mathcal{C}} & \xrightarrow{\simeq} & \bar{\mathcal{N}} \otimes_{\bar{\mathcal{N}}} \overline{\mathcal{C}} & \xrightarrow{\bar{g}} & \overline{\mathcal{C}} \\
\| & & \| & & \| \\
\overline{\mathcal{C}_A} & \xrightarrow{\simeq} & \bar{A} \otimes_{\bar{A}} \overline{\mathcal{C}_A} & \xrightarrow{\bar{g}_A} & \overline{\mathcal{C}_A}
\end{array}
$$

We also notice that g_A preserves filtration, thus we conclude that g_A is a $\bar{A}$ structure on $\mathcal{C}_A$. $\square$

Definition 2.10. We call h_A the **canonical** $\bar{A}$ structure on A. We call $\mathcal{C}_A$ the **restriction of scalars** of $\mathcal{C}$ and we call g_A the **canonical** $\bar{A}$ structure on $\mathcal{C}_A$.

2.2. **Cyclic unital A_∞ algebras.** Let $\mathcal{R}$ be a filtered dga over a field $\Bbbk$ with filtration $\varsigma_{\mathcal{R}}$ and let $\mathcal{C}$ be a filtered $\mathcal{R}$ bimodule with filtration $\varsigma_{\mathcal{C}}$ such that if $\varsigma_{\mathcal{C}}(a) = \infty$ then $a = 0$. That is, $\mathcal{C}$ is a filtered module.

The following extends [36, Definition 1.1] for the case that $\mathcal{R}$ is not commutative.

Definition 2.11. An n-dimensional (curved) **cyclic unital A_∞ structure** over $\mathcal{R}$ on $\mathcal{C}$ is a triple $(\{\mathfrak{m}_k\}_{k \geq 0}, \prec, \succ, \mathbf{e})$ of maps $\mathfrak{m}_k : \mathcal{C}^{\otimes k} \to \mathcal{C}[2-k]$, a pairing $\prec, \succ : \mathcal{C} \otimes \mathcal{C} \to \mathcal{R}[-n]$, and an element $\mathbf{e} \in \mathcal{C}$ with $|\mathbf{e}| = 0$, satisfying the following properties. We denote by α, possibly with subscripts, an element of $\mathcal{C}$, and by a an element of $\mathcal{R}$.

(1) The operations $\mathfrak{m}_k$ are $\mathcal{R}$-multilinear in the sense that

$$\mathfrak{m}_k(a \cdot \alpha_1, \dots, \alpha_k) = (-1)^{|a|} a \cdot \mathfrak{m}_k(\alpha_1, \dots, \alpha_k) + \delta_{1,k} \cdot da \cdot \alpha_1,$$

and

$$\mathfrak{m}_k(\alpha_1, \dots, \alpha_k \cdot a) = \mathfrak{m}_k(\alpha_1, \dots, \alpha_k) \cdot a + (-1)^{|\alpha_1|} \delta_{1,k} \cdot \alpha_1 \cdot da,$$

and

$$\mathfrak{m}_k(\alpha_1, \dots, \alpha_{i-1}, a \cdot \alpha_i, \dots, \alpha_k) = \mathfrak{m}_k(\alpha_1, \dots, \alpha_{i-1} \cdot a, \alpha_i, \dots, \alpha_k).$$

(2) The A_∞ relations hold:

$$\sum_{\substack{k_1+k_2=k+1 \\ 1 \leq i \leq k_1}} (-1)^{\sum_{j=1}^{i-1}(|\alpha_j|+1)} \mathfrak{m}_{k_1}(\alpha_1, \dots, \alpha_{i-1}, \mathfrak{m}_{k_2}(\alpha_i, \dots, \alpha_{i+k_2-1}), \alpha_{i+k_2}, \dots, \alpha_k) = 0.$$

(3) $\varsigma_{\mathcal{C}}(\mathfrak{m}_k(\alpha)) \geq \varsigma_{\mathcal{C}}(\alpha)$ and $\varsigma_{\mathcal{C}}(\mathfrak{m}_0) > 0$.

(4) $\mathfrak{m}_k(\alpha_1, \dots, \alpha_{i-1}, \mathbf{e}, \alpha_{i+1}, \dots, \alpha_k) = 0 \quad \forall k \neq 0, 2.$

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(5) $\mathfrak{m}_2(\mathbf{e}, \alpha) = \alpha = (-1)^{|\alpha|} \mathfrak{m}_2(\alpha, \mathbf{e})$.

(6) The pairing $\prec,\succ$ is $\mathcal{R}$-bilinear in the sense that

$$\prec \alpha_1, a \cdot \alpha_2 \succ = \prec \alpha_1 \cdot a, \alpha_2 \succ, \qquad \forall a \in \mathcal{R},$$

and

$$\prec b \cdot \alpha_1, \alpha_2 \succ = b \cdot \prec \alpha_1, \alpha_2 \succ, \qquad \forall b \in Z(\mathcal{R}).$$

(7) $\varsigma_{\mathcal{R}}(\prec \alpha_1, \alpha_2 \succ) \geq \varsigma_{\mathcal{C}}(\alpha_1) + \varsigma_{\mathcal{C}}(\alpha_2)$.

(8) $\prec \alpha_1, \alpha_2 \succ = (-1)^{(|\alpha_1|+1)(|\alpha_2|+1)+1} \prec \alpha_2, \alpha_1 \succ$.

(9) The pairing is cyclic:

$$\prec \mathfrak{m}_k(\alpha_1, \ldots, \alpha_k), \alpha_{k+1} \succ =$$

$$= (-1)^{(|\alpha_{k+1}|+1)\sum_{j=1}^{k}(|\alpha_j|+1)} \prec \mathfrak{m}_k(\alpha_{k+1}, \alpha_1, \ldots, \alpha_{k-1}), \alpha_k \succ + \delta_{1,k} \cdot d \prec \alpha_1, \alpha_2 \succ.$$

(10) $\prec \mathfrak{m}_0, \mathbf{e} \succ = 0$.

Definition 2.12. A cyclic unital A_∞ algebra $(\mathcal{C}, \{\mathfrak{m}_k\}_{k \geq 0}, \prec, \succ, \mathbf{e})$ **arises from a dga** if

$$\mathfrak{m}_k = 0, \quad k \neq 1, 2.$$

In this case the **underlying dga** structure on $\mathcal{C}$ is given by

$$d\alpha_1 = \mathfrak{m}_1(\alpha_1), \qquad \alpha_1 \alpha_2 = (-1)^{|\alpha_1|} \mathfrak{m}_2(\alpha_1, \alpha_2), \qquad \alpha_1, \alpha_2 \in \mathcal{C}.$$

Let Υ denote the algebra obtained by equipping $\mathcal{C}$ with the product above. We say that $(\mathcal{C}, \{\mathfrak{m}_k\}_{k \geq 0}, \prec, \succ, \mathbf{e})$ arises from the dga (Υ, d).

Recall the notion of the residue module from Definition 2.3. By property (3), the maps $\mathfrak{m}_k$ descend to maps on the residue module,

$$\bar{\mathfrak{m}}_k : \overline{\mathcal{C}}^{\otimes k} \longrightarrow \overline{\mathcal{C}}.$$

For $\bar{\alpha}_1, \ldots, \bar{\alpha}_k \in \overline{\mathcal{C}}$, we have

$$\bar{\mathfrak{m}}_k(\bar{\alpha}_1, \ldots, \bar{\alpha}_k) = \overline{\mathfrak{m}_k(\alpha_1, \ldots, \alpha_k)},$$

where $\alpha_i \in \mathcal{C}$ is a representative for $\bar{\alpha}_i$. Similarly, for $k \geq 1$ we define operators

$$\mathfrak{m}_{k,E} : \left(\mathcal{C}^{\otimes k}\right)_E \longrightarrow \mathcal{C}_E.$$

For $[\alpha_1 \otimes \ldots \otimes \alpha_k] \in \left(\mathcal{C}^{\otimes k}\right)_E$, we define

$$\mathfrak{m}_{k,E}([\alpha_1 \otimes \ldots \otimes \alpha_k]) = [\mathfrak{m}_k(\alpha_1 \otimes \ldots \otimes \alpha_k)],$$

where $\alpha_1 \otimes \ldots \otimes \alpha_k \in F^E \mathcal{C}^{\otimes k}$ is a representative for $[\alpha_1 \otimes \ldots \otimes \alpha_k]$. If $\mathfrak{m}_0 \in F^E \mathcal{C}$, then we define $\mathfrak{m}_{0,E}$ to be its image in $\mathcal{C}_E$. We sometime extend the domain of the operators to $F^E \mathcal{C}^{\otimes k}$ by precomposing them with the quotient map.

Proposition 2.13. *Let $(\mathcal{C}, \{\mathfrak{m}_k\}_{k \geq 0}, \prec, \succ, \mathbf{e})$ be a cyclic unital A_∞ algebra over a commutative dga $\mathcal{R}$. Assume that $\mathcal{C}$ has an $\bar{\mathcal{R}}$ structure $g : \mathcal{R} \otimes \bar{\mathcal{C}} \xrightarrow{\sim} \mathcal{C}$ and let f be its inverse. Assume also that $(\bar{\mathcal{C}}, \{\bar{\mathfrak{m}}_k\}_{k \geq 0}, \prec, \succ, \mathbf{e})$ arises from a dga. So, $\mathcal{R} \otimes \bar{\mathcal{C}}$ inherits a dga structure as well. Let $\alpha_1, \ldots, \alpha_k \in \mathcal{C}$ be homogeneous elements such that $\sum_{i=1}^{k} \varsigma_{\mathcal{C}}(\alpha_i) = E$. For $k = 1$ we have*

$$f(\mathfrak{m}_1(\alpha_1)) \equiv df(\alpha_1) \pmod{F^{>E} \mathcal{C}}.$$

If in addition the left and right $\mathcal{R}$ module structures on $\mathcal{C}[1]$ coincide, for $k = 2$ we have

$$f\left(\mathfrak{m}_2(\alpha_1, \alpha_2)\right) \equiv (-1)^{|\alpha_1|} f(\alpha_1) f(\alpha_2) \quad (\mathrm{mod}\ F^{>E}\mathcal{C}).$$

If $k > 2$, then

$$\mathfrak{m}_k(\alpha_1, \ldots, \alpha_k) \equiv 0 \quad (\mathrm{mod}\ F^{>E}\mathcal{C}).$$

Proof. Assume with out loss of generality there exists $a_j \in \mathcal{R}$ and $b_j \in \bar{\mathcal{C}}$, such that $g(a_j \otimes b_j) = \alpha_j$ for $1 \le j \le k$. For $k = 1$, we have $\varsigma_{\mathcal{R}}(a_1) = E$. We calculate

$$df(\alpha_1) = d(a_1 \otimes b_1)$$
$$= (-1)^{|a_1|} a_1 \otimes db_1 + da_1 \otimes b_1$$
$$= (-1)^{|a_1|} a_1 \otimes \bar{m}_1(b_1) + da_1 \otimes b_1$$
$$= (-1)^{|a_1|} a_1 (1 \otimes \bar{\mathfrak{m}}_1(b_1)) + da_1 \otimes b_1.$$

Using Lemma 2.5, we proceed

$$= (-1)^{|a_1|} a_1 \left(1 \otimes \overline{\mathfrak{m}_1(g(1 \otimes b_1))}\right) + da_1 \otimes b_1. \tag{8}$$

Using Lemma 2.6, we deduce that

$$1 \otimes \overline{\mathfrak{m}_1(g(1 \otimes b_1))} \equiv f(\mathfrak{m}_1(g(1 \otimes b_1))) \quad (\mathrm{mod}\ F^{>0}\mathcal{C}).$$

Remembering that $\mathcal{C}$ is a filtered module, we combine the preceding equation with equation (8) to deduce

$$df(\alpha_1) \equiv (-1)^{|a_1|} a_1 f\left(\mathfrak{m}_1(g(1 \otimes b_1))\right) + da_1 \otimes b_1 \equiv$$
$$\equiv f\left(\mathfrak{m}_1(g(a_1 \otimes b_1))\right) - f\left(da_1 \cdot g(1 \otimes b_1)\right) + da_1 \otimes b_1 \equiv f(\mathfrak{m}_1(\alpha_1)) \quad (\mathrm{mod}\ F^{>E}\mathcal{C}).$$

For $k = 2$ we have

$$f(\mathfrak{m}_2(\alpha_1, \alpha_2)) = (-1)^{|a_1|} a_1 f\left(\mathfrak{m}_2(g(1 \otimes b_1) a_2, g(1 \otimes b_2))\right). \tag{9}$$

Remembering that the left and right $\mathcal{R}$ module structures on $\mathcal{C}[1]$ coincide, we continue

$$= (-1)^{|a_1| + |a_2||b_1|} a_1 a_2 f\left(\mathfrak{m}_2(g(1 \otimes b_1), g(1 \otimes b_2))\right). \tag{10}$$

We use Lemma 2.6 and Lemma 2.5 to deduce the equivalence

$$f\left(\mathfrak{m}_2(g(1 \otimes b_1), g(1 \otimes b_2))\right) \equiv 1 \otimes \overline{\mathfrak{m}_2(g(1 \otimes b_1), g(1 \otimes b_2))} \equiv$$
$$\equiv 1 \otimes \bar{\mathfrak{m}}_2(\overline{g(1 \otimes b_1)}, \overline{g(1 \otimes b_2)}) \equiv 1 \otimes \bar{\mathfrak{m}}_2(b_1, b_2) \quad (\mathrm{mod}\ F^{>0}\mathcal{C}).$$

By the definition of the dga on $\bar{\mathcal{C}}$ and the property $\varsigma_{\mathcal{R}}(a_1 a_2) \ge \varsigma_{\mathcal{R}}(a_1) + \varsigma_{\mathcal{R}}(a_2)$, we conclude

$$f(\mathfrak{m}_2(\alpha_1, \alpha_2)) \equiv (-1)^{|a_1| + |a_2||b_1|} (a_1 a_2) \left(1 \otimes \bar{\mathfrak{m}}_2(b_1, b_2)\right) \equiv$$
$$\equiv (-1)^{|a_1| + |b_1| + |a_2||b_1|} (a_1 a_2) \otimes (b_1 b_2) \equiv (-1)^{|\alpha_1|} f(\alpha_1) f(\alpha_2) \quad (\mathrm{mod}\ F^{>E}\mathcal{C}).$$

We omit the proof of the cases $k > 2$ as it is similar. $\square$

The following is a generalization of [36, Proposition 4.20], which formulates the A_∞ relations with respect to the cyclic pairing.

Proposition 2.14 (Cyclic structure equations). *Let $(\mathcal{C}, \{\mathfrak{m}_k\}_{k\geq 0}, \prec, \succ, \mathbf{e})$ be a cyclic unital A_∞ algebra, then for $k \geq 0$,*

$$d\prec\mathfrak{m}_k(\alpha_1, \ldots, \alpha_k), \alpha_{k+1}\succ =$$

$$= \sum_{\substack{k_1+k_2=k+1 \\ k_1\geq 1, k_2\geq 0 \\ 1\leq i\leq k_1}} (-1)^{\nu(\alpha;k_1,k_2,i)} \prec\mathfrak{m}_{k_1}(\alpha_{i+k_2}, \ldots, \alpha_{k+1}, \alpha_1, \ldots, \alpha_{i-1}), \mathfrak{m}_{k_2}(\alpha_i, \ldots, \alpha_{k_2+i-1})\succ$$

with

$$\nu(\alpha; k_1, k_2, i) := \sum_{j=1}^{i-1}(|\alpha_j| + 1) + \sum_{j=i+k_2}^{k+1}(|\alpha_j| + 1)\left(\sum_{\substack{m\neq j \\ 1\leq m\leq k+1}}(|\alpha_m| + 1) + 1 \right) + 1.$$

Proof. As our pairing $\prec\cdot, \cdot\succ$ is cyclic, the following equation holds for $\xi, \eta \in \mathcal{C}$:

$$\prec\mathfrak{m}_1(\xi), \eta\succ = (-1)^{(|\eta|+1)(|\xi|+1)}\prec\mathfrak{m}_1(\eta), \xi\succ + d\prec\xi, \eta\succ.$$

We can plug in $\mathfrak{m}_k(\alpha_1, \ldots \alpha_k)$ and α_{k+1} instead of ξ and η in the equation above to get

$$d\prec\mathfrak{m}_k(\alpha_1, \ldots, \alpha_k), \alpha_{k+1}\succ =$$

$$= \prec\mathfrak{m}_1\left(\mathfrak{m}_k(\alpha_1, \ldots, \alpha_k)\right), \alpha_{k+1}\succ - (-1)^{(|\alpha_{k+1}|+1)(|\mathfrak{m}_k(\alpha_1, \ldots, \alpha_k)|+1)}\prec\mathfrak{m}_1(\alpha_{k+1}), \mathfrak{m}_k(\alpha_1, \ldots, \alpha_k)\succ.$$

Using the A_∞ structure and cyclic symmetry we continue,

$$d\prec\mathfrak{m}_k(\alpha_1, \ldots, \alpha_k), \alpha_{k+1}\succ =$$

$$= -\sum_{\substack{k_1+k_2=k+1 \\ 1<k_1 \\ 1\leq i\leq k_1}}(-1)^{\sum_{j=1}^{i-1}(|\alpha_j|+1)}\prec\mathfrak{m}_{k_1}(\alpha_1, \ldots, \alpha_{i-1}, \mathfrak{m}_{k_2}(\alpha_i, \ldots, \alpha_{i+k_2-1}), \alpha_{i+k_2}, \ldots, \alpha_k), \alpha_{k+1}\succ +$$

$$+ (-1)^{(|\alpha_{k+1}|+1)(|\mathfrak{m}_k(\alpha_1, \ldots, \alpha_k)|+1)+1}\prec\mathfrak{m}_1(\alpha_{k+1}), \mathfrak{m}_k(\alpha_1, \ldots, \alpha_k)\succ$$

$$= \sum_{\substack{k_1+k_2=k+1 \\ 1<k_1 \\ 1\leq i\leq k_1}}(-1)^{1+\sum_{j=1}^{i-1}(|\alpha_j|+1)+\nu}\prec\mathfrak{m}_{k_1}(\alpha_{i+k_2}, \ldots, \alpha_k, \alpha_{k+1}, \alpha_1, \ldots, \alpha_{i-1}), \mathfrak{m}_{k_2}(\alpha_i, \ldots, \alpha_{i+k_2-1})\succ +$$

$$+ (-1)^{(|\alpha_{k+1}|+1)(\sum_{j=1}^{k}(|\alpha_j|+1)+1)+1}\prec\mathfrak{m}_1(\alpha_{k+1}), \mathfrak{m}_k(\alpha_1, \ldots, \alpha_k)\succ,$$

with the sign ν as follows:

$$\nu = \sum_{j=i+k_2}^{k+1}(|\alpha_j| + 1)\left(\sum_{\substack{m\neq j \\ 1\leq m\leq k+1 \\ m\notin\{i,\ldots,k_2+i-1\}}}(|\alpha_m| + 1) + (|\hat{\mathfrak{m}}(\alpha_i, \ldots, \alpha_{i+k_2-1})| + 1) \right)$$

$$\equiv \sum_{j=i+k_2}^{k+1}(|\alpha_j| + 1)\left(\sum_{\substack{m\neq j \\ 1\leq m\leq k+1}}(|\alpha_m| + 1) + 1 \right) \pmod{2}.$$

The result now follows since

$$\nu(\alpha; k_1, k_2, i) = \nu(\alpha; k_1, k_2, i) + 1 + \sum_{j=1}^{i-1}(|\alpha_j| + 1), \qquad \text{for } k_1 > 1,$$

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and

$$\nu(\alpha; k_1, k, k_1) = (|\alpha_{k+1}| + 1) \left(\sum_{j=1}^{k} (|\alpha_j| + 1) + 1 \right) + 1, \qquad \text{for} \quad k_1 = 1.$$

$\square$

2.3. **Extension of scalars.** Let $\mathcal{R}$ be a differential graded algebra over a field $\Bbbk$ with filtration $\varsigma_\mathcal{R}$ and let $\mathcal{C}$ be an $\mathcal{R}$ bimodule with filtration $\varsigma_\mathcal{C}$. Let $(\mathcal{C}, \{\mathfrak{m}_k\}_{k\geq 0}, \prec, \succ, \mathbf{e})$ be an n-dimensional curved cyclic unital A_∞ algebra, and let (S, e_S) be a unital associative differential graded algebra over $\Bbbk$, where $e_S \in S$ is the unit. Let $F : S \to S$ be a graded cyclic $\Bbbk$ linear map, that is also a $Z(S)$ module homomorphism.

Define $\hat{\mathcal{R}} := S \otimes \mathcal{R}$, which is naturally a differential graded algebra, and define $\hat{\mathcal{C}} := S \otimes \mathcal{C}$, which is naturally a graded bimodule over $\hat{\mathcal{R}}$. We use the grading of $\mathcal{C}[1]$ to determine the bimodule structure. So, for $t, s \in S$, $\alpha \in \mathcal{C}$ and $b \in \mathcal{R}$ we have

$$(t \otimes b)(s \otimes \alpha) = (-1)^{|b||s|} ts \otimes b\alpha, \qquad (s \otimes \alpha)(t \otimes b) = (-1)^{t(|\alpha|+1)} st \otimes \alpha b.$$

Equip $\hat{\mathcal{R}}$ with the filtration $\varsigma_{\hat{\mathcal{R}}}$ induced by $\varsigma_\mathcal{R}$ and the trivial filtration on S, and equip $\hat{\mathcal{C}}$ with the filtration $\varsigma_{\hat{\mathcal{C}}}$ induced by $\varsigma_\mathcal{C}$ and the trivial filtration on S. Define maps $\hat{\mathfrak{m}}_k : \hat{\mathcal{C}}^{\otimes k} \to \hat{\mathcal{C}}[2-k]$ by

$$\hat{\mathfrak{m}}_k (s_1 \otimes \alpha_1, \dots, s_k \otimes \alpha_k) =$$
$$= (-1)^{\sum_{j=1}^{k} |s_j| + \sum_{j=1}^{k} |s_j| \sum_{i=1}^{j-1} (|\alpha_i|+1)} s_1 \cdots s_k \otimes \mathfrak{m}_k (\alpha_1, \dots, \alpha_k) + \delta_{1,k} \cdot ds_1 \otimes \alpha_1,$$

and

$$\hat{\mathfrak{m}}_0 = e_S \otimes \mathfrak{m}_0.$$

Define a pairing $\prec, \succ_F : \hat{\mathcal{C}} \otimes \hat{\mathcal{C}} \to \hat{\mathcal{R}}[-n]$ by

$$\prec s_1 \otimes \alpha_1, s_2 \otimes \alpha_2 \succ_F = (-1)^{|s_2|(|\alpha_1|+1)} F(s_1 s_2) \otimes \prec \alpha_1, \alpha_2 \succ.$$

Proposition 2.15. *The triple* $(\{\hat{\mathfrak{m}}_k\}_{k\geq 0}, \prec, \succ_F, \mathbf{e}_s \otimes \mathbf{e})$ *is a n-dimensional cyclic unital A_∞ structure over $\hat{\mathcal{R}}$ on $\hat{\mathcal{C}}$.*

Definition 2.16. The cyclic unital A_∞ algebra $(\hat{\mathcal{C}}, \{\hat{\mathfrak{m}}_k\}_{k\geq 0}, \prec, \succ_F, \mathbf{e}_S \otimes \mathbf{e})$ is called the (S, F) **extension of scalars** of the cyclic unital A_∞ algebra $(\mathcal{C}, \{\mathfrak{m}_k\}_{k\geq 0}, \prec, \succ, \mathbf{e})$.

The proof of Proposition 2.15 is given in the end of this section based on the series of lemmas stated below.

Lemma 2.17. *The operations $\hat{\mathfrak{m}}_k$ are $\hat{\mathcal{R}}$-multilinear in the sense that for $a \in \hat{\mathcal{R}}$,*

$$\hat{\mathfrak{m}}_k(a \cdot \hat{\alpha}_1, \dots, \hat{\alpha}_k) = (-1)^{|a|} a \cdot \mathfrak{m}_k(\hat{\alpha}_1, \dots, \hat{\alpha}_k) + \delta_{1,k} \cdot da \cdot \hat{\alpha}_1,$$
$$\hat{\mathfrak{m}}_k(\hat{\alpha}_1, \dots, \hat{\alpha}_k \cdot a) = \mathfrak{m}_k(\hat{\alpha}_1, \dots, \hat{\alpha}_k) \cdot a + (-1)^{|\hat{\alpha}_1|} \delta_{1,k} \cdot \hat{\alpha}_1 \cdot da,$$

and

$$\hat{\mathfrak{m}}_k(\hat{\alpha}_1, \dots, \hat{\alpha}_{i-1}, a \cdot \hat{\alpha}_i, \dots, \hat{\alpha}_k) = \hat{\mathfrak{m}}_k(\hat{\alpha}_1, \dots, \hat{\alpha}_{i-1} \cdot a, \hat{\alpha}_i, \dots, \hat{\alpha}_k).$$

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Proof. We will prove the first and second equations equation only for $k = 1$, as the other cases are similar but easier. Let $t, s \in S, \alpha \in \mathcal{C}$ and $b \in \mathcal{R}$ be homogeneous elements. We compute,

$$\hat{\mathfrak{m}}_1\left((t \otimes b) \cdot (s \otimes \alpha)\right) =$$

$$= (-1)^{|b||s|}\hat{\mathfrak{m}}_1\left(ts \otimes b \cdot \alpha\right)$$

$$= (-1)^{|b||s|+|t|+|s|}ts \otimes \mathfrak{m}_1(b \cdot \alpha) + (-1)^{|b||s|}d(ts) \otimes b \cdot \alpha$$

$$= (-1)^{|b||s|+|t|+|s|}ts \otimes \left((-1)^{|b|}b \cdot \mathfrak{m}_1(\alpha) + db \cdot \alpha\right) + (-1)^{|b||s|}d(ts) \otimes b \cdot \alpha$$

$$= (-1)^{|b||s|+|t|+|s|}ts \otimes \left((-1)^{|b|}b \cdot \mathfrak{m}_1(\alpha) + db \cdot \alpha\right) +$$

$$+ (-1)^{|b||s|}dt \cdot s \otimes b \cdot \alpha + (-1)^{|b||s|+|t|}tds \otimes b \cdot \alpha$$

$$= (-1)^{|t|+|s|+|b|}(t \otimes b)(s \otimes \mathfrak{m}_1(\alpha)) + (-1)^{|t|}(t \otimes db) \cdot (s \otimes \alpha)+$$

$$+ (dt \otimes b) \cdot (s \otimes \alpha) + (-1)^{|b|+|t|}(t \otimes b) \cdot (ds \otimes \alpha)$$

$$= (-1)^{|t \otimes b|}(t \otimes b) \cdot \hat{\mathfrak{m}}_1(s \otimes \alpha) + d(t \otimes b) \cdot (s \otimes \alpha).$$

We also show the second equality by direct calculations:

$$\hat{\mathfrak{m}}_1\left((s \otimes \alpha) \cdot (t \otimes b)\right) = (-1)^{|t||\alpha|}\hat{\mathfrak{m}}_1\left(st \otimes \alpha \cdot b\right)$$

$$= (-1)^{|t||\alpha|+|t|+|s|}st \otimes \mathfrak{m}_1(\alpha \cdot b) + (-1)^{|t||\alpha|}d(st) \otimes \alpha \cdot b$$

$$= (-1)^{|t||\alpha|+|t|+|s|}st \otimes \left(\mathfrak{m}_1(\alpha) \cdot b + (-1)^{|\alpha|}\alpha \cdot db\right) +$$

$$+ \left((-1)^{|\alpha||t|+|s|}sdt + (-1)^{|\alpha||t|}(ds)t\right) \otimes \alpha \cdot b$$

$$= (-1)^{|s|}(s \otimes \mathfrak{m}_1(\alpha) + ds \otimes \alpha) \cdot (t \otimes b) + (-1)^{|\alpha|+|s|}s \otimes \alpha \cdot d(t \otimes b)$$

$$= \hat{\mathfrak{m}}_1(s \otimes \alpha) \cdot (t \otimes b) + (-1)^{|s|+|\alpha|}s \otimes \alpha \cdot (d(t \otimes b)).$$

We will prove the third equality stated. Let $t, s_1, \ldots, s_k \in S, \alpha_1, \ldots, \alpha_k \in \mathcal{C}$ and $b \in \mathcal{R}$ be homogeneous elements. We compute

$$\hat{\mathfrak{m}}_k(s_1 \otimes \alpha_1, \ldots, t \otimes b \cdot s_i \otimes \alpha_i, \ldots, s_k \otimes \alpha_k) =$$

$$= (-1)^{|b||s_i|}\hat{\mathfrak{m}}_k(s_1 \otimes \alpha_1, \ldots, ts_i \otimes b \cdot \alpha_i, \ldots, s_k \otimes \alpha_k)$$

$$= (-1)^{|b||s_i|+|t|+\sum_{j=1}^k |s_k|+\sum_{j=1}^k |s_j| \sum_{i=1}^{j-1}(|\alpha_i|+1)+|t|\sum_{s=1}^{i-1}(|\alpha_s|+1)}s_1 \cdots s_{i-1}ts_i \cdots s_k \otimes$$

$$\otimes \mathfrak{m}_k(\alpha_1, \ldots, \alpha_{i-1} \cdot b, \alpha_i, \ldots, \alpha_k)$$

$$= (-1)^{|t|(|\alpha_{i-1}|+1)}\hat{\mathfrak{m}}_k(s_1 \otimes \alpha_1, \ldots, s_{i-1}t \otimes \alpha_{i-1}b, \ldots, s_k \otimes \alpha_k)$$

$$= \hat{\mathfrak{m}}_k(s_1 \otimes \alpha_1, \ldots, (s_{i-1} \otimes \alpha_{i-1}) \cdot (t \otimes b), \ldots, s_k \otimes \alpha_k).$$

$\square$

Lemma 2.18. *The operations $\{\hat{\mathfrak{m}}_k^\gamma\}_{k \geq 0}$ define an A_∞ structure on $\hat{C}$. That is,*

$$\sum_{\substack{k_1+k_2=k+1 \\ 1 \leq i \leq k_1}} (-1)^{\sum_{j=1}^{i-1}(|\alpha_j|+1)}\hat{\mathfrak{m}}_{k_1}(\hat{\alpha}_1, \ldots, \hat{\alpha}_{i-1}, \hat{\mathfrak{m}}_{k_2}(\hat{\alpha}_i, \ldots, \hat{\alpha}_{i+k_2-1}), \hat{\alpha}_{i+k_2}, \ldots, \hat{\alpha}_k) = 0.$$

Proof. Let $s_1, \ldots, s_k \in S$, $\alpha_1, \ldots, \alpha_k \in \mathcal{C}$, be homogeneous elements. For $k_1 = k$ and $k_2 = 1$, we compute

$$\hat{\mathfrak{m}}_k(s_1 \otimes \alpha_1, \ldots, s_{i-1} \otimes \alpha_{i-1}, \hat{\mathfrak{m}}_1(s_i \otimes \alpha_i), \ldots, s_k \otimes \alpha_k) =$$
$$= \hat{\mathfrak{m}}_k(s_1 \otimes \alpha_1, \ldots, s_{i-1} \otimes \alpha_{i-1}, (-1)^{|s_i|} s_i \otimes \mathfrak{m}_1(\alpha_i) + ds_i \otimes \alpha_i, \ldots, s_k \otimes \alpha_k)$$
$$= (-1)^{\sum_{j=1}^{i-1} |s_j| + \sum_{j=1}^k |s_j| \sum_{s=1}^{j-1}(|\alpha_s|+1)} s_1 \cdots s_k \otimes \mathfrak{m}_k(\alpha_1, \ldots, \mathfrak{m}_1(\alpha_i), \ldots, \alpha_k) +$$
$$+ (-1)^{1 + \sum_{j=1}^k |s_j| + \sum_{j=1}^k |s_j| \sum_{s=1}^{j-1}(|\alpha_s|+1) + \sum_{j=1}^{i-1}(|\alpha_j|+1)} s_1 \cdots ds_i \cdots s_k \otimes \mathfrak{m}_k(\alpha_1, \ldots, \alpha_k). \quad (11)$$

We will now compute for $k_1 = 1$ and $k_2 = k$,

$$\hat{\mathfrak{m}}_1\left(\hat{\mathfrak{m}}_k(s_1 \otimes \alpha_1, \ldots s_k \otimes \alpha_k)\right) =$$
$$= (-1)^{\sum_{j=1}^k |s_j| + \sum_{j=1}^k |s_j| \sum_{i=1}^{j-1}(|\alpha_i|+1)} \hat{\mathfrak{m}}_1(s_1 \cdots s_k \otimes \mathfrak{m}_k(\alpha_1, \ldots, \alpha_k)) =$$
$$= (-1)^{\sum_{j=1}^k |s_j| \sum_{i=1}^{j-1}(|\alpha_i|+1)} s_1 \cdots s_k \otimes \mathfrak{m}_1(\mathfrak{m}_k(\alpha_1, \ldots, \alpha_k)) +$$
$$+ (-1)^{\sum_{j=1}^k |s_j| + \sum_{j=1}^k |s_j| \sum_{i=1}^{j-1}(|\alpha_i|+1)} d(s_1 \cdots s_k) \otimes \mathfrak{m}_k(\alpha_1, \ldots, \alpha_k)$$
$$= (-1)^{\sum_{j=1}^k |s_j| \sum_{i=1}^{j-1}(|\alpha_i|+1)} s_1 \cdots s_k \otimes \mathfrak{m}_1(\mathfrak{m}_k(\alpha_1, \ldots, \alpha_k)) +$$
$$+ (-1)^{\sum_{j=1}^k |s_j| + \sum_{j=1}^k |s_j| \sum_{i=1}^{j-1}(|\alpha_i|+1)} \sum_{i=1}^k (-1)^{\sum_{j=1}^{i-1} |s_j|} s_1 \cdots ds_i \cdots s_k \otimes \mathfrak{m}_k(\alpha_1, \ldots, \alpha_k) =$$
$$= (-1)^{\sum_{j=1}^k |s_j| \sum_{i=1}^{j-1}(|\alpha_i|+1)} s_1 \cdots s_k \otimes \mathfrak{m}_1(\mathfrak{m}_k(\alpha_1, \ldots, \alpha_k)) +$$
$$+ \sum_{i=1}^k (-1)^{\sum_{j=i}^k |s_j| + \sum_{j=1}^k |s_j| \sum_{i=1}^{j-1}(|\alpha_i|+1)} s_1 \cdots ds_i \cdots s_k \otimes \mathfrak{m}_k(\alpha_1, \ldots, \alpha_k). \quad (12)$$

Adding up (11) and (12) with the sign coming from the A_∞ relations, we obtain

$$\sum_{i=1}^k (-1)^{\sum_{j=1}^{i-1}(|s_j \otimes \alpha_j|+1)} \hat{\mathfrak{m}}_k(s_1 \otimes \alpha_1, \ldots, s_{i-1} \otimes \alpha_{i-1}, \hat{\mathfrak{m}}_1(s_i \otimes \alpha_i), \ldots, s_k \otimes \alpha_k) +$$
$$+ \hat{\mathfrak{m}}_1\left(\hat{\mathfrak{m}}_k(s_1 \otimes \alpha_1, \ldots s_k \otimes \alpha_k)\right) =$$
$$= (-1)^{\sum_{j=1}^k |s_j| \sum_{i=1}^{j-1}(|\alpha_i|+1)} s_1 \cdots s_k \otimes \mathfrak{m}_1(\mathfrak{m}_k(\alpha_1, \ldots, \alpha_k)) +$$
$$+ \sum_{i=1}^k (-1)^{\sum_{j=1}^{i-1}(|\alpha_j|+1) + \sum_{j=1}^k |s_j| \sum_{s=1}^{j-1}(|\alpha_s|+1)} s_1 \cdots s_k \otimes \mathfrak{m}_k(\alpha_1, \ldots, \mathfrak{m}_1(\alpha_i), \ldots, \alpha_k). \quad (13)$$

For $s_1, \ldots, s_k \in S$, $\alpha_1, \ldots, \alpha_k \in \mathcal{C}$, homogeneous elements, and $k_1, k_2 \neq 1$, compute

$$\hat{\mathfrak{m}}_{k_1}(s_1 \otimes \alpha_1, \ldots, s_{i-1} \otimes \alpha_{i-1}, \hat{\mathfrak{m}}_{k_2}(s_i \otimes \alpha_i, \ldots, s_{i+k_2-1} \otimes \alpha_{i+k_2-1}), \ldots, s_k \otimes \alpha_k) =$$
$$= \hat{\mathfrak{m}}_{k_1}\left(s_1 \otimes \alpha_1, \ldots, s_{i-1} \otimes \alpha_{i-1}, (-1)^* \prod_{j=i}^{i+k_2-1} s_j \otimes \mathfrak{m}_{k_2}(\alpha_i, \ldots, \alpha_{i+k_2-1}), \ldots, s_k \otimes \alpha_k\right)$$
$$= (-1)^*(-1)^{**} \prod_{j=1}^k s_j \otimes \mathfrak{m}_{k_1}(\alpha_1, \ldots, \alpha_{i-1}, \mathfrak{m}_{k_2}(\alpha_i, \ldots, \alpha_{i+k_2-1}), \alpha_{i+k_2}, \ldots, \alpha_k), \quad (14)$$

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with

$$* = \sum_{j=i}^{i+k_2-1} |s_j| + \sum_{j=i}^{i+k_2-1} |s_j| \sum_{s=i}^{j-1} (|\alpha_s| + 1),$$

$$** = \sum_{j=1}^{k} |s_j| + \sum_{j=1}^{i-1} |s_j| \sum_{s=1}^{j-1} (|\alpha_i| + 1) + \left(\sum_{j=i}^{i+k_2-1} |s_j| \right) \sum_{s=1}^{i-1} (|\alpha_i| + 1) +$$

$$+ \sum_{s=i+k_2}^{k} |s_s| \left(1 + \sum_{m=1}^{s-1} (|\alpha_m| + 1) \right).$$

Thus we get,

$$* + ** \equiv \sum_{j=1}^{i-1} |s_j| + \sum_{j=1}^{k} |s_j| \sum_{s=1}^{j-1} (|\alpha_s| + 1) \quad \mod 2.$$

Note that $\sum_{j=1}^{k} |s_j| \sum_{s=1}^{j-1} (|\alpha_s| + 1)$ is not a function of i, k_1, k_2. We also observe that

$$\sum_{j=1}^{i-1} (|s_j \otimes \alpha_j| + 1) = \sum_{j=1}^{i-1} |s_j| + \sum_{j=1}^{i-1} (|\alpha_j| + 1).$$

If we abbreviate $\hat{\alpha}_i = s_i \otimes \alpha_i$, we get that adding (14) with the signs coming from the A_∞ relations to (13) results,

$$\sum_{\substack{k_1+k_2=k+1 \\ 1 \le i \le k_1}} (-1)^{\sum_{j=1}^{i-1} (|\hat{\alpha}_j|+1)} \hat{\mathfrak{m}}_{k_1}(\hat{\alpha}_1, \ldots, \hat{\alpha}_{i-1}, \hat{\mathfrak{m}}_{k_2}(\hat{\alpha}_i, \ldots, \hat{\alpha}_{i+k_2-1}), \hat{\alpha}_{i+k_2}, \ldots, \hat{\alpha}_k) =$$

$$= (-1)^{\sum_{j=1}^{k} |s_j| \sum_{s=1}^{j-1} (|\alpha_s|+1)} s_1 \cdots s_k \otimes \sum_{\substack{k_1+k_2=k+1 \\ 1 \le i \le k_1}} (-1)^{\sum_{j=1}^{i-1} (|\alpha_j|+1)}$$

$$\mathfrak{m}_{k_1}(\alpha_1, \ldots, \alpha_{i-1}, \mathfrak{m}_{k_2}(\alpha_i, \ldots, \alpha_{i+k_2-1}), \alpha_{i+k_2}, \ldots, \alpha_k) = 0.$$

The last equality follows from the A_∞ relations of $\mathcal{C}$. □

Lemma 2.19. *The pairing $\prec, \succ_F$ is $\hat{\mathcal{R}}$-bilinear in the sense that*

$$\prec \alpha_1, a \cdot \alpha_2 \succ_F = \prec \alpha_1 \cdot a, \alpha_2 \succ_F, \qquad \forall a \in \hat{\mathcal{R}},$$

and

$$\prec a \cdot \alpha_1, \alpha_2 \succ_F = a \cdot \prec \alpha_1, \alpha_2 \succ_F, \qquad \forall a \in Z(\hat{\mathcal{R}}).$$

Proof. The first equation is deduced from the definition of $\prec, \succ_F$. We will prove the second equation. As both $\mathcal{R}$ and S are algebras over the field $\Bbbk$, we have

$$Z(S \otimes_{\Bbbk} \mathcal{R}) \simeq Z(S) \otimes_{\Bbbk} Z(\mathcal{R}).$$

Thus it suffices to compute for $s_1, s_2 \in S, \alpha_1, \alpha_2 \in \mathcal{C}, a \in Z(S)$ and $b \in Z(\mathcal{R})$,

$$
\begin{aligned}
\prec a \otimes b \cdot s_1 \otimes \alpha_1, s_2 \otimes \alpha_2 \succ_F &= \prec (-1)^{|b||s_1|} a s_1 \otimes b\alpha_1, s_2 \otimes \alpha_2 \succ_F = \\
&= (-1)^{|b||s_1|+|s_2|(|\alpha_1|+|b|+1)} F(as_1 \cdot s_2) \otimes \prec b\alpha_1, \alpha_2 \succ = \\
&= (-1)^{|b||s_1|+|s_2|(|\alpha_1|+|b|+1)} a F(s_1 \cdot s_2) \otimes b \prec \alpha_1, \alpha_2 \succ = \\
&= (-1)^{|s_2|(|\alpha_1|+1)} a \otimes b \cdot F(s_1 \cdot s_2) \otimes \prec \alpha_1, \alpha_2 \succ = \\
&= a \otimes b \cdot \prec s_1 \otimes \alpha_1, s_2 \otimes \alpha_2 \succ_F.
\end{aligned}
$$

$\square$

Lemma 2.20.

$$
\prec \hat{\alpha}_1, \hat{\alpha}_2 \succ_F = (-1)^{(|\hat{\alpha}_1|+1)(|\hat{\alpha}_2|+1)+1} \prec \hat{\alpha}_2, \hat{\alpha}_1 \succ_F.
$$

Proof. For $s_1, s_2 \in S$ and $\alpha_1, \alpha_2 \in \mathcal{C}$ homogeneous, we compute,

$$
\begin{aligned}
\prec s_1 \otimes \alpha_1, s_2 \otimes \alpha_2 \succ_F &= (-1)^{|s_2|(|\alpha_1|+1)} F(s_1 \cdot s_2) \prec \alpha_1, \alpha_2 \succ = \\
&= (-1)^{|s_1||s_2|+|s_2|(|\alpha_1|+1)+(|\alpha_1|+1)(|\alpha_2|+1)+1} F(s_2 \cdot s_1) \prec \alpha_2, \alpha_1 \succ = \\
&= (-1)^{(|\alpha_1|+|s_1|+1)(|\alpha_2|+|s_2|+1)+1} \prec s_2 \otimes \alpha_2, s_1 \otimes \alpha_1 \succ_F.
\end{aligned}
$$

$\square$

Lemma 2.21. *For any* $\hat{\alpha}_1, \ldots, \hat{\alpha}_{k+1} \in \hat{\mathcal{C}}$

$$
\begin{aligned}
&\prec \hat{\mathfrak{m}}_k(\hat{\alpha}_1, \ldots, \hat{\alpha}_k), \hat{\alpha}_{k+1} \succ_F = \\
&\quad = (-1)^{(|\hat{\alpha}_{k+1}|+1) \sum_{j=1}^k (|\hat{\alpha}_j|+1)} \prec \hat{\mathfrak{m}}_k(\hat{\alpha}_{k+1}, \hat{\alpha}_1, \ldots, \hat{\alpha}_{k-1}), \hat{\alpha}_k \succ_F + \delta_{1,k} \cdot d \prec \hat{\alpha}_1, \hat{\alpha}_2 \succ_F.
\end{aligned}
$$

Proof. Let $s_1, \ldots, s_k \in S, \alpha_1, \ldots, \alpha_k \in \mathcal{C}$, be homogeneous elements. Assume first $k \neq 1$, and calculate

$$
\begin{aligned}
\prec \hat{\mathfrak{m}}_k(s_1 \otimes \alpha_1, \ldots, s_k \otimes \alpha_k), s_{k+1} \otimes \alpha_{k+1} \succ_F &= \\
&= (-1)^{\eta_1} \prec s_1 \cdots s_k \mathfrak{m}_k(\alpha_1, \ldots, \alpha_k), s_{k+1} \otimes \alpha_{k+1} \succ_F \\
&= (-1)^{\eta_1+\eta_2} F(s_1 \cdots s_{k+1}) \prec \mathfrak{m}_k(\alpha_1, \ldots, \alpha_k), \alpha_{k+1} \succ \qquad (15) \\
&= (-1)^{\eta_1+\eta_2+\eta_3} F(s_{k+1} \cdots s_k) \prec \mathfrak{m}_k(\alpha_1, \ldots, \alpha_k), \alpha_{k+1} \succ.
\end{aligned}
$$

Employing the cyclic symmetry property (9) of $\mathcal{C}$, we have

$$
= (-1)^{\eta_1+\eta_2+\eta_3+\eta_4} F(s_{k+1} \cdots s_k) \prec \mathfrak{m}_k(\alpha_{k+1}, \alpha_1, \ldots, \alpha_{k-1}), \alpha_k \succ,
$$

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with

$$\eta_1 = \sum_{j=1}^{k} |s_j| + \sum_{j=1}^{k} |s_j| \sum_{i=1}^{j-1} (|\alpha_i| + 1),$$

$$\eta_2 = |s_{k+1}| \sum_{j=1}^{k} (|\alpha_j| + 1) + |s_{k+1}|,$$

$$\eta_3 = |s_{k+1}| \sum_{j=1}^{k} |s_j|,$$

$$\eta_4 = (|\alpha_{k+1}| + 1) \sum_{j=1}^{k} (|\alpha_j| + 1).$$

Similarly to (15) we compute,

$$\prec \hat{\mathfrak{m}}_k (s_{k+1} \otimes \alpha_{k+1}, s_1 \otimes \alpha_1 \ldots, s_{k-1} \otimes \alpha_{k-1}), s_k \otimes \alpha_k \succ_F =$$
$$= (-1)^{\eta_5 + \eta_6} F(s_{k+1} \cdots s_k) \prec \mathfrak{m}_k (\alpha_{k+1}, \alpha_1, \ldots, \alpha_{k-1}), \alpha_k \succ,$$

with

$$\eta_5 = \sum_{j=1}^{k-1} |s_j| + |s_{k+1}| + \sum_{j=1}^{k-1} |s_j| \left(|\alpha_{k+1}| + 1 + \sum_{i=1}^{j-1} (|\alpha_i| + 1) \right).$$

$$\eta_6 = |s_k| \sum_{j=1}^{k-1} (|\alpha_j| + 1) + |s_k|(|\alpha_{k+1}| + 1) + |s_k|$$

Those computations imply that

$$\prec \hat{\mathfrak{m}}_k (s_1 \otimes \alpha_1, \ldots, s_k \otimes \alpha_k), s_{k+1} \otimes \alpha_{k+1} \succ_F =$$
$$= (-1)^{\eta_1 + \eta_2 + \eta_3 + \eta_4 + \eta_5 + \eta_6} \prec \hat{\mathfrak{m}}_k (s_{k+1} \otimes \alpha_{k+1}, s_1 \otimes \alpha_1 \ldots, s_{k-1} \otimes \alpha_{k-1}), s_k \otimes \alpha_k \succ_F.$$

Observe that

$$\eta_1 + \eta_2 + \eta_5 + \eta_6 = |s_{k+1}| \sum_{j=1}^{k} (|\alpha_j| + 1) + (|\alpha_{k+1}| + 1) \sum_{j=1}^{k} |s_j|.$$

So,

$$\eta_1 + \eta_2 + \eta_3 + \eta_4 + \eta_5 + \eta_6 = (|\alpha_{k+1}| + |s_{k+1}| + 1) \sum_{j=1}^{k} (|\alpha_j| + |s_j| + 1).$$

Thus we have showed the result for $k > 1$.

For $k = 1$ we compute

$$\prec \hat{\mathfrak{m}}_1 (s_1 \otimes \alpha_1), s_2 \otimes \alpha_2 \succ_F =$$
$$= \prec (-1)^{|s_1|} s_1 \otimes \mathfrak{m}_1(\alpha_1) + ds_1 \otimes \alpha_1, s_2 \otimes \alpha_2 \succ_F$$
$$= (-1)^{|s_1| + |s_2||\alpha_1|} F(s_1 \cdot s_2) \otimes \prec \mathfrak{m}_1(\alpha_1), \alpha_2 \succ + (-1)^{|s_2|(|\alpha_1| + 1)} F(ds_1 \cdot s_2) \otimes \prec \alpha_1, \alpha_2 \succ.$$

We proceed using cyclic symmetry property (9) of $\mathcal{C}$ to develop the first summand of the equation above.

$$(-1)^{|s_1|+|s_2||\alpha_1|}F(s_1 \cdot s_2) \otimes \prec \mathfrak{m}_1(\alpha_1), \alpha_2 \succ =$$
$$= (-1)^{|s_1|+|s_2||\alpha_1|}F(s_1 \cdot s_2) \otimes \left((-1)^{(|\alpha_1|+1)(|\alpha_2|+1)}\prec \mathfrak{m}_1(\alpha_2), \alpha_1 \succ + d \prec \alpha_1, \alpha_2 \succ\right)$$
$$= (-1)^{|s_1|+|s_2||\alpha_1|+|s_1||s_2|+(|\alpha_1|+1)(|\alpha_2|+1)}F(s_2 \cdot s_1) \otimes \prec \mathfrak{m}_1(\alpha_2), \alpha_1 \succ +$$
$$+ (-1)^{|s_1|+|s_2||\alpha_1|}F(s_1 \cdot s_2) \otimes d \prec \alpha_1, \alpha_2 \succ$$
$$= (-1)^{(|s_1|+|\alpha_1|+1)(|s_2|+|\alpha_2|+1)}\prec (-1)^{|s_2|}s_2 \otimes \mathfrak{m}_1(\alpha_2), s_1 \otimes \alpha_1 \succ_F +$$
$$+ (-1)^{|s_1|+|s_2||\alpha_1|}F(s_1 \cdot s_2) \otimes d \prec \alpha_1, \alpha_2 \succ.$$

We use the equality

$$(-1)^{|s_2|}s_2 \otimes \mathfrak{m}_1(\alpha_2) = \hat{\mathfrak{m}}_1(s_2 \otimes \alpha_2) - ds_2 \otimes \alpha_2,$$

to derive

$$(-1)^{(|s_1|+|\alpha_1|+1)(|s_2|+|\alpha_2|+1)}\prec (-1)^{|s_2|}s_2 \otimes \mathfrak{m}_1(\alpha_2), s_1 \otimes \alpha_1 \succ_F =$$
$$= (-1)^{(|s_1|+|\alpha_1|+1)(|s_2|+|\alpha_2|+1)}\prec \hat{\mathfrak{m}}_1(s_2 \otimes \alpha_2) - ds_2 \otimes \alpha_2, s_1 \otimes \alpha_1 \succ_F$$
$$= (-1)^{(|s_1|+|\alpha_1|+1)(|s_2|+|\alpha_2|+1)}\left(\prec \hat{\mathfrak{m}}_1(s_2 \otimes \alpha_2), s_1 \otimes \alpha_1 \succ_F - \prec ds_2 \otimes \alpha_2, s_1 \otimes \alpha_1 \succ_F\right).$$

Using Lemma 2.20, we develop the second term further,

$$-(-1)^{(|s_1|+|\alpha_1|+1)(|s_2|+|\alpha_2|+1)}\prec ds_2 \otimes \alpha_2, s_1 \otimes \alpha_1 \succ_F =$$
$$= (-1)^{|s_1|+|\alpha_1|+1}\prec s_1 \otimes \alpha_1, ds_2 \otimes \alpha_2 \succ_F$$
$$= (-1)^{1+|s_1|+|\alpha_1|+(|\alpha_1|+1)(|s_2|+1)}F(s_1 \cdot ds_2) \otimes \prec \alpha_1, \alpha_2 \succ$$
$$= (-1)^{|s_1|+|s_2|(|\alpha_1|+1)}F(s_1 \cdot ds_2) \otimes \prec \alpha_1, \alpha_2 \succ.$$

Combining the computations above we get,

$$\prec \hat{\mathfrak{m}}_1(s_1 \otimes \alpha_1), s_2 \otimes \alpha_2 \succ_F = (-1)^{(|s_1|+|\alpha_1|+1)(|s_2|+|\alpha_2|+1)}\prec \hat{\mathfrak{m}}_1(s_2 \otimes \alpha_2), s_1 \otimes \alpha_1 \succ_F +$$
$$+ (-1)^{|s_2|(|\alpha_1|+1)}F(ds_1 \cdot s_2) \otimes \prec \alpha_1, \alpha_2 \succ +$$
$$+ (-1)^{|s_1|+|s_2|(|\alpha_1|+1)}F(s_1 \cdot ds_2) \otimes \prec \alpha_1, \alpha_2 \succ +$$
$$+ (-1)^{|s_1|+|s_2|+|s_2|(|\alpha_1|+1)}F(s_1 \cdot s_2) \otimes d \prec \alpha_1, \alpha_2 \succ$$
$$= (-1)^{(|s_1|+|\alpha_1|+1)(|s_2|+|\alpha_2|+1)}\prec \hat{\mathfrak{m}}_1(s_2 \otimes \alpha_2), s_1 \otimes \alpha_1 \succ_F +$$
$$+ d \prec s_1 \otimes \alpha_1, s_2 \otimes \alpha_2 \succ_F.$$

$\square$

Proof of Proposition 2.15. We will show that $(\{\hat{\mathfrak{m}}_k\}_{k\geq0}, \prec, \succ_F, \mathbf{e}_s \otimes \mathbf{e})$ satisfies the properties required to define a cyclic unital A_∞ structure on $\hat{\mathcal{C}}$. Lemma 2.17 proves that the operators $\hat{\mathfrak{m}}_k$ satisfy property (1) of Definition 2.11. The operators $\hat{\mathfrak{m}}_k$ satisfy the A_∞ relation property (2), as shown in Lemma 2.18. The properties (3),(7) hold as we chose the filtrations on $\hat{\mathcal{C}}$ and $\hat{\mathcal{R}}$ to be given by the filtrations on $\mathcal{C}$ and $\mathcal{R}$. Property (4) clearly follows from the corresponding property of $\mathcal{C}$. Property (5) follows from the corresponding property of $\mathcal{C}$. Property (6) holds for $\prec\cdot,\cdot\succ_F$ by Lemma 2.19. Property (8) holds for $\prec\cdot,\cdot\succ_F$ by Lemma 2.20. Property (9)

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holds by Lemma 2.21. Property (10) follows from the definition of the extension and the corresponding property for $\mathcal{C}$. $\qquad\square$

Lemma 2.22. *There is a canonical $\bar{\mathcal{R}}$ module isomorphism $\bar{\hat{\mathcal{C}}} \simeq S \otimes \bar{\mathcal{C}}$ given by $\overline{s \otimes \alpha} \mapsto s \otimes \bar{\alpha}$.*

Proof. We observe the exact sequence
$$0 \longrightarrow F^{>0}\mathcal{C} \longrightarrow \mathcal{C} \longrightarrow \bar{\mathcal{C}} \longrightarrow 0.$$
Thus we obtain an exact sequence
$$S \otimes_{\Bbbk} F^{>0}\mathcal{C} \xrightarrow{a} S \otimes_{\Bbbk} \mathcal{C} \longrightarrow S \otimes_{\Bbbk} \bar{\mathcal{C}} \longrightarrow 0.$$
To conclude we show that
$$\operatorname{Im} a = F^{>0}(S \otimes_{\Bbbk} \mathcal{C}).$$
Indeed, by definition, the filtration on the tensor product is given by
$$\nu(\beta) = \sup_{\beta = \sum_i s_i \otimes \alpha_i} \min_i \left(\nu_S(s_i) + \nu_{\mathcal{C}}(\alpha_i) \right).$$
As ν_S is trivial, $\beta \in F^{>0}(S \otimes_{\Bbbk} \mathcal{C})$ if and only if we can write $\beta = \sum_i s_i \otimes \alpha_i$ with $\nu(\alpha_i) > 0$, which is equivalent to $\beta \in \operatorname{Im}(a)$. $\qquad\square$

Proposition 2.23. *Let $(\hat{\mathcal{C}}, \{\hat{\mathfrak{m}}_k\}_{k \geq 0}, \prec, \succ_F, \mathbf{e}_S \otimes \mathbf{e})$ be an (S, F) extension of a cyclic unital A_∞ algebra $(\mathcal{C}, \{\mathfrak{m}_k\}_{k \geq 0}, \prec, \succ, \mathbf{e})$. If $(\bar{\mathcal{C}}, \{\bar{\mathfrak{m}}_k\}_{k \geq 0}, \prec, \succ, \mathbf{e})$ arises from a dga (Υ, d), then $(\bar{\hat{\mathcal{C}}}, \{\bar{\hat{\mathfrak{m}}}_k\}_{k \geq 0}, \prec, \succ_F, \mathbf{e}_S \otimes \mathbf{e})$ arises from the dga $(S \otimes \Upsilon, d \otimes id + id \otimes d)$ using the canonical isomorphism $\bar{\hat{\mathcal{C}}} \to S \otimes \Upsilon$ given in Lemma 2.22.*

Proof. Let $\alpha_1, \alpha_2 \in \mathcal{C}$ and $s_1, s_2 \in S$ be homogeneous elements. First, recall that
$$\hat{\mathfrak{m}}_1(s_1 \otimes \alpha_1) = (-1)^{|s_1|} s_1 \otimes \mathfrak{m}_1(\alpha_1) + ds_1 \otimes \alpha_1.$$
It follows that under the identification $\overline{s \otimes \alpha} \mapsto s \otimes \bar{\alpha}$ we have
$$\bar{\hat{\mathfrak{m}}}_1(\overline{s_1 \otimes \alpha_1}) := \overline{\hat{\mathfrak{m}}_1(s_1 \otimes \alpha_1)} = (-1)^{|s_1|} s_1 \otimes \overline{\mathfrak{m}_1(\alpha_1)} + ds_1 \otimes \bar{\alpha}_1.$$
As $(\bar{\mathcal{C}}, \{\bar{\mathfrak{m}}_k\}_{k \geq 0}, \prec, \succ, \mathbf{e})$ arises from the dga (Υ, d) we showed that
$$\bar{\hat{\mathfrak{m}}}_1(s_1 \otimes \bar{\alpha}_1) = (-1)^{|s_1|} s_1 \otimes d\bar{\alpha}_1 + ds_1 \otimes \bar{\alpha}_1.$$
In a similar way, we can show that
$$(s_1 \otimes \bar{\alpha}_1)(s_2 \otimes \bar{\alpha}_2) = (-1)^{|s_1 \otimes \bar{\alpha}_1|} \bar{\hat{\mathfrak{m}}}_2(s_1 \otimes \bar{\alpha}_1, s_2 \otimes \bar{\alpha}_2), \qquad \bar{\hat{\mathfrak{m}}}_k = 0, \quad k \neq 1, 2.$$
$\qquad\square$

Definition 2.24. Let $(\hat{\mathcal{C}}, \{\hat{\mathfrak{m}}_k\}_{k \geq 0}, \prec, \succ_F, \mathbf{e}_S \otimes \mathbf{e})$ be an (S, F) extension of a cyclic unital A_∞ algebra $(\mathcal{C}, \{\mathfrak{m}_k\}_{k \geq 0}, \prec, \succ, \mathbf{e})$ over a commutative dga $\mathcal{R}$. Suppose $\mathcal{C}$ has an $\bar{\mathcal{R}}$ structure $g : \mathcal{R} \otimes_{\bar{\mathcal{R}}} \bar{\mathcal{C}} \to \mathcal{C}$ with inverse f. The **induced $\bar{\mathcal{R}}$ structure** on $(\hat{\mathcal{C}}, \{\hat{\mathfrak{m}}_k\}_{k \geq 0}, \prec, \succ_F, \mathbf{e}_S \otimes \mathbf{e})$ is
$$\hat{g} : \mathcal{R} \otimes_{\bar{\mathcal{R}}} \bar{\hat{\mathcal{C}}} \to \hat{\mathcal{C}}$$
given by the composition
$$\mathcal{R} \otimes_{\bar{\mathcal{R}}} \bar{\hat{\mathcal{C}}} = \mathcal{R} \otimes_{\bar{\mathcal{R}}} \overline{(S \otimes_{\Bbbk} \mathcal{C})} \xrightarrow{\sim} \mathcal{R} \otimes_{\bar{\mathcal{R}}} \left(S \otimes_{\Bbbk} \bar{\mathcal{C}} \right) \xrightarrow{\sim} S \otimes_{\Bbbk} \left(\mathcal{R} \otimes_{\bar{\mathcal{R}}} \bar{\mathcal{C}} \right) \xrightarrow[id \otimes g]{\sim} \hat{\mathcal{C}}.$$

Recall the definition of the modules $\mathcal{C}_E$ in Definition 2.3.

Lemma 2.25. *Let $(\mathcal{C}, \{\mathfrak{m}_k\}_{k \geq 0}, \prec\ , \succ, \mathbf{e})$ be a cyclic unital A_∞ algebra over $\mathcal{R}$ such that the residue algebra $(\bar{\mathcal{C}}, \{\bar{\mathfrak{m}}_k\}_{k \geq 0}, \prec\ , \succ, \mathbf{e})$ arises from a dga (Υ, d). Assume that $\mathcal{C}$ has an $\bar{\mathcal{R}}$ structure given by $g : \mathcal{R} \otimes \bar{\mathcal{C}} \to \mathcal{C}$ and let f be its inverse. Let $(\hat{\mathcal{C}}, \{\hat{\mathfrak{m}}_k\}_{k \geq 0}, \prec\ , \succ_F, 1 \otimes \mathbf{e})$ be an (S, F) extension of $\mathcal{C}$, and let $E \in \mathbb{R}_{\geq 0}$. Then the induced map $\hat{\mathfrak{m}}_1 : \hat{\mathcal{C}}_E \to \hat{\mathcal{C}}_E$ squares to zero.*

Proof. We first note that
$$(\hat{\mathcal{C}}, \{\hat{\mathfrak{m}}_k\}_{k \geq 0}, \prec\ , \succ_F, 1 \otimes \mathbf{e})$$
is a cyclic A_∞ algebra over $\mathcal{R}$. Let $\hat{g}$ be the induced $\bar{\mathcal{R}}$ structure on it, and let $\hat{f}$ be its inverse. As $(\bar{\mathcal{C}}, \{\bar{\mathfrak{m}}_k\}_{k \geq 0}, \prec\ , \succ, \mathbf{e})$ arises from the dga (Υ, d), Proposition 2.23 implies that $(\hat{\bar{\mathcal{C}}}, \{\hat{\bar{\mathfrak{m}}}_k\}_{k \geq 0}, \prec\ , \succ_F, \mathbf{e}_S \otimes \mathbf{e})$ arises from the dga $(\hat{\Upsilon}, d \otimes id + id \otimes d)$. We will abbreviate $d \otimes id + id \otimes d$ by $\hat{d}$. We see that all the conditions of Proposition 2.13 are satisfied, and we can deduce
$$\hat{f} \circ \hat{\mathfrak{m}}_1 \circ \hat{\mathfrak{m}}_1(\alpha) = \hat{d} \circ \hat{f} \circ \hat{\mathfrak{m}}_1(\alpha) = \hat{d} \circ \hat{d} \circ \hat{f}(\alpha) = 0,$$
for all $\alpha \in \hat{\mathcal{C}}_{E_i}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3. Pseudo-completeness and the sababa property

3.1. Definitions. In the following section, we introduce the property of pseudo-completeness of an A_∞ algebra and show that relevant classes of A_∞ algebras are pseudo-complete. Before plunging into the definition, a word of explanation is in order. The proof of Theorems 2 and 3 relies on an inductive argument based on obstruction theory applied to an (S, F) extension of the de Rham Fukaya A_∞ algebra of the Lagrangian submanifold L. As explained in Remarks 4.18, and 6.13 below, the analysis of the obstruction classes requires us to consider S with the trivial filtration. Consequently, unit bounding cochains have trivial filtration, and a priori there is no reason the Maurer-Cartan equation should converge. Pseudo-completeness guarantees convergence of the Maurer-Cartan equation and related infinite sums that appear in the obstruction theory computation. Precise statements are formulated in Remark 3.2 Lemma 3.9, and Corollary 4.16 below.

Definition 3.1. Let $(\mathcal{C}, \mathfrak{m}_k)$ be an A_∞ algebra over $\mathcal{R}$, complete with respect to its filtration ς_1. Let ς_2 be another filtration with $\varsigma_2 \geq \varsigma_1$. We say that $(\mathcal{C}, \mathfrak{m}_k)$ is **pseudo-complete** with respect to ς_2, if for all $E > 0$ and $D \in \mathbb{Z}$ there exists $K > 0$ such that for $\alpha_1, \ldots, \alpha_k \in \mathcal{C}$, we have
$$\sum_{i=1}^{k} \varsigma_2(\alpha_i) > K, \ |\mathfrak{m}_k(\alpha_1, \ldots, \alpha_k)| \geq D \implies \varsigma_1(\mathfrak{m}_k(\alpha_1, \ldots, \alpha_k)) > E.$$

Remark 3.2. Endow $\mathcal{C}^{\otimes k}$ and $\mathcal{C}$ with the topology induced by the filtration $\varsigma_2^{\otimes k}$ and ς_1 respectively. Then pseudo-completeness is equivalent to the continuity of $\mathfrak{m}_k : (\mathcal{C}^{\otimes k})_{j+k} \to \mathcal{C}$ for each degree j uniformly in k.

As mentioned above, we will use pseudo-completeness to show convergence of the Maurer-Cartan equation in an (S, F) extension of the Fukaya A_∞ algebra of the Lagrangian submanifold L. However, as explained in Remark 4.17 below, if we (S, F) extend the whole algebra, the result is not pseudo-complete. Instead, we focus on a subalgebra that satisfies the sababa property in the following definition. Lemma 3.8 below guarantees that the (S, F) extension

of the sababa subalgebra is pseudocomplete. The sababa property for a subalgebra is closely related to the sababa property for a monoid used in [35].

Definition 3.3. A filtered differential graded algebra $\mathcal{R}$ is **sababa** if it is generated as an algebra by elements $\{\lambda_i\}_{i \in \mathbb{N}}$ with $\lim_{i \to \infty} \varsigma_{\mathcal{R}}(\lambda_i) = \infty$. A cyclic unital A_∞ algebra $(\mathcal{C}, \{\mathfrak{m}_k\}_{k \geq 0}, \prec, \succ, \mathbf{e})$ over $\mathcal{R}$ is called **sababa** if $\mathcal{R}$ is sababa, $\mathcal{C}$ has an $\bar{\mathcal{R}}$ structure, and there exists $h \in \mathbb{Z}$ such that if $\bar{\alpha} \in \bar{\mathcal{C}}$ then $|\bar{\alpha}| \leq h$. The constant h is called the sababa constant.

3.2. Preliminary lemmas.

Lemma 3.4. *Let* $(\mathcal{C}, \{\mathfrak{m}_k\}_{k \geq 0}, \prec, \succ, \mathbf{e})$ *be a sababa cyclic unital A_∞ algebra over a sababa* $\mathcal{R}$. *For all $E > 0$ there exists $m \in \mathbb{Z}$ such that if α is a homogeneous element in $\mathcal{C}$ then*

$$\varsigma_{\mathcal{C}}(\alpha) < E \implies |\alpha| \leq m.$$

Proof. We will use the $\bar{\mathcal{R}}$ structure to identify $\mathcal{C}$ with $\mathcal{R} \otimes_{\bar{\mathcal{R}}} \bar{\mathcal{C}}$. Let $r \in \mathcal{R}$ and $\bar{\alpha} \in \bar{\mathcal{C}}$ be homogeneous elements with $\varsigma_{\mathcal{C}}(r \otimes \bar{\alpha}) < E$. We pick a set of generators $\{\lambda_i\}_{i \in \mathbb{N}}$ of $\mathcal{R}$ with $\lim_{i \to \infty} \varsigma_{\mathcal{R}}(\lambda_i) = \infty$, and we assume without loss of generality that we can write $r = \lambda_{i_1} \ldots \lambda_{i_k}$. We can also assume that $\varsigma_{\mathcal{R}}(\lambda_{i_j}) > 0$ for all $1 \leq j \leq k$. Indeed, if for some $1 \leq j \leq k$ we have $\varsigma_{\mathcal{R}}(\lambda_{i_j}) = 0$, we can replace

$$r \mapsto \lambda_{i_1} \ldots \lambda_{i_{j-1}} \lambda_{i_{j+1}} \ldots \lambda_{i_k}, \qquad \bar{\alpha} \mapsto \pm \bar{\lambda}_{i_j} \bar{\alpha}.$$

Since $\lim_{i \to \infty} \varsigma_{\mathcal{R}}(\lambda_i) = \infty$, the set

$$A := \left\{ r = \lambda_{i_1} \ldots \lambda_{i_k} : k \in \mathbb{N}, \lambda_{i_j} \in \{\lambda_i\}_{i \in \mathbb{N}}, \varsigma_{\mathcal{R}}(\lambda_{i_j}) > 0, \varsigma_{\mathcal{R}}(r) < E \right\}$$

is finite. Thus we have a bound $a > 0$ such that if $r \in A$ then $|r| < a$. Thus we have

$$|r \otimes \bar{\alpha}| = |r| + |\bar{\alpha}| \leq a + h,$$

where h is the sababa constant of $\mathcal{C}$. Hence $m = a + h$ is the desired constant. $\square$

Lemma 3.5. *Let* (S, e_S) *be a filtered unital associative differential graded algebra over $\mathbb{k}$ with a filtration ς_S. Let* $(\hat{\mathcal{C}}, \{\hat{\mathfrak{m}}_k\}_{k \geq 0}, \prec, \succ_F, \mathbf{e}_S \otimes \mathbf{e})$ *be an (S, F) extension of a cyclic unital A_∞ algebra* $(\mathcal{C}, \{\mathfrak{m}_k\}_{k \geq 0}, \prec, \succ, \mathbf{e})$. *Then for all $\alpha_1, \ldots \alpha_k \in \hat{\mathcal{C}}$ we have*

$$\varsigma_S \otimes \varsigma_{\mathcal{C}}(\hat{\mathfrak{m}}_k(\alpha_1, \ldots \alpha_k)) \geq \sum_{i=1}^{k} \varsigma_S \otimes \varsigma_{\mathcal{C}}(\alpha_i).$$

Proof. Assume by contradiction that the lemma is false. Then there exist $\alpha_1, \ldots, \alpha_k \in \hat{\mathcal{C}}$ such that

$$\sum_{i=1}^{k} \varsigma_S \otimes \varsigma_{\mathcal{C}}(\alpha_i) > \varsigma_S \otimes \varsigma_{\mathcal{C}}(\hat{\mathfrak{m}}_k(\alpha_1, \ldots \alpha_k)).$$

In particular, there exist $\{s_{ij}\}_{j \in \mathbb{N}} \subset S$, and $\{\eta_{ij}\}_{j \in \mathbb{N}} \subset \mathcal{C}$, such that

$$\alpha_i = \sum_{j \in \mathbb{N}} s_{ij} \otimes \eta_{ij},$$

and that

$$\sum_{i=1}^{k} \min_{j \in \mathbb{N}} \varsigma_S(s_{ij}) + \varsigma_{\mathcal{C}}(\eta_{ij}) > \varsigma_S \otimes \varsigma_{\mathcal{C}}(\hat{\mathfrak{m}}_k(\alpha_1, \ldots \alpha_k)).$$

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As $S \otimes \mathcal{C}$ is a filtered module we have

$$\varsigma_S \otimes \varsigma_{\mathcal{C}}(x+y) \geq \min(\varsigma_S \otimes \varsigma_{\mathcal{C}}(x), \varsigma_S \otimes \varsigma_{\mathcal{C}}(y)), \qquad x, y \in S \otimes \mathcal{C}.$$

Thus we have

$$\begin{aligned}
\varsigma_S \otimes \varsigma_{\mathcal{C}}(\hat{\mathfrak{m}}_k(\alpha_1, \ldots \alpha_k)) &\geq \min_{(j_1, \ldots, j_k) \in \mathbb{N}^k} \varsigma_S \otimes \varsigma_{\mathcal{C}}(\hat{\mathfrak{m}}_k(s_{1j_1} \otimes \eta_{ij_1}, \ldots s_{kj_k} \otimes \eta_{kj_k})) \\
&= \min_{(j_1, \ldots, j_k) \in \mathbb{N}^k} \varsigma_S \otimes \varsigma_{\mathcal{C}}(s_{1j_1} \ldots s_{kj_k} \otimes \mathfrak{m}_k(\eta_{1j_1}, \ldots, \eta_{kj_k})) \\
&\geq \min_{(j_1, \ldots, j_k) \in \mathbb{N}^k} \varsigma_S(s_{1j_1} \ldots s_{kj_k}) + \varsigma_{\mathcal{C}}(\mathfrak{m}_k(\eta_{1j_1}, \ldots, \eta_{kj_k})) \\
&\geq \min_{(j_1, \ldots, j_k) \in \mathbb{N}^k} \sum_{i=1}^{k} \varsigma_S(s_{ij_i}) + \sum_{i=1}^{k} \varsigma_{\mathcal{C}}(\eta_{ij_i}),
\end{aligned}$$

where the last inequality follows from S being a filtered ring and property (3) of the A_∞ algebra $\mathcal{C}$. Thus we have a contradiction as

$$\sum_{i=1}^{k} \min_{j \in \mathbb{N}} \varsigma_S(s_{ij}) + \varsigma_{\mathcal{C}}(\eta_{ij}) \leq \min_{(j_1, \ldots, j_k) \in \mathbb{N}^k} \sum_{i=1}^{k} \varsigma_S(s_{ij_i}) + \sum_{i=1}^{k} \varsigma_{\mathcal{C}}(\eta_{ij_i}) \leq \varsigma_S \otimes \varsigma_{\mathcal{C}}(\hat{\mathfrak{m}}_k(\alpha_1, \ldots \alpha_k)).$$

$\square$

3.3. Descending filtrations.

Definition 3.6. A filtered unital associative differential graded algebra (S, e_S) over $\mathbb{k}$ with a filtration ς_S is called **descending** if for all $s \in S$,

A filtered unital associative differential graded algebra (S, e_S) over $\mathbb{k}$ with a filtration ς_S is called **descending** if for all $s \in S$,

$$|s| \leq -\varsigma_S(s).$$

Let $(\hat{\mathcal{C}}, \{\hat{\mathfrak{m}}_k\}_{k \geq 0}, \prec, \succ_F, \mathbf{e}_s \otimes \mathbf{e})$ be an (S, F) extension of a cyclic unital A_∞ algebra over $\mathcal{R}$. For $m \in \mathbb{Z}$, we define the map $\pi_{S,m} : \hat{\mathcal{C}} \to \bigoplus_{i \leq m} S_i \otimes \mathcal{C}$ to be the projection. Explicitly, for $s \in S_j$, and $\alpha \in \mathcal{C}$, the map is defined by

$$\pi_{S,m}(s \otimes \alpha) = \begin{cases} s \otimes \alpha, & j \leq m \\ 0, & otherwise, \end{cases}$$

and extended linearly for arbitrary elements.

Lemma 3.7. *Let (S, e_s) be a descending unital differential graded algebra over $\mathbb{k}$. Let*

$$(\hat{\mathcal{C}}, \{\hat{\mathfrak{m}}_k\}_{k \geq 0}, \prec, \succ_F, \mathbf{e}_S \otimes \mathbf{e})$$

be an (S, F) extension of a cyclic unital A_∞ algebra $(\mathcal{C}, \{\mathfrak{m}_k\}_{k \geq 0}, \prec, \succ, \mathbf{e})$ over $\mathcal{R}$. Let $E \in \mathbb{R}_{\geq 0} \cup \{\infty\}$, and let $\{\alpha_n\}_{n \in \mathbb{N}} \in \hat{\mathcal{C}}$ be a sequence such that, $\varsigma_{\hat{\mathcal{C}}}(\alpha_n) \leq E$ and

$$\varsigma_S \otimes \varsigma_{\mathcal{C}}(\alpha_n) - \varsigma_{\hat{\mathcal{C}}}(\alpha_n) \xrightarrow{n \to \infty} \infty.$$

Then for any $m \in \mathbb{Z}$, we have for n large enough

$$\pi_{S,m}(\alpha_n) \not\equiv 0 \pmod{F^{>E}\hat{\mathcal{C}}},$$

Proof. With out loss of generality we can assume $m < 0$. Pick N such that for all $n > N$ we have

$$\varsigma_S \otimes \varsigma_{\mathcal{C}}(\alpha_n) - \varsigma_{\hat{\mathcal{C}}}(\alpha_n) \geq -m + 1.$$

Thus there exist $\{\eta_i\}_{i \in \mathbb{N}} \subset \mathcal{C}$, and $\{s_i\}_{i \in \mathbb{N}} \subset S$ such that

$$\varsigma_{\mathcal{C}}(\eta_i) \xrightarrow{i \to \infty} \infty, \qquad \alpha_n = \sum_{i=1}^{\infty} s_i \otimes \eta_i,$$

and

$$\min_{1 \leq i < \infty} \varsigma_S(s_i) + \varsigma_{\mathcal{C}}(\eta_i) \geq \varsigma_S \otimes \varsigma_{\mathcal{C}}(\alpha_n) - 1.$$

Then,

$$\min_{1 \leq i < \infty} \varsigma_S(s_i) + \varsigma_{\mathcal{C}}(\eta_i) \geq -m + \varsigma_{\hat{\mathcal{C}}}(\alpha_n) \geq -m + \min_{1 \leq j < \infty} \varsigma_{\mathcal{C}}(\eta_j).$$

Let i_0 and j_0 be indices that realize the minima above. Then

$$\varsigma_S(s_{j_0}) + \varsigma_{\hat{\mathcal{C}}}(\alpha_n) \geq \varsigma_S(s_{j_0}) + \varsigma_{\mathcal{C}}(\eta_{j_0}) \geq \varsigma_S(s_{i_0}) + \varsigma_{\mathcal{C}}(\eta_{i_0}) > -m + \varsigma_{\hat{\mathcal{C}}}(\alpha_n).$$

From the preceding inequality we get that $\varsigma_S(s_{j_0}) > -m$. Since S is descending we deduce $|s_{j_0}| < m$. In the case that the projection of α_n to $S_{|s_{j_0}|} \otimes \mathcal{C}$ is non-trivial modulo $F^{>E}\hat{\mathcal{C}}$ we are done. Otherwise, we have

$$\alpha_n = \sum_{\substack{i \in \mathbb{N}: \\ |s_i| \neq |s_{j_0}|}} s_i \otimes \eta_i \quad (\mathrm{mod}\ F^{>E}\hat{\mathcal{C}}).$$

In the same manner as before, we can find another index $j_1 \neq j_0$ such that $\varsigma_{\hat{\mathcal{C}}}(\alpha_n) \geq \varsigma_{\mathcal{C}}(\eta_{j_1})$ and $|s_{j_1}| < m$. If the projection of α_n to $S_{|s_{j_1}|} \otimes \mathcal{C}$ is trivial modulo $F^{>E}\hat{\mathcal{C}}$ we repeat the argument and find another index j_2. Because $\varsigma_{\mathcal{C}}(\eta_i) \xrightarrow{i \to \infty} \infty$ there has to be some index j_l, such that $\varsigma_{\hat{\mathcal{C}}}(\alpha_n) \geq \varsigma_{\mathcal{C}}(\eta_{j_l})$, $|s_{j_l}| < m$ and the projection of α_n to $S_{|s_{j_l}|} \otimes \mathcal{C}$ is modulo $F^{>E}\hat{\mathcal{C}}$ non-trivial. We find that $\pi_{S,m}(\alpha_n) \neq 0$ $(\mathrm{mod}\ F^{>E}\hat{\mathcal{C}})$ for all $n > N$. $\qquad\square$

3.4. **Pseudo-completeness of extension.**

Proposition 3.8. *Let (S, e_s) be a descending unital differential graded algebra over $\Bbbk$. Let $(\hat{\mathcal{C}}, \{\hat{\mathfrak{m}}_k\}_{k \geq 0}, \prec, \succ_F, \mathbf{e}_S \otimes \mathbf{e})$ be an (S, F) extension of an A_∞ algebra $(\mathcal{C}, \{\mathfrak{m}_k\}_{k \geq 0}, \prec, \succ, \mathbf{e})$ over $\mathcal{R}$. Assume also that $(\mathcal{C}, \{\mathfrak{m}_k\}_{k \geq 0}, \prec, \succ, \mathbf{e})$ is also a sababa cyclic unital A_∞ algebra over some diffrential graded algebra $\mathcal{R}'$. Then the A_∞ algebra $(\hat{\mathcal{C}}, \{\hat{\mathfrak{m}}_k\}_{k \geq 0}, \prec, \succ_F, \mathbf{e}_S \otimes \mathbf{e})$ is pseudo-complete with respect to $\varsigma_S \otimes \varsigma_{\mathcal{C}}$.*

Proof. Assume by contradiction that the proposition is false. Then there exist $E > 0$ and $D \in \mathbb{Z}$ such that for all $K > 0$ there exist $\alpha_1^K, \ldots, \alpha_{k_K}^K \in \hat{\mathcal{C}}$ with

$$\sum_{i=1}^{k_K} \varsigma_S \otimes \varsigma_{\mathcal{C}}(\alpha_i^K) > K, \quad |\hat{\mathfrak{m}}_{k_K}(\alpha_1^K, \ldots, \alpha_{k_K}^K)| \geq D, \quad \varsigma_{\hat{\mathcal{C}}}(\hat{\mathfrak{m}}_k(\alpha_1^K, \ldots, \alpha_{k_K}^K)) \leq E.$$

It follows from our assumption and Lemma 3.5 that

$$\varsigma_S \otimes \varsigma_{\mathcal{C}}(\hat{\mathfrak{m}}_k(\alpha_1^K, \ldots, \alpha_{k_K}^K)) - \varsigma_{\hat{\mathcal{C}}}((\hat{\mathfrak{m}}_k(\alpha_1^K, \ldots, \alpha_{k_K}^K)) \xrightarrow{K \to \infty} \infty.$$

For all $m \in \mathbb{Z}$ we can use Lemma 3.7 to find some N, such that for $K > N$ we have

$$\pi_{S,m}(\hat{\mathfrak{m}}_k(\alpha_1^K, \ldots, \alpha_{k_K}^K)) \not\equiv 0 \quad (\mathrm{mod}\ F^{>E}\hat{\mathcal{C}}).$$

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Then for any representation

$$\pi_{S,m}(\hat{\mathfrak{m}}_k(\alpha_1^K, \ldots, \alpha_{k_K}^K)) = \sum_{i \in \mathbb{N}} s_i \otimes \eta_i,$$

such that for all $i \in \mathbb{N}$ the elements $s_i \in S$ and $\eta_i \in \mathcal{C}$ are homogeneous, there exists $j \in \mathbb{N}$ such that $\varsigma_{\mathcal{C}}(\eta_j) < E$. By Lemma 3.4 there exists a bound M on the degree of homogeneous elements η of $\mathcal{C}$ with $\varsigma_{\mathcal{C}}(\eta) < E$. Thus we get

$$|s_j \otimes \eta_j| \leq m + M.$$

Choosing m sufficiently small, this contradicts the lower bound

$$D \leq |s_j \otimes \eta_j|.$$

$\square$

3.5. **Convergence results.** Let $(\mathcal{C}, \{\mathfrak{m}_k\}_{k \geq 0})$ be an A_∞ algebra over $\mathcal{R}$, complete with respect to its filtration ς_1 . Let ς_2 be another filtration defined on $\mathcal{C}$, and assume that $(\mathcal{C}, \{\mathfrak{m}_k\}_{k \geq 0})$ is pseudo-complete with respect to ς_2.

Lemma 3.9. *Let $b \in \mathcal{C}$ with $\varsigma_2(b) > 0$, and $|b| \geq 1$ then for any $x_1, \ldots, x_k \in \mathcal{C}$ homogeneous,*

$$\lim_{\ell \to \infty} \min_{\ell_0 + \cdots + \ell_k = \ell} \varsigma_1(\mathfrak{m}_{k+\ell_0 + \cdots + \ell_k}(\underbrace{b, \cdots, b}_{\ell_0}, x_1, \underbrace{b, \cdots, b}_{\ell_1}, \cdots, \underbrace{b, \cdots, b}_{\ell_{k-1}}, x_k, \underbrace{b, \cdots, b}_{\ell_k})) = \infty.$$

Proof. We will show that for $E > 0$ there is only finitely many instances of $\ell_0, \cdots, \ell_k \in \mathbb{Z}_{\geq 0}$ with

$$\varsigma_1\left(\mathfrak{m}_{k+\ell_0 + \cdots + \ell_k}(\underbrace{b, \cdots, b}_{\ell_0}, x_1, \underbrace{b, \cdots, b}_{\ell_1}, \cdots, \underbrace{b, \cdots, b}_{\ell_{k-1}}, x_k, \underbrace{b, \cdots, b}_{\ell_k})\right) \leq E.$$

By pseudo-completeness of $(\mathcal{C}_1, \mathfrak{m}_k)$ with respect to ς_2, for $D := \sum_{i=1}^k |x_i|$ there exists $K > 0$ such that for $\alpha_1, \ldots \alpha_m \in \mathcal{C}_1$, we have

$$\sum_{i=1}^m \varsigma_2(\alpha_i) > K, \ |\mathfrak{m}_m(\alpha_1, \ldots, \alpha_m)| \geq D \implies \varsigma_1(\mathfrak{m}_k(\alpha_1, \ldots, \alpha_m)) > E.$$

From the assumption on b there exists $\ell \in \mathbb{N}$ such that $\sum_{i=1}^{\ell} \varsigma_2(b) > K$, and

$$\left| \mathfrak{m}_{k+\ell_0 + \cdots + \ell_k}(\underbrace{b, \cdots, b}_{\ell_0}, x_1, \underbrace{b, \cdots, b}_{\ell_1}, \cdots, \underbrace{b, \cdots, b}_{\ell_{k-1}}, x_k, \underbrace{b, \cdots, b}_{\ell_k}) \right| \geq D.$$

Thus, if $\sum_{i=0}^k \ell_i \geq \ell$ then

$$\varsigma_1\left(\mathfrak{m}_{k+\ell_0 + \cdots + \ell_k}(\underbrace{b, \cdots, b}_{\ell_0}, x_1, \underbrace{b, \cdots, b}_{\ell_1}, \cdots, \underbrace{b, \cdots, b}_{\ell_{k-1}}, x_k, \underbrace{b, \cdots, b}_{\ell_k})\right) > E.$$

$\square$

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Let $\overset{\circ}{\mathcal{C}}$ be the filtered $\mathcal{R}$ module given by the non completed direct sum $\bigoplus_{m\in\mathbb{Z}}(\mathcal{C})_m$. For $b\in\mathcal{C}$ with $\varsigma_2(b)>0$ and $|b|=1$, the above lemma, completeness of $\mathcal{C}$, and property (3) of the A_∞ algebra allows us to define a bounded multilinear map $\overset{\circ}{\mathfrak{m}}{}_k^b:\prod_{i=1}^k\overset{\circ}{\mathcal{C}}\to\mathcal{C}$ by

$$\overset{\circ}{\mathfrak{m}}{}_k^b(x_1,\cdots,x_k)=\sum_{\ell_0,\cdots,\ell_k\in\mathbb{Z}_{\geq0}}\mathfrak{m}_{k+\ell_0+\cdots+\ell_k}(\underbrace{b,\cdots,b}_{\ell_0},x_1,\underbrace{b,\cdots,b}_{\ell_1},\cdots,\underbrace{b,\cdots,b}_{\ell_{k-1}},x_k,\underbrace{b,\cdots,b}_{\ell_k}).$$

Definition 3.10. For $b\in\mathcal{C}$ with $\varsigma_2(b)>0$ and $|b|=1$, a b **deformation** of the operator $\mathfrak{m}_k:\mathcal{C}^{\otimes k}\to\mathcal{C}[2-k]$ is a bounded $\mathcal{R}$ linear map $\mathfrak{m}_k^b:\mathcal{C}^{\otimes k}\to\mathcal{C}[2-k]$ such that the following diagram commutes

$$\begin{array}{ccc}\prod_{i=1}^k\overset{\circ}{\mathcal{C}}&\xrightarrow{\ \tau\ }&\mathcal{C}^{\otimes k}\\&{\scriptstyle\overset{\circ}{\mathfrak{m}}{}_k^b}\searrow&\Big\downarrow{\scriptstyle\mathfrak{m}_k^b}\\&&\mathcal{C}\end{array},$$

where $\tau:\prod_{i=1}^k\mathcal{C}\to\mathcal{C}^{\otimes k}$ is given by $(x_1,\ldots,x_k)\mapsto x_1\otimes\ldots\otimes x_k$.

Lemma 3.11. *For $b\in\mathcal{C}$ with $\varsigma_2(b)>0$ and $|b|=1$, there exists a unique b deformation of $\mathfrak{m}_k:\mathcal{C}^{\otimes k}\to\mathcal{C}$ for each $k\geq0$.*

Proof. For $k=0$, the statement follows from Lemma 3.9 and completeness of $\mathcal{C}$. For $k=1$ the statement follows from continuity of $\overset{\circ}{\mathfrak{m}}{}_1^b$, and completeness of $\mathcal{C}$. For $k=2$, the statement follows by Proposition 1 in [3, Section 2.1.7], concerning bounded bilinear maps. The case $k>2$ is obtained by adjusting Proposition 1 in [3, Section 2.1.7] to bounded multilinear maps. $\qquad\square$

Lemma 3.12. *Let $x_1,\ldots x_k\in\mathcal{C}$ then*

$$\mathfrak{m}_k^b(x_1,\cdots,x_k)=\sum_{\ell_0,\cdots,\ell_k\in\mathbb{Z}_{\geq0}}\mathfrak{m}_{k+\ell_0+\cdots+\ell_k}(\underbrace{b,\cdots,b}_{\ell_0},x_1,\underbrace{b,\cdots,b}_{\ell_1},\cdots,\underbrace{b,\cdots,b}_{\ell_{k-1}},x_k,\underbrace{b,\cdots,b}_{\ell_k}).$$

Proof. Write $x_i=\sum_{j=1}^\infty\alpha_j^i$ with $\alpha_j^i\in\overset{\circ}{\mathcal{C}}$. As boundness implies continuity for filtered bilinear maps, we have

$$\mathfrak{m}_k^b(x_1,\cdots,x_k)=$$

$$=\sum_{j_1,\ldots j_k\in\mathbb{N}}\mathfrak{m}_k^b(\alpha_{j_1}^1,\ldots,\alpha_{j_k}^k)$$

$$=\sum_{j_1,\ldots j_k\in\mathbb{N}}\sum_{\ell_0,\cdots,\ell_k\in\mathbb{Z}_{\geq0}}\mathfrak{m}_{k+\ell_0+\cdots+\ell_k}(\underbrace{b,\cdots,b}_{\ell_0},\alpha_{j_1}^1,\underbrace{b,\cdots,b}_{\ell_1},\cdots,\underbrace{b,\cdots,b}_{\ell_{k-1}},\alpha_{j_k}^k,\underbrace{b,\cdots,b}_{\ell_k}).$$

By Proposition 2 in [3, Section 1.1.8] we can change the order of summation, and get

$$=\sum_{\ell_0,\cdots,\ell_k\in\mathbb{Z}_{\geq0}}\sum_{j_1,\ldots j_k\in\mathbb{N}}\mathfrak{m}_{k+\ell_0+\cdots+\ell_k}(\underbrace{b,\cdots,b}_{\ell_0},\alpha_{j_1}^1,\underbrace{b,\cdots,b}_{\ell_1},\cdots,\underbrace{b,\cdots,b}_{\ell_{k-1}},\alpha_{j_k}^k,\underbrace{b,\cdots,b}_{\ell_k})$$

$$=\sum_{\ell_0,\cdots,\ell_k\in\mathbb{Z}_{\geq0}}\mathfrak{m}_{k+\ell_0+\cdots+\ell_k}(\underbrace{b,\cdots,b}_{\ell_0},x_1,\underbrace{b,\cdots,b}_{\ell_1},\cdots,\underbrace{b,\cdots,b}_{\ell_{k-1}},x_k,\underbrace{b,\cdots,b}_{\ell_k}).$$

To obtain the last equality, we used the boundedness of the operators $\{\mathfrak{m}_k\}_{k\geq 0}$. $\qquad\square$

The Lemma 3.12 allows us to use the axioms of the n-dimensional curved cyclic unital A_∞ algebra $(\mathcal{C}, \{\mathfrak{m}_k\}_{k\geq 0}, \prec, \succ, \mathbf{e})$ and deduce the corollary below.

Corollary 3.13. *Let $(\mathcal{C}, \{\mathfrak{m}_k\}_{k\geq 0}, \prec, \succ, \mathbf{e})$ be an n-dimensional curved cyclic unital A_∞ algebra. Assume that the left and right module structures of $\mathcal{C}[1]$ over $Z(\mathcal{R})$ coincide. Let b be an element of $\mathcal{C}$ with $\varsigma_2(b) > 0$ and $|b| = 1$. Then, $(\mathcal{C}, \{\mathfrak{m}_k^b\}_{k\geq 0}, \prec, \succ, \mathbf{e})$ is a n-dimensional curved cyclic unital A_∞ algebra over $Z(\mathcal{R})$ if $\overline{\mathfrak{m}_0^b} = 0$.*

Remark 3.14. We do not exclude the case that $\varsigma_1(b) = 0$. So, the condition $\overline{\mathfrak{m}_0^b} = 0$ is not trivial.

Let (S, e_s) be a descending unital differential graded algebra over $\Bbbk$. Let

$$(\hat{\mathcal{C}}, \{\hat{\mathfrak{m}}_k\}_{k\geq 0}, \prec, \succ_F, \mathbf{e}_S \otimes \mathbf{e})$$

be an (S, F) extension of a cyclic unital A_∞ algebra $(\mathcal{C}, \{\mathfrak{m}_k\}_{k\geq 0}, \prec, \succ, \mathbf{e})$ over $\mathcal{R}$. Assume also that $(\mathcal{C}, \{\mathfrak{m}_k\}_{k\geq 0}, \prec, \succ, \mathbf{e})$ is also a sababa cyclic unital A_∞ algebra over some differential graded algebra $\mathcal{R}'$. By Proposition 3.8 $(\hat{\mathcal{C}}, \{\hat{\mathfrak{m}}_k\}_{k\geq 0}, \prec, \succ_F, \mathbf{e}_S \otimes \mathbf{e})$ is pseudo-complete with respect to $\varsigma_S \otimes \varsigma_{\mathcal{C}}$. Thus, given $b \in \hat{\mathcal{C}}$ with $\varsigma_S \otimes \varsigma_{\mathcal{C}}(b) > 0$ and $|b| = 1$, Lemma 3.11 guarantees the existence and uniqueness of the b deformed operators $\{\hat{\mathfrak{m}}_k^b\}_{k\geq 0}$.

Corollary 3.15. *Let (S, e_s) be a descending unital differential graded algebra over $\Bbbk$. Let $(\hat{\mathcal{C}}, \{\hat{\mathfrak{m}}_k\}_{k\geq 0}, \prec, \succ_F, \mathbf{e}_S \otimes \mathbf{e})$ be an (S, F) extension of a cyclic unital A_∞ algebra $(\mathcal{C}, \{\mathfrak{m}_k\}_{k\geq 0}, \prec, \succ, \mathbf{e})$ over $\mathcal{R}$ that is sababa over a subalgebra $\mathcal{R}' \subset \mathcal{R}$. Assume that the left and right module structures of $\mathcal{C}[1]$ over $\mathcal{R}$ coincide. Let $b \in \hat{\mathcal{C}}$, with $\varsigma_S \otimes \varsigma_{\mathcal{C}}(b) > 0$, and $|b| = 1$. Then, $(\hat{\mathcal{C}}, \{\hat{\mathfrak{m}}_k^b\}_{k\geq 0}, \prec, \succ_F, \mathbf{e}_S \otimes \mathbf{e})$ is cyclic unital A_∞ algebra over $Z\left(\hat{\mathcal{R}}\right)$ if $\overline{\hat{\mathfrak{m}}_0^b} = 0$.*

Proposition 3.16. *Let (S, e) be a descending unital differential graded algebra over $\Bbbk$. Let $(\mathcal{C}, \{\mathfrak{m}_k\}_{k\geq 0}, \prec, \succ, \mathbf{e})$ be a cyclic unital A_∞ algebra over $\mathcal{R}$ such that the residue algebra $(\bar{\mathcal{C}}, \{\bar{\mathfrak{m}}_k\}_{k\geq 0}, \prec, \succ, \mathbf{e})$ arises from a dga (Υ, d). Assume that the left and right module structures of $\mathcal{C}[1]$ over $\mathcal{R}$ coincide. Assume also that $\mathcal{C}$ has an $\bar{\mathcal{R}}$ structure given by $g : \mathcal{R} \otimes \bar{\mathcal{C}} \to \mathcal{C}$ and let f be its inverse. Let $(\hat{\mathcal{C}}, \{\hat{\mathfrak{m}}_k\}_{k\geq 0}, \prec, \succ_F, 1 \otimes \mathbf{e})$ be an (S, F) extension of $\mathcal{C}$, let $\hat{g}$ be the induced $\bar{\mathcal{R}}$ structure on it, and let $\hat{f}$ be its inverse. Let $b \in \hat{\mathcal{C}}$ with $|b| = 1$, and $\varsigma_S \otimes \varsigma_{\mathcal{C}}(b) > 0$. Let $\alpha \in \hat{\mathcal{C}}_E$ be a homogeneous element, then*

$$\hat{f}\left(\hat{\mathfrak{m}}_1^b(\alpha)\right) = d\hat{f}(\alpha) - [\hat{f}(b), \hat{f}(\alpha)],$$

where the product on the right hand side is induced by the algebra structure of $\mathcal{R} \otimes \hat{\Upsilon}$ given by the isomorphism $\hat{\bar{\mathcal{C}}} \to \hat{\Upsilon}$ of Lemma 2.22.

Proof. We first note that

$$(\hat{\mathcal{C}}, \{\hat{\mathfrak{m}}_k\}_{k\geq 0}, \prec, \succ_F, \mathbf{e}_S \otimes \mathbf{e})$$

is a cyclic A_∞ algebra over $\mathcal{R}$. As the left and right $\mathcal{R}$ module structures on $\mathcal{C}[1]$ coincide, so do the left and right $\mathcal{R}$ module structures on $\hat{\mathcal{C}}[1]$. As $(\bar{\mathcal{C}}, \{\bar{\mathfrak{m}}_k\}_{k\geq 0}, \prec, \succ, \mathbf{e})$ arises from the dga (Υ, d), Proposition 2.23 implies that $(\hat{\bar{\mathcal{C}}}, \{\hat{\bar{\mathfrak{m}}}_k\}_{k\geq 0}, \prec, \succ_F, \mathbf{e}_S \otimes \mathbf{e})$ arises from the dga $(\hat{\Upsilon}, id \otimes d)$. Thus the conditions of Proposition 2.13 are satisfied, and we can deduce

$$\hat{f}(\hat{\mathfrak{m}}_1^b(\alpha)) \equiv df(\alpha) + (-1)^{|b|}\hat{f}(b)\hat{f}(\alpha) + (-1)^{|\alpha|}\hat{f}(\alpha)\hat{f}(b) \pmod{F^{>E}\mathcal{R} \otimes_{\bar{\mathcal{R}}} \bar{\hat{\mathcal{C}}}}$$

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where the product is given by the natural product on $\mathcal{R} \otimes_{\bar{\mathcal{R}}} \bar{\hat{\mathcal{C}}}$. The equation above is equivalent to the desired formula as $|b| = 1$. $\qquad\square$

3.6. **Opposites and duality.** The following section is based on [34]. Let $\mathcal{C}, \mathcal{D}$, be graded complete filtered vector spaces. An A_∞ pre-homomorphism

$$\mathfrak{f} : \mathcal{C} \to \mathcal{D}$$

is a collection of maps

$$\mathfrak{f}_k : \mathcal{C}^{\otimes k} \to \mathcal{D}, \quad k \geq 0$$

of degree $1 - k$ such that $\inf_{v \neq 0}(\varsigma_\mathcal{D} \mathfrak{f}_k(v)) - \varsigma_{\mathcal{C}^{\otimes k}}(v) \geq 0$ with strict inequality for $k = 0$.

Let $(\mathcal{C}, \mathfrak{m}^\mathcal{C})$, $(\mathcal{D}, \mathfrak{m}^\mathcal{D})$, be A_∞ algebras. An A_∞ pre-homomorphism $\mathfrak{f} : \mathcal{C} \to \mathcal{D}$ is called an A_∞ homomorphism if

$$\sum_{\substack{l \\ r_1 + \cdots + r_l = k}} \mathfrak{m}_l^D \left(\mathfrak{f}_{r_1}(\alpha_1, \ldots, \alpha_{r_1}), \ldots, \mathfrak{f}_{r_l}(\alpha_{k-r_l+1}, \ldots, \alpha_k)\right) =$$

$$= \sum_{\substack{k_1 + k_2 = k+1 \\ 1 \leq i \leq k_1}} (-1)^{\sum_{j=1}^{i-1}(|\alpha_j|+1)} \mathfrak{f}_{k_1}\left(\alpha_1, \ldots, \alpha_{i-1}, \mathfrak{m}_{k_2}^C(\alpha_i, \ldots, \alpha_{i+k_2-1}), \ldots, \alpha_k\right).$$

Let $\alpha = (\alpha_1, \ldots, \alpha_k)$ be a list of homogeneous elements $\alpha_j \in \mathcal{C}$. Let σ be a permutation of the set $\{1, \ldots, k\}$. We use the notation

$$s_\sigma(\alpha) = \sum_{\substack{i<j \\ \sigma(i)>\sigma(j)}} \left(\left|\alpha_{\sigma(i)}\right| + 1\right)\left(\left|\alpha_{\sigma(j)}\right| + 1\right).$$

For the rest of the paper, we fix τ to be the permutation $(1, \ldots, k) \mapsto (k, k-1, \ldots, 1)$. Given an A_∞ algebra $(\mathcal{C}, \mathfrak{m})$, the opposite A_∞ algebra structure $\mathfrak{m}^{op}$ on $\mathcal{C}$ is defined by

$$\mathfrak{m}_k^{op}(\alpha_1, \ldots, \alpha_k) = (-1)^{s_\tau(\alpha)+k+1} \mathfrak{m}_k(\alpha_k, \ldots, \alpha_1).$$

A straightforward calculation shows that $\mathfrak{m}_k^{op}$ is indeed an A_∞ algebra.

Definition 3.17. Let $\mathcal{C}$ be a graded complete filtered $\mathcal{R}$ vector space and let $\mathfrak{d} : \mathcal{C} \to \mathcal{C}$ be an involution, that is, a linear map of degree zero such that $\mathfrak{d}^2 = \mathrm{Id}$. In particular, $\mathfrak{d}$ is a strict A_∞ prehomomorphism. An A_∞ structure $\mathfrak{m}$ on $\mathcal{C}$ is called $\mathfrak{d}$ **self-dual** if $\mathfrak{d} : (\mathcal{C}, \mathfrak{m}^{op}) \to (\mathcal{C}, \mathfrak{m})$ is an A_∞ homomorphism.

For a set automorphism $\kappa : Z \to Z$, let $Z^\kappa \subset Z$ denote the subset fixed pointwise by κ. Let $\mathfrak{d} : \mathcal{C} \to \mathcal{C}$ be an involution, and assume that $(\mathcal{C}, \mathfrak{m})$ is $\mathfrak{d}$ self-dual.

Lemma 3.18. Let $b \in (\mathcal{C}_1)^{-\mathfrak{d}}$ with $\varsigma_2(b) > 0$. Then $\mathfrak{m}_0^b \in (\mathcal{C}_2)^{-\mathfrak{d}}$.

Proof. The fact that $(\mathcal{C}, \mathfrak{m})$ is $\mathfrak{d}$-self-dual combined with the assumption that $|b| = 1$ implies that

$$\mathfrak{d} \circ \mathfrak{m}_k(b, \ldots, b) = (-1)^{k+1+s_\tau(\alpha_1, \ldots, \alpha_k)} \mathfrak{m}_k(\mathfrak{d}(b), \ldots, \mathfrak{d}(b)) = -\mathfrak{m}_k(b, \ldots, b),$$

and that $\mathfrak{d}\mathfrak{m}_0 = -\mathfrak{m}_0$. Thus, the result follows from Lemma 3.12. $\qquad\square$

Lemma 3.19. Let $(\mathcal{C}, \mathfrak{m})$ be a $\mathfrak{d}$-self-dual A_∞ algebra, and let $b \in (\mathcal{C}_1)^{-\mathfrak{d}}$ with $\varsigma_2(b) > 0$. Then the deformed operator $\mathfrak{m}_1^b$ is $\mathfrak{d}$-equivariant, i.e.,

$$\mathfrak{d} \circ \mathfrak{m}_1^b = \mathfrak{m}_1^b \circ \mathfrak{d}.$$

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Proof. For any $x \in \mathcal{C}$, using Lemma 3.12 and the fact that $(\mathcal{C}, \mathfrak{m})$ is $\mathfrak{d}$-self-dual combined with the assumption that $|b| = 1$:

$$\mathfrak{d} \circ \mathfrak{m}_1^b(x) = \sum_{\ell_0, \ell_1 \geq 0} \mathfrak{d} \circ \mathfrak{m}_{1+\ell_0+\ell_1}(\underbrace{b, \ldots, b}_{\ell_0}, x, \underbrace{b, \ldots, b}_{\ell_1})$$

$$= \sum_{\ell_0, \ell_1 \geq 0} (-1)^{\ell_0+\ell_1} \mathfrak{m}_{1+\ell_0+\ell_1}(\underbrace{-b, \ldots, -b}_{\ell_1}, \mathfrak{d}(x), \underbrace{-b, \ldots, -b}_{\ell_0})$$

$$= \mathfrak{m}_1^b(\mathfrak{d}(x)).$$

$\square$

Corollary 3.20. *The subspace* $(\mathcal{C})^{-\mathfrak{d}}$ *is preserved by* $\mathfrak{m}_1^b$.

Proof. If $\mathfrak{d}(\eta) = -\eta$, then by Lemma 3.19:

$$\mathfrak{d}(\mathfrak{m}_1^b(\eta)) = \mathfrak{m}_1^b(\mathfrak{d}(\eta)) = \mathfrak{m}_1^b(-\eta) = -\mathfrak{m}_1^b(\eta).$$

$\square$

Corollary 3.21. $\mathfrak{m}_1^b(C)^{-\mathfrak{d}} = \mathfrak{m}_1^b(C^{-\mathfrak{d}})$.

Proof. If $\mathfrak{m}_1^b(x) = \eta$ for some $x \in \mathcal{C}$, then by Lemma 3.19, $\mathfrak{m}_1^b(\mathfrak{d}(x)) = \mathfrak{d}(\eta) = -\eta$. Thus $y = \frac{x - \mathfrak{d}(x)}{2} \in \mathcal{C}$ satisfies $\mathfrak{d}(y) = -y$ and

$$\mathfrak{m}_1^b(y) = \frac{\mathfrak{m}_1^b(x) - \mathfrak{m}_1^b(\mathfrak{d}(x))}{2} = \frac{\eta - (-\eta)}{2} = \eta.$$

$\square$

3.7. **Obstruction theory.** Let $(\mathcal{C}, \{\mathfrak{m}_k\}_{k \geq 0}, \prec, \succ, \mathbf{e})$ be an n-dimensional curved cyclic unital A_∞ algebra complete with respect to its filtration ς_1. Let ς_2 be another filtration defined on $\mathcal{C}$, and assume that $(\mathcal{C}, \{\mathfrak{m}_k\}_{k \geq 0}, \prec, \succ, \mathbf{e})$ is pseudo-complete with respect to ς_2. Recall the definitions of the modules $\mathcal{C}_E$ from Definition 2.3. Assume that the left and right module structures of $\mathcal{C}[1]$ over $Z(\mathcal{R})$ coincide. Given b of degree 1 with $\varsigma_2(b) > 0$, then $\mathfrak{m}_1^b$ is a well defined degree 1 operator on $\mathcal{C}$. It follows from the A_∞ relation that for $\alpha \in \mathcal{C}$ we have $\mathfrak{m}_1^b \circ \mathfrak{m}_1^b(\alpha) = \mathfrak{m}_2^b(\mathfrak{m}_0^b, \alpha) + (-1)^{(|\alpha|+1)(|\mathfrak{m}_0^b|+1)} \mathfrak{m}_2^b(\alpha, \mathfrak{m}_0^b)$. So, if $\mathfrak{m}_0^b = 0$ then $\mathfrak{m}_1^b \circ \mathfrak{m}_1^b$ is a boundary operator. Thus it is natural to consider the **Maurer-Cartan equation**,

$$\mathfrak{m}_0^b = 0, \qquad b \in \mathcal{C}^1, \qquad \varsigma_2(b) > 0.$$

Define the $\mathcal{R}$ module

$$\mathcal{C}_{\not e} := \frac{\mathcal{C}}{Z(\ker d_{\mathcal{R}}) \cdot e}.$$

Lemma 3.22. *Let* $(\mathcal{C}, \{\mathfrak{m}_k\}_{k \geq 0}, \prec, \succ, \mathbf{e})$ *be an n-dimensional curved cyclic unital A_∞ algebra. Then the induced operators* $\mathfrak{m}_k : \mathcal{C}_{\not e}^{\otimes k} \to \mathcal{C}_{\not e}$ *given by*

$$\mathfrak{m}_k([\alpha_1], \ldots, [\alpha_k]) = [\mathfrak{m}_k(\alpha_1, \ldots, \alpha_k)],$$

are well defined for $k \neq 2$.

Proof. Follows form the properties of a cyclic A_∞ algebra given in Definition 2.11. $\square$

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Definition 3.23. Let $(\mathcal{C}, \{\mathfrak{m}_k\}_{k\geq 0}, \prec, \succ, \mathbf{e})$ be an n-dimensional curved cyclic A_∞ algebra. An element $b \in \mathcal{C}$ of degree 1 with $\varsigma_2(b) > 0$ is a **bounding cochain** if it solves the Maurer-Cartan equation in the quotient modules $\mathcal{C}_{\not\varsigma}$ and $\overline{\mathcal{C}}$.

Remark 3.24. In the literature the term bounding cochain is sometimes reserved for a solution of the Maurer-Cartan equation in $\mathcal{C}$ and the definition above corresponds to weak bounding chains. For $b \in \mathcal{C}^1$ a bounding chain, we do not assume that $\varsigma_1(b) > 0$. So, the condition that b solves the Maurer-Cartan equation in the quotient module $\overline{\mathcal{C}}$ is not automatic. If the filtration ς_1 is trivial, then the requirement for a bounding cochain to solve the Maurer-Cartan equation in the quotient module $\mathcal{C}_{\not\varsigma}$ is trivial.

Let $\mathfrak{d} : \mathcal{C} \to \mathcal{C}$ be an involution, and assume that $(\mathcal{C}, \mathfrak{m})$ is $\mathfrak{d}$ self-dual. Let κ denote $-\mathfrak{d}$ or the identity map on $\mathcal{C}$. Let $b \in \mathcal{C}^\kappa$ of degree 1 with $\varsigma_2(b) > 0$ be a solution to the Maurer-Cartan equation in the quotient in the quotient module $\overline{\mathcal{C}}$. Recall the definitions of $\mathfrak{m}_{k,E}^b$ from Section 2.2.

Lemma 3.25. *Let $b \in \mathcal{C}^\kappa$ be a bounding cochain for the induced A_∞ algebra $\overline{\mathcal{C}}$. Then, the induced operator $\mathfrak{m}_{1,E}^b$ on $\mathcal{C}_E^\kappa$ is a boundary operator.*

Proof. By the assumption $\overline{\mathfrak{m}_0^b} = 0$. Let α be an element of $F^{\geq E}\mathcal{C}$ representing an element of $\mathcal{C}_E^\kappa$. It follows from properties (1),(2),(5) of the cyclic A_∞ algebra that

$$-\mathfrak{m}_1^b(\mathfrak{m}_1^b(\alpha)) = \mathfrak{m}_2^b(\mathfrak{m}_0^b, \alpha) + (-1)^{(|\alpha|+1)}\mathfrak{m}_2^b(\alpha, \mathfrak{m}_0^b).$$

Observe that $\varsigma_1(\mathfrak{m}_0^b) > 0$. So it follows from property (3) of the cyclic A_∞ structure of $\mathcal{C}$ that

$$\varsigma_1\left(\mathfrak{m}_2^b(\mathfrak{m}_0^b, \alpha) + (-1)^{(|\alpha|+1)}\mathfrak{m}_2^b(\alpha, \mathfrak{m}_0^b)\right) > E.$$

The claim now follows from the definition of $\mathcal{C}_E$, when κ is the identity. When $\kappa = -\mathfrak{d}$, we also need to show that $\mathfrak{m}_1^b$ preserves $\mathcal{C}_E^\kappa$, and this follows from Corollary 3.20. $\qquad\square$

Observe that property (3) implies that the induced operators $\{\mathfrak{m}_k\}_{k\geq 0}$ on the quotient module $\frac{\mathcal{C}}{F^{>E_\ell}\mathcal{C}}$ are well defined, and they define an A_∞ algebra on $\frac{\mathcal{C}}{F^{>E_\ell}\mathcal{C}}$. Property (5) combined with property (3) of Definition 2.11 implies that the induced A_∞ algebra on $\frac{\mathcal{C}}{F^{>E_\ell}\mathcal{C}}$ is cyclic unital. Assume that $\mathrm{Im}\,\varsigma_1$ can be ordered $\{E_0 < E_1 < E_2 < \ldots\} \subset \mathbb{R}$. Let $b \in \mathcal{C}^\kappa$ be a bounding cochain for the induced A_∞ algebra $\frac{\mathcal{C}}{F^{>E_\ell}\mathcal{C}}$. It follows from the discreteness of $\mathrm{Im}\,\varsigma_1$ and Lemma 3.18 that $\mathfrak{m}_0^b \in F^{E_{\ell+1}}\mathcal{C}_{\not\varsigma}^\kappa$. Let $\mathfrak{m}_{0,E_{\ell+1}}^b \in \mathcal{C}_{\not\varsigma,E_{\ell+1}}^\kappa$ denote the image of $\mathfrak{m}_0^b$.

Lemma 3.26. *We have $\mathfrak{m}_{1,E_{\ell+1}}^b(\mathfrak{m}_{0,E_{\ell+1}}^b) = 0$.*

Proof. It follows from the A_∞ relations that $\mathfrak{m}_1^b(\mathfrak{m}_0^b) = 0$. So, by the definition of the induced operator $\mathfrak{m}_{1,E_{\ell+1}}^b(\mathfrak{m}_{0,E_{\ell+1}}^b) = 0$. $\qquad\square$

Let $b \in \mathcal{C}^\kappa$ be a bounding cochain for the induced A_∞ algebra $\frac{\mathcal{C}}{F^{>E_\ell}\mathcal{C}}$. Abbreviate by $\mathfrak{o}$ the class of $\mathfrak{m}_{0,E_{\ell+1}}^b$ in $H^2\left(\mathcal{C}_{\not\varsigma,E_{\ell+1}}^\kappa, \mathfrak{m}_1^b\right)$.

Lemma 3.27. *Suppose $\mathfrak{o} = 0$. Then there exists $\xi \in F^{E_{\ell+1}}\mathcal{C}^\kappa$ of degree 1 such that if we set*

$$b' = b + \xi,$$

then $b' \in \mathcal{C}^\kappa$ is a bounding cochain for the induced A_∞ algebra $\frac{\mathcal{C}}{F^{>E_{\ell+1}}}$.

Proof. Since $\mathfrak{o} = 0$, we can choose $\xi \in F^E \mathcal{C}^\kappa$ such that $\mathfrak{m}_1^b(\xi) \equiv -\mathfrak{m}_0^b \pmod{F^{>E}\mathcal{C}_{\mathcal{C}}}$. It follows from Property (3) of Definition 2.11 that

$$\mathfrak{m}_0^{b+\xi} \equiv \mathfrak{m}_0^b + \mathfrak{m}_1^b(\xi) \equiv 0 \pmod{F^{>E}\mathcal{C}_{\mathcal{C}}}.$$

$\square$

Lemma 3.28. *Let $(\mathcal{C}, \{\mathfrak{m}_k\}_{k\geq 0}, \prec, \succ, \mathbf{e})$ be a cyclic unital A_∞ algebra over $\mathcal{R}$ such that the residue algebra $(\bar{\mathcal{C}}, \{\bar{\mathfrak{m}}_k\}_{k\geq 0}, \prec, \succ, \mathbf{e})$ arises from a dga. Assume that $\mathcal{C}$ has an $\bar{\mathcal{R}}$ structure given by $g : \mathcal{R} \otimes \bar{\mathcal{C}} \to \mathcal{C}$ and let f be its inverse. Let b be a bounding cochain for the induced A_∞ algebra $\bar{\mathcal{C}}$. Then, f induces an isomorphism of cochain complexes between $\left(\mathcal{C}_E, \mathfrak{m}_{1,E}^b\right)$ and $\left(\mathcal{R}_E \otimes \bar{\mathcal{C}}, d_{\mathcal{R}} \otimes \mathrm{Id} + \mathrm{Id} \otimes (\bar{\mathfrak{m}}_1)^{\bar{b}}\right).$*

Proof. It follows from the definition of an $\bar{\mathcal{R}}$ structure, Definition 2.4, that f induces an isomorphism of $\bar{\mathcal{R}}$-modules

$$\mathcal{C}_E \to \mathcal{R}_E \otimes \bar{\mathcal{C}}.$$

It follows from the definition of the residue algebra and Lemma 3.12 that $\overline{\mathfrak{m}_k^b} = (\bar{\mathfrak{m}}_k)^{\bar{b}}$. We can use the explicit formula given in Lemma 3.12 to deduce that $(\bar{\mathfrak{m}}_k)^{\bar{b}} = \bar{\mathfrak{m}}_k$ for $k \neq 1$, as $(\bar{\mathcal{C}}, \{\bar{\mathfrak{m}}_k\}_{k\geq 0}, \prec, \succ, \mathbf{e})$ arises from a dga. It follows that the residue algebra of $(\mathcal{C}, \{\mathfrak{m}_k^b\}_{k\geq 0}, \prec, \succ, \mathbf{e})$ arises from a dga. So, Proposition 2.13 implies that f is a chain map, and the result follows. $\square$

3.8. **The ring $\mathbb{R}\langle s \rangle$.** Define $S := \mathbb{R}\langle s \rangle$ to be the ring of non-commutative polynomials in one variable where the grading on S is given by $|s| = 1 - n$. Define a filtration $\nu_S : S \to \mathbb{R}_{0\geq} \cup \{\infty\}$ by

$$\nu_S\left(\sum_{i=0}^m a_i s^i\right) = \inf_{a_i \neq 0} -|s^i| = \inf_{a_i \neq 0} i(n-1).$$

Recall that $n > 1$, so this is not the trivial filtration.

Define a $\mathbb{Z}_2$ grading on S by, for homogeneous elements $a \in S$,

$$|a|_{\mathbb{Z}_2} = |a| \mod 2.$$

Let Υ be a graded commutative algebra, and write $\hat{\Upsilon} := S \otimes \Upsilon$. We equip $\hat{\Upsilon}$ with the $\mathbb{Z}_2$ grading induced by the $\mathbb{Z}_2$ grading of S, and we denote it also by $|\cdot|_{\mathbb{Z}_2}$. For example if s is a homogeneous element in S, then for any $r \in \Upsilon$ we have

$$|s \otimes r|_{\mathbb{Z}_2} \equiv |s| \mod 2.$$

Lemma 3.29. *Let $s_1, s_2 \in S$ and $\alpha, \xi \in \Upsilon$ homogeneous elements. Then the multiplication on $\hat{\Upsilon}$ satisfies,*

$$(s_1 \otimes \alpha)(s_2 \otimes \xi) = (-1)^{(|\alpha \otimes s_1|)(|\xi \otimes s_2|) + |s_1|_{\mathbb{Z}_2}|s_2|_{\mathbb{Z}_2}} (s_2 \otimes \xi)(s_1 \otimes \alpha).$$

Proof. We compute the product of the elements $s_1 \otimes \alpha, s_2 \otimes \xi \in S \otimes \Upsilon$,

$$(s_1 \otimes \alpha)(s_2 \otimes \xi) = (-1)^{|\alpha||s_2|} s_1 s_2 \otimes (\alpha\xi) = (-1)^{|\alpha||s_2| + |\xi||\alpha|} s_1 s_2 \otimes (\xi\alpha).$$

If we change the order of multiplication, we get

$$(s_2 \otimes \xi)(s_1 \otimes \alpha) = (-1)^{|\xi||s_1|} s_2 s_1 \otimes (\xi\alpha).$$

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Remembering that in S we have $s_1 s_2 = s_2 s_1$, we conclude that

$$(s_1 \otimes \alpha)(s_2 \otimes \xi) = (-1)^{(|\alpha \otimes s_1|)(|\xi \otimes s_2|) + |s_1|_{\mathbb{Z}_2}|s_2|_{\mathbb{Z}_2}}(s_2 \otimes \xi)(s_1 \otimes \alpha).$$

$\square$

Corollary 3.30. *For* $\alpha_1, \alpha_2 \in \hat{\Upsilon}$

$$[\alpha_1, \alpha_2] = \begin{cases} 0, & |\alpha_1|_{\mathbb{Z}_2} \cdot |\alpha_2|_{\mathbb{Z}_2} = 0 \\ 2\alpha_1 \cdot \alpha_2, & otherwise. \end{cases} \tag{16}$$

It follows that the center of $\hat{\Upsilon}$ *is given by*

$$Z(\hat{\Upsilon}) = \{x \in \hat{\Upsilon} \mid |x|_{\mathbb{Z}_2} = 0\}.$$

We define the $\mathbb{R}$ module homomorphism $F : S \to S$ to be the projection to the odd part. Concretely, the morphism is given on homogeneous elements by

$$F(as^i) = \begin{cases} 0, & i \equiv 0 \mod 2, \\ as^i, & otherwise. \end{cases}$$

Lemma 3.31. *The homomorphism* $F : S \to S$ *is a graded cyclic* $Z(S)$ *module homomorphism.*

Proof. Let $s_1, \ldots, s_n \in S$ be homogeneous elements. If $s_1 \cdots s_n$ is an odd element in S then $|s_1 \cdots s_{n-1}|_{\mathbb{Z}_2}|s_n|_{\mathbb{Z}_2} = 0$. Thus

$$F(s_1 \cdots s_n) = s_1 \cdots s_n = s_n \cdot s_1 \cdots s_{n-1} = (-1)^{|s_n| \sum_{i=1}^{n-1} |s_i|} F(s_n \cdot s_1 \cdots s_{n-1}).$$

If $s_1 \cdots s_n$ is an even element in S then the equality above also holds. Hence $F : S \to S$ is a graded cyclic homomorphism. We use Corollary 3.30 to deduce that $F : S \to S$ is a $Z(S) - \mod$ homomorphism. $\square$

4. FUKAYA A_∞ ALGEBRAS

In this section we recall definitions and results concerning Fukaya A_∞-algebras from [36]. The notation, as well as sign and orientation conventions, are the same as in [36]. Then, using the results of Section 2.3, we extend scalars by the non-commutative ring $S = R\langle s \rangle$ in Section 4.3. In Section 4.3.3, we restrict scalars to a sababa A_∞ algebra so that the scalar extension by S becomes pseudo-complete (Proposition 3.8). This pseudo-completeness is the key property that allows us in Section 4.4 to use the notions of bounding cochains developed in Section 3.7 in the setting of Fukaya A_∞-algebras. Section 4.5 introduces the discrete filtration and finite energy quotients, which provide the technical framework needed to adapt the abstract obstruction theory from Section 3.7 to the Fukaya A_∞ context of Section 6. When an anti-symplectic involution is present, Section 4.6 establishes the self-duality properties required for the real case.

4.1. **Construction.** Throughout this work (X, ω) is a symplectic manifold, $L \subset X$ is a connected Lagrangian submanifold with relative spin structure $\mathfrak{s}$, and J is an ω-tame almost complex structure. Let $\dim_{\mathbb{R}} X = 2n$. Write $\beta_0 := 0 \in \Pi$.

A J-holomorphic genus-0 open stable map to (X, L) of degree $\beta \in \Pi$ with one boundary component, $k + 1$ boundary marked points, and l interior marked points, is a quadruple

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$(\Sigma, u, \vec{z}, \vec{w})$ as follows. The domain Σ is a genus-0 nodal Riemann surface with boundary consisting of one connected component. The map

$$u : (\Sigma, \partial\Sigma) \to (X, L)$$

is continuous, J-holomorphic on each irreducible component of Σ, and $[u_*([\Sigma, \partial\Sigma])] = \beta$. The marked points are denoted by

$$\vec{z} = (z_0, \ldots, z_k), \qquad \vec{w} = (w_1, \ldots, w_l),$$

with $z_j \in \partial\Sigma$, $w_j \in int(\Sigma)$, distinct. The labeling of the marked points z_j respects the cyclic order given by the orientation of $\partial\Sigma$ induced by the complex orientation of Σ. Stability means that if Σ_i is an irreducible component of Σ, then either $u|_{\Sigma_i}$ is non-constant, or the combined number of marked points and nodal points on Σ_i is at least 3. An isomorphism of open stable maps $(\Sigma, u, \vec{z}, \vec{w})$ and $(\Sigma', u', \vec{z}', \vec{w}')$ is a homeomorphism $\theta : \Sigma \to \Sigma'$, biholomorphic on each irreducible component, such that

$$u = u' \circ \theta, \qquad z'_j = \theta(z_j), \quad j = 0, \ldots, k, \qquad w'_j = \theta(w_j), \quad j = 1, \ldots, l.$$

Let $\mathcal{M}_{k+1,l}(\beta) = \mathcal{M}_{k+1,l}(\beta; J)$ denote the moduli space of J-holomorphic genus zero open stable maps to (X, L) of degree β with one boundary component, $k + 1$ boundary marked points, and l internal marked points. Let the evaluation maps

$$evb_j^\beta : \mathcal{M}_{k+1,l}(\beta) \to L, \qquad j = 0, \ldots, k,$$
$$evi_j^\beta : \mathcal{M}_{k+1,l}(\beta) \to X, \qquad j = 1, \ldots, l,$$

be given by $evb_j^\beta((\Sigma, u, \vec{z}, \vec{w})) = u(z_j)$ and $evi_j^\beta((\Sigma, u, \vec{z}, \vec{w})) = u(w_j)$. We may omit the superscript β when the omission does not create ambiguity.

We assume that $\mathcal{M}_{k+1,l}(\beta)$ is a smooth orbifold with corners and that evb_0^β is a proper submersion.

Let $\gamma \in \mathcal{I}_Q D$ be a closed form with $|\gamma| = 2$. For example, given closed differential forms $\gamma_j \in A^*(X, L)$ for $j = 0, \ldots, N$, take t_j of degree $2 - |\gamma_j|$ and $\gamma := \sum_{j=0}^N t_j \gamma_j$. Define

$$\mathfrak{m}_k^\gamma : C^{\otimes k} \longrightarrow C$$

by

$$\mathfrak{m}_k^\gamma(\alpha_1, \ldots, \alpha_k) :=$$

$$= \delta_{k,1} \cdot d\alpha_1 + (-1)^{\varepsilon(\alpha)} \sum_{\substack{\beta \in \Pi \\ l \geq 0}} T^\beta \frac{1}{l!} evb_{0*}^\beta (\bigwedge_{j=1}^l (evi_j^\beta)^* \gamma \wedge \bigwedge_{j=1}^k (evb_j^\beta)^* \alpha_j).$$

with

$$\varepsilon(\alpha) := \sum_{j=1}^k j(|\alpha_j| + 1) + 1.$$

The push-forward $(evb_0)_*$ is defined by integration over the fiber; it is well-defined because evb_0 is a proper submersion.

Denote by $\langle\ ,\ \rangle$ the signed Poincaré pairing

$$\langle \xi, \eta \rangle := (-1)^{|\eta|} \int_L \xi \wedge \eta. \tag{17}$$

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Equip C with the diagonal bimodule structure over R with respect to grading of $C[1]$.

Theorem 4.1 ([36, Theorem 1]). $(C, \{\mathfrak{m}_k^\gamma\}_{k \geq 0}, \langle \, , \, \rangle, 1)$ *is a cyclic unital A_∞ algebra in the sense of Definition 2.11.*

For $\alpha \in C$ and $k \in \mathbb{N}$ recall from Section 1.3 the notation $(\alpha)_k$ for the component of differential form degree k.

Proposition 4.2 (Top degree [36, Proposition 3.12]). *Let $\alpha_1, \ldots \alpha_k \in C$ then*

$$(\mathfrak{m}_k^\gamma(\alpha_1, \ldots, \alpha_k))_n = \begin{cases} (\gamma|_L)_n, & k = 0, \\ (d\alpha_1)_n, & k = 1, \\ ((-1)^{|\alpha_1|}\alpha_1 \wedge \alpha_2)_n, & k = 2, \\ 0, & otherwise. \end{cases}$$

In light of Definition 2.12, we can reformulate [36, Theorem 3-(3)] as follows.

Proposition 4.3. *The A_∞ algebra $(\overline{C}, \{\bar{\mathfrak{m}}_k^\gamma\}_{k \geq 0}, \langle \, , \, \rangle, 1)$ (resp. $(\widetilde{\mathfrak{C}}, \{\tilde{\bar{\mathfrak{m}}}_k^{\tilde\gamma}\}_{k \geq 0}, \langle \, , \, \rangle, 1)$) arises from the dga of differential forms $(A^*(L), d)$ (resp. $(A^*(I \times L), d)$).*

4.2. Pseudoisotopies. Set

$$\mathfrak{C} := A^*(I \times L; R), \qquad \mathfrak{D} := A^*(I \times X, I \times L; Q), \qquad \mathfrak{R} := A^*(I; R). \tag{18}$$

The filtration ν induces filtrations on $\mathfrak{R}, \mathfrak{C}$, and $\mathfrak{D}$, which we still denote by ν. For $t \in [0, 1]$ and letting M be either L or the point, denote by

$$j_t : M \to [0, 1] \times M$$

the inclusion $j_t(p) = (t, p)$. We construct a family of A_∞ structures on $\mathfrak{C}$. Let $\{J_t\}_{t \in I}$ be a path in $\mathcal{J}$ from $J = J_0$ to $J' = J_1$. For each β, k, l, set

$$\widetilde{\mathcal{M}}_{k+1,l}(\beta) := \{(t, u, \vec{z}, \vec{w}) \,|\, (u, \vec{z}, \vec{w}) \in \mathcal{M}_{k+1,l}(\beta; J_t)\}.$$

We have evaluation maps

$$\widetilde{evb}_j : \widetilde{\mathcal{M}}_{k+1,l}(\beta) \longrightarrow I \times L, \quad j \in \{0, \ldots, k\},$$

$$\widetilde{evb}_j(t, u, \vec{z}, \vec{w}) := (t, u(z_j)),$$

and

$$\widetilde{evi}_j : \widetilde{\mathcal{M}}_{k+1,l}(\beta) \longrightarrow I \times X, \quad j \in \{1, \ldots, l\}$$

$$\widetilde{evi}_j(t, u, \vec{z}, \vec{w}) := (t, u(w_j)).$$

It follows from the assumption on $\mathcal{J}$ that all $\widetilde{\mathcal{M}}_{k+1,l}(\beta)$ are smooth orbifolds with corners, and $\widetilde{evb}_0$ is a proper submersion.

Example 4.4. In the special case when $J_t = J = J'$ for all $t \in I$, we have $\widetilde{\mathcal{M}}_{k+1,l}(\beta) = I \times \mathcal{M}_{k+1,l}(\beta; J)$. The evaluation maps in this case are $\widetilde{evb}_j = \mathrm{Id} \times evb_j$ and $\widetilde{evi}_j = \mathrm{Id} \times evi_j$.

Let

$$p : I \times L \longrightarrow I, \qquad p_{\mathcal{M}} : \widetilde{\mathcal{M}}_{k+1,l}(\beta) \longrightarrow I$$

denote the projections.

For each closed $\tilde{\gamma} \in \mathcal{I}_Q \mathfrak{D}$ with $|\tilde{\gamma}| = 2$, define structure maps

$$\tilde{\mathfrak{m}}_k^{\tilde{\gamma}} : \mathfrak{C}^{\otimes k} \longrightarrow \mathfrak{C}$$

by

$$\tilde{\mathfrak{m}}_k^{\tilde{\gamma}}(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_k) :=$$

$$= \delta_{k,1} \cdot d\tilde{\alpha}_1 + (-1)^{\varepsilon(\tilde{\alpha})} \sum_{\substack{\beta \in \Pi \\ l \geq 0}} T^\beta \frac{1}{l!} \left(\widetilde{evb}_0^\beta \right)_* (\bigwedge_{j=1}^l (\widetilde{evi}_j^\beta)^* \tilde{\gamma} \wedge \bigwedge_{j=1}^k (\widetilde{evb}_j^\beta)^* \tilde{\alpha}_j).$$

Define a pairing on $\mathfrak{C}$:

$$\langle\!\langle \, , \, \rangle\!\rangle : \mathfrak{C} \otimes \mathfrak{C} \longrightarrow \mathfrak{R}, \qquad \langle\!\langle \tilde{\xi}, \tilde{\eta} \rangle\!\rangle := (-1)^{|\tilde{\eta}|} p_*(\tilde{\xi} \wedge \tilde{\eta}).$$

Note that

$$\langle\!\langle \tilde{\xi}, \tilde{\eta} \rangle\!\rangle = (-1)^{|\tilde{\eta}|} p_*(\tilde{\xi} \wedge \tilde{\eta}) = (-1)^{|\tilde{\eta}| + |\tilde{\eta}| \cdot |\tilde{\xi}|} p_*(\tilde{\eta} \wedge \tilde{\xi}) = (-1)^{(|\tilde{\eta}|+1)(|\tilde{\xi}|+1)+1} \langle\!\langle \tilde{\eta}, \tilde{\xi} \rangle\!\rangle. \qquad (19)$$

Equip $\mathfrak{C}$ with the diagonal bimodule structure over $\mathfrak{R}$ with respect to grading of $\mathfrak{C}[1]$. In the proof of [36, Theorem 2], the following theorem is proved.

Theorem 4.5. $(\mathfrak{C}, \{\tilde{\mathfrak{m}}_k^{\tilde{\gamma}}\}_{k \geq 0}, \langle\!\langle \, , \, \rangle\!\rangle, 1)$ *is a cyclic unital* A_∞ *algebra for each closed* $\tilde{\gamma} \in \mathcal{I}_Q \mathfrak{D}$ *with* $|\tilde{\gamma}| = 2$.

Definition 4.6. Let $S_1 = (\mathfrak{m}, \prec \, , \succ, \mathbf{e})$ and $S_2 = (\mathfrak{m}', \prec \, , \succ', \mathbf{e}')$ be cyclic unital A_∞ structures on C. A cyclic unital **pseudoisotopy** from S_1 to S_2 is a cyclic unital A_∞ structure $(\tilde{\mathfrak{m}}, \preccurlyeq \, , \succcurlyeq, \tilde{\mathbf{e}})$ on the $\mathfrak{R}$-module $\mathfrak{C}$ such that for all $\tilde{\alpha}_j \in \mathfrak{C}$ and all $k \geq 0$,

$$j_0^* \tilde{\mathfrak{m}}_k(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_k) = \mathfrak{m}_k(j_0^* \tilde{\alpha}_1, \ldots, j_0^* \tilde{\alpha}_k),$$
$$j_1^* \tilde{\mathfrak{m}}_k(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_k) = \mathfrak{m}'_k(j_1^* \tilde{\alpha}_1, \ldots, j_1^* \tilde{\alpha}_k),$$

and

$$j_0^* \preccurlyeq \tilde{\alpha}_1, \tilde{\alpha}_2 \succcurlyeq \, = \, \prec j_0^* \tilde{\alpha}_1, j_0^* \tilde{\alpha}_2 \succ, \qquad j_0^* \tilde{\mathbf{e}} = \mathbf{e},$$
$$j_1^* \preccurlyeq \tilde{\alpha}_1, \tilde{\alpha}_2 \succcurlyeq \, = \, \prec j_1^* \tilde{\alpha}_1, j_1^* \tilde{\alpha}_2 \succ', \qquad j_1^* \tilde{\mathbf{e}} = \mathbf{e}'.$$

Write $\mathfrak{m}^{\gamma'}$ for the operations defined in Section 4.1 using the almost complex structure J' and a closed form $\gamma' \in \mathcal{I}_Q D$ with $\deg_D \gamma' = 2$.

Theorem 4.7 ([36, Theorem 2]). *Assume* $[\gamma] = [\gamma'] \in \widehat{H}^*(X, L; Q)$, *and let* $\xi \in (\mathcal{I}_Q D)_1$ *be such that*

$$\gamma' - \gamma = d\xi$$

and define

$$\tilde{\gamma} := \gamma + t(\gamma' - \gamma) + dt \wedge \xi \in (\mathcal{I}_Q \mathfrak{D})_2. \qquad (20)$$

Then, $(\tilde{\mathfrak{m}}^{\tilde{\gamma}}, \langle\!\langle \, , \, \rangle\!\rangle, 1)$ *is a pseudoisotopy from* $(\mathfrak{m}^\gamma, \langle \, , \, \rangle, 1)$ *to* $(\mathfrak{m}^{\gamma'}, \langle \, , \, \rangle, 1)$.

Proposition 4.8 (Top degree [36, Proposition 4.17]). *Let* $\tilde{\alpha}_1, \ldots \tilde{\alpha}_k \in C$ *then*

$$\left(\tilde{\mathfrak{m}}_k^{\tilde{\gamma}}(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_k) \right)_{n+1} = \begin{cases} (\tilde{\gamma}|_{I \times L})_{n+1}, & k = 0, \\ (d\tilde{\alpha}_1)_{n+1}, & k = 1, \\ \left((-1)^{|\tilde{\alpha}_1|} \tilde{\alpha}_1 \wedge \tilde{\alpha}_2 \right)_{n+1}, & k = 2, \\ 0, & otherwise. \end{cases}$$

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Proposition 4.9 ([36, Proposition 4.15]). *The A_∞ algebra $(\overline{\mathfrak{C}}, \{\bar{\tilde{\mathfrak{m}}}_k^{\tilde{\gamma}}\}_{k\geq 0}, \langle\ ,\ \rangle, 1)$ arises from the dga of differential forms $(A^*(I \times L), d)$.*

Lemma 4.10 ([36, Lemma 4.9]). *For any $\tilde{\xi}, \tilde{\eta} \in A^*(I \times L)$,*

$$(-1)^{\deg \tilde{\xi} + \deg \tilde{\eta} + n} \int_I d\langle\!\langle \tilde{\xi}, \tilde{\eta} \rangle\!\rangle = \langle j_1^* \tilde{\xi}, j_1^* \tilde{\eta} \rangle - \langle j_0^* \tilde{\xi}, j_0^* \tilde{\eta} \rangle.$$

4.3. Extension of scalars.

4.3.1. *Definitions.* Recall the definitions of $R, C, \mathfrak{R}, \mathfrak{C}$ from Section 1.2.1, and (18). Also recall the Definition 2.4 of an $\bar{\mathcal{N}}$ structure on a filtered commutative unital ring $\mathcal{N}$, and an $\bar{\mathcal{N}}$ structure on a filtered $\mathcal{N}$ module $\mathcal{C}$. In order to define an $\bar{R}$ structure on R, observe that we have an isomorphism $\mathbb{R} \xrightarrow{\sim} \bar{R}$ given by $a \mapsto \overline{aT^{\beta_0}}$. As R is a unital algebra over $\mathbb{R}$ there exist a canonical map $k : \mathbb{R} \to R$. It is straightforward to check that

$$\bar{R} \simeq \mathbb{R} \xrightarrow{k} R$$

defines a $\bar{R}$ structure on R. We have an isomorphism $A^*(L) \xrightarrow{\sim} \overline{C}$ given by the inclusion, and the composition

$$R \otimes_{\bar{R}} \overline{C} \simeq R \otimes_{\mathbb{R}} A^*(L) \simeq C$$

defines a $\bar{R}$ structure on C. We may also think of $(\mathfrak{C}, \{\tilde{\mathfrak{m}}_k^{\tilde{\gamma}}\}_{k\geq 0}, \langle\!\langle\ ,\ \rangle\!\rangle, 1)$ as a curved cyclic A_∞ algebra over R using the homomorphism $R \to \mathfrak{R}$. With this ground diffrential graded algebra on $\mathfrak{C}$. So, with the same $\bar{R}$ structure on R we define the $\bar{R}$ structure on $\mathfrak{C}$ by

$$R \otimes_{\bar{R}} \overline{\mathfrak{C}} \simeq R \otimes_{\mathbb{R}} A^*(I \times L) \simeq \mathfrak{C}.$$

4.3.2. *Extension of scalars.* Recall the Definition 2.16 concerning (S, F) extensions and take S and F as in Section 3.8. Recall the definition of the cyclic unital A_∞ algebra $(\hat{C}, \{\hat{\mathfrak{m}}_k^\gamma\}_{k\geq 0}, \langle\ ,\ \rangle_F, 1 \otimes 1)$ from Section 1.2.1 and observe it is the (S, F) extension of $(C, \{\mathfrak{m}_k^\gamma\}_{k\geq 0}, \langle\ ,\ \rangle, 1)$ from Theorem 4.1. Similarly, define $(\hat{C}, \{\hat{\mathfrak{m}}_k^{\gamma'}\}_{k\geq 0}, \langle\ ,\ \rangle_F, 1 \otimes 1)$ to be the (S, F) extension of $(C, \{\mathfrak{m}_k^{\gamma'}\}_{k\geq 0}, \langle\ ,\ \rangle, 1)$ from Theorem 4.7, and define $(\hat{\mathfrak{C}}, \{\hat{\tilde{\mathfrak{m}}}_k^{\tilde{\gamma}}\}_{k\geq 0}, \langle\!\langle\ ,\ \rangle\!\rangle_F, 1)$ to be the (S, F) extension of $(\mathfrak{C}, \{\tilde{\mathfrak{m}}_k^{\tilde{\gamma}}\}_{k\geq 0}, \langle\!\langle\ ,\ \rangle\!\rangle, 1)$ from Theorem 4.5. Then, the following is a corollary of Proposition 2.15.

Proposition 4.11. $(\hat{C}, \{\hat{\mathfrak{m}}_k^\gamma\}_{k\geq 0}, \langle\ ,\ \rangle_F, 1)$ *and* $(\hat{\mathfrak{C}}, \{\hat{\tilde{\mathfrak{m}}}_k^{\tilde{\gamma}}\}_{k\geq 0}, \langle\!\langle\ ,\ \rangle\!\rangle_F, 1)$ *are n-dimensional cyclic unital A_∞ algebras.*

The following is a corollary of Proposition 2.15 and Theorem 4.7.

Lemma 4.12. $(\hat{\mathfrak{C}}, \{\hat{\tilde{\mathfrak{m}}}_k^{\tilde{\gamma}}\}_{k\geq 0}, \langle\!\langle\ ,\ \rangle\!\rangle_F, 1 \otimes 1)$ *is a pseudoisotopy from* $(\hat{C}, \{\hat{\mathfrak{m}}_k^\gamma\}_{k\geq 0}, \langle\ ,\ \rangle_F, 1 \otimes 1)$ *to* $(\hat{C}, \{\hat{\mathfrak{m}}_k^{\gamma'}\}_{k\geq 0}, \langle\ ,\ \rangle_F, 1 \otimes 1)$.

4.3.3. *Sababa property.* Let Θ be a compact metric space and $\Theta \to \mathcal{J}(X, \omega) : t \mapsto J_t$ be a continuous function with respect to the C^∞-topology on $\mathcal{J}(X, \omega)$. By Gromov compactness, for a fixed $E \in \mathbb{R}$ the set

$$\Pi_E := \{\beta \mid \omega(\beta) \leq E, \exists t \in \Theta : \mathcal{M}_{3,0}(\beta, J_t) \neq \emptyset\}$$

is finite. Thus we can order $\Pi_\infty = \cup_E \Pi_E$ as a list $\beta_0, \beta_1, \ldots$, where $i < j$ implies $\omega(\beta_i) \leq \omega(\beta_j)$. As above β_0 denotes the zero element. Recall Definition 3.3 concerning the sababa property. Let $R^\diamond \subset R$ be a sababa sub-algebra of R, and assume also that $\{T^\beta\}_{\beta \in \Pi_\infty}, \{t_i\}_{i=1}^N \subset R^\diamond$.

Since $\bar{R}$ is isomorphic to $\mathbb{R}$, it follows that $\bar{R}^\diamond$ is also isomorphic to $\mathbb{R}$. Recall Definition 2.10, and define $C^\diamond := C_{R^\diamond}$ to be the $R^\diamond$ restriction of scalars of C. Let $g : R^\diamond \otimes_{\bar{R}^\diamond} \overline{C^\diamond} \to C^\diamond$ be the canonical $\bar{R}^\diamond$ structure on $C^\diamond$. Define $\mathfrak{C}^\diamond := \mathfrak{C}_{\mathfrak{R}^\diamond}$ to be the $R^\diamond$ restriction of scalars of $\mathfrak{C}$. Let $g : R^\diamond \otimes_{\bar{R}^\diamond} \overline{\mathfrak{C}^\diamond} \to \mathfrak{C}^\diamond$ be the canonical $\bar{R}^\diamond$ structure on $\mathfrak{C}^\diamond$.

Lemma 4.13. $(C^\diamond, \{\mathfrak{m}_k^\gamma\}_{k\geq 0}, \langle\,,\,\rangle, 1)$ (resp. $(\mathfrak{C}, \{\tilde{\mathfrak{m}}_k^{\tilde{\gamma}}\}_{k\geq 0}, \langle\!\langle\,,\,\rangle\!\rangle, 1)$) is a sababa cyclic unital A_∞ algebra over $R^\diamond$ (resp. $R^\diamond$).

Proof. Since, $\{T^\beta\}_{\beta\in\Pi_\infty}, \{t_i\}_{i=1}^N \subset R^\diamond$ we have $\mathfrak{m}_k^\gamma(C^{\diamond\otimes k}) \subset C^\diamond$. Thus $(C^\diamond, \{\mathfrak{m}_k^\gamma\}_{k\geq 0}, \langle\,,\,\rangle, 1)$ is a cyclic unital A_∞ algebra. We observe that the degrees of elements in $\overline{C} = A^*(L)$ are bounded by n. Thus, we have
$$|\alpha| \leq n.$$
In particular, n is a sababa constant for $C^\diamond$. The proof for $(\mathfrak{C}, \{\tilde{\mathfrak{m}}_k^{\tilde{\gamma}}\}_{k\geq 0}, \langle\!\langle\,,\,\rangle\!\rangle, 1)$ is the same but with sababa constant $n+1$. $\square$

Recall the definition of ν from Section 1.3, and also denote by ν the induced filtration on $\hat{C}$ and $\hat{\mathfrak{C}}$. Let $\Upsilon = \hat{C}$ or $\Upsilon = \hat{R}$. Decompose Υ as $\Upsilon = S \otimes_\mathbb{R} R \otimes_\mathbb{R} \Upsilon'$ where $\Upsilon' = A^*(L)$ or $\Upsilon' = \mathbb{R}$, respectively. Let $\alpha_1, \ldots, \alpha_n \in \hat{C}$. For $i = 1, \ldots, n$, write

$$\alpha_i = \sum_{j=0}^\infty s^{k_{ij}} \lambda_{ij} \alpha_{ij}, \qquad \alpha_{ij} \in \Upsilon', \quad \lambda_{ij} = T^{\beta_{ij}} \prod_{a=0}^N t_a^{l_{aij}}, \quad \lim_i \nu(\lambda_{ij}) = \infty. \qquad (21)$$

Notation 4.14. Denote by $R(\alpha_1, \ldots, \alpha_n)$ the sababa subalgebra of R generated by

$$\{\lambda_{ij}\}_{i\in[n], j\in\mathbb{N}}, \ \{T^\beta\}_{\beta\in\Pi_\infty}, \{t_i\}_{i=1}^N.$$

4.3.4. *Pseudo-completeness.*

Corollary 4.15. The A_∞ algebras $(\hat{C}^\diamond, \{\hat{\mathfrak{m}}_k^\gamma\}_{k\geq 0}, \langle\,,\,\rangle_F, 1\otimes 1)$ and $(\hat{\mathfrak{C}}^\diamond, \{\hat{\tilde{\mathfrak{m}}}_k^{\tilde{\gamma}}\}_{k\geq 0}, \langle\!\langle\,,\,\rangle\!\rangle_F, 1)$ are pseudo-complete with respect to $\nu_S \otimes \nu$ when $n > 1$.

Proof. If $n > 1$, then $(S, 1)$ is a descending unital differential graded algebra over $\mathbb{R}$. Apply Proposition 3.8. $\square$

Define $\mathfrak{R}^\diamond := R^\diamond \otimes A^*(I)$. It follows from Theorem 4.5 that $(\hat{\mathfrak{C}}^\diamond, \{\tilde{\mathfrak{m}}_k^{\tilde{\gamma}}\}_{k\geq 0}, \langle\!\langle\,,\,\rangle\!\rangle, 1)$ is a cyclic unital A_∞ algebra over $\mathfrak{R}^\diamond$. Recall Definition 3.10 concerning b deformations of an A_∞ algebra.

Corollary 4.16. Let $b \in \hat{C}^\diamond$ (resp. $b \in \hat{\mathfrak{C}}^\diamond$) with $\nu_S \otimes \nu(b) > 0$ and $|b| = 1$. Then $\hat{\mathfrak{m}}_k^{\gamma,b}(\eta_1, \ldots, \eta_k)$ (resp. $\hat{\tilde{\mathfrak{m}}}_k^{\tilde{\gamma},b}(\eta_1, \ldots, \eta_k)$) converges with respect to ν for all $\eta_1, \ldots, \eta_k \in \hat{C}^\diamond$ (resp. $\hat{\mathfrak{C}}^\diamond$). Thus, $\hat{\mathfrak{m}}_k^{\gamma,b} : \hat{C}^{\diamond\otimes k} \to \hat{C}^\diamond$ (resp. $\hat{\tilde{\mathfrak{m}}}_k^{\tilde{\gamma},b} : \hat{\mathfrak{C}}^{\diamond\otimes k} \to \hat{\mathfrak{C}}^\diamond$) is well defined. Moreover, $(\hat{C}^\diamond, \{\hat{\mathfrak{m}}_k^{\gamma,b}\}_{k\geq 0}, \langle\,,\,\rangle_F, 1\otimes 1)$ (resp. $(\hat{\mathfrak{C}}^\diamond, \{\hat{\tilde{\mathfrak{m}}}_k^{\tilde{\gamma},b}\}_{k\geq 0}, \langle\!\langle\,,\,\rangle\!\rangle_F, 1)$) is a cyclic unital A_∞ algebra over $Z(\hat{R}^\diamond)$ (resp. $Z(\hat{\mathfrak{R}}^\diamond)$).

Proof. This follows from Corollary 3.15. $\square$

Remark 4.17. If $R^\diamond$ is not sababa then $C^\diamond$ is not necessarily pseudo-complete as seen in the following example. Let $\beta_1, \beta_2, \beta_3 \in \Pi$ and assume they satisfy $\omega(\beta_1) = -\omega(\beta_2)$ and $\omega(\beta_3) > 0$. Also assume that $\mu(\beta_1) = 2n - 2$, $\mu(\beta_2) = 0$ and $\mu(\beta_3) = 2$. Then for $m \in \mathbb{N}$ we have $T^{m(\beta_1+\beta_2)+\beta_3}$ is an element of $\mathcal{I}_{\hat{R}}$. For each m we have $\nu(T^{m(\beta_1+\beta_2)+\beta_3}) = \omega(\beta_3)$, so

$\{T^{m(\beta_1+\beta_2)+\beta_3}\}_{m\in\mathbb{N}}$ is not a subset of any sababa algebra. Define $\lambda_m := s^{2m}T^{m(\beta_1+\beta_2)+\beta_3} \in \hat{R}$, and observe that $\lim_{m\to\infty} \nu_S \otimes \nu(\lambda_m) = \infty$ and $|\lambda_m| = 2$. Let η'_k be an element of $C^{\otimes k}$ with $|\eta'_k| \geq k - 2$ and $\nu(\mathfrak{m}_k^\gamma(\eta'_k)) \leq M$ for some $M \in \mathbb{R}$. We define $\eta_k := \lambda_k \otimes \eta'_k$, so

$$\lim_{k\to\infty} \nu_S \otimes \nu(\eta_k) = \infty, \qquad |\eta_k| \geq k, \qquad \nu(\hat{\mathfrak{m}}_k^\gamma(\eta_k)) \leq M + \omega(\beta_3).$$

This contradicts pseudo-completeness. Moreover, the set $\{\hat{\mathfrak{m}}_k^\gamma(\eta_k)\}_{k\in\mathbb{N}}$ is linearly independent because $s^{2m}|\hat{\mathfrak{m}}_k^\gamma(\eta_k)$ if and only if $m = k$. Thus, $\hat{\mathfrak{m}}^\gamma(\eta)$ does not converge with respect to ν.

Remark 4.18. Without Corollary 4.16, we would have $\hat{\mathfrak{m}}_0^{\gamma,b_{(l)}} \in \hat{\hat{C}}^\diamond$, where $\hat{\hat{C}}^\diamond$ is defined to be the completion with respect to $\nu_S \otimes \nu$. Then the inductive construction in the proof of Proposition 6.1 would give $b \in \hat{\hat{C}}^\diamond$, so (γ, b) would not be a unit bounding pair. If we were to relax the definition of unit bounding pairs and allow $b \in \hat{\hat{C}}^\diamond$, the inductive constructions of Section 6.2 would not work. See Remark 6.13.

4.4. Bounding cochains and gauge equivalence.

4.4.1. *Bounding pairs and b-deformations.* Recall Definition 1.1 from Section 1.2.1, and Definition 3.23. Observe that for a bounding pair (γ, b) where $b \in \hat{C}^\diamond$ we have $\nu_S \otimes \nu(b) > 0$, and $|b| = 1$. So, Corollary 4.16 implies that the deformed operator $\hat{\mathfrak{m}}_0^{\gamma,b}$ is well defined. We can use Lemma 3.12 to rewrite equation (2) as

$$\hat{\mathfrak{m}}_0^{\gamma,b} = c \cdot 1, \qquad c \in Z(\mathcal{I}_{\hat{R}}), \qquad |c| = 2.$$

All the generators of $\mathcal{I}_{\hat{R}}$ have strictly positive ν filtration. So, $\nu(\hat{\mathfrak{m}}_0^{\gamma,b}) > 0$. Hence, b is a bounding cochain for the A_∞ algebra $(\hat{C}^\diamond, \{\hat{\mathfrak{m}}_k^\gamma\}_{k\geq 0}, \langle\,,\,\rangle_F, 1)$.

In the other direction, if b is a bounding cochain for the A_∞ algebra $(\hat{C}^\diamond, \{\hat{\mathfrak{m}}_k^\gamma\}_{k\geq 0}, \langle\,,\,\rangle_F, 1)$, then (γ, b) is a bounding pair as in Definition 1.1.

Remark 4.19. The assumption that $b \in \hat{C}^\diamond$ is not restrictive. We can always write $\hat{R}^\diamond = R(b)$ in our construction of $\hat{C}^\diamond$.

4.4.2. *Gauge equivalence.*

Definition 4.20. We say a bounding pair (γ, b) with respect to J is **gauge equivalent** to a bounding pair (γ', b') with respect to J', if there exist a path $\{J_t\}$ in $\mathcal{J}$ from J to J', $\tilde{\gamma} \in (\mathcal{I}_Q\mathfrak{D})_2$ closed, and $\tilde{b} \in (\mathcal{I}_R\mathfrak{C})_1$ such that

(1) the triple $(\tilde{\mathfrak{m}}^{\tilde{\gamma}}, \langle\!\langle\,,\,\rangle\!\rangle, 1)$ is a pseudoisotopy from $(\mathfrak{m}^\gamma, \langle\,,\,\rangle, 1)$ to $(\mathfrak{m}^{\gamma'}, \langle\,,\,\rangle, 1)$;
(2) $j_0^*\tilde{b} = b, \quad j_1^*\tilde{b} = b'$;
(3) $\tilde{b}$ is a bounding cochain for $\tilde{\mathfrak{m}}^{\tilde{\gamma}}$.

In this case, we say that $(\tilde{\mathfrak{m}}^{\tilde{\gamma}}, \tilde{b})$ is a **pseudoisotopy** from $(\mathfrak{m}^\gamma, b)$ to $(\mathfrak{m}^{\gamma'}, b')$ and write $(\gamma, b) \sim (\gamma', b')$. In the special case $J_t = J = J'$, $\gamma = \gamma'$ and $\tilde{\gamma} = \pi^*\gamma$, we say b is gauge equivalent to b' as a bounding cochain for $\mathfrak{m}^\gamma$.

Remark 4.21. Suppose $(\tilde{\mathfrak{m}}^{\tilde{\gamma}}, \tilde{b})$ is a pseudoisotopy from $(\hat{\mathfrak{m}}^\gamma, b)$ to $(\hat{\mathfrak{m}}^{\gamma'}, b')$ with $\hat{\mathfrak{m}}_0^{\gamma,b} = c \cdot 1$, $\hat{\mathfrak{m}}_0^{\gamma',b'} = c' \cdot 1$ and $\tilde{\mathfrak{m}}_0^{\tilde{\gamma},\tilde{b}} = \tilde{c} \cdot 1$. The assumption $d\tilde{c} = 0$ implies $\tilde{c} = \hat{c} + f(t)\,dt$ with $\hat{c}$ constant. We get

$$j_0^*\tilde{\mathfrak{m}}_0^{\tilde{\gamma},\tilde{b}} = \hat{\mathfrak{m}}_0^{\gamma,b}, \qquad j_1^*\tilde{\mathfrak{m}}_0^{\tilde{\gamma},\tilde{b}} = \hat{\mathfrak{m}}_0^{\gamma',b'}.$$

Therefore, $c = j_0^*\tilde{c} = \hat{c} = j_1^*\tilde{c} = c'$.

We prove that the map ϱ given by (3) is well defined. The following is Lemma 3.15 of [35].

Lemma 4.22. *Let M be a manifold with $\partial M = \emptyset$ and let $\tilde{\xi} \in A^*(I \times M)$ such that $d\tilde{\xi} = 0$. Then*

$$[j_0^* \tilde{\xi}] = [j_1^* \tilde{\xi}] \in H^*(M).$$

Lemma 4.23. *Assume $n > 0$. If $(\gamma, b) \sim (\gamma', b')$, then $[\gamma] = [\gamma']$ and $\int_L b \equiv \int_L b'$ (mod $(\mathcal{I}_{\hat{R}}^{odd})^2$).*

Proof. By definition of gauge equivalence, there exists a form $\tilde{\gamma} \in \mathcal{I}_Q \mathfrak{D}$ with $d\tilde{\gamma} = 0$ such that $j_0^* \tilde{\gamma} = \gamma$ and $j_1^* \tilde{\gamma} = \gamma'$. Lemma 4.22 therefore implies that $[\gamma] = [\gamma']$.

Choose $\tilde{b}$ as in the definition of gauge equivalence. Then Proposition 4.8 with $\tilde{\gamma}|_{I \times L} = 0$ and the definition of a bounding cochain imply

$$(d\tilde{b} - \tilde{b} \cdot \tilde{b})_{n+1} = (\tilde{\mathfrak{m}}_0^{\tilde{\gamma}, \tilde{b}})_{n+1} = (\tilde{c})_{n+1},$$

where $\tilde{c}$ is an element of $(\mathcal{I}_{\hat{\mathfrak{R}}} \cdot 1)_2$. Thus, Lemma 3.29 gives

$$(d\tilde{b})_{n+1} = (\tilde{c} + (\tilde{b}^0 + \tilde{b}^1) \cdot (\tilde{b}^0 + \tilde{b}^1))_{n+1} = (\tilde{c} + \tilde{b}^1 \cdot \tilde{b}^1)_{n+1}.$$

We compute

$$\int_L b' - \int_L b = \int_{\partial(I \times L)} \tilde{b} = \int_{I \times L} d(\tilde{b}) = \int_{I \times L} (\tilde{c} + \tilde{b}^1 \cdot \tilde{b}^1)_{n+1} = \int_{I \times L} \tilde{b}^1 \cdot \tilde{b}^1.$$

For the last equality, we have used the fact that for $n > 0$, we have $(\tilde{c})_{n+1} = 0$. Hence, we have shown that $\int_L b \equiv \int_L b'$ (mod $(\mathcal{I}_{\hat{R}}^{odd})^2$). $\qquad \square$

4.4.3. *Bounding cochains of the residue algebra.* Assume $n > 1$, so ν_S is non-trivial.

Lemma 4.24. *Let $b \in \hat{C}^\diamond$ (resp. $b \in \hat{\mathfrak{C}}^\diamond$) be a degree 1 element such that $\nu_S \otimes \nu(b) > 0$. Then there exists an element $\alpha \in A^n(L)$ (resp. $A^n(I \times L)$) such that $\bar{b} = s \otimes \alpha$.*

Proof. Notice that $\bar{b} \in s \cdot \overline{\hat{C}^\diamond}$. Otherwise for all representation $b = \sum s_i \otimes \alpha_i$, where $s_i \in S$ and $\alpha_i \in C$, there exists i with $\nu_S(s_i) = 0$ and $\nu(\alpha_i) = 0$. This is a contradiction to the assumption that $\nu_S \otimes \nu(b) > 0$. Hence, by Proposition 4.3 and Lemma 2.22, there exists an element $\alpha \in A^n(L)$ such that $\bar{b} = s \otimes \alpha$. The proof for $b \in \hat{\mathfrak{C}}^\diamond$ is the same but exchanging Proposition 4.3 with Proposition 4.9 $\qquad \square$

Lemma 4.25. *Let $b \in \hat{C}^\diamond$ be a degree 1 element such that $\nu_S \otimes \nu(b) > 0$. Then, $\bar{b}$ is a bounding cochain for the A_∞ algebra $(\overline{\hat{C}^\diamond}, \{\bar{\mathfrak{m}}_k^\gamma\}_{k \geq 0})$, and $\bar{\mathfrak{m}}_1^{\gamma, b}$ is a boundary operator on $\overline{\hat{C}^\diamond}$.*

Proof. It follows from Proposition 4.3 and Proposition 2.23 that $\bar{\mathfrak{m}}_0^{\gamma, b} = d\bar{b} - \bar{b} \cdot \bar{b}$. By Lemma 4.24 $\bar{b} = s \otimes \alpha$ for some $\alpha \in A^n(L)$. It follows that,

$$d\bar{b} - \bar{b} \cdot \bar{b} = 0.$$

So, $\bar{b}$ is a bounding cochain for the A_∞ algebra $(\overline{\hat{C}^\diamond}, \{\bar{\mathfrak{m}}_k^\gamma\}_{k \geq 0})$, and by Lemma 3.25 $\bar{\mathfrak{m}}_1^{\gamma, b}$ is a boundary operator on $\overline{\hat{C}^\diamond}$. $\qquad \square$

Corollary 4.26. *Assume $b \in \hat{C}^\diamond$ (resp. $b \in \hat{\mathfrak{C}}^\diamond$) is a bounding cochain for the A_∞ algebra $(\overline{\hat{C}^\diamond}, \{\bar{\mathfrak{m}}_k^\gamma\}_{k \geq 0})$ (resp. $(\overline{\hat{\mathfrak{C}}^\diamond}, \{\bar{\tilde{\mathfrak{m}}}_k^\gamma\}_{k \geq 0})$). Then, for homogeneous elements $\eta \in \overline{\hat{C}^\diamond}$ (resp. $\eta \in \overline{\hat{\mathfrak{C}}^\diamond}$), we have $\bar{\mathfrak{m}}_1^{\gamma, b}(\eta) = \mathrm{Id} \otimes d\eta - [\bar{b}, \eta]$ (resp. $\bar{\tilde{\mathfrak{m}}}_1^{\gamma, b}(\eta) = \mathrm{Id} \otimes d\eta - [\bar{b}, \eta]$) is a boundary operator.*

Proof. By Proposition 4.3, the A_∞ algebra $(\overline{C^\diamond}, \{\bar{\mathfrak{m}}_k^\gamma\}_{k\geq 0}, \langle\ ,\ \rangle, 1)$ arises from the dga of differential forms $(A^*(L), d)$, and the assumptions of Proposition 3.16 hold. That is

    (1) $(S, 1)$ is a descending unital differential graded algebra over $\mathbb{R}$.
    (2) $(\overline{C^\diamond}, \{\bar{\mathfrak{m}}_k^\gamma\}_{k\geq 0}, \langle\ ,\ \rangle, 1)$ is a cyclic unital A_∞ algebra that arises from the dga $(A^*(L), d)$.
    (3) The left and right module structure of $C^\diamond[1]$ over $R^\diamond$ concide.
    (4) $C^\diamond$ has an $\bar{R}^\diamond$ structure g with inverse f.

By definition of an $\bar{R}^\diamond$ structure, $\hat{f}$ induces the identity on $\widehat{\overline{C^\diamond}}$. So, the result follows. The proof for $(\widehat{\bar{\mathfrak{C}}^\diamond}, \{\bar{\tilde{\mathfrak{m}}}_k^{\tilde{\gamma}}\}_{k\geq 0})$ is the same. $\square$

4.4.4. *Bounding cochains of the residue algebra of a pseudoisotopy.* Throughout this section, we assume that $(\tilde{\mathfrak{m}}^{\tilde{\gamma}}, \langle\!\langle\ ,\ \rangle\!\rangle, 1)$ is a pseudoisotopy from $(\mathfrak{m}^\gamma, \langle\ ,\ \rangle, 1)$ to $(\mathfrak{m}^{\gamma'}, \langle\ ,\ \rangle, 1)$.

We let $b, b' \in \hat{C}^\diamond$ denote bounding cochains for the A_∞ algebras $(\hat{C}, \{\hat{\mathfrak{m}}_k^\gamma\}_{k\geq 0}, \langle\ ,\ \rangle_F, 1 \otimes 1)$ and $(\hat{C}, \hat{\mathfrak{m}}^{\gamma'}, \langle\ ,\ \rangle_F, 1\otimes 1)$, respectively, and assume that $\int_L \hat{f}(\bar{b}) = \int_L \hat{f}(\bar{b}')$.

Lemma 4.27. *There exists an element $\tilde{b}_0 \in \left(\hat{\mathfrak{C}}^\diamond\right)_1$, such that $\bar{\tilde{b}}_0$ is a bounding cochain with respect to the A_∞ algebra $(\widehat{\bar{\mathfrak{C}}}, \{\bar{\tilde{\mathfrak{m}}}_k^{\tilde{\gamma}}\}_{k\geq 0}, \langle\!\langle\ ,\ \rangle\!\rangle_F, 1)$, and $j_0^*(\tilde{b}_0) = b$, $j_1^*(\tilde{b}_0) = b'$.*

Proof. It follows from Proposition 4.3 and Proposition 2.23 that $\bar{\tilde{\mathfrak{m}}}_0^{\tilde{\gamma}, \tilde{b}} = d\bar{\tilde{b}} - \bar{\tilde{b}} \cdot \bar{\tilde{b}}$ for all $\tilde{b} \in \left(\hat{\mathfrak{C}}^\diamond\right)_1$. Lemma 4.24 gives $\bar{b} = s \otimes \alpha$ for some $\alpha \in A^n(L)$ and $\bar{b}' = s \otimes \alpha'$ for some $\alpha' \in A^n(L)$. By Poincaré duality and the assumption $\int_L \hat{f}(\bar{b}) = \int_L \hat{f}(\bar{b}')$, there exists $\eta \in A^{n-1}(L)$ such that $d\eta = (-1)^{1-n}(\alpha - \alpha')$. Define

$$\tilde{b}_0 := tb - (1-t)b' + \hat{g}\left(1 \otimes s \otimes \eta \wedge dt\right).$$

Using Lemma 2.5 and recalling that $n > 1$, it follows that

$$d\bar{\tilde{b}}_0 - \bar{\tilde{b}}_0 \cdot \bar{\tilde{b}}_0 = -s \otimes ((\alpha - \alpha') \wedge dt) + d(s \otimes \eta \wedge dt) = 0.$$

$\square$

Define the submodule $\mathfrak{B} \subset \hat{\mathfrak{C}}^\diamond$ over the dga $Z(\hat{\mathfrak{R}}^\diamond)$ by

$$\mathfrak{B} := \left\{a \in \hat{\mathfrak{C}}^\diamond : j_i^*(a) \in Z(\hat{R}^\diamond) \cdot 1 \text{ for } i = 0, 1\right\}.$$

Lemma 4.28. *Let $\tilde{b}_0 \in \left(\hat{\mathfrak{C}}^\diamond\right)_1$ be as in Lemma 4.27. Then the operators $\{\tilde{\mathfrak{m}}_k^{\tilde{\gamma}, \tilde{b}_0}\}_{k\geq 0}$ preserve $\mathfrak{B}$ and thus define an A_∞ algebra structure $(\mathfrak{B}, \{\tilde{\mathfrak{m}}_k^{\tilde{\gamma}, \tilde{b}_0}\}_{k\geq 0}, \langle\!\langle\ ,\ \rangle\!\rangle_F, 1)$ over the dga $Z(\hat{\mathfrak{R}}^\diamond)$.*

Proof. Corollary 3.15 implies that $\{\tilde{\mathfrak{m}}_k^{\tilde{\gamma}, \tilde{b}_0}\}_{k\geq 0}$ is an A_∞ algebra over $\hat{\mathfrak{C}}^\diamond$. It remains to show that the operators $\{\tilde{\mathfrak{m}}_k^{\tilde{\gamma}, \tilde{b}_0}\}_{k\geq 0}$ preserve $\mathfrak{B}$. Let $a_1, \ldots, a_k \in \mathfrak{B}$. By Lemma 4.12,

$$j_0^* \tilde{\mathfrak{m}}_k^{\tilde{\gamma}, \tilde{b}_0}(a_1, \ldots, a_k) = \hat{\mathfrak{m}}_k^{\gamma, b}\left(j_0^*(a_1), \ldots, j_0^*(a_k)\right).$$

Since b is a bounding cochain, it follows that

$$j_0^* \tilde{\mathfrak{m}}_0^{\tilde{\gamma}, \tilde{b}_0} \in Z(\hat{R}^\diamond) \cdot 1.$$

Since the $\hat{\mathfrak{m}}_k^{\gamma, b}$ are operators of an A_∞ algebra over $Z(R^\diamond)$, by the definition of $\mathfrak{B}$ and the unit property,

$$j_0^* \tilde{\mathfrak{m}}_k^{\tilde{\gamma}, \tilde{b}_0}(a_1, \ldots, a_k) = 0, \qquad k \neq 0, 2.$$

Also, the unit property implies

$$j_0^* \tilde{\mathfrak{m}}_2^{\tilde{\gamma}, \tilde{b}_0}(a_1, a_2) = j_0^*(a_1) j_0^*(a_2) \in Z(\hat{R}^\diamond) \cdot 1.$$

Similarly, $j_1^* \tilde{\mathfrak{m}}_k^{\tilde{\gamma}, \tilde{b}_{(l)}}(a_1, \dots, a_k) \in Z(\hat{R}^\diamond) \cdot 1$, and the result follows. $\qquad\square$

4.5. **Discrete filtration and finite energy quotients.** Because $(C^\diamond, \{\mathfrak{m}_k^\gamma\}_{k \geq 0}, \langle \ , \ \rangle, 1)$ (resp. $(\mathfrak{C}^\diamond, \{\tilde{\mathfrak{m}}_k^{\tilde{\gamma}}\}_{k \geq 0}, \langle\!\langle \ , \ \rangle\!\rangle, 1)$) is sababa, the set $\nu(\hat{C}^\diamond) = \nu(C^\diamond) \subset \mathbb{R}$ ($\nu(\hat{\mathfrak{C}}^\diamond) = \nu(\mathfrak{C}^\diamond) \subset \mathbb{R}$) can be written

$$\nu(\hat{C}^\diamond) = \{0 = E_0 < E_1 < E_2 < \dots\}, \tag{22}$$

with $\lim_{i \to \infty} E_i = \infty$.

Remark 4.29. The above discreteness of ν is important for inductive constructions. Consider the following general problem. Let $\mathfrak{p} : \mathcal{N} \to \mathcal{M}$ be a morphism of complete $\mathbb{R}$ filtered modules. Suppose we are searching for solutions to the equation

$$\mathfrak{p}(\eta) = c. \tag{23}$$

Assume that equation (23) has the property that if η is a solution modulo $F^{>E}\mathcal{M}$ for all $E < E'$, then there exists $\eta' \in F^{E'}\mathcal{N}$ such that

$$\mathfrak{p}(\eta + \eta') \equiv c \pmod{F^{E'}\mathcal{M}}.$$

If $\mathcal{M}$ has a discrete filtration $\{0 = E_0 < E_1 < E_2 < \dots\}$, this property of $\mathfrak{p}$ assures us that if we are given $\eta_0 \in \mathcal{N}$ that is a solution to equation (23) modulo $F^{>0}\mathcal{M}$, then we can find a genuine solution to the equation. Indeed the property of $\mathfrak{p}$ implies the existence of elements $\eta_i \in F^{E_i}\mathcal{N}$ such that $\eta_{(l)} = \sum_{i=0}^l \eta_i$ is a solution to equation (23) modulo $F^{E_l}\mathcal{M}$, for all $l \in \mathbb{N}$. Hence, since $\mathcal{N}$ is complete, we have that $\eta_\infty := \lim_{l \to \infty} \eta_{(l)}$ is a genuine solution to equation (23).

For $i \in \mathbb{N}$ recall the definitions of $R_{E_i}^\diamond, \hat{R}_{E_i}^\diamond, \hat{C}_{E_i}^\diamond$ (resp. $\mathfrak{R}_{E_i}^\diamond, \hat{\mathfrak{R}}_{E_i}^\diamond, \hat{\mathfrak{C}}_{E_i}^\diamond$) from Definition 2.3. Let $\hat{g} : R^\diamond \otimes_{\bar{R}^\diamond} \overline{\hat{C}^\diamond} \to \hat{C}^\diamond$ (resp. $\hat{g} : R^\diamond \otimes_{\mathfrak{R}^\diamond} \overline{\hat{\mathfrak{C}}^\diamond} \to \hat{\mathfrak{C}}^\diamond$) be the induced $\bar{R}^\diamond$ structure on $\hat{C}^\diamond$ (resp. $\hat{\mathfrak{C}}^\diamond$) as in Definition 2.24, and let $\hat{f}$ be its inverse. Let $b \in \hat{C}^\diamond$ (resp. $b \in \hat{\mathfrak{C}}^\diamond$) be a bounding cochain for the residue A_∞ algebra. By Corollary 4.16 the map $\hat{\mathfrak{m}}_1^{\gamma, b} : \hat{C}^\diamond \to \hat{C}^\diamond$ (resp. $\hat{\tilde{\mathfrak{m}}}_1^{\tilde{\gamma}, b} : \hat{\mathfrak{C}}^\diamond \to \hat{\mathfrak{C}}^\diamond$) is well defined. By Lemma 3.25 the induced map $\hat{\mathfrak{m}}_1^{\gamma, b} : \hat{C}_{E_i}^\diamond \to \hat{C}_{E_i}^\diamond$ (resp. $\hat{\tilde{\mathfrak{m}}}_1^{\tilde{\gamma}, b} : \hat{\mathfrak{C}}_{E_i}^\diamond \to \hat{\mathfrak{C}}_{E_i}^\diamond$) is a boundary operator. Define the integral $\int_L : R^\diamond \otimes \overline{\hat{C}^\diamond} \to \hat{R}^\diamond$ by the composition

$$R^\diamond \otimes \overline{\hat{C}^\diamond} \xrightarrow{\ \text{Id}\ } R^\diamond \otimes S \otimes A^*(L) \xrightarrow{\ \text{Id} \otimes \int_L\ } R^\diamond \otimes S \otimes \mathbb{R} \cong \hat{R}^\diamond \ .$$

Define the integral $\int_{I \times L} : R^\diamond \otimes \overline{\hat{\mathfrak{C}}^\diamond} \to \hat{R}^\diamond$ by the composition

$$R^\diamond \otimes \overline{\hat{\mathfrak{C}}^\diamond} \xrightarrow{\ 2.22\ } R^\diamond \otimes S \otimes A^*(I \times L) \xrightarrow{\ \text{Id} \otimes \int_{I \times L}\ } R^\diamond \otimes S \otimes \mathbb{R} \cong \hat{R}^\diamond \ .$$

In what follows let $\mathcal{C} = \hat{C}^\diamond, \hat{\mathfrak{C}}^\diamond$ and $\mathfrak{m}_k = \mathfrak{m}_k^\gamma, \tilde{\mathfrak{m}}_k^{\tilde{\gamma}}$. Consider $\mathbb{R}$ as a graded module concentrated in degree 0. Then, the module $\hom_\mathbb{R}(R_E^\diamond, \mathbb{R})$ has a natural grading. Denote the module $\hom_\mathbb{R}(R_E^\diamond, \mathbb{R})$ by $R_E^{\diamond \vee}$. For a homogeneous element $r \in R_E^{\diamond \vee}$, let $\hat{\pi}_r : R_E^\diamond \otimes \overline{\mathcal{C}} \to \overline{\mathcal{C}}[|r|]$ be given by $\lambda \otimes \alpha \mapsto r(\lambda)\alpha$. Let

$$\pi_r : \mathcal{C}_E \to \overline{\mathcal{C}}[|r|] \tag{24}$$

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be given by $\hat{\pi}_r \circ \hat{f}$. Observe that the map extend naturally to a map on the quotients $\pi_r : \mathcal{C}_{\mathcal{G},E} \to \overline{\mathcal{C}_{\mathcal{G}}}[\|r\|]$.

Lemma 4.30. *Let r be a homogeneous element in $R_E^{\diamond\vee}$. Then,*

$$\pi_r : (\mathcal{C}_E, \mathfrak{m}_{1,E}^b) \to (\overline{\mathcal{C}}[\|r\|], (-1)^{|r|}\mathfrak{m}_1^b)$$

is a cochain map. The same result holds for the map on the quotients $\pi_r : \mathcal{C}_{\mathcal{G},E} \to \overline{\mathcal{C}_{\mathcal{G}}}[\|r\|]$.

Proof. Let $\mathcal{C}$ be a cochain complex. Then there exists a natural cochain complex isomorphism $\mathbb{R}[k] \otimes \mathcal{C} \to \mathcal{C}[k]$ given on pure tensors by $a \otimes b \mapsto ab$. As the differential on $R_E^\diamond$ is trivial, the assignment $a \mapsto r(a)$ defines a cochain map $\eta : R_E^\diamond \to \mathbb{R}[\|r\|]$. So, $\eta \otimes Id$ is a cochain map from $R_E^\diamond \otimes \overline{\mathcal{C}} \to \mathbb{R}[\|r\|] \otimes \overline{\mathcal{C}}$. By Lemma 3.28 (possibly with b trivial) $\hat{f}$ is a cochain complex isomorphism. So, π_r is a cochain complex map, as it can be defined by composing cochain maps. $\qquad\square$

Lemma 4.31. *A cochain $\alpha \in (\mathcal{C}_{\mathcal{G},E})_*$ is $\mathfrak{m}_1^b$ exact if and only if for all homogeneous $r \in R_E^{\diamond\vee}$, the cochain $\pi_r(\alpha) \in \overline{\mathcal{C}_{\mathcal{G}}}_{*+|r|}$ is $\bar{\mathfrak{m}}_1^b$ exact.*

Proof. Pick a basis $\{r_i\}_{i=1}^m$ for $R_E^\diamond$ as a real vector space and let $\{r_i^\vee\}_{i=1}^m$ be its dual. Let η^i be the primitives of the cochains $\pi_{r_i^\vee}(\alpha)$. Then, $(-1)^{|r_i|}r_i \otimes \eta^i$ is a primitive for $\hat{f}(\alpha)$. By Lemma 3.28 it follows that α is exact. The other direction follows from Lemma 4.30. $\qquad\square$

Lemma 4.32. $\alpha \in ((\mathfrak{B}_{\mathcal{G}}^\kappa)_E)_*$ *is* $\tilde{\mathfrak{m}}_1^{\tilde{\gamma},\tilde{b}} : \mathfrak{B}_E^\kappa \mapsto \mathfrak{B}_E^\kappa$ *exact, if and only if for all homogeneous $r \in R_E^{\diamond\vee}$, the element $\pi_r(\alpha) \in \overline{\mathfrak{B}_{\mathcal{G}}}^*[\|r\|]^\kappa$ is $\bar{\tilde{\mathfrak{m}}}_1^{\bar{\tilde{b}}}$ exact.*

Proof. We observe that induced $\bar{R}^\diamond$ structure sends $\mathfrak{B}$ to $R^\diamond \otimes \overline{\mathfrak{B}}$. Thus, the proof is the same as 4.31. $\qquad\square$

Remark 4.33. Define the operator $\mathfrak{p} : \left(\mathcal{I}_{\hat{R}^\diamond}\hat{C}^\diamond\right)_1 \to \left(\hat{C}_{\mathcal{G}}^\diamond\right)_2$ by

$$\eta \mapsto \hat{\mathfrak{m}}_0^{\gamma,b+\eta}.$$

In Section 6, our goal is to establish a bounding cochain by resolving $\mathfrak{p}(\eta) = 0$. As described in Remark 4.29, we aim to demonstrate that the problem operator $\mathfrak{p} : \left(\mathcal{I}_{\hat{R}^\diamond}\hat{C}^\diamond\right)_1 \to \left(\hat{C}_{\mathcal{G}}^\diamond\right)_2$ exhibits the characteristic that, if η is a solution modulo $F^{>E}\left(\hat{C}_{\mathcal{G}}^\diamond\right)_2$, then for all $E < E'$, then there exists $\eta' \in F^{E'}\left(\mathcal{I}_{\hat{R}^\diamond}\hat{C}^\diamond\right)_1$ such that

$$\mathfrak{p}(\eta + \eta') \equiv 0 \pmod{F^{E'}\left(\hat{C}_{\mathcal{G}}^\diamond\right)_2}.$$

Given Lemma 3.27, this property ultimately becomes a cohomological issue within the cochain complexes $(\hat{C}_{\mathcal{G},E}^\diamond, \hat{\mathfrak{m}}_{1,E}^{\gamma,b})$. The findings presented in this section indicate that our primary concern is to understand $H^*\left(\widehat{C}_{\mathcal{G}}^\diamond, \bar{\hat{\mathfrak{m}}}_1^{\gamma,b}\right)$.

4.6. **Anti-symplectic involutions and self-duality.** Let $\phi : X \to X$ be an anti-symplectic involution, that is, $\phi^*\omega = -\omega$. Let $L \subset fix(\phi)$ and $J \in \mathcal{J}_\phi$. In particular, $\phi^*J = -J$. For the entire Section 4.6, we take $S_L \subset H_2(X, L; \mathbb{Z})$ with $\text{Im}(Id + \phi_*) \subset S_L$, so ϕ_* acts on $\Pi_L = H_2(X, L; Z)/S_L$ as $-Id$. Also, we take $\deg t_j \in 2\mathbb{Z}$ for all $j = 0, \ldots, N$.

Recall the action of ϕ^* on Λ, Q, R, C, D, defined in equation (4) and the definition of real elements (5). For a group Z on which a function κ acts, let $Z^\kappa \subset Z$ denote the elements fixed by κ. Recall Definition 3.17 of a self dual A_∞ algebra.

Proposition 4.34 ([35, Proposition 4.5]). *Suppose the relative spin structure $\mathfrak{s}$ on L is in fact a spin structure, and $\gamma \in \mathcal{I}_Q D$ is real. Then, for all $\alpha = (\alpha_1, \ldots, \alpha_k)$ with $\alpha_j \in C$, we have*

$$\phi^* \mathfrak{m}_k^\gamma(\alpha_1, \ldots, \alpha_k) = (-1)^{1+k+s_\tau^{[1]}(\alpha)} \mathfrak{m}_k^\gamma(\phi^* \alpha_k, \ldots, \phi^* \alpha_1).$$

That is, $(C, \mathfrak{m}^\gamma)$ is self-dual.

Define $\phi^* : S \to S$ by

$$\phi^*(a) = (-1)^{\frac{|a|(|a|-1)}{2}} a$$

Lemma 4.35. *If $a, b \in S$, then $\phi^*(ab) = (-1)^{|a||b|} \phi^*(b) \cdot \phi^*(a)$, that is, ϕ^* is a homomorphism from S to S^{op}.*

Proof. We use the algebraic identity for $a, b \in \mathbb{Z}$,

$$\frac{(a+b)(a+b-1)}{2} + \frac{a(a-1)}{2} + \frac{b(b-1)}{2} = ab,$$

to deduce the equation. $\square$

Corollary 4.36. *For $x_1, \ldots, x_k \in S$ we have $\phi^*(x_1 \cdots x_k) = (-1)^{\sum_{i<j} |x_i||x_j|} \phi^*(x_k) \cdots \phi^*(x_1)$.*

Proposition 4.37. *Suppose the relative spin structure $\mathfrak{s}$ on L is a spin structure, and $\gamma \in \mathcal{I}_Q D$ is real. Then, for all $\alpha = (\alpha_1, \ldots, \alpha_k)$ with $\alpha_j \in C$, we have*

$$\phi^* \hat{\mathfrak{m}}_k^\gamma(\alpha_1, \ldots, \alpha_k) = (-1)^{1+k+s_\tau^{[1]}(\alpha)} \hat{\mathfrak{m}}_k^\gamma(\phi^* \alpha_k, \ldots, \phi^* \alpha_1).$$

That is, $(\hat{C}, \hat{\mathfrak{m}}^\gamma)$ is ϕ^* self-dual.

Proof. Let $x_1 \otimes \alpha_1, \ldots, x_k \otimes \alpha_k \in \hat{C}$. By definition of the extension of scalars and Corollary 4.36,

$$\phi^* \left(\hat{\mathfrak{m}}_k^\gamma(x_1 \otimes \alpha_1, \ldots, x_k \otimes \alpha_k) \right) =$$

$$= (-1)^{\sum_{i=1}^k |x_i| + \sum_{i=1}^k |x_i| \sum_{j=1}^{i-1}(|\alpha_j|+1)} \phi^*(x_1 \cdots x_k) \otimes \phi^* \left(\mathfrak{m}_k^\gamma(\alpha_1, \ldots, \alpha_k) \right)$$

$$= (-1)^{\sum_{i<j} |x_i||x_j| + \sum_{i=1}^k |x_i| + \sum_{i=1}^k |x_i| \sum_{j=1}^{i-1}(|\alpha_j|+1)} \phi^*(x_k) \cdots \phi^*(x_1) \otimes \phi^* \left(\mathfrak{m}_k^\gamma(\alpha_1, \ldots, \alpha_k) \right).$$

Abbreviating $a = \sum_{i<j} |x_i||x_j| + \sum_{i=1}^k |x_i| + \sum_{i=1}^k |x_i| \sum_{j=1}^{i-1}(|\alpha_j|+1)$ and invoking Proposition 4.34, we continue

$$= (-1)^{a+1+k+s_\tau^{[1]}(\alpha)} \phi^*(x_k) \cdots \phi^*(x_1) \otimes \mathfrak{m}_k^\gamma(\phi^* \alpha_k, \ldots, \phi^* \alpha_1).$$

On the other hand, we have

$$\hat{\mathfrak{m}}_k^\gamma(\phi^*(x_k) \otimes \phi^* \alpha_k, \ldots, \phi^*(x_1) \otimes \phi^* \alpha_1) =$$

$$(-1)^{\sum_{i=1}^k |x_i| + \sum_{i=1}^k |x_i| \sum_{j=i+1}^k(|\alpha_j|+1)} \phi^*(x_k) \cdots \phi^*(x_1) \otimes \mathfrak{m}_k^\gamma(\phi^* \alpha_k, \ldots, \phi^* \alpha_1).$$

Combining the preceding computations,

$$\phi^* \left(\hat{\mathfrak{m}}_k^\gamma(x_1 \otimes \alpha_1, \ldots, x_k \otimes \alpha_k) \right) =$$

$$(-1)^{a+1+k+x_\tau^{[1]}(\alpha)+\sum_{i=1}^k |x_i| + \sum_{i=1}^k |x_i| \sum_{j=i+1}^k(|\alpha_j|+1)} \hat{\mathfrak{m}}_k^\gamma(\phi^*(x_k) \otimes \phi^* \alpha_k, \ldots, \phi^*(x_1) \otimes \phi^* \alpha_1).$$

We simplify the sign as follows.

$$a + 1 + k + s_\tau^{[1]}(\alpha) + \sum_{i=1}^{k} |x_i| + \sum_{i=1}^{k} |x_i| \sum_{j=i+1}^{k} (|\alpha_j| + 1) \equiv$$

$$\equiv \sum_{i<j} |x_i||x_j| + \sum_{i=1}^{k} |x_i| \sum_{j=1}^{i-1} (|\alpha_j| + 1) + 1 + k + s_\tau^{[1]}(\alpha) + \sum_{i=1}^{k} |x_i| \sum_{j=i+1}^{k} (|\alpha_j| + 1) \quad (\text{mod } 2).$$

Changing the order of summation, we continue

$$\equiv 1 + k + s_\tau^{[1]}(\alpha) + \sum_{i<j} |x_i|(|x_j \otimes \alpha_j| + 1) + \sum_{j<i} (|\alpha_j| + 1)|x_i|$$

$$\equiv 1 + k + \sum_{i<j} |x_i|(|x_j \otimes \alpha_j| + 1) + \sum_{j<i} (|\alpha_j| + 1)(|x_i| + |\alpha_i| + 1)$$

$$\equiv 1 + k + \sum_{i<j} (|x_i \otimes \alpha_i| + 1)(|x_j \otimes \alpha_j| + 1) \quad (\text{mod } 2).$$

The proposition follows. □

Corollary 4.38. *Let* $b \in \left(\hat{C}^\diamond\right)^{-\phi^*}$ *with* $\nu_S \otimes \nu(b) > 0$ *and* $|b| = 1$. *Then,* $\hat{\mathfrak{m}}_1^{\gamma,b}(\eta) \in \left(\hat{C}^\diamond\right)^{-\phi^*}$ *for all* $\eta \in \left(\hat{C}^\diamond\right)^{-\phi^*}$. *So, if* b *is a bounding cochain for the residue algebra, then the real elements* $\left(\hat{C}_E^\diamond\right)^{-\phi^*}$ *form a sub-complex of* $\left(\hat{C}_E^\diamond, \hat{\mathfrak{m}}_{E,1}^{\gamma,b}\right)$.

Proof. This follows from Corollary 3.20. □

Lemma 4.39. *A homogeneous element* $a \in \hat{R}$ *is real if and only if* $\deg a \equiv 2$ *or* $3 \pmod 4$

Proof. For $t \in R$ and $x \in S$ homogeneous we compute

$$\phi^*(x) \otimes \phi^*(t) = (-1)^{\frac{|x|(|x|-1)}{2}} x \otimes \left((-1)^{\frac{|t|(|t|-1)}{2}} t\right) = (-1)^{\frac{|x|(|x|-1)}{2} + \frac{|t|(|t|-1)}{2}} x \otimes t$$

We recall that

$$\frac{|x|(|x|-1)}{2} + \frac{|t|(|t|-1)}{2} = \frac{(|x|+|t|)(|x|+|t|-1)}{2} + |t||x|.$$

As the elements of R are of even degree, we can deduce that,

$$\phi^* \otimes \phi^*(x \otimes t) = (-1)^{\frac{(|x \otimes t|)(|s \otimes t|-1)}{2}} x \otimes t.$$

Thus $x \otimes t$ are real if and only if $|x \otimes t| \equiv 2$ or $3 \pmod 4$ □

For $v \in \mathbb{Z}$ and $a \in S$ homogeneous, denote by $a_v \in S[v]$ the element corresponding to a.

Definition 4.40. Define $\phi^* : S[v] \to S[v]$ by

$$\phi^*(a_v) = (-1)^{\frac{(|a|-v)(|a|-v-1)}{2}} a_v = (-1)^{\frac{(|a_v|)(|a_v|-1)}{2}} a_v.$$

For $\mathbb{R}$ algebras Υ define $\phi^* : \hat{\Upsilon}[v] \to \hat{\Upsilon}[v]$ by identifying $\hat{\Upsilon}[v] \cong S[v] \otimes \Upsilon$ and tensoring ϕ^* on $S[v]$ with the identity map on Υ.

For homogeneous $r \in R_E^{\diamond\vee}$, recall the definition of π_r from equation (24). The preceding definition of ϕ^* is motivated by the following lemma.

Lemma 4.41. $\pi_r : \hat{C}_E^\diamond \to \overline{\hat{C}^\diamond}[|r|]$ *is ϕ^* equivariant.*

Proof. By the definition of $\hat{C}^\diamond$, for a homogeneous element $\alpha \in \hat{C}_E^\diamond$, we can write

$$\alpha = \left[\sum_{j=0}^m s^{k_j} \otimes \lambda_j \otimes \eta\right],$$

where s is the generator of S, and $\lambda_j \in R^\diamond$ and $\eta \in A^*(L)$, are homogeneous elements. Recalling that ϕ^* acts trivially on forms on L, we compute

$$\pi_r\left(\phi^*(\alpha)\right) = \pi_r\left(\left[\sum_{i=0}^m \phi^*(s^{k_j}) \otimes \phi^*(\lambda_j) \otimes \eta\right]\right)$$

$$= \sum_{i=0}^m (-1)^{|s^{k_j}||\lambda_j|} r\left([\phi^*(\lambda_j)]\right) \phi^*(s^{k_j})_{|r|} \otimes \eta$$

$$= \sum_{i=0}^m (-1)^{|s^{k_j}||\lambda_j| + \frac{|s^{k_j}|(|s^{k_j}|-1)}{2} + \frac{|\lambda_j|(|\lambda_j|-1)}{2}} r\left([\lambda_j]\right) (s^{k_j})_{|r|} \otimes \eta$$

$$= \sum_{i=0}^m (-1)^{\frac{(|s^{k_j}|+|\lambda_j|)(|s^{k_j}|+|\lambda_j|-1)}{2}} r\left([\lambda_j]\right) (s^{k_j})_{|r|} \otimes \eta.$$

Observe that for j such that $r([\lambda_j]) \neq 0$ then $|\lambda_j| = -|r|$. So, $r\left([\lambda_j]\right) (s^{k_j})_{|r|} \in S[|r|]_{|s^{k_j}|+|\lambda_j|}$. Hence, we continue

$$= \sum_{i=0}^m \phi^*\left(r\left([\lambda_j]\right) (s^{k_j})_{|r|} \otimes \eta\right).$$

$\square$

Lemma 4.42. *Let $s \in S$ be a homogeneous element of degree $-p$, and let $r \in R_E^{\diamond\vee}$ be a homogeneous element. Assume that $s \otimes \alpha \in \hat{C}_E^\diamond$ is real, where $\alpha \in C_E^\diamond$. Then,*

$$\pi_r(s \otimes \alpha) \neq 0 \implies |r| + p \equiv 1, 2 \pmod 4.$$

Proof. Write

$$\alpha = \left[\sum_i f_i \otimes \alpha_i\right],$$

where $0 \neq f_i \in R^\diamond$ and $\alpha_i \in A^*(L)$, are homogeneous. Assume that $\pi_r(s \otimes \alpha) \neq 0$. Then there exists i such that $r([f_i]) \neq 0$. It follows that $|f_i| = -|r|$. Since ϕ^* acts trivially on forms on L and $s \otimes \alpha$ is real, it follows that $s \otimes f_i$ is real. Thus, Lemma 4.39 implies that

$$|r| + p = -|f_i| + p \equiv 1, 2 \pmod 4.$$

$\square$

Lemma 4.43. *Assume that b is a real bounding cochain of the residue A_∞ algebra. Then $\phi^* : (\overline{\hat{C}^\diamond}[v], \hat{\mathfrak{m}}_1^{\gamma,b}[v]) \to (\overline{\hat{C}^\diamond}[v], \hat{\mathfrak{m}}_1^{\gamma,b}[v])$ is a cochain map.*

Proof. We observe that by Corollary 4.26 the result will follow if
$$\phi^*([\bar{b}, \eta]) = [\bar{b}, \phi^*(\eta)],$$
for all $\eta \in \overline{\hat{C}^\diamond}[v] \cong S \otimes A^*(L)[v]$. Assume first that n is odd. Observe that $|s| = 1 - n$, and thus S is graded by $2\mathbb{Z}$. It follows from Corollary 3.30 that
$$\phi^*([\bar{b}, \eta]) = \phi^*(0) = 0 = [\bar{b}, \phi^*(\eta)].$$
So, the results holds for n odd.

Assume that n is even. By Lemma 4.24 there exists $\alpha \in A^n(L)$ such that $\bar{b} = s \otimes \alpha$. By Lemma 4.39 α is trivial if $n \equiv 0 \pmod 4$. Thus, the result holds for $n \equiv 0 \pmod 4$.

We now specialize to the remaining case that $n \equiv 2 \pmod 4$. It's enough to assume that $\eta = s^j \otimes \tilde{\eta}$ with $\tilde{\eta} \in A^*(L)$. Corollary 3.30 implies that the above equality holds if $|\eta|_{\mathbb{Z}_2} = 0$. So, assume that j is odd. Observe that
$$3j + v \equiv 1 \pmod 4 \implies 3(j+1) + v \equiv 0 \pmod 4,$$
and
$$3j + v \equiv 3 \pmod 4 \implies 3(j+1) + v \equiv 2 \pmod 4.$$
It follows that $s\phi^*(s^j) = \phi^*(s^{j+1})$. Thus, ϕ^* changes the sign of $[\bar{b}, \eta]$ if and only if it changes the sign of η, and the result follows. $\square$

As a corollary of Lemma 4.41, and Lemma 4.43 we have the following lemma.

Lemma 4.44. *Let $r \in \hat{R}_E^\diamond$ be a homogeneous element, then the restriction*
$$\pi_r : \left((\hat{C}_E^\diamond)^{-\phi^*}, \hat{\mathfrak{m}}_{E,1}^{\gamma,b} \right) \to \left(\overline{\hat{C}^\diamond}[|r|]^{-\phi^*}, \bar{\hat{\mathfrak{m}}}_1^{\gamma,b} \right).$$
defines a chain map. Moreover, the map descends to the quotients $\pi_r : \left(\hat{C}_{\mathscr{C},E}^\diamond \right)^{-\phi^*} \to$ $\overline{\hat{C}_{\mathscr{C}}^\diamond}[|r|]^{-\phi^*}$.

Proposition 4.45 ([35, Proposition 4.13]). *Suppose the relative spin structure $\mathfrak{s}$ on L is a spin structure, and $\tilde{\gamma} \in \mathcal{I}_Q\mathfrak{D}$ is real. Then, for all $\tilde{\alpha} = (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_k)$ with $\tilde{\alpha}_j \in \mathfrak{C}$, we have*
$$\phi^*\tilde{\mathfrak{m}}_k^{\tilde{\gamma}}(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_k) = (-1)^{1+k+s_\tau^{[1]}(\tilde{\alpha})}\tilde{\mathfrak{m}}_k^{\tilde{\gamma}}(\phi^*\tilde{\alpha}_k, \ldots, \phi^*\tilde{\alpha}_1).$$

We now extend this self-duality to the extension of scalars $(\hat{\mathfrak{C}}^\diamond, \hat{\tilde{\mathfrak{m}}}^{\tilde{\gamma}})$.

Proposition 4.46. *Suppose the relative spin structure $\mathfrak{s}$ on L is a spin structure, and $\tilde{\gamma} \in \mathcal{I}_Q\mathfrak{D}$ is real. Then, for all $\tilde{\alpha} = (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_k)$ with $\tilde{\alpha}_j \in \hat{\mathfrak{C}}^\diamond$, we have*
$$\phi^*\hat{\tilde{\mathfrak{m}}}_k^{\tilde{\gamma}}(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_k) = (-1)^{1+k+s_\tau^{[1]}(\tilde{\alpha})}\hat{\tilde{\mathfrak{m}}}_k^{\tilde{\gamma}}(\phi^*\tilde{\alpha}_k, \ldots, \phi^*\tilde{\alpha}_1).$$

Proof. The proof is identical to that of Proposition 4.37, utilizing the properties of the involution on S and substituting Proposition 4.34 with Proposition 4.45. $\square$

Corollary 4.47. *Let $\tilde{b} \in \left(\hat{\mathfrak{C}}^\diamond \right)^{-\phi^*}$ with $\nu_S \otimes \nu(\tilde{b}) > 0$ and $|\tilde{b}| = 1$. Then, $\hat{\mathfrak{m}}_1^{\tilde{\gamma}, \tilde{b}}(\eta) \in \left(\hat{\mathfrak{C}}^\diamond \right)^{-\phi^*}$ for all $\eta \in \left(\hat{\mathfrak{C}}^\diamond \right)^{-\phi^*}$. So, if $\tilde{b}$ is a bounding cochain for the residue algebra, then the real elements $\left(\hat{\mathfrak{C}}_E^\diamond \right)^{-\phi^*}$ form a sub-complex of $\left(\hat{\mathfrak{C}}_E^\diamond, \hat{\mathfrak{m}}_{1,E}^{\tilde{\gamma}, \tilde{b}} \right)$.*

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Proof. This follows from Corollary 3.20 applied to the algebra $(\hat{\mathfrak{C}}^\diamond, \hat{\tilde{\mathfrak{m}}}^{\tilde{\gamma}})$. $\qquad\square$

We observe that the maps $j_i^* : \mathfrak{C} \to C$ preserve the ϕ^* invariant subspaces.

5. Spectral sequence

In this section, we introduce spectral sequences associated with the filtered differential graded modules derived from our A_∞ algebras. These spectral sequences are crucial tools for analyzing the obstruction theory in Section 6, both for constructing bounding pairs (Theorem 5.2) and for constructing gauge equivalences via pseudoisotopies (Theorem 5.3).

We recall several definitions from earlier sections. The modules $\hat{C}^\diamond$ over $\hat{R}^\diamond$ and $\hat{\mathfrak{C}}^\diamond$ over $\hat{\mathfrak{R}}^\diamond$ are defined in Section 4.3.3, and the $\hat{\mathfrak{R}}^\diamond$ module $\mathfrak{B}$ is defined in Section 4.4.4. Corollary 4.26 endows $\overline{\hat{C}^\diamond}$, and $\overline{\hat{\mathfrak{C}}^\diamond}$ with a cochain complex structure over $\mathbb{R}$, if they are given bounding cochains. For an $\mathcal{R}$ module $\mathcal{C}$, the notation $\mathcal{C}_{\not{\mathfrak{e}}}$ was introduced in Section 3.7. Note that while the unit elements of both $\mathfrak{B}$ equal 1, we will continue to denote them by $\mathbf{e}$ for consistency with earlier notation.

Throughout this section, we maintain the same assumptions as in Section 4.4.4: the triple $(\tilde{\mathfrak{m}}^{\tilde{\gamma}}, \langle\!\langle\ ,\ \rangle\!\rangle, 1)$ is a pseudoisotopy from $(\mathfrak{m}^\gamma, \langle\ ,\ \rangle, 1)$ to $(\mathfrak{m}^{\gamma'}, \langle\ ,\ \rangle, 1)$; $b, b' \in \hat{C}^\diamond$ are bounding cochains for the A_∞ algebras $(\hat{C}, \{\hat{\mathfrak{m}}_k^\gamma\}_{k\geq 0}, \langle\ ,\ \rangle_F, 1\otimes 1)$ and $(\hat{C}, \hat{\mathfrak{m}}^{\gamma'}, \langle\ ,\ \rangle_F, 1\otimes 1)$, respectively; and $\int_L \hat{f}(\bar{b}) = \int_L \hat{f}(\bar{b}')$. We also choose $\tilde{b} = \tilde{b}_0 \in \left(\hat{\mathfrak{C}}^\diamond\right)_1$ as in Lemma 4.27, which ensures that $H^*\left(\overline{\mathfrak{B}}, \bar{\tilde{\mathfrak{m}}}_1^{\tilde{\gamma},\tilde{b}}\right)$ is well-defined by Lemma 4.28.

Understanding the cohomology groups $H^*\left(\overline{\hat{C}_{\not{\mathfrak{e}}}^\diamond}, \bar{\hat{\mathfrak{m}}}_1^{\gamma,b}\right)$ and $H^*\left(\overline{\mathfrak{B}}, \bar{\tilde{\mathfrak{m}}}_1^{\tilde{\gamma},\tilde{b}}\right)$ computed by these spectral sequences allows us to determine whether obstructions to constructing bounding cochains or constructing pseudoisotopies vanish, particularly by examining specific components identified by the filtration (see Corollaries 5.29 and 5.34).

Definition 5.1. A spectral sequence (of cohomological type) is a collection of differential bigraded modules $\{E_r^{*,*}, d_r\}$, where $r = 0, 1, \ldots$ The differentials are of bidegree $(r, 1-r)$ and for all p, q, r, we have $E_{r+1}^{p,q} \simeq H^{p,q}\left(E_r^{*,*}, d_r\right)$.

We state the main theorems first and defer their proofs to Section 5.3, after presenting the necessary supporting lemmas and general spectral sequence machinery.

Theorem 5.2. *Let $b \in \hat{C}^\diamond$ be a degree 1 element such that $\nu_S \otimes \nu(b) > 0$. There exists a spectral sequence $(E_r^{p,q}\left(\overline{\hat{C}_{\not{\mathfrak{e}}}^\diamond}\right), d_r)$ that converges to $H^*\left(\overline{\hat{C}_{\not{\mathfrak{e}}}^\diamond}, \bar{\hat{\mathfrak{m}}}_1^{\gamma,b}\right)$ and satisfies*

$$E_1^{p,q} \simeq \begin{cases} 0, & (p \in 2\mathbb{N} \bigwedge q + 2p = 0), \\ S_{-p} \otimes H^{q+2p}(L, d), & \text{otherwise.} \end{cases}$$

If $\int_L \bar{b} = \overline{\int_L \hat{f}(b)} \neq 0$ and n is even, then

$$E_\infty^{p,q} = \begin{cases} 0, & (q + 2p = 0) \bigvee (p \in 2\mathbb{N} + 2 \bigwedge q + 2p = n), \\ E_1^{p,q}, & \text{otherwise.} \end{cases}$$

Otherwise (if $\int_L \bar{b} = 0$ or n is odd), the spectral sequence degenerates at the E_1 page, so

$$E_\infty^{p,q} = E_1^{p,q}.$$

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Theorem 5.3. *There exists a spectral sequence $(E_r^{p,q}(\overline{\mathfrak{B}}), d_r)$ that converges to $H^*\left(\overline{\mathfrak{B}}, \bar{\tilde{\mathfrak{m}}}_1^{\tilde{\gamma},\tilde{b}}\right)$.*

Its E_r page, for $r \geq 1$, is related to the E_r page of the spectral sequence for $(\hat{C}_{\mathcal{G}}^{\diamond}, \bar{\tilde{\mathfrak{m}}}_1^{\gamma,b})$ (from Theorem 5.2) by:

$$E_r^{p,q}\left(\overline{\mathfrak{B}}\right) \cong \begin{cases} E_r^{p,q-1}\left(\overline{\hat{C}_{\mathcal{G}}^{\diamond}}\right), & 2p+q \neq 0, \text{ or } p \in 2\mathbb{N}+1, \\ S_{-p}, & \text{otherwise.} \end{cases}.$$

In particular, if $\int_L \bar{b} = \overline{\int_L \hat{f}(b)} \neq 0$ and n is even, then

$$E_\infty^{p,q}\left(\overline{\mathfrak{B}}\right) \cong \begin{cases} 0, & (q+2p=1) \bigvee (q+2p=n+1 \bigwedge p \in 2\mathbb{N}+2), \\ E_1^{p,q}\left(\overline{\mathfrak{B}}\right), & \text{otherwise.} \end{cases}$$

Otherwise (if $\int_L \bar{b} = \overline{\int_L \hat{f}(b)} \neq 0$ or n is odd), the spectral sequence degenerates at the E_1 page.

5.1. Spectral sequence preliminaries.
We recall the standard construction and properties of spectral sequences associated with filtered complexes.

5.1.1. Spectral sequence of filtered complex. Let $(A, d, F^\bullet)$ be a filtered differential graded module. Following the proof of [31, Theorem 2.6], for $r \geq -1$, let

$$\begin{aligned} Z_r^{p,q} &= F^p A^{p+q} \cap d^{-1}(F^{p+r} A^{p+q+1}) \\ B_r^{p,q} &= F^p A^{p+q} \cap d(F^{p-r} A^{p+q-1}) \\ Z_\infty^{p,q} &= \ker d \cap F^p A^{p+q} \\ B_\infty^{p,q} &= \operatorname{Im} d \cap F^p A^{p+q}. \end{aligned} \tag{25}$$

For $r \geq 0$, let

$$E_r^{p,q} = Z_r^{p,q}/(Z_{r-1}^{p+1,q-1} + B_{r-1}^{p,q}) \tag{26}$$

and let $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ be the map induced by d.

The following is a consequence of the proof of [31, Theorem 2.6].

Theorem 5.4. *The sequence of differential bigraded modules $\{E_r^{*,*}, d_r\}$ above is a spectral sequence. Moreover,*

$$E_0^{p,q} = F^p A^{p+q}/F^{p+1} A^{p+q},$$

and $d_0 : E_0^{p,q} \to E_0^{p,q+1}$ is the map induced by d. Suppose further that the filtration is bounded, that is, for each dimension n, there are values $s = s(n)$ and $t = t(n)$ such that

$$\{0\} \subset F^s A^n \subset F^{s-1} A^n \subset \cdots \subset F^{t+1} A^n \subset F^t A^n = A^n.$$

Then, the spectral sequence converges to $H(A, d)$, that is,

$$E_\infty^{p,q} \cong F^p H^{p+q}(A, d)/F^{p+1} H^{p+q}(A, d).$$

Let $(A_1, d_1, F_1^\bullet), (A_2, d_2, F_2^\bullet)$ be filtered differential graded modules. A map $f : (A_1, d_1, F_1^\bullet) \to (A_2, d_2, F_2^\bullet)$ is a morphism of filtered differential graded modules if f is a cochain map and $f(F_1^p A_1) \subseteq F_2^p A_2$. Let $\{(E_r^{*,*}, d_r)\}$ and $\{(\bar{E}_r^{*,*}, \bar{d}_r)\}$ be two spectral sequences. A morphism of spectral sequences is a collection of maps of differential bigraded modules

$$f_r : E_r^{*,*} \longrightarrow \bar{E}_r^{*,*}, \quad r = 0, 1, 2, \ldots,$$

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such that for each $r \geq 0$, the map $f_{r+1} : E_{r+1}^{*,*} \cong H^{*,*}(E_r*, *, d_r) \to \bar{E}_{r+1}^{*,*} \cong H^{*,*}(\bar{E}_r^{*,*}, \bar{d}_r)$ is the map induced by f_r on cohomology.

Theorem 5.5 ([31, Theorem 3.5]). *A morphism of filtered differential graded modules*

$$\phi : (A_1, d_1, F_1^\bullet) \to (A_2, d_2, F_2^\bullet),$$

determines a morphism of the associated spectral sequences. If, for some n, $\phi_n : E_n^{*,*}(A_1) \to E_n^{*,*}(A_2)$ is an isomorphism of bigraded modules, then for $n \leq \ell \leq \infty$ the bigraded morphism $\phi_\ell : E_\ell^{*,*}(A_1) \to E_\ell^{*,*}(A_2)$ is an isomorphism. If the filtrations are bounded, then ϕ induces an isomorphism $H(\phi) : H(A_a, d_1) \to H(A_2, d_2)$.

Proposition 5.6. *Let $(A_i, d_i, F_i^\bullet)$ be filtered differential graded modules for $i = 1, 2, 3$. Suppose there exists a short exact sequence of filtered differential graded modules:*

$$0 \to (A_1, d_1, F_1^\bullet) \xrightarrow{f} (A_2, d_2, F_2^\bullet) \xrightarrow{h} (A_3, d_3, F_3^\bullet) \to 0.$$

Assume f, and h is strictly filtered, i.e., $f^{-1}(F_2^p A_2) = F_1^p A_1$ for all p. Then we obtain a short exact sequence of bigraded differential modules

$$0 \to (E_0^{*,*}(A_1), d_{1,0}) \xrightarrow{f_0} (E_0^{*,*}(A_2), d_{2,0}) \xrightarrow{h_0} (E_0^{*,*}(A_3), d_{3,0}) \to 0.$$

In particular the assumption hold if if the filtration on A_1 is the induced filtration from A_2 and the filtration on A_3 is the quotient filtration.

If in addition for all $r \geq 1$ and all p, q, the map induced by h on the cycles $Z_r^{p,q}(A_2) \to Z_r^{p,q}(A_3)$ is surjective, then the induced maps f_r and h_r on the E_r-pages of the spectral sequences form a short exact sequence of bigraded differential modules:

$$0 \to (E_r^{*,*}(A_1), d_{1r}) \xrightarrow{f_r} (E_r^{*,*}(A_2), d_{2r}) \xrightarrow{h_r} (E_r^{*,*}(A_3), d_{3r}) \to 0.$$

Proof. By the assumption that f and h are strictly filtered and [38, Tag 0124,Tag 0127], we obtain a short exact sequence

$$0 \to E_0^{p,q}(A_1) \xrightarrow{f_0^{p,q}} E_0^{p,q}(A_2) \xrightarrow{h_0^{p,q}} E_0^{p,q}(A_3) \to 0.$$

This is the case $r = 0$ of the proposition.

For $r \geq 1$, if the map induced by h on the cycles $Z_r^{p,q}(A_2) \to Z_r^{p,q}(A_3)$ is surjective, then so is the induced map $h_r : E_r^{p,q}(A_2) \to E_r^{p,q}(A_3)$. By induction we have an exact sequence,

$$0 \to (E_{r-1}^{p,q}(A_1), d_{1r}) \xrightarrow{f_{r-1}} (E_{r-1}^{p,q}(A_2), d_{2r}) \xrightarrow{h_{r-1}} (E_{r-1}^{p,q}(A_3), d_{3r}) \to 0.$$

Using the long exact sequence in cohomology with respect to d_{r-1} together with surjectivity of $h_r = H(h_{r-1})$ gives case r of the proposition. $\qquad\square$

5.1.2. *Twisted cohomology.* Let A be a unital dga. Define

$$[\cdot, \cdot] : A \times A \to A,$$

by $[a, x] = ax - (-1)^{|a||x|} xa$. For $a \in A_1$, define

$$d_{twist} : A \to A[1]$$

by $d_{twist}(x) = dx - [a, x]$.

Lemma 5.7. *If $da - \frac{[a,a]}{2} \in Z(A)$, then d_{twist} is a boundary map.*

Proof. For any $x \in A$ the graded Jacobi identity implies

$$(-1)^{|x|}[a, [a, x]] - [a, [x, a]] + (-1)^{|x|}[x, [a, a]] = 0.$$

By graded anti-symmetry this can be rewritten

$$(-1)^{|x|}[a, [a, x]] + (-1)^{|x|}[a, [a, x]] + (-1)^{|x|}[x, [a, a]] = 0.$$

The above equation implies that

$$[a, [a, x]] = -[x, \frac{[a, a]}{2}].$$

We compute

$$
\begin{aligned}
d_{twist} \circ d_{twist}(x) &= d_{twist}\left(dx - [a, x]\right) \\
&= -[a, dx] - d[a, x] + [a, [a, x]] \\
&= -(da)x + x(da) + [a, [a, x]] \\
&= -[da, x] + [a, [a, x]] \\
&= -[da, x] + [\frac{[a, a]}{2}, x] \\
&= 0.
\end{aligned}
$$

$\square$

Lemma 5.8. *If* $da - \frac{[a, a]}{2} \in Z(A)$, *then* $(A, \wedge, d_{twist})$ *is a unital dga.*

Proof. Let $\alpha, \beta \in A$, then by definition

$$d_{twist}(\alpha \wedge \beta) = d(\alpha \wedge \beta) - [a, \alpha \wedge \beta]$$

$$= d\alpha \wedge \beta + (-1)^{|\alpha|}\alpha \wedge d\beta - \left(a \wedge (\alpha \wedge \beta) + (-1)^{|\alpha|+|\beta|+1}(\alpha \wedge \beta) \wedge a\right).$$

We procced with the computations

$$= d\alpha \wedge \beta + (-1)^{|\alpha|}\alpha \wedge d\beta - \left([a, \alpha] \wedge \beta + (-1)^{|\alpha|}\alpha \wedge (a \wedge \beta) + (-1)^{|\alpha|+|\beta|+1}\alpha \wedge (\beta \wedge a)\right)$$

$$= d\alpha \wedge \beta + (-1)^{|\alpha|}\alpha \wedge d\beta - \left([a, \alpha] \wedge \beta + (-1)^{|\alpha|}\alpha \wedge [a, \beta]\right)$$

$$= d_{twist}(\alpha) \wedge \beta + (-1)^{|\alpha|}\alpha \wedge d_{twist}(\beta)$$

$\square$

Let $(S, \mathbf{e})$ be a unital graded $\Bbbk$ algebra with $S_0 = \Bbbk \cdot \mathbf{e}$ and $S_i = \{0\}$ for $i > 0$. Observe that any element in $s \in S$ can be written $s = \sum_{i=n}^{0} s_i$ where n is a negative integer and $s_i \in S_i$. Define a filtration ν_S on S by

$$\nu_S(\sum_{i=0}^{-\infty} s_i) = -\min_{s_i \neq 0} i.$$

So, if $S \not\cong \Bbbk$ then ν_S defines a non trivial filtration. For a $\Bbbk$ vector space $\mathcal{N}$, also denote by ν_S the filtration on $S \otimes \mathcal{N}$ defined by tensoring ν_S with the trivial filtration on $\mathcal{N}$. Let A be a differential graded algebra over $\Bbbk$. Let $\hat{A}$ be the S extension of scalars of A. Endow $\hat{A}$ with the filtration ν_S. We give $\hat{A}$ the obvious filtered differential graded structure over $\Bbbk$. Observe that $Z(\hat{A}) = Z(S) \otimes_{\Bbbk} Z(A)$.

Let $b \in \hat{A}_1$ be an element such that $\nu_S(b) > 0$ and $db - \frac{[b,b]}{2} \in Z(\hat{A})$. By Lemma 5.7 the differential d_{twist} defined by b is a boundary map. Assume that A is bounded from above by $m \in \mathbb{N}$. So, $\hat{A}_1 = \bigoplus_{i=0}^{m-1} S_{-i} \otimes A_{1+i}$, and we can write $b = \sum_{i=0}^{m-1} b_i$ with $b_i \in S_{-i} \otimes A_{1+i}$. The assumption $\nu_S(b) > 0$ implies that $b_0 = 0$.

Theorem 5.9. *There exists a spectral sequence that converges to $H^*(\hat{A}, d_{twist})$ with $E_1^{p,q} \cong S_{-p} \otimes H^{2p+q}(A)$. Moreover, each element $\alpha \in E_r^{p,q}$ can be represented by an element $x \in Z_r^{p,q}$ such that*

$$x = \sum_{i=0}^{\infty} x_i, \qquad x_i \in S_{-p-i} \otimes A_{2p+i+q},$$

$$d \sum_{i=0}^{r-1} x_i - \sum_{\substack{1 \leq j \leq m-1 \\ 0 \leq i < r-j}} [b_j, x_i] = 0.$$

Furthermore,

$$d_r(\alpha) = d_r([x]) = \left[- \sum_{0 \leq i \leq r-1} [b_{r-i}, x_i] + F^{p+r+1}\hat{A} \right],$$

where $b_{r-i} = 0$ if $r - i < 1$ or $r - i > m - 1$.

Proof. Observe the long exact sequence

$$\cdots \xrightarrow{k} H^{p+q}\left(F^{p+1}\hat{A}\right) \xrightarrow{i} H^{p+q}\left(F^p\hat{A}\right) \xrightarrow{j} H^{p+q}\left(F^p\hat{A}/F^{p+1}\hat{A}\right)$$

$$\xrightarrow{k}$$

$$H^{p+q+1}\left(F^{p+1}\hat{A}\right) \xrightarrow{i} H^{p+q+1}\left(F^p\hat{A}\right) \xrightarrow{j} H^{p+q+1}\left(F^p\hat{A}/F^{p+1}\hat{A}\right) \xrightarrow{k} \cdots$$

where k is the connecting homomorphism. Define the bigraded modules

$$E_1^{p,q} = H^{p+q}\left(F^p\hat{A}/F^{p+1}\hat{A}\right),$$

and $D^{p,q} = H^{p+q}\left(F^p\hat{A}\right)$. This gives an exact couple from the long exact sequences:

$$\begin{array}{ccc}
D^{p+1,q-1} & \xrightarrow{\quad i \quad} & D^{p,q} \\
& & \Big\downarrow{\scriptstyle j} \quad . \\
D^{p+1,q} & & \\
\scriptstyle k \nwarrow & & \\
& E_1^{p,q} &
\end{array}$$

By [31, Theorem 2.6] This yields a spectral sequence $(E_r^{*,*}, d_r)$. By [31, Proposition 2.11] the spectral sequence is the same as Theorem 5. By Theorem 5.4 d_0 is induced form d_{twist}. We observe that $d_0 = d$ as $\nu_S(b) > 0$. So, $E_1^{p,q} \cong S_{-p} \otimes H^{2p+q}(A)$, and the spectral sequence converges.

By [31, Proposition 2.9]

$$E_r^{p,*} \cong \frac{k^{-1}\left(\operatorname{im} i^{r-1} : D^{p+r,*} \longrightarrow D^{p+1,*}\right)}{j\left(\ker i^{r-1} : D^{p,*} \longrightarrow D^{p-r+1,*}\right)}$$

with differential induced on the representation by $j \circ k$.

Suppose z is in $H^{p+q}\left(F^p\hat{A}/F^{p+1}\hat{A}\right) = E_1^{p,q}$. Then $z = \left[x + F^{p+1}\hat{A}\right]$ with x in $F^p\hat{A}$ and $d_{twist}(x)$ in $F^{p+1}\hat{A}$. The boundary homomorphism k in the long exact sequence that is part of the exact couple can be described explicitly by

$$k\left(\left[x + F^{p+1}\hat{A}\right]\right) = [d_{twist}(x)] \quad \text{in} \quad H^{p+q+1}\left(F^{p+1}\hat{A}\right).$$

Thus $\left[x + F^{p+1}\hat{A}\right]$ is in $k^{-1}\left(\operatorname{im} i^{r-1}\right)$ if and only if $[d_{twist}(x)]$ is in $\operatorname{im} i^{r-1}$ if and only if $d_{twist}(x)$ is in $F^{p+r}\hat{A}_{p+q+1}$. For $x \in F^p\hat{A}$ a representative of an element in $k^{-1}\left(\operatorname{im} i^{r-1}\right)$ we decompose as

$$x = \sum_{i=0}^{\infty} x_i, \qquad x_i \in S_{-p-i} \otimes A_{2p+i+q}.$$

We have

$$d_{twist}(x) = d\sum_{i=0}^{\infty} x_i - [\sum_{j=1}^{m-1} b_j, \sum_{i=0}^{\infty} x_i].$$

So, the condition that $d_{twist}(x) \in F^{p+r}\hat{A}_{p+q+1}$ is equivalent to

$$d\sum_{i=0}^{r-1} x_i - \sum_{\substack{1 \le j \le m-1 \\ 0 \le i < r-j}} [b_j, x_i] = 0.$$

It follows that

$$d_{twist}(x) \in dx_r - \sum_{0 \le i \le r-1} [b_{r-i}, x_i] + F^{p+r+1}\hat{A}.$$

Thus,

$$j \circ k\left(\left[x + F^{p+1}\hat{A}\right]\right) = \left[-\sum_{0 \le i \le r-1} [b_{r-i}, x_i] + F^{p+r+1}\hat{A}\right],$$

where $b_{r-i} = 0$ if $r - i < 1$ or $r - i > m - 1$. $\qquad \square$

Definition 5.10. A spectral sequence of algebras over R is a spectral sequence, $\{E_r^{*,*}, d_r\}$ together with algebra structures $\psi_r : E_r \otimes_R E_r \to E_r$ for each r, such that ψ_{r+1} can be written as the composite

$$\psi_{r+1} : E_{r+1} \otimes_R E_{r+1} \overset{\cong}{\Rightarrow} H\left(E_r\right) \otimes_R H\left(E_r\right)$$

$$\underset{p}{\longrightarrow} H\left(E_r \otimes E_r\right) \xrightarrow[H(\psi_r)]{} H\left(E_r\right) \underset{\cong}{\longrightarrow} E_{r+1},$$

where the homomorphism p is given by $p([u] \otimes [v]) = [u \otimes v]$.

Theorem 5.11. *[31, Theorem 2.14] Suppose (A, d, F) is a filtered differential graded algebra over R with product $\psi : A \otimes_R A \to A$. Suppose that the product satisfies the condition for all p, q,*

$$\psi \left(F^p A \otimes_R F^q A \right) \subset F^{p+q} A.$$

Then, the spectral sequence associated with (A, d, F) is a spectral sequence of algebras. If the filtration on A is bounded, then the spectral sequence converges to $H(A, d)$ as an algebra.

Lemma 5.12. *Assume $b \in Z_1^{\ell, 1-\ell}$, then $E_\ell = E_1$. Moreover, the differential $d_\ell : E_\ell^{p,q} \to E_\ell^{p+\ell, q+1-\ell}$ is given by*

$$d_\ell(\alpha) = -[[b], \alpha],$$

where $[b]$ denotes the class represented by b in $E_\ell^{\ell, 1-\ell} = E_1^{\ell, 1-\ell}$, and the commutator $[\cdot, \cdot]$ is taken with respect to the algebra structure on the E_1 page (induced by the product ψ on A).

Proof. We observe that Theorem 5.9 implies that if $b \in F^\ell \hat{A}_1$, then $d_i = 0$ for $1 \le i \le \ell - 1$. It follows that an element $\alpha \in E_\ell^{p,q}$ can be represented by $x \in S_{-p} \otimes A_{2p+q}$ such that $dx = 0$. Therefore,

$$
\begin{aligned}
d_\ell(\alpha) = d_\ell([x]) &= [dx - [b, x]] \\
&= \left[-\psi(b, x) + (-1)^{|x|} \psi(x, b) \right] \\
&= -\psi_1([b], [x]) + (-1)^{|x|} \psi_1([x], [b]),
\end{aligned}
$$

where the last equality follows from Theorem 5.11. $\qquad\square$

5.1.3. Mapping cones.

Definition 5.13. Let (C, d_C) and (D, d_D) be two cochain complexes. Let $f : C \to D$ be a cochain map. The **mapping cone** of f, denoted by $\mathrm{Cone}(f)$ is a cochain complex defined as follows. For each integer n, the module in degree n is the direct sum:

$$\mathrm{Cone}(f)_n = D_{n-1} \oplus C_n.$$

The differential $d_{Cone} : \mathrm{Cone}(f)_n \to \mathrm{Cone}(f)_{n+1}$ is defined for an element $(d, c) \in D_{n-1} \oplus C_n$ by,

$$d_{Cone}(d, c) = (-d_D^{n-1}(d) + f^n(c), d_C^n(c)).$$

The mapping cone fits into a short exact sequence of cochain complexes:

$$0 \longrightarrow D[-1] \overset{h}{\to} \mathrm{Cone}(f) \overset{p}{\to} C \longrightarrow 0, \tag{27}$$

where $h(d) = (d, 0)$ for $d \in D[-1]_n$, $p(d, c) = c$ for $(d, c) \in \mathrm{Cone}(f)_n = D_{n-1} \oplus C_n$.

For filtered cochain complexes (A, d, F) and (B, d', F') we define the filtration on $A \oplus B$ by $F_{A \oplus B}^p (A \oplus B)_n = F^p A \oplus F'^p B$. We also define $F^p A[n]_m = F^p A_{n+m}$. For $f : A \to B$ define the filtration on $\mathrm{Cone}(f)$ by

$$F^p \mathrm{Cone}(f) = F'^p B[-1] \oplus F^p A.$$

Observe that

$$E_r^{p,q}(A[n]) = E_r^{p, q+n}(A) \tag{28}$$

because $Z_r^{p,q}(A[n]) = Z_r^{p, q+n}(A)$ and $B_r^{p,q}(A[n]) = B_r^{p, q+n}(A)$.

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Lemma 5.14. *Let $f : C \to D$ be a morphism of filtered chain complexes. Let $f_r : E_r^{p,q}(C) \to E_r^{p,q}(D)$ be the map induced by f on the r-th pages of the associated spectral sequences. If f_r is injective for all $r \geq 1$, then there exists a short exact sequence of complexes:*

$$0 \longrightarrow E_r^{p,q}(C) \xrightarrow{\ f_r\ } E_r^{p,q}(D) \xrightarrow{\ h[1]_r\ } E_r^{p,q}(\mathrm{Cone}(f)[1]) \longrightarrow 0.$$

Proof. By definition of the filtration on $Cone(f)$, the morphisms in the exact sequence (27) are strictly filtered. By Proposition 5.6, we obtain a long exact sequence

$$\ldots \longrightarrow E_1^{p,q}(C) \xrightarrow{\ \partial\ } E_1^{p,q+1}(D[-1]) \xrightarrow{\ h_1\ } E_1^{p,q+1}(\mathrm{Cone}(f)) \xrightarrow{\ p_1\ } E_1^{p,q+1}(C) \longrightarrow \ldots,$$

By definition of $Cone(f)$ and a short computation, the boundary homomorphism ∂ coincides with the induced map $f_1 : E_1^{p,q}(C) \to E_1^{p,q}(D) = E_1^{p,q+1}(D[-1])$ given by Theorem 5.5. Since f_1 is injective by assumption, keeping in mind equation (28), we obtain a short exact sequence

$$0 \longrightarrow E_1^{p,q}(C) \xrightarrow{\ f_1\ } E_1^{p,q}(D) \xrightarrow{\ h[1]_1\ } E_1^{p,q}(\mathrm{Cone}(f)[1]) \longrightarrow 0,$$

and the base of the induction holds.

By the induction hypothesis for $r - 1$, there arises a long exact sequence from the short exact sequence of the $r - 1$ pages,

By assumption, f_r is injective. Thus, the long exact sequence splits into short exact sequences, and the result follows by induction. $\qquad\square$

5.2. Application to ν_S filtration.

We apply the general theory to the specific filtration ν_S on $\widehat{C}^\diamond$ and $\overline{\widehat{\mathfrak{C}}}^\diamond$. By abuse of notation, we write $\tilde{\mathfrak{m}}^{\tilde{\gamma}}$ instead of $\hat{\tilde{\mathfrak{m}}}^{\tilde{\gamma}}$. The relevant differentials for the spectral sequence are $\hat{\bar{\mathfrak{m}}}_1^{\gamma,b}$ and $\bar{\tilde{\mathfrak{m}}}_1^{\tilde{\gamma},\tilde{b}}$ respectively. By Corollary 4.26, we have

$$\hat{\bar{\mathfrak{m}}}_1^{\gamma,b}(\eta) = (\mathrm{Id} \otimes d)\eta - [\bar{b}, \eta] \tag{29}$$

and similarly for $\bar{\tilde{\mathfrak{m}}}_1^{\tilde{\gamma},\tilde{b}}$. Thus, these differentials are a special case of the differential d_{twist} from Section 5.1.2.

5.2.1. *Base calculations.*

Lemma 5.15. *The canonical isomorphism of Lemma 2.22 induces an isomorphism*

$$F^p_{\nu_S} \overline{\hat{C}^\diamond}_h \simeq \bigoplus_{\substack{\ell \geq p \\ -\ell+m=h \\ 0 \leq m \leq n}} S_{-\ell} \otimes A^m(L) = \bigoplus_{\substack{\ell \geq p \\ -h \leq \ell \leq n-h}} S_{-\ell} \otimes A^{h+\ell}(L)$$

$$F^p_{\nu_S} \overline{\hat{\mathfrak{C}}^\diamond}_h \simeq \bigoplus_{\substack{\ell \geq p \\ -\ell+m=h \\ 0 \leq m \leq n+1}} S_{-\ell} \otimes A^m(I \times L) = \bigoplus_{\substack{\ell \geq p \\ -h \leq \ell \leq n+1-h}} S_{-\ell} \otimes A^{h+\ell}(I \times L)$$

Proof. We calculate,

$$F^p_{\nu_S} \overline{\hat{C}^\diamond}_h \simeq \bigoplus_{\substack{\ell \geq p \\ -\ell+m=h \\ 0 \leq m \leq n}} S_{-\ell} \otimes A^m(L) = \bigoplus_{\substack{\ell \geq p \\ -h \leq \ell \leq n-h}} S_{-\ell} \otimes A^{h+\ell}(L).$$

$\square$

Lemma 5.16. *For every $h \in \mathbb{Z}$,*

$$F^{-h}_{\nu_S} \overline{\hat{C}^\diamond}_h = \overline{\hat{C}^\diamond}_h, \qquad F^{n-h}_{\nu_S} \overline{\hat{C}^\diamond}_h \simeq S_{h-n} \otimes A^n(L), \qquad F^{n-h+1}_{\nu_S} \overline{\hat{C}^\diamond}_h = 0,$$

and

$$F^{-h}_{\nu_S} \left(\overline{\hat{C}^\diamond_{\not{\mathfrak{C}}}} \right)_h = \left(\overline{\hat{C}^\diamond_{\not{\mathfrak{C}}}} \right)_h, \qquad F^{n-h}_{\nu_S} \left(\overline{\hat{C}^\diamond_{\not{\mathfrak{C}}}} \right)_h \simeq S_{h-n} \otimes A^n(L), \qquad F^{n-h+1}_{\nu_S} \left(\overline{\hat{C}^\diamond_{\not{\mathfrak{C}}}} \right)_h = 0.$$

In particular, the filtration induced by ν_S on $\overline{\hat{C}^\diamond_{\not{\mathfrak{C}}}}$ is bounded in all degrees. Similar statements hold for $\overline{\hat{\mathfrak{C}}^\diamond}$ and $\overline{\mathfrak{B}}$.

Proof. The first line is obtained by inserting $p = -h, n - h, n - h + 1$ in the isomorphism of Lemma 5.15 above. The second line follows by doing the same computation in the quotient $\hat{C}^\diamond_{\not{\mathfrak{C}}}$. $\square$

Observe that $Z(S) \cdot e \in \ker\left(\bar{\mathfrak{m}}^{\gamma,b}_1 \right)$ by the Properties (2) and (4) of Definition 2.11.

Lemma 5.17. *For $0 \leq \ell \leq \infty$ we have $E^{p,q}_\ell (Z(S) \cdot 1) = S_{-p} \cdot 1$, if $2p + q = 0$ and $p \in 2\mathbb{N}$. Otherwise, it is trivial.*

Proof. By equation (29) the differential on $Z(S) \cdot 1$ is trivial. By Corollary 3.30 $Z(S) = \bigoplus_{i \in \mathbb{N}} S_{-2i}$. Theorem 5.4 implies that

$$E^{p,q}_\ell (Z(S) \cdot 1) = E^{p,q}_0 (Z(S) \cdot 1) = \frac{F^p_{\nu_S} (Z(S) \cdot 1)_{p+q}}{F^{p+1}_{\nu_S} (Z(S) \cdot 1)_{p+q}} \cong Z(S)_{-p} \otimes (\mathbb{R})_{2p+q}.$$

The claim follows by combining the above observations. $\square$

Let $\overline{\mathfrak{B}}_0$ be the submodule of $\left(\overline{\hat{\mathfrak{C}}^\diamond} \right)$ defined by

$$\left\{ a \in \overline{\hat{\mathfrak{C}}^\diamond} : j^*_i(a) = 0 \text{ for } i = 0, 1 \right\}.$$

The submodule $\overline{\mathfrak{B}}_0$ is preserved by $\tilde{\mathfrak{m}}_1^{\tilde{\gamma},\tilde{b}}$ and thus is a subcomplex. Recall Definition 4.40 concerning the action of an anti-symplectic involution ϕ on shifted complexes. In particular, taking the subcomplex fixed by $-\phi^*$ does not commute with shifts. For $v \in \mathbb{Z}$ and $\kappa = Id, -\phi^*$, let $\mathcal{A}$ denote the cochain complex $\widehat{\widehat{C}}^{\diamond}[v]^{\kappa}, \widehat{\tilde{\mathfrak{C}}}^{\diamond}[v]^{\kappa}$ or $\overline{\mathfrak{B}}_0[v]^{\kappa}$ with boundary operator $\mathfrak{m}_1 = \bar{\hat{\mathfrak{m}}}_1^{\gamma,b}[v], \bar{\tilde{\mathfrak{m}}}_1^{\tilde{\gamma},\tilde{b}}[v]$ or $\tilde{\mathfrak{m}}_1^{\tilde{\gamma},\tilde{b}}[v]$. When $\kappa = -\phi^*$, we additionally assume that b and $\tilde{b}$ are real, so that Lemma 4.43 ensures $\mathcal{A}$ is indeed a complex. Let $\mathcal{L} = L, I \times L$ or $(I \times L, \partial I \times L)$. Let $(E_0^{p,q}(\mathcal{A}), d_0)$ be the E_0 page of the spectral sequence associated with $(\mathcal{A}, \mathfrak{m}_1, F_{\nu_S})$. Define $A^{p,q}(\mathcal{L}) := S_{-p}[v]^{\kappa} \otimes A^{q+2p}(\mathcal{L})$. Let

$$\iota_{p,q} : A^{p,q}(\mathcal{L}) \to E_0^{p,q}(\mathcal{A})$$

denote the map induced by the inverse of the canonical isomorphism of Lemma 2.22 composed with the quotient map $\mathcal{A} \to E_0(\mathcal{A})$.

Lemma 5.18. *The operator $d_0 : E_0^{p,q}(\mathcal{A}) \to E_0^{p,q}(\mathcal{A})$ agrees with the map induced from* $\mathfrak{m}_1 : F_{\nu_S}^p(\mathcal{A})_{p+*} \to F_{\nu_S}^p(\mathcal{A})_{p+*+1}$.

Proof. We write the proof for $\mathcal{A} = \widehat{\widehat{C}}^{\diamond}$. The other cases are similar. Let $\alpha' \in \widehat{C}^{\diamond}$ represent an element $\alpha \in E_0$. As $\nu_S \otimes \nu(b) > 0$, it follows from Lemma 3.12 and property (3) of Definition 2.11 that

$$[\overline{\hat{\mathfrak{m}}_1^{\gamma,b}(\alpha')}] = [\bar{\hat{\mathfrak{m}}}_1^{\gamma}(\overline{\alpha'})] .$$

By Theorem 5.4 $d_0(\alpha) = [\overline{\hat{\mathfrak{m}}_1^{\gamma,b}(\alpha')}]$, and the result follows. $\square$

Lemma 5.19. *The map $\iota_{p,\bullet} : (A^{p,\bullet}(\mathcal{L}), \mathrm{Id} \otimes d) \to (E_0^{p,\bullet}(\mathcal{A}), d_0)$ is a cochain isomorphism.*

Proof. We write the proof for $\mathcal{A} = \widehat{\widehat{C}}^{\diamond}$. The other cases are similar. It follows from Lemma 5.15 that $\iota_{p,q}$ is an isomorphism of modules. By Lemma 5.18, the differential $d_0 : E_0^{p,*} \to E_0^{p,*+1}$ is the map induced from $\bar{\hat{\mathfrak{m}}}_1^{\gamma} : F_{\nu_S}^p \widehat{\widehat{C}}^{\diamond}_{p+*} \to F_{\nu_S}^p \widehat{\widehat{C}}^{\diamond}_{p+*+1}$. It follows from Proposition 2.23 that

$$d_0 \circ \iota_{p,\bullet} = \iota_{p,\bullet+1} \circ (\mathrm{Id} \otimes d) .$$

So, $\tilde{\iota}_{p,\bullet}$ is a cochain map. $\square$

5.2.2. *The spectral sequence for twisted relative cohomology.* Here, we collect results that are used to analyze the spectral sequence for $(\overline{\mathfrak{B}}_0, \bar{\tilde{\mathfrak{m}}}_1^{\tilde{\gamma},\tilde{b}})$ culminating in Corollary 5.24.

Corollary 5.20. *Let $i = 1$ or $i = 0$. The map $j_i^* : \left(\widehat{\tilde{\mathfrak{C}}}^{\diamond}[v]^{\kappa}, \bar{\tilde{\mathfrak{m}}}_1^{\tilde{\gamma},\tilde{b}}\right) \to \left(\widehat{\widehat{C}}^{\diamond}[v]^{\kappa}, \bar{\tilde{\mathfrak{m}}}_1^{\gamma,j_i^*(\tilde{b})}\right)$ induces an isomorphism $E_1^{p,q}(\widehat{\tilde{\mathfrak{C}}}^{\diamond}[v]^{\kappa}) \to E_1^{p,q}(\widehat{\widehat{C}}^{\diamond}[v]^{\kappa})$.*

Proof. Consider the commutative diagram

$$
\begin{array}{ccc}
H^q\left(A^{p,\bullet}(I \times L)\right) & \xrightarrow{\quad j_i^* \quad} & H^q\left(A^{p,\bullet}(L)\right) \\
\downarrow {\scriptstyle (\iota_{p,q})_*} & & \downarrow {\scriptstyle (\iota_{p,q})_*} \\
E_1^{p,q}(\widehat{\tilde{\mathfrak{C}}}^{\diamond}) & \xrightarrow{\quad j_i^* \quad} & E_1^{p,q}(\widehat{\widehat{C}}^{\diamond})
\end{array}
$$

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The vertical maps are isomorphisms by Lemma 5.19. Observe that j_i is a deformation retract of topological spaces. Thus, the top horizontal map is an isomorphism. It follows that the bottom horizontal map is also an isomorphism. $\qquad\square$

Corollary 5.21. *For $\ell \geq 1$, the map $j_i^* : \left(\widehat{\overline{\mathfrak{C}^\diamond}}[v]^\kappa, \bar{\tilde{\mathfrak{m}}}_1^{\tilde{\gamma},\tilde{b}}\right) \to \left(\widehat{\overline{C^\diamond}}[v]^\kappa, \bar{\hat{\mathfrak{m}}}_1^{\gamma, j_i^*(\tilde{b})}\right)$ induces an isomorphism $E_\ell^{p,q}(\widehat{\overline{\mathfrak{C}^\diamond}}[v]^\kappa) \to E_\ell^{p,q}(\widehat{\overline{C^\diamond}}[v]^\kappa)$.*

Proof. Corollary 5.20 and Theorem 5.5 imply the result. $\qquad\square$

In light of Corollary 5.21, the spectral sequence associated with $(\widehat{\overline{C^\diamond}}, \bar{\hat{\mathfrak{m}}}_1^{\gamma, j_i^*(\tilde{b})}, F_{\nu_S})$ is independent up to isomorphism of the choice of $i \in \{0,1\}$ for $\ell \geq 1$. Thus, by abuse of notation, we write $E_\ell^{p,q}(\widehat{\overline{C^\diamond}})$ for both.

Corollary 5.22. *For the filtered cochain complex map*

$$(j_0^*, j_1^*) : \left(\widehat{\overline{\mathfrak{C}^\diamond}}[v]^\kappa, \bar{\tilde{\mathfrak{m}}}_1^{\tilde{\gamma},\tilde{b}}\right) \to \left(\widehat{\overline{C^\diamond}}[v]^\kappa \oplus \widehat{\overline{C^\diamond}}[v]^\kappa, \bar{\hat{\mathfrak{m}}}_1^{\gamma, j_0^*(\tilde{b})}[v] \oplus \bar{\hat{\mathfrak{m}}}_1^{\gamma, j_1^*(\tilde{b})}[v]\right),$$

we have for $1 \leq \ell \leq \infty$,

$$E_\ell^{p,q+1}\left(\text{Cone}\left((j_0^*, j_1^*)\right)\right) \cong E_\ell^{p,q}\left(\widehat{\overline{C^\diamond}}[v]^\kappa\right).$$

Proof. The filtration on $\widehat{\overline{\mathfrak{C}^\diamond}}[v]$ and $\widehat{\overline{C^\diamond}}[v]$ is bounded in each degree by Lemma 5.16, and hence the spectral sequence converges for each p and q for some r. By Corollary 5.21, the assumptions of Lemma 5.14 are satisfied. Hence, there exists an exact sequence of modules

$$0 \longrightarrow E_\ell^{p,q}\left(\widehat{\overline{\mathfrak{C}^\diamond}}[v]^\kappa\right) \xrightarrow{(j_0^*, j_1^*)} E_\ell^{p,q}\left(\widehat{\overline{C^\diamond}}[v]^\kappa \oplus \widehat{\overline{C^\diamond}}[v]^\kappa\right) \longrightarrow E_\ell^{p,q+1}\left(\text{Cone}\left((j_0^*, j_1^*)\right)\right) \longrightarrow 0.$$

The results now follow from Corollary 5.21. $\qquad\square$

Lemma 5.23. *The inclusion $\iota : \overline{\mathfrak{B}_0}[v]^\kappa \to \text{Cone}\left((j_0^*, j_1^*)\right)$ is a filtered morphism. Moreover, it induces an isomorphism $E_r^{p,q}\left(\overline{\mathfrak{B}_0}[v]^\kappa\right) \cong E_\ell^{p,q}\left(\text{Cone}\left((j_0^*, j_1^*)\right)\right)$, for all $\ell \geq 1$.*

Proof. Fix p and let $\rho_t^* : A^{p,\bullet}(I \times L) \to A^{p,\bullet}(L)$ for $t = 0,1$ denote the restriction maps. Consider the following commutative diagram.

$$\begin{array}{ccc}
A^{p,\bullet}(I \times L, \partial I \times L) & \xrightarrow{\iota_{p,\bullet}} & E_0^{p,\bullet}\left(\overline{\mathfrak{B}_0}\right) \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} \\
\text{Cone}\left((\rho_0^*, \rho_1^*)\right) & \xrightarrow[\iota_{p,\bullet}[-1]\oplus\iota_{p,\bullet}[-1]\oplus\iota_{p,\bullet}]{} & E_0^{p,\bullet}\left(\text{Cone}\left((j_0^*, j_1^*)\right)\right)
\end{array}$$

It follows from Lemma 5.19 that the horizontal arrows are isomorphisms. The left vertical arrow is the standard quasi-isomorphism taking a relative form to the mapping cone of the restrictions. Consequently, the right vertical arrow is a quasi-isomorphism. The result follows by applying Theorem 5.5 to the morphism $i : \overline{\mathfrak{B}_0} \to \text{Cone}\left((j_0^*, j_1^*)\right)$. $\qquad\square$

Corollary 5.24. *For all $1 \leq \ell \leq \infty$, there is an isomorphism*

$$E_\ell^{p,q}\left(\overline{\mathfrak{B}_0}[v]^\kappa\right) \cong E_\ell^{p,q-1}\left(\widehat{\overline{C^\diamond}}[v]^\kappa\right).$$

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Proof. This follows from Lemma 5.23 and Corollary 5.22. $\qquad\square$

Remark 5.25. It is possible to calculate the spectral sequence for $(\overline{\mathfrak{B}}_0, \tilde{\tilde{\mathfrak{m}}}_1^{\tilde{\gamma},\tilde{b}})$ using Corollary 5.21 and the short exact sequence $0 \to \overline{\mathfrak{B}}_0 \to \overline{\widehat{\mathfrak{C}}^\diamond} \to \widehat{C}^\diamond \oplus \widehat{C}^\diamond \to 0$. The boundary homomorphism of the associated long exact sequence reflects the twisted differential. We use the cone complex instead to avoid explicitly calculating this boundary homomorphism.

5.2.3. *Relations between the spectral sequences of the pseudoisotopy.*

Lemma 5.26. *We have a short exact sequence*

$$0 \longrightarrow \overline{\mathfrak{B}}_0[v]^\kappa \overset{\varkappa}{\hookrightarrow} \overline{\mathfrak{B}}[v]^\kappa \overset{(j_0^*, j_1^*)}{\longrightarrow} Z(S) \cdot e[v]^\kappa \oplus Z(S) \cdot e[v]^\kappa \longrightarrow 0$$

Proof. By Corollary 4.26, we have $\tilde{\tilde{\mathfrak{m}}}_1^{\gamma,\tilde{b}}(\eta) = (\mathrm{Id} \otimes d)\eta - [\tilde{\tilde{b}}, \eta]$. It follows from Corollary 3.30 that $Z(S) \cdot e \oplus Z(S) \cdot e$ is a subcomplex of $\widehat{C}^\diamond \oplus \widehat{C}^\diamond$. So, the short exact sequence of modules

$$0 \longrightarrow \overline{\mathfrak{B}}_0^\kappa \overset{\varkappa}{\hookrightarrow} \overline{\mathfrak{B}}^\kappa \overset{(j_0^*, j_1^*)}{\longrightarrow} Z(S)^\kappa \cdot e \oplus Z(S)^\kappa \cdot e \longrightarrow 0,$$

is a short exact sequence of cochain complexes. $\qquad\square$

Lemma 5.27. *Let $\ell \geq 1$ be an integer. The map $\varkappa_\ell^{p,q} : E_\ell^{p,q}\left(\overline{\mathfrak{B}}_0[v]^\kappa\right) \to E_\ell^{p,q}\left(\overline{\mathfrak{B}}[v]^\kappa\right)$ is an isomorphism if $2p + q + v \neq 0, 1$ or p is odd. Moreover $E_\ell^{p,q}\left(\overline{\mathfrak{B}}[v]^\kappa\right) = 0$, if p is even and $2p + q + v = 1$. If p is even and $2p + q + v = 0$, then $E_\ell^{p,q}\left(\overline{\mathfrak{B}}[v]^\kappa\right) \simeq (S[v]^\kappa)_{-p-v} \otimes 1$.*

Proof. For simplicity, we assume that $v = 0$. The proof for $v \neq 0$ is similar. We proceed by induction on ℓ.

Base case $\ell = 1$: We determine the behavior of $\varkappa_1^{p,q}$ using the long exact sequence associated with the short exact sequence in Lemma 5.26:

$$\cdots \to E_1^{p,q-1}(K) \overset{\partial}{\to} E_1^{p,q}\left(\overline{\mathfrak{B}}_0^\kappa\right) \overset{\varkappa_1}{\to} E_1^{p,q}\left(\overline{\mathfrak{B}}^\kappa\right) \overset{(j_0^*, j_1^*)}{\longrightarrow} E_1^{p,q}(K) \to \cdots$$

where $K = Z(S)^\kappa \cdot e \oplus Z(S)^\kappa \cdot e$. We analyze the map $\varkappa_1^{p,q}$ in four cases:

- If p is odd: The terms $E_1^{p,q}(K)$ and $E_1^{p,q-1}(K)$ vanish by Lemma 5.17. Thus, $\varkappa_1^{p,q}$ is an isomorphism.
- If p is even and $2p + q \neq 0, 1$: The condition $2p + q \neq 0$ ensures $E_1^{p,q}(K) = 0$ by Lemma 5.17. Similarly, the condition $2p + q \neq 1$ implies $2p + (q - 1) \neq 0$, so $E_1^{p,q-1}(K) = 0$. Consequently, $\varkappa_1^{p,q}$ is an isomorphism.
- If p is even and $2p + q = 0$: Lemma 5.15 implies $E_0^{p,q-1}\left(\overline{\mathfrak{B}}^\kappa\right) = E_0^{p,q-1}\left(\overline{\mathfrak{C}}^\kappa\right) = 0$, so

$$E_1^{p,q}\left(\overline{\mathfrak{B}}^\kappa\right) = \ker d_0|_{\overline{\mathfrak{B}}} \subset \ker d_0 = E_1^{p,q}\left(\overline{\mathfrak{C}}^\kappa\right) \subset E_0^{p,q}\left(\overline{\mathfrak{C}}^\kappa\right).$$

  By Lemma 5.19, we have $E_1^{p,q}\left(\overline{\mathfrak{C}}^\kappa\right) = \iota_{p,q}((S^\kappa)_{-p} \otimes 1) \subset E_1^{p,q}\left(\overline{\mathfrak{B}}^\kappa\right)$. Thus, $E_1^{p,q}\left(\overline{\mathfrak{B}}^\kappa\right) \simeq (S^\kappa)_{-p} \otimes 1$.
- If p is even and $2p + q = 1$: By Corollary 5.24, and Lemma 5.19 we identify $E_1^{p,q}\left(\overline{\mathfrak{B}}_0^\kappa\right) \cong (S^\kappa)_{-p} \otimes 1$ and $E_1^{p,q-1}\left(\overline{\mathfrak{B}}_0^\kappa\right) = 0$. By Lemma 5.17, we have $E_1^{p,q-1}(K) \cong (S^\kappa)_{-p} \otimes 1 \oplus (S^\kappa)_{-p} \otimes 1$ and by the result of the preceding case, $E_1^{p,q-1}\left(\overline{\mathfrak{B}}^\kappa\right) \cong (S^\kappa)_{-p} \otimes 1$. Substituting these calculations into the relevant segment of the exact sequence yields:

$$0 \to (S^\kappa)_{-p} \otimes 1 \overset{(j_0^*, j_1^*)}{\longrightarrow} (S^\kappa)_{-p} \otimes 1 \oplus (S^\kappa)_{-p} \otimes 1 \overset{\partial}{\to} (S^\kappa)_{-p} \otimes 1 \overset{\varkappa_1}{\to} E_1^{p,q}\left(\overline{\mathfrak{B}}^\kappa\right) \to 0.$$

Exactness and finite dimensionality implies $E_1^{p,q}\left(\overline{\mathfrak{B}}^\kappa\right) = 0$.

This establishes the base case.

Inductive step: Assume by induction that $\varkappa_\ell^{p,q}$ is an isomorphism if $2p + q \neq 0, 1$ or if p is odd. We assume $2p + q \geq 0$, otherwise $E_\ell^{p,q}(\overline{\mathfrak{B}}_0) = E_\ell^{p,q}(\overline{\mathfrak{B}}) = 0$ by Lemma 5.15 and Theorem 5.4.

By [32, Lemma 1.2.3], $\varkappa_{\ell+1}^{p,q}$ is an epimorphism if $\varkappa_\ell^{p+\ell,q+1-\ell}$ is a monomorphism. By the induction hypothesis $\varkappa_\ell^{p+\ell,q+1-\ell}$ is a monomorphism if $2(p+\ell) + (q + 1 - \ell) \neq 0, 1$. This is indeed the case since

$$2(p + \ell) + (q + 1 - \ell) = 2p + q + 1 + \ell > 1.$$

Thus, $\varkappa_{\ell+1}^{p,q}$ is an epimorphism.

We now show that $\varkappa_{\ell+1}^{p,q}$ is a monomorphism. Let $(p', q') = (p - \ell, q - 1 + \ell)$. We distinguish two cases based on the domain of the relevant differential.

- Case 1: p' is even and $2p' + q' = 0$. It follows from the base of the induction that if $(S^\kappa)_{-p'} \otimes 1 \neq 0$, then $s^{\frac{p'}{n-1}} \otimes 1$ is a generator of $E_1^{p',q'}(\overline{\mathfrak{B}})$. We calculate using equation (29)

$$\bar{\bar{\mathfrak{m}}}_1^{\tilde{\gamma},\tilde{b}}(s^{\frac{p'}{n-1}} \otimes 1) = d(s^{\frac{p'}{n-1}} \otimes 1) - [\bar{\bar{b}}, s^{\frac{p'}{n-1}} \otimes 1] = 0.$$

  Thus $d_\ell : E_1^{p',q'}(\overline{\mathfrak{B}}) \to E_1^{p,q}(\overline{\mathfrak{B}})$ is trivial. Since the differential entering $E_\ell^{p,q}$ vanishes, no new relations are introduced in the quotient, and the isomorphism $\varkappa_\ell^{p,q}$ descends to a monomorphism $\varkappa_{\ell+1}^{p,q}$.
- Case 2: All other cases. Here, either $2p' + q' \neq 0$ or p' is odd. The induction hypothesis applies, meaning $\varkappa_\ell^{p',q'}$ is an epimorphsim. By [32, Lemma 1.2.3], the surjectivity of the map on the domain of the differential implies that $\varkappa_{\ell+1}^{p,q}$ is a monomorphism.

We have proven that $\varkappa_{\ell+1}^{p,q}$ is both a monomorphism and an epimorphism if $2p + q + v \neq 0, 1$ or p is odd. The proof also explicitly verified the cases for the vanishing and specific module structure when $2p + q + v \in \{0, 1\}$ and p is even. $\qquad\square$

Lemma 5.28. *Let $Q : \overline{\mathfrak{B}}[v]^\kappa \to \overline{\mathfrak{B}}_{\mathcal{G}}[v]^\kappa$ be the quotient map, and let $1 \leq \ell \leq \infty$. Then, for p odd, or $2p + q + v \neq 0$, the map $Q_\ell^{p,q} : E_\ell^{p,q}\left(\overline{\mathfrak{B}}[v]^\kappa\right) \to E_\ell^{p,q}\left(\overline{\mathfrak{B}}_{\mathcal{G}}[v]^\kappa\right)$ is an isomorphism. For p even and $2p + q + v = 0$, the map $Q_\ell^{p,q} : E_\ell^{p,q}\left(\overline{\mathfrak{B}}[v]^\kappa\right) \to E_\ell^{p,q}\left(\overline{\mathfrak{B}}_{\mathcal{G}}[v]^\kappa\right)$ is trivial.*

Proof. For simplicity, we assume that $v = 0$. The proof for $v \neq 0$ is similar. By definition, we have a short exact of modules

$$0 \to (\overline{\ker(d_{\hat{\mathfrak{R}}^\diamond})} \cdot \mathbf{e})^\kappa \hookrightarrow \overline{\mathfrak{B}}^\kappa \xrightarrow{Q} \overline{\mathfrak{B}}_{\mathcal{G}}^\kappa \to 0.$$

By Lemma 3.22 the sequence above is an exact sequence of complexes where the differential on $\overline{\ker(d_{\hat{\mathfrak{R}}^\diamond})} \cdot \mathbf{e}$ is trivial.

Recall that S has a trivial differential. Using Proposition 2.23 with $\mathcal{C} = \mathfrak{R}^\diamond$, we deduce that $\overline{\ker(d_{\hat{\mathfrak{R}}^\diamond})} \simeq S \otimes \overline{\ker(d_{R^\diamond})}$. As both S and $\ker(d_{R^\diamond})$ are algebras over a field, we obtain:

$$Z(\overline{\ker(d_{\hat{\mathfrak{R}}^\diamond})}) = Z(S) \otimes \ker(d_{A^*(I)}).$$

Since $Z(S) = \bigoplus_{i \in \mathbb{N}} S_{-2i}$, Theorem 5.4 implies that for $0 \le \ell \le \infty$:

$$E_\ell^{p,q}\left(\overline{\ker\left(d_{\hat{\mathfrak{R}}^\diamond}\right) \cdot \mathbf{e}}\right) \cong \frac{F_{\nu_S}^p\left(Z(S) \otimes \ker(d_{A^*(I)})\right)_{p+q}}{F_{\nu_S}^{p+1}\left(Z(S) \otimes \ker(d_{A^*(I)})\right)_{p+q}}$$

$$\cong \begin{cases} S_{-p} \otimes 1, & p \in 2\mathbb{N}, \ 2p+q = 0, \\ S_{-p} \otimes A^1(I), & p \in 2\mathbb{N}, \ 2p+q = 1, \\ 0, & \text{otherwise.} \end{cases} \tag{30}$$

We use this computation to analyze the map $Q_1^{p,q}$. The short exact sequence of complexes induces a long exact sequence on the E_1 page:

$$E_1^{p,q-1}\left(\overline{\mathfrak{B}}_{\mathcal{C}}\right) \xrightarrow{\partial} E_1^{p,q}\left(\overline{\ker\left(d_{\hat{\mathfrak{R}}^\diamond}\right) \cdot \mathbf{e}}\right) \to E_1^{p,q}\left(\overline{\mathfrak{B}}\right) \xrightarrow{Q_1^{p,q}} E_1^{p,q}\left(\overline{\mathfrak{B}}_{\mathcal{C}}\right) \xrightarrow{\partial} E_1^{p,q+1}\left(\overline{\ker\left(d_{\hat{\mathfrak{R}}^\diamond}\right) \cdot \mathbf{e}}\right)$$

We analyze $Q_1^{p,q}$ by distinguishing four cases distinguished by $2p + q$, which is the differential form degree, and the parity of p.

- Case 1: $2p + q < 0$. By Lemma 5.15 with $h = p + q$ and Theorem 5.4 we have $E_1^{p,q}\left(\overline{\mathfrak{B}}_{\mathcal{C}}\right) = E_1^{p,q}\left(\overline{\mathfrak{B}}\right) = 0$.
- Case 2: p is even and $2p + q = 0$. From Lemma 5.27, we know $E_1^{p,q}(\overline{\mathfrak{B}}^\kappa) \cong \mathbb{R}$. By the above computation, $E_1^{p,q}(\overline{\ker(d_{\hat{\mathfrak{R}}^\diamond}) \cdot \mathbf{e}}) \cong \mathbb{R}$ as well. $E_1^{p,q-1}\left(\overline{\mathfrak{B}}_{\mathcal{C}}\right) = 0$ as we have $2p + q - 1 < 0$. So, the map between $E_1^{p,q}((\overline{\ker(d_{\hat{\mathfrak{R}}^\diamond}) \cdot \mathbf{e}})^\kappa)$ and $E_1^{p,q}(\overline{\mathfrak{B}}^\kappa)$ is an isomorphism, which forces the map $Q_1^{p,q}$ to be the zero map.
- Case 3: p is even and $2p + q = 1$. We have $E_1^{p,q+1}\left(\overline{\ker\left(d_{\hat{\mathfrak{R}}^\diamond}\right) \cdot \mathbf{e}}\right) \cong 0$, so $Q_1^{p,q}$ is surjective. From Lemma 5.27, we have $E_1^{p,q}(\overline{\mathfrak{B}}^\kappa) = 0$. Consequently, $Q_1^{p,q}$ is an isomorphism of trivial modules.
- Case 4: All other cases (p odd, or p even with $2p + q > 1$). In this range, the calculation above shows that $E_1^{p,q}(\overline{\ker(d_{\hat{\mathfrak{R}}^\diamond}) \cdot \mathbf{e}}) = 0$ and $E_1^{p,q+1}(\overline{\ker(d_{\hat{\mathfrak{R}}^\diamond}) \cdot \mathbf{e}}) = 0$. Thus, $Q_1^{p,q}$ is an isomorphism.

This establishes the result for $\ell = 1$. The argument for general ℓ proceeds by induction as in the proof of Lemma 5.27. $\qquad \square$

5.3. Proofs of main theorems.

Proof of Theorem 5.2. First, we establish the E_1 page of the spectral sequence. From Theorem 5.9, we obtain

$$E_1^{p,q}\left(\widehat{C}^\diamond\right) \cong S_{-p} \otimes H^{q+2p}(L, d).$$

Next, we show that $E_1^{p,q} = E_{n-1}^{p,q}$, i.e., the differentials d_r for $1 \le r < n - 1$ are all trivial. This follows from Lemma 4.24 and Lemma 5.12.

We then demonstrate that $E_n^{p,q} = E_\infty^{p,q}$, i.e., the spectral sequence degenerates at the E_n page. Observe that for any pair (p, q) where $q + 2p \ne 0$, either $E_1^{p,q} = 0$ or $E_1^{p+n-1,q+2-n} = 0$. This is because $E_1^{p+n-1,q+2-n} \cong S_{-(p+n-1)} \otimes H^{q+2p+n}(L, d)$, which requires $0 \le q + 2p + n \le n$ to be non-zero, implying $q + 2p = 0$.

A similar argument shows that for any $r \ge n$, the differential $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ must be trivial since the target involves a cohomology degree of $q + 2p + r + 1 > n$, which exceeds the dimension of L. Hence $E_n^{p,q} = E_\infty^{p,q}$.

The critical step is computing the differential $d_{n-1} : E_{n-1}^{p,q} \to E_{n-1}^{p+n-1,q+2-n}$. For pairs (p,q) where $2p + q \neq 0$, our earlier discussion shows that d_{n-1} is trivial.

Let us focus on the case where $2p + q = 0$. From Theorem 5.9 and Corollary 3.30, we can deduce that d_{n-1} is trivial if either n is odd or p is even. For the remaining case (when n is even and p is odd), Lemma 5.12 and Corollary 3.30 tell us that the differential is given by:

$$d_{n-1}(a) = -2[\bar{b}] \cup a.$$

To determine whether this map is an isomorphism, we consider $h : S_{-p} \to S_{-p+1-n}$ defined by $a \mapsto a \cdot (-2 \int_L \bar{b})$. This map can be expressed as a composition:

$$a \mapsto a \otimes 1 \xrightarrow{d_{n-1}} -2a \cdot 1 \cup [\bar{b}] \xrightarrow{\cap [L]} -2a \int_L \bar{b}.$$

Since multiplication by s is bijective, all maps in this composition are isomorphisms except possibly d_{n-1}. Therefore, h is an isomorphism if and only if d_{n-1} is. We conclude that d_{n-1} is an isomorphism precisely when $\int_L \bar{b} \neq 0$, and otherwise, it is the zero map.

Finally, note that the exact sequence

$$0 \to Z(S) \cdot 1 \to \overline{\hat{C}^\diamond} \to \overline{\hat{C}_{\phi}^\diamond} \to 0$$

satisfies the assumptions of Proposition 5.6. The result now follows from Lemma 5.17. $\square$

Proof of Theorem 5.3. Convergence follows Theorem 5.11 as by Lemma 5.16 the filtration is bounded. Lemma 5.27 and Corollary 5.24 imply that the E_r page, for $r \geq 1$, is given by:

$$E_r^{p,q}(\mathfrak{B}) \cong \begin{cases} E_r^{p,q-1}\left(\overline{\hat{C}^\diamond}\right), & 2p+q \neq 0, 1, \text{ or } p \in 2\mathbb{N} + 1, \\ 0, & 2p + q = 1, p \in 2\mathbb{N}, \\ S_{-p}, & \text{otherwise.} \end{cases}.$$

Theorem 5.2 implies the result. $\square$

5.4. Corollaries.

Corollary 5.29. *There exists a natural isomorphism $F_{\nu_S}^{n-h} H^h\left(\overline{\hat{C}_{\phi}^\diamond}, \bar{\hat{\mathfrak{m}}}_1^{\gamma,b}\right) \cong E_\infty^{n-h,2h-n}$. Moreover, if $\int_L \bar{b} \neq 0$, or n is odd, then*

$$F_{\nu_S}^{n-h} H^h\left(\overline{\hat{C}_{\phi}^\diamond}, \bar{\hat{\mathfrak{m}}}_1^{\gamma,b}\right) = \begin{cases} 0, & h \in 2\mathbb{Z}\backslash\{n\}, \\ S_{h-n} \otimes H^n(L,d), & \text{otherwise.} \end{cases}$$

Proof. We have an exact sequence

$$F_{\nu_s}^{n-h+1} H^h\left(\overline{\hat{C}_{\phi}^\diamond}, \hat{\mathfrak{m}}_1^{\gamma,\bar{b}}\right) \longrightarrow F_{\nu_S}^{n-h} H^h\left(\overline{\hat{C}_{\phi}^\diamond}, \hat{\mathfrak{m}}_1^{\gamma,\bar{b}}\right) \longrightarrow E_\infty^{n-h,2h-n}.$$

It follows from Lemma 5.16 that $F_{\nu_S}^{n-h} H^h\left(\overline{\hat{C}_{\phi}^\diamond}, \hat{\mathfrak{m}}_1^{\gamma,b}\right) \cong E_\infty^{n-h,2h-n}$. Observe that $n = 2h - n + 2(n-h)$. If n is odd then $S_{h-n} = 0$ for $h \in 2\mathbb{Z}\backslash\{n\}$ Thus, Theorem 5.2 implies the result, if $\int_L \bar{b} \neq 0$, or n is odd. $\square$

Lemma 5.30. *Let p, m be integers such that $0 < p + m < n$, then*

$$E_\infty^{p,m-p} \cong S_{-p} \otimes H^{m+p}(L,d).$$

Proof. $q := m - p$. We have $q + 2p = 2p + m - p = p + m$. By the assumption $0 < p + m < n$, and thus $0 < q + 2p < n$. It now follows from Theorem 5.2 that $E_\infty^{p,q} = S_{-p} \otimes H^{q+2p}(L, d)$. So, the result follows. $\qquad\square$

Lemma 5.31. *If $H^i(L; \mathbb{R}) = 0$ for $i \neq 0, n$, then the natural map $F_{\nu_S}^{n-m} H^m(\overline{\widehat{C}_{\mathbf{\not{\mathbf{c}}}}^\diamond}, \bar{\hat{\mathfrak{m}}}_1^{\gamma,b}) \hookrightarrow$*
$H^m(\overline{\widehat{C}_{\mathbf{\not{\mathbf{c}}}}^\diamond}, \bar{\hat{\mathfrak{m}}}_1^{\gamma,b})$ *is surjective.*

Proof. Observe the exact sequence

$$F_{\nu_s}^{\ell+1} H^m(\overline{\widehat{C}_{\mathbf{\not{\mathbf{c}}}}^\diamond}, \bar{\mathfrak{m}}_1^{\gamma,b}) \longrightarrow F_{\nu_s}^{\ell} H^m(\overline{\widehat{C}_{\mathbf{\not{\mathbf{c}}}}^\diamond}, \bar{\mathfrak{m}}_1^{\gamma,b}) \longrightarrow E_\infty^{\ell,m-\ell}.$$

By Lemma 5.16 $F_{\nu_s}^{-m} H^m(\overline{\widehat{C}_{\mathbf{\not{\mathbf{c}}}}^\diamond}, \bar{\hat{\mathfrak{m}}}_1^{\gamma,b}) = H^m(\overline{\widehat{C}_{\mathbf{\not{\mathbf{c}}}}^\diamond}, \bar{\hat{\mathfrak{m}}}_1^{\gamma,b})$. So, to prove the result, it's enough to show that $E_\infty^{\ell,m-\ell} = 0$, when $-m \leq \ell < n - m$.

If $\ell = -m$, then $2(\ell) + (m - \ell) = 2\ell - 2\ell = 0$. Hence, Theorem 5.2 implies that $E_\infty^{\ell,m-\ell} = 0$. If $-m < \ell < n - m$, then the cohomological assumption on L and Lemma 5.30 imply that $E_\infty^{\ell,m-\ell} = 0$. $\qquad\square$

Corollary 5.32. *If $H^i(L; \mathbb{R}) = 0$ for $i \neq 0, n$, then the natural map*

$$F_{\nu_S}^{n-(2+v)} H^2(\overline{\widehat{C}_{\mathbf{\not{\mathbf{c}}}}^\diamond}[v], (-1)^v \bar{\hat{\mathfrak{m}}}_1^{\gamma,b}) \hookrightarrow H^2\left(\overline{\widehat{C}_{\mathbf{\not{\mathbf{c}}}}^\diamond}[v], (-1)^v \bar{\hat{\mathfrak{m}}}_1^{\gamma,b}\right)$$

is surjective.

Proof. Immediate from Lemma 5.31. $\qquad\square$

Corollary 5.33. *If $H^i(L; \mathbb{R}) = 0$ for $i \neq 0, n$, then the natural map*

$$F_{\nu_S}^{n-(2+v)} H^2(\overline{\mathfrak{B}_{\mathbf{\not{\mathbf{c}}}}}[v], (-1)^v \tilde{\bar{\mathfrak{m}}}_1^{\tilde{\gamma},\bar{b}}) \hookrightarrow F_{\nu_S}^{1-(2+v)} H^2\left(\overline{\mathfrak{B}_{\mathbf{\not{\mathbf{c}}}}}[v], (-1)^v \tilde{\bar{\mathfrak{m}}}_1^{\tilde{\gamma},\bar{b}}\right)$$

is surjective.

Proof. For simplicity assume that $v = 0$. For $-2 < \ell < n - 2$ we identify $E_\infty^{\ell,2-\ell}(\mathfrak{B}_{\mathbf{\not{\mathbf{c}}}})$ with $E_\infty^{\ell,1-\ell}(\widehat{C}_{\mathbf{\not{\mathbf{c}}}}^\diamond)$ by using Theorem 5.3 and Lemma 5.28. Then we proceed just as in the proof of Lemma 5.31. $\qquad\square$

Corollary 5.34. *There exists a natural isomorphism*

$$F_{\nu_S}^{1+n-h} H^h\left(\overline{\mathfrak{B}_{\mathbf{\not{\mathbf{c}}}}}, \tilde{\mathfrak{m}}_1^{\tilde{\gamma},\bar{b}}\right) \cong E_\infty^{1+n-h,2h-n-1}\left(\overline{\mathfrak{B}_{\mathbf{\not{\mathbf{c}}}}}\right).$$

Proof. The first isomorphism is standard convergence from Theorem 5.4 together with Lemma 5.16. $\qquad\square$

5.5. **Real spectral sequence.** For this section we assume that we are in the real setting. Let $b \in \hat{C}^\diamond$ be a bounding cochain for the residue algebra and a real element. It follows from Lemma 4.43 that $\overline{\hat{C}^\diamond}[v]^{-\phi^*}$ and $\overline{\hat{C}^\diamond}[v]^{\phi^*}$ are subcomlexes of $(\overline{\hat{C}^\diamond}[v], \bar{\hat{\mathfrak{m}}}_1^{\gamma,b}[v])$. We abuse the notation and write $\frac{Id+\phi^*}{2}$ as a map from $S[v]$ to itself and as the extended map from $\overline{\hat{C}^\diamond}[v]$ to itself.

Lemma 5.35. *For every $p, q \in \mathbb{Z}$, and $r \geq 0$, there exists an exact sequence*

$$0 \longrightarrow E_r^{p,q}\left(\overline{\hat{C}^\diamond}[v]^{-\phi^*}\right) \hookrightarrow E_r^{p,q}\left(\overline{\hat{C}^\diamond}[v]\right) \xrightarrow{\frac{Id+\phi^*}{2}} E_r^{p,q}\left(\overline{\hat{C}^\diamond}[v]^{\phi^*}\right) \longrightarrow 0.$$

Proof. We apply Proposition 5.6 to the short exact sequence of filtered differential graded modules,

$$0 \longrightarrow \overline{\hat{C}^\diamond}[v]^{-\phi^*} \lhook\joinrel\longrightarrow \overline{\hat{C}^\diamond}[v] \xrightarrow{\frac{Id+\phi^*}{2}} \overline{\hat{C}^\diamond}[v]^{\phi^*} \longrightarrow 0.$$

First, we verify the surjectivity of the map $\frac{Id+\phi^*}{2}$. Let $\xi \in Z_r^{p,q}\left(\overline{\hat{C}^\diamond}[v]^{\phi^*}\right)$. Then, by definition, $\phi^*(\xi) = \xi$. So, $\xi = \frac{Id+\phi^*}{2}(\xi)$. The filtration condition of Proposition 5.6 is satisfied because the filtration on $\overline{\hat{C}^\diamond}[v]^{-\phi^*}$ is the filtration induced as a submodule of $\overline{\hat{C}^\diamond}[v]$. The lemma follows. $\qquad\square$

Corollary 5.36. *For every $p, q \in \mathbb{Z}$, and $\ell \geq 0$, there is an exact sequence*

$$0 \longrightarrow E_\ell^{p,q}\left(Z(S)[v]^\kappa \cdot 1\right) \lhook\joinrel\longrightarrow E_\ell^{p,q}\left(\overline{\hat{C}^\diamond}[v]^\kappa\right) \longrightarrow\joinrel\twoheadrightarrow E_\ell^{p,q}\left(\frac{\overline{\hat{C}^\diamond}[v]^\kappa}{Z(S)[v]^\kappa \cdot 1}\right) \longrightarrow 0 \ ,$$

where $\kappa = -\phi^*, \phi^*, Id$. It follows that there is an exact sequence,

$$0 \longrightarrow E_\ell^{p,q}\left(\frac{\overline{\hat{C}^\diamond}[v]^{-\phi^*}}{Z(S)[v]^{-\phi^*} \cdot 1}\right) \lhook\joinrel\longrightarrow E_\ell^{p,q}\left(\frac{\overline{\hat{C}^\diamond}[v]}{Z(S)[v] \cdot 1}\right) \xrightarrow{\frac{Id+\phi^*}{2}}\joinrel\twoheadrightarrow E_\ell^{p,q}\left(\frac{\overline{\hat{C}^\diamond}[v]^{\phi^*}}{Z(S)[v]^{\phi^*} \cdot 1}\right) \longrightarrow 0 \ .$$

Proof. Observe that by Corollary 3.30 the differentials on $Z(S)[v] \cdot 1$, $Z(S)[v]^{-\phi^*} \cdot 1$ and on $Z(S)[v]^{\phi^*} \cdot 1$ are trivial. It follows that for $\kappa = \phi^*, -\phi^*, Id$, the map $Z_\ell^{p,q}\left(\overline{\hat{C}^\diamond}[v]^\kappa\right) \to Z_\ell^{p,q}\left(\frac{\overline{\hat{C}^\diamond}[v]^\kappa}{Z(S)[v]^\kappa \cdot 1}\right)$ is surjective. Hence, the assumptions of Proposition 5.6 hold, and the first claim follows. So, the rows in the following diagram are exact.

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 \to & E_\ell^{p,q}(Z(S)[v]^{-\phi^*} \cdot 1) & \to & E_\ell^{p,q}\left(\overline{\hat{C}^\diamond}[v]^{-\phi^*}\right) & \to & E_\ell^{p,q}\left(\frac{\overline{\hat{C}^\diamond}[v]^{-\phi^*}}{Z(S)[v]^{-\phi^*} \cdot 1}\right) & \to & 0 \\
& \downarrow & & \downarrow & & \downarrow & & \\
0 \longrightarrow & E_\ell^{p,q}(Z(S)[v] \cdot 1) & \longrightarrow & E_\ell^{p,q}\left(\overline{\hat{C}^\diamond}[v]\right) & \longrightarrow & E_\ell^{p,q}\left(\frac{\overline{\hat{C}^\diamond}[v]}{Z(S)[v] \cdot 1}\right) & \longrightarrow & 0 \\
& \downarrow & & \downarrow & & \downarrow & & \\
0 \longrightarrow & E_\ell^{p,q}(Z(S)[v]^{\phi^*} \cdot 1) & \longrightarrow & E_\ell^{p,q}\left(\overline{\hat{C}^\diamond}[v]^{\phi^*}\right) & \longrightarrow & E_\ell^{p,q}\left(\frac{\overline{\hat{C}^\diamond}[v]^{\phi^*}}{Z(S)[v]^{\phi^*} \cdot 1}\right) & \longrightarrow & 0 \\
& \downarrow & & \downarrow & & \downarrow & & \\
& 0 & & 0 & & 0 & &
\end{array}
$$

It follows from the Lemma 5.35 that the middle column is exact. The differentials on $Z(S)[v] \cdot 1$, $Z(S)[v]^{-\phi^*} \cdot 1$ and on $Z(S)[v]^{\phi^*} \cdot 1$ are trivial. So, the left column is exact. Hence, by the nine lemma, the right column is also exact. $\qquad\square$

Define $\overline{\hat{C}^\diamond_{\not{g}}[v]}^{-\phi^*} := \frac{\overline{\hat{C}^\diamond[v]}^{-\phi^*}}{Z(S)[v]^{-\phi^*}\cdot 1}$.

Corollary 5.37.

$$E^{p,q}_\infty\left(\overline{\hat{C}^\diamond_{\not{g}}[v]}^{-\phi^*}\right) = \begin{cases} E^{p,q}_\infty\left(\overline{\hat{C}^\diamond_{\not{g}}[v]}\right), & -p-v = 2,3 \pmod 4, \\ 0, & otherwise. \end{cases}$$

Proof. It follows from Lemma 5.15 with the fact that ϕ^* acts trivially on differential forms that

$$E^{p,q}_0\left(\overline{\hat{C}^\diamond[v]}^{-\phi^*}\right) = \begin{cases} E^{p,q}_0\left(\overline{\hat{C}^\diamond[v]}\right), & -p-v = 2,3 \pmod 4, \\ 0, & otherwise, \end{cases}$$

and that

$$E^{p,q}_0\left(\overline{\hat{C}^\diamond[v]}^{\phi^*}\right) = \begin{cases} E^{p,q}_0\left(\overline{\hat{C}^\diamond[v]}\right), & -p-v = 0,1 \pmod 4, \\ 0, & otherwise. \end{cases}$$

So, if $-p-v = 0,1 \pmod 4$, then $E^{p,q}_\ell\left(\overline{\hat{C}^\diamond[v]}^{-\phi^*}\right) = 0$, and consequently by Corollary 5.36 above $E^{p,q}_\ell\left(\overline{\hat{C}^\diamond_{\not{g}}[v]}^{-\phi^*}\right) = 0$. Otherwise, $E^{p,q}_\ell\left(\overline{\hat{C}^\diamond[v]}^{\phi^*}\right) = 0$, and thus $E^{p,q}_\ell\left(\frac{\overline{\hat{C}^\diamond[v]}^{\phi^*}}{Z(S)[v]^{\phi^*}\cdot 1}\right) = 0$. So, if $-p-v = 2,3 \pmod 4$, the injective map that is given in the above corollary

$$E^{p,q}_\ell\left(\overline{\hat{C}^\diamond_{\not{g}}[v]}^{-\phi^*}\right) \hookrightarrow E^{p,q}_\ell\left(\overline{\hat{C}^\diamond_{\not{g}}[v]}\right)$$

is surjective. Hence, the boundedness of the filtration implies the result. $\qquad\square$

Lemma 5.38. *Let $v \in 2\mathbb{Z}$. Assume $H^k(L;\mathbb{R}) = 0$ for $k \equiv 0,3 \pmod 4$, $k \neq 0$. Then, the natural map $F^{n-(2+v)}_{\nu_S} H^2(\overline{\hat{C}^\diamond_{\not{g}}[v]}^{-\phi^*}, \bar{\tilde{\mathfrak{m}}}^{\gamma,b}_1) \hookrightarrow H^2(\overline{\hat{C}^\diamond_{\not{g}}[v]}^{-\phi^*}, \bar{\tilde{\mathfrak{m}}}^{\gamma,b}_1)$ is surjective.*

Proof. By Lemma 5.16, we know that

$$F^{-2-v}_{\nu_S} H^2(\overline{\hat{C}^\diamond_{\not{g}}[v]}^{-\phi^*}, \bar{\tilde{\mathfrak{m}}}^{\gamma,b}_1) = H^2(\overline{\hat{C}^\diamond_{\not{g}}[v]}^{-\phi^*}, \bar{\tilde{\mathfrak{m}}}^{\gamma,b}_1).$$

Consider the exact sequence:

$$F^{\ell+1}_{\nu_s} H^2(\overline{\hat{C}^\diamond_{\not{g}}[v]}^{-\phi^*}, \bar{\tilde{\mathfrak{m}}}^{\gamma,b}_1) \longrightarrow F^{\ell}_{\nu_s} H^2(\overline{\hat{C}^\diamond_{\not{g}}[v]}^{-\phi^*}, \bar{\tilde{\mathfrak{m}}}^{\gamma,b}_1) \longrightarrow E^{\ell,2-\ell}_\infty\left(\overline{\hat{C}^\diamond_{\not{g}}[v]}^{-\phi^*}\right).$$

To establish our claim, we need to show that $E^{\ell,2-\ell}_\infty\left(\overline{\hat{C}^\diamond_{\not{g}}[v]}^{-\phi^*}\right) = 0$ for all ℓ in the range $-2-v \leq \ell < n-2-v$. Let us first examine the case when $\ell = -2-v$. Let $p = \ell$ and $q = 2-\ell$. We have

$$E^{p,q}_\infty\left(\overline{\hat{C}^\diamond_{\not{g}}[v]}\right) \cong E^{p,q+v}_\infty\left(\overline{\hat{C}^\diamond_{\not{g}}}\right).$$

For our choice of p and q, we have $2p+q+v = 0$. Since $v \in 2\mathbb{Z}$, we know p is even. Hence, by Theorem 5.2, $E^{p,q}_\infty\left(\overline{\hat{C}^\diamond_{\not{g}}[v]}\right) = 0$. It now follows from Corollary 5.37 that $E^{\ell,2-\ell}_\infty\left(\overline{\hat{C}^\diamond_{\not{g}}[v]}^{-\phi^*}\right) = 0$ when $\ell = -2-v$.

Now consider when $-2 - v < \ell < n - 2 - v$. With $p = \ell$ and $q = 2 - \ell$ as before, the pair (p, q) satisfies $0 < q + 2p + v < n$. Theorem 5.2 tells us:

$$E_\infty^{p,q}\left(\overline{\hat{C}_{\not{\mathcal{G}}}^\diamond}[v]\right) \cong E_\infty^{p,q+v}\left(\widehat{C}_{\not{\mathcal{G}}}^\diamond\right)$$
$$= E_1^{p,q+v}\left(\widehat{C}_{\not{\mathcal{G}}}^\diamond\right)$$
$$\cong S_{-p} \otimes H^{2p+q+v}(L).$$

Applying Corollary 5.37 and using the fact that $q = 2 - p$, we obtain:

$$\bigoplus_{-2-v<p<n-2-v} E_\infty^{p,q}\left(\overline{\hat{C}_{\not{\mathcal{G}}}^\diamond}[v]^{-\phi^*}\right) = \bigoplus_{\substack{-2-v<p<n-2-v \\ -p-v\equiv 2,3 \pmod 4}} E_\infty^{p,q}\left(\overline{\hat{C}_{\not{\mathcal{G}}}^\diamond}[v]\right)$$

$$= \bigoplus_{\substack{-2-v<p<n-2-v \\ -p-v\equiv 2,3 \pmod 4}} S_{-p} \otimes H^{2p+q+v}(L)$$

$$= \bigoplus_{\substack{-2-v<p<n-2-v \\ -p-v\equiv 2,3 \pmod 4}} S_{-p} \otimes H^{p+2+v}(L).$$

When $-p - v \equiv 2, 3 \pmod 4$, we have $p + 2 + v \equiv 0, 3 \pmod 4$. Our assumption states that $H^k(L; \mathbb{R}) = 0$ for $k \equiv 0, 3 \pmod 4$, $k \neq 0$. Therefore, all these cohomology groups vanish, and we have:

$$\bigoplus_{\substack{-2-v<p<n-2-v \\ -p-v\equiv 2,3 \pmod 4}} S_{-p} \otimes H^{p+2+v}(L) = 0.$$

This shows that $E_\infty^{\ell,2-\ell}\left(\overline{\hat{C}_{\not{\mathcal{G}}}^\diamond}[v]^{-\phi^*}\right) = 0$ for all ℓ in the range $-2 - v \leq \ell < n - 2 - v$. This completes the proof. $\qquad\square$

Lemma 5.39. *Let $v \in 2\mathbb{Z}$. Assume $H^k(L; \mathbb{R}) = 0$ for $k \equiv 2, 3 \pmod 4$, $k \neq 0$. Then the natural map $F_{\nu_S}^{n+1-(2+v)} H^2(\overline{\mathfrak{B}_{\not{\mathcal{G}}}}[v]^{-\phi^*}, \bar{\bar{\mathfrak{m}}}_1^{\tilde{\gamma},\tilde{b}}) \hookrightarrow F_{\nu_S}^{1-(2+v)} H^2(\overline{\mathfrak{B}_{\not{\mathcal{G}}}}[v]^{-\phi^*}, \bar{\bar{\mathfrak{m}}}_1^{\tilde{\gamma},\tilde{b}})$ is surjective.*

Proof. Consider the exact sequence:

$$F_{\nu_s}^{\ell+1} H^2(\overline{\mathfrak{B}_{\not{\mathcal{G}}}}[v]^{-\phi^*}, \bar{\bar{\mathfrak{m}}}_1^{\tilde{\gamma},\tilde{b}}) \longrightarrow F_{\nu_s}^{\ell} H^2(\overline{\mathfrak{B}_{\not{\mathcal{G}}}}[v]^{-\phi^*}, \bar{\bar{\mathfrak{m}}}_1^{\tilde{\gamma},\tilde{b}}) \longrightarrow E_\infty^{\ell,2-\ell}\left(\overline{\mathfrak{B}_{\not{\mathcal{G}}}}[v]^{-\phi^*}\right).$$

To establish our claim, we need to show that $E_\infty^{\ell,2-\ell}\left(\overline{\mathfrak{B}_{\not{\mathcal{G}}}}[v]^{-\phi^*}\right) = 0$ for all ℓ in the range $-2 - v < \ell < n + 1 - 2 - v$.

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Write $p = \ell$ and $q = 2 - \ell$. We have that the pair (p, q) satisfies $0 < q + 2p + v < n + 1$.

$$\bigoplus_{-2-v<p<n+1-2-v} E_\infty^{p,q}\left(\overline{\mathfrak{B}}_{\mathpalette\c@ncel{\varphi}}[v]^{-\phi^*}\right) \overset{5.28}{=} \bigoplus_{-2-v<p<n-1-v} E_\infty^{p,q}\left(\overline{\mathfrak{B}}[v]^{-\phi^*}\right)$$

$$\overset{5.27,5.23,5.22}{=} \bigoplus_{-2-v<p<n-1-v} E_\infty^{p,q-1}\left(\hat{C}^\diamond[v]_{\mathpalette\c@ncel{\varphi}}^{-\phi^*}\right)$$

$$\overset{5.37}{=} \bigoplus_{\substack{-2-v<p<n-1-v \\ -p-v\equiv 2,3 \pmod 4}} E_\infty^{p,q-1}\left(\overline{\hat{C}^\diamond_{\mathpalette\c@ncel{\varphi}}}[v]\right)$$

$$\overset{5.2}{=} \bigoplus_{\substack{-2-v<p<n-1-v \\ -p-v\equiv 2,3 \pmod 4}} S_{-p} \otimes H^{2p+q-1+v}(L)$$

$$\overset{q+p=2}{=} \bigoplus_{\substack{-2-v<p<n-1-v \\ -p-v\equiv 2,3 \pmod 4}} S_{-p} \otimes H^{p+1+v}(L).$$

When $-p - v \equiv 2, 3 \pmod 4$, we have $p + 1 + v \equiv 0, 3 \pmod 4$. Our assumption states that $H^k(L; \mathbb{R}) = 0$ for $k \equiv 2, 3 \pmod 4$, $k \neq 0$. Therefore, all these cohomology groups vanish, and we have:

$$\bigoplus_{\substack{-2-v<p<n-1-v \\ -p-v\equiv 2,3 \pmod 4}} S_{-p} \otimes H^{p+1+v}(L) = 0.$$

This shows that $E_\infty^{\ell, 2-\ell}\left(\overline{\mathfrak{B}}_{\mathpalette\c@ncel{\varphi}}[v]^{-\phi^*}\right) = 0$ for all ℓ in the range $-2 - v < \ell < n + 1 - 2 - v$. This completes the proof. $\qquad\square$

6. Classification of bounding pairs

6.1. Existence of bounding cochains.

6.1.1. *Statement.* Recall the notion of a bounding pair (γ, b) given in Definition 1.1. Define $\mathcal{I}_S = (s) \triangleleft S$ to be the ideal of S generated by s. It is our objective to prove the following result.

Proposition 6.1. *Consider the following two cases.*

(1) *We assume $H^i(L; \mathbb{R}) = 0$ for $i \neq 0, n$ and take $\kappa = \mathrm{Id}$.*

(2) *We assume (X, L, ω, ϕ) is a real setting, $\mathfrak{s}$ is a spin structure, R is concentrated in even degree, $H^k(L; \mathbb{R}) = 0$ for $k \equiv 0, 3 \pmod 4$, $k \neq 0, n$, and take $\kappa = -\phi^*$.*

Then, for any closed $\gamma \in (\mathcal{I}_Q D)_2^\kappa$ and any $a \in (\mathcal{I}_S + \mathcal{I}_{\hat{R}})_{1-n}^\kappa$ such that $a^1 \in s \cdot \hat{R}^*$, there exists a bounding cochain b for $\hat{\mathfrak{m}}^\gamma$ such that

$$\int_L b = a.$$

Remark 6.2. When $\kappa = -\phi^*$, the condition $a^1 \in s \cdot \hat{R}^*$ imposes restrictions on the dimension of L. Specifically, $-\phi^*(s) = s$ if and only if $n \equiv 2 \pmod 4$ or $n \equiv 3 \pmod 4$.

The proof of Proposition 6.1 is based on the obstruction theory developed below building on ideas of [8, 36].

6.1.2. *Obstruction cochains.* Recall the definitions of $R, C,$ and S, and the decomposition of tensor products of S to odd and even parts from Section 1.2.1. Also, recall the definition of ν from Section 1.3. Fix a sababa sub-algebra $R^\diamond \subset R$ and the corresponding restriction of scalars $C^\diamond$ as in Section 4.3.3. By Lemma 4.13 $(C^\diamond, \mathfrak{m}_k^\gamma, \langle\,,\,\rangle, 1)$ is a sababa cyclic unital A_∞ algebra. Let S and ν_S be as in Section 3.8. Let $\left(\hat{C}^\diamond, \{\hat{\mathfrak{m}}_k^\gamma\}_{k\geq 0}, \langle,\rangle_F, 1\right)$ be the (S, F) extension of $(C^\diamond, \mathfrak{m}_k^\gamma, \langle\,,\,\rangle, 1)$, which by Corollary 4.15 is pseudo-complete with respect to $\nu_S \otimes \nu$. Let $\hat{g} : R^\diamond \otimes_{\bar{R}^\diamond} \hat{C}^\diamond \to \hat{C}^\diamond$ be the induced $\bar{R}^\diamond$ structure on $\hat{C}^\diamond$ as in Definition 2.24, and let $\hat{f}$ be its inverse. Recall from Section 4.3.3 that $\bar{R}^\diamond$ is isomorphic to $\mathbb{R}$. Henceforward, all tensor products will be taken over $\mathbb{R}$. By Lemma 2.22 and Proposition 4.3, we have $\overline{\hat{C}^\diamond} \simeq S \otimes A^*(L)$.

Let $E_0 < E_1 < \dots$ be defined as in (22). For $i \in \mathbb{N}$ recall the definitions of $R_{E_i}^\diamond, \hat{R}_{E_i}^\diamond, \hat{C}_{E_i}^\diamond$ from Definition 2.3. Also recall the definition of $\pi_r : \hat{C}_E^\diamond \to \overline{\hat{C}^\diamond}[|r|]$ from Section 4.5.

Let $b \in \left(\hat{C}^{\diamond\,\kappa}\right)_1$ with $\nu_s \otimes \nu(b) > 0$ be a bounding cochain in $\frac{\hat{C}^\diamond}{F^{>E_l}\hat{C}^\diamond}$, as in Definition 3.23. Then, Lemma 3.26 implies that $\hat{\mathfrak{m}}_0^{\gamma,b} \in F^{E_{l+1}}\hat{C}_{\not\!{c}}^\diamond$ and that $\hat{\mathfrak{m}}_0^{\gamma,b}$ represents a cocycle in $(\hat{C}_{\not\!{c},E_{l+1}}^\diamond)^\kappa$, which we denote by $\hat{\mathfrak{m}}_{0,E_{l+1}}^{\gamma,b}$. For the rest of this section, we abbreviate

$$\mathfrak{o} = [\hat{\mathfrak{m}}_{0,E_{l+1}}^{\gamma,b}] \in H^2\left((\hat{C}_{\not\!{c},E_{l+1}}^\diamond)^\kappa, \hat{\mathfrak{m}}_1^{\gamma,b}\right).$$

Lemma 6.3. *Let $r \in R_E^{\diamond\,\vee}$ and $\alpha \in F^E\left(\hat{C}_{\not\!{c}}^{\diamond\,\kappa}\right)_m$ such that $\bar{\hat{\mathfrak{m}}}_1^{\gamma,b}(\pi_r(\alpha)) = 0$. If $[\pi_r(\alpha)] \in F_{\nu_S}^\ell H^m(\overline{\hat{C}_{\not\!{c}}^\diamond}[|r|]^\kappa, \bar{\hat{\mathfrak{m}}}_1^{\gamma,b})$, then there exists $\eta \in F^E\left(\hat{C}_{\not\!{c}}^{\diamond\,\kappa}\right)_{m-1}$ such that*

$$\pi_r\left(\alpha + \hat{\mathfrak{m}}_1^{\gamma,b}(\eta)\right) \in F_{\nu_S}^\ell\left(\overline{\hat{C}_{\not\!{c}}^\diamond}\right)[|r|]_m^\kappa.$$

Proof. Assume without loss of generality that $r \neq 0$. Let $\bar{\xi} \in \overline{\hat{C}_{\not\!{c}}^\diamond}$. If $r^\vee$ is any element of $R_E^\diamond$ that solves the equation $r(r^\vee) = 1$, then $\bar{\xi} = \hat{\pi}_r(r^\vee \otimes \bar{\xi})$. Since $\hat{f}$ is an isomorphism, it follows that π_r is surjective. By the assumption of the lemma, there exists $\bar{\eta} \in \left(\overline{\hat{C}_{\not\!{c}}^\diamond}[|r|]^\kappa\right)_{m-1}$ such that $\bar{\hat{\mathfrak{m}}}_1^{\gamma,\bar{b}}(\bar{\eta}) + \pi_r(\alpha) \in F_{\nu_S}^\ell\left(\overline{\hat{C}_{\not\!{c}}^\diamond}[|r|]^\kappa\right)_m$. Let η' be a homogenoues element in $\pi_r^{-1}(\bar{\eta})$, and define $\eta = \frac{\eta' + f(\eta')}{2}$. Then, by Lemma 4.30 (and Lemma 4.41 if $\kappa = -\phi^*$), we have

$$\pi_r\left(\alpha + \hat{\mathfrak{m}}_1^{\gamma,b}(\eta)\right) = \pi_r(\alpha) + \hat{\mathfrak{m}}_1^{\gamma,\bar{b}}(\pi_r(\eta)) = \pi_r(\alpha) + \bar{\hat{\mathfrak{m}}}_1^{\gamma,\bar{b}}(\bar{\eta}) \in F_{\nu_S}^\ell\left(\overline{\hat{C}_{\not\!{c}}^\diamond}[|r|]^\kappa\right)_m.$$

$\square$

Lemma 6.4. *Assume that n is odd, or $\int f(\bar{b}) \neq 0$. Let $r \in R_{E_{l+1}}^{\diamond\,\vee}$ be homogeneous. If $(\pi_r)_*(\mathfrak{o})$ is in the image of the inclusion*

$$F_{\nu_S}^{n-(2+|r|)}H^2(\overline{\hat{C}_{\not\!{c}}^\diamond}[|r|]^\kappa, \bar{\hat{\mathfrak{m}}}_1^{\gamma,b}) \hookrightarrow H^2\left(\overline{\hat{C}_{\not\!{c}}^\diamond}[|r|]^\kappa, \bar{\hat{\mathfrak{m}}}_1^{\gamma,b}\right),$$

then $(\pi_r)_*(\mathfrak{o}) = 0$.

Proof. Let $h = 2 + |r|$. It follows from Lemma 6.3 that we can choose $b^\dagger \in \left(\hat{C}^{\diamond\,\kappa}\right)_1$ such that

(1) $b^\dagger \equiv b \pmod{F^{>E_l}\hat{C}_{\not\!{c}}^\diamond}$;

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(2) $\pi_r\left(\hat{\mathfrak{m}}_0^{\gamma,b\dagger}\right) \in F^{n-h}\overline{\hat{C}_{\cancel{\mathscr{C}}}^\diamond}[|r|]^\kappa$;

(3) and $[\hat{\mathfrak{m}}_{0,E_{l+1}}^{\gamma,b\dagger}] = [\hat{\mathfrak{m}}_{0,E_{l+1}}^{\gamma,b}] = \mathfrak{o}$ in $H^2\left(\hat{C}_{\cancel{\mathscr{C}},E_{l+1}}^\diamond, \hat{\mathfrak{m}}_1^{\gamma,b}\right)$.

Let $(E_r^{p,q}, d_r)$ be the spectral sequence of Theorem 5.2. Corollary 5.29, and Corollary 3.21 imply that

$$F_{\nu_S}^{n-h}H^2(\overline{\hat{C}_{\cancel{\mathscr{C}}}^\diamond}[|r|]^\kappa, \bar{\hat{\mathfrak{m}}}_1^{\gamma,b}) \hookrightarrow F_{\nu_S}^{n-h}H^2(\overline{\hat{C}_{\cancel{\mathscr{C}}}^\diamond}[|r|], \bar{\hat{\mathfrak{m}}}_1^{\gamma,b}) \cong F_{\nu_S}^{n-h}H^h(\overline{\hat{C}_{\cancel{\mathscr{C}}}^\diamond}, \bar{\hat{\mathfrak{m}}}_1^{\gamma,b}) \cong E_\infty^{n-h,2h-n}.$$

Assume first $E_1^{n-h,2h-n} = E_\infty^{n-h,2h-n}$. Then, by Theorem 5.2, we have that n is odd, or $n-h$ is odd, or $n-h = 0$. Additionally, $(\pi_r)_*(\mathfrak{o})$ is trivial if $[\pi_r\left(\hat{\mathfrak{m}}_{0,E_{l+1}}^{\gamma,b\dagger}\right)] \in E_1^{n-h,2h-n}$ is trivial. Lemma 5.16 and (2) imply that

$$\pi_r\left(\hat{\mathfrak{m}}_{0,E_{l+1}}^{\gamma,b\dagger}\right) \in S_{h-n} \otimes A^n(L). \tag{31}$$

Equation (31) together with Lemma 5.19 implies that

$$\int_L \pi_r\left(\hat{\mathfrak{m}}_{0,E_{l+1}}^{\gamma,b\dagger}\right) = 0 \implies (\pi_r)_*(\mathfrak{o}) = 0.$$

Since either n is odd, in which case $\mathcal{I}_S^{odd}$ is trivial, or $h-n$ is zero or odd, in fact

$$\int_L \pi_r\left(\hat{\mathfrak{m}}_{0,E_{l+1}}^{\gamma,b\dagger}\right) \equiv 0 \pmod{\left(\mathcal{I}_S^{odd}\right)^2} \implies (\pi_r)_*(\mathfrak{o}) = 0.$$

Recalling that $\gamma|_L = 0$, it follows from Proposition 4.2 that

$$\left(\hat{f}(\hat{\mathfrak{m}}_0^{\gamma,b\dagger})\right)_n = \left(d\hat{f}(b^\dagger) - \hat{f}(b^\dagger)\cdot\hat{f}(b^\dagger)\right)_n =$$

$$\left(d\hat{f}(b^\dagger) - \frac{1}{2}[\hat{f}(b^\dagger), \hat{f}(b^\dagger)]\right)_n \equiv (d\hat{f}(b^\dagger))_n \pmod{\left(\mathcal{I}_S^{odd}\right)^2},$$

where the equivalence follows from Corollary 3.30. Thus, using $[\cdot]$ to denote the equivalence class in the quotient $\hat{R}_{E_{l+1}}^\diamond = F^{E_{l+1}}\hat{R}^\diamond/F^{>E_{l+1}}\hat{R}^\diamond$, we have

$$\int_L \pi_r\left(\hat{\mathfrak{m}}_{0,E_{l+1}}^{\gamma,b\dagger}\right) = (r \otimes \mathrm{Id})\left(\left[\int_L \hat{f}\left(\hat{\mathfrak{m}}_0^{\gamma,b\dagger}\right)\right]\right) \equiv (r \otimes \mathrm{Id})\left(\left[\int_L d\hat{f}\left(b^\dagger\right)\right]\right) \pmod{\left(\mathcal{I}_S^{odd}\right)^2}.$$

It follows from Stokes' Theorem that $(\pi_r)_*(\mathfrak{o}) = 0$.

Now assume that $E_1^{n-h,2h-n} \neq E_\infty^{n-h,2h-n}$. Theorem 5.2 implies that $E_\infty^{n-h,2h-n} = 0$, so $(\pi_r)_*(\mathfrak{o}) = 0$. $\qquad\square$

Proposition 6.5. *In the two cases of Proposition 6.1, if n is odd, or $\int f(\bar{b}) \neq 0$, then $\mathfrak{o} = 0$.*

Proof. The assumptions of Corollary 5.32 are satisfied in case (1), and the assumption of Lemma 5.38 are satisfied in case (2). Hence for any homogeneous element $r \in R_E^{\diamond\vee}$ the element $(\pi_r)_*(\mathfrak{o})$ is in the image of

$$F_{\nu_S}^{n-(2+|r|)}H^2(\overline{\hat{C}_{\cancel{\mathscr{C}}}^\diamond}[|r|]^\kappa, \bar{\hat{\mathfrak{m}}}_1^{\gamma,b}) \hookrightarrow H^2\left(\overline{\hat{C}_{\cancel{\mathscr{C}}}^\diamond}[|r|], \bar{\hat{\mathfrak{m}}}_1^{\gamma,b}\right).$$

Thus, the assumptions of Lemma 6.4 are satisfied, and $(\pi_r)_*(\mathfrak{o}) = 0$. It now follows from Lemma 4.31 that $\hat{\mathfrak{m}}_{0,E_{l+1}}^{\gamma,b}$ is exact. So, $\mathfrak{o} = 0$. $\qquad\square$

Corollary 6.6. *In the two cases of Proposition 6.1, there exists $\eta \in F^{E_l}\left(\hat{C}_{\text{\cancel{\phi}}}^{\diamond \kappa}\right)_1$ such that $b + \eta \in \left(\hat{C}_{\text{\cancel{\phi}}}^{\diamond \kappa}\right)_1$ is a solution to the Maurer-Cartan equation in $\frac{\hat{C}_{\text{\cancel{\phi}}}^{\diamond}}{F^{>E_{l+1}}\hat{C}_{\text{\cancel{\phi}}}^{\diamond}}$, and $\int_L \hat{f}(b + \eta) = \int_L \hat{f}(b).$*

Proof. By Proposition 6.5 we are in the setting of Lemma 3.27, so we can choose $\xi \in F^{E_{l+1}}\left(\hat{C}_{\text{\cancel{\phi}}}^{\diamond \kappa}\right)_1$ such that $b + \xi$ is a bounding cochain in $\frac{\hat{C}^{\diamond}}{F^{>E_{l+1}}\hat{C}^{\diamond}}$. Let $P_n : \overline{\hat{C}^{\diamond}} \to S \otimes A^n(L)$ be the projection. Then by Corollary 4.26, $\ker\left(\bar{\mathfrak{m}}_1^{\gamma,b} \circ P_n\right) = \overline{\hat{C}^{\diamond}}$. So, $\eta := \hat{g}\left(\hat{f}(\xi) - \text{Id} \otimes P_n \circ \hat{f}(\xi)\right)$ is a primitive for $\hat{\mathfrak{m}}_{0,E_{l+1}}^{\gamma,b}$ by Lemma 3.28. It follows that $b + \eta$ is a solution to the Maurer-Cartan equation in $\frac{\hat{C}_{\text{\cancel{\phi}}}^{\diamond}}{F^{>E_{l+1}}\hat{C}_{\text{\cancel{\phi}}}^{\diamond}}$, and

$$\int_L \hat{f}(b + \hat{g}(h)) = \int_L \hat{f}(b) + \int_L \hat{f}(\xi) - \text{Id} \otimes \pi_n \circ \hat{f}(\xi) = \int_L \hat{f}(b).$$

$\square$

Proof of Proposition 6.1. Let $a \in (\mathcal{I}_S + \mathcal{I}_{\hat{R}})_{1-n}$ such that $a^1 \in s \cdot \hat{R}^*$. Define $R^{\diamond} := R(a)$ as in Notation 4.14. Take $\bar{b}_0 \in A^n(L)$ any representative of the Poincaré dual of a point, and let $b_{(0)} := a \otimes \bar{b}_0$. By Lemma 4.25 $b_{(0)}$ is a bounding cochain for $\overline{\hat{C}^{\diamond}}$. Applying Corollary 6.6 repeatedly, we get a sequence of elements $\{\eta_i\}_{i=1}^{\infty} \subset \left(\hat{C}^{\diamond}\right)_1$ such that $\eta_i \in F^{E_i}\hat{C}^{\diamond}$, and the element $b_{(\ell)} := b_{(0)} + \sum_{i=1}^{\ell} \eta_i$ is a solution to the Maurer-Cartan equation in $\frac{\hat{C}_{\text{\cancel{\phi}}}^{\diamond}}{F^{>E_\ell}\hat{C}_{\text{\cancel{\phi}}}^{\diamond}}$, and

$$\int_L \hat{f}\left(b_{(\ell)}\right) = \int_L \hat{f}(b_{(0)}) = a.$$

Since $\lim_{i \to \infty} \nu(\eta_i) = \infty$, it follows from completeness of $\hat{C}^{\diamond}$ that there exists a unique limit $b := \lim_{\ell \to \infty} b_{(\ell)}$ and $\int_L b = \lim_{\ell \to \infty} \int_L b_{(\ell)} = a$. By construction b is a solution to the Maurer-Cartan equation in $\hat{C}_{\text{\cancel{\phi}}}^{\diamond}$. Moreover, $\nu(\hat{\mathfrak{m}}_0^{\gamma,b}) > 0$ as b is a solution to the Maurer-Cartan equation in $\overline{\hat{C}^{\diamond}}$. We conclude that b is a bounding cochain for γ. $\square$

6.2. Gauge equivalence of bounding pairs.

6.2.1. *Formulation.* In the following, we use the notation of Section 4.2. In particular, we write $\mathfrak{m}^{\gamma'}$ for operations defined using the almost complex structure J' and a closed form $\gamma' \in \mathcal{I}_Q D$ with $\deg_D \gamma' = 2$. Recall also the definition (18) of $\mathfrak{R} = A^*(I; R)$. Denote by $\pi : I \times X \to X$ the projection. By abuse of notation, we write $\tilde{\mathfrak{m}}^{\gamma}$ instead of $\hat{\tilde{\mathfrak{m}}}^{\tilde{\gamma}}$.

Proposition 6.7. *Consider the following two cases.*
  (1) *We assume $H^i(L; \mathbb{R}) = 0$ for $i \neq 0, n$, and take $\kappa = \text{Id}$.*
  (2) *We assume (X, L, ω, ϕ) is a real setting, $\mathfrak{s}$ is a spin structure, R is concentrated in even degree, and $H^k(L; \mathbb{R}) = 0$ for $k \equiv 0, 3 \pmod{4}$ and $k \neq 0, n$, and take $\kappa = -\phi^*$.*
Let (γ, b) be a unit bounding pair with respect to J and let (γ', b') be a unit bounding pair with respect to J' such that $\varrho([\gamma, b]) = \varrho([\gamma', b'])$. Assume further that $\gamma, \gamma' \in (\mathcal{I}_Q D)^{\kappa}$ and $b, b' \in (\hat{C}^{\diamond})^{\kappa}$. Then $(\gamma, b) \sim (\gamma', b')$.

The proof of Proposition 6.7 is given toward the end of the section based on the construction detailed in the following.

6.2.2. *Obstruction theory.* Fix a sababa sub-algebra $R^\diamond \subset R$, and define $\mathfrak{R}^\diamond := R^\diamond \otimes A^*(I)$ and $\mathfrak{C}^\diamond := R^\diamond \otimes A^*(I \times L)$. Then, by Lemma 4.13 we have $(\mathfrak{C}^\diamond, \{\tilde{\mathfrak{m}}_k^{\tilde{\gamma}}\}_{k \geq 0}, \langle\langle \, , \, \rangle\rangle, 1)$ is a sababa cyclic unital A_∞ algebra over $R^\diamond$. As in Section 4.5, ν gives a discrete filtration on $\hat{\mathfrak{C}}^\diamond$. To prove the gauge equivalence of Proposition 6.7, we construct a bounding cochain $\tilde{b} \in \hat{\mathfrak{C}}^\diamond$ following the strategy of Remark 4.29. Thus, the bounding cochain $\tilde{b}$ is defined as a limit of a sequence $\tilde{b}_{(\ell)} = \sum_{i=0}^\ell \tilde{b}_i$, where $\tilde{b}_{(\ell)}$ satisfies

$$\tilde{\mathfrak{m}}^{\tilde{\gamma}, \tilde{b}_{(\ell)}} \equiv \tilde{c}_{(\ell)} \cdot 1 \quad (\mathrm{mod} \ F^{E_l} \hat{\mathfrak{C}}^\diamond), \qquad \tilde{c}_{(\ell)} \in (\mathcal{I}_R \hat{\mathfrak{R}}^\diamond)_2, \tag{32}$$

and $d\tilde{c}_{(\ell)} \equiv 0 \ (\mathrm{mod} \ F^{E_l} \hat{\mathfrak{R}}^\diamond)$.

Denote by $\hat{g} : R^\diamond \otimes \overline{\hat{\mathfrak{C}}^\diamond} \to \hat{\mathfrak{C}}^\diamond$ the induced $\bar{R}^\diamond$ structure on $\hat{\mathfrak{C}}^\diamond$ from Section 4.3.1 and by $\hat{f}$ its inverse. Let $b, b' \in \hat{C}^\diamond$ be bounding cochains for the A_∞ algebras $(\hat{C}^\diamond, \{\hat{\mathfrak{m}}_k^\gamma\}_{k \geq 0}, \langle \, , \, \rangle_F, 1 \otimes 1)$ and $\left(\hat{C}^\diamond, \hat{\mathfrak{m}}^{\gamma'}, \langle \, , \, \rangle_F, 1 \otimes 1\right)$ respectively. Assume that $\int_L \hat{f}(b) - \int_L \hat{f}(b') \in \left(\mathcal{I}_{\hat{R}}^{odd}\right)^2$. Let $\tilde{b}_0$ be as in Lemma 4.27. By Lemma 4.28 $\tilde{b}_0$ induces an A_∞ algebra $(\mathfrak{B}, \{\tilde{\mathfrak{m}}_k^{\tilde{\gamma}, \tilde{b}_0}\}_{k \geq 0}, \langle\langle \, , \, \rangle\rangle_F, 1)$. Let $\tilde{b} \in (\mathfrak{B}^\kappa)$ be a bounding cochain with respect to the induced A_∞ algebra

$$\left(\frac{\mathfrak{B}}{F^{>E_l}\mathfrak{B}}, \{\tilde{\mathfrak{m}}_k^{\tilde{\gamma}, \tilde{b}_0}\}_{k \geq 0}, \langle\langle \, , \, \rangle\rangle_F, 1\right).$$

It can be shown that $\left(\tilde{\mathfrak{m}}_0^{\tilde{\gamma}, \tilde{b}_0}\right)^{\tilde{b}} = \tilde{\mathfrak{m}}_0^{\tilde{\gamma}, \tilde{b}_0 + \tilde{b}}$. Then, the definition of a bounding cochain implies that $\left(\tilde{\mathfrak{m}}_0^{\tilde{\gamma}, \tilde{b}_0}\right)^{\tilde{b}} \in F^{E_{l+1}}\mathfrak{B}_{\phi}$. So, Lemma 3.26 implies that $\tilde{\mathfrak{m}}_0^{\tilde{\gamma}, \tilde{b}_0 + \tilde{b}}$ represents a cocycle in $(\mathfrak{B}_{\phi, E_{l+1}})^\kappa$, which we denote by $\tilde{\mathfrak{m}}_{0, E_{l+1}}^{\tilde{\gamma}, \tilde{b}_0 + \tilde{b}}$. For the rest of this section, we abbreviate

$$\mathfrak{o} = [\tilde{\mathfrak{m}}_{0, E_{l+1}}^{\tilde{\gamma}, \tilde{b}_0 + \tilde{b}}] \in H^2\left(\mathfrak{B}_{\phi, E_{l+1}}^\kappa, \hat{\mathfrak{m}}_1^{\gamma, b}\right).$$

Lemma 6.8. *Let r be an element of $R_E^{\diamond \vee}$, and $\alpha \in F^E\left(\mathfrak{B}_{\phi}^\kappa\right)_m$ such that $\tilde{\mathfrak{m}}_1^{\tilde{\gamma}, \tilde{b}_0 + \tilde{b}}(\alpha) = 0$. If $[\pi_r(\alpha)] \in F_{\nu_S}^\ell H^m\left(\overline{\mathfrak{B}_{\phi}}[|r|]^\kappa, \tilde{\mathfrak{m}}_0^{\tilde{\gamma}, \tilde{b}_0 + \tilde{b}}\right)$, then there exists $\eta \in F^{E_l}\mathfrak{B}^1$, such that*

$$\pi_r\left(\alpha + \tilde{\mathfrak{m}}_1^{\tilde{\gamma}, \tilde{b}}(\eta)\right) \in F_{\nu_S}^\ell\left(\overline{\mathfrak{B}_{\phi}}\right)_m [|r|]^\kappa.$$

Proof. Same as the proof of Lemma 6.3, but using Lemma 4.32 instead of Lemma 4.31. □

Lemma 6.9.

$$\pi_r(\mathfrak{o}) \in F_{\nu_S}^{1-(2+|r|)} H^2\left(\overline{\mathfrak{B}_{\phi}}[|r|]^\kappa, \bar{\bar{\mathfrak{m}}}_1^{\gamma, \tilde{b}}\right)$$

Proof. By Lemma 5.16 we have that

$$\pi_r(\mathfrak{o}) \in F_{\nu_S}^{-(2+|r|)} H^2\left(\overline{\mathfrak{B}_{\phi}}[|r|]^\kappa, \bar{\tilde{\mathfrak{m}}}_1^{\gamma, \tilde{b}}\right).$$

To obtain the stronger bound asserted by the lemma, it suffices to demonstrate that the class represented by $\pi_r(\mathfrak{o})$ vanishes in the corresponding graded component of the spectral sequence, specifically

$$[\pi_r(\mathfrak{o})] = 0 \in E_\infty^{-(2+|r|), 4+|r|}\left(\overline{\mathfrak{B}_{\phi}}[|r|]^\kappa\right).$$

Since $\tilde{b}$ is a bounding cochain in $\frac{\mathfrak{B}}{F^{>E_l}\mathfrak{B}}$, there exists $\tilde{c} \in Z\left(\ker d_{\widehat{\mathfrak{R}^\diamond}}\right)1$ such that $\tilde{\mathfrak{m}}_0^{\tilde{\gamma}, \tilde{b}_0 + \tilde{b}} - \tilde{c} \in$ $F^{E_{l+1}}\mathfrak{B}$. Properties (2) and (4) of the A_∞ algebra $\left(\mathfrak{B}, \{\tilde{\mathfrak{m}}^{\tilde{\gamma}, \tilde{b}_0 + \tilde{b}}\}_{k \geq 0}\right)$ imply that $[\tilde{\mathfrak{m}}_0^{\tilde{\gamma}, \tilde{b}_0 + \tilde{b}} - \tilde{c}]$ is a cocycle in $(\mathfrak{B}_{E_{l+1}})^\kappa$. So, $\pi_r\left([\tilde{\mathfrak{m}}_0^{\tilde{\gamma}, \tilde{b}_0 + \tilde{b}} - \tilde{c}]\right)$ is a cocycle in $\left(\overline{\mathfrak{B}}[|r|]^\kappa\right)_2$. Let $Q : \overline{\mathfrak{B}}[|r|]^\kappa \to$ $\overline{\mathfrak{B}_{\not\phi}}[|r|]^\kappa$ and $Q' : \mathfrak{B} \to \mathfrak{B}_{\not\phi}$ be the quotients map. We observe that

$$Q \circ \pi_r\left([\tilde{\mathfrak{m}}_0^{\tilde{\gamma}, \tilde{b}_0 + \tilde{b}} - \tilde{c}]\right) = \pi_r\left([Q'\left(\tilde{\mathfrak{m}}_0^{\tilde{\gamma}, \tilde{b}_0 + \tilde{b}} - \tilde{c}\right)]\right) =$$
$$= \pi_r\left([Q'\left(\tilde{\mathfrak{m}}_0^{\tilde{\gamma}, \tilde{b}_0 + \tilde{b}}\right) - Q'(\tilde{c})]\right) = \pi_r\left([Q'\left(\tilde{\mathfrak{m}}_0^{\tilde{\gamma}, \tilde{b}_0 + \tilde{b}}\right)]\right) = \pi_r(\tilde{\mathfrak{m}}_{0, E_{l+1}}^{\tilde{\gamma}, \tilde{b}_0 + \tilde{b}}).$$

Consequently, the class $[\pi_r(\mathfrak{o})]$ lies in the image of the map induced by Q on the E_∞ page

$$[\pi_r(\mathfrak{o})] \in Q_\infty^{-2-|r|, 4+|r|}\left(E_\infty^{-2-|r|, 4+|r|}\left(\overline{\mathfrak{B}}[|r|]^\kappa\right)\right).$$

We set $p = -(2 + |r|)$ and $q = 4 + |r|$. To complete the proof we distinguish two cases based on the parity of p:

- Case p is even. The indices satisfy $2p + q + |r| = 0$. By Lemma 5.28, $Q_\infty^{p,q}$ is trivial, implying $[\pi_r(\mathfrak{o})] = 0$.
- Case p is odd. By Theorem 5.2 and Theorem 5.3 $E_\infty^{p,q}\left(\overline{\mathfrak{B}}[|r|]\right) = 0$, which forces $[\pi_r(\mathfrak{o})] = 0$.

$\square$

Lemma 6.10. *Assume that n is odd or $\int_L \hat{f}(\bar{b}) \neq 0$. Let $r \in R_{E_{l+1}}^{\diamond}{}^\vee$ be homogeneous. Assume $(\pi_r)_*(\mathfrak{o})$ is in the image of $F_{\nu_S}^{1+n-(2+|r|)} H^2(\overline{\mathfrak{B}_{\not\phi}}[|r|]^\kappa, \tilde{\mathfrak{m}}_1^{\gamma, \tilde{b}_0}) \hookrightarrow H^2(\overline{\mathfrak{B}_{\not\phi}}[|r|]^\kappa, \tilde{\mathfrak{m}}_1^{\gamma, \tilde{b}_0})$. Then, $(\pi_r)_*(\mathfrak{o}) = 0$.*

Proof. Let $h = 2 + |r|$. It follows from Lemma 6.8 that we can choose $\tilde{b}^\dagger \in (\mathfrak{B}^\kappa)_1$ such that

(1) $\tilde{b}^\dagger \equiv \tilde{b} \pmod{F^{>E_l}\mathfrak{B}^\kappa}$;
(2) $\pi_r\left(\tilde{\mathfrak{m}}_{0, E_{l+1}}^{\tilde{\gamma}, \tilde{b}_0 + \tilde{b}^\dagger}\right) \in F_{\nu_S}^{1+n-h}\overline{\mathfrak{B}_{\not\phi}}^\kappa$;
(3) and $[\tilde{\mathfrak{m}}_{0, E_{l+1}}^{\tilde{\gamma}, \tilde{b}_0 + \tilde{b}^\dagger}] = [\tilde{\mathfrak{m}}_{0, E_{l+1}}^{\tilde{\gamma}, \tilde{b}_0 + \tilde{b}}] = \mathfrak{o}$ in $H^2\left((\mathfrak{B}_{\not\phi})_{E_l}, \tilde{\mathfrak{m}}_1^{\tilde{\gamma}, \tilde{b}_0}\right)$.

Let $(E_r^{p,q}, d_r)$ be the spectral sequence of Theorem 5.3. By Lemma 5.28 we have

$$E_\infty^{n+1-h, 2h-n-1}(\overline{\mathfrak{B}_{\not\phi}}) \cong E_\infty^{n+1-h, 2h-n-1}.$$

Thus, Corollary 3.21, and Corollary 5.34 imply that

$$F_{\nu_S}^{n+1-h} H^2(\overline{\mathfrak{B}_{\not\phi}}[|r|]^\kappa, \tilde{\mathfrak{m}}_1^{\tilde{\gamma}, \tilde{b}_0}) \hookrightarrow F_{\nu_S}^{n+1-h} H^2(\overline{\mathfrak{B}_{\not\phi}}[|r|], \tilde{\mathfrak{m}}_1^{\tilde{\gamma}, \tilde{b}_0})$$
$$\cong F_{\nu_S}^{n+1-h} H^h(\overline{\mathfrak{B}_{\not\phi}}, \tilde{\mathfrak{m}}_1^{\tilde{\gamma}, \tilde{b}_0}) \cong E_\infty^{n+1-h, 2h-n-1}.$$

Assume first that $E_1^{n+1-h, 2h-n-1} = E_\infty^{n+1-h, 2h-n-1}$. Then, by Theorem 5.3, we have n is odd, or $n + 1 - h$ is odd, or $n + 1 - h = 0$. Additionally, $(\pi_r)_*(\mathfrak{o})$ is trivial if and only if $[\pi_r\left(\tilde{\mathfrak{m}}_{0, E_{l+1}}^{\tilde{\gamma}, \tilde{b}_0 + \tilde{b}^\dagger}\right)] \in E_1^{n+1-h, 2h-n-1}$ is trivial. Lemma 5.16 with (2) implies that

$$\pi_r\left(\tilde{\mathfrak{m}}_0^{\tilde{\gamma}, \tilde{b}_0 + \tilde{b}^\dagger}\right) \in S_{h-n-1} \otimes A^{n+1}(I \times L). \tag{33}$$

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Equation (33) together with Lemma 5.19 implies that

$$\int_{I\times L} \pi_r\left(\tilde{\mathfrak{m}}_{0,E_{l+1}}^{\tilde{\gamma},\tilde{b}_0+\tilde{b}^\dagger}\right) = 0 \implies (\pi_r)_*(\mathfrak{o}) = 0.$$

Since either n is odd, in which case $\mathcal{I}_S^{odd}$ is trivial, or $h-n-1$ is zero or odd, in fact

$$\int_{I\times L} \pi_r\left(\tilde{\mathfrak{m}}_{0,E_{l+1}}^{\tilde{\gamma},\tilde{b}_0+\tilde{b}^\dagger}\right) \equiv 0 \pmod{\left(\mathcal{I}_S^{odd}\right)^2} \implies (\pi_r)_*(\mathfrak{o}) = 0.$$

Recalling that $\tilde{\gamma}|_{I\times L} = 0$, it follows from Proposition 4.8 that

$$\left(\hat{f}(\tilde{\mathfrak{m}}_0^{\tilde{\gamma},\tilde{b}_0+\tilde{b}^\dagger})\right)_n = \left(d\hat{f}(\tilde{b}_0+\tilde{b}^\dagger) - \hat{f}(\tilde{b}_0+\tilde{b}^\dagger)\cdot\hat{f}(\tilde{b}_0+\tilde{b}^\dagger)\right)_n =$$

$$\left(d\hat{f}(\tilde{b}_0+\tilde{b}^\dagger) - \frac{1}{2}[\hat{f}(\tilde{b}_0+\tilde{b}^\dagger),\hat{f}(\tilde{b}_0+\tilde{b}^\dagger)]\right)_n \equiv (d\hat{f}(b^\dagger))_n \pmod{\left(\mathcal{I}_S^{odd}\right)^2},$$

where the equivalence follows from Corollary 3.30. Thus, using $[\cdot]$ to denote the equivalence class in the quotient $\hat{R}_{E_{l+1}}^\diamond = F^{E_{l+1}}\hat{R}^\diamond/F^{>E_{l+1}}\hat{R}^\diamond$, we have

$$\int_{I\times L} \pi_r\left(\tilde{\mathfrak{m}}_{0,E_{l+1}}^{\tilde{\gamma},\tilde{b}_0+\tilde{b}^\dagger}\right) = (r\otimes\mathrm{Id})\left(\left[\int_{I\times L}\hat{f}\left(\tilde{\mathfrak{m}}_0^{\tilde{\gamma},\tilde{b}_0+\tilde{b}^\dagger}\right)\right]\right) \equiv$$

$$\equiv (r\otimes\mathrm{Id})\left(\left[\int_{I\times L} d\hat{f}\left((\tilde{b}_0+\tilde{b}^\dagger)\right)\right]\right) \pmod{\left(\mathcal{I}_S^{odd}\right)^2}.$$

Recall that $n > 0$ and that $j_i^*(\tilde{b}^\dagger) \in Z(\hat{R}^\diamond)\cdot 1$ for $j = 0,1$. Thus, Stokes' Theorem implies

$$\int_{I\times L} d\hat{f}\left((\tilde{b}_0+\tilde{b}^\dagger)\right) = \int_L \hat{f}(b) - \int_L \hat{f}(b') \equiv 0 \pmod{\left(\mathcal{I}_{\hat{R}}^{odd}\right)^2},$$

where the equivalence is by assumption. Let $q_S : S \to \frac{S}{\left(\mathcal{I}_S^{odd}\right)^2}$ be the quotient map of vector spaces. We factorize $q_S \circ (r\otimes\mathrm{Id}_S)$ through $\left(\frac{\hat{R}^\diamond}{\left(\mathcal{I}_{\hat{R}}^{odd}\right)^2}\right)_E$ as in the diagram,

$$\begin{array}{ccc}
\hat{R}_E^\diamond & \longrightarrow & \left(\dfrac{\hat{R}^\diamond}{\left(\mathcal{I}_{\hat{R}}^{odd}\right)^2}\right)_E \\
{\scriptstyle r\otimes\mathrm{Id}_S}\Big\downarrow & & \Big\downarrow \\
S & \xrightarrow{\quad q_S \quad} & \dfrac{S}{\left(\mathcal{I}_S^{odd}\right)^2}
\end{array} \quad ,$$

and deduce that

$$q_S \circ (r\otimes\mathrm{Id})\left(\left[\int_{I\times L} d\hat{f}\left((\tilde{b}_0+\tilde{b}^\dagger)\right)\right]\right) = 0.$$

It follows that $(\pi_r)_*(\mathfrak{o}) = 0$.

Now assume that $E_1^{n+1-h,2h-n-1} \neq E_\infty^{n+1-h,2h-n-1}$. By Theorem 5.3, we have

$$E_\infty^{n+1-h,2h-n-1} = 0,$$

so $(\pi_r)_*(\mathfrak{o}) = 0$. $\qquad\square$

Proposition 6.11. *In the two cases of Proposition 6.7, if n is odd or $\int_L \hat{f}(\bar{b}) \neq 0$, then $\mathfrak{o} = 0$.*

Proof. The assumption of Corollary 5.33 are satisfied in the case (1). In case (2), the assumptions of Lemma 5.39. Hence, by Lemma 6.9 for any homogeneous element $r \in R_E^{\diamond \vee}$ the element $(\pi_r)_*(\mathfrak{o})$ is in the image of $F_{\nu_S}^{n-(2+|r|)} H^2(\overline{\mathfrak{B}}_{\mathscr{C}}[|r|]^\kappa, \tilde{\mathfrak{m}}_1^{\tilde{\gamma}, \tilde{b}_0}) \hookrightarrow H^2(\overline{\mathfrak{B}}_{\mathscr{C}}[|r|]^\kappa, \tilde{\mathfrak{m}}_1^{\tilde{\gamma}, \tilde{b}_0})$. Thus, the assumptions of Lemma 6.10 are satisfied, and $(\pi_r)_*(\mathfrak{o})$ is an exact cochain for any homogeneous element $r \in R_E^{\diamond \vee}$. It follows from Lemma 4.32 that $\mathfrak{o} = 0$. $\qquad\square$

Corollary 6.12. *In the two cases of Proposition 6.7, assume that n is odd, or $\int_L \hat{f}(\bar{b}) \neq 0$. Then, there exists $\eta \in F^{E_l}(\mathfrak{B})_1$ such that*

(1) $\tilde{b} + \eta$ *is a bounding cochain for* $\frac{\mathfrak{B}}{F^{>E_l}\mathfrak{B}}$;
(2) $j_i^*(\tilde{b} + \eta) = j_i^*(\tilde{b})$ *for $i = 0, 1$.*

Proof. By Proposition 6.11 we are in the setting of Lemma 3.27, and we can choose $\xi \in F^{E_l}(\mathfrak{B}^\kappa)_1$ such that $\tilde{b}+\xi$ is a bounding cochain for $\frac{\mathfrak{B}}{F^{>E_l}\mathfrak{B}}$. Define $\eta := \xi - ((1-t)j_0^*(\xi) + tj_1^*(\xi))$. Observe that

$$((1-t)j_0^*(\xi) + tj_1^*(\xi)) \in Z(\hat{\mathfrak{R}}^\diamond) \cdot 1.$$

Hence, by the properties (1) and (4) of the A_∞ algebra

$$\tilde{\mathfrak{m}}_1^{\tilde{\gamma}, \tilde{b}}(\eta) = \tilde{\mathfrak{m}}_1^{\tilde{\gamma}, \tilde{b}}(\xi) - d\left((1-t)j_0^*(\xi) + tj_1^*(\xi)\right).$$

As $d\left((1-t)j_0^*(\xi) + tj_1^*(\xi)\right) \in Z(\ker d_{\hat{\mathfrak{R}}^\diamond}) \cdot 1$, we deduce that $\tilde{b} + \eta$ is a solution to the Maurer-Cartan equation in $\frac{\mathfrak{B}}{F^{>E_l}\mathfrak{B}}$. $\qquad\square$

Proof of Proposition 6.7. By assumption $\varrho([\gamma, b]) = \varrho([\gamma', b'])$. Thus, we have $\int_L \hat{f}(b) - \int_L \hat{f}(b') \in \left(\mathcal{I}_{\hat{R}}^{odd}\right)^2$, and we can define $\tilde{\gamma}$ by equation (20). Define $R^\diamond := R(b, b')$ as in Notation 4.14.

By Lemma 4.27 there exists an element $\tilde{b}_0 \in (\hat{\mathfrak{C}}^\diamond)^\kappa$ with $\nu_s \otimes \nu(\tilde{b}) > 0$, such that $\bar{\tilde{b}}_0$ is a bounding cochain for $\bar{\bar{\mathfrak{C}}}$, and $j_0^*(\tilde{b}_0) = b$, $j_1^*(\tilde{b}_0) = b'$. Recall that b is a unit bounding cochain, so

$$\int_L \hat{f}(\bar{b}) \neq 0.$$

Therefore, we can apply Corollary 6.12 successively and get a sequence of elements $\{\eta_i\}_{i=1}^\infty \subset \mathfrak{B}^\kappa$ such that $\eta_i \in F^{E_i}\mathfrak{B}$, and the element $\tilde{x}_{(\ell)} := \sum_{i=1}^\ell \eta_i$ is a solution to the Maurer-Cartan equation in $\frac{\mathfrak{B}}{F^{>E_l}\mathfrak{B}}$. Since $\lim_{i \to \infty} \nu(\eta_i) = \infty$, it follows from completeness of $\hat{\mathfrak{C}}^\diamond$ that there exists a unique limit $\mathfrak{b} := \tilde{b}_0 + \lim_{\ell \to \infty} \tilde{x}_{(\ell)}$. By construction $\mathfrak{b}$ is a solution to the Maurer-Cartan equation in $\hat{\mathfrak{C}}_{\mathscr{C}}^\diamond$, and $j_0^*(\mathfrak{b}) = b$, $j_1^*(\mathfrak{b}) = b'$. Moreover, $\nu(\tilde{\mathfrak{m}}_0^{\tilde{\gamma}, \mathfrak{b}}) > 0$ as $\mathfrak{b}$ is a solution to the Maurer-Cartan equation in $\frac{\hat{\mathfrak{C}}^\diamond}{F^{>0}\hat{\mathfrak{C}}^\diamond}$. We conclude that $(\tilde{\mathfrak{m}}^{\tilde{\gamma}}, \mathfrak{b})$ is a pseudoisotopy from $(\hat{\mathfrak{m}}^\gamma, b)$ to $(\hat{\mathfrak{m}}^{\gamma'}, b')$. $\qquad\square$

Remark 6.13. Here it is important that b and b' are elements of $\hat{C}$ and not only the $\nu_S \otimes \nu$ completion of $\hat{R} \otimes A^*(L)$ which we will denote by $\hat{\hat{C}}$. Indeed, to construct the sababa algebra $R(b, b')$, we decomposed the chains b, b' as in equation (21), this decomposition does not always exist for elements of $\hat{\hat{C}}$. For example $\sum_{k \in \mathbb{N}} \lambda_k \cdot 1$ where $\lambda_k \in \hat{R}$ are defined in

Remark 4.17 can not be decomposed as in equation 21. Working with the sababa A_∞ algebra $\hat{\mathfrak{C}}^\diamond$ constructed from $R^\diamond$ is crucial for our construction of the bounding cochain $\tilde{b}$ as explained in Remark 4.29.

7. Superpotential

7.1. Invariance under pseudoisotopy. Here we restrict ourself to $n > 0$ and even.

Proof of Theorem 1. Let $\tilde{\gamma}$ and $\tilde{b}$ be as in the definition of gauge equivalence, Definition 4.20. Recall from [36, Remark 2.4] that Stokes' theorem for differential forms with coefficients in a graded ring comes with a sign,

$$\int_M d\xi = (-1)^{\dim M + |\xi| + 1} \int_{\partial M} \xi.$$

We use this for the computation of the sign below. We also remember that the induced chain map by a continues function is of degree 0 in our computation of the signs. Hence by Lemma 4.12, Lemma 4.10, and Proposition 2.14,

$$\Omega_{J'}(\gamma', b') - \Omega_J(\gamma, b) = (-1)^n \Big(\sum_{k \geq 0} \frac{1}{k+1} \Big(\langle \mathfrak{m}_k^{\gamma'}(b'^{\otimes k}), b' \rangle - \langle \mathfrak{m}_k^{\gamma}(b^{\otimes k}), b \rangle \Big) \Big)$$

$$= (-1)^{n+n+1} \Big(pt_* \sum_{k \geq 0} \frac{1}{k+1} d \langle\!\langle \tilde{\mathfrak{m}}_k^{\tilde{\gamma}}(\tilde{b}^{\otimes k}), \tilde{b} \rangle\!\rangle \Big)$$

$$= pt_* \sum_{\substack{k_1 + k_2 = k+1 \\ k \geq 0, k_1 \geq 1, k_2 \geq 0}} \frac{k_1}{k+1} \cdot \langle\!\langle \tilde{\mathfrak{m}}_{k_1}^{\tilde{\gamma}}(\tilde{b}^{\otimes k_1}), \tilde{\mathfrak{m}}_{k_2}^{\tilde{\gamma}}(\tilde{b}^{\otimes k_2}) \rangle\!\rangle.$$

We may relax the limit of summation to $k_1 \geq 0$ since k_1 multiplies the summands. Interchanging the summation indices k_1 and k_2 and using the symmetry of the pairing (8), we continue

$$= pt_* \frac{1}{2} \sum_{\substack{k_1 + k_2 = k+1 \\ k, k_1, k_2 \geq 0}} \frac{k_1}{k+1} \cdot \langle\!\langle \tilde{\mathfrak{m}}_{k_1}^{\tilde{\gamma}}(\tilde{b}^{\otimes k_1}), \tilde{\mathfrak{m}}_{k_2}^{\tilde{\gamma}}(\tilde{b}^{\otimes k_2}) \rangle\!\rangle +$$

$$+ pt_* \frac{1}{2} \sum_{\substack{k_1 + k_2 = k+1 \\ k, k_1, k_2 \geq 0}} \frac{k_2}{k+1} \cdot \langle\!\langle \tilde{\mathfrak{m}}_{k_1}^{\tilde{\gamma}}(\tilde{b}^{\otimes k_1}), \tilde{\mathfrak{m}}_{k_2}^{\tilde{\gamma}}(\tilde{b}^{\otimes k_2}) \rangle\!\rangle$$

We can add $0 = pt_* \frac{1}{2} \langle\!\langle \tilde{\mathfrak{m}}_0^{\tilde{\gamma}}, \tilde{\mathfrak{m}}_0^{\tilde{\gamma}} \rangle\!\rangle$ because $(\tilde{\mathfrak{m}}_0^{\tilde{\gamma}})^1 = 0$, thus we derive

$$= pt_* \frac{1}{2} \sum_{k_1, k_2 \geq 0} \langle\!\langle \tilde{\mathfrak{m}}_{k_1}^{\tilde{\gamma}}(\tilde{b}^{\otimes k_1}), \tilde{\mathfrak{m}}_{k_2}^{\tilde{\gamma}}(\tilde{b}^{\otimes k_2}) \rangle\!\rangle.$$

By $\tilde{b}$ being a bounding cochain

$$= pt_* \frac{1}{2} \langle\!\langle \tilde{c} \cdot 1, \tilde{c} \cdot 1 \rangle\!\rangle.$$

Using $n > 0$ and the bilinearity of $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ over $\hat{\mathfrak{R}}$ from Property (6), we conclude

$$= 0.$$

Hence we conclude that the superpotential is invariant under gauge equivalence. $\qquad\square$

7.2. Comparison with Welschinger invariants.

Proposition 6.1 in the real setting, can be simplified in the case $n = 2, 3$.

Proposition 7.1. *If $n = 2$, and $\gamma \in (\mathcal{I}_Q D)_2$ is real and closed, any form $b \in (\mathcal{I}_{\hat{R}})_{1-n} \otimes A^n(L)$ is a real bounding cochain for $\hat{\mathfrak{m}}^\gamma$.*

Proof. Since $\dim L = 2$, it follows that $\mathfrak{s}$ is a spin structure. Lemma 4.39 implies that b is real. Assume by contradiction that b is not a bounding chain, and let i be the minimal index such that
$$\hat{\mathfrak{m}}_0^{\gamma,b} \not\equiv 0 \quad (\mathrm{mod}\ F^{>E_i}\hat{C}_{\not\mathcal{C}}^\diamond).$$
So, there exists $r \in (R_{E_i}^\diamond)^\vee$ homogeneous, such that $\pi_r\left(\hat{\mathfrak{m}}_0^{\gamma,b}\right) \notin Z(S)[|r|] \cdot 1$.

As $|\hat{\mathfrak{m}}_0^{\gamma,b}| = 2$ it follows from Lemma 4.42, and Proposition 4.3 that
$$\pi_r\left(\hat{\mathfrak{m}}_0^{\gamma,b}\right) \in \bigoplus_{j \equiv 0,3 \ (\mathrm{mod}\ 4)} (S[|r|])_{2-j} \otimes \left(\overline{C^\diamond}\right)_j = \bigoplus_{j=0,3} (S[|r|])_{2-j} \otimes \left(\overline{C^\diamond}\right)_j.$$

Since $n = 2$, then $\left(\overline{C^\diamond}\right)_3 \cong A^3(L) = 0$. So
$$\pi_r\left(\hat{\mathfrak{m}}_0^{\gamma,b}\right) \in (S[|r|])_2 \otimes \left(\overline{C^\diamond}\right)_0 \cong (S[|r|])_2 \otimes A^0(L).$$

It follows from Lemma 4.25 that $i > 0$, and that $\hat{\mathfrak{m}}_1^{\gamma,b}$ is a boundary operator on $\hat{C}_{\not\mathcal{C},E_i}^\diamond$. The choice of i implies that $\hat{\mathfrak{m}}_{0,E_i}^{\gamma,b}$ is a cocycle. Recall that π_r is a cochain map. So, Lemma 4.26 implies that $d\left(\pi_r\left(\hat{\mathfrak{m}}_0^{\gamma,b}\right)\right) = 0$.

Since $\pi_r\left(\hat{\mathfrak{m}}_0^{\gamma,b}\right) \in (S[|r|])_2 \otimes A^0(L)$, we can write $\pi_r\left(\hat{\mathfrak{m}}_0^{\gamma,b}\right) = \lambda \otimes 1$ where $\lambda \in (S[|r|])_2$ and $1 \in A^0(L)$ is the constant function.

Moreover, we know that $\pi_r\left(\hat{\mathfrak{m}}_0^{\gamma,b}\right)$ is a $Z(S)[|r|]$ multiple of 1. Since $S[|r|]_2 \subset Z(S)[|r|]$ (as elements of even degree in S are in the center), we have that $\lambda \in Z(S)[|r|]$ and $\pi_r\left(\hat{\mathfrak{m}}_0^{\gamma,b}\right) = \lambda \otimes 1$ is a $Z(S)[|r|]$ multiple of 1, a contradiction. $\qquad\square$

Proof of Theorem 4. Assume without loss of generality that $[\gamma_N] = A$. Abbreviate $k = k_{d,l}$. By Proposition 7.1 we may choose $b = s \cdot \bar{b}$ where $\bar{b}$ is a representative of the Poincaré dual of a point. By definition, for k odd
$$\mathrm{OGW}_{\beta,k}(\gamma_N^{\otimes l}) = k!l! \cdot [\text{the coefficient of } T^\beta s^k t_N^l \text{ in the formal power series } \Omega]$$
$$= k!l!\left(\frac{1}{l!k}\langle evb_{0*}^\beta(\bigwedge_{j=1}^{l}(evi_j^\beta)^*\gamma_N \wedge \bigwedge_{j=1}^{k-1}(evb_j^\beta)^*\bar{b}), \bar{b}\rangle\right)$$
$$= -(k-1)!\left(\int_{\mathcal{M}_{k,l}(\beta)} \wedge_{j=1}^l(evi_j)^*\gamma_N \wedge \wedge_{j=0}^{k-1}(evb_j)^*\bar{b}\right).$$

Denote by $\mathcal{M}_{k,l}^S(\beta)$ the moduli space of genus zero J-holomorphic open stable maps $u : (\Sigma, \partial\Sigma) \to (X, L)$ of degree $\beta \in \Pi$ with one boundary component, k unordered boundary points, and l interior marked points. It comes with evaluation maps $evb_j^\beta : \mathcal{M}_{k,l}^S(\beta) \to L$, $j = 1, \ldots, k$, and $evi_j^\beta : \mathcal{M}_{k,l}^S(\beta) \to X$, $j = 1, \ldots, l$. The space $\mathcal{M}_{k,l}^S(\beta)$ carries a natural orientation induced by the spin structure on L as in [8, Chapter 8] and [33]. The diffeomorphism

of $\mathcal{M}_{k,l}^S(\beta)$ corresponding to relabeling boundary marked points by a permutation $\sigma \in S_k$ was defined to preserve orientation

$$\mathrm{OGW}_{\beta,k}(\gamma_N^{\otimes l}) = -\int_{\mathcal{M}_{k,l}^S(\beta)} \wedge_{j=1}^l (evi_j)^* \gamma_N \wedge \wedge_{j=1}^k (evb_j)^* \bar{b}.$$

Thus, the theorem follows from [33, Theorem 1.8]: The factor of 2^{1-l} arises from two independent sources. First, as explained in [33, Theorem 1.8], each real holomorphic sphere corresponds to two holomorphic disks. This gives a factor of 2. Second, each interior constraint γ_N is Poincaré dual to the homology class of point. On the other hand, Welschinger [40, 41] considers constraints that are pairs of conjugate points, thus representing twice the homology class of a point. This gives an additional factor of 2^{-l}. □

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INSTITUTE OF MATHEMATICS, THE HEBREW UNIVERSITY OF JERUSALEM
Email address: `eladkosloff@gmail.com`

INSTITUTE OF MATHEMATICS, THE HEBREW UNIVERSITY OF JERUSALEM
Email address: `jake@math.huji.ac.il`