

# Bipartiteness in Progressive Second-Price Multi-Auction Networks with Perfect Substitute

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## Abstract

We consider a bipartite network of buyers and sellers, where the sellers run locally independent Progressive Second-Price (PSP) auctions, and buyers may participate in multiple auctions, forming a multi-auction market with perfect substitute. The paper develops a projection-based influence framework for decentralized PSP auctions. We formalize primary and expanded influence sets using projections on the active bid index set and show how partial orders on bid prices govern allocation, market shifts, and the emergence of saturated one-hop shells. Our results highlight the robustness of PSP auctions in decentralized environments by introduc-

ing saturated components and a structured framework for phase transitions in multi-auction dynamics. This structure ensures deterministic coverage of the strategy space, enabling stable and truthful embedding in the larger game. We further model intra-round dynamics using an index  $\tau_k$  to capture coordinated asynchronous seller updates coupled through buyers' joint constraints. Together, these constructions explain how local interactions propagate across auctions and gives premise for coherent equilibria—without requiring global information or centralized control.

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## 1 Introduction

The Progressive Second Price (PSP) auction, introduced by Lazar and Semret [12], and later expanded upon in Semret's dissertation [19], presented a full theoretical framework for distributed resource pricing and demonstrated the linkage between PSP and VCG-type efficiency results. The PSP auction is a decentralized mechanism characterized by truthfulness, individual rationality, and social welfare maximization. Unlike traditional centralized auctions, PSP allows buyers and sellers to iteratively interact through local bidding rounds, dynamically allocating consumable resources such as network bandwidth and other communicative and computational resources. In PSP auctions, winners pay a cost determined by the externality that is imposed on others, calculated from the distribution of allocations and the bid. This ensures truthful reporting of valuation through incentive compatibility as was shown in the foundational work of Vickrey, Clarke,



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and Groves [8, 10, 22]. The resulting equilibria adhere to the exclusion-compensation principle, preventing unilateral improvement without harming another participant.

Our focus is on developing adaptive auction mechanisms, like the Progressive Second Price (PSP) auction, that respond to market dynamics by allowing agents to adjust their bids based on local information gathered from their network neighbors. This motivates the study of influence sets, dynamic participation, and the role of network effects in shaping bidding behavior. In these settings, agents lack full market information and are affected by network dependencies.

Maillé et al. [9] build directly on Lazar and Semret’s 1999 PSP framework by addressing the one remaining free parameter in the model — the reserve price. They demonstrate that while PSP guarantees convergence, efficiency, and incentive compatibility, the seller’s reserve price can be optimized by simple numerical methods, allowing PSP markets to balance efficiency with revenue maximization.

These iterative updates operate as strategic interactions in a decentralized framework, where the PSP auction converges, perhaps astonishingly, deterministically to an  $\varepsilon$ -Nash equilibrium. This has been shown to be true on the networks of 20 years ago, when bandwidth and bandwidth allocation was perhaps a different game. A real-world, modern network faces significant obstacles; it is a game of partial information played in a web of interconnected decisions, dynamic participation, and evolving market constraints. This motivates a graph-theoretic treatment of information flow and motivates the introduction of the concept of market saturation.

The structure of this paper is as follows. Section 2 introduces the foundations necessary to model influence propagation in decentralized auctions. In Section 3, we present the Progressive Second Price (PSP) auction mechanism, outlining its bidding rules, participation logic, and price allocation behavior. Section 4 defines and explores the dynamics of influence sets, establishing a framework for analyzing how strategy updates propagate across the market. Our approach adopts and extends these concepts through graph-based methods, specifically leveraging the bipartite graph to systematically represent buyer–seller interactions. Section 5 introduces the concept of saturation as the limit of influence propagation, characterizing a locally evolving equilibrium structure. The simulation framework and implementation are discussed in Section 6, and this paper’s conclusion and future work are presented in Section 7.

## **2 Background and Related Work**

This paper introduces a graph-based analytical framework to examine the dynamics of Progressive Second Price (PSP) auctions within decentralized market structures. Our approach builds on foundational concepts in auction theory, network influence propagation, and graph analysis, while situating the PSP model among several related domains.

The Progressive Second Price (PSP) auction, initially proposed by Lazar and Semret [12], extends classical second-price mechanisms [8, 10, 22] into decentralized contexts. Earlier studies such as Maillé and Tuffin [14] and Semret’s dissertation [19] provided a full system-level model of distributed market control and the theoretical grounding for the PSP auction mechanism, analyzing network-based PSP equilibria and pricing strategies. Subsequent formal analyses such as Qu, Jia, and Caines [17] presented key results on the Uniformly Quantized PSP (UQ-PSP) mechanism, showing that it guarantees convergence to a unique limit price independent of initial conditions, achieves  $\gamma$ -incentive compatibility, and extends naturally to network topologies where equilibria depend on local information exchange. Their framework provided the first rigorous quantized extension of the PSP model, establishing discrete convergence proofs that later generalizations such as those developed in this work, Qu, Jia, and Caines [18] further extended these results to networked PSP convergence, introducing asynchronous coordination and bounded-delay

convergence. Subsequent work has investigated distributed or multi-resource variants, including privacy-preserving and differential frameworks in data and spectrum markets [7, 23], expanding PSP-like mechanisms to new computational settings.

Local coordination rules, when combined with bounded delays and limited information exchange, can achieve global properties similar to those in consensus and averaging protocols. Beyond traditional equilibrium analysis, distributed consensus and coordination models offer insight into asynchronous bidding and update rules. Aguilera and Toueg [1] and Lynch [13] describe protocols ensuring eventual consistency under partial information, concepts that are applicable to asynchronous PSP updates. These works demonstrate that convergent systems operating under bounded delay result in deterministic convergence guarantees in decentralized markets.

In decentralized markets, agents' strategies depend on local interactions but propagate indirectly through shared participation and local coordination. This connects PSP analysis to the broader literature on influence diffusion and cascading behavior, as in Kleinberg [11], Oki et al. [16], and Osvaldo and Queen [2], which examine network-driven contagion and adaptive decision processes. The theoretical foundation of influence sets also aligns with the study of sphere-of-influence graphs [15, 21] and dynamic graph structures that represent iterative strategic dependencies.

Graph-based approaches are central to understanding multi-agent optimization. Baur [4], Barrett [3], and related work on planar and dynamic graphs illustrate how reachability, closure, and resistance distance can capture evolving connectivity. In the PSP context, our use of projection operators extends these methods by linking graph reachability to economic stability, enabling a deterministic interpretation of market influence propagation.

This work examines decentralized auction theory, distributed coordination, influence propagation, and graph-theoretic modeling to provide a coherent analytical framework for PSP auctions. This expanded foundation motivates the later sections on local saturation and asynchronous market dynamics.

### 3 The PSP Auction Mechanism

The Progressive Second Price (PSP) auction is a decentralized mechanism in which buyers iteratively submit bids to sellers, and sellers update reserve prices based on received bids. Each auction operates locally, and coordination emerges through repeated interactions across the market graph. The mechanism rules first appears in [12], defining the bid structure, auction dynamics, pricing rules, allocation strategies, and participation behavior. In what follows, we define the bid structure, auction dynamics, pricing rules, allocation strategies, and participation behavior that govern the PSP mechanism. Let  $\mathcal{I} = \mathcal{B} \cup \mathcal{L}$  denote the set of all agents, partitioned into buyers and sellers. Each seller  $j \in \mathcal{L}$  manages a local auction for a divisible resource, and each buyer  $i \in \mathcal{B}$  may submit bids to a subset of sellers. The bid profile of auction  $j$  is given by the column vector  $s^j$  with entries  $s_i^j$ , where  $(i, j) \in \mathcal{B} \times \mathcal{L}$ . A bid

$$s_i^j = (q_i^j, p_i^j) \in S_i^j = [0, Q^j] \times [0, \infty)$$

represents a single interaction between buyer  $i$  and seller  $j$ , where  $q_i^j$  is the quantity requested by the buyer and  $p_i^j$  is the unit price offered.

In decentralized markets governed by distributed Progressive Second Price (PSP) auctions, agents submit bids in the form of price-quantity pairs at discrete time steps. These bids are locally observable: buyers receive feedback from auctions in which they participate, and sellers observe aggregate demand over time. However, the global structure of the market—including overlapping buyer influence, competition externalities, and inferred network effects—must be reconstructed from these partial, temporally indexed signals.

Object	Single auction $j$	Across all auctions
quantity	$Q^j$	$(Q^1, \dots, Q^J)$
Player $i$ 's bid pair	$s_i^j = (q_i^j, p_i^j)$	$s_i = (s_i^1, \dots, s_i^J)$
Strategy space of player $i$	$S_i^j = [0, Q^j] \times [0, \infty)$	$S_i = \prod_{j=1}^J S_i^j$
<i>Opposing bids w.r.t. player <math>i</math></i>	$s_{-i}^j = (s_1^j, \dots, s_{i-1}^j, s_{i+1}^j, \dots, s_n^j)$	$s_{-i} = (s_{-i}^1, \dots, s_{-i}^J)$
Profile in auction $j$	$s^j = (s_1^j, \dots, s_n^j)$	$s = (s^1, \dots, s^J)$
Grand strategy space	$S^j = \prod_{i=1}^n S_i^j$	$S = \prod_{j=1}^J S^j$

■ **Table 1** Basic sets and notation for a bundle of  $J$  independent PSP auctions

### 3.1 Bounded Participation

Each buyer will know the available quantity for each market in which they bid. Buyers act strategically by selecting sellers, adjusting bid quantities, and choosing whether to participate based on their expected ability to satisfy demand. In the PSP framework buyers cannot reveal their entire valuation functions in a single step; instead they must request allocations iteratively. To regulate this behavior we introduce a bounded participation rule, which endogenously limits the set of sellers a buyer engages with, and can be seen as an analogue of the opt-out behavior given in [6].

Fix buyer  $i$  at time  $t$  and let  $p^*$  denote the common marginal price identified from opponents' bids. For each seller  $j$  let  $c_j = \text{cap}_j(p^*)$  be the residual quantity available to  $i$  at price  $p^*$ . Define the desired total quantity

$$z_i^* = \min \left\{ \bar{q}_i(t), \sum_j c_j \right\}. \quad (1)$$

► **Definition 1** (Bounded participation rule). *Buyer  $i$  selects a minimal-cost subset of sellers  $\mathcal{L}_i(t) \subseteq \mathcal{L}$ , ordered by nondecreasing price  $p_{(n)}^j(t)$ , such that*

$$\sum_{j \in \mathcal{L}_i(t)} c_j(t) \geq z_i^*. \quad (2)$$

*The buyer allocates requests sequentially to the least expensive sellers until the desired total quantity  $z_i^*$  is reached, subject to residual capacities  $c_j(t)$ . For  $j \notin \mathcal{L}_i(t)$ , set  $q_i^j = 0$ .*

This rule formalizes bounded participation at fixed  $t$ : each buyer interacts only with the fewest necessary sellers to realize  $z^*$ , in an attempt to minimize the cost of participation. The resulting allocation targets allocations at a common marginal price  $p^*(t)$  under residual quantity constraints.

### 3.2 Residual Quantity and Allocation

As a market with perfect but incomplete information, sellers can only gain information about demand by observing buyer behavior, determined by the connectivity of the auction graph. In each iteration, every seller completes one update of its local auction.

For each seller  $j$ , the reserve price  $p_*^j(t)$  is the price at which seller  $j$  is indifferent between selling her final unit of resource and keeping it. Equivalently, the seller may be viewed as submitting

an internal bid  $(Q^j, p_*^j(t))$  on her own auction. At the end of each round  $t$ , the reserve price is updated with information from the set of active bids, where  $\mathcal{B}^j(t)$  is the set of buyers who win strictly positive allocations at seller  $j$  in round  $t$ , and  $\epsilon > 0$ .

We define the clearing price at seller  $j$  to be the smallest price at which aggregate awarded quantity meets available quantity:

$$\chi^j(t) = \min \left\{ y : \sum_{k: p_k^j(t) > y} a_k^j(t) \geq Q^j(t) \right\}. \quad (3)$$

Any residual supply must therefore be allocated among bids that tie at prices just above  $\chi^j(t)$ , after higher-priced bids are filled. Let

$$\underline{p}^j(t) := \min \{ p_i^j(t) : i \in \mathcal{B}^j(t) \}, \quad \bar{p}^j(t) := \max \{ p_i^j(t) : i \notin \mathcal{B}^j(t) \}, \quad (4)$$

be the lowest winning and highest losing bid prices at seller  $j$ , and where buyers *not* in  $\mathcal{B}^j(t)$  receive zero allocation at seller  $j$ . The clearing price satisfies

$$\bar{p}^j(t) < \bar{p}^j(t) + \epsilon \leq \chi^j(t) \leq \underline{p}^j(t) - \epsilon < \underline{p}^j(t)$$

whenever there is at least one winning and one losing bidder at seller  $j$ . In particular,  $\chi^j(t)$  lies in the open interval between the highest losing and lowest winning bid. At equilibrium, the reserve price  $p_*^j(t)$  coincides with the clearing price at seller  $j$ , i.e., the clearing price implied by the PSP allocation rule.

Buyers at higher prices are therefore always served in full, whereas buyers at the threshold price may be rationed. At each price level  $y$ , the residual quantity is given by

$$R^j(y, t) = \left[ Q^j(t) - \sum_{k: p_k^j(t) > y} a_k^j(t) \right]^+. \quad (5)$$

When multiple buyers tie at  $p_i^j(t) = y$ , the awarded allocation respects both the buyer's request and the residual supply. We refer to the tie-splitting rule originated in the analysis of quantized PSP auctions by Qu, Jia, and Caines [17],

$$a_i^j(s(t)) = \min \left\{ q_i^j(t), \frac{q_i^j(t)}{\sum_{\ell: p_\ell^j(t) = y} q_\ell^j(t)} R^j(y, t) \right\}. \quad (6)$$

The bid quantity  $q_i^j(t)$  and the allocation  $a_i^j(t)$  are complementary. In fact, the buyer strategy is the first term in the minimum, the second term being owned by the seller. For each buyer-seller pair  $(i, j)$  at time  $t$ ,  $a_i^j(t)$  is the *awarded* amount that seller  $j$  allocates to buyer  $i$  once the allocation rule has been applied. By construction,

$$a_i^j(t) \leq q_i^j(t), \quad (7)$$

with equality holding when residual supply at the buyer's price suffices to satisfy all tied requests. The mechanism therefore never awards more than requested and may award less when quantity is limited.

We remark that the reserve price  $p_*^j(t)$  that lies in the margin interval determined by the bids

$$\bar{p}^j(t) < p_*^j(t) < \underline{p}^j(t), \quad (8)$$

whenever both  $\bar{p}^j(t)$  and  $\underline{p}^j(t)$  are defined, and we deliberately leave the precise rule for selecting  $p_*^j(t)$  within the interval (8) unspecified. In particular, admissible choices include

$$p_*^j(t) = \chi^j(t), \quad p_*^j(t) = \bar{p}^j(t) + \epsilon, \quad p_*^j(t) = \underline{p}^j(t) - \epsilon,$$

provided that reserve price updates lie within  $\epsilon$  and the resulting sequence  $\{p_*^j(t)\}_t$  is nondecreasing.

### 3.3 Exclusion–Compensation

Each buyer's payment follows a second-price externality principle, this is the “social opportunity cost” of the PSP pricing rule. The exclusion–compensation payment to buyer  $i$  equals the loss imposed on other buyers at that seller when  $i$  participates. For a fixed auction  $j$  we use the opposing buyers' piecewise-constant marginal price function  $P^j(\cdot, s_{-i}^j)$  built from  $s_{-i}^j$ ,

$$c_i^j(s) = \int_0^{a_i^j(s)} P^j(z, s_{-i}^j) dz, \quad (9)$$

which holds true locally at each auction, where the opposing bids are calculated against the allocated resource to buyer  $i$ . The amount of resource available at price  $p_{(n)}^j$  is  $\xi_{n-1}^j - \xi_n^j \geq 0$ . The local inverse price function is then

$$P^j(z, s_{-i}^j) = p_{(n)}^j \quad \text{for } z \in (\xi_n^j, \xi_{n-1}^j]. \quad (10)$$

For each ordered price  $y$ , we have that  $P_i(z, s_{-i})$  is defined for the range of  $z$  corresponding to the total resource available from all sellers at that price, i.e.,

$$z \in \left( \sum_{p_{(n)}^j > y} (\xi_{n-1}^j - \xi_n^j), \sum_{p_{(n)}^j \geq y} (\xi_{n-1}^j - \xi_n^j) \right]. \quad (11)$$

Define the aggregate residual quantity

$$Q_i(y, s_{-i}) = \sum_{j=1}^{\mathcal{L}} Q_i^j(y, s_{-i}^j), \quad P_i(z, s_{-i}) = \inf \{ y \geq 0 : Q_i(y, s_{-i}) \geq z \}, \quad (12)$$

where because  $Q_i(y, s_{-i})$  is a right-continuous, nondecreasing step function with finitely many jumps at  $\{p_{(m)}^j\}$ , the infimum is attained.

### 3.4 Valuation and Utility

Each buyer  $i$  has an elastic valuation function  $\theta_i : [0, Q_i] \rightarrow [0, \infty)$  with strictly decreasing derivative  $\theta'_i$ . The valuation depends on the total awarded quantity across all sellers:

$$V_i(a) = \theta_i \left( \sum_{j=1}^J a_i^j(t) \right) = \int_0^{\sum_{j=1}^J a_i^j(t)} \theta'_i(z) dz. \quad (13)$$

Given a strategy profile  $s$ , the utility of buyer  $i$  for potential allocation  $a$  is dependent on the cost,  $c_i(s)$ , where the cost to buyer  $i$  as a function of the entire strategy profile  $s$ .

In the dynamic setting this profile evolves with iteration  $t$ , where  $c_i(s)$  may represent total participation costs, including membership fees, per-round overhead, and per-auction message costs. Utility is given by

$$u_i(s) = V_i(a) - c_i(s), \quad (14)$$

where  $c_i(s)$  is a dynamic cost function that evolves over time with bid updates.

The buyers' utility functions implicitly define a potential over the allocation space, as buyers seek to maximize their utility through strategic allocation requests. We note that a uniform (coordinated) bid price from buyers across active sellers upholds strategic simplicity and second-price incentives, which are rational under quasi-linear utilities, as shown in the original PSP framework [12].

Following Lazar and Semret [12], updates occur only when the buyer's utility improvement exceeds a small positive threshold, ensuring asynchronous convergence under bounded delay. In a single-auction market, buyer  $i$  accepts a new bid  $s'_i$  only if  $u_i(s'_i; s_{-i}) - u_i(s_i; s_{-i}) > \varepsilon$ . In the multi-auction setting, buyer  $i$  posts a vector of bids that share a common marginal price  $p_i^*$  across all connected sellers. The utility comparison therefore becomes an aggregate test, where, in terms of the opposing bid vector  $s_{-i}$ , any gain in utility at time  $t$  depends on the current state of play. Information propagation across the market affects how the vector of opposing bids  $s_{-i}$  is formed, and thus how externalities are computed. The realized utility improvement  $\Delta u_i(t)$  is evaluated relative to the previous round to determine if a new bid exceeds the cost of participation.

The discussion of externality under multiple auctions running asynchronously and a formal convergence analysis of the PSP mechanism to a single, unique, global  $\varepsilon$ -Nash network equilibrium, as was given in [19], is outside the scope of this paper. We instead focus on the iterative application of a uniform marginal price and the localized pricing structure resulting from progressive bid updates on connected network components consisting of multiple sellers sharing multiple buyers under an assumed bipartite structure. A formal analysis of the effects of latency on a PSP auction is given in [5].

## 4 Influence Sets

We model the behavior of vertices (buyers and sellers) in this bipartite structure using influence sets. Each vertex's strategy space is influenced by neighboring vertices and evolves over time. Influence sets restrict the strategy space of buyers and sellers within bounded regions, stabilizing auction dynamics and supporting predictable market equilibria. As rational agents, buyers and sellers do not optimize perfectly but instead operate within acceptable thresholds of cost and utility. Influence propagation determines the flow of information, where bidding saturation occurs once influence sets stabilize. At saturation, there is no vertex that, upon calculating his measure of utility, suffers a changing set of opponent bids, and all subsequent bid updates are calculated on locally stable subgraphs of the market.

Thus, influence sets are subsets of the auction graph that represent the scope of influence a particular vertex (buyer or seller) has on others over a finite number of auction iterations. These sets structure interactions into subsets of the auction graph where local equilibria form dense regions where bid updates are stabilized.

### 4.1 Primary (Direct) Influence Sets

Following the original definition from [6], the *primary influence set*, denoted  $\Lambda$ , for a given seller  $j$  at time  $t$ , is defined set-theoretically,

$$\Lambda_{\mathcal{L}}(j, t) = \bigcup_{i \in \mathcal{B}^j(t)} \mathcal{L}_i(t), \quad (15)$$

where  $\mathcal{B}^j(t)$  is the set of buyers bidding on seller  $j$ , and  $\mathcal{L}_i(t)$  is the set of sellers that buyer  $i$  bids on. Thus,  $\Lambda_{\mathcal{L}}^{(1)}(j, t)$  represents all sellers directly connected to auction  $j$  via shared buyers at time  $t$ . This definition captures the notion that influence propagates across the auction graph through buyer-seller connections. The superscript (1) denotes the first layer of influence anchored at seller  $j$  expanding through buyer-mediated connections.

To illustrate, for a buyer  $i$ , the relevant bids  $s_i^j$  flow *from* the buyer to sellers. For a seller  $j$ , we reverse this; bids flow *into* the seller from buyers. The base case captures this directionality, which we get from market theory: buyers have positive demand, and sellers have negative demand

(otherwise known as surplus). The direct influence set for buyer  $i$ , denoted  $\Lambda_{\mathcal{B}}^{(1)}(i, t)$ , includes buyers directly connected to  $i$  through shared sellers,

$$\Lambda_{\mathcal{B}}(i, t) = \bigcup_{j \in \mathcal{L}_i(t)} \mathcal{B}^j(t). \quad (16)$$

We now have the first layer of *buyer-to-buyer* influence induced by common seller participation. It serves as a foundation for constructing buyer–buyer influence graphs and identifying bid coordination structures within the network. This expression gathers the buyers indirectly connected to buyer  $i$  through shared sellers, filtered by active bids at the given iteration. It provides a way to trace buyer–buyer influence mediated through seller auctions.

We extend the definition from [6] in a theoretical and practical sense, defining the base case explicitly as the vertex itself,

$$\Lambda^{(0)}(x, t) = \{x\}, \quad (17)$$

emphasizing that at the zeroth level, the influence set represents only the vertex itself. This represents a measure of “self-influence”, such as reserve prices (for sellers) or initial valuations (for buyers), economically aligning with the idea that a seller starts from a reserve price reflecting their own valuation, while a buyer’s self-valuation corresponds to their initial maximum willingness-to-pay.

## 4.2 Expanded (Indirect) Influence Sets

For any vertex  $x$  in the auction graph (buyer or seller), and for  $n \geq 1$ , the primary influence set is expanded from the  $(n - 1)$ –step influence set by aggregating direct neighbors at the next layer:

$$\Lambda^{(n)}(x, t) = \bigcup_{y \in \Lambda^{(n-1)}(x, t)} \Lambda^{(1)}(y, t). \quad (18)$$

Where we define a two-hop projection operator

$$\Lambda^{(1)}(y, t) := \begin{cases} \bigcup_{i \in \mathcal{B}^y(t)} \mathcal{L}_i(t), & \text{if } y \in \mathcal{L}, \\ \bigcup_{j \in \mathcal{L}_y(t)} \mathcal{B}^j(t), & \text{if } y \in \mathcal{B}, \end{cases}$$

which always returns vertices of the *same type* as  $y$  after one buyer–seller alternation. For sellers, the  $n$ –step influence set may be computed recursively,

$$\Lambda_{\mathcal{L}}^{(n)}(j, t) = \bigcup_{i \in \Lambda_{\mathcal{B}}^{(n-1)}(j, t)} \mathcal{L}_i(t),$$

where  $\Lambda_{\mathcal{B}}^{(n-1)}(j, t)$  is the set of buyers reachable from seller  $j$  in  $n - 1$  steps. This returns the set of sellers that receive nonzero bids from buyers who are indirectly connected to auction  $j$  via shared bidding activity across  $n$  rounds of the PSP auction. It describes how seller  $j$ ’s influence propagates through buyer behavior across seller neighborhoods. For buyers,

$$\Lambda_{\mathcal{B}}^{(n)}(i, t) = \bigcup_{j \in \Lambda_{\mathcal{L}}^{(n-1)}(i, t)} \mathcal{B}^j(t),$$

where  $\Lambda_{\mathcal{L}}^{(n-1)}(i, t)$  collects sellers indirectly connected to buyer  $i$ .



Each new layer  $\Lambda^{(n)}(x, t)$  therefore adds the direct neighbors of all vertices in the previous layer, producing a breadth-first expansion in the auction graph. This recursive expansion therefore builds a “growing influence ball” centered at  $x$ , where the secondary set acts as a generalized neighborhood closure or hull around the initial primary set. At each step  $n$ , the influence set  $\Lambda^{(n)}(x, t)$  forms an outer boundary surrounding the influence set  $\Lambda^{(1)}(x, t)$ , recursively aggregating direct neighborhoods around previously identified influence vertices.

### Pathwise Characterization.

In graph theory, this structure parallels the  $n$ -hop neighborhood closure or a breadth-first expansion of distance- $n$  shells. We characterize  $\Lambda^{(n)}(x, t)$  pathwise as the set of all vertices reachable from  $x$  by paths alternating between buyers and sellers, of length up to  $2n$ . Formally, let  $\mathcal{G} = (\mathcal{I}, E)$  denote the bipartite auction graph, where  $\mathcal{I}$  is the set of agents (buyers and sellers), and an edge  $(i, j) \in E$  exists if buyer  $i$  bids on seller  $j$ . The graph alternates between buyers and sellers by construction: no two buyers or two sellers are directly connected. Because of the bipartiteness we have the parity rule, and consequently

$$\Lambda^{(n)}(x, t) = \{y \in \mathcal{I} \mid \text{dist}_{\mathcal{G}}(x, y) = 2n\}.$$

From a strategic perspective, the expanded influence set  $\Lambda^{(n)}(x, t)$  describes the scope of anticipated externalities: the agents whose actions may not affect  $x$  directly, but may impact  $x$ 's incentives via shared neighbors. These influence chains emerge in environments with incomplete information and approximate the region of the market that affects the *expected utility gradient* of vertex  $x$ . In equilibrium analysis, these indirect sets are crucial for understanding stability, coordination potential, and susceptibility to shock propagation (e.g., strategic manipulation or correlated noise). As noted in [6] and echoed in broader decentralized market theory (e.g., [20]), indirect influence plays a key role in shaping convergence. While  $\Lambda(x, t)$  governs observed interaction,  $\Lambda^{(n)}(x, t)$  governs inferred or mediated interdependence—and together, they define the full strategic visibility of a vertex.

## 4.3 Projection Domains and Influence Operations

The influence set framework captures cascading dependencies and forms the foundation for our graph-theoretic analysis. To analyze the propagation of influence in the auction network, we construct influence sets using a sequence of projection operations on the underlying bid graph. Each active bid is indexed by a pair  $(i, j) \in \mathcal{B} \times \mathcal{L}$ , where buyer  $i$  submits a bid to seller  $j$ . These interactions collectively form the strategy space  $S(t)$ , which consists of the full collection of price-quantity bids  $s_i^j$ . We extract the active subgame by identifying  $\mathcal{I}_{\text{active}}(t) \subseteq \mathcal{B} \times \mathcal{L}$ , the set of observed interactions between buyers and sellers at time  $t$ . These pairs serve as both an interaction graph and an index set for the time-dependent strategy array  $\mathbf{s}(t)$ .

## 4.4 Projection-Based Influence Propagation

Let  $\mathcal{I}_{\text{active}}(t) \subseteq \mathcal{B} \times \mathcal{L}$  be the set of buyer-seller pairs that submit positive bids at time  $t$ . Each pair  $(i, j)$  indexes the strategy array  $\mathbf{s}(t) \in S(t)$ , so  $\mathcal{I}_{\text{active}}(t)$  is both a graph on  $\mathcal{B} \cup \mathcal{L}$  and an index set for the strategy space.

### Projection maps.

$$\pi : \mathcal{I}_{\text{active}}(t) \longrightarrow \mathcal{B}, \quad \pi(i, j) = i, \quad \varpi : \mathcal{I}_{\text{active}}(t) \longrightarrow \mathcal{L}, \quad \varpi(i, j) = j.$$

- *Structural role.* Alternating compositions  $\varpi \circ \pi^{-1} \circ \pi \circ \varpi^{-1} \dots$  trace paths through the bipartite auction graph, giving  $n$ -hop neighborhoods.

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- *Strategic role.* Acting on  $S(t)$ , the same maps carve out partial strategy profiles (e.g. all bids of a given buyer).

**Full pre-images.** We take the composition of the the projections in order to restrict and vectorize the space  $S(t)$ . For any buyer  $i$  and seller  $j$  we write

$$\varpi^{-1}(i, t) = \{(i, j') \in \mathcal{I}_{\text{active}}(t)\}, \quad \pi^{-1}(j, t) = \{(i', j) \in \mathcal{I}_{\text{active}}(t)\},$$

so that

$$\varpi \circ \pi^{-1}(i, t) = \{j' \mid (i, j') \in \mathcal{I}_{\text{active}}(t)\} = \mathcal{L}_i(t), \quad \pi \circ \varpi^{-1}(j, t) = \{i' \mid (i', j) \in \mathcal{I}_{\text{active}}(t)\} = \mathcal{B}^j(t).$$

**$n$ -step influence.** Because the graph is bipartite, two successive projections always return a vertex of the *same* type. These projections serve dual purposes: structurally, they trace paths through the auction graph, alternating between buyers and sellers; strategically, they extract subspaces of  $S(t)$  that represent partial strategies or responses and may evolve as patterns in the form of active bids sets.

### Connected components via iterated projections

Because the auction graph is bipartite, two successive projections return a vertex of the same type. Define the composition operators

$$P := \varpi \circ \pi^{-1} \quad (\text{buyer} \rightarrow \text{seller}), \quad Q := \pi \circ \varpi^{-1} \quad (\text{seller} \rightarrow \text{buyer}).$$

Starting from a seller  $j$ , one step of influence expansion is

$$\Lambda_{\mathcal{L}}^{(1)}(j, t) = P Q(j) = \varpi \circ \pi^{-1} \circ \pi \circ \varpi^{-1}(j, t),$$

which moves seller  $\rightarrow$  buyers  $\rightarrow$  sellers. Analogously, for a buyer  $i$  we set  $\Lambda_{\mathcal{B}}^{(1)}(i, t) = Q P(i)$ .

The  $n$ -hop neighborhoods follow by simple iteration,

$$\Lambda_{\mathcal{L}}^{(n)}(j, t) = (P Q)^n(j), \quad \Lambda_{\mathcal{B}}^{(n)}(i, t) = (Q P)^n(i), \quad n \geq 1,$$

with the base case  $\Lambda^{(0)}(x, t) = \{x\}$ .

Each application of  $PQ$  (or  $QP$ ) adds exactly one buyer–seller alternation, so  $\Lambda^{(n)}(x, t)$  is the breadth-first shell lying  $n$  hops away from  $x$ . Iterating until  $\Lambda^{(n)}(x, t) = \Lambda^{(n-1)}(x, t)$  closes the connected component containing  $x$ . Thus the recursive projection operator captures both direct and indirect influence flows. Thus,  $\Lambda^{(n)}$  may be interpreted as the  $n$ -step neighborhood in the auction graph and as a dynamic closure of best-response behavior. Each expansion layer captures not just structural proximity but strategic influence—the transmission of incentive, information, and utility across the network. In this way, we convert local participation patterns into global influence propagation, formalized as graph-theoretic expansions over the projected structure of the strategy space.

## 4.5 Partial Ordering and Market Shifts

While the projection mappings  $\pi$ ,  $\varpi$  and their compositions produce index sets (subsets of  $\mathcal{B}$  or  $\mathcal{L}$ ), these sets are characterized by the underlying strategy space  $S(t)$ . These indices correspond directly to elements of the strategy space  $S(t)$ , which contains structured bid information  $s_i^j(t)$ . That is, the projection  $\varpi^{-1}(i, t)$  retrieves all bid tuples  $(i, j)$  in the index set, but equivalently defines the subspace of  $s_i(t)$  consisting of all bid array submitted by buyer  $i$  at time  $t$ .

Each element  $s^j(t) \in S^j(t)$  is a bid array, and the collection  $\pi \circ \varpi^{-1}(j, t)$ . Aside from being an index set of buyers, we have a set of bid arrays  $\{s_i^j(t)\}_{i \in \mathcal{B}^j(t)}$  that can be partially ordered by their prices  $p_{(n)}^j(t)$ . While the projection operators isolate buyers or sellers structurally, the *functional influence* between market participants is mediated through the comparison of bid prices.

This introduces a natural partial order among bidders for each seller at a fixed time, and we define a partial ordering on  $S^j(t) \subset S(t)$  by  $p_i^j(t) < p_k^j(t)$  if buyer  $i$  bids less than buyer  $k$  for the same seller  $j$ . Given the set of buyers  $\mathcal{B}^j(t) = \pi \circ \varpi^{-1}(j, t)$ , we may impose a partial order structure based on the associated price bids  $\{p_i^j(t)\}$ . This ordering determines which buyers are accepted by seller  $j$  (those with highest prices until the resource is exhausted), and the sensitivity of a potential equilibrium to shifts in bidding behavior.

As shown in [6], a market shift occurs precisely when a buyer outside of  $\mathcal{B}^j(t)$  improves their relative position in this order, causing  $\mathcal{B}^j(t)$  to be recomputed. Such shifts reflect structural changes in functional influence, as they alter the competitive hierarchy among bids and propagate through the multi-auction environment. Specifically, a market shift in auction  $j$  occurs when the partial order of bids at seller  $j$  changes in a way that affects allocation. From [6], two cases are critical:

1. **Demand Shortfall:** A buyer  $i \in \mathcal{B}^j(t)$  reduces their bid quantity so that total demand falls below available supply:

$$\sum_{i \in \mathcal{B}^j(t)} a_i^j(t) < Q^j(t).$$

The auction must recompute its reserve price or reallocate supply among remaining buyers.

2. **Bid Overtake:** A buyer  $i^* \notin \mathcal{B}^j(t)$  improves their valuation so that

$$p_{i^*}^j(t) < p_*^j(t),$$

where  $p_*^j(t)$  is the minimum accepted bid price at auction  $j$ . The buyer  $i^*$  displaces the marginal buyer, triggering a shift in  $\mathcal{B}^j(t)$ .

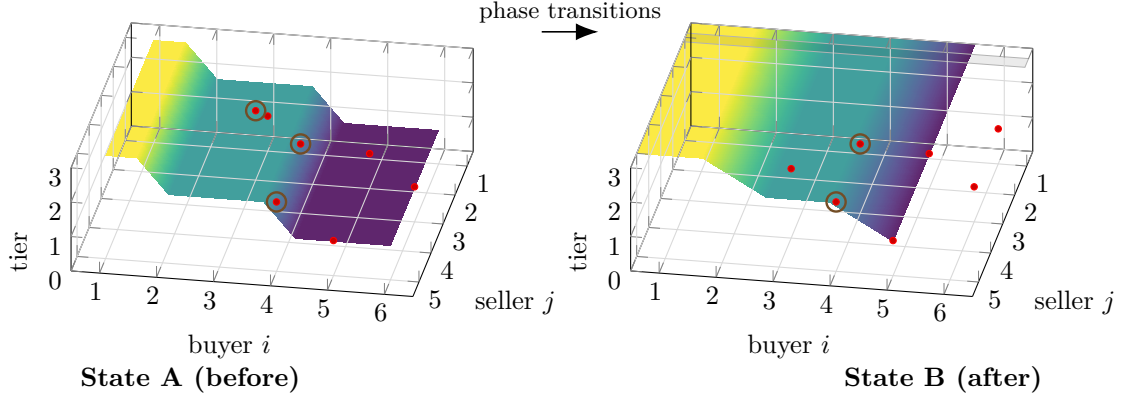
Either case changes the minimal winning price, breaking the partial ordering within the projected sets, and forces the auction to recompute  $\mathcal{B}^j(t)$ . As the seller frontier  $\Lambda_{\mathcal{L}}^{(1)}(j, t)$  consists of sellers connected to  $j$  through shared buyers, the reallocation thereby propagates influence through layers of the the expansion  $\Lambda_{\mathcal{L}}^{(n)}(j, t) = (\varpi \circ \pi^{-1} \circ \pi \circ \varpi^{-1})^n(j)$ .

## 5 Influence Shells and Local Saturation

The projection and ordering framework developed above allows us to describe how influence propagates through the auction network. We now ask under what conditions this propagation stabilizes. As buyers and sellers iteratively adjust bids, certain neighborhoods of the market reach a state in which no participant can improve their utility through unilateral deviation. These locally stable regions form influence shells—bounded subgraphs within which allocations, prices, and bid updates remain consistent under further iterations. When every buyer and seller in such a region satisfies this best-response property, the shell is said to be saturated. The following gives the formal notation and enumerates the assumptions that we have made in the generalization of influence sets as were defined in [6].

► **Definition 2 (Saturated Influence Shell).** A primary influence set  $\Lambda_{\mathcal{L}}^{(1)}(j, t)$  associated with seller  $j$  at time  $t$  is said to be saturated if no buyer or seller within this set can improve their utility by unilaterally altering their bids. Formally, for every buyer  $i \in \mathcal{B}^j(t)$  and every seller  $\ell \in \Lambda_{\mathcal{L}}^{(1)}(j, t)$ , the following holds,

$$u_i(t) \geq u_i'(t), \quad \text{for any feasible alternative strategy } s_i'(t).$$



■ **Figure 1** 3D matrix view: rows (buyers), columns (sellers), and  $z$  encodes price tiers. The colored surface shows the buyer price; filled markers are active bids; open circles show marginal winners. The right panel shows a transition where a new high-tier participant appears at  $j=1$ , a demand shortfall removes a low cell, and a reconfiguration shifts activity at  $j=2$ .

Global market equilibrium decomposes into interconnected saturated shells, each functioning as stable subsystems. We establish conditions for the existence of a saturated influence shell.

- (i.) **Countable and Locally Finite Graph.** The sets of buyers  $\mathcal{B}$  and sellers  $\mathcal{L}$  are at most countably infinite. Each participant engages in only finitely many transactions, ensuring finite degree at every instant. This guarantees that all projection maps  $(\pi, \varpi)$  encounter only finite fibres and that the influence operator (18) perform finite unions.  
Buyers and sellers participate in locally finite networks, enabling stable equilibrium convergence within compact, bounded strategy spaces. Market rules explicitly limit resources and interactions, ensuring finite dimensionality.
- (ii.) **Bounded Influence and Bids.** Influence propagation strength remains bounded, preventing divergence. Each seller's fixed endowment  $Q^j$  and each buyer's fixed demand cap  $Q_i$  are finite and time-independent. Hence every non-zero bid quantity  $q_i^j(t)$  lies in the compact interval  $[0, Q_i]$ , and every realized allocation is in  $[0, Q^j]$ .
- (iii.) **Partial Ordering Stability.** The partial ordering induced by the bid structure must satisfy stable bid threshold rankings for all relevant buyers and sellers within the shell, thus establishing clear marginal price tiers.

## 5.1 Ordering and Influence Propagation

We recall the ordering relationship from [6] that holds for any seller within a saturated primary influence set  $\Sigma := \Lambda_{\mathcal{L}}^{(1)}(j, t)$ .

► **Lemma 3** (Local Price Ladder [6, Thm. 2.3, proof]). *Let the market be at time  $t$  with seller  $j$  and its saturated primary influence set  $\Sigma := \Lambda_{\mathcal{L}}^{(1)}(j, t)$ . Pick any neighbor  $k \in \Sigma$  and two buyers*

$$i \in \mathcal{B}^j(t), \quad \ell \in \Lambda_{\mathcal{B}}^{(1)}(i, t) \setminus \mathcal{B}^j(t),$$

*i.e.,  $i$  bids on  $j$ ,  $k$  is another seller reached from  $i$ , and  $\ell$  is a buyer that bridges further to  $k$  but not to  $j$ .*

*If the shell  $\Lambda_{\mathcal{L}}^{(1)}(j, t)$  is saturated—no profitable deviation exists for any vertex in this set—because  $\ell$  does not bid on  $j$  while  $i$  bids on both  $j$  and  $k$ , the marginal thresholds must nest as follows*

$$\sup(\bar{p}^k(t), \underline{p}^k(t)) \leq p_\ell^*(t) < \inf(\bar{p}^j(t), \underline{p}^j(t)) \leq p_i^*(t), \quad (19)$$

where the marginal intervals are chosen as in (4), and in particular, every choice of reserve prices  $p_*^k(t) \in (\bar{p}^k(t), \underline{p}^k(t))$  and  $p_*^j(t) \in (\bar{p}^j(t), \underline{p}^j(t))$  satisfies

$$p_*^k(t) \leq p_\ell^*(t) < p_*^j(t) \leq p_i^*(t). \quad (20)$$

**Proof.** The argument for (20) is contained within the proof of [6, Thm. 2.3]. Left-to-right the chain reads

- (i.)  $p_*^k$  seller  $k$ 's reserve price;
- (ii.)  $p_\ell^*$  buyer  $\ell$ 's bid that clears the marginal tier in *both* auctions;
- (iii.)  $p_*^j$  seller  $j$ 's reserve price;
- (iv.)  $p_i^*$  the highest active bid of buyer  $i$  on seller  $j$ .

The left inequality holds because  $k$  clears at the minimum of the two buyers' bids; the strict middle inequality follows from  $\ell$  placing no bid on  $j$ ; the right inequality is enforced by buyer  $i$ 's cross-auction participation. In a saturated shell, any profitable deviation by  $k$  toward  $j$  or by  $i$  away from  $j$  is ruled out. Hence buyer  $\ell$ 's marginal valuation must lie weakly above all prices at which *newk* can clear its final unit, while buyer  $i$ 's valuation must lie weakly below all prices at which  $j$  can still clear. This forces the margin intervals to nest as in (19). Since  $p_*^k(t) \in (\bar{p}^k(t), \underline{p}^k(t))$  and  $p_*^j(t) \in (\bar{p}^j(t), \underline{p}^j(t))$  by the definition of the reserve price  $p_*^j$ , the pointwise ladder (20) follows immediately.  $\blacktriangleleft$

Economically, this implies that higher prices at neighboring sellers prevent buyers from deviating profitably, ensuring that no participant has an incentive to alter their bidding strategy unilaterally. Hence, the ordering captures a stable distribution of resources and prices, reflecting locally optimal market conditions.

### Weak (local) Monotonicity

The projected influence sets, together with the induced partial orders, thus form the dynamic framework for market evolution, where projections identify which vertices are connected, and partial orders determine how influence is transmitted via price shifts. Market shifts occur when the partial order structure is perturbed beyond certain thresholds, forcing recomputation of  $\mathcal{B}^j(t)$  or reserve prices.

► **Proposition 4** (Local Monotonicity). *Let  $\Sigma = \Lambda_{\mathcal{L}}^{(1)}(j, t)$  be the saturated one-hop shell of seller  $j$  at time  $t$ . For each seller  $k \in \Sigma$ , let the marginal intervals be chosen as in (4), and let the reserve price  $p_*^k(t)$  be selected inside the interval  $(\bar{p}^k(t), \underline{p}^k(t))$ .*

*Now, consider the vector of seller reserves and buyer marginal valuations restricted to  $\Sigma$ . Under elastic, strictly decreasing valuation functions  $\theta'_i(z)$  and any reserve-update rule that*

- (i) *selects  $p_*^k(t+1) \in (\bar{p}^k(t+1), \underline{p}^k(t+1))$  for each  $k \in \Sigma$ , and*
- (ii) *is nondecreasing in the bids  $\{p_i^k(t)\}_i$  and the previous reserve  $p_*^k(t)$ ,*

*the PSP price-update map*

$$p_*^k(t) \mapsto \tilde{p}_*^k(t+1) \quad \text{and} \quad p_i^k(t) \mapsto \tilde{p}_i^k(t+1) \quad (21)$$

*is locally monotone on  $\Sigma$ .*

**Proof.** Elasticity of the valuation functions implies that each buyer's marginal value  $p_i^*(t) = \theta'_i(z_i(t))$  is strictly decreasing in its own allocation. Increasing any bid or reserve inside the saturated shell  $\Sigma$  may raise some allocations and lower others, but it cannot create a reversal of

the bid ordering that defines the interval  $(\bar{p}^k(t), \underline{p}^k(t))$ . In particular, all ladder relations (20) remain invariant under such updates.

For sellers, the reserve-update rule is assumed nondecreasing in both the previous reserve  $p_*^k(t)$  and the local bids  $\{p_i^k(t)\}_i$ . Thus any componentwise increase in  $(p_*^k(t), p_i^k(t))$  cannot decrease any updated reserve  $p_*^k(t+1)$ . By construction, each updated reserve remains inside its interval  $(\bar{p}^k(t+1), \underline{p}^k(t+1))$ , and, because the shell  $\Sigma$  is saturated, these intervals evolve compatibly with the ordering relations and cannot induce a downward jump that violates the ladder.

Therefore, every coordinate of the updated pair  $(p_*^k(t+1), p_i^k(t+1))$  is weakly increasing in the corresponding coordinate of  $(p_*^k(t), p_i^k(t))$ . Hence the update rule satisfies the order-preserving property (21), and the PSP price-update map is locally monotone on  $\Sigma$ . ◀

The partial ordering structure induced by bidding behavior is essential in analyzing and predicting the direction and magnitude of market shifts resulting from influence dynamics. A local allocation triggers global bid adaptation, reinforcing that while seller auctions operate independently, buyer strategy space remains tightly coupled. In integrated markets (scarce supply), the partial orders are dense and tightly coupled, making markets highly sensitive and globally coordinated. In fragmented markets (abundant supply), the partial orders become sparse and disconnected, leading to localized equilibria and insulating submarkets from external shocks. Thus, the transition from integrated to fragmented equilibrium is not just a graph phenomenon—it is a transition in the connectivity of the partial order structure induced by bidding.

► **Remark 5** (On the ordering of the PSP price map). Although the local update map is written on the pair  $(p^j, p_i^j)$ , the PSP rule actually *acts* only on the first coordinate: the buyer's bid  $p_i^j$  is simply carried forward (or set to 0 if  $j \notin \mathcal{L}_i$ ).

## 5.2 Asynchronous Sellers and Coupled Buyers

To represent the fine-grained dynamics of bid selection and displacement within an auction round, we introduce an internal index  $\tau_k$  to describe local progression steps. Each  $\tau_k$  denotes a partial-ordering resolution event—an allocation decision at auction  $j$  followed by an update to the reserve price and potentially to the projected sets. The index acts as a local time variable inside the global iteration  $t$ , allowing us to separate micro-adjustments from round-to-round evolution.

► **Definition 6** (Allocation Step  $\tau_k$ ). *At each  $\tau_k$  within round  $t$ , seller  $j$  selects the highest bidder in  $\pi \circ \varpi^{-1}(j)$  not yet fulfilled, allocates a feasible amount  $a_i^j(\tau_k)$ , updates the reserve price  $p_*^j(\tau_{k+1})$ , and recomputes  $\mathcal{B}^j$  and  $\Lambda_{\mathcal{L}}(j)$  as needed.*

Each  $\tau_k$  inside a global round  $t$  is therefore a local ordering-resolution event: seller  $j$  picks the highest unfilled bidder, allocates a feasible amount  $a_i^j(\tau_k)$ , updates its reserve, and recomputes the bidder set  $\mathcal{B}^j(t)$  as well as the seller shell  $\Lambda_{\mathcal{L}}^{(1)}(j, t)$ . The sequence  $\tau_1, \tau_2, \dots$  terminates when no remaining buyer meets the current reserve or when supply is exhausted.

Sellers operate independently: each seller's  $\tau_k$  sequence proceeds without synchronization with others. However, buyers must maintain a consistent strategy across all sellers they bid on. Since the buyer's bid array  $\sigma_i(t)$  is defined jointly over  $\mathcal{L}_i(t)$ , any change to the outcome of one auction requires coordinated updates across all components. This coupling between independent seller threads through shared buyers produces the feedback mechanism responsible for market coherence. From a game-theoretic perspective, each seller executes a local best-response process, while each buyer enforces cross-auction consistency of marginal valuation.

We define the buyer update rule as

$$q_i^j(\tau_{k+1}) = Q_i(t) - \sum_{j' \in \mathcal{L}_i(t)} a_i^{j'}(\tau_k), \quad \forall j \in \mathcal{L}_i(t), \quad (22)$$

indicating that the buyer updates all bids simultaneously based on observed allocations. Equation (22) ensures that a buyer's total requested quantity never exceeds its available resource  $Q_i(t)$  and redistributes residual demand across the active seller set  $\mathcal{L}_i(t)$ . The rule formalizes how buyers translate local allocation feedback into revised offers, maintaining a form of budget balance across asynchronous auctions.

Because sellers run their  $\tau_k$  threads asynchronously while buyers must update all bids coherently according to (22), local price changes propagate through the partial orders defined above, layer by layer. Each seller's reserve adjustment initiates a chain of bid updates in the neighborhoods that share its buyers. This extends the projection-ordering framework, enabling intra-round modeling of bid dynamics, bid-induced reordering, and precise tracking of influence propagation through updates to the projected domains. In this sense, the  $\tau_k$  sequence acts as a micro-time resolution that reveals how local saturation unfolds inside each global auction round.

### Stability under asynchronous evolution

The following proposition shows that, even under these independent update threads, the relative ordering of bids remains stable and the local price ladder is preserved.

► **Proposition 7 (Saturated Shell).** *Let  $\Sigma = \Lambda_{\mathcal{L}}^{(1)}(j, t)$  be saturated at  $\tau_k$ . Assume that for every seller  $k \in \Sigma$  the reserve price lies in the interval,*

$$\bar{p}^k(\tau_k) < p_*^k(\tau_k) < \underline{p}^k(\tau_k),$$

*and that each  $\tau$ -update preserves all ladder relations (20) inside  $\Sigma$ . If a local resolution step  $\tau_k \rightarrow \tau_{k+1}$  modifies the strategy space only inside  $\Sigma$ , then:*

- (i) *The ladder (20) is preserved: no ordering reversal is possible, and every affected marginal or reserve price weakly increases; a strict increase occurs whenever the winning bid or reserve at some seller in  $\Sigma$  rises.*
- (ii) *If every buyer or seller that first appears in  $\Sigma_{n+1} \setminus \Sigma_n$  has no profitable deviation given the preserved ladder, then the expanded shell  $\Sigma_{n+1}$  is saturated.*

**Proof.** If a  $\tau$ -update reallocates quantity within  $\Sigma$ , the clearing price or reserve at the affected seller can only move upward within its margin interval. Because all reserves and winning bids in  $\Sigma$  lie inside their intervals  $(\bar{p}^k, \underline{p}^k)$ , no update can create a reversal of the ordering that defines the ladder (20). Thus each affected component moves weakly upward, and whenever the winning bid or reserve at some seller in  $\Sigma$  increases, at least one of the four prices in the ladder strictly increases.

Saturation implies every  $\tau$ -update inside  $\Sigma$  preserves best-response conditions given the ladder; hence extending  $\Sigma$  by including agents in  $\Sigma_{n+1} \setminus \Sigma_n$  yields a saturated larger shell whenever those newly added agents also have no profitable deviation under the same ordering. ◀

Saturation implies every  $\tau_k$  update inside  $\Sigma$  is either a demand-shortfall or bid-overtake event; both raise the marginal price they touch, propagating weakly upward along every chain of the form (20). A strict increase occurs whenever the winning bid or reserve at some auction in  $\Sigma$  is lifted. Thus, local “saturation” is a best-response property of a one-hop influence shell.

By an inductive test we extend saturation shell-by-shell.

► **Corollary 8.** *The new shell  $\Sigma_{n+1}$  inherits the price-ordering ladder (20): all its marginal prices are no smaller than those in  $\Sigma_n$ ; in particular,  $p_*^k(\tau_{k+1}) \geq p_*^k(\tau_k)$  for all  $k \in \Sigma_{n+1}$ .*

**Proof.** Consider any buyer-seller quadruple  $(k, \ell, j, i)$  whose seller  $k$  lies in the freshly revealed layer  $\Sigma_{n+1} \setminus \Sigma_n$ . Applying Proposition 4 to that quadruple shows that the ladder inequality (20)



is preserved and all four prices weakly increase. Repeating this argument for every such quadruple that touches  $\Sigma_{n+1}$  completes the extension of the monotone ladder one hop outward. For every edge  $(i, k)$  with  $k \in \Sigma_n$ , Proposition 7 guarantees that a local  $\tau$ -update cannot decrease the marginal price  $p_*^k$ . Hence  $(p_*^k)_{k \in \Sigma_n}$  is component-wise non-decreasing from  $\tau_k$  to  $\tau_{k+1}$ . ◀

The asynchronous update model developed above provides the conceptual foundation for our simulation studies. It captures the essential features of decentralized PSP dynamics: sellers acting independently on local information, buyers coordinating across overlapping auctions, and influence propagating through partially ordered interactions. In the following section, we use these principles to construct an event-driven simulation framework that allows us to observe how local saturation emerges in practice.

## 6 Simulation Framework and Implementation

This section summarizes the simulation code used to study the PSP markets with multiple sellers and buyers. The simulation architecture explores the practical realizations of decentralized coordination. The event-driven approach reproduces the iterative best-response behavior implied by the mechanism and allows examination of convergence properties, price dispersion, and efficiency loss due to network coupling.

Following Semret and Lazar [12], each buyer's valuation is given by a parabolic curve of the form

$$\theta_i(z) = \kappa_i(\bar{q}_i - z/2)z \quad \text{for } z \in [0, \bar{q}_i]$$

where  $\bar{q}_i$  represents the maximum quantity of goods desired and  $\kappa_i = \bar{p}_i/\bar{q}_i$  has dimensions marginal price per unit where  $\bar{p}_i$  is the maximum marginal value that buyer  $i$  would ever place on the resource.

### 6.1 Event-Driven Algorithm and Asynchronous Updates

The simulation operates as a discrete-time event system. Events are scheduled and processed in a priority queue, advancing the simulation clock  $t$  to the next event. Two event types exist:

1. **Buyer Compute:** Buyer  $i$  evaluates its local state, computes updated bids  $(z_i^j, p_i^j)$  on each connected seller  $j$ , and schedules bid events when meaningful changes occur.
2. **Post Bid:** Seller  $j$  clears its auction, applying second-price allocation and updating quantities, payments, and revenues.

Buyer and seller events may reschedule each other (e.g., clears triggered after meaningful bid changes). The loop halts when no effective changes remain or a step limit is reached. The simplified pseudocode is shown in Algorithm 1.

#### Algorithm 1 Event-driven PSP simulation

---

```

1: Initialize market state  $M$ ; schedule all buyers.
2: while queue not empty and not converged do
3:    $(t, \text{type}, \text{payload}) \leftarrow \text{pop}()$ 
4:   if type = BUYER_COMPUTE then
5:     Update  $(z_i^j, p_i^j)$  for buyer  $i$  on feasible links.
6:     Schedule POST_BID events for affected sellers.
7:   else if type = POST_BID then
8:     Seller  $j$  clears auction, enforcing  $Q^j$ , opponent ordering, and payments.
9:   end if
10: end while

```

---



Each buyer  $i$  computes a uniform (or per-seller) bid price using a valuation-based update  $w = \theta'_i(\sum_j z_i^j)$ . Quantities are apportioned across incident sellers using a local best-response step. Buyers are sorted by descending unit price  $p_{(n)}^j$ . The clearing process accumulates allocations until total demand equals available resource  $Q^j$ . The threshold price

$$p_*^j = \min\{p_i^j : \sum_{k: p_k^j \geq p_i^j} q_k^j \geq Q^j\} \quad (23)$$

identifies the seller's marginal (clearing) price.

For each seller  $j$ , the clear routine builds a partial ordering by posted marginal prices (bid prices), serves opponents until available resource  $Q^j$  is exhausted, and charges the price incurred by the externality of participation to all served buyers for that seller. The routine updates  $\mathbf{a}$ , seller revenue, and per-buyer costs. We define the set of active buyers with positive bids as

$$\mathcal{I}_j = \{i : q_i^j > 0, a_{ij} \in \mathbf{A}\}, \quad (24)$$

where  $\mathbf{A}$  represents the biadjacency matrix captures direct buyer-seller interactions:

$$\mathbf{A}(t) \in 0, 1^{|\mathcal{B}| \times |\mathcal{L}|},$$

where  $\mathbf{A}_{ij}(t) = 1$  if buyer  $i$  bids on seller  $j$  at time  $t$ , and  $\mathbf{A}_{ij}(t) = 0$  otherwise. Rows of  $\mathbf{A}(t)$  identify each buyer's active sellers; columns identify all buyers bidding on a given seller.

Experiments are conducted using randomized networks of  $I = |\mathcal{B}|$  buyers and  $J = |\mathcal{L}|$  sellers. The connectivity matrix  $\mathbf{A}_{ij}$  determines which buyers may interact with which sellers. Each run uses the following protocol:

1. Initialize market state  $M$  with parameters  $(I, J, Q^j, \varepsilon, \text{reserve})$ .
2. Assign buyer valuations  $(\bar{q}_i, \kappa_i)$  and budgets  $b_i$  from uniform ranges.
3. Generate random adjacency matrices with varying density (percentage of shared buyers).
4. Execute the event-driven simulation for a fixed iteration limit.
5. Record convergence statistics, prices, allocations, and revenues.

The event scheduler supports both deterministic and stochastic updates, allowing controlled comparison between synchronous and asynchronous dynamics.

### Experimental Setup.

Each experiment initializes a market with  $I$  buyers and  $J$  sellers. Seller capacities are fixed at  $Q_{\max} = [60.0, 40.0]$ , with buyers distributed across both sellers according to a connectivity percentage that varies from 0% (fully isolated) to 100% (fully connected) in increments of 10%. For each connectivity level, a base random seed (`base_seed = 20405008`) ensures reproducibility while allowing controlled stochastic variation across runs.

Buyers' valuation parameters are drawn independently from uniform ranges:

$$\bar{q}_i \sim \mathcal{U}(10, 60), \quad \kappa_i \sim \mathcal{U}(1.0, 3.5).$$

Noise and perturbation effects are controlled by  $\epsilon = 2.5$ . For each seed and connectivity level, the simulation executes until convergence, measuring clearing prices, allocations, and bid prices.

A sequence of derived seeds `seed = base_seed + s` is used for each connectivity level  $s$ , ensuring comparable random draws while preserving independence across runs.

## 6.2 Price Ladder Verification

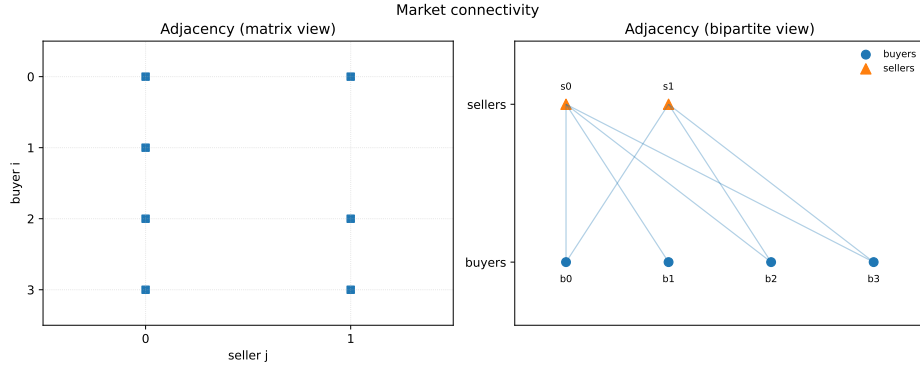
The simulation presented here focuses on verifying the *price ladder condition* across interconnected sellers. This experiment represents a localized instance of the broader PSP market, designed to test whether clearing prices obey a monotonic relationship when sellers share buyers through overlapping influence sets.

The experiment initializes a small market composed of two sellers ( $j = 1$  and  $\ell = 0$ ) and four buyers ( $i = 0, 1, 2, 3$ ). The adjacency structure allows some buyers to connect to both sellers, while others remain local. Sellers have distinct capacities,  $Q^1 = 8$  and  $Q^0 = 15$ , reflecting asymmetric market sizes. The buyer valuation and bid initialization follow:

$$\begin{aligned} (0, 1) : q = 8, p = 40, & & (0, 0) : q = 8, p = 40, \\ (1, 0) : q = 2, p = 4, & & (2, 0) : q = 6, p = 1. \end{aligned}$$

We have the connectivity of the market, in alignment with Lemma 3.

■ **Figure 2** Adjacency structure showing market connectivity between buyers and sellers.



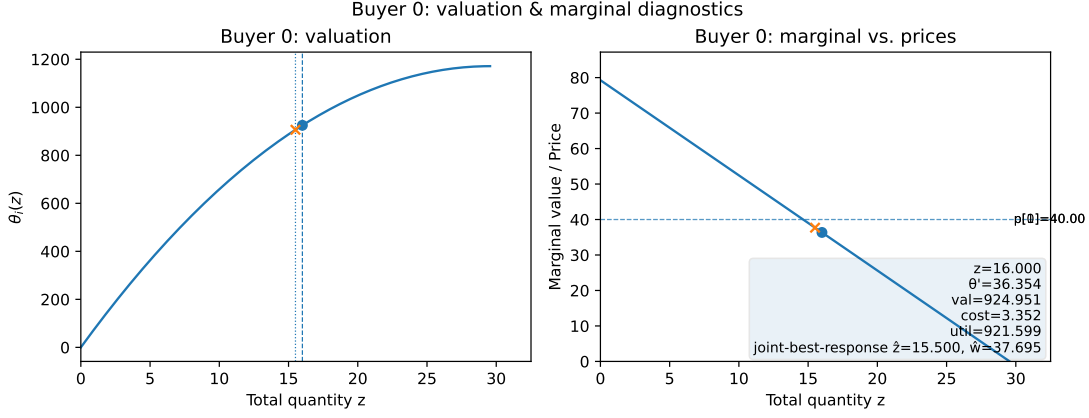
The resulting market has a highly skewed valuation distribution, allowing one buyer to dominate both sellers, while others form the marginal tiers that define second-price boundaries.

The algorithm scans all sellers and their one-hop neighbors to evaluate tuples  $(\ell, k, j, i)$  where Buyer  $i$  connects the two sellers. It tests the three inequalities defining the ladder ordering  $p_\ell^* < p_k < p_j^* \leq p_i$ . If these inequalities hold for all tuples, the market satisfies the monotone price ladder condition. Violations are reported with detailed tuple traces to aid in diagnosing market inconsistencies.

In this configuration, the ladder tuples satisfy all three inequalities, confirming a monotone relationship among clearing and bid prices. The system outputs a detailed report including, number of valid tuples and unique seller pairs  $(j, \ell)$ , margins between successive price tiers:  $(p_k - p_\ell^*)$ ,  $(p_j^* - p_k)$ , and  $(p_i - p_j^*)$ , and a summary of any violations detected. For this experiment, the output indicates no violations and consistent monotonicity, demonstrating that the PSP clearing mechanism maintains a globally ordered price structure when local competition and influence overlap exist.

This controlled experiment provides an analytical validation of the price ladder lemma in a simplified setting, and is intended to act as a unit test. It confirms that bid prices across connected sellers obey the expected inequalities implied Lemma 3. More generally, it shows that when buyers bridge multiple sellers, the second-price mechanism induces a coherent ordering of marginal prices, and provides an analytical tool for extending this verification to larger graphs. In the tuples

■ **Figure 3** Buyer 0 valuation curve and marginal diagnostics.



$(\ell, k, j, i) = (0, 1, 1, 0)$  and  $(0, 2, 1, 0)$  we observe

$$p_\ell^* = 1.0, \quad p_k \in \{4.0, 1.0\}, \quad p_j^* = 40.0, \quad p_i = 40.0.$$

Thus the high-tier buyer at seller  $j$  sits at the clearing price, while mid-tier competitors remain strictly below  $p_j^*$ . The reported margins  $(p_k - p_\ell^*) = 0$ ,  $(p_j^* - p_k) = 36$ , and  $(p_i - p_j^*) = 0$  reveal a wide central gap: a single dominant tier clears seller  $j$ , whereas seller  $\ell$  is anchored by low-tier participation at a much smaller price.

The monotone relationship validated here provides empirical confirmation of Lemma 3. The experiment illustrates how buyers bridging sellers stabilize the market through consistent price ordering, even when capacities and bid magnitudes differ substantially. The asymmetry in seller revenues and capacities demonstrates how equilibrium adapts to network structure, with high-valuation buyers dominating smaller auctions and lower-tier participants anchoring larger ones.

Our next experiment allows us to observe the propagation of equilibrium constraints across overlapping influence shells, offering empirical evidence for Propositions 7 of this paper.

### 6.3 Connectivity

Further experimental results are aggregated as functions of the overlap percentage between buyer–seller pairs, revealing how market interdependence affects stability, bid prices, and efficiency. The framework also enables sensitivity analysis under perturbations to parameters such as  $\varepsilon$ , budget distributions, and the structure of influence sets.

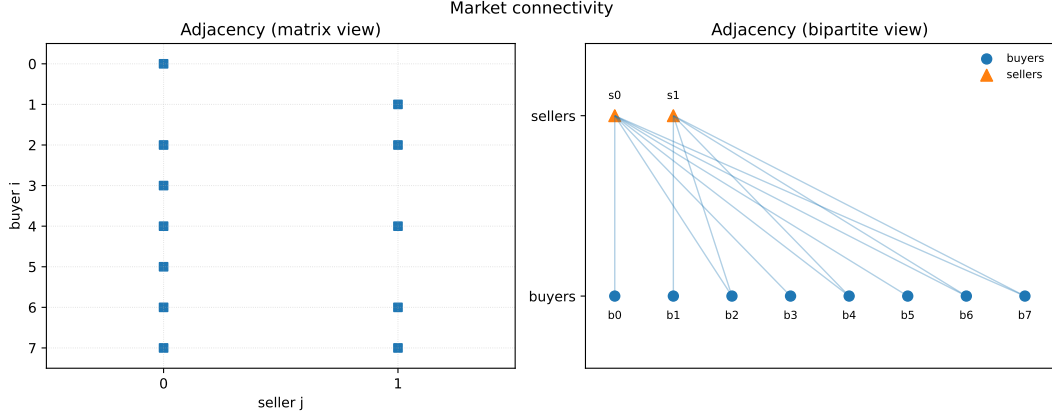
In this experiment, connectivity was gradually increased to observe how equilibrium formation and price alignment change as the market transitions from isolated to coupled seller networks. Starting from a sparse adjacency structure, buyers were allowed to participate in multiple auctions, creating overlaps that induced cross-seller influence and coupling of price dynamics.

Figure 4 illustrates the adjacency structure used in the experiment. Connectivity defines the feasible market domain  $I_{\text{active}}(t) \subset \mathcal{B} \times \mathcal{L}$  that bounds all strategic interactions.

Figure 5 shows the utility surface for a single buyer–seller pair, as was presented in [12], here buyer 6 and seller 0, plotted over bid quantity  $z_i^j$  and price  $w_i^j$ . The surface depicts the buyer’s instantaneous utility  $u_i(z_i^j, w_i^j) = \theta_i(z_i^j) - z_i^j w_i^j$  given the opponent bids and current market reserve. The concave ridge indicates the buyer’s optimal quantity at the current price

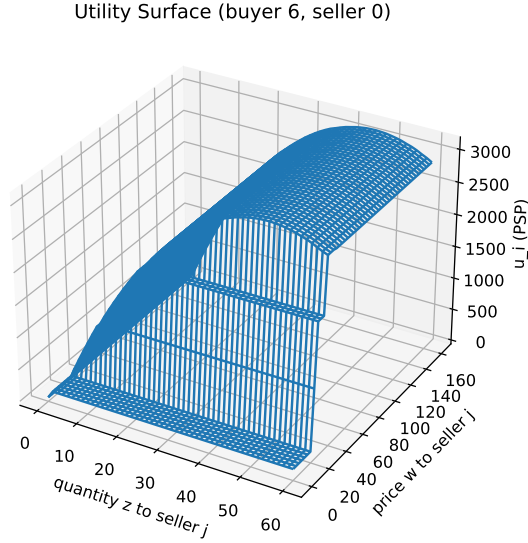
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■ **Figure 4** Adjacency and market connectivity for the  $8 \times 2$  experiment. Connectivity is set at 50%.



level, while the lower regions show diminishing returns and cost-dominated outcomes. We see a stable interior optimum: movements along the quantity axis correspond to allocation changes, whereas movements along the price axis reflect valuation gradients.

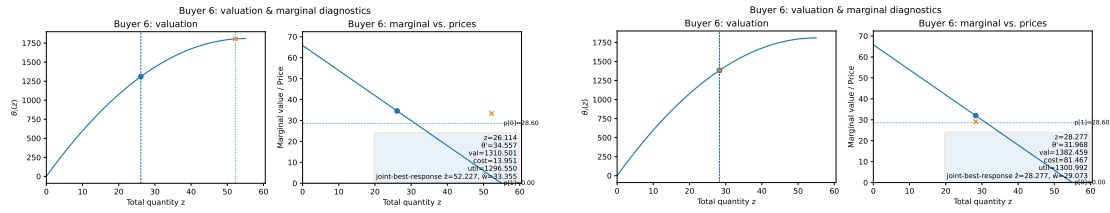
■ **Figure 5** Single buyer–seller utility surface for buyer 6 at seller 0. The surface plots  $u_i(z_i^j, w_i^j) = \theta_i(z_i^j) - z_i^j w_i^j$  over quantity  $z_i^j$  and unit price  $w_i^j$ , holding the opposing bids fixed at the snapshot.



Now, for Buyer 6 we evaluate the staircase  $P_i(\cdot, s_{-i})$ , compute  $(v_i, w_i)$ , perform the minimal-cost fill, and then read off the realized total  $\hat{z}_i$  and price  $p^* := P_i(\hat{z}_i, s_{-i})$ . The left panel shows  $(\hat{z}_i, \theta_i(\hat{z}_i))$  on the concave valuation curve; the right panel shows  $\theta'_i(z)$  together with the dashed price level  $p^*$ . In the runs shown, the valuation is quadratic,

$$\theta_i(z) \approx az - \frac{b}{2}z^2, \quad \theta'_i(z) = a - bz, \quad a \approx 66, \quad b \approx 1.1,$$

so marginal value declines approximately linearly from  $\theta'_i(0) \approx 66$  to near zero around  $z \approx 60$ .



(a) Constraint-limited: the joint best response lies on a feasibility boundary;  $\theta'_i(\hat{z}_i) > p^*$  so the buyer would expand if capacity at  $p^*$  were available.

(b) Interior optimum: here  $\theta'_i(z_i) > p$ , so the buyer would expand if possible. The joint-best-response is the point where  $\theta'_i(z) = p$ .

**Figure 6** Buyer-level diagnostics under the Progressive Second Price (PSP) joint best response. Each panel shows the valuation  $\theta_i(z)$  with the realized point  $(\hat{z}_i, \theta_i(\hat{z}_i))$  and the marginal curve  $\theta'_i(z)$  with a dashed line at  $p^*$ , illustrating the transition from constraint-limited to price-limited behavior along improvement paths.

Two sellers induce a two-step staircase in  $P_i$ ; the three snapshots correspond to marginal price levels near  $p^* \approx 32.1$  with  $\hat{z}_i \approx 28.2$  (interior),  $p^* \approx 28.6$  with  $\hat{z}_i \approx 28.3$  (price-limited), and a high-availability case with  $\hat{z}_i \approx 52.2$  where the buyer is constrained by feasibility at that price. These values are taken directly from the algorithm's output and no post-hoc smoothing is applied.

When the  $X$  marker in the marginal panel lies above the dashed line, the realized point satisfies  $\theta'_i(\hat{z}_i) > p^*$ ; the buyer would buy more at the prevailing price, but the minimal-cost fill has saturated feasible capacity at that price, so the joint best response is attained on a boundary of the feasible region rather than at marginal equality. When the marker sits on the dashed line,  $\theta'_i(\hat{z}_i) = p^*$  holds and the allocation is locally efficient; here the construction returns an interior maximizer of  $U_i(z) = \theta_i(z) - c_i(z)$ . When the marker lies below the dashed line,  $\theta'_i(\hat{z}_i) < p^*$  and any further increase in quantity would decrease utility; the best response is therefore at or near a participation boundary even though the valuation point on the left panel is well inside the curve.

**Table 2** Buyer regimes and their economic interpretation.

Regime	Relation	Economic meaning
Constraint-limited	$\theta'_i(z^*) > p^*$	Buyer wants more, but supply or externality restrict.
Equilibrium (interior)	$\theta'_i(z^*) = p^*$	Marginally efficient allocation — true local equilibrium.
Price-limited	$\theta'_i(z^*) < p^*$	Buyer withdraws — market too expensive.

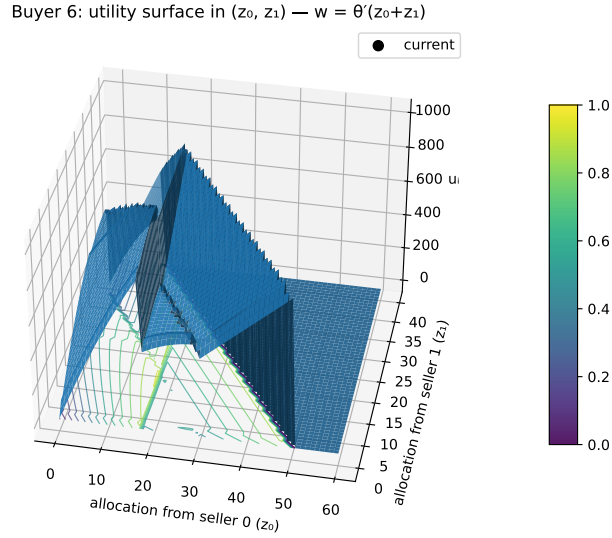
Figure 7 extends the analysis to the joint allocation space of the two sellers, each point on the surface corresponds to a feasible distribution across sellers 0 and 1 under a uniform price  $w = \theta'_i(z_{\text{total}})$ . The height of the surface indicates utility under the split given the opposing bids present in the snapshot. The ridge along constant  $z_{\text{total}}$  identifies the efficient split between sellers: solutions on the plateau indicate a local optimum; as the solution shifts below a ridge the buyer could improve utility by increasing its bid quantity, and the solution shifts toward the seller facing weaker opposing demand. Therefore, even with a uniform price tied to  $z_{\text{total}}$ , the allocation decision remains two-dimensional due to how opponent demand and residual available resource shape the intersection of feasible pricing and allocations.

To summarize our results, seller 0, with greater available resource ( $Q_{\max} = 60$ ), cleared at a slightly higher price  $p_0^* = 30.084$  than Seller 1 ( $p_1^* = 28.597$ ). Despite this asymmetry, both sellers exhibited similar expected revenues ( $E_0 = 44.76$ ,  $E_1 = 31.16$ ) and low variance, attributed to 40% of buyers participating in both markets. The shared influence among these buyers synchronized

■ **Table 3** Interpretation of ridges in the buyer's utility surface.

Ridge type	What it corresponds to
Sharp rise in $u_i$	A new seller step becomes available (increase in supply).
Sharp drop	Another buyer's bid dominates $\rightarrow$ PSP second-price step kicks in.
Plateau	Both sellers saturated or prices equalize (local equilibrium).

■ **Figure 7** Shared-seller utility surface where buyer 6's utility is a function of total requested quantity  $z_{\text{total}} = z_0 + z_1$ ;  $u_i(z_0, z_1) = \theta_i(z_{\text{total}}) - w(z_0, z_1)(z_{\text{total}})$  and  $w = \theta'(z_{\text{total}})$ : the feasible participation surface.



seller behavior, leading to a nearly uniform price surface.

Buyer-level data shows that bridging buyers—particularly buyers 6 and 7—maintained bids across both sellers with marginal valuations (32.118, 33.355) close to Seller 0's clearing price. Their dual participation enforced cross-market coherence, ensuring that no single auction could deviate significantly from the shared equilibrium. Buyers at the margins (e.g., 1 and 2) contributed to shaping intermediate prices, stabilizing the monotone progression across tuples.

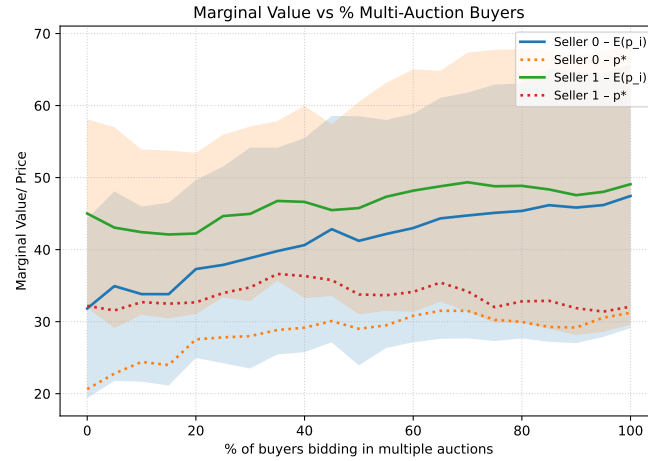
Furthermore, at each equilibrium or stopping point, the following quantities were collected,

$$E_j = \frac{1}{A_j} \sum_{i \in B^j} a_i^j p_i \quad \text{and} \quad V_j = \frac{1}{A_j} \sum_{i \in B^j} a_i^j (p_i - E_j)^2 \quad \text{where} \quad A_j = \sum_{i \in B^j} a_i^j$$

where  $E_j$  is the expected seller revenue,  $V_j$  is the variance of seller revenue across realizations. In addition, buyer classification (winners, zero allocation, opt-out behavior) and network statistics (fraction of shared buyers, influence overlap) are collected.

Figure 8 shows that as connectivity increases, both sellers exhibit convergence in marginal value and clearing price. Seller 0 maintains a higher marginal value throughout due to its larger available resource ( $Q_j = 60$ ), but the gap between sellers narrows with increasing overlap, confirming that multi-auction buyers mediate price synchronization across markets. The alignment of  $E(p_i)$  and  $p^*$  demonstrates how shared influence accelerates equilibrium formation and reduces price dispersion.

Overall, increasing connectivity transforms the market from a set of independent price islands



■ **Figure 8** Marginal value versus the percentage of buyers participating in multiple auctions.

into a unified utility surface. Sparse configurations produce local equilibria with greater variance between sellers, while denser networks encourage faster convergence through influence propagation. These findings validate the theoretical expectation that shared influence sets promote global coordination and equilibrium alignment.

### Statement on Supplementary Material

The code for the experiments presented in this paper can be found at:

- <https://drive.google.com/drive/folders/193g7ZbiIKGZRsutuuSB-fwf-jBnmYHo9>
- <https://github.com/jkblazek/TGDK-2025>

## 7 Conclusion and Future Work

This paper establishes a graph-theoretic framework for analyzing Progressive Second-Price (PSP) auctions, connecting decentralized market dynamics to structural properties of influence graphs. We formalized and expanded the concepts of influence sets and saturation, which together bound strategy spaces deterministically and ensure stable, truthful convergence in decentralized settings.

Our analysis relies on two levels of saturation, linked by the partial-ordering structure of bids; local saturation is a set-level best-response property. Establishing a formal fixed-point characterization of this process—perhaps using lattice or order-theoretic methods—remains an important direction for future work.

Our present analysis instead focuses on the constructive evolution of influence shells and the preservation of local monotonicity. Specifically, our approach demonstrates how recursive expansions of influence sets reveal market interactions across successive auction rounds. By introducing intra-round resolution via the  $\tau_k$  steps, we provide a finer-grained analytical tool to model the internal dynamics of auction iterations, clarifying the subtle interactions between buyer strategies and seller pricing rules.

The establishment of monotonicity in bid updates via induced partial ordering ensures stable, non-oscillatory convergence under realistic, elastic valuation conditions. Our framework provides a robust method to anticipate market shifts, characterize equilibrium thresholds, and ensure consistent propagation of influence across dynamic network topologies.



Future research will explore several promising extensions in reserve price optimization. Could there be an optimal coordinated reserve vector, chosen using buyer feedback, that upholds Lemma 3? We start by defining an admissible reserve price region, where for fixed  $t$ , the admissible set of reserve profiles is

$$R(t) = \{p_* \in \mathbb{R}^J : p_*^j \in (\bar{p}^j(t), \underline{p}^j(t)) \forall j, \text{ and Lemma 3 holds}\}.$$

Thus, we may determine the existence of *coordinated reserve profile*, where, given seller-side weights  $w_j \geq 0$  at time  $t$ , is defined as any  $p_*^{\text{coord}}(t) \in R(t)$  that maximizes

$$\Phi(p_*) = \sum_{j=1}^J w_j Q^j(t) p_*^j$$

over  $R(t)$ . We conjecture that at least one coordinated reserve profile  $p_*^{\text{coord}}(t)$  exists for every  $\epsilon > 0$ . Moreover, every such profile preserves the interval ladder inequalities (19) and hence is consistent with local saturation of primary influence shells.

Finally, integrating stochastic perturbations and noise into the PSP framework will deepen the realism of the model, allowing exploration of market resilience under uncertainty. Additionally, applying resistance distance [3] to the reserve profiles and diffusion-based influence models could yield deeper insights into influence propagation and market stability. Finally, empirical validation through simulation and real-world decentralized applications, such as spectrum and bandwidth auctions, will be critical to validate and refine theoretical predictions and improve practical mechanism design.

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## A

 Market Shift Revealed by Partial Ordering

► **Example 9** (Market Shift Revealed by Partial Ordering). In this example we model a simple reactive reserve update,

$$p_*^j(t+1) = \max \left\{ p_*^j(t), \max_{i \notin \mathcal{B}^j(t)} p_i^j(t) + \epsilon \right\},$$

with  $\epsilon > 0$  providing strict improvement for convergence. Thus the seller always keeps its internal bid strictly above the highest losing buyer, and the reserve price is nondecreasing in  $t$ . Alternative clearing-price rules that set  $p_*^j$  to the threshold  $\chi^j(t)$  are equivalent for our results.

Consider a PSP auction market with two sellers  $L_1, L_2$  and four buyers  $B_3, B_4, B_5, B_6$ . Initial buyer-seller connections are represented by the adjacency matrix:

$$\mathbf{A}^{(0)} = \begin{array}{c|cc} & L_1 & L_2 \\ \hline B_3 & 1 & 0 \\ B_4 & 1 & 1 \\ B_5 & 1 & 1 \\ B_6 & 0 & 1 \end{array}$$

### Auction Iteration $t = 1$

**Seller  $L_1$**  initially receives bids from  $B_3, B_4, B_5$ . Suppose initial bid prices are ordered as follows:

$$p_{B_3}^{L_1}(1) = 2.0 > p_{B_4}^{L_1}(1) = 1.8 > p_{B_5}^{L_1}(1) = 1.5.$$

Progressive allocation steps for  $L_1$ :

- $\tau_1$  :  $B_3$  allocated requested quantity, pays second-highest price 1.8. Seller updates reserve to  $1.8 + \epsilon$ .
- $\tau_2$  :  $B_4$  allocated next available quantity, pays 1.5. Reserve updates to  $1.5 + \epsilon$ .
- $\tau_3$  :  $B_5$  receives allocation, pays reserve ( $1.5 + \epsilon$ ).

**Seller  $L_2$**  has bids from  $B_4, B_5, B_6$ , with initial ordering:

$$p_{B_5}^{L_2}(1) = 1.9 > p_{B_4}^{L_2}(1) = 1.7 > p_{B_6}^{L_2}(1) = 1.4.$$

Progressive allocation steps for  $L_2$ :

- $\tau_1$  :  $B_5$  allocated, pays second-highest price 1.7. Reserve updates to  $1.7 + \epsilon$ .
- $\tau_2$  :  $B_4$  allocated next, pays 1.4. Reserve updates to  $1.4 + \epsilon$ .
- $\tau_3$  :  $B_6$  allocated, pays reserve ( $1.4 + \epsilon$ ).

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**Partial Ordering and Initial Influence Sets** Initially, influence projections:

$$\pi \circ \varpi^{-1}(L_1) = \{B_3, B_4, B_5\}, \quad \pi \circ \varpi^{-1}(L_2) = \{B_4, B_5, B_6\}$$

Both sellers share buyers  $B_4, B_5$ , forming an integrated influence structure.

**Market Shift at  $t = 2$ : Buyer  $B_4$  increases bid on  $L_1$**  Buyer  $B_4$  increases their bid on seller  $L_1$  to overtake  $B_3$ :

$$p_{B_4}^{L_1}(2) = p_{B_3}^{L_1}(1) + \delta, \quad \delta > 0.$$

This triggers an immediate, asynchronous allocation decision at seller  $L_1$ :

$\tau_1$  : Seller  $L_1$  allocates to the new highest bidder  $B_4$ , who pays the second-highest bid  $p_{B_3}^{L_1}(1)$ .

Reserve updates accordingly.

The new partial ordering on  $L_1$ :

$$p_{B_4}^{L_1}(2) > p_{B_3}^{L_1}(2) > p_{B_5}^{L_1}(2).$$

**Coupled Buyer Rebid** Buyer  $B_4$  observes this new allocation outcome and immediately updates their residual demand. Since buyers maintain consistent bid strategies across all sellers, buyer  $B_4$  must now adjust their bid quantity for seller  $L_2$  simultaneously:

$$\sigma_{B_4}^{L_2}(\tau_2) = Q_{B_4}(2) - a_{B_4}^{L_1}(\tau_1),$$

and submits this updated bid quantity at price  $p_{B_4}^{L_2}(2)$ .

Seller  $L_2$ , asynchronously and independently from  $L_1$ , now processes this rebid at its next local step:

$\tau_2$  : Seller  $L_2$  allocates quantity to buyer  $B_4$ , charging the next-highest price among competing bidders (e.g., buyer  $B_5$ 's previous bid).

**Propagation of Influence via Projection Mappings:** The shift at  $L_1$  updates the projection and partial ordering structure, immediately affecting the shared buyer set with seller  $L_2$ . The updated projections remain:

$$\begin{aligned} \pi \circ \varpi^{-1}(L_1) &= B_3, B_4, B_5, \\ \pi \circ \varpi^{-1}(L_2) &= B_4, B_5, B_6, \end{aligned}$$

but buyer  $B_4$ 's strategic rebid triggers a recomputation of reserve prices and rebidding decisions at  $L_2$ , influencing buyer allocations in subsequent  $\tau_k$  steps.

Thus, a local change in buyer  $B_4$ 's bid on one seller ( $L_1$ ) creates a cascading effect through the partial ordering structure, inducing market shifts and influencing allocation outcomes on another seller ( $L_2$ ). The explicit recomputation of partial orderings demonstrates clearly how strategic perturbations propagate through interconnected auction markets.