

# Optimizing Weak Orders via Integer Linear Programming

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## Abstract

Rank aggregation problems aim to combine multiple individual orderings of a common set of items into a consensus ranking that best reflects the collective preferences. This paper introduces a general Integer Linear Programming (ILP) framework to model and solve, in an exact way, problems whose solutions are weak orders (a.k.a. bucket orders). Within this framework, we consider additional relevant constraints to produce the consensus bucket order, considering configurations with a fixed number of buckets, predefined bucket sizes, top- $k$  type problems, and fairness constraints. All these formulations are developed in a general setting, allowing their application to different rank aggregation contexts. One of these problems is the Optimal Bucket Order Problem (OBOP), for which we propose for the first time an exact formulation and test the variants proposed. The computational study includes, on the one hand, a comparison between the exact results obtained by our models and the heuristic methods proposed by Aledo et al. (2018), and on the other hand, an additional evaluation of their performance on a representative set of instances from the PrefLib library. The results confirm the validity and efficiency of the proposed approach, providing a solid foundation for future research on rank aggregation problems with weak orders as consensus rankings.

**Keywords:** Integer Programming, Rank Aggregation Problems, Fairness, Optimal Bucket Order Problem (OBOP)

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## 1. Introduction

Rank aggregation problems consist in combining multiple individual rankings into a single consensus ranking that best reflects the collective preference or ordering. These problems arise in diverse domains, such as social choice (Dwork et al., 2001), information retrieval (Farah and Vanderpooten, 2007), meta-search engines (Dwork et al., 2001), bioinformatics (Kolde et al., 2012), and recommendation systems (Balchanowski and Boryczka, 2023), where different sources or evaluators produce potentially conflicting orderings of the same set of items. The central challenge is to design principled methods that reconcile such discrepancies while satisfying desirable properties of fairness, robustness, and computational efficiency (Balestra et al., 2024; Dong et al., 2019; Kuhlman and Rundensteiner, 2020; Aledo et al., 2025).

A variety of formulations emerge depending on both the measures of dissimilarity employed and the type of rankings considered as input or required as output. Among the most studied dissimilarity measures are the Kendall tau distance, which counts the number of pairwise disagreements between two rankings; the Spearman footrule distance, based on absolute position differences; and more general

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Kemeny–Snell distances, which provide axiomatic frameworks for defining consensus (Critchlow, 1985; Kumar and Vassilvitskii, 2010). The nature of the rankings also introduces important distinctions: one may deal with complete rankings of all items, partial or incomplete rankings where only a subset of items is ordered, or rankings with ties, leading to richer but more complex aggregation models (Lin, 2010; Alcaraz et al., 2022; Franceschini et al., 2022).

Rank aggregation problems can involve additional structural constraints on the consensus ranking. For instance, restrictions such as top- $k$  aggregation (where only the top portion of the ranking is required Chen and Suh, 2015; Lin, 2010) or positional constraints (where certain items must appear in pre-specified positions of the ranking). Such settings arise naturally in applications such as college admissions, hiring committees, or recommendation systems (Chakraborty et al., 2024).

Within this general framework, the Kemeny Rank aggregation problem (KRP) emerges as one of the fundamental issues in the field. It seeks the ranking minimizing the sum of Kendall distances to the input rankings (Kemeny, 1962). Variants of the KRP permit, for instance, the incorporation of incomplete rankings and/or rankings containing ties. Although it is an NP-hard problem, the KRP has been handled using Integer Linear Programming (ILP) (Ali and Meilă, 2012), also considering the existence of ties in the output (Brancotte et al., 2015; Yoo and Escobedo, 2021) (see Section 2).

In general, for many combinatorial optimization problems that cannot be solved efficiently, both exact optimization models and approximate solution algorithms are commonly employed. The former, which yield optimal solutions, are typically applied to small or moderately sized instances, where solver runtimes remain reasonable. The latter, in contrast, are used to tackle large-scale instances where exact methods become computationally infeasible.

In this work, we propose several optimization models to address aggregation problems whose solutions are weak orderings, that is, complete rankings allowing for ties. These models enable the computation of optimal solutions, as opposed to the approximate results typically obtained through (meta)heuristic approaches.

It is worth noting that, while there exists an extensive body of literature on optimization models for linear orderings (see Alcaraz et al. (2022) for a survey), the literature concerning weak orderings is comparatively scarce. Beyond contributing to narrowing this gap, the formulation of optimization models makes it possible to obtain optimal solutions within feasible subspaces. In other words, such formulations allow us to determine the best possible ranking among those satisfying specific conditions, by suitably modifying or incorporating constraints and/or variables. For instance, in the context of linear orders, additional variables and constraints have been introduced to handle cyclic orders (García-Nové et al. (2017)), to rank cluster representatives (Alcaraz et al. (2020)), or to establish consistent orderings among sets of elements (Labbé et al. (2023)). Moreover, alternative objective functions have also been proposed to reduce the number of optimal solutions (Benito-Marimon et al., 2025), and structural properties of optimal solutions have been studied in Ceberio et al. (2015).

Thus, in this paper, we provide a general setting to tackle different rank aggregation problems of  $n$  items using ILP. We focus on instances where the output is a weak order (also called bucket order). In particular, we consider the following variants:

- Models with a fixed number of buckets (Section 4). First of all, we seek consensus rankings in which the number of buckets is predetermined. As particular cases, we consider settings where all buckets have the same size, or where bucket sizes are prescribed (and possibly different) depending on their position in the output ranking.

- Top- $k$  type models (Section 5), where the goal is to minimize the fitness function under the assumption that there exists a worst bucket containing the remaining  $n - k$  items. As we will see, this approach also allows the top- $k$  items to be enriched by providing them with a weak ordering. This formulation should not be confused with the classical top- $k$  problem, as we will discuss in detail. We call this variant the Tail-Collapsed Upper- $k$ .
- Models which incorporate fairness constraints (Section 6), enabling group-aware consensus ranking responses.

As a case study, we adapt the above variants to a well-known rank aggregation problem, the Optimal Bucket Order Problem (OBOP) (Gionis et al., 2006; Ukkonen et al., 2009; Kenkre et al., 2011; Aledo et al., 2017). The OBOP is a distance-based rank aggregation task in which the input is a pairwise order matrix  $C$  of order  $n$ , typically encoding the precedence relations among  $n$  items, and the output is a bucket order, namely complete ranking that allows ties (see Section 3 for the details). To date, the OBOP has been addressed using greedy methods, such as the Bucket Pivot Algorithm (BPA) (Gionis et al., 2006; Ukkonen et al., 2009) and its improved variants (Aledo et al., 2017), as well as other approaches designed to handle a large number of items (Aledo et al., 2021). Evolutionary algorithms have also been applied (Aledo et al., 2018; Lorena et al., 2021). In addition, specialized algorithms have been proposed to produce as output a bucket order with a predetermined number of buckets (D’Ambrosio et al., 2019). However, to the best of our knowledge no integer formulation has been introduced for this problem and the variants hereby considered.

Thus, we conduct the following computational experiments (see Section 7):

- We tackle the OBOP using ILP on a representative benchmark of instances with sizes (i.e., number  $n$  of ranked items) ranging from 50 to 250, obtained from the online libraries PrefLib<sup>1</sup> and MovieLens<sup>2</sup>. This benchmark coincides with the one used in Aledo et al. (2018), where evolutionary methods were applied. As noted earlier, only non-exact approaches have been employed to solve the OBOP. In this study, we provide exact solutions for these widely used instances, thereby enabling a precise evaluation of the performance of algorithms previously applied to this problem.
- To illustrate the different variants considered in this study, we use a second benchmark of 30 instances with  $n$  ranging from 5 to 55, which is also obtained from PrefLib. The reason for using this simpler benchmark is that a reduced number of items allows for a clearer illustration of the variants introduced in this study. For this benchmark, we tackle the three problems described above: the OBOP for a fixed number of buckets ( $p$ -OBOP), the Tail-Collapsed Upper- $k$  (TCU- $k$  OBOP), and the OBOP incorporating fairness constraints (Fair OBOP). In particular, in Section 7, we conduct an experimental study to evaluate the differences regarding solutions, fitness, and computational performance between the different variants and the base case (OBOP solution).

The paper concludes with some final remarks and directions for future research (Section 8). We also include two final appendices that address more technical aspects of the  $p$ -OBOP model.

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<sup>1</sup><http://www.preflib.org/>

<sup>2</sup><https://grouplens.org/datasets/movielens/100k/>

## 2. Integer Programming Framework

In this section, we introduce the integer programming framework that characterizes problems whose solutions are weak orders, following [Fiorini and Fishburn \(2004\)](#). Recall that a *weak order* (or *bucket order*) is a complete preorder: every pair of items is comparable, ties are allowed, and the relation is transitive, so items may be grouped into ranked equivalence classes.

We now encode a weak order on the set  $[[n]] = \{1, 2, \dots, n\}$  through the binary variables

$$x_{rs} = \begin{cases} 1 & \text{if } r \text{ comes before or is tied with } s, \\ 0 & \text{otherwise,} \end{cases} \quad \forall r, s \in [[n]] : r \neq s.$$

Hence,  $x_{rs} = x_{sr} = 1$  indicates a tie between  $r$  and  $s$ , while  $x_{rs} = 1$  and  $x_{sr} = 0$  implies that item  $r$  is ranked ahead of item  $s$ .

The standard constraints ensuring that the resulting relation is a weak order are (see [Brancotte et al. \(2015\)](#), [Yoo and Escobedo \(2021\)](#)):

$$\min f(\Pi, \mathbf{x}) \tag{1a}$$

$$\text{s.t. } x_{rs} + x_{sr} \geq 1 \quad \forall r, s \in [[n]] : r < s, \tag{1b}$$

$$x_{rs} + x_{st} \leq 1 + x_{rt} \quad \forall r, s, t \in [[n]] : r \neq s \neq t \neq r, \tag{1c}$$

$$x_{rs} \in \{0, 1\} \quad \forall r, s \in [[n]] : r \neq s. \tag{1d}$$

For any aggregation problem  $\Pi$ , the objective function is represented by  $f(\Pi, \mathbf{x})$  defined in terms of the variables  $x_{rs}$ . The concrete expression of this function depends on the nature of  $\Pi$ . For example, for the *generalized Kemeny aggregation* problem, that is, the KRP in which the input rankings may be complete or partial and may contain ties,  $f(\text{KRP}, \mathbf{x}) = -\sum_{r=1}^n \sum_{s=1}^n a_{rs}^{\text{KRP}} (2x_{rs} - 1)$ , where  $a_{rs}^{\text{KRP}}$  is computed in different ways, depending on whether these input rankings are complete or not (see [Yoo and Escobedo, 2021](#), for details). For the *Linear Ordering Problem* (LOP), we have  $f(\text{LOP}, \mathbf{x}) = -\sum_{r=1}^n \sum_{s \neq r} a_{rs}^{\text{LOP}} x_{rs}$ , where  $a_{rs}^{\text{LOP}}$  is the number of times  $r$  precedes  $s$ . In both cases, the negative sign in front of the summations indicates that the problem is formulated as a maximization problem (see [Alcaraz et al. \(2022\)](#) for a description and application of the LOP).

Constraint (1b) guarantees that every pair of items is comparable, while (1c) enforces transitivity. Together, they ensure that the resulting relation is a weak order.

If ties are not allowed, the inequality in the comparability constraint can be replaced by the equality

$$x_{rs} + x_{sr} = 1 \quad \forall r, s \in [[n]] : r < s, \tag{1b'}$$

resulting in a formulation whose feasible solutions correspond to linear orders, that is, permutations or complete rankings without ties. Moreover, if one wishes to obtain a permutation consistent with a given weak order solution  $\hat{\mathbf{x}}$ , as defined by [Aledo et al. \(2016\)](#), one must add constraints (1b') together with the following variable fixings:

$$x_{rs} = 1 \quad \forall r, s \in [[n]], r \neq s, \quad \hat{x}_{rs} = 1, \quad \hat{x}_{sr} = 0.$$

Finally, to check whether multiple optimal solutions exist, an exclusion constraint can be added to

remove a previously found solution  $\hat{\mathbf{x}}$  from the feasible region:

$$1 \leq \sum_{r=1}^n \sum_{\substack{s=1 \\ s \neq r}}^n ((1 - \hat{x}_{rs})x_{rs} + \hat{x}_{rs}(1 - x_{rs})).$$

This constraint forces at least one binary variable to differ from the current optimal solution, thus excluding it while preserving all other feasible solutions. Resolving the model with this restriction allows for the identification of alternative optima, if any exist.

### 3. The Optimal Bucket Order Problem

As introduced in Section 2, a *bucket order*  $\mathcal{B}$  (also called a *weak order* or a *complete ranking with ties*) on  $n$  elements is an ordered partition of  $[[n]] = \{1, 2, \dots, n\}$ . That is, it consists of disjoint subsets (buckets)  $B_1, B_2, \dots, B_p$  with  $1 \leq p \leq n$ , such that  $\bigcup_{u=1}^p B_u = [[n]]$ . We represent  $\mathcal{B}$  by listing the elements with commas between items in the same bucket and vertical bars between buckets. For instance,  $4 \mid 1 3 \mid 2 5$  denotes a bucket order on  $[[5]]$  with three buckets, expressing that 4 is preferred to all others, 1 and 3 are tied and ranked ahead of 2 and 5, which are also tied. Each bucket order  $\mathcal{B}$  can be represented by a *bucket matrix*  $B = (b_{rs})_{n \times n}$ , where  $b_{rs} = 1$  if  $r$  is preferred to  $s$ ,  $b_{rs} = 0$  if  $s$  is preferred to  $r$ , and  $b_{rs} = 0.5$  when  $r$  and  $s$  are tied. Thus,  $b_{rr} = 0.5$  for all  $r$ , and  $b_{rs} + b_{sr} = 1$  for all  $r \neq s$ .

On the other hand, a *pair order matrix* is defined as a matrix  $C = (c_{rs})_{n \times n}$  satisfying that  $c_{rs} \in [0, 1]$ ,  $c_{rs} + c_{sr} = 1$  for all  $r \neq s$ , and  $c_{rr} = 0.5$  for all  $r$ . Therefore, every bucket matrix is a pair order matrix, but not every pair order matrix corresponds to a bucket order, since transitivity may fail and the numerical restrictions on its entries are weaker.

The *Optimal Bucket Order Problem* (OBOP) (Gionis et al., 2006; Ukkonen et al., 2009) is a distance-based rank aggregation problem. Specifically, given a pair order matrix  $C$ , the goal of the OBOP is to find a bucket order  $\mathcal{B}$  (or equivalently, a bucket matrix  $B$ ) minimizing

$$D(B, C) = \sum_{r=1}^n \sum_{s=1}^n |b_{rs} - c_{rs}|.$$

Since the OBOP is NP-complete (Gionis et al., 2006), several greedy and (meta)heuristic algorithms have been proposed (Ukkonen et al., 2009; Kenkre et al., 2011; Aledo et al., 2017, 2018; D'Ambrosio et al., 2019; Aledo et al., 2021), but no exact methods have yet been applied to solve it — a gap this work aims to address.

#### 3.1. Integer Linear Programming formulation for the OBOP

As established in Section 2, the constraints guaranteeing that any feasible solution represents a weak order are already known. The only remaining element for a complete formulation of the OBOP is its objective function, whose expression in terms of the  $x$ -variables is presented in the following proposition, and whose linearized counterpart will be introduced later in the paper.

**Proposition 1.** *Consider the Integer Programming model (1) described in Section 2, which ensures that any feasible solution  $\mathbf{x}$  corresponds to a weak order. If the objective function is defined as*

$$f(\text{OBOP}, \mathbf{x}) = \sum_{r=1}^n \sum_{\substack{s=1 \\ s \neq r}}^n \left| \frac{x_{rs} - x_{sr} + 1}{2} - c_{rs} \right|, \quad (2)$$

then the resulting optimization model solves the OBOP, since  $f(\text{OBOP}, \mathbf{x})$  coincides with the distance  $D(B, C)$  that the OBOP seeks to minimize. Therefore, by combining the constraints in (1) with the objective function above, we obtain an exact Integer Programming formulation of the OBOP.

*Proof.* Since  $b_{rr} = c_{rr} = 0.5$  for all  $r \in [[n]]$ , the diagonal terms of  $D(B, C)$  vanish. For  $r \neq s$ , recall that  $b_{rs} = 1$  if  $r$  precedes  $s$ ,  $b_{rs} = 0.5$  if  $r$  and  $s$  are tied, and  $b_{rs} = 0$  otherwise. According to model (1), these cases correspond respectively to  $(x_{rs}, x_{sr}) = (1, 0)$ ,  $(1, 1)$ , and  $(0, 1)$ , yielding in all situations

$$b_{rs} = \frac{x_{rs} - x_{sr} + 1}{2}.$$

Hence,  $f(\text{OBOP}, \mathbf{x})$  in (2) coincides with  $D(B, C)$ , proving the equivalence.  $\square$

However, this function can be simplified, as we state in the following corollary.

**Corollary 1.** *The objective function (2), which models the OBOP, coincides with*

$$f(\text{OBOP}, \mathbf{x}) = 2 \sum_{r=1}^n \sum_{s=r+1}^n \left| \frac{x_{rs} - x_{sr} + 1}{2} - c_{rs} \right|. \quad (3)$$

*Proof.* For  $r \neq s$ , note that  $\frac{x_{rs}-x_{sr}+1}{2} = 1 - \frac{x_{sr}-x_{rs}+1}{2}$  and  $c_{rs} = 1 - c_{sr}$ . Hence the summands for  $(r, s)$  and  $(s, r)$  coincide, so the sum over all ordered pairs equals twice the sum over  $r < s$ , yielding (3) and the same objective value as (2).  $\square$

Once established that the OBOP can be formulated with the above objective function (3), it remains to linearize it. To this end, we introduce auxiliary decision variables  $d_{rs}$  for all  $r, s \in [[n]]$  with  $r < s$ , representing the absolute deviation between  $b_{rs}$  and  $c_{rs}$  (which coincides with that between  $b_{sr}$  and  $c_{sr}$ ). Using a well-known linearization technique, we may rewrite the objective function for any weak order model as

$$f(\text{OBOP}, \mathbf{x}) = 2 \sum_{r=1}^n \sum_{s=r+1}^n d_{rs}, \quad (4)$$

together with the constraints

$$d_{rs} \geq \frac{x_{rs} - x_{sr} + 1}{2} - c_{rs} \quad \forall r, s \in [[n]], r < s, \quad (5a)$$

$$d_{rs} \geq c_{rs} - \frac{x_{rs} - x_{sr} + 1}{2} \quad \forall r, s \in [[n]], r < s. \quad (5b)$$

Incorporating the objective function (4) and constraints (5a) and (5b) into the weak order model (1) yields a complete Integer Linear Programming formulation of the OBOP, given by:

$$\min \quad 2 \sum_{r=1}^n \sum_{s=r+1}^n d_{rs} \quad (6a)$$

$$\text{s.t. } (1b) - (1c) \quad (6b)$$

$$(5a) - (5b) \quad (6c)$$

$$x_{rs} \in \{0, 1\} \quad \forall r, s \in [[n]], r \neq s. \quad (6d)$$

*Remark 1.* It is clear that the problem could also be solved with the objective function (3) without the factor 2. However, in order to keep our results consistent with those in the literature, we will retain this factor.

In order to illustrate the proposed model for the OBOP, we consider the following example.

**Example 1.** Let us consider the following instance of the OBOP with 8 items, whose pair order matrix is given in Figure 1.

$$\frac{1}{100} \begin{pmatrix} 50 & 52 & 46 & 72 & 60 & 70 & 82 & 90 \\ 48 & 50 & 10 & 42 & 60 & 90 & 22 & 78 \\ 54 & 90 & 50 & 76 & 98 & 85 & 82 & 80 \\ 28 & 58 & 24 & 50 & 80 & 76 & 65 & 80 \\ 40 & 40 & 2 & 20 & 50 & 55 & 0 & 20 \\ 30 & 10 & 15 & 24 & 45 & 50 & 50 & 0 \\ 18 & 78 & 18 & 35 & 100 & 50 & 50 & 90 \\ 10 & 22 & 20 & 20 & 80 & 100 & 10 & 50 \end{pmatrix}$$

Figure 1: Pairwise order matrix used in Example 1

Solving this instance with model (6) yields the unique optimum

$$1 \ 3 \mid 2 \ 4 \ 7 \mid 8 \mid 5 \ 6$$

with an objective value of 10.78.

### 3.2. Utopian matrix

To conclude this section, let us revise the notion of *utopian matrix* introduced in [Aledo et al. \(2017\)](#). Given a pair order matrix  $C$  of order  $n$ , the utopian matrix  $U^C = (u_{rs}^C)_{n \times n}$  is defined as

$$u_{rs}^C = \begin{cases} 1 & \text{if } c_{rs} > 0.75, \\ 0.5 & \text{if } 0.25 \leq c_{rs} \leq 0.75, \\ 0 & \text{if } c_{rs} < 0.25. \end{cases}$$

The utopian matrix  $U^C$  provides a superoptimal solution to the OBOP, satisfying  $D(B, C) \geq D(U^C, C)$  for any bucket matrix  $B$ . However,  $U^C$  may violate transitivity and thus not belong to the feasible set of the OBOP.

In fact,  $U^C$  can be obtained from model (6) by omitting the transitivity constraints (1c), as the resulting formulation adjusts the variables to best approximate  $c_{rs}$ , yielding

$$x_{rs} = u_{rs}^C \quad \forall r, s \in [[n]], r \neq s,$$

which precisely defines the utopian matrix. For Example 1, we obtain  $D(U^C, C) = 7.22$ , which shows that the utopian matrix achieves a smaller value than the optimal feasible solution.

## 4. Models with a fixed number $p$ of Buckets

In this section, we extend the ranking problem to the case where the number of buckets is fixed in advance to a number  $p$ ,  $1 \leq p \leq n$ . Hereafter, we refer to this problem as the  $p$ -bucket problem. A similar idea was examined by [D'Ambrosio et al. \(2019\)](#), who proposed a heuristic median ranking constrained to a given number of buckets. Here, we instead develop exact formulations within a general optimization framework capable of handling diverse rank aggregation criteria.

This setting arises naturally in classification and rating systems where items are grouped into a limited number of categories. For example, movie platforms such as IMDb or Rotten Tomatoes rate films using one to five stars, each star level representing a bucket of tied items. Likewise, university rankings often divide institutions into a few tiers (e.g., top, high, medium, low), forming a weak order with a fixed number of buckets.

We introduce two alternative formulations: one based on item–bucket assignment variables and another using representative items. We also discuss extensions that impose fixed bucket capacities or specify the size of each bucket according to its rank position. The objective function is left general, allowing the framework to accommodate different aggregation or consensus measures.

For the first model, we introduce the following variables:

$$y_{ru} = \begin{cases} 1 & \text{if item } r \text{ is assigned to bucket } u, \\ 0 & \text{otherwise,} \end{cases} \quad \forall r \in [[n]], \forall u \in [[p]].$$

Unless otherwise specified, when referring to bucket  $u$  we mean the bucket that occupies the  $u$ –th position in the ranking.

We propose the following model as our first formulation:

$$\min f(\Pi, \mathbf{x}) \tag{7a}$$

$$\text{s.t.} \quad \sum_{u=1}^p y_{ru} = 1 \quad \forall r \in [[n]], \tag{7b}$$

$$\sum_{r=1}^n y_{ru} \geq 1 \quad \forall u \in [[p]], \tag{7c}$$

$$y_{ru} + y_{su} \leq x_{rs} + x_{sr} \quad \forall r, s \in [[n]], \forall u \in [[p]] : r < s, \tag{7d}$$

$$x_{sr} + \sum_{v=1}^u y_{rv} + \sum_{v=u+1}^p y_{sv} \leq 2 \quad \forall r, s \in [[n]], \forall u \in [[p-1]] : r \neq s, \tag{7e}$$

$$x_{rs} \in \{0, 1\} \quad \forall r, s \in [[n]] : r \neq s. \tag{7f}$$

$$y_{ru} \in \{0, 1\} \quad \forall r \in [[n]], \forall u \in [[p]]. \tag{7g}$$

Constraints (7b) ensure that every item belongs to exactly one bucket, (7c) guarantee that each bucket contains at least one item, constraints (7d) enforce that all items assigned to the same bucket are tied, and constraints (7e) forbid an item from being ranked before (or tied with) another that is assigned to a later bucket, ensuring that the global ranking remains consistent with the ordering of the buckets.

For completeness, we note that constraints from the base weak-order model (1) are redundant in this formulation; they remain valid inequalities for model (7). We formalize this result below.

**Proposition 2.** *The base inequalities (1b) and (1c), defining the weak order structure, are valid inequalities for model (7).*

*Proof.* Let  $r, s \in [[n]]$  with  $r < s$ . By (7b), there exists some  $u$  such that  $y_{ru} = 1$ . Applying (7d) to  $(r, s, u)$  yields  $1 + y_{su} \leq x_{rs} + x_{sr}$ , hence  $x_{rs} + x_{sr} \geq 1$ .

For distinct  $r, s, t$ , if  $x_{rs} = x_{st} = 1$ , then by (7e) item  $r$  cannot be placed after item  $s$ , and  $s$  cannot be placed after  $t$ , which together prevent  $r$  from being placed after  $t$ . Hence  $x_{rt} = 1$ , and constraint (1c) is satisfied.  $\square$

Moreover, the integrality of variables  $\mathbf{x}$  can be relaxed, as formalized below.

**Proposition 3.** *The integrality constraints (7f) on variables  $\mathbf{x}$  can be relaxed.*

*Proof.* Let  $r, s \in [[n]]$  with  $r \neq s$ . By (7b), there exist  $u, v \in [[p]]$  such that  $y_{ru} = y_{sv} = 1$ . If  $u = v$ , then from (7d) and the fact that the variables satisfy  $x_{rs} \leq 1$  and  $x_{sr} \leq 1$ , we obtain  $x_{rs} = x_{sr} = 1$ . If  $u < v$ , constraint (7e) implies  $x_{sr} = 0$ , and by (1b) we must have  $x_{rs} = 1$ . Similarly, if  $u > v$ , we get  $x_{rs} = 0$  and  $x_{sr} = 1$ . In all cases, the variables  $x_{rs}$  take integer values, so their integrality can be relaxed.  $\square$

The above formulation fully characterizes the  $p$ -bucket problem using assignment variables. However, an alternative model can be obtained by introducing representative-based variables, offering a different but equivalent perspective of the problem.

In this second formulation, we assume that the representative of each bucket is the item with the highest index among its members. Accordingly, if item  $r$  belongs to a bucket whose representative is  $s$ , then  $r < s$ . Based on this convention, instead of using the assignment variables  $y_{ru}$ , we work with the following decision variables:

$$\alpha_r = \begin{cases} 1 & \text{if item } r \text{ is chosen as the representative of a bucket,} \\ 0 & \text{otherwise,} \end{cases} \quad \forall r \in [[n]],$$

$$\beta_{rs} = \begin{cases} 1 & \text{if item } r \text{ belongs to a bucket whose representative is } s, \\ 0 & \text{otherwise,} \end{cases} \quad \forall r, s \in [[n]] : r < s.$$

Based on these variables, we propose the following formulation as our second model for the  $p$ -bucket problem:

$$\min f(\Pi, \mathbf{x}) \tag{8a}$$

$$\text{s.t. } (1b) - (1c) \tag{8b}$$

$$\sum_{r=1}^n \alpha_r = p \tag{8c}$$

$$\beta_{rs} \leq \alpha_s \quad \forall r, s \in [[n]] : r < s, \tag{8d}$$

$$\sum_{s=r+1}^n \beta_{rs} + \alpha_r = 1 \quad \forall r \in [[n]], \tag{8e}$$

$$x_{rs} \geq \beta_{rs} \quad \forall r, s \in [[n]] : r < s, \tag{8f}$$

$$x_{sr} \geq \beta_{rs} \quad \forall r, s \in [[n]] : r < s, \tag{8g}$$

$$x_{rs} + x_{sr} + \alpha_r \leq 2 \quad \forall r, s \in [[n]] : r < s, \tag{8h}$$

$$x_{rs} \in \{0, 1\} \quad \forall r, s \in [[n]] : r \neq s, \tag{8i}$$

$$\alpha_r \in \{0, 1\} \quad \forall r \in [[n]], \tag{8j}$$

$$\beta_{rs} \in \{0, 1\} \quad \forall r, s \in [[n]] : r < s. \tag{8k}$$

Constraint (8c) ensures that exactly  $p$  representatives are selected and therefore  $p$  buckets are created. Constraints (8d) enforce that an item can only be placed in a bucket whose representative has actually been selected. Constraints (8e) require that every item is either a representative itself or is assigned to the bucket of a representative with a higher index. Constraints (8f) and (8g) ensure that, whenever

an element belongs to a bucket represented by another one, both must be tied. By transitivity, this implies that all elements within the same bucket are tied. Finally, constraints (8h) guarantee that each bucket has exactly one representative: the largest-indexed item in that bucket.

In addition, we note that the following inequality, although not explicitly included in model (8), is always satisfied by any feasible solution and can therefore be regarded as a valid inequality.

**Proposition 4.** *The following constraints are valid inequalities for model (8):*

$$\beta_{rs} + \alpha_s \leq x_{rs} + x_{sr} \quad \forall r < s \quad (9)$$

*Proof.* If  $\beta_{rs} + \alpha_s \leq 1$ , the result follows from (1b). If  $\beta_{rs} + \alpha_s = 2$ , then  $\beta_{rs} = \alpha_s = 1$ , and by (8f)–(8g),  $x_{rs} = x_{sr} = 1$ . Hence, (9) holds in all cases.  $\square$

**Proposition 5.** *By adding constraint (9) to model (8) and removing constraints (8f) and (8g), we obtain an alternative formulation of the problem.*

*Proof.* Fix  $r < s$ . If  $\beta_{rs} = 0$ , then the removed constraints hold trivially. If  $\beta_{rs} = 1$ , constraint (8d) implies  $\alpha_s = 1$ , and (9) gives  $x_{rs} = x_{sr} = 1$ , which satisfies both (8f) and (8g). Hence, the added constraint reproduces the effect of the removed ones.  $\square$

To illustrate the effect of prescribing the number of buckets, we extend the previous example as follows.

**Example 2.** *Continuing with Example 1, we now consider the version of the OBOP in which the ranking must be partitioned into a prescribed number of buckets, denoted as the  $p$ -OBOP. This problem is solved using the optimization models (7) and (8) adapted to the OBOP, that is, by incorporating the auxiliary variables  $\mathbf{d}$ , replacing the objective function with (4), and adding constraints (5a) and (5b). For  $p = 2$  and  $p = 5$ , both models yield the unique optima*

$$1\ 2\ 3\ 4\ 7 \mid 5\ 6\ 8 \quad \text{and} \quad 1\ 3 \mid 4\ 7 \mid 2 \mid 8 \mid 5\ 6$$

*with objective values of 12.14 and 11.34, respectively. When  $p = 4$ , the solution coincides with that of the original OBOP, as expected.*

#### 4.1. Equal bucket sizes

If, in addition, all buckets are required to contain the same number  $q$  of items (with  $p \cdot q = n$ ), the models can be adapted as follows. In formulation (7), the set of constraints (7c) is replaced by

$$\sum_{r \in [[n]]} y_{ru} = q \quad \forall u \in [[p]], \quad (10)$$

whereas in formulation (8), the following constraints are added:

$$\sum_{r=1}^{s-1} \beta_{rs} = (q-1) \cdot \alpha_s \quad \forall s \in [[n]]. \quad (11)$$

A natural application of this setting arises in tournaments or ranking systems where participants must be grouped into divisions of equal size. For example, in a sports league, teams may be divided into groups of the same number of members for a preliminary stage, based on their overall performance

scores. Each group corresponds to a bucket of identical capacity, and the resulting partition defines a weak order among divisions rather than among individual teams.

We next provide an example to show how the models behave when all buckets are required to have the same size.

**Example 3.** *Revisiting Example 1, we now impose equal bucket sizes. For two buckets of capacity four and four buckets of capacity two, the unique optima are*

$$1 \ 3 \ 4 \ 7 \mid 2 \ 5 \ 6 \ 8 \quad \text{and} \quad 1 \ 3 \mid 4 \ 7 \mid 2 \ 8 \mid 5 \ 6$$

with objective values of 13.14 and 11.46, respectively. The former is worse (in terms of fitness) than that in Example 2, while the latter differs from the optimal solution of the standard OBOP in Example 1, even though the number of buckets is the same.

#### 4.2. Prescribed bucket sizes by rank position

This setting models situations where the partition of the ranking is completely specified in terms of bucket sizes (e.g., the top bucket must contain exactly three items, the second bucket four items, etc.). The optimization problem then consists only in deciding which items are assigned to each bucket, while the relative order between buckets is fully predetermined.

In this variant, the exact size of each bucket is fixed in advance according to its position in the ranking. That is, for each  $u \in [[p]]$ , the number of items that must be assigned to bucket  $u$  is a prescribed parameter  $q_u$ , with  $\sum_{u \in [[p]]} q_u = n$ . Such a configuration may arise in selection or allocation processes where the number of elements per category is predetermined. For instance, in a scholarship or award distribution problem, the number of recipients for each prize level (say three gold, four silver, and five bronze awards) is fixed in advance. The relative order of these categories is therefore known, and the optimization focuses solely on determining which candidates should be assigned to each level.

It should be noted that, although the representative-based model (8) could in principle be extended to this setting by introducing additional variables, such an adaptation would no longer preserve the essence of a representative-based formulation. The key advantage of that approach is that the order of the buckets does not need to be specified, whereas in the present framework the order is fixed in advance, and this benefit is therefore lost. For this reason, we restrict our attention to adapting model (7) to this setting.

With this interpretation, the model is adapted as follows. In formulation (7), replace (7c) by:

$$\sum_{r=1}^n y_{ru} = q_u \quad \forall u \in [[p]]. \tag{12}$$

Constraints (12) share similarities to those from Section 4.1, and guarantee that each bucket has its prescribed capacity  $q_u$ . The resulting model is hereafter referred to as model (12).

The main theoretical results derived for model (7) remain valid in this formulation. In particular, the inequalities established in Proposition 2 also hold for model (12), since their proofs rely only on the structural relations between variables  $x$  and  $y$ , which are preserved here. Moreover, the integrality of variables  $x$  can be relaxed for the same reason as before.

Let us see how this variant operates through our toy example.

**Example 4.** *Continuing with Example 1, we now fix both the order and capacity of each bucket. For three ordered buckets of capacities one, three, and four, and for four ordered buckets of capacities one, two, two, and three, the unique optima are*

$$3 | 1 4 7 | 2 5 6 8 \quad \text{and} \quad 3 | 1 4 | 2 7 | 5 6 8$$

with objective values of 13.66 and 13.78, respectively. When four ordered buckets of capacities two, three, one, and two are imposed, the solution coincides with that of the original OBOP, confirming the model's consistency.

## 5. Tail-Collapsed Upper- $k$

Identifying which subset of  $k$  items stands out within a collection of  $n$  is often more informative than producing a complete ranking. Focusing exclusively on the upper part of the structure is also computationally advantageous, since the number of feasible configurations is substantially reduced when the lower portion is collapsed into a single bucket containing exactly  $n - k$  items. This leaves the  $k$  potentially relevant items unconstrained, allowing different upper configurations to be compared. Furthermore, it is highly relevant that the solutions show the strongest overall agreement when the last  $n - k$  items are considered to be of no interest.

This approach is not equivalent to the classical top- $k$  problem. Several heuristic and non-exact algorithms have been proposed for this task (see, for instance, [Lin \(2010\)](#)). An exact procedure in this spirit would consist of solving the problem with the model (1) and then keeping only the first  $k$  positions of the resulting order. However, even such an exact approach raises an additional issue: if the  $k$ -th position lies within a bucket of tied items, one must decide how to truncate that bucket, thereby breaking a tie that the optimisation process considered meaningful. This truncation-based methodology can become ambiguous when the top- $k$  boundary falls inside a block of tied or nearly tied items, since an additional rule is needed to break that tie. Another possible way to obtain a top- $k$  ordering would be to first solve the OBOP to optimality, then construct any permutation consistent with its bucket structure, and finally truncate this permutation after the first  $k$  positions. This alternative does not suffer from the truncation ambiguity, since the permutation can be chosen after the bucketed solution is known, but it is computationally more expensive because it requires solving the full OBOP before producing the desired top- $k$  order. Moreover, neither the option of truncating a bucket nor that of choosing the best ones in the consistent permutation has to correspond to the one with the highest consensus (i.e., the lowest value of the objective function, if we are minimizing) among all those that treat the last  $n - k$  elements as equal.

Our approach is also not equivalent to imposing exactly two fixed blocks, with an upper block of size  $k$ , since that formulation, unlike our proposal, forces the top  $k$  ranked elements to form a single undifferentiated group. Again, the solution in which the first  $k$  elements are tied does not have to be the one with the highest consensus if the goal is only to highlight the order among the top elements.

The Tail-Collapsed Upper- $k$  problem addresses these issues by fixing the lower block of size  $n - k$  and treating its elements as interchangeable, while allowing the upper region to retain whatever structure the optimization requires, that is, by selecting among multiple configurations the one that achieves the highest consensus. This formulation is computationally simpler than fully ordering all  $n$  items, since only the top- $k$  region is modeled in full detail. The  $k$  selected items may form one or several levels, depending on what yields the highest objective value.

To formulate the Tail-Collapsed Upper- $k$  problem, we introduce the following variables:

$$z_r = \begin{cases} 1 & \text{if item } r \text{ belongs to the last bucket,} \\ 0 & \text{otherwise,} \end{cases} \quad \forall r \in [[n]].$$

Our formulation is then given by

$$\min f(\Pi, \mathbf{x}) \quad (13a)$$

$$\text{s.t. } (1b) - (1c) \quad (13b)$$

$$\sum_{r \in [[n]]} z_r = n - k \quad (13c)$$

$$x_{rs} + z_r \leq 1 + z_s \quad \forall r, s \in [[n]] : r \neq s, \quad (13d)$$

$$x_{rs} \geq z_s \quad \forall r, s \in [[n]] : r \neq s, \quad (13e)$$

$$x_{rs} \in \{0, 1\} \quad \forall r, s \in [[n]] : r \neq s, \quad (13f)$$

$$z_r \in \{0, 1\} \quad \forall r \in [[n]]. \quad (13g)$$

Constraint (13c) fixes the size of the last bucket to  $n - k$ , while (13d) and (13e) ensure that every item in the last bucket appears after those outside it.

We illustrate the behavior of this formulation through our toy example.

**Example 5.** Using the same data as in Example 1, we now consider the version of the OBOP in which only the top  $k$  positions are preserved while the remaining items are collapsed into a single tail bucket. We refer to this problem as the TCU- $k$  OBOP, and solve it using formulation (13) adapted to the OBOP. For  $k = 4$  and  $k = 7$ , the unique optima are

$$1 \ 3 \ | \ 4 \ 7 \ || \ 2 \ 5 \ 6 \ 8 \quad \text{and} \quad 1 \ 3 \ | \ 2 \ 4 \ 7 \ | \ 8 \ | \ 5 \ || \ 6$$

with objective values of 12.66 and 11.58, respectively. Hereafter, the symbol  $||$  denotes the separation between the head (the first  $k$  positions) and the collapsed tail in the bucket notation. For  $k = 6$ , the model reproduces the optimal ranking of the OBOP.

We next provide an example illustrating that the previously mentioned problems and the TCU- $k$  approach are not equivalent.

**Example 6.** Consider an instance with 10 items, whose pairwise order matrix is shown in Figure 2.

$$\frac{1}{100} \begin{pmatrix} 50 & 82 & 12 & 77 & 5 & 91 & 24 & 80 & 15 & 76 \\ 18 & 50 & 88 & 22 & 79 & 10 & 7 & 95 & 25 & 79 \\ 88 & 12 & 50 & 81 & 23 & 90 & 75 & 14 & 5 & 83 \\ 23 & 78 & 19 & 50 & 80 & 0 & 25 & 85 & 11 & 97 \\ 95 & 21 & 77 & 20 & 50 & 8 & 89 & 76 & 2 & 75 \\ 9 & 90 & 10 & 100 & 92 & 50 & 78 & 18 & 84 & 20 \\ 76 & 93 & 25 & 75 & 11 & 22 & 50 & 79 & 9 & 85 \\ 20 & 5 & 86 & 15 & 24 & 82 & 21 & 50 & 77 & 13 \\ 85 & 75 & 95 & 89 & 98 & 16 & 91 & 23 & 50 & 80 \\ 24 & 21 & 17 & 3 & 25 & 80 & 15 & 87 & 20 & 50 \end{pmatrix}$$

Figure 2: Pairwise order matrix used in Example 6

Solving the OBOP yields two optima

$$6 \ | \ 9 \ | \ 5 \ | \ 3 \ 7 \ | \ 1 \ | \ 4 \ | \ 2 \ | \ 10 \ | \ 8 \quad \text{and} \quad 6 \ | \ 9 \ | \ 5 \ | \ 3 \ | \ 7 \ | \ 1 \ | \ 4 \ | \ 2 \ | \ 10 \ | \ 8$$

both with an objective value of 13.14; the only difference is whether items 3 and 7 are tied or strictly ordered.

When applying the TCU- $k$  OBOP for  $k = 6$ , the optimal solution obtained is

$$9 \mid 3 \mid 1 6 7 \mid 4 \parallel 2 5 8 10$$

with an objective value of 28.9. This is somewhat unexpected when compared with the optimal OBOP solutions reported above, since the six best-ranked items do not coincide with those appearing in this solution. In fact, the best rankings whose tails coincide with the tail of the standard OBOP are

$$6 \mid 9 \mid 5 \mid 3 7 \mid 1 \parallel 2 4 8 10 \quad \text{and} \quad 6 \mid 9 \mid 5 \mid 3 \mid 7 \mid 1 \parallel 2 4 8 10,$$

both with a higher objective value of 29.1.

Moreover, for  $k = 3$ , the TCU- $k$  OBOP yields

$$6 \mid 9 \mid 5 \parallel 1 2 3 4 7 8 10$$

with an objective value of 30.1. For comparison, the two-bucket variant (with a first bucket of size  $k = 3$ ) admits the optimal solution

$$6 7 9 \mid 1 2 3 4 5 8 10$$

with an objective value of 31.38 while the result of visually splitting the OBOP ranking into two sections, the first with  $3 (= k)$  items and the second with  $7 (= n - k)$ , is

$$5 6 9 \mid 1 2 3 4 7 8 10$$

with an objective value of 32.06.

*Remark 2.* In this section, we have observed that the result of grouping the elements of a list into homogeneous subgroups (referred to as buckets in our context) can differ substantially when the number or size of these homogeneous groups is modified. This peculiarity is not exclusive to weak ordering methods, which constitute the main focus of this study, but also arises in any non-hierarchical clustering method, such as the  $k$ -means algorithm, and in facility location problems (see [Laporte et al. \(2019\)](#) for a survey) like the  $p$ -median problem. In both non-hierarchical clustering and the  $p$ -median problem, the way elements are grouped changes with the number and size of the groups, prioritizing configurations that optimize an objective function, namely the sum of squared distances in clustering and the transportation cost in location problems.

## 6. Incorporating fairness constraints

Fairness is a key concern in ranking and recommendation systems, where it is often desirable to regulate how different item groups are represented throughout the ranking, either by promoting the visibility of some or limiting the dominance of others. In response, recent works in rank aggregation have introduced mechanisms to handle protected attributes (e.g., gender, race, religion), enabling group-aware consensus ranking generation ([Chakraborty et al., 2022](#); [Pitoura et al., 2022](#)). For example, in an academic ranking, one might require a minimum share of students from underrepresented schools among the top positions, or restrict any single school from exceeding a certain proportion.

Following the idea introduced by [Chakraborty et al. \(2024\)](#), who define fairness for linear orderings by controlling the proportion of items from each group within the top positions, we extend this notion

to our bucket-based setting, where ties are allowed. Specifically, we incorporate fairness constraints into the basic model (1), using the assignment variables  $\mathbf{y}$  from model (7) with up to potential  $n$  buckets in total. Empty buckets are forced to the end of the ranking to preserve the natural order of filled ones.

Let  $G_1, G_2, \dots, G_g \subseteq [[n]]$  be item groups that form a partition of  $[[n]]$ , and let  $\lambda_{i\ell}, \mu_{i\ell} \in [0, 1]$  denote the lower and upper bounds for the desired proportion of items from group  $G_i$  within the top  $\ell$  buckets, whenever such bounds are specified by the decision maker in a logically consistent manner; otherwise, the default values  $\lambda_{i\ell} = 0$  and  $\mu_{i\ell} = 1$  apply. The corresponding integer constraints are:

$$\left\lfloor \lambda_{i\ell} \sum_{r=1}^n \sum_{u=1}^{\ell} y_{ru} \right\rfloor \leq \sum_{s \in G_i} \sum_{u=1}^{\ell} y_{su} \leq \left\lceil \mu_{i\ell} \sum_{r=1}^n \sum_{u=1}^{\ell} y_{ru} \right\rceil, \quad \forall i \in [[g]], \forall \ell \in [[n]].$$

The use of floor and ceiling operators is essential. Without them, proportional lower bounds imposed simultaneously for all groups would effectively require each group to contribute at least one item to the top- $\ell$  buckets, regardless of whether the prefix is large enough to justify such a requirement. The lower bound becomes nonbinding when the top- $\ell$  segment contains too few items to enforce meaningful representation, and it activates only once the cumulative size is sufficient. A symmetric rationale applies to the upper bound through the use of the ceiling operator.

These fairness constraints are written on the cumulative top- $\ell$  portion rather than on individual buckets. This is important because, depending on the bucket structure of the ranking, per-bucket proportional constraints may become trivially slack. For example, if a bucket contains very few items, such as a single one, then any proportional lower bound would immediately round down to zero. By working with cumulative prefixes, the constraints preserve their intended effect in a discrete ranking setting, in line with the approach commonly adopted in the literature (see, for instance, Chakraborty et al. (2024); Wei et al. (2022)). The need for the floor and ceiling operators and the cumulative constraints is further justified in the following example.

**Example 7.** Consider items  $\{1, \dots, 6\}$  and one protected group,  $G_1 = \{1, 2, 3\}$ , which we wish to represent proportionally in the first two buckets, that is, with  $\lambda_{1\ell} = \mu_{1\ell} = 0.5$  for  $\ell \in \{1, 2\}$ ,  $\lambda_{1\ell} = 0$ ,  $\mu_{1\ell} = 1$  for  $\ell \in \{3, \dots, 6\}$ . All remaining items constitute the complementary group, for which no bounds are imposed.

If the fairness constraints were imposed without the floor and ceiling operators, the feasible bucket orders would be limited to configurations where the number of items in the first two buckets is even, such as:

$$1 \ 2 \ 3 \ 4 \ 5 \ 6 \quad 1 \ 2 \ 4 \ 5 \mid 3 \ 6 \quad 1 \ 4 \mid 2 \ 3 \ 5 \ 6 \quad 1 \ 4 \mid 2 \ 5 \mid 3 \ 6$$

These are some of the arrangements that could satisfy the proportional requirements exactly in integers, which is overly restrictive. To see why, note that proportional fairness applies to the cumulative top- $\ell$  portion of the ranking. When few items are included in the prefix, achieving an exact ratio such as 0.5 is often impossible. For instance, if the first two buckets jointly contain three items, the desired proportion requires  $0.5 \times 3 = 1.5$  items from each group, which is meaningless in a discrete setting. The floor and ceiling operators correct this by enforcing integer-consistent bounds, requiring at least one and at most two items from group  $G_1$  in that prefix. For instance, the ranking  $1 \ 4 \mid 2 \mid 3 \ 5 \ 6$  now becomes valid. In this case, the first two buckets together contain three items,  $\{1, 2, 4\}$ . The proportional bounds for this prefix are  $\lfloor 0.5 \cdot 3 \rfloor = 1$  and  $\lceil 0.5 \cdot 3 \rceil = 2$ , meaning that between one and two

of these items must belong to group  $G_1$ . Since items 1 and 2 are in  $G_1$  and item 4 is not, the condition is satisfied. Without rounding, however, the fractional target of 1.5 items from group  $G_1$  would make this configuration infeasible. Thus, the floor and ceiling operators ensure that fairness is consistent in a discrete setting, enlarging the feasible region while excluding only clearly unfair rankings such as 1 2 3 | 4 5 6 or 4 5 | 1 2 | 3 6.

Once rounding is introduced, the feasible set expands considerably. As a drawback, when the number of items in the bucket is low the constraints can be overly relaxed and have a lower bound of  $\lfloor 0.5 \cdot 1 \rfloor = 0$  or an upper bound of  $\lceil 0.5 \cdot 1 \rceil = 1$ . Cumulative bounds are then essential to force the desired proportionality in the long term. For instance, if the fairness constraints were imposed with the floor and ceiling operators but without the cumulative constraints, we might obtain configurations such as:

$$4 | 5 | 6 | 1 | 2 | 3 \quad 4 | 5 | 1 2 | 3 4 \quad 1 | 2 | 3 | 4 | 5 | 6 \quad 1 2 4 | 3 | 5 6$$

In the first two configurations, group  $G_1$  is entirely omitted from early buckets, whereas in the last two configurations group  $G_1$  is overrepresented. In both cases, the intended 50% proportion is not achieved. Cumulative bounds prevent this by enforcing fairness over prefixes of the ranking, regardless of how many items are tied in each level.

Finally, with the floor and ceiling operators and cumulative bounds, configurations like

$$1 | 4 | 5 | 2 | 3 | 6 \quad 4 | 2 | 1 3 5 6 \quad 1 | 2 4 | 3 | 5 | 6 \quad 1 4 | 2 3 5 | 6 \quad 1 2 4 | 3 5 6$$

where the 50% proportion is respected regardless of the number of items per bucket, are all feasible.

The previous example illustrates that the proposed constraints behave as intended. However, the choice of the parameters  $\lambda$  and  $\mu$  cannot be arbitrary, since inappropriate values may lead to infeasibility. These parameters must be selected consistently with both the total number of items and the desired group proportions. For instance, it is clear that the cumulative bounds imposed by  $\lambda$  across all buckets should not exceed 1, that is,  $\sum_{i \in [[g]]} \lambda_{i\ell} \leq 1$  for  $\ell \in [[n]]$ . Moreover, if for some group  $G_i$  we set  $\lambda_{i\ell} > |G_i|/n$ , this bounds the number of items that can be placed in buckets 1 to  $\ell$  to at most  $\lfloor \frac{|G_i|}{\lambda_{i\ell}} \rfloor$ , since the total proportion of elements of group  $G_i$  cannot exceed its overall share in the dataset. Consequently, the final value  $\lambda_{in}$  must necessarily satisfy  $\lambda_{in} \leq |G_i|/n$ . A similar reasoning applies to the parameters  $\mu$ , which determine the lower bounds on the presence of each group in the top-ranked buckets. In this case,  $\sum_{i \in [[g]]} \mu_{i\ell} \geq 1$  for  $\ell \in [[n]]$ , the number of items that can be placed in buckets 1 to  $\ell$  must be at most  $\lfloor \frac{n - |G_i|}{\mu_{i\ell}} \rfloor$  and  $\mu_{in} \geq |G_i|/n$  for  $i \in [[g]]$ . Both  $\lambda$  and  $\mu$  should therefore be calibrated to reflect realistic group proportions while ensuring the feasibility of the optimization problem.

When no ties are allowed, we have  $\sum_{r=1}^n \sum_{u=1}^{\ell} y_{ru} = \ell$ , and the above constraints reduce to their classical form, which requires no linearization. In the bucket-based setting, however, ties may occur. To keep the overall formulation linear, we replace the floor and ceiling operators with equivalent systems of linear inequalities. Each proportional target is expressed as a reduced rational number,  $\lambda_{i\ell} = \eta_{i\ell}/\rho_{i\ell}$  and  $\mu_{i\ell} = \kappa_{i\ell}/\tau_{i\ell}$ , with  $\rho_{i\ell}, \tau_{i\ell} > 0$ . This rewriting allows us to encode the same integer bounds using only linear constraints, thus ensuring that the model remains a valid linear (mixed-integer) formulation.

**Lemma 1.** *The integer fairness constraints*

$$\left\lfloor \lambda_{i\ell} \sum_{r=1}^n \sum_{u=1}^{\ell} y_{ru} \right\rfloor \leq \sum_{s \in G_i} \sum_{u=1}^{\ell} y_{su} \leq \left\lceil \mu_{i\ell} \sum_{r=1}^n \sum_{u=1}^{\ell} y_{ru} \right\rceil, \quad \forall i \in [[g]], \forall \ell \in [[n]],$$

are equivalent, under integer variables, to the following linear inequalities:

$$\eta_{i\ell} \sum_{r=1}^n \sum_{u=1}^{\ell} y_{ru} \leq \rho_{i\ell} \sum_{s \in G_i} \sum_{u=1}^{\ell} y_{su} + (\rho_{i\ell} - 1), \quad \forall i \in [[g]], \forall \ell \in [[n]] \quad (14a)$$

$$\tau_{i\ell} \sum_{s \in G_i} \sum_{u=1}^{\ell} y_{su} - (\tau_{i\ell} - 1) \leq \kappa_{i\ell} \sum_{r=1}^n \sum_{u=1}^{\ell} y_{ru}, \quad \forall i \in [[g]], \forall \ell \in [[n]]. \quad (14b)$$

where  $\lambda_{i\ell} = \eta_{i\ell}/\rho_{i\ell}$  and  $\mu_{i\ell} = \kappa_{i\ell}/\tau_{i\ell}$ , expressed as irreducible fractions with positive denominators.

*Proof.* Fix  $i \in [[g]]$  and  $\ell \in [[n]]$ , and set  $T := \sum_{r=1}^n \sum_{u=1}^{\ell} y_{ru} \in \mathbb{Z}$  and  $S := \sum_{s \in G_i} \sum_{u=1}^{\ell} y_{su} \in \mathbb{Z}$ . For the lower bound,

$$\lfloor \lambda_{i\ell} T \rfloor \leq S \iff \lambda_{i\ell} T < S + 1 \iff \eta_{i\ell} T < \rho_{i\ell} S + \rho_{i\ell} \iff \eta_{i\ell} T \leq \rho_{i\ell} S + (\rho_{i\ell} - 1),$$

where the last step uses integrality. For the upper bound,

$$S \leq \lceil \mu_{i\ell} T \rceil \iff S < \mu_{i\ell} T + 1 \iff \tau_{i\ell} S < \kappa_{i\ell} T + \tau_{i\ell} \iff \tau_{i\ell} S - (\tau_{i\ell} - 1) \leq \kappa_{i\ell} T,$$

again by integrality. This yields the claimed linear inequalities, proving equivalence under integer variables.  $\square$

These two linear inequalities serve as exact linear counterparts of the integer fairness constraints introduced earlier. They guarantee that the representation of each group within the top- $\ell$  buckets adheres to the intended proportional bounds while preserving the linear structure of the formulation.

We are now ready to integrate these fairness conditions into the basic version of our bucket-based framework. In particular, they are incorporated into model (1), which employs the assignment variables  $\mathbf{y}$  defined in model (7), now considering  $n$  buckets in total. By this we do not mean that the solution must contain  $n$  distinct ranking positions. Rather, this formulation allows for up to  $n$  buckets, with the understanding that many of them may remain empty. This device is simply a modelling convenience: it enables ties to be represented without fixing in advance the number of non-empty buckets, which will be determined endogenously by the optimisation process.

$$\min f(\Pi, \mathbf{x}) \quad (15a)$$

$$\text{s.t. } \sum_{u=1}^n y_{ru} = 1 \quad \forall r \in [[n]], \quad (15b)$$

$$\sum_{s=1}^n y_{su} \geq \sum_{v=u+1}^n y_{rv} \quad \forall r, u \in [[n]] : u < n, \quad (15c)$$

$$y_{ru} + y_{su} \leq x_{rs} + x_{sr} \quad \forall r, s, u \in [[n]] : r < s, \quad (15d)$$

$$x_{sr} + \sum_{v=1}^u y_{rv} + \sum_{v=u+1}^n y_{sv} \leq 2 \quad \forall r, s, u \in [[n]] : r \neq s, u < n, \quad (15e)$$

$$(14a) - (14b) \quad (15f)$$

$$x_{rs} \in \{0, 1\} \quad \forall r, s \in [[n]] : r \neq s, \quad (15g)$$

$$y_{ru} \in \{0, 1\} \quad \forall r, u \in [[n]]. \quad (15h)$$

All constraints are similar to those in formulation (7), except for (15c), (14a), and (14b). Con-

straint (15c) ensures that any empty buckets, if any, appear only at the end of the ranking, since the number of nonempty buckets is not known beforehand. Constraints (14a)–(14b) correspond to the fairness conditions introduced earlier, ensuring that the distribution of items across buckets respects the prescribed bounds for each group. By tuning these parameters, the decision maker can control the relative visibility of different groups within the ranking.

For completeness, we note that the main theoretical results derived for model (7) remain valid in this formulation, as was the case for model (12).

It is worth noting that specific choices of these parameters recover well-known fairness definitions from the literature. For instance, in Wei et al. (2022) a ranking is deemed fair when both bounds are fixed to the global proportion of each group in the population, that is, when  $\lambda_{i\ell} = \mu_{i\ell} = |G_i|/n$  for all  $i$  and  $\ell$ , and where the groups  $G_i$  form a disjoint partition of the item set. This highlights that our formulation subsumes such definitions as particular cases while allowing for overlapping groups and more flexible proportional bounds.

To illustrate how these fairness constraints affect the resulting rankings, let us examine a simple example.

**Example 8.** Continuing with the data from Example 1, consider two groups  $G_1 = \{1, 3, 4, 8\}$  and  $G_2 = \{2, 5, 6, 7\}$ . Assume that  $\lambda_{i\ell} = \mu_{i\ell} = \frac{|G_i|}{n}$  for all  $i \in [[g]]$  and  $\ell \in [[n]]$ . The Fair OBOP, that is, the OBOP with fairness constraints, is then solved using model (15) adapted to the OBOP, yielding the unique optimum

$$3 \mid 1 \ 2 \ 4 \ 7 \mid 5 \ 8 \mid 6$$

with an objective value of 11.86. Note that the optimal solution of the OBOP given in Example 1 is not feasible under these fairness constraints. In that unfair solution, the first bucket contains two items and both of them belong to  $G_1$ , whereas proportionality requires that at least one of them must belong to  $G_2$ . Hence, the original OBOP optimum violates the fairness conditions and is therefore infeasible for the Fair OBOP.

More generally, fairness requirements can be expressed through explicit lower and upper bounds, as detailed next.

*Remark 3.* Lower and upper bounds on the number of items of a group  $G_i$  assigned to a bucket  $u$  (resp. to the top- $\ell$  buckets) can be imposed by adding

$$L_{iu} \leq \sum_{r \in G_i} y_{ru} \leq U_{iu} \quad (\text{resp. } L_{i\ell} \leq \sum_{r \in G_i} \sum_{u=1}^{\ell} y_{ru} \leq U_{i\ell}),$$

where  $L_{iu}$  and  $U_{iu}$  (resp.  $L_{i\ell}$  and  $U_{i\ell}$ ) denote the corresponding bounds whenever such bounds are specified; otherwise, the default values  $L_{iu} = 0$ ,  $U_{iu} = |G_i|$  (resp.  $L_{i\ell} = 0$ ,  $U_{i\ell} = |G_i|$ ) apply.

It is also possible to impose an upper bound  $\nu$  on the number of buckets considered. In that case, the definition of the assignment variables must be adjusted so that  $u \in [[\nu]]$ , and every summation or constraint involving bucket indices is updated accordingly by replacing  $[[n]]$  with  $[[\nu]]$ . Similarly, a lower bound can be imposed by requiring that the first buckets contain at least one element, thus ensuring that the initial part of the ranking is always populated. It is clear that this fairness model can also be applied to the version of the problem with a fixed number of buckets. In particular, we can take  $p$  buckets and include the non-emptiness constraints (7c), while removing (15c). Furthermore,

bucket capacities could be incorporated by adding constraints (12). Recall that  $q_u$  denotes the exact number of elements assigned to bucket  $u$ . In this case, the linearization of the floor and ceiling functions becomes redundant, as the total number of items in the first  $\ell$  buckets is fixed to  $\sum_{u=1}^{\ell} q_u$ , and the fairness constraints can therefore be directly expressed in linear form.

Alternatively, one could consider adapting model (13) to include fairness requirements. A natural approach would be to impose lower and upper bounds,  $L_i$  and  $U_i$ , on the number of elements from each subset  $G_i$  appearing in the last bucket. This leads to the additional constraints

$$L_i \leq \sum_{r \in G_i} z_r \leq U_i, \quad \forall i \in [[g]].$$

## 7. Computational experiment

Next, we present the computational experiment, which is divided into two parts. The first part compares our approach with the heuristics proposed in [Aledo et al. \(2018\)](#), while the second part involves solving a set of selected instances using our models under different parameter configurations, which will be described in detail later in this section.<sup>3</sup>

All tests were performed using the commercial IP solver *Xpress Mosel* on a server equipped with an AMD Ryzen 9 7950X processor (3.4 GHz, 16 cores and 32 threads) and  $4 \times 48$  GB of DDR5 RAM (196 GB in total). The system is configured to allow up to 16 single-threaded tasks to run concurrently, each with approximately 16 GB of allocated memory, and relies on  $2 \times 2$  TB Western Digital Black SN770 NVMe SSDs in RAID 1 to ensure data reliability during execution. In our experiments, each Mosel instance was assigned 4 computational threads, and we executed 4 jobs in parallel, a configuration that balanced solver performance with controlled resource usage across the benchmark set.

### 7.1. Comparison with heuristic methods for OBOP

In this section, we compare the performance of our exact formulation for the OBOP with the heuristic methods proposed by [Aledo et al. \(2018\)](#). The goal of this experiment is to evaluate the quality of the heuristic solutions and to assess the computational performance of our exact approach across the 51 dataset used in [Aledo et al. \(2018\)](#). Specifically, the set of benchmark instances used in this comparison is detailed in this section to ensure consistency with the experimental setup described by the authors. Table 1 reports, for each instance, the number of items ( $n$ ) and voters ( $m$ ), that is, input rankings used to obtain the pair order matrix  $C$ , the objective value achieved by the heuristic and by the exact method (Obj. value), the linear relaxation (LR) and the utopic (Uto.) bounds, the total computation time in seconds (Time), the optimality gap (Gap), and the number of optimal solutions found (#Optima). For #Optima we only consider the values 1, 2, 3 or  $> 3$  when at least four solutions have been found.

As shown in Table 1, the heuristic methods match the objective value obtained by our exact model in nearly all instances, highlighting their strong practical performance. However, unlike the heuristics, our exact formulation provides a mathematical guarantee of optimality. In a few instances, it surpasses the heuristic results, confirming its ability to identify the true global optimum.

The computation times are reasonable for a combinatorial problem with such a vast number of feasible configurations. For the largest instances, the heuristic is able to produce a feasible solution

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<sup>3</sup>All instances used in this study can be downloaded from the following repository: <https://github.com/juande-jaimie-alcantara/Optimizing-Weak-Orders-via-MILP.git>.

Instance	Items	Voters	Heuristic	Exact				#Optima	
				Obj. value	Obj. value	LR Bound	Uto. Bound		
ED-10-02	51	9	524.97	524.97	164.61	65.59	7.26	0.00	1
ED-10-03	54	10	513.38	513.38	175.18	54.09	6.57	0.00	1
ED-10-11	50	11	558.01	558.01	216.50	89.64	6.28	0.00	1
ED-10-14	62	15	744.64	744.64	250.37	160.16	10.60	0.00	1
ED-10-15	52	14	576.96	576.96	206.95	117.64	6.64	0.00	1
ED-10-16	57	16	585.07	585.07	174.11	137.10	9.63	0.00	1
ED-10-17	61	17	683.76	683.76	199.81	168.25	12.12	0.00	1
ED-10-50	170	4	<b>5469.83</b>	5742.17	2197.17	261.17	172800	12.79	—
ED-11-01	240	5	5560.00	<b>5556.20</b>	0.00	4522.40	39801.98	0.00	2
ED-11-02	242	5	10913.40	10913.40	0.00	6693.80	77452.66	0.00	1
ED-11-03	103	5	1653.00	1653.00	0.00	1238.00	229.51	0.00	2
ED-14-02	100	5000	1953.24	1953.24	272.04	1148.20	1017.29	0.00	1
ED-15-07	110	4	1443.00	1443.00	0.00	920.00	565.47	0.00	> 3
ED-15-08	99	4	1304.50	1304.50	0.00	1015.50	367.55	0.00	> 3
ED-15-09	115	4	1684.00	1684.00	0.00	1091.00	536.54	0.00	> 3
ED-15-10	71	4	666.50	666.50	0.00	452.50	101.04	0.00	> 3
ED-15-11	63	4	484.00	484.00	0.00	325.00	34.35	0.00	> 3
ED-15-13	93	4	1381.50	1381.50	0.00	1011.50	211.81	0.00	> 3
ED-15-14	163	4	3505.50	<b>3502.50</b>	0.00	2273.50	2945.57	0.00	2
ED-15-15	69	4	728.00	728.00	0.00	461.00	107.15	0.00	2
ED-15-16	70	4	673.00	673.00	0.00	487.00	88.73	0.00	> 3
ED-15-17	127	4	2304.00	2304.00	0.00	1591.00	848.89	0.00	3
ED-15-19	87	4	999.50	999.50	0.00	707.50	271.52	0.00	> 3
ED-15-20	122	4	2382.50	2382.50	0.00	1562.50	794.26	0.00	1
ED-15-21	96	4	1246.50	1246.50	0.00	865.50	271.38	0.00	1
ED-15-22	112	4	1720.50	1720.50	0.00	1257.50	530.64	0.00	> 3
ED-15-23	142	4	2840.00	<b>2838.00</b>	0.00	1915.00	6148.12	0.00	3
ED-15-24	91	4	1028.50	1028.50	0.00	658.50	147.32	0.00	2
ED-15-26	82	4	844.00	844.00	0.00	551.00	42.49	0.00	2
ED-15-27	95	4	1220.50	1220.50	0.00	847.50	61.23	0.00	> 3
ED-15-28	102	4	1518.50	1518.50	0.00	1107.50	125.28	0.00	2
ED-15-29	106	4	1410.00	1410.00	0.00	985.00	85.32	0.00	> 3
ED-15-30	64	4	560.50	560.50	0.00	377.50	17.88	0.00	> 3
ED-15-31	67	4	589.50	589.50	0.00	403.50	14.25	0.00	> 3
ED-15-32	153	4	3112.50	3112.50	0.00	2175.50	604.38	0.00	2
ED-15-33	128	4	2298.50	2298.50	0.00	1563.50	230.87	0.00	3
ED-15-34	55	4	444.50	444.50	0.00	324.50	4.30	0.00	> 3
ED-15-35	68	4	674.50	674.50	0.00	392.50	15.37	0.00	1
ED-15-39	89	4	922.00	922.00	0.00	629.00	43.33	0.00	1
ED-15-40	131	4	2512.00	2512.00	0.00	1543.00	495.29	0.00	3
ED-15-51	77	4	777.00	777.00	0.00	538.00	105.69	0.00	2
ED-15-54	60	4	410.50	410.50	0.00	281.50	5.84	0.00	> 3
ED-15-57	73	4	821.50	821.50	0.00	556.50	22.73	0.00	3
ED-15-60	72	4	658.50	658.50	0.00	394.50	28.99	0.00	1
ED-15-69	81	4	809.00	809.00	0.00	580.00	30.53	0.00	> 3
ED-15-77	56	4	498.67	498.67	0.00	297.00	13.36	0.00	1
ML-050-6	50	936	418.21	418.21	69.97	231.91	4.57	0.00	1
ML-100-1	100	951	1869.63	1869.63	303.39	970.85	166.02	0.00	1
ML-200-3	200	936	8421.68	8421.68	—	2625.44	4079.50	0.00	—
ML-250-4	250	942	12281.83	12281.83	—	2516.49	104304.45	0.00	—
ML-250-5	250	939	12257.52	12257.52	—	2461.64	121672.11	0.00	—

Table 1: Results obtained for the heuristic and exact approaches. From left to right, the columns indicate the instance name, the number of items ( $n$ ) and voters ( $m$ ), the objective value achieved by the heuristic, the objective value obtained by the exact method, the linear relaxation (LR) and utopic (Uto.) bounds, the computation time in seconds, the optimality gap (Gap) between the best bound and the best solution, and the number of optimal solutions found (#Optima). All numerical values are rounded to two decimals.

even when the exact model struggles due to the exponential complexity of the problem. Nevertheless, the overall results confirm the robustness and reliability of our exact approach as a reference benchmark. After extending the running time, the model was able to solve all instances except one, which illustrates both the inherent difficulty of the problem and the strength of the formulation. Values marked with – indicate data that could not be computed within the imposed time or memory limits.

## 7.2. Testing our models for the OBOP variants

In this section, we describe the computational experiments conducted to evaluate our proposed models. We first present the set of 30 benchmark instances, which were taken from the PrefLib<sup>4</sup> library, and explain how the corresponding pair order matrix was derived. This includes the preprocessing steps and any transformations applied to adapt the data to our framework. In the subsequent subsections, we focus on specific model variants and parameter configurations to analyse their performance under different experimental conditions.

Within *PrefLib*, we selected the instances corresponding to *Poland Local Elections* (LP-68-xx), *Boxing* (BOX-42-xx), and *Seasons Power Ranking* (SPR-56-xx). Each instance provides a set of items to be ranked, denoted by  $n$ , a number of voters, denoted by  $m$ , and a collection of individual rankings without ties, which are not necessarily complete. Let  $A = (a_{rs})_{n \times n}$  denote the matrix whose entry  $a_{rs}$  represents the number of voters who prefer item  $r$  to item  $s$ . Based on this, the pair order matrix  $C = (c_{rs})_{n \times n}$  is defined as follows:

$$c_{rs} = \begin{cases} \frac{a_{rs}}{a_{rs} + a_{sr}}, & \text{if } a_{rs} + a_{sr} \neq 0, \\ 0.5, & \text{if } a_{rs} + a_{sr} = 0. \end{cases}$$

### 7.2.1. OBOP

To analyse the impact of our models, we first solved the selected instances with model (6). The numerical results are summarized in Table 2. From each solution, we retrieved the optimal value (Obj. value), the bound of the linear relaxation (LR Bound), and the computation time in seconds (Time), all rounded to two decimal places. In addition, we ran the model again after removing the transitivity constraints (1c) in order to obtain the bound provided by the utopian matrix (Uto. Bound), which corresponds to the optimal value of that relaxed problem, also rounded to two decimal places. Finally, we computed the number of optimal solutions (#Optima) using the approach described in Section 2 and recorded the number of buckets in the optimal solutions (#Buckets).

In Table 2, we observe that all instances are solved almost instantly, each in less than five seconds. Although we could investigate the maximum instance size that the model can handle, our main goal here is to compare the overall performance of the method as additional constraints are imposed on the consensus ranking. The results also show that the bounds provided by the linear relaxation and by the utopian matrix are not necessarily comparable: in some cases one is tighter, while in others the opposite occurs. Moreover, we observe that the number of buckets in the optimal solutions always consists of consecutive numbers, which might suggest that this is a general property; however, we show in Appendix B that this is not necessarily the case.

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<sup>4</sup><http://www.preflib.org/>

Instance	$n$	$m$	OBOP					
			Obj. value	LR Bound	Uto. Bound	Time	#Optima	#Buckets
LPE-68-01	10	23	15.63	3.53	6.15	0.01	1	4
LPE-68-02	14	15	34.14	11.64	3.05	0.03	2	3,4
LPE-68-03	9	15	12.29	4.0	1.29	0.01	1	4
LPE-68-04	17	15	29.78	8.78	1.78	0.05	1	3
LPE-68-05	5	15	3.94	0.00	3.29	0.00	1	1
LPE-68-06	9	15	11.25	3.00	0.92	0.01	1	3
LPE-68-07	13	15	25.73	11.11	1.73	0.02	1	3
LPE-68-08	7	15	6.38	0.00	6.12	0.01	1	4
LPE-68-09	10	21	6.99	2.98	3.20	0.01	2	7,8
LPE-68-10	7	15	4.97	0.00	1.77	0.00	1	2
BOX-42-01	17	31	14.27	7.08	2.47	0.03	3	14,15,15
BOX-42-02	23	46	33.7	5.57	12.14	0.09	1	9
BOX-42-03	20	36	22.94	7.14	10.21	0.06	2	11,12
BOX-42-04	21	36	32.69	7.59	13.86	0.07	1	9
BOX-42-05	22	37	27.18	11.3	5.73	0.08	2	13,14
BOX-42-06	22	37	17.38	8.31	4.38	0.08	1	14
BOX-42-07	25	41	51.58	29.57	10.10	0.12	2	10,11
BOX-42-08	22	38	28.01	16.03	7.56	0.08	1	12
BOX-42-09	18	9	3.89	0.00	2.89	0.04	1	13
BOX-42-10	16	31	1.45	0.00	1.45	0.03	1	13
SPR-56-01	31	5	76.07	22.53	25.87	0.28	1	13
SPR-56-03	41	5	155.97	53.03	44.30	1.76	1	10
SPR-56-08	35	5	83.13	10.63	46.33	0.41	1	9
SPR-56-12	31	5	73.3	25.87	27.90	0.28	2	11,12
SPR-56-15	29	5	62.17	12.93	38.03	0.19	1	16
SPR-56-16	42	15	252.98	77.83	67.30	1.54	1	7
SPR-56-19	55	15	402.95	138.18	79.20	4.78	1	5
SPR-56-29	40	15	180.51	54.57	56.50	1.12	1	7
SPR-56-30	43	14	266.56	92.87	66.31	1.58	1	6
SPR-56-94	47	17	305.39	85.89	75.87	2.11	1	4

Table 2: Results obtained for the standard OBOP model (6). From left to right, the columns show the instance name, the number of items ( $n$ ) and voters ( $m$ ), the optimal value, the linear relaxation (LR) and utopic (Uto.) bounds, the computation time in seconds, the number of optimal solutions, and the number of buckets in each optimal solution. All numerical values are rounded to two decimals.

### 7.2.2. $p$ -OBOP

We now compare the optimal value obtained in the standard OBOP with that of the model including a fixed number of buckets, setting  $p$  equal to 2, 5, the number of buckets in the optimal solution with the fewest buckets minus one (denoted by  $\underline{p}$ ), and the number of buckets in the optimal solution with the most buckets plus one (denoted by  $\bar{p}$ ). In addition, we also consider a configuration in which we fix the number of buckets to  $p = 3$  and fix the capacities of the first two buckets to  $q_1 = \lceil n/10 \rceil$  and  $q_2 = \lceil n/5 \rceil$ .

Both the assignment-based formulation (7) and the representative-based formulation (8) are valid approaches and yield the same optimal values. We discuss further implementation details and carry out a comparative study between the two formulations in [Appendix A](#), where we show that the assignment-based model equipped with the additional valid inequalities (1b) and (1c) exhibits somewhat better computational performance than the representative-based formulation. For this reason, we adopt this strengthened assignment-based model as the reference formulation for the subsequent analysis.

Table 3 summarises the optimal values obtained for the  $p$ -OBOP model, all rounded to two decimal

<sup>5</sup>Since the optimal OBOP solution for instance LPE-68-05 has a single bucket, there is no feasible configuration with  $p - 1 = 0$ .

Instance	$n$	$m$	Opt. OBOP	$p$ -OBOP				
				$p = 2$	$p = 5$	$p = \underline{p} - 1$	$p = \bar{p} + 1$	$p = 3$ $q_1 = \lceil n/10 \rceil, q_2 = \lceil n/5 \rceil$
LPE-68-01	10	23	15.63	17.07	15.68	15.92	15.68	17.13
LPE-68-02	14	15	34.14	35.14	36.14	35.14	36.14	42.14
LPE-68-03	9	15	12.29	13.29	13.29	13.29	13.29	13.29
LPE-68-04	17	15	29.78	35.67	37.44	35.67	32.44	56.11
LPE-68-05	5	15	3.94	4.01	6.06	— <sub>5</sub>	4.01	5.01
LPE-68-06	9	15	11.25	11.75	14.25	11.75	13.25	16.25
LPE-68-07	13	15	25.73	31.26	26.71	31.26	26.26	25.73
LPE-68-08	7	15	6.38	6.73	7.33	6.38	7.33	8.93
LPE-68-09	10	21	6.99	16.71	8.14	7.14	7.54	18.49
LPE-68-10	7	15	4.97	4.97	8.60	9.40	5.77	5.77
BOX-42-01	17	31	14.27	56.12	27.85	15.01	15.17	54.12
BOX-42-02	23	46	33.70	87.50	37.39	33.83	34.57	77.90
BOX-42-03	20	36	22.94	76.07	32.98	22.98	23.40	73.22
BOX-42-04	21	36	32.69	82.99	40.20	33.10	32.95	73.28
BOX-42-05	22	37	27.18	85.95	35.00	27.42	27.57	74.74
BOX-42-06	22	37	17.38	89.81	34.50	17.62	17.78	80.71
BOX-42-07	25	41	51.58	123.71	64.17	51.71	52.16	132.32
BOX-42-08	22	38	28.01	95.75	41.00	28.21	28.06	96.12
BOX-42-09	18	9	3.89	64.33	18.33	4.00	4.22	67.11
BOX-42-10	16	31	1.45	52.64	15.90	2.29	1.87	48.90
SPR-56-01	31	5	76.07	172.07	83.87	76.27	76.67	163.27
SPR-56-03	41	5	155.97	293.30	173.23	156.97	156.10	288.63
SPR-56-08	35	5	83.13	200.53	92.73	83.33	83.53	242.93
SPR-56-12	31	5	73.30	181.70	87.17	73.70	73.37	172.50
SPR-56-15	29	5	62.17	140.63	77.37	62.30	62.50	175.03
SPR-56-16	42	15	252.98	304.23	254.99	253.39	253.70	277.53
SPR-56-19	55	15	402.95	486.56	402.95	405.35	403.42	463.24
SPR-56-29	40	15	180.51	265.43	182.01	180.76	180.85	248.48
SPR-56-30	43	14	266.56	306.13	267.87	267.87	266.90	316.11
SPR-56-94	47	17	305.39	366.17	306.41	313.16	306.41	356.20

Table 3: Results obtained for the extended  $p$ -OBOP model. From left to right, the columns show the instance name, the number of items ( $n$ ) and voters ( $m$ ), the optimal value of the standard OBOP model (6), and the optimal values of the extended model (7) for different numbers of buckets  $p$ . Here,  $\underline{p}$  and  $\bar{p}$  denote respectively the smallest and largest numbers of buckets yielding an optimal OBOP solution. The last column corresponds to model (12), with  $p = 3$  and bucket capacities  $\lceil n/10 \rceil$ ,  $\lceil n/5 \rceil$ , and the remaining items in the last bucket. All numerical values are rounded to two decimals.

places. As expected, the smallest deviations from the optimal OBOP value occur when the number of buckets differs by only one from the optimal configuration, that is, when  $p = \underline{p} - 1$  or  $p = \bar{p} + 1$ . In contrast, when  $p$  takes more distant values, the resulting objective values tend to deteriorate significantly, as one may expect. Although the maximum computation time observed was below 500 seconds, the vast majority of instances were solved in under 30 seconds.

We selected instances BOX-42-01, BOX-42-02, BOX-42-03, BOX-42-04, and BOX-42-05, and further analysed the results by considering all possible values for the number of buckets, from 1 to  $n$ . The optimal value obtained for each  $p$  and each instance is shown in Figure 3, where the corresponding minima are highlighted in a darker shade; these correspond precisely to the numbers of buckets in the optimal solutions of the standard OBOP. The case  $p = 1$  was omitted to improve visual clarity, since it represents a trivial solution (all items tied) whose objective values are considerably higher and would otherwise distort the scale of the plot. Although the overall shape of the curves might appear predictable at first sight, it is not guaranteed that these profiles are unimodal; indeed, we provide in Appendix B a counterexample showing that this unimodal behaviour does not hold in general.

### 7.2.3. TCF- $k$ OBOP

We now examine how the optimal values change under the TCU- $k$  formulation. To this end, we solved the same family of instances for several values of  $k$ , namely 3, 5,  $\lceil n/10 \rceil$ ,  $\lceil n/5 \rceil$  and  $\lceil n/2 \rceil$ , using

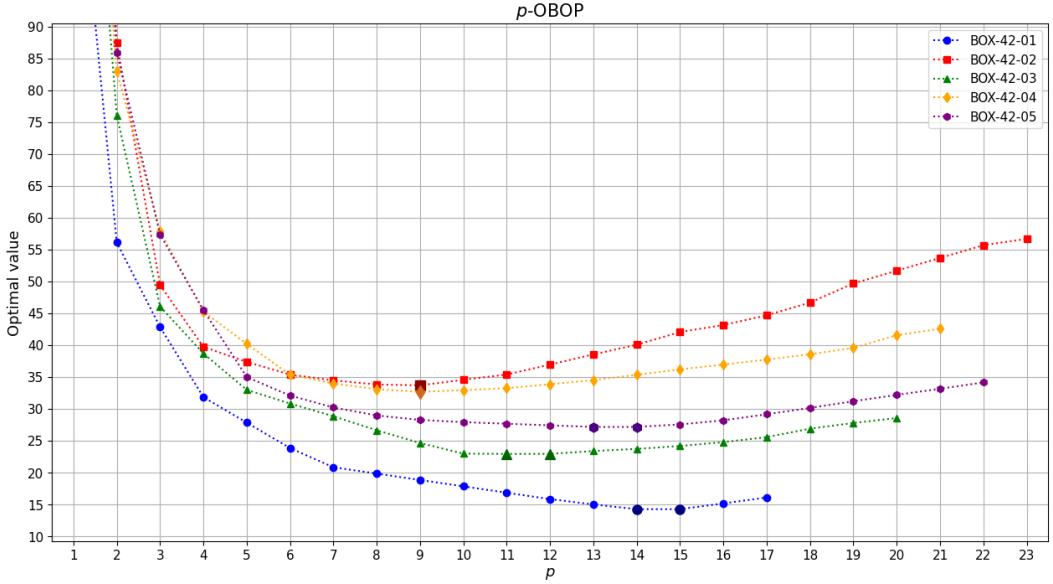


Figure 3: Plot of the  $p$ -OBOP results across instances BOX-42-01 to BOX-42-05

model (13) now interpreted as the TCU- $k$  OBOP.

Table 4 reports the optimal values obtained by the TCU- $k$  OBOP. For small  $k$ , the model focuses on distinguishing only a few top items, while the remaining  $n - k$  items are collapsed into a single tail bucket. As  $k$  increases, more flexibility is allowed in the upper region, and the objective value decreases accordingly. As expected, for each instance, the minimum value is attained when  $k$  equals  $n$  minus the size of the worst bucket in an optimal solution of the standard OBOP, thereby reaffirming the consistency of the TCU- $k$  model with respect to the OBOP.

To further illustrate this behaviour, we computed the optimal values of the TCU- $k$  OBOP for every  $k \in \{1, \dots, n\}$  on five representative instances (BOX-42-01 to BOX-42-05). The results are displayed in Figure 4, where the global minima are highlighted in a darker shade. In every case, one minimum occurs at  $k = n$ , since collapsing an empty tail recovers the standard OBOP, and another appears exactly at  $k = n - |\text{worst bucket}|$ , in line with the previous observation.

#### 7.2.4. Fair OBOP

We complete the testing of our models with the Fair OBOP. For this purpose, we selected the groups as shown in Table 5, without following any specific criterion other than ensuring that they form a valid partition. One configuration includes groups of disparate sizes, while the other consists of groups of approximately equal size. In both cases, proportional bounds were applied with  $\lambda_{i\ell} = \mu_{i\ell} = |G_i|/n$ . Furthermore, the number of buckets was fixed to lie between the number of buckets in the optimal solution with the fewest buckets and that in the optimal solution with the most buckets, in order to obtain a fairness-consistent solution with respect to the original one.

Table 5 summarises the optimal values obtained for the Fair OBOP model, all rounded to two decimal places. In most instances, the introduction of fairness constraints produces only minor variations with respect to the standard OBOP values. However, in certain cases the optimal value increases substantially, reflecting the additional restrictions imposed on the ranking structure.

Figure 5 illustrates the fairness behaviour for the instance BOX-42-10, considering three groups. The horizontal axis represents the index  $\ell$  of the top buckets, while the vertical axis shows, for each

Instance	$n$	$m$	OBOP		TCU- $k$ OBOP				
			Opt. Value	$n -  \text{Worst bucket} $	$k = 3$	$k = 5$	$k = \lceil n/10 \rceil$	$k = \lceil n/5 \rceil$	$k = \lceil n/2 \rceil$
LPE-68-01	10	23	15.63	9	16.63	15.92	18.70	17.07	15.92
LPE-68-02	14	15	34.14	13	36.14	41.14	35.14	36.14	42.14
LPE-68-03	9	15	12.29	6	13.29	13.29	15.71	13.29	13.29
LPE-68-04	17	15	29.78	16	43.67	51.11	40.44	47.11	56.11
LPE-68-05	5	15	3.94	0	4.57	3.94	4.60	4.60	4.57
LPE-68-06	9	15	11.25	8	15.25	16.25	11.75	13.25	16.25
LPE-68-07	13	15	25.73	5	29.26	25.73	31.26	29.26	30.26
LPE-68-08	7	15	6.38	5	8.33	6.38	11.11	10.21	6.73
LPE-68-09	10	21	6.99	8	17.49	9.55	27.46	22.49	9.55
LPE-68-10	7	15	4.97	3	4.97	7.27	6.97	6.77	6.77
BOX-42-01	17	31	14.27	16	80.58	56.74	94.32	67.58	30.27
BOX-42-02	23	46	33.70	16	142.39	107.74	142.39	107.74	52.64
BOX-42-03	20	36	22.94	19	108.59	78.59	125.59	94.37	42.71
BOX-42-04	21	36	32.69	17	115.80	90.55	115.80	90.55	49.26
BOX-42-05	22	37	27.18	15, 21	139.68	108.11	139.68	108.11	41.02
BOX-42-06	22	37	17.38	17	144.63	111.36	144.63	111.36	39.26
BOX-42-07	25	41	51.58	20, 21	188.07	161.83	188.07	161.83	79.10
BOX-42-08	22	38	28.01	19	153.80	121.92	153.80	121.92	59.52
BOX-42-09	18	9	3.89	16	98.33	73.11	113.33	86.11	32.89
BOX-42-10	16	31	1.45	15	74.64	53.97	87.64	63.03	27.29
SPR-56-01	31	5	76.07	24	306.87	263.87	283.87	226.87	117.67
SPR-56-03	41	5	155.97	29	570.90	509.50	509.50	396.10	185.43
SPR-56-08	35	5	83.13	10	411.73	360.53	385.93	314.73	128.53
SPR-56-12	31	5	73.30	24, 26	317.70	273.90	293.90	235.10	118.03
SPR-56-15	29	5	62.17	17	274.63	236.03	274.63	219.03	93.43
SPR-56-16	42	15	252.98	40	476.83	410.56	410.56	319.73	254.68
SPR-56-19	55	15	402.95	28	853.27	764.83	730.55	570.81	407.43
SPR-56-29	40	15	180.51	23	449.42	387.32	413.65	316.37	187.37
SPR-56-30	43	14	266.56	23	496.68	460.83	460.83	385.58	269.20
SPR-56-94	47	17	305.39	46	587.28	489.54	519.48	396.89	319.13

Table 4: Results obtained for the TCU- $k$  OBOP model (13). From left to right, the columns show the instance name, the number of items ( $n$ ) and voters ( $m$ ), the optimal value of the standard OBOP model (6), and the optimal values of the Top- $k$  variant for different values of  $k$ . The first two cases ( $k = 3$  and  $k = 5$ ) use fixed values, whereas the remaining ones scale with  $n$ , namely  $k = \lceil n/10 \rceil$ ,  $k = \lceil n/5 \rceil$ , and  $k = \lceil n/2 \rceil$ . All values are rounded to two decimals.

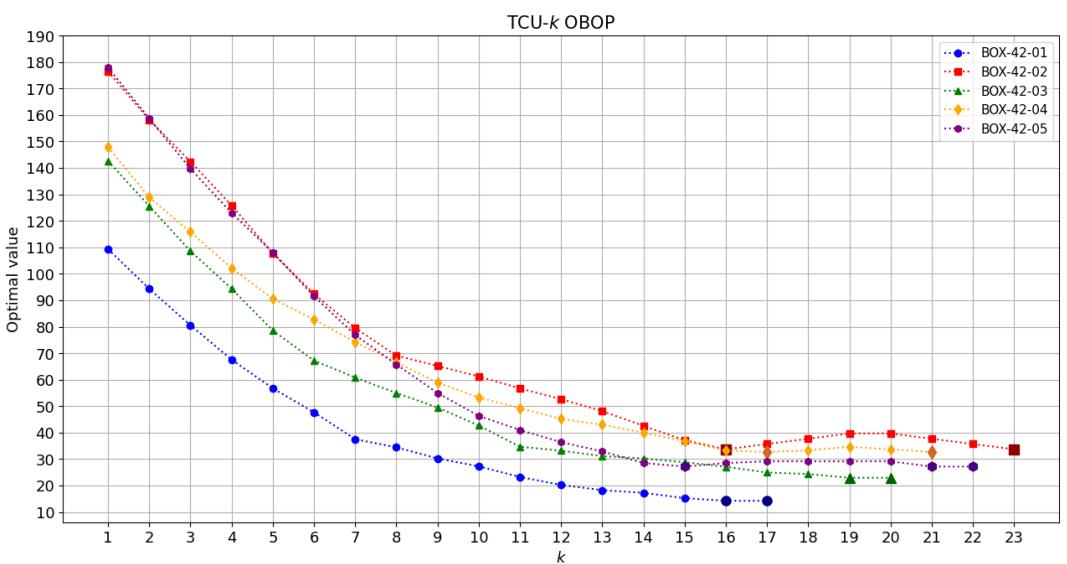


Figure 4: Plot of the TCU- $k$  OBOP results across instances BOX-42-01 to BOX-42-05

Instance	$n$	$m$	Opt. OBOP	Fair OBOP		
				$G_1 = \{r \in [n] : r \leq 0.8n\}$	$G_1 = \{r \in [n] : r \bmod 3 = 0\}$	
				$G_2 = \{r \in [n] : r > 0.8n\}$	$G_2 = \{r \in [n] : r \bmod 3 = 1\}$	$G_3 = \{r \in [n] : r \bmod 3 = 2\}$
LPE-68-01	10	23	15.63	15.63		15.70
LPE-68-02	14	15	34.14	34.14		34.14
LPE-68-03	9	15	12.29	13.29		12.29
LPE-68-04	17	15	29.78	29.78		29.78
LPE-68-05	5	15	3.94	3.94		3.94
LPE-68-06	9	15	11.25	11.25		11.25
LPE-68-07	13	15	25.73	25.73		30.73
LPE-68-08	7	15	6.38	6.53		7.56
LPE-68-09	10	21	6.99	6.99		9.14
LPE-68-10	7	15	4.97	4.97		4.97
BOX-42-01	17	31	14.27	14.27		24.68
BOX-42-02	23	46	33.70	34.72		34.83
BOX-42-03	20	36	22.94	24.72		25.72
BOX-42-04	21	36	32.69	38.28		34.23
BOX-42-05	22	37	27.18	39.73		36.73
BOX-42-06	22	37	17.38	42.78		29.57
BOX-42-07	25	41	51.58	51.58		60.79
BOX-42-08	22	38	28.01	29.64		42.13
BOX-42-09	18	9	3.89	10.22		12.89
BOX-42-10	16	31	1.45	6.68		42.32
SPR-56-01	31	5	76.07	90.87		93.27
SPR-56-03	41	5	155.97	162.57		176.83
SPR-56-08	35	5	83.13	86.60		130.27
SPR-56-12	31	5	73.30	84.17		92.03
SPR-56-15	29	5	62.17	63.10		68.83
SPR-56-16	42	15	252.98	258.41		268.87
SPR-56-19	55	15	402.95	406.90		431.70
SPR-56-29	40	15	180.51	182.59		186.59
SPR-56-30	43	14	266.56	266.56		278.56
SPR-56-94	47	17	305.39	308.46		310.37

Table 5: Results for the Fair OBOP model (15). Columns report, from left to right, the instance name,  $n$ ,  $m$ , the optimal value of the standard OBOP model (6), and the optimal values obtained under two fairness settings: one splitting the top 80% and bottom 20% of ranks, and another partitioning items by  $r \bmod 3$ . In both cases, proportional bounds  $\lambda_{i\ell} = \mu_{i\ell} = |G_i|/n$  are applied.

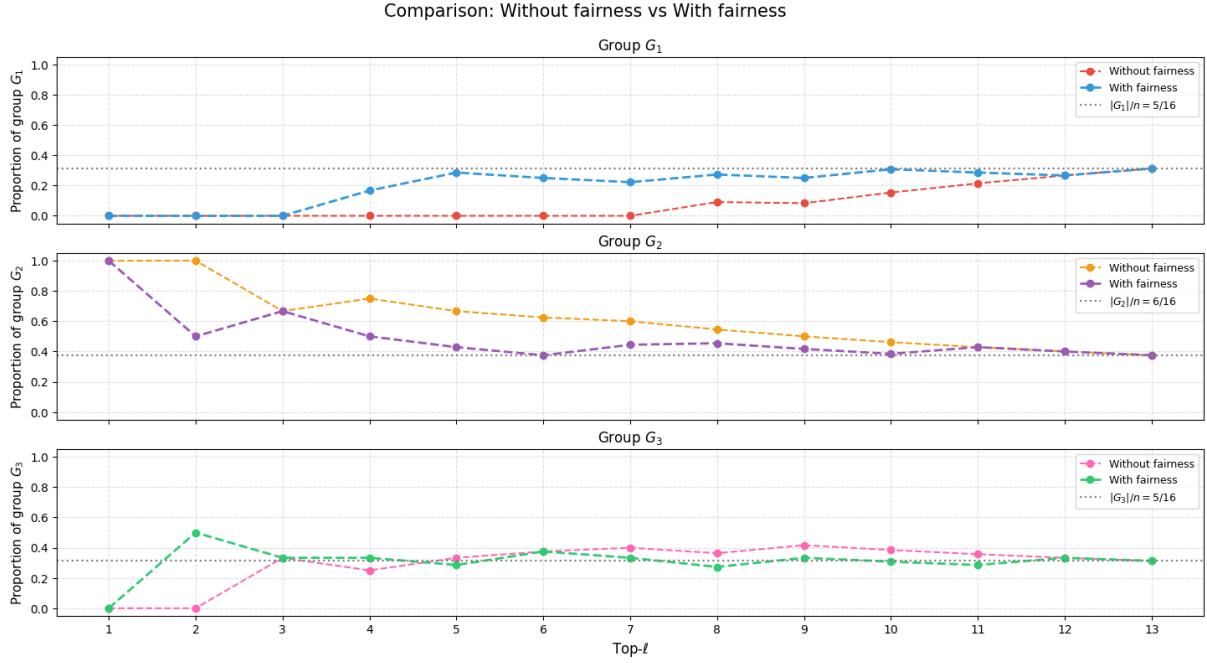


Figure 5: Comparison of fairness and non-fairness results for instance BOX-42-10

group, the proportion of its items contained within the top- $\ell$  buckets relative to the total number of items in those buckets. The goal of the proportional fairness constraints is to keep this proportion close to  $|G_i|/n$ , within a small tolerance margin. The figure is divided into three subplots, one for each group, comparing the behaviour with and without fairness (that is, the optimal solution of the Fair OBOP and the optimal solution of the standard OBOP, respectively). It can be observed that Group 1 is not represented until the eighth bucket in the standard OBOP solution, but fairness constraints enforce earlier representation. Conversely, Group 2 is prevented from being overrepresented, while Group 3 already exhibits approximately proportional representation without the need for additional constraints.

## 8. Conclusions and future research

This paper has introduced a unified Integer Linear Programming (ILP) framework for modeling and exactly solving rank aggregation problems whose outputs are weak orders or bucket orders. The proposed formulations encompass a wide range of variants, including configurations with a fixed number of buckets, top- $k$  type problems, and fairness-aware consensus rankings, each of which is strongly motivated by practical applications. In this regard, the integration of fairness constraints within the optimization process provides a formal mechanism for ensuring equitable consensus rankings, a growing concern in socially sensitive applications such as recommendation systems, hiring processes, or resource allocation. These models extend and generalize existing approaches while establishing a flexible and rigorous foundation for future research on rank aggregation with weak orders.

As a case of study, we have addressed the Optimal Bucket Order Problem (OBOP), providing the first exact formulations for these rank aggregation problem. The computational experiments, conducted on benchmarks from the PrefLib repository, have demonstrated that the ILP approach is capable of producing exact solutions for instances of moderate size while offering a valuable reference for assessing the accuracy and efficiency of existing heuristic and evolutionary methods.

Beyond validating the effectiveness of the ILP-based approach, this study underscores the potential of optimization models to deepen our understanding of the structural properties of consensus rankings

across different problem settings. By providing exact formulations, the proposed framework enables a systematic evaluation of heuristic and approximate algorithms and establishes a bridge between combinatorial optimization and preference aggregation theory.

Future research could focus on extending the scalability of the proposed formulations through decomposition techniques, relaxation strategies, or hybrid ILP–metaheuristic approaches. Another promising line involves adapting the models to dynamic or uncertain environments, where input preferences may evolve over time or be subject to noise. Additionally, exploring the trade-offs between fairness, optimality, and computational performance could provide new insights into the design of transparent and robust consensus mechanisms. Altogether, the proposed ILP framework lays the groundwork for both theoretical and applied advancements in rank aggregation, inviting future studies to build upon its foundations toward more efficient, interpretable, and equitable solutions.

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## Appendix A. Comparison of $p$ models

This section compares several equivalent formulations of model (7) for solving the  $p$ -OBOP. All formulations are mathematically valid and represent the same problem, but they differ in the additional constraints considered or in the relaxation of the binary variables  $\mathbf{x}$  to the continuous domain  $[0, 1]$ . The objective of this comparison is to evaluate the impact of these variations on computational performance, particularly in terms of solving time and final gap. Results are reported for different instances and values of  $p$ , allowing an assessment of the robustness and efficiency of each formulation under diverse settings.

As shown in Table A.6, although all formulations are equivalent from a modeling perspective, their computational behavior varies considerably. The inclusion of additional structural constraints strongly influences both solving time and solution quality. In particular, the formulation that combines the transitivity constraints with the before-or-after constraints without relaxing the binary variables  $\mathbf{x}$  (namely (7) + (1b) + (1c)) achieves the best overall performance. This variant consistently provides exact solutions within reasonable computation times, demonstrating a favorable balance between model strength and numerical efficiency in the resolution of the  $p$ -OBOP.

Additionally, a second set of experiments was conducted to evaluate the impact of specific inequalities on the performance of model (8). In this case, the objective is to determine which combinations or substitutions of inequalities lead to stronger formulations and faster convergence.

Instance	$n$	$m$	$p$	(7)		(7) + (1b)		(7) + (1c)		(7) + (1b) + (1c)		(7) + Relax $\mathbf{x}$		(7) + (1b) + Relax $\mathbf{x}$		(7) + (1c) + Relax $\mathbf{x}$		(7) + (1b) + (1c) + Relax $\mathbf{x}$	
				Time	Gap	Time	Gap	Time	Gap	Time	Gap	Time	Gap	Time	Gap	Time	Gap	Time	Gap
ED-10-02	51	9	2	<b>4.59</b>	0.00	10.01	0.00	1124.23	0.00	38.99	0.00	9.35	0.00	11.34	0.00	2680.83	0.00	3066.90	0.00
ED-10-03	54	10	2	<b>5.19</b>	0.00	7.83	0.00	1054.20	0.00	14.57	0.00	9.91	0.00	10.97	0.00	3202.51	0.00	3600	3.53
ED-10-11	50	11	2	<b>7.39</b>	0.00	9.85	0.00	1051.74	0.00	22.82	0.00	13.58	0.00	12.63	0.00	3486.87	0.00	3600	4.17
ED-10-14	62	15	2	<b>8.90</b>	0.00	9.85	0.00	3161.16	0.00	22.62	0.00	22.10	0.00	21.25	0.00	3600	117.32	3600	116.52
ED-10-15	52	14	2	<b>8.23</b>	0.00	10.66	0.00	1578.47	0.00	33.63	0.00	11.31	0.00	15.43	0.00	3600	14.23	3600	14.68
ED-15-20	122	4	2	25.31	0.00	<b>22.16</b>	0.00	3600	249.02	3600	25.44	2417.31	0.00	3600	4.97	3600	—	3600	—
ED-15-21	96	4	2	3.76	0.00	<b>2.38</b>	0.00	3600	278.45	3212.55	0.00	147.83	0.00	124.00	0.00	3600	—	3600	—
ED-15-22	112	4	2	6.06	0.00	<b>4.28</b>	0.00	3600	248.51	3600	44.75	325.79	0.00	976.62	0.00	3600	—	3600	—
ED-15-23	142	4	2	9.98	0.00	<b>6.73</b>	0.00	3600	285.59	3600	42.92	1398.02	0.00	1114.80	0.00	3600	—	3600	—
ED-15-24	91	4	2	3.16	0.00	<b>1.83</b>	0.00	3600	363.25	2169.20	0.00	63.17	0.00	142.08	0.00	3600	—	3600	—
ED-10-02	51	9	5	3600	269.43	3600	127.69	3600	84.64	<b>42.90</b>	0.00	3600	273.35	3600	198.51	3600	404.79	3600	229.71
ED-10-03	54	10	5	3600	208.06	3600	107.92	3600	175.98	<b>160.25</b>	0.00	3600	208.06	3600	182.37	3600	336.39	3600	368.91
ED-10-11	50	11	5	3600	198.60	3600	126.83	3600	70.60	<b>11.42</b>	0.00	3600	197.32	3600	225.44	3600	221.34	3600	272.88
ED-10-14	62	15	5	3600	286.95	3600	132.28	3600	87.52	3600	<b>0.15</b>	3600	266.93	3600	268.48	3600	457.14	3600	315.48
ED-10-15	52	14	5	3600	199.46	3600	90.75	3600	74.91	<b>3015.40</b>	0.00	3600	200.31	3600	187.88	3600	412.97	3600	279.67
ED-15-20	122	4	5	3600	763.11	3600	119.42	3600	84.80	3600	<b>0.59</b>	3600	763.11	3600	795.55	3600	—	3600	—
ED-15-21	96	4	5	3600	356.73	3600	26.36	3600	87.69	3600	<b>8.64</b>	3600	356.73	3600	337.02	3600	—	3600	213017.65
ED-15-22	112	4	5	3600	430.48	3600	23.40	3600	—	3600	<b>11.16</b>	3600	429.15	3600	394.22	3600	—	3600	—
ED-15-23	142	4	5	3600	741.91	3600	91.94	3600	88.04	3600	<b>16.56</b>	3600	741.91	3600	621.63	3600	—	3600	—
ED-15-24	91	4	5	3600	259.77	3600	28.21	3600	114.81	<b>1342.87</b>	0.00	3600	277.02	3600	280.64	3600	—	3600	330337.37

Table A.6: Comparison of model (7) variants for the  $p$ -OBOP. The term “Relax  $\mathbf{x}$ ” denotes the relaxation of binary variables  $\mathbf{x}$  to the continuous domain  $[0, 1]$ .

Instance	$n$	$m$	$p$	(8)		(8) – (8f) – (8g) + (9)		(8) + (9)	
				Time	Gap	Time	Gap	Time	Gap
ED-10-02	51	9	2	995.41	0.00	1016.43	0.00	<b>493.93</b>	0.00
ED-10-03	54	10	2	<b>196.27</b>	0.00	869.89	0.00	366.32	0.00
ED-10-11	50	11	2	<b>10.56</b>	0.01	182.56	0.00	11.66	0.00
ED-10-14	62	15	2	<b>11.27</b>	0.00	21.31	0.00	12.07	0.00
ED-10-15	52	14	2	<b>781.46</b>	0.00	1172.56	0.00	889.97	0.00
ED-15-20	122	4	2	3600	<b>23.02</b>	3600	87.40	3600	23.02
ED-15-21	96	4	2	3600	32.54	3600	69.36	3600	<b>29.61</b>
ED-15-22	112	4	2	3600	<b>37.40</b>	3600	41.68	3600	37.41
ED-15-23	142	4	2	3600	—	3600	111.18	3600	<b>23.15</b>
ED-15-24	91	4	2	3600	<b>32.42</b>	3600	62.11	3600	61.43
ED-10-02	51	9	5	<b>51.86</b>	0.01	323.51	0.01	302.15	0.01
ED-10-03	54	10	5	<b>6.01</b>	0.00	47.08	0.00	34.54	0.00
ED-10-11	50	11	5	<b>5.93</b>	0.00	27.07	0.00	34.78	0.00
ED-10-14	62	15	5	<b>11.00</b>	0.00	71.60	0.00	66.62	0.00
ED-10-15	52	14	5	<b>7.70</b>	0.01	43.33	0.01	35.95	0.01
ED-15-20	122	4	5	3600	0.11	3600	3.40	3600	<b>0.07</b>
ED-15-21	96	4	5	3600	<b>1.26</b>	3600	2.75	3600	2.13
ED-15-22	112	4	5	3600	9.12	3600	98.29	3600	<b>7.91</b>
ED-15-23	142	4	5	3600	23.25	3600	46.12	3600	<b>22.27</b>
ED-15-24	91	4	5	<b>95.49</b>	0.00	3600	2.63	230.99	0.00
ED-10-02	51	9	20	3600	15.46	1015.58	0.00	<b>433.73</b>	0.00
ED-10-03	54	10	20	3600	4.59	860.85	0.00	<b>342.96</b>	0.00
ED-10-11	50	11	20	3600	15.78	181.26	0.00	<b>21.13</b>	0.00
ED-10-14	62	15	20	3600	0.60	21.28	0.00	<b>12.96</b>	0.00
ED-10-15	52	14	20	3600	10.19	1400.03	0.00	<b>883.35</b>	0.00
ED-15-20	122	4	20	3600	52.28	3600	87.40	3600	<b>23.02</b>
ED-15-21	96	4	20	<b>2145.80</b>	0.04	3600	69.36	3600	29.61
ED-15-22	112	4	20	3600	<b>37.40</b>	3600	41.68	3600	37.41
ED-15-23	142	4	20	3600	—	3600	111.18	3600	<b>23.15</b>
ED-15-24	91	4	20	3600	<b>32.42</b>	3600	62.11	3600	61.43

Table A.7: Comparison of model (8) variants for the  $p$ -OBOP.

From Table A.7, we observe that the inclusion of additional inequalities (9) can significantly enhance the model's performance, particularly for smaller values of  $p$ . In general, the strengthened formulations improve convergence and reduce the optimality gap, confirming the positive effect of incorporating well-chosen valid inequalities. These findings, together with the previous analysis, provide valuable insight into which modeling strategies, both in terms of constraint structure and variable relaxation, yield the most efficient formulations for the  $p$ -OBOP.

## Appendix B. Behaviour of the $p$ -OBOP model

A natural question arises as to whether the existence of optimal solutions for two distinct numbers of buckets implies the existence of an optimal solution for every intermediate number of buckets. The following example demonstrates that this property does not necessarily hold.

**Example 9.** Consider the following instance of the OBOP with 4 items, whose pairwise order matrix is shown in Figure B.6.

$$\frac{1}{100} \begin{pmatrix} 50 & 55 & 90 & 100 \\ 45 & 50 & 20 & 80 \\ 10 & 80 & 50 & 65 \\ 0 & 20 & 35 & 50 \end{pmatrix}$$

Figure B.6: Pairwise order matrix used in Example 9.

Solving this instance with model (7) for values of  $p$  ranging from 1 to 4 yields the optimal values shown in Figure B.7.

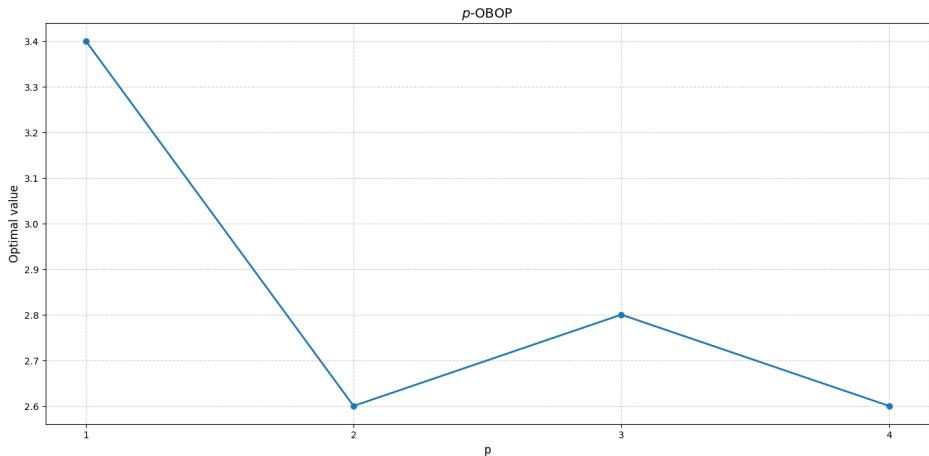


Figure B.7: Optimal values of the  $p$ -OBOP for different values of  $p$

This example illustrates that the curve of optimal values for the  $p$ -OBOP as a function of  $p$  is not necessarily unimodal. In particular, the existence of optimal solutions for the standard OBOP for  $p = n - 1$  and  $p = n + 1$  does not imply the existence of one for  $p = n$ .

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