On the core of the Shapley-Scarf housing market model with full preferences*

Jun Zhang†

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Abstract

We examine core concepts in the classical model of Shapley and Scarf (1974) under full preferences. Among the standard notions, the strong core may be empty, whereas the weak core, though always nonempty, can be overly large and include unreasonable allocations. Our main findings are: (1) The exclusion core of Balbuzanov and Kotowski (2019)—a recent concept shown to outperform standard cores in complex environments under strict preferences—can also be empty. We establish a necessary and sufficient condition for its nonemptiness, showing that it is more often nonempty than the strong core. (2) We introduce two new core concepts, built on the exclusion core and the strong core respectively, by refining the assumptions on how indifferent agents may block. Both are nonempty and Pareto efficient, and coincide with the strong core whenever the latter is nonempty. (3) These core concepts are ordered by set inclusion, with the strong core as the smallest and the weak core as the largest.

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†Institute for Social and Economic Research, Nanjing Audit University. Email: zhangjun404@gmail.com

1 Introduction

Understanding market design models from a cooperative perspective has been a tradition in the field, dating back to the seminal contributions of Gale and Shapley (1962) and Shapley and Scarf (1974). Gale and Shapley (1962) introduce the concept of stable matching in the two-sided marriage model and establish its existence. Shapley and Scarf (1974) introduce the one-sided housing market model and prove the nonemptiness of the core, latter referred to as the *weak core*. The weak core is based on strong domination and may include Pareto inefficient allocations. In contrast, the *strong core*, defined via weak domination, has been favored in the field because it guarantees Pareto efficiency, a key desideratum in many market design environments. However, the strong core may be empty when agents have weak preferences. Since the housing market model has been extensively studied and played a foundational role in modeling various market design problems, this paper investigates these issues and seeks appropriate core concepts that ensure both nonemptiness and Pareto efficiency in the model.

The weak core and the strong core differ in their assumptions on the blocking behavior of indifferent agents. In the weak core, an agent joins a blocking coalition only if he strictly benefits from the deviation, whereas the strong core permits indifferent agents to participate. This difference is inconsequential in transferable utility environments, since transfers within a blocking coalition can make all members strictly better off. However, it becomes crucial in market design settings where transfers are typically prohibited. Allowing indifferent agents to join blocking coalitions is essential for achieving Pareto efficiency, yet doing so may render the strong core empty. Because Pareto efficiency is central in market design, the literature often assumes strict preferences, under which the strong core is nonempty and includes a unique element in the housing market model, and can be computed via the top trading cycle (TTC) mechanism. Nevertheless, weak preferences are pervasive in practice and have been considered since the original work of Shapley and Scarf (1974). For example, in kidney exchange, it is standard to assume that patients have weak preferences over compatible kidneys. In allocation problems with multi-unit objects, agents are naturally indifferent among identical copies of the same object. Additional arguments for studying weak preferences can be found in Bogomolnaia and Moulin (2004), Bogomolnaia et al. (2005), and Erdil and Ergin (2017), among others. We present the housing market model in Section 2 and define standard core concepts in Section 3.1.

We approach the problem by examining the behavioral foundations of coalition formation under standard core concepts. The weak core adopts the standard economics assumption that agents are self-interested. In contrast, the strong core assumes that agents are "altruistic": they may join a blocking coalition to benefit others as long as they are not harmed. This interpretation, however, is controversial, since indifferent agents may help someones in the coalition while potentially harming outside agents. We note that the notion of *exclusion rights*, recently introduced by Balbuzanov and Kotowski (2019) (hereafter BK), offers a new and more compelling interpretation. In the housing market model, a group of agents directly controls their endowments and holds the right to evict others who occupy their endowments. By threatening eviction, the group gains indirect control over those agents' endowments. Leveraging these indirect control rights, the group can expand its control over more objects. Under strict preferences, in any blocking coalition that does not include redundant indifferent agents,[1] the object received by each indifferent agent must be directly or indirectly controlled by the strictly better-off agents. Thus, indifferent agents can be viewed as being compelled to join the coalition under the threat imposed by the better-off agents. This insight motivates us to build on exclusion rights to understand coalition formation under weak preferences.

We first examine the *exclusion core*, a core concept based on exclusion rights, which is originally defined by BK for their general model that subsumes the housing market model as a special case. The exclusion core consists of allocations in which no coalition can strictly benefit all of its members by evicting others from the objects they control. BK show that, under strict preferences, the exclusion core is more effective than the strong core in eliminating unintuitive allocations in their model, and the two cores coincide in the housing market model. BK do not address weak preferences. Thus, we examine this concept as a potential new solution in the housing market model under weak preferences.

Our examination of the exclusion core under weak preferences is new to the literature. We obtain the following results. First, it no longer coincides with the strong core and may also be empty. Second, we identify a necessary and sufficient condition for its nonemptiness (Proposition 1), which is strictly weaker than the counterpart for the strong core (Quint and Wako, 2004). So, the exclusion core is more often nonempty than the strong core. Moreover, whenever the strong core is nonempty, they coincide. Finally, the exclusion core fails to satisfy an intuitive property we call *equivalence-closedness* (Proposition 2), which requires that, for any two allocations that are indifferent to all agents, if one belongs to a solution, the other also belongs to the solution. This property is intuitive, as agents' blocking decisions should depend only on their welfare. Both the strong core and the weak core satisfy this property. These results are presented in Section 3.2.

The inadequacy of the exclusion core suggests us to reevaluate its definition under

[1] In a blocking coalition, a group of indifferent agents is redundant if they allocate their endowments among themselves and thus contribute nothing to the others in the coalition.

weak preferences, which leads to our revision of the concept in Section 4. Our first observation is that the notion of control rights needs to be revised under weak preferences. In BK's definition, if an agent receives an object owned by a group, the group indirectly controls the agent's endowment via threatening to evict him from that object. This threat is credible under strict preferences, since the agent cannot find other indifferent objects. Under weak preferences, however, since the agent may view several objects as indifferent, the threat becomes not credible if there exist other indifferent objects not owned by the group. We therefore redefine control rights as follows: a group directly controls their own endowments, and indirectly controls another agent's endowment if and only if all objects that the agent views as indifferent to his assignment are controlled by the group.

Our second observation is that the exercise of exclusion rights under weak preferences is more subtle than under strict preferences, which requires the inclusion of indifferent agents in blocking coalitions. This is in contrast to BK's original definition, which precludes indifferent agents from joining blocking coalitions, to ensure the nonemptiness of the exclusion core in their model. In the housing market model, under weak preferences, subtle cases arise when a group of agents I does not control another group J's endowments but can still effectively evict others from J's endowments. In these cases, all objects that J views as indifferent to their current assignments but are not owned by J are owned by I. In one case, if I evicts J from I's endowments, J is able to revert to their own endowments to maintain their welfare, which effectively allows I to indirectly evict others from J's endowments. In another case, if I evicts J from I's endowments, J is unable to maintain their welfare by reverting to their own endowments. That is, J relies on I's endowments to maintain their welfare. This dependence gives I leverage to compel J to join their coalition. Our new definition unifies these cases by requiring that, if a blocking coalition includes indifferent agents, then, for each indifferent agent, all indifferent objects not owned by their group must be exclusively owned by the group of strictly better-off agents in the coalition. Equivalently, all indifferent objects for each indifferent agent in the coalition are owned by the coalition.

Our first solution concept, the *rectified exclusion core*, requires that if a coalition blocks an allocation via another one, then (1) at least one member of the coalition is strictly better off and no member is worse off, (2) every agent harmed by the blocking must be evicted from an object controlled by the coalition, and (3) if the coalition includes indifferent agents, all indifferent objects for indifferent agents must be owned by the coalition. This definition differs from the exclusion core in that we redefine controlled objects and allow indifferent agents to join a blocking coalition under condition (3). The rectified exclusion core is nonempty, Pareto efficient, and equivalence-closed (Theorem 1). It is a superset of the exclusion core and a subset of the weak core, and coincides with the strong core whenever the strong core is nonempty. To prove its nonemptiness, we propose a generalization of TTC to find its elements (Lemma 1). Our algorithm follows the procedures of the existing generalizations of TTC in the literature (Alcalde-Unzu and Molis, 2011; Jaramillo and Manjunath, 2012; Aziz and De Keijzer, 2012; Saban and Sethuraman, 2013; Plaxton, 2013; Ahmad, 2021). To preserve strategy-proofness of TTC, these algorithms need to carefully select pointing rules for agents. Since our purpose is only to find elements of our solution, we do not select a specific pointing rule, only requiring that at least one trading cycle is generated in each step. This implies that the outcomes of these existing algorithms all belong to the rectified exclusion core. Previously, the literature only shows that the outcomes of these algorithms belong to the weak core and are in the strong core when the strong core is nonempty.

The notion of exclusion rights is a useful conceptual tool for understanding coalition formation under weak preferences, which forms the basis for our modification of the exclusion core. However, different from the exclusion core and our modification that view endowments as a distribution of exclusion rights, standard core concepts view endowments as something that agents must consume by themselves or exchange with others. So, they require that a blocking coalition must deviate by reallocating their endowments among themselves. By imposing this requirement while preserving the above condition for indifferent agents in a blocking coalition, we introduce another solution concept, the *rectified strong core*, in Section 5. It differs from the strong core in one critical aspect: while the strong core assumes that a *single* indifferent object suffices to justify the participation of an indifferent agent in a blocking coalition, the rectified strong core requires that *all* indifferent objects for the agent must be owned by the coalition. This distinction vanishes under strict preferences, but under weak preferences, it reflects opposing extreme assumptions on the blocking behavior of indifferent agents. The rectified strong core is nonempty, Pareto efficient, and equivalence-closed (Proposition 3). It is a superset of the rectified exclusion core and a subset of the weak core. It coincides with the strong core whenever the latter is nonempty.

Thus, we obtain the following relationships between the various core concepts:

$$\text{Strong core} \subseteq \text{Exclusion core} \subseteq \text{Rectified exclusion core} \subseteq \text{Rectified strong core} \subseteq \text{Weak core}.$$

Except for the weak core, all others coincide with the strong core whenever the latter is nonempty. In particular, they coincide under strict preferences.

We further examine a special case of our model in Section 6, where agents' indifferent preferences arise from the existence of multiple copies of objects. So, an agent views any two objects as indifferent if and only if all agents view the two objects as indifferent. In this special model, we show that the strong core may still be empty, whereas the exclusion core is nonempty and coincides with the set of outcomes of TTC in an artificial priority-based allocation model where each object ranks its owners by an arbitrary order. However, in this special model, the exclusion core is still not equivalence-closed. So, the rectified exclusion core can still be strictly larger than the exclusion core.

We conclude the paper in Section 7 by discussing other potential solution concepts. First, Wako et al. (2007) have shown that, in general, the von Neumann-Morgenstern (vNM) stable set, defined based on either strong domination or weak domination, does not exist in the housing market model under weak preferences. Second, while Demuynck et al. (2019) have shown that their myopic stable set, defined based on weak domination, coincides with the strong core in the housing market model under strict preferences, we show that, under weak preferences, this solution may include Pareto inefficient allocations. Finally, although the bargaining set introduced by Yılmaz and Yılmaz (2022) is nonempty, Pareto efficient, and lies between the strong core and the weak core, we show that it can be strictly larger than the strong core when the latter is nonempty. So, it may include unintuitive allocations when obviously intuitive allocations exist.

Related literature. The housing market model has been extensively studied. Although Shapley and Scarf (1974) and Roth and Postlewaite (1977), the two seminal papers on this model, both consider full preferences, follow-up papers have mostly focused on strict preferences; see Afacan et al. (2024) for a survey. Among the papers that have addressed weak preferences, some focus on generalizing TTC to maintain its Pareto efficiency and strategy-proofness properties that hold under strict preferences, and others are interested in the set of competitive allocations, because it is nonempty and lies between the strong core and the weak core. However, we do not think the set of competitive allocations is a desirable solution, since it often includes Pareto inefficient elements, can be strictly larger than the strong core when the latter is nonempty, and has anomalies that have been pointed out by Roth and Postlewaite (1977).[2]

Specifically, Wako (1984) proves that a nonempty strong core may be a strict subset of the set of competitive allocations. Wako (1991) proves that the strong core coincides with the set of competitive allocations if and only if any two competitive allocations are indifferent to all agents. Quint and Wako (2004) provide a necessary and sufficient condition for the strong core to be nonempty, which implies that a nonempty strong core is

[2] In some markets, all Pareto efficient allocations in the weak core are not competitive, all competitive allocations are not Pareto efficient, and there exist two allocations that are indifferent to all agents, but one is competitive yet the other is not.

essentially single-valued. Biró et al. (2023) prove that the strong core satisfies the respect-improvement property if being nonempty, and the set of competitive allocations satisfies the property in a stochastic-dominance sense.[3]

Several papers have provided cooperative foundations for the set of competitive allocations. Wako (1999) shows that this set coincides with a modification of the strong core based on antisymmetric weak domination.[4] Toda (1997) shows that this set is the unique vNM stable set based on Wako's antisymmetric weak domination. Kawasaki (2010) and Klaus et al. (2010) respectively prove that this set is the unique vNM stable set based on a farsighted version of Wako's antisymmetric weak domination, or based on a farsighted version of strong domination if no agent is indifferent between his endowment and any other object. In these results, additional conditions must be imposed on the domination relation or on agents' preferences because a vNM-stable set generally does not exist (Wako et al., 2007). Ehlers (2004) shows that the strong core is not Nash implementable and the minimal monotonic extension of the strong core is the set of competitive allocations.[5]

Alcalde-Unzu and Molis (2011) and Jaramillo and Manjunath (2012) independently propose the first generalizations of TTC to full preferences that preserve Pareto efficiency and strategy-proofness. Aziz and De Keijzer (2012) study the computational complexity of these algorithms, and Plaxton (2013) proposes another algorithm with enhanced computational efficiency. Saban and Sethuraman (2013) unify these generalizations by defining a family of algorithms and derive sufficient conditions on pointing rules to preserve strategy-proofness. Ahmad (2021) study weak group strategy-proofness of these algorithms. Our generalization of TTC to find elements of our proposed cores follow the common formats of these existing algorithms. But since we do not pursue strategy-proofness, our description shows more flexibility in selecting pointing rules.

There is a literature studying standard cores in the housing market model with externalities (Mumcu and Saglam, 2007; Graziano et al., 2020; Hong and Park, 2022; Doğan et al., 2011; Aslan and Lainé, 2020; Klaus and Meo, 2023). It shows that even the weak core can be empty under unrestricted strict preferences. So, it typically focuses on restricted preference domains on which the strong core or other solutions are nonempty.

Finally, following BK, a few papers have further studied the exclusion core. Balbuzanov and Kotowski (2024) generalize their concept from their original model to a more general one that incorporates production. Ishida and Park (2025) study a special

[3] The respect-improvement property requires that an agent must receive a weakly better assignment when his endowment becomes more desirable for others.

[4] Indifferent agents in a blocking coalition must receive the same object in the old and new allocations

[5] A monotonic extension of the strong core is a superset of the strong core that satisfies Maskin's Monotonicty, which is necessary for Nash implementation.

case of BK's model, where agents are partitioned into groups and each group collectively owns a partitioned group of objects, and show that the strong core and the exclusion core have a nonempty intersection, though they do not include each other. Zhang (2025) studies refinements of the exclusion core in BK's model. All those papers assume strict preferences. In future research we may explore how to employ the approach in this paper to address weak preferences in those papers.

2 The housing market model

In the housing market model, a market is represented by a tuple $\Gamma = (I, O, \omega, \succsim_I)$, where I is a finite set of agents, O is a finite set of objects, with $|O| = |I|$, ω is a one-to-one mapping from I to O, and $\succsim_I = (\succsim_i)_{i \in I}$ is a preference profile of agents. For each $i \in I$, $\omega(i)$ is the object owned by i. Each i has a preference relation $\succsim_i$ on O, which is transitive and complete but might not be strict. For any distinct $o, o' \in O$, if $o \succsim_i o'$ but $o' \not\succsim_i o$, then i strictly prefers o over o', denoted by $o \succ_i o'$; if $o \succsim_i o'$ and $o' \succsim_i o$, then i is indifferent between o and o', denoted by $o \sim_i o'$. An agent i **accepts** an object o if $o \succsim_i \omega(i)$. Every nonempty $C \subseteq I$ is called a **coalition**. A coalition C' is a **subcoalition** of C if $C' \subseteq C$. For convenience, we often denote a market by $M(\omega, \succsim_I)$.

An **allocation** is a one-to-one mapping μ from I to O where $\mu(i)$ denotes the object received by i. An allocation μ is **individually rational** if, $\forall i \in I$, $\mu(i) \succsim_i \omega(i)$. Two allocations μ and μ' are **(welfare) equivalent** if, $\forall i \in I$, $\mu(i) \sim_i \mu'(i)$. An allocation σ is a **Pareto improvement** over μ for C if, $\forall i \in C$, $\sigma(i) \succsim_i \mu(i)$, and for some $j \in C$, $\sigma(j) \succ_j \mu(j)$. An allocation is **Pareto efficient** if it does not have a Pareto improvement for all agents.

We define some useful notations. Given any $i \in I$, for every $O' \subseteq O$, let $B_i(O') = \{o \in O' : o \succsim_i o' \text{ for all } o' \in O'\}$, the set of i's favorite objects among O'. For every $o \in O$, let $\mathcal{I}_i(o) = \{o' \in O : o' \sim_i o\}$, the set of objects i views as indifferent with o. For any coalition C and any allocation μ, let $\mu(C) = \cup_{i \in C}\{\mu(i)\}$, the set of objects assigned to C. Then, $\omega(C)$ is the set of objects owned by C. For any two allocations μ and σ, we call an agent **unaffected** from μ to σ if he receives indifferent objects in the two allocations. Then, we let $C_{\sigma \succ \mu} = \{i \in C : \sigma(i) \succ_i \mu(i)\}$ and $C_{\sigma \sim \mu} = \{i \in C : \sigma(i) \sim_i \mu(i)\}$, respectively the set of members of C who become strictly better off from μ to σ and the set of members of C whose welfare remains unaffected.

Let $\mathcal{E}$ denote the set of markets. For each $\Gamma \in \mathcal{E}$, let $\mathcal{A}(\Gamma)$ denote the set of allocations. Let $2^{\mathcal{A}(\Gamma)}$ denote the power set of $\mathcal{A}(\Gamma)$. A **solution** is a correspondence $f : \mathcal{E} \to \bigcup_{\Gamma \in \mathcal{E}} 2^{\mathcal{A}(\Gamma)}$ such that, for every $\Gamma \in \mathcal{E}$, $f(\Gamma) \in 2^{\mathcal{A}(\Gamma)}$. We allow $f(\Gamma)$ to be empty for some Γ. A solution f is *Pareto efficient* if, for every $\Gamma \in \mathcal{E}$, if $f(\Gamma)$ is nonempty, all elements of $f(\Gamma)$ are Pareto efficient allocations in Γ.

We introduce the following property as a criterion to assess solution concepts. It requires that a solution is "closed" regarding equivalent allocations: if one allocation belongs to the solution, any other equivalent allocation also belongs to the solution. This property appears intuitive for coalition blocking, as agents should base their blocking decisions solely on their welfare.

Definition 1. *A solution f is **equivalence-closed** if, for any market Γ and any two equivalent allocations μ and μ' in Γ, $\mu \in f(\Gamma)$ if and only if $\mu' \in f(\Gamma)$.*

3 Existing solutions: standard core and exclusion core

3.1 Strong core and weak core

In the standard definition, a coalition blocks an allocation if its members can benefit from a reallocation of their endowments among themselves. Depending on whether all members must be strictly better off in the deviation, the definition has two variants.

Definition 2. *In a market $M(\omega, \succsim_I)$, an allocation μ is **weakly blocked** by a coalition C via another allocation σ if*

1. *$\forall i \in C$, $\sigma(i) \succsim_i \mu(i)$ and $\exists j \in C$, $\sigma(j) \succ_j \mu(j)$;*
2. *$\sigma(C) = \omega(C)$.*

*The **strong core** consists of allocations that are not weakly blocked.*

*In the above definition, if $\forall i \in C$, $\sigma(i) \succ_i \mu(i)$, then we say that μ is **strongly blocked** by C via σ. The **weak core** consists of allocations that are not strongly blocked.*

When an allocation is weakly (strongly) blocked by a coalition via another allocation, we say that the first allocation is weakly (strongly) dominated by the second.

The strong core is a subset of the weak core. Under strict preferences, the strong core is nonempty and a singleton. Under weak preferences, however, the strong core may be empty, but if nonempty, it must be Pareto efficient. No matter preferences are strict or not, the weak core is nonempty but may not be Pareto efficient. This difference arises because unaffected agents are excluded from strong blocking coalitions, yet their participation is necessary for ensuring Pareto efficiency. See the following example for an illustration.

	1	2	3
ω:	a	b	c
μ:	b	a	c
σ:	c	a	b
σ':	a	c	b

(a) Allocations

$\succsim_1$	$\succsim_2$	$\succsim_3$
c	a	b
b	b	c
a	c	a

(b) $\succsim_I$

$\succsim'_1$	$\succsim'_2$	$\succsim_3$
b	a,b,c	b
a		c
c		a

(c) $\succsim'_I$

Example 1 (Weak core and Strong core)**.** *Three agents* $1,2,3$ *respectively own three objects* a,b,c*. In the following tables, the left one lists agents' endowments and the allocations under our examination. The other two tables represent two preference profiles.*

*(**Strict preferences**) In* $\succsim_I$*, since agents' favorite objects are distinct,* σ *is the unique Pareto efficient allocation. The weak core* $= \{\mu,\sigma\}$*, while the strong core* $= \{\sigma\}$*.* μ *is weakly blocked by* $\{1,2,3\}$ *via* σ*, and* 2 *is unaffected from* μ *to* σ*. In contrast,* μ *is not strongly blocked because, given that* 2 *cannot be made strictly better off and thus cannot join any blocking coalition,* $\{1,3\}$ *have no rights to reallocate* b*.*

*(**Weak preferences**) In* $\succsim'_I$*, both* 1 *and* 3 *most prefer* 2*'s endowment and then prefer their own endowments, while* 2 *is indifferent between all objects. Thus,* μ *and* σ' *are the individually rational and Pareto efficient allocations for* $\succsim'_I$*. In* μ*,* 1 *and* 2 *exchange endowments, while in* σ'*,* 3 *and* 2 *exchange endowments.*

The weak core $= \{\omega,\mu,\sigma'\}$*.* ω *is not Pareto efficient, but it is not strongly blocked because* 2 *cannot be made strictly better off and thus cannot join any blocking coalition.*

In contrast, the strong core $= \emptyset$*.* μ *and* σ' *are the only candidates for the strong core. But* μ *is weakly blocked by* $\{2,3\}$ *via* σ'*, while* σ' *is weakly blocked by* $\{1,2\}$ *via* μ*. Because* 2 *is indifferent between* a *and* c*,* 2 *alternates between* 1 *and* 3 *in forming weak blocking coalitions.*

Under weak preferences, Quint and Wako (2004) provide a necessary and sufficient condition for the strong core to be nonempty. Their condition uses the following algorithm to partition agents into groups.

Partition by minimal self-mapped sets (PMSS)

- **Step** $t \geq 1$: Denote the set of remaining agents by V_t. Generate a directed graph (V_t,Γ_t) in which every $i \in V_t$, represented by a node, points to all of the other agents who own i's most preferred objects among the remaining. Denote by $\Gamma_t(i)$ the set of agents pointed by i. A set of agents is called a **minimal self-mapped set**, denoted by T_t, if $\cup_{i\in T_t}\Gamma_t(i) = T_t$ and there does not exist $\emptyset \neq T' \subsetneq T_t$ such that $\cup_{i\in T'}\Gamma_t(i) = T'$. Find a minimal self-mapped set T_t. Let $V_{t+1} = V_t \backslash T_t$. If $V_{t+1} = \emptyset$, stop the algorithm; otherwise, go to the next step.

A minimal self-mapped set is a group of agents for whom all of their most preferred objects are owned by group, and no strict subset of the group satisfies the same condition. Given a market, let t^* denote the last step of PMSS and $T^* = (T_1, T_2, \dots, T_{t^*})$ denote the sequence of minimal self-mapped sets generated in PMSS.

Definition 3 (Quint and Wako, 2004)**.** *In a market $M(\omega, \succsim_I)$, a PMSS $T^* = (T_1, T_2, \dots, T_{t^*})$ is a **top trading segmentation** (TTS) if, for every $T_k \in T$, there exists a one-to-one mapping μ_k from T_k to $\omega(T_k)$ such that, for every $i \in T_k$, $\mu_k(i) \in B_i\Big(O \setminus \bigcup_{\ell=1}^{k-1} \omega(T_\ell)\Big)$.*

In words, a TTS is a sequence of minimal self-mapped sets $(T_1, T_2, \dots, T_{t^*})$ in which each T_k can distribute their objects among themselves in a way such that each $i \in T_k$ obtains one of his most preferred objects among those owned by $T_k \cup T_{k+1} \cup \cdots \cup T_{t^*}$.

The existence of a TTS is necessary and sufficient for the strong core to be nonempty.

Proposition 0 (Quint and Wako, 2004)**.** *In any market $M(\omega, \succsim_I)$, an allocation μ is in the strong core if and only if there exists a TTS $T^* = (T_1, T_2, \dots, T_{t^*})$ such that, for every $T_k \in T^*$, $\mu(T_k) = \omega(T_k)$ and, for every $i \in T_k$, $\mu(i) \in B_i(O \setminus \bigcup_{\ell=1}^{k-1} \omega(T_\ell))$.*

A corollary of this result is that, when the strong core is nonempty, all of its elements are equivalent, and any allocation equivalent to the elements of the strong core belongs to the strong core. This implies that the strong core is equivalence-closed. It is easy to prove that the weak core is also equivalence-closed (see Proposition 2).

3.2 Balbuzanov and Kotowski's exclusion core

In a general model encompassing the housing market model as a special case, assuming strict preferences, Balbuzanov and Kotowski (2019) propose the exclusion core as a new solution concept. The exclusion core and the strong core do not include each other in the general model, but they coincide in the housing market model under strict preferences. When preferences are weak, however, the performance of the exclusion core and its relationship with the strong core is unknown. We examine these questions in this subsection.

We first define the exclusion core for the housing market model. Given an allocation μ, every coalition C directly controls their endowments $\omega(C)$. Moreover, C indirectly controls the endowments of those who occupy $\omega(C)$, and this control right can be extended to more agents. Formally, the set of objects (directly or indirectly) controlled by a coalition C in an allocation μ is defined to be

$$\Omega(C|\omega, \mu) = \omega(\cup_{k=0}^{\infty} \tilde{C}^k)$$

where $\tilde{C}^0 = C$ and $\tilde{C}^k = \tilde{C}^{k-1} \cup \{i \in I \backslash \tilde{C}^{k-1} : \mu(i) \in \omega(\tilde{C}^{k-1})\}$ for every $k \geq 1$.

The exclusion core consists of allocations where no coalition can strictly benefit all of its members by evicting others from their controlled objects.

Definition 4 (Balbuzanov and Kotowski, 2019). *In any market $M(\omega, \succsim_I)$, an allocation μ is **exclusion blocked** by a coalition C via another allocation σ if*

1. $\forall i \in C, \sigma(i) \succ_i \mu(i)$;
2. $\forall j \in I \backslash C, \mu(j) \succ_j \sigma(j) \implies \mu(j) \in \Omega(C|\omega, \mu)$.

*The **exclusion core** consists of allocations that are not exclusion blocked.*

Unlike the strong core, unaffected agents are precluded from joining exclusion blocking coalitions. Balbuzanov and Kotowski (2019) argue that this is necessary to ensure the nonemptiness of the exclusion core in their model. However, unlike the weak core, this restriction does not prevent the exclusion core from achieving Pareto efficiency, since every Pareto inefficient allocation is exclusion blocked by the set of agents who are strictly better off in a Pareto improvement.

Our first finding is that, the exclusion core and the strong core no longer coincide in the housing market model under weak preferences.

Example 1 revisited ($\emptyset$ = Strong core $\subsetneq$ Exclusion core). *In $\succsim_I'$, the strong core $= \emptyset$, while the exclusion core $= \{\mu, \sigma'\}$. In either μ or σ, because* 2 *cannot be made strictly better off, he cannot join any exclusion blocking coalition. Then,* 1 *and* 3 *cannot evict each other.*

Second, we find that the exclusion core can also be empty under weak preferences.

Example 2 (Exclusion core = $\emptyset$). *Consider the following market with three agents.*

	1	2	3
ω:	a	b	c
μ:	b	c	a
σ:	c	a	b

$\succsim_1$	$\succsim_2$	$\succsim_3$
b	a, c	b
c	b	a
a		c

In any individually rational and Pareto efficient allocation, the three agents must trade objects in a cycle, leading to μ or σ. Thus, in either μ or σ, every agent controls all objects. So, 3 *can exclusion block μ via σ, and* 1 *can exclusion block σ via μ. So, the exclusion core $= \emptyset$.*

Finally, the exclusion core is not equivalence-closed.

	1	2	3
ω:	a	b	c
μ:	a	c	b
σ:	b	a	c
δ:	b	c	a
η:	c	a	b

$\succsim_1$	$\succsim_2$	$\succsim_3$
b	a, c	b
a, c	b	a, c

Example 3 (Exclusion core is not equivalence-closed)**.** *Consider the following market, which differs from Example 2 in that agents* 1 *and* 3 *become indifferent between a and c.*

The exclusion core $= \{\mu, \sigma\}$. *While* σ *and* δ *are equivalent,* δ *is not in the exclusion core, because it is exclusion blocked by* 3 *via* μ. *Similarly, while* μ *and* η *are equivalent,* η *is not in the exclusion core, because it is exclusion blocked by* 1 *via* σ.

It is clear that agents have equal blocking incentives in equivalent allocations. Thus, the exclusion core's violation of equivalence-closedness must be due to the possibility that agents have unequal exclusion rights in equivalent allocations. That is, for a coalition C and two equivalent allocations μ and μ', it might be that $\Omega(C|\omega, \mu) \neq \Omega(C|\omega, \mu')$. In contrast, in the standard core concepts, agents' blocking rights are determined solely by their endowments, which are independent of allocations.

We provide a necessary and sufficient condition for the exclusion core to be nonempty.

Proposition 1. *In any market* $M(\omega, \succsim_I)$, *an allocation* μ *is in the exclusion core if and only if* μ *is Pareto efficient, and there exists a partition of agents* $T = (T_1, T_2, \ldots, T_t)$ *such that, for every* $T_k \in T$, $\mu(T_k) = \omega(T_k)$, *and, for every* $i \in T_k$, $\mu(i) \in B_i\Big(O \backslash \bigcup_{\ell=1}^{k-1} \omega(T_\ell)\Big)$.

The partition $T = (T_1, T_2, \ldots, T_t)$ in Proposition 1 does not need to be a TTS. That is, the most preferred objects of $T_k \in T$ among $O \backslash \bigcup_{\ell=1}^{k-1} \omega(T_\ell)$ do not need to be owned exclusively by T_k. Therefore, the above condition is more relaxed than the corresponding condition for the strong core in Proposition 0. This implies that the exclusion core is more often nonempty than the strong core. Example 1 has shown that when the strong core is empty, the exclusion core may be nonempty.

Proposition 2 is the main result of this section.

Proposition 2. *(1) The exclusion core is a superset of the strong core and a subset of the weak core. (2) It coincides with the strong core when the strong core is nonempty. (3) However, the strong core and the weak core are equivalence-closed, whereas the exclusion core is not.*

4 Rectified exclusion core

The possibility for the strong core and the exclusion core to lack predictive power under weak preferences motivates us to analyze the underlying causes and seek potential rectifications. This leads to our proposal of the rectified exclusion core.

We begin our analysis by examining the behavioral foundation for unaffected agents in a weak blocking coalition. The cooperative approach does not explain why an unaffected agent may be willing to join such a coalition. In the absence of side payments, a seemingly intuitive explanation is altruism: unaffected agents are willing to help the others in the coalition without incurring any cost to themselves. However, this so-called altruism is inherently biased: it benefits agents within the coalition but may harm those outside it. In Example 1, under $\succsim_I'$, agent 2 is willing to help either agent 1 or agent 3 and alternates between them in forming blocking coalitions, which results in an empty strong core.

Balbuzanov and Kotowski's concept of exclusion rights offers a more plausible explanation for the behavior of unaffected agents. Consider strict preferences. Suppose that a coalition C weakly blocks an allocation μ via another σ, and $C_{\sigma\sim\mu} \neq \emptyset$. Therefore, for every $i \in C_{\sigma\sim\mu}$, $\sigma(i) = \mu(i)$. If there exists a subcoalition $C' \subseteq C_{\sigma\sim\mu}$ such that $\mu(C') = \omega(C')$, let C' be the largest subcoalition satisfying this condition.[6] So, C' does not contribute objects to the others in the coalition under both μ and σ. Thus, they can removed from the coalition without any loss. For every $i \in C_{\sigma\sim\mu} \backslash C'$, we must be able to find a chain, denoted by $i \to i_1 \to i_2 \to \cdots \to j$, in which every agent receives the next agent's endowment under μ and the end of the chain is an agent from $C_{\sigma>\mu}$. This means that the objects received by $C_{\sigma\sim\mu} \backslash C'$ in μ are directly or indirectly controlled by $C_{\sigma>\mu}$. Therefore, the participation of $C_{\sigma\sim\mu} \backslash C'$ in the coalition can be viewed as the result of threat exerted by $C_{\sigma>\mu}$, who hold exclusion rights over $C_{\sigma\sim\mu} \backslash C'$. In fact, this is the argument used by Balbuzanov and Kotowski to prove the equivalence between the strong core and the exclusion core in the housing market model under strict preferences.

Since exclusion rights can effectively explain coalition formation in standard core concepts under strict preferences, we are motivated to examine exclusion rights and the associated concept of the exclusion core under weak preferences.

Reformulating exclusion right The idea behind the exclusion core is that agents have the right to evict others from the objects they control. Our first observation is that, under weak preferences, the definition of exclusion rights needs to be reformulated. Regardless of whether preferences are strict or not, it is natural that every coalition C directly

[6] This largest C' must be unique, because for any two subcoalitions C^a and C^b such that $\mu(C^a) = \omega(C^a)$ and $\mu(C^b) = \omega(C^b)$, we have $\mu(C^a \cup C^b) = \omega(C^a \cup C^b)$.

controls their endowments $\omega(C)$ and has the right to evict any others from $\omega(C)$ in any allocation. Under strict preferences, C can indirectly control the endowments of the agents who occupy $\omega(C)$ by threatening to evict them from $\omega(C)$. This threat is credible because those agents cannot find other objects as good as $\omega(C)$. Under weak preferences, however, this threat might become incredible, because those agents may find other objects as good as $\omega(C)$. For example, if an agent is indifferent between all objects, then he is not fear of any threat because he can remain unaffected by consuming his own endowment. Therefore, in our new definition, C indirectly controls an agent i's endowment in an allocation μ if, not only $\mu(i)$, but also all the objects that i views as good as $\mu(i)$ are controlled by C.

Formally, in an allocation μ, the set of objects controlled by a coalition C is

$$\Omega^*(C|\omega,\mu) = \omega(\cup_{k=0}^{\infty} C^k),$$

where $C^0 = C$, and $C^k = C^{k-1} \cup \{i \in I \setminus C^{k-1} : \mathcal{I}_i(\mu(i)) \subseteq \omega(C^{k-1})\}$ for every $k \geq 1$. Under strict preferences, $\Omega^*(C|\omega,\mu)$ reduces to the definition used by Balbuzanov and Kotowski.

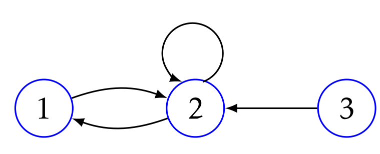
By replacing $\Omega(C|\omega,\mu)$ with $\Omega^*(C|\omega,\mu)$ in Definition 4, we would obtain a new core concept, which is larger than the exclusion core. In Example 2, this new core would be equal to $\{\mu,\sigma\}$. Agent 3 cannot exclusion block μ via σ, because 3 directly controls his endowment c but does not indirectly control 2's endowment, since 2 is indifferent between a and c. Similarly, 1 cannot exclusion block σ via μ. However, we observe that, under weak preferences, the exercise of exclusion rights also becomes subtle, which requires us to allow unaffected agents to join exclusion blocking coalitions.

Reformulating exclusion blocking coalition The following two examples illustrate the scenarios in which we allow unaffected agents to join exclusion blocking coalitions.

Example 4. *Consider the following market with three agents. For convenience, we use a graph to indicate each agent's favorite object. This convention is followed by other examples in this section.*

	1	2	3
ω:	a	b	c
μ:	c	a	b
σ:	b	a	c

$\succsim_1$	$\succsim_2$	$\succsim_3$
b	a,b	b
c	c	c
a		a



Since agent 1 most prefers agent 2's endowment, and 2 views 1's endowment as one of his favorite objects but does not accept agent 3's endowment, the intuitive allocation in this market is σ, in which 1 and 2 exchange endowments. The strong core = the exclusion core = $\{\sigma\}$.

Consider the allocation μ, in which agents exchange endowments along the cycle $1 \to 3 \to 2 \to 1$. To rule out μ, we must allow 1 to evict 3 from b. However, if we only replace $\Omega(C|\omega,\mu)$ with $\Omega^*(C|\omega,\mu)$ in Definition 4, 1 would not indirectly control b by threatening to evict 2 from a, since 2 is indifferent between a and b.

Nevertheless, if we examine the market more carefully, we will see that 1 can actually achieve the goal of evicting 3 from b by exercising his direct exclusion right. Specifically, since 1 directly controls a, if 1 evicts 2 from a, 2 must reclaim his endowment b to remain unaffected. Then, 3 must be evicted from b.

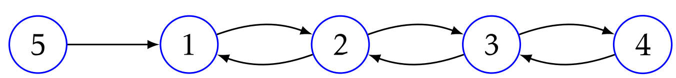
Example 4 illustrates one scenario in which we allow unaffected agents to join an exclusion blocking coalition: $\bigcup_{i \in C_{\sigma\sim\mu}} \mathcal{I}_i(\mu(i)) \backslash \omega(C_{\sigma\sim\mu}) \subseteq \omega(C_{\sigma>\mu})$, and there exists a one-to-one mapping $g : C_{\sigma\sim\mu} \to \omega(C_{\sigma\sim\mu})$ such that for every $i \in C_{\sigma\sim\mu}$, $g(i) \sim_i \mu(i)$.

In other words, $C_{\sigma>\mu}$ does not control the endowments of $C_{\sigma\sim\mu}$, because $C_{\sigma\sim\mu}$ can remain unaffected by allocating their endowments among themselves. However, aside from their endowments, all other objects that make $C_{\sigma\sim\mu}$ unaffected are owned by $C_{\sigma>\mu}$. Thus, if $C_{\sigma>\mu}$ evicts $C_{\sigma\sim\mu}$ from $\omega(C_{\sigma>\mu})$, $C_{\sigma\sim\mu}$ must revert to $\omega(C_{\sigma\sim\mu})$ to remain unaffected, which must evict others from $\omega(C_{\sigma\sim\mu})$. Thus, $C_{\sigma>\mu}$ can indirectly evict others from $\omega(C_{\sigma\sim\mu})$ by exercising their direct exclusion rights. To demonstrate this exclusion right, we allow $C_{\sigma\sim\mu}$ and $C_{\sigma>\mu}$ to jointly form a coalition. In Example 4, we allow 1 and 2 to form a coalition to evict 3 from b.

Example 5. *Consider the following market with five agents.*

	1	2	3	4	5
ω:	a	b	c	d	e
μ:	b	c	d	e	a
σ:	b	a	d	c	e

$\succsim_1$	$\succsim_2$	$\succsim_3$	$\succsim_4$	$\succsim_5$
b	a, c	b, d	c	a
a	b	c	e	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	



The four agents $\{1,2,3,4\}$ can obtain one of their favorite objects by allocating their endowments among themselves: 1 and 2 exchange endowments, and 3 and 4 exchange endowments. This produces the allocation σ. The strong core = the exclusion core = $\{\sigma\}$.

In the allocation μ, the five agents exchange endowments along the cycle $5 \to 1 \to 2 \to 3 \to 4 \to 5$. To rule out μ, we must allow 4 to evict 5 from a. However, if we only replace $\Omega(C|\omega,\mu)$ with $\Omega^*(C|\omega,\mu)$ in Definition 4, 4 would not indirectly control 3's endowment by threatening to evict 3 from d, since 3 is indifferent between b and d. Therefore, 4 would not indirectly control the remaining agents' endowments and would have no right to evict 5.

Nevertheless, if we examine the market more carefully, we will find that 4 *can achieve the goal of evicting* 5 *by exercising his direct exclusion right. Specifically, in μ, if* 4 *evicts* 3 *from d,* 3 *has to obtain b to remain unaffected. Then, there are two cases.*

- *If* 3 *can successfully obtain b after being evicted from d,* 1 *will be evicted from b, which is his unique favorite object. Then,* 4 *can leverage this indirect threat to obtain the control right of* 1*'s endowment a, and then evict* 5.

- *If* 3 *cannot obtain b after being evicted from d,* 4*'s threat towards* 3 *becomes credible. Leveraging this threat,* 4 *can obtain the control right of c and threaten to evict* 2 *from c. To counteract this threat,* 2 *will need to obtain a to remain unaffected. Similarly as above, if* 2 *can successfully obtain a,* 5 *will be evicted from a. If* 2 *cannot obtain a,* 4*'s indirect threat towards* 2 *becomes credible, and then* 4 *can obtain the control right of b. Then,* 4 *can threaten* 1 *to obtain the control right of a, and then evict* 5.

Example 5 illustrates another scenario in which we allow unaffected agents to join an exclusion blocking coalition: although $C_{\sigma>\mu}$ does not control the endowments of $C_{\sigma\sim\mu}$, we have $\bigcup_{i\in C_{\sigma\sim\mu}}\mathcal{I}_i(\mu(i))\backslash\omega(C_{\sigma\sim\mu})\subseteq\omega(C_{\sigma>\mu})$, and there does not exist a one-to-one mapping $g: C_{\sigma\sim\mu}\to\omega(C_{\sigma\sim\mu})$ such that, for every $i\in C_{\sigma\sim\mu}$, $g(i)\sim_i\mu(i)$.

In other words, $C_{\sigma\sim\mu}$ must rely on the endowments of $C_{\sigma>\mu}$ to remain unaffected. By leveraging the threat of evicting $C_{\sigma\sim\mu}$ from $\omega(C_{\sigma>\mu})$, $C_{\sigma>\mu}$ can indirectly control $\omega(C_{\sigma\sim\mu})$ and pressure $C_{\sigma\sim\mu}$ to join the coalition. In Example 5, $\{1,2,3\}$ relies on 4's endowment to remain unaffected. Thus, we allow $\{1,2,3,4\}$ to form a coalition to exclusion block μ.

To unify the above scenarios, our definition imposes the following requirement on an exclusion blocking coalition C: if $C_{\sigma\sim\mu}\neq\emptyset$, then $\forall i\in C_{\sigma\sim\mu}$, $\mathcal{I}_i(\mu(i))\backslash\omega(C_{\sigma\sim\mu})\subseteq\omega(C_{\sigma>\mu})$, or equivalently, $\mathcal{I}_i(\mu(i))\subseteq\omega(C)$. In words, all the objects that make every $i\in C_{\sigma\sim\mu}$ unaffected are owned by C. In this case, if $C_{\sigma>\mu}$ evicts $C_{\sigma\sim\mu}$ from $\omega(C_{\sigma>\mu})$, then, either $C_{\sigma\sim\mu}$ can revert to their own endowments to remain unaffected, which evicts others from $\omega(C_{\sigma\sim\mu})$, or $C_{\sigma\sim\mu}$ cannot rely on their own endowments to remain unaffected, thereby justifying their participation in the coalition.[7]

Definition 5. *In any market $M(\omega,\succsim_I)$, an allocation μ is* ***rectification exclusion blocked*** *by a coalition C via another allocation σ if*

1. $\forall i\in C$, $\sigma(i)\succsim_i\mu(i)$, *and* $\exists j\in C$, $\sigma(j)\succ_j\mu(j)$;

[7] Our definition accommodates the case that, for every $i\in C_{\sigma\sim\mu}$, $\mathcal{I}_i(\mu(i))\subseteq\omega(C_{\sigma>\mu})$. So, $C_{\sigma>\mu}$ indirectly controls the endowments of $C_{\sigma\sim\mu}$. In this case, we have flexibility to define the blocking coalition: either $C_{\sigma>\mu}$ alone forms a coalition to evict others from $\omega(C_{\sigma\sim\mu})$, or $C_{\sigma>\mu}$ and $C_{\sigma\sim\mu}$ jointly form a coalition to evict others from $\omega(C_{\sigma\sim\mu})$. A difference is that, if $C_{\sigma\sim\mu}$ does not join the blocking coalition, they are allowed to be harmed by the blocking, but if they join the blocking coalition, they cannot be harmed.

2. $\forall k \in I \backslash C$, $\mu(k) \succ_k \sigma(k) \implies \mu(k) \in \Omega^*(C|\omega, \mu)$;

3. $\forall i \in C_{\sigma \sim \mu}$, $\mathcal{I}_i(\mu(i)) \subseteq \omega(C)$.

The ***rectified exclusion core*** *consists of allocations that are not rectifyingly exclusion blocked.*

Our main result in this section is the following theorem.

Theorem 1. *(1) The rectified exclusion core is nonempty, Pareto efficient, and equivalence-closed. (2) It is a superset of the exclusion core and a subset of the weak core. (3) It coincides with the strong core (and with the exclusion core) when the strong core core is nonempty.*

We remark on the equivalence-closedness property. As shown in Proposition 2, the strong core and the weak core satisfy this property, whereas the exclusion core violates it. The issue with the exclusion core is that a coalition C may have different exclusion rights in equivalent allocations. After reformulating exclusion rights, we ensure that $\Omega^*(C|\omega, \mu) = \Omega^*(C|\omega, \mu')$ whenever μ and μ' are equivalent. As a result, the rectified exclusion core satisfies the property.

4.1 An algorithm to find elements of the rectified exclusion core

We present an algorithm called **generalized top trading cycle** (GTTC), and prove that all allocations it finds belong to the rectified exclusion core. This implies that the rectified exclusion core is nonempty.

Under strict preferences, TTC is an individually rational, Pareto efficient, and strategy-proof mechanism, and its outcome is the unique element of the strong core. Under weak preferences, the literature has proposed several generalizations of TTC to preserve Pareto efficiency and strategy-proofness. The procedures of these algorithms are similar in that their each step essentially consists of three stages. In the **departure** stage, a group of agents are removed with their assignments, if there are no beneficial trades involving them in subsequent steps. In the **pointing** stage, a rule selects a unique pointee for each agent such that at least one beneficial trading cycle is formed. These selection rules are crucial for ensuring strategy-proofness. In the **trading** stage, cycles formed in the pointing stage are traded.

Our definition of GTTC is similar to those procedures, but it does not specify a selection rule for the pointing stage. It only requires that at least one beneficial trading cycle is formed . We prove that this requirement is sufficient to ensure that the outcome of the algorithm belongs to the rectified exclusion core. If someone wants to achieve strategy-proofness, a specific selection rule in the existing algorithms can be employed.

Generalized Top Trading Cycle

Step $t \geq 1$: Every step includes three stages.

- **Departure**: Among the remaining agents, a group of agents is chosen to depart with the objects they hold if, for every agent in the group, two conditions are met:

  1. He holds one of his most preferred objects among the remaining ones;
  2. All of his most preferred objects among remaining ones are held by the group.

  Once a group departs, there may exist another group that satisfies the above two conditions. Choose one of such groups and let it depart. Repeat this operation until there are no groups that are chosen to depart. At any point in this process, an agent is said to become **satisfied** if the object he holds becomes one of his most preferred objects among the remaining ones.

  At the end of the departure stage, denote the set of remaining agents by I_t and the set of remaining objects by O_t. If I_t is empty, stop the algorithm. Otherwise, denote the current allocation by μ_t such that, for every $i \in I_t$, $\mu_t(i)$ is the object held by i. Any $i \in I_t$ is said to be **satisfied** if $\mu_t(i)$ is among i's most preferred objects among O_t. Otherwise, i is said to be **unsatisfied**.

- **Pointing**: Let each agent point to one of the others who hold his favorite objects such that at least one beneficial trading cycle is generated; that is, at least one agent in the cycle strictly prefers the object held by the pointee to the object he holds.

- **Trading**: Clear the cycles generated in the Pointing stage such that every agent in every cycle obtains the object held by the pointee. Go to the next step.

In each step of GTTC, the pointing stage generates at least one beneficial trading cycle. By trading the cycle, at least one agent obtains a strictly better object than the object he holds. Since there are finite agents and finite objects, GTTC must stop in finite steps. When GTTC stops, all agents are involved in departing groups. These groups can be ordered as a sequence $(G_1, G_2, \ldots, G_k)$ such that each group G_ℓ departs before the group $G_{\ell+1}$. An agent may become satisfied at some point in the algorithm but departs in a later step. Once an agent becomes satisfied, he remains satisfied in subsequent steps, until he departs with his assignment.

Lemma 1. *Every outcome of GTTC belongs to the rectified exclusion core.*

GTTC may not find all elements of the rectified exclusion core. In Example 3, δ and η are in the rectified exclusion core, but they cannot be found by GTTC.[8]

5 Rectified strong core

Similar to our motivation for the rectified exclusion core, this section proposes a modification of the strong core to address its emptiness issue. Section 4 has shown that BK's exclusion right provides a useful conceptual tool for explaining coalition formation under the strong core, based on which we propose the condition "$\forall i \in C_{\sigma\sim\mu}, \mathcal{I}_i(\mu(i)) \subseteq \omega(C)$" to regulate the blocking behavior of unaffected agents under weak preferences. However, exclusion blocking does not need to satisfy the feasibility condition required by standard core concepts, where a blocking coalition must redistribute their endowments among themselves (i.e., "$\sigma(C) = \omega(C)$"). By replacing condition (2) of Definition 5 with this feasibility condition, we obtain the following concept.

Definition 6. *In a market $M(\omega, \succsim_I)$, an allocation μ is **rectification blocked** by a coalition C via another allocation σ if*

1. $\forall i \in C$, $\sigma(i) \succsim_i \mu(i)$, *and* $\exists j \in C$, $\sigma(j) \succ_j \mu(j)$;
2. $\sigma(C) = \omega(C)$;
3. $\forall i \in C_{\sigma\sim\mu}$, $\mathcal{I}_i(\mu(i)) \subseteq \omega(C)$.

*The **rectified strong core** consists of allocations that are not rectification blocked.*

Compared to the definition of the strong core, the above definition adds the condition (3). We formulate their difference as follows: the strong core assumes that an unaffected agent is willing to join a blocking coalition if and only if the coalition owns **one** object that makes him unaffected and assigns the object to him, whereas the rectified strong core assumes that an unaffected agent is willing to join a blocking coalition if and only if the coalition owns **all** the objects that make him unaffected and assigns one of them to him. Thus, the strong core is a subset of the rectified strong core.

Under strict preferences, this difference vanishes, and the rectified strong core collapses to the strong core. As discussed, $C_{\sigma\sim\mu}$ can be viewed as being compelled to join the coalition due to the threat posed by $C_{\sigma\succ\mu}$. Condition (3) of Definition 6 generalizes this

[8] Specifically, in step one of GTTC, 1 and 3 must point to 2 and 2 must point to one of them. Thus, either $\{1,2\}$ or $\{2,3\}$ form a cycle and exchange their endowments. After that, there is no more beneficial trading cycle. So, GTTC finds either μ or σ.

interpretation to settings with weak preferences, which has been discussed in Section 4. In contrast, the conventional definition of the strong core literally applies the same condition on unaffected agents from strict preferences to weak preferences.

We summarize our results about the rectified strong core in the following proposition.

Proposition 3. *(1) The rectified strong core is nonempty, Pareto efficient, and equivalence-closed. (2) It is a superset of the rectified exclusion core and a subset of the weak core. (3) It coincides with the strong core when the strong core is nonempty.*

Theorem 1 and Proposition 3 together imply the following relationships between the various core concepts discussed in the paper:

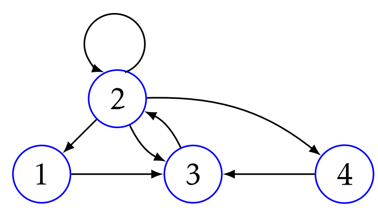
Strong core $\subseteq$ Exclusion core $\subseteq$ Rectified exclusion core $\subseteq$ Rectified strong core $\subseteq$ Weak core.

The set inclusion relation between any two core concepts can be strict. In $\succsim'_I$ of Example 1, the strong core is a strict subset of the exclusion core. In Example 3, the nonempty exclusion core is a strict subset of the rectified exclusion core, which equals $\{\mu, \sigma, \delta, \eta\}$. In Example 6 below, the rectified exclusion core is a strict subset of the rectified strong core. In $\succsim_I$ of Example 1, the rectified strong core, coinciding with the strong core, is a strict subset of the weak core.

Example 6 (Rectified exclusion core $\subsetneq$ rectified strong core). *Consider the following market.*

	1	2	3	4
ω:	a	b	c	d
μ:	b	a	d	c
σ:	c	a	b	d

$\succsim_1$	$\succsim_2$	$\succsim_3$	$\succsim_4$
c	a, b, c, d	b	c
b		d	d
a		c	a
d		a	b



We show that μ is in the rectified strong core, but not in the rectified exclusion core. In μ, the two pairs $\{1,2\}$ and $\{3,4\}$ respectively exchange their endowments.

To show that μ is in the rectified strong core, suppose that μ is rectification blocked by a coalition C via another μ'. Since μ is Pareto efficient, it must be that $C \subsetneq I$. Since 2 is indifferent between all objects, it does not hold that $\mathcal{I}_2(\mu(2)) \subseteq \omega(C)$. Therefore, $2 \notin C$. Since 1 and 3 are the only agents who have not received their most preferred objects in μ, either $1 \in C_{\mu' > \mu}$ or $3 \in C_{\mu' > \mu}$ (or both). However, because $2 \notin C$, if $3 \in C$, it is impossible for 3 to receive b in μ'. Therefore, it must be that $1 \in C_{\mu' > \mu}$. Then, it must be that $\mu'(1) = c$, which requires $3 \in C$ and $4 \notin C$, since 4 must be worse off in μ'. Therefore, $C = \{1,3\}$. But this means that $\mu'(3) = a$, which is a contradiction, since 3 is worse off in μ'.

We then show that μ is rectification exclusion blocked by $C = \{1, 3\}$ via σ. Note that both agents in the coalition are strictly better off in σ, and 4 is the only agent who is worse off in σ. It is easy to verify that $\mu(4) = c \in \{a, c\} = \Omega^*(\{1, 3\}|\omega, \mu)$.

6 Housing market with multiple copies of objects

This section examines a special case of the housing market model in which agents' indifferent preferences stem from the existence of multiple copies of objects. Formally, we consider a setting in which there exists a finite set of object types, denoted by $\mathcal{O}$. For each $x \in \mathcal{O}$, let $O_x \subseteq O$ denote the set of the copies of x. Therefore, $O = \cup_{x \in \mathcal{O}} O_x$. Each $i \in I$ owns an object $\omega(i) \in O$ and has a preference relation $\succsim_i$ over O such that, for any distinct $o, o' \in O$, $o \sim_i o'$ if and only if $o, o' \in O_x$ for some $x \in \mathcal{O}$; that is, o and o' are different copies of the same object type. Let I_x denote the set of owners of the copies of x.

In this special model, the strong core may still be empty. In Example 3, a and c can be viewed as two copies of the same object type. The strong core is empty in the example.

However, we prove that the exclusion core is nonempty. Notably, the exclusion core coincides with the set of outcomes of TTC in an artificial priority-based allocation model.

Specifically, let each $x \in \mathcal{O}$ rank the agents in I_x using a priority order $\rhd_x$, where $i \rhd_x j$ means that i is ranked above j. Given a priority structure $(\rhd_x)_{x \in \mathcal{O}}$, TTC proceeds as follows: in each step, among the remaining agents and object types, let each agent point to his most preferred object type, and each object type point to its highest-priority owner; in each generated cycle, let each agent receive the object owned by the highest-priority owner of the object type he points to, and then remove them.

This algorithm is a refinement of the generalization of TTC introduced by Abdulkadiroğlu and Sönmez (2003) for the school choice model. Their algorithm does not distinguish between the different copies of the same object type, whereas to obtain our result, we need to specify the copy of the object type each agent receives. By varying the priority structure, we obtain different outcomes of TTC. We prove that an allocation belongs to the exclusion core if and only if it can be found by TTC under some priority structure.

Proposition 4. *In the housing market model with multiple copies of objects, the exclusion core equals the set of the outcomes of TTC in the artificial priority-based allocation model.*

We illustrate this proposition by revisiting Example 3.

Example 3 revisited (Finding the exclusion core)**.** *We view a and c as two copies of an object type x, and view b as the unique copy of another object type y.*

In the first step of our TTC, 1 *and* 3 *point to* y*, and* 2 *points to* x*. If* 1 *is ranked above* 3 *in the priority order for* x*, then* 1 *and* 2 *form a cycle, with* 1 *receiving* b *and* 2 *receiving* a*. This finds the allocation* σ *in the exclusion core. If, instead,* 3 *is ranked above* 1*, we then find the other allocation* μ *in the exclusion core.*

However, if we assign arbitrary copies of object types to agents involved in cycles, we then also find δ *and* η*, which are not in the exclusion core but are equivalent to its elements.*

In this special model, the exclusion core is still not equivalence-closed (see Example 3). Since the rectified exclusion core is equivalence-closed, it must include all allocations that are equivalent to the elements of the exclusion core. These allocations are exactly those found by Abdulkadiroğlu and Sönmez (2003)'s TTC where agents can be assigned arbitrary copies of the object type they point to when involved in cycles. The set of these allocations can be viewed as the "equivalence closure" of the exclusion core. However, Example 7 below shows that this closure may still be strictly smaller than the rectified exclusion core; that is, the rectified exclusion core may have elements that are not equivalent to any element of the exclusion core.

Example 7 (Closure of exclusion core $\subsetneq$ rectified exclusion core)**.** *Consider the following market with three object types and five agents. Object* a *is the only copy of an object type. Objects* b *and* b' *are two distinct copies of another type, and* c *and* c' *are two distinct copies of the third type.*

	1	2	2′	3	3′
ω:	a	b	b'	c	c'
μ:	b	c	c'	a	b'
σ:	b	a	c	b'	c'

$\succsim_1$	$\succsim_2$	$\succsim_{2'}$	$\succsim_3$	$\succsim_{3'}$
b, b'	a	a	a	a
a	c, c'	c, c'	b, b'	b, b'
c, c'	b	b	c, c'	c, c'

We show that μ *is in the rectified exclusion core, but it is not equivalent to any element of the exclusion core.*

Suppose that μ *is rectification exclusion blocked by a coalition* C*. In* μ*, only one agent in* $\{2, 2', 3'\}$ *can be made strictly better off, and the agent must receive object* a *to be better off. Thus, to block* μ*,* 3 *must be evicted from* a*. If* $1 \in C$*, since he must be unaffected, all owners of his indifferent objects must belong to* C*. So,* $\{2, 2'\} \subseteq C$*. Since at least one of* 2 *and* 2′ *must be unaffected, all owners of his indifferent objects must belong to* C*. So,* $\{3, 3'\} \subseteq C$*, which, however, is impossible. Thus,* $1 \notin C$*, which implies that* C *must indirectly control* a*. So,* C *must directly or indirectly control both* b *and* b'*. If* C *directly controls* b *and* b'*, it must be that* $\{2, 2'\} \subseteq C$*. Then, as above, it must be that* $\{3, 3'\} \subseteq C$*, which is impossible. If* C *indirectly controls* b *and* b'*, it must be that* $\{3, 3'\} \subseteq C$*, which again is impossible.*

In TTC, under any priority structure, in the first step, 1 *points to the object type of* b *and* b', *and both* 2 *and* $2'$ *point to* a. *Thus, one of* 2 *and* $2'$ *must receive* a, *depending on whose priority is higher. For instance, if* 2 *is ranked above* $2'$ *in the priority order, we obtain the allocaiton* σ. *However,* μ, *in which neither of* 2 *and* $2'$ *receives* a, *cannot be generated by TTC and is not equivalent to any element of the exclusion core.*

7 Concluding remark

This section concludes the paper by examining other potential solution concepts for the housing market model. The first solution we consider is the **vNM stable set**. A set of allocations is a vNM stable set if each allocation inside the set is not dominated by any other allocation inside the set (internal stability), and each allocation outside the set is dominated by an allocation inside the set (external stability). Wako (1991) shows that the strong core, when it is nonempty, is the unique vNM stable set based on weak domination. However, Wako et al. (2007) shows that, in general, a vNM stable set based on either strong or weak domination does not exist. Several papers modify the definition of weak domination and then prove that the corresponding vNM stable set coincides with the set of competitive allocations. However, as discussions in the related literature section, the set of competitive allocations has undesirable features.

Demuynck et al. (2019) introduce the **myopic stable set** (MSS) for a general class of social environments. A MSS always exists and is unique for a social environment with a finite state space. In the housing market model, a weak domination MSS is a set of allocations satisfying three conditions: (1) no coalition of agents can benefit from deviating from an allocation inside the set to an allocation outside the set (deterrence of external deviations); (2) from any allocation outside the set there is a finite sequence of coalition blocking that leads to an allocation inside the set (iterated external stability); (3) no strict subset of the set satisfies the former two conditions (minimality). Demuynck et al. (2019) have proved that, in the housing market model under strict preferences, the weak domination MSS coincides with the strong core. However, we show that, when preferences are weak, the weak domination MSS may include Pareto inefficient allocations.

Example 2 revisited (MSS includes Pareto inefficient elements)**.** *The allocations* μ *and* σ *are Pareto efficient in the example. Other than* ω, *there exist three other allocations:* $\mu'(1,2,3) = (a,c,b)$, $\sigma'(1,2,3) = (b,a,c)$, *and* $\delta(1,2,3) = (c,b,a)$.

The weak domination MSS $= \{\mu,\mu',\sigma,\sigma'\}$. *So, it has Pareto inefficient elements,* μ' *and* σ'. *To verify that this set is a MSS, note that* μ *is weakly dominated by* μ', μ' *is weakly dominated* *by σ and σ', σ is weakly dominated by σ', σ' is weakly dominated by μ and μ', δ is weakly dominated by μ and σ, and ω is weakly dominated by any other allocation.*

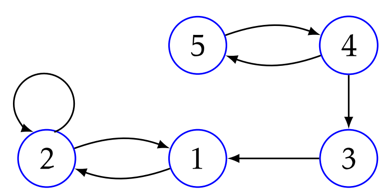
Note that both the vNM stable set and the MSS are setwise solutions; whether an allocation belongs to the set depends on other allocations in the set. In contrast, the various core concepts discussed in our paper are pointwise solutions; whether an allocation belongs to a core is a property of the allocation, irrespective of other allocations in the core.

Yılmaz and Yılmaz (2022) propose a solution based on the **bargaining set** developed by Aumann and Maschler (1964). Their idea is that, when a coalition blocks an allocation to reach a new allocation, it should consider the possible counter-blocking of the new allocation by other coalitions. An allocation is in the bargaining set if whenever it is blocked by coalition, there exists a counter-blocking of the new allocation by another coalition that overlaps with the original coalition, and the new coalition blocks the new allocation by claiming their welfare in the original allocation. When a coalition C weakly blocks an allocation μ via another σ, there is a degree of freedom to select σ. Yılmaz and Yılmaz impose the selection rule such that, in σ, all agents outside C who are affected by the blocking receive their own endowments, and the others who are unaffected keep their assignments in μ. The bargaining set is nonempty and Pareto efficient, and lies between the strong core and the weak core. However, it may be strictly larger than the strong core when the latter is nonempty (Example 8). The bargaining set and our solutions do not include each other.

Example 8 ($\emptyset \neq$ strong core $\subsetneq$ bargaining set)**.** *Consider a market with five agents.*

	1	2	3	4	5
ω:	a	b	c	d	e
μ:	e	b	a	c	d
δ:	b	a	c	e	d

$\succsim_1$	$\succsim_2$	$\succsim_3$	$\succsim_4$	$\succsim_5$
b	a, b	a	c, e	d
e	$\vdots$	c	$\vdots$	$\vdots$
$\vdots$		$\vdots$		



The strong core = $\{\delta\}$. We show that μ, which is not in the strong core, is in the bargaining set. In μ, 2 receives his endowment, and the remaining agents exchange endowments along a cycle $1 \to 5 \to 4 \to 3 \to 1$. There exists only one coalition $\{1,2\}$ that weakly blocks μ via an allocation σ in which 1 and 2 exchange endowments. By the selection rule of Yılmaz and Yılmaz (2022), the other agents receive their own endowments in σ. Then, σ is weakly blocked by $\{2,4,5\}$ via an allocation μ' in which 2 receives his endowment, and 4 and 5 exchange their endowments. In μ', all members of $\{2,4,5\}$ receive objects indifferent to their assignments in μ, and the counter-blocking coalition includes one member of the initial blocking coalition (i.e., 2). Therefore, μ is in the bargaining set defined by Yılmaz and Yılmaz (2022).

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Appendix

A Proofs of Propositions 1, 2, 3, and 4

Proof of Proposition 1. (If) Suppose that μ is a Pareto efficient allocation, and there exists a partition of agents $T = (T_1, T_2, \ldots, T_t)$ such that, for every $T_k \in T$, $\mu(T_k) = \omega(T_k)$, and, for every $i \in T_k$, $\mu(i) \in B_i\Big(O \backslash \bigcup_{\ell=1}^{k-1} \omega(T_\ell)\Big)$. We prove that μ is in the exclusion core.

Suppose that μ is exclusion blocked by some coalition C via some σ. Since μ is Pareto efficient, there must exist an agent who becomes worse off in σ compared to μ. Thus, C must evict some agent. However, we prove that this is impossible. First, all agents in T_1 cannot join C, because all of them have obtained their favorite objects in μ and thus cannot be made strictly better off. Given this, because all agents in T_2 have obtained their favorite objects among $O \backslash \omega(T_1)$, if any agent in T_2 joins C, the agent must become strictly better off by obtaining an object owned by some agent in T_1. However, because $\mu(T_1) = \omega(T_1)$, C does not control $\omega(T_1)$ and thus cannot evict any agent in T_1. So, T_2 cannot join C. This argument can be inductively applied to all remaining groups in T to conclude that no agents can join C. Therefore, μ is not exclusion blocked.

(Only if) Suppose that the exclusion core is nonempty, and μ is any allocation in it. So, μ is Pareto efficient. Moreover, in μ, agents can be partitioned into disjoint groups such that the agents in each group trade their endowments along a cycle represented by

$$i_1 \to i_2 \to i_3 \to \cdots \to i_k \to i_1,$$

where $i_\ell \to i_{\ell+1}$ means that $\mu(i_\ell) = \omega(i_{\ell+1})$.[9] Denote the set of agents in a typical group by T_k. Therefore, for every T_k, $\mu(T_k) = \omega(T_k)$, and no strict subset of T_k satisfies this condition. We prove that these groups can be arranged into an order $T = (T_1, T_2, \ldots, T_t)$ such that, for every $T_k \in T$ and every $i \in T_k$, $\mu(i) \in B_i\Big(O \backslash \bigcup_{\ell=1}^{k-1} \omega(T_\ell)\Big)$.

First, for every $T_k \in T$, we prove that every $i \in T_k$ most prefers his assignment $\mu(i)$ among all objects in $\omega(T_k)$. Because the agents in T_k trade their endowments along a cycle represented above, every agent in T_k controls all objects in $\omega(T_k)$. Thus, if any $i \in T_k$ strictly prefers some object in $\omega(T_k)$ to $\mu(i)$, then i would be able to exclusion block μ, which is a contradiction. So, every $i \in T_k$ must most prefer $\mu(i)$ among all objects in $\omega(T_k)$.

Second, we prove that there must exist a group in which all agents most prefer their assignments among all objects. We label this group T_1. To find T_1, start with any group T_a. If some $i_a \in T_a$ strictly prefers an object owned by another group T_b to his assignment

[9] If an agent obtains his own endowment, he forms a cycle with himself.

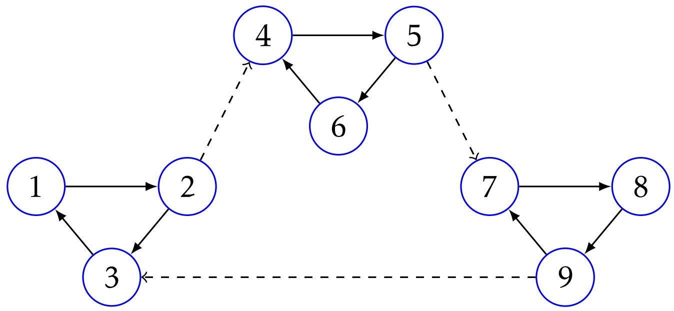

Figure 1: Suppose there are three groups $\{1,2,3\}$, $\{4,5,6\}$, and $\{7,8,9\}$. The agents in each group trade their endowments along the cycle and most prefer their assignments among the objects owned by the group. Suppose that 2 strictly prefers 4's endowment to his assignment, 5 strictly prefers 7's endowment to his assignment, and 9 strictly prefers 3's endowment of his assignment. Then, $\{2,5,9\}$ can exclusion block the allocation because they control all objects in the three cycles.

$\mu(i_a)$, we then examine T_b. If some $i_b \in T_b$ strictly prefers an object owned by another group T_c to his assignment $\mu(i_b)$, we then examine T_c. Continuing this search process, because there are finite groups, we must either find a group in which all agents most prefer their assignments among all objects, or find a group in which some i strictly prefers an object owned by a group we have examined to his assignment $\mu(i)$. In the former case, we label the group T_1. We prove that the latter case is impossible. In the latter case, there exists a sequence of groups $T' = (T'_1, T'_2, \ldots, T'_m)$ such that, in every $T'_k \in T' \backslash T_m$, some i_k strictly prefers the object owned by some agent in T'_{k+1} to his assignment $\mu(i_k)$, while some $i_m \in T'_m$ strictly prefers the object owned by some agent in T'_1 to his assignment $\mu(i_m)$. Then, $\{i_1, i_2, \ldots, i_m\}$ can form a coalition to exclusion block μ, because these agents control all of the objects in $\cup_{T_k \in T'} \omega(T_k)$ (see the illustration in Figure 1). This is a contradiction.

After finding T_1, we can repeat the above argument to find the group T_2 in which all agents most prefer their assignments among $O \backslash \omega(T_1)$. Inductively applying the above argument, we can find the desired order of groups $T = (T_1, T_2, \ldots, T_t)$. ■

Proof of Proposition 2. (1) By Proposition 0, when the strong core is nonempty, there exists a TTS $T^* = (T_1, T_2, \ldots, T_{t^*})$ such that, for every allocation μ in the strong core, every $T_k \in T^*$, and every $i \in T_k$, $\mu(i) \in B_i(O \backslash \bigcup_{\ell=1}^{k-1} \omega(T_\ell))$. By Proposition 1, μ is in the exclusion core. Thus, the exclusion core is a superset of the strong core. Theorem 1 proves that the exclusion core is a subset of the rectified exclusion core, which is a subset of the weak core. Thus, the exclusion core is a subset of the weak core.

(2) Theorem 1 proves that the rectified exclusion core coincides with the strong core when the latter is nonempty. Since the exclusion core lies between the strong core and the rectified exclusion core, it also coincides with the strong core when the latter is nonempty.

(3) Example 3 shows that the exclusion core is not equivalence-closed. In contrast, both the strong core and the weak core are equivalence-closed: for any two equivalent allocations μ and μ', if μ is weakly (strongly) blocked by C via another σ, since $\sigma(C) = \omega(C)$ and for every $i \in C$, $\mu'(i) \sim_i \mu(i)$, μ' is also weakly (strongly) blocked by C via σ. ■

Proof of Proposition 3. (1) Since the rectified exclusion core is nonempty, and the rectified strong core is a superset of the rectified exclusion core, which is proved below, the rectified strong core is nonempty.

All elements of the rectified strong core are Pareto efficient, because any Pareto inefficient allocation is rectification blocked by the grand coalition I via a Pareto improvement.

To prove equivalence-closedness, consider any two equivalent allocations μ and μ' in any market. We prove that, if μ is in the rectified strong core, μ' is also in the rectified strong core. Suppose that μ' is rectification blocked by a coalition C via another allocation σ. Then, since $C_{\sigma\sim\mu} = C_{\sigma\sim\mu'}$, $C_{\sigma>\mu} = C_{\sigma>\mu'}$, and for every $i \in C$, $\mathcal{I}_i(\mu(i)) = \mathcal{I}_i(\mu'(i))$, μ is also rectification blocked by C via σ, which is a contradiction.

(2) We first prove that the rectified strong core is a superset of the rectified exclusion core. In any market, let μ be any allocation not in the rectified strong core. Suppose that μ is rectification blocked by a coalition C via another allocation σ. We then prove that μ is rectification exclusion blocked by C via some allocation σ'.

Define $I_1 = \{i \in I\backslash C : \mu(i) \in \omega(C)\}$ and $I_2 = \{i \in I\backslash C : \omega(i) \in \mu(C)\}$. The two sets may not be disjoint. It is evident that $|I_1| = |I_2|$. Let σ' be any allocation such that, $\forall i \in I\backslash(I_1 \cup C)$, $\sigma'(i) = \mu(i)$; $\forall i \in C$, $\sigma'(i) = \sigma(i)$; and $\forall i \in I_1$, $\sigma'(i) \in \omega(I_2)$. Then, for any $j \in I\backslash C$ such that $\mu(j) \succ_j \sigma'(j)$, it must be that $j \in I_1$ and $\mu(j) \in \omega(C)$. Because $C_{\sigma'\sim\mu} = C_{\sigma\sim\mu}$, for every $i \in C_{\sigma'\sim\mu}$, it holds that $\mathcal{I}_i(\mu(i)) \subseteq \omega(C)$. So, μ is rectification exclusion blocked by C via σ'.

The rectified strong core is a subset of the weak core, because any strong blocking coalition is a rectification blocking coalition.

(3) When the strong core is nonempty, by Proposition 0, there exists a TTS $T^* = (T_1, T_2, \dots, T_{t^*})$. Let μ be any element of the rectified strong core. In the following, we prove that, for every $T_k \in T^*$ and every $i \in T_k$, $\mu(i) \in B_i\left(O\backslash \bigcup_{\ell=1}^{k-1} \omega(T_\ell)\right)$. Then, by Proposition 0, μ is in the strong core. Since the strong core is a subset of the rectified exclusion core, which is a subset of the rectified strong core, the three cores must coincide.

We first prove that, for every $i \in T_1$, $\mu(i) \in B_i(O)$. Suppose that this is not true. Let μ_1 be a one-to-one mapping from T_1 to $\omega(T_1)$ such that, for every $i \in T_1$, $\mu_1(i) \in B_i(O)$. Then, T_1 can rectification block μ via an allocation μ' in which, for every $i \in T_1$, $\mu'(i) = \mu_1(i)$. The key is to verify condition (3) of Definition 6. The condition is satisfied because all of the most preferred objects for each agent in T_1 are owned by T_1.

Similarly, if it is not true that, for every $i \in T_2$, $\mu(i) \in B_i(O \backslash \omega(T_1))$, we then let μ_2 be a one-to-one mapping from T_2 to $\omega(T_2)$ such that, for every $i \in T_2$, $\mu_2(i) \in B_i(O \backslash \omega(T_1))$. Then, $T_1 \cup T_2$ can rectification block μ via an allocation μ' in which, for every $i \in T_1$, $\mu'(i) = \mu(i)$, and for every $i \in T_2$, $\mu'(i) = \mu_2(i)$. The key is to verify condition (3) of Definition 6. The condition is satisfied because all of the most preferred objects for each agent in T_1 are owned by T_1, and all of the most preferred objects for each agent in T_2 among $O \backslash \omega(T_1)$ are owned by T_2.

The above argument can be inductively applied to each remaining $T_k \in T^*$ to prove that, for each $i \in T_k$, $\mu(i) \in B_i\big(O \backslash \bigcup_{\ell=1}^{k-1} \omega(T_\ell)\big)$. ■

Proof of Proposition 4. In any market, let μ be the outcome of TTC in the priority-based allocation model under some priority structure $(\rhd_a)_{a \in \mathcal{O}}$. Let $T = (T_1, T_2, \ldots, T_t)$ denote the order of cycles removed in the procedure of TTC, where each T_k represents the set of agents involved in the corresponding cycle. If multiple cycles are removed in the same step of TTC, their relative ranking can be arbitrary in the above order. Then, it is obvious that for each $T_k \in T$, $\mu(T_k) = \omega(T_k)$, and for each $i \in T_k$, $\mu(i) \in B_i\big(O \backslash \bigcup_{\ell=1}^{k-1} \omega(T_\ell)\big)$. By Proposition 1, μ is in the exclusion core.

Let μ be any allocation in the exclusion core. By Proposition 1, there exists a partition of agents $T = (T_1, T_2, \ldots, T_t)$ such that, for every $T_k \in T$, $\mu(T_k) = \omega(T_k)$, and for every $i \in T_k$, $\mu(i) \in B_i\big(O \backslash \bigcup_{\ell=1}^{k-1} \omega(T_\ell)\big)$. Moreover, the proof of the "only if" part of Proposition 1 shows that each $T_k \in T$ can be chosen such that the agents in T_k trade their endowments along a cycle. Therefore, each T_k consists of agents who own copies of distinct object types. Then, for each $a \in \mathcal{O}$, we create a priority order $\rhd_a$ such that, for any distinct $i, j \in I_a$, $i \rhd_a j$ if $i \in T_k$, $j \in T_{k'}$, and $k < k'$. Since for every $T_k \in T$ and every $i \in T_k$, $\mu(i) \in B_i\big(O \backslash \bigcup_{\ell=1}^{k-1} \omega(T_\ell)\big)$, the outcome of TTC under the created priority structure must be μ. ■

B Proofs of Lemma 1 and Theorem 1

Proof of Lemma 1. Let μ be an outcome of GTTC. We prove that μ is in the rectified exclusion core.

<u>(Pareto efficiency)</u> We first prove that μ is Pareto efficient. Let $(G_1, G_2, \ldots, G_K)$ be the order of the departing groups in the procedure of GTTC that generates μ. Every member of G_1 obtains one of his most preferred objects among O. So, they cannot be made strictly better off. All of their most preferred objects are also held by G_1. After G_1 departs with their assignments, every member of G_2 obtains one of his most preferred objects among $O \backslash \mu(G_1)$. Thus, they cannot be made strictly better off without making any member of

G_1 worse off. Applying this argument inductively to the remaining groups, we conclude that μ is Pareto efficient.

(Unblock) Since μ is Pareto efficient, it cannot be rectification exclusion blocked by the grand coalition I. Now, we prove that μ cannot be rectification exclusion blocked by any coalition $C \subsetneq I$.

Suppose that μ is rectification exclusion blocked by a coalition C via another allocation σ. Without loss of generality, let C include all agents who are strictly better off in σ; that is, $C_{\sigma > \mu} = \{i \in I : \sigma(i) \succ_i \mu(i)\}$. Among the agents in $C_{\sigma>\mu}$, let j^* be an agent who first becomes satisfied in the procedure of GTTC. If there are multiple agents who become satisfied simultaneously, let j^* be one of them. Since $\sigma(j^*) \succ_{j^*} \mu(j^*)$, $\sigma(j^*)$ must be removed in the algorithm before j^* becomes satisfied. Let $(G_1, G_2, \dots, G_L)$ be the order of groups that depart before j^* becomes satisfied. Then, it must be that $\sigma(j^*) \in \mu(G_1 \cup G_2 \cup \cdots \cup G_L)$.

We first prove that there exists $i^* \in G_1 \cup G_2 \cup \cdots \cup G_L$ such that $\mu(i^*) \succ_{i^*} \sigma(i^*)$. Since $\sigma(j^*) \in \mu(G_1 \cup G_2 \cup \cdots \cup G_L)$ and $j^* \notin G_1 \cup G_2 \cup \cdots \cup G_L$, there must exist $i^* \in G_1 \cup G_2 \cup \cdots \cup G_L$ such that $\sigma(i^*) \notin \mu(G_1 \cup G_2 \cup \cdots \cup G_L)$. However, since for every $i \in G_1 \cup G_2 \cup \cdots \cup G_L$, $\{o \in O : o \succsim_i \mu(i)\} \subseteq \mu(G_1 \cup G_2 \cup \cdots \cup G_L)$, it must be that $\mu(i^*) \succ_{i^*} \sigma(i^*)$.

Since μ is rectification exclusion blocked by C via σ, $\mu(i^*) \in \Omega^*(C|\omega, \mu)$. Recall that $\Omega^*(C|\omega,\mu) = \omega(\cup_{k=0}^{\infty} C^k)$, where $C^0 = C$, and $C^k = C^{k-1} \cup \{i \in I \backslash C^{k-1} : \mathcal{I}_i(\mu(i)) \subseteq \omega(C^{k-1})\}$ for every $k \geq 1$. Since $i^* \notin C$, there exists $\ell \geq 1$ such that $i^* \in C^{\ell}$ but $i^* \notin C^{\ell-1}$. So, $\mu(i^*)$ is the endowment of some $i' \in C^{\ell-1}$. However, Claim 2 below implies that $i^* \in C^{\ell-1}$, which is a contradiction. So, μ cannot be rectification exclusion blocked by C, meaning that it is in the rectified exclusion core.

The proof of Claim 2 relies on Claim 1.

Claim 1. *In GTTC, for any $k \geq 0$, and for any cycle that is generated before j^* becomes satisfied, if an agent in the cycle belongs to C^k, then all agents in the cycle belong to C^k.*

Claim 2. *In GTTC, for any $k \geq 0$, if an agent i holds the endowment of some $i' \in C^k$ at some point before j^* becomes satisfied, then $i \in C^k$.*

Proof of Claim 1. For any $k \geq 0$, let $(Y_1, Y_2, \dots, Y_K)$ denote the order of cycles that are generated before j^* becomes satisfied and that involve an agent from C^k, where each Y_k represents the set of agents involved in the corresponding cycle. If several cycles are generated in the same step, their relative ranking is arbitrary in the order. Without loss of generality, we represent the first cycle Y_1 by

$$i_1 \to i_2 \to i_3 \to \cdots \to i_k \to i_1,$$

where $i_\ell \to i_{\ell+1}$ means that after clearing the cycle, i_ℓ obtains the object held by $i_{\ell+1}$. We assume that $i_1 \in C^k$. Let o_2 be the object held by i_2 in the cycle. After clearing the cycle, i_1 obtains o_2. So, $o_2 \sim_{i_1} \mu(i_1)$. There are two cases.

- Case 1: $k = 0$. It means that $i_1 \in C$. Because Y_1 is generated before j^* becomes satisfied and j^* is the first agent among $C_{\sigma>\mu}$ who becomes satisfied, it must be that $i_1 \in C_{\mu\sim\sigma}$. Therefore, $\mathcal{I}_{i_1}(\mu(i_1)) \subseteq \omega(C)$. This means that the owner of o_2 must belong to C. If i_2 is not the owner of o_2, then the owner of o_2 must be involved in a cycle before Y_1 is generated. However, this contradicts the definition of Y_1. So, i_2 must the owner of o_2. Then, similar to i_1, it must be that $i_2 \in C_{\mu\sim\sigma}$. Applying this argument inductively to the other agents in the cycle, we conclude that all agents in the cycle belong to $C_{\mu\sim\sigma}$.

- Case 2: $k > 0$. Since $i_1 \in C^k$, $\mathcal{I}_{i_1}(\mu(i_1)) \subseteq \omega(C^{k-1})$. So, the owner of o_2 must belong to C^{k-1}. If i_2 is not the owner of o_2, then the owner of o_2 must be involved in a cycle before Y_1 is generated. However, because $C^{k-1} \subseteq C^k$, the existence of such a cycle contradicts the definition of Y_1. So, i_2 must be the owner of o_2. It implies that $i_2 \in C^{k-1} \subseteq C^k$. Applying this argument inductively to the other agents in the cycle, we conclude that all agents in the cycle belong to C^k.

We then consider the second cycle Y_2. Without loss of generality, we still represent the cycle by

$$i_1 \to i_2 \to i_3 \to \cdots \to i_k \to i_1,$$

and assume that $i_1 \in C^k$. Let o_2 be the object held by i_2. After clearing the cycle, i_1 obtains o_2. There are two cases.

- Case 1: $k = 0$. It means that $i_1 \in C$. Because Y_2 is generated before j^* becomes satisfied and j^* is the first agent among $C_{\sigma>\mu}$ who becomes satisfied, it must be that $i_1 \in C_{\mu\sim\sigma}$. Therefore, $\mathcal{I}_{i_1}(\mu(i_1)) \subseteq \omega(C)$. So, the owner of o_2 must belong to C. If i_2 is not the owner of o_2, then the owner of o_2 and i_2 must be involved in cycles that are generated earlier than Y_2, and after clearing these cycles, i_2 obtains o_2. By the definition of Y_1, the owner of o_2 and i_2 must be involved in Y_1. In this case, we have proved that $i_2 \in C_{\mu\sim\sigma}$. If i_2 is the owner of o_2, then similar to i_1, $i_2 \in C_{\mu\sim\sigma}$. So, in any case, we have $i_2 \in C_{\mu\sim\sigma}$. Applying these arguments inductively to the other agents in the cycle, we conclude that all agents in the cycle belong to $C_{\mu\sim\sigma}$.

- Case 2: $k > 0$. Since $i_1 \in C^k$, $\mathcal{I}_{i_1}(\mu(i_1)) \subseteq \omega(C^{k-1})$. So, the owner of o_2 belongs to C^{k-1}. If i_2 is not the owner of o_2, then the owner of o_2 and i_2 must be involved in cycles that are generated earlier than Y_2, and after clearing these cycles, i_2 obtains o_2. Because $C^{k-1} \subseteq C^k$, by the definition of Y_1, the owner of o_2 and i_2 must be involved in Y_1. In this case, we have proved that $i_2 \in C^k$. If i_2 is the owner of o_2, we directly obtain that $i_2 \in C^{k-1} \subseteq C^k$. So, in any case, we have $i_2 \in C^k$. Applying these arguments inductively to the other agents in the cycle, we conclude that all agents in the cycle belong to C^k.

Applying the arguments inductively to the remaining cycles, we prove the claim. □

Proof of Claim 2. If $i = i'$, obviously $i \in C^k$. If $i \neq i'$, since i holds $\omega(i')$ at some point before j^* becomes satisfied, i and i' must be respectively involved in a sequence of cycles such that an agent i_1 first obtains $\omega(i')$ from i' in a cycle Z_1, then an agent i_2 obtains $\omega(i')$ from i_1 in a cycle Z_2, and so on, until i obtains $\omega(i')$ from an agent i_k in a cycle Z_{k+1}. By Claim 1, since $i' \in C^k$, all agents in Z_1 belong to C^k. Thus, $i_1 \in C^k$. Again, by Claim 1, all agents in Z_2 belong to C^k. Thus, $i_2 \in C^k$. By applying Claim 1 inductively to all these cycles, we conclude that all agents in these cycles belong to C^k. Thus, $i \in C^k$. □

This completes the proof of Lemma 1. ■

Proof of Theorem 1. (1) Lemma 1 implies that the rectified exclusion core is nonempty. By Proposition 3, the rectified exclusion core is a subset of the rectified strong core, which is Pareto efficient. So, it is Pareto efficient. Below, we prove that it is equivalence-closed.

Let μ and μ' be two equivalent allocations in a market. Suppose that μ is in the rectified exclusion core, yet μ' is not. Let μ' be rectification exclusion blocked by a coalition C via another σ. We then prove that μ is also rectification exclusion blocked by C via σ. We verify the following conditions.

(i) $\forall i \in C$, $\sigma(i) \succsim_i \mu'(i) \sim_i \mu(i)$, and $\exists j \in C$, $\sigma(j) \succ_j \mu'(j) \sim_j \mu(j)$.

(ii) $\forall k \in I \backslash C$, $\mu(k) \succ_k \sigma(k)$ if and only if $\mu'(k) \succ_k \sigma(k)$. Given $\mu'(k) \in \Omega^*(C|\omega, \mu')$, to prove that $\mu(k) \in \Omega^*(C|\omega, \mu)$, we prove that $\Omega^*(C|\omega, \mu) = \Omega^*(C|\omega, \mu')$.

Let $\Omega^*(C|\omega, \mu) = \omega(\cup_{k=0}^{\infty} C^k)$, where $C^0 = C$, and $C^k = C^{k-1} \cup \{i \in I \backslash C^{k-1} : \mathcal{I}_i(\mu(i)) \subseteq \omega(C^{k-1})\}$ for every $k \geq 1$. Let $\Omega^*(C|\omega, \mu') = \omega(\cup_{k=0}^{\infty} \tilde{C}^k)$, where $\tilde{C}^0 = C$, and $\tilde{C}^k = \tilde{C}^{k-1} \cup \{i \in I \backslash \tilde{C}^{k-1} : \mathcal{I}_i(\mu'(i)) \subseteq \omega(\tilde{C}^{k-1})\}$ for every $k \geq 1$. Since $\mathcal{I}_i(\mu(i)) = \mathcal{I}_i(\mu'(i))$ for all $i \in I$, it holds that $C^k = \tilde{C}^k$ for all $k \geq 0$. Thus, $\Omega^*(C|\omega, \mu) = \Omega^*(C|\omega, \mu')$.

(iii) Since $C_{\sigma\sim\mu} = C_{\sigma\sim\mu'}$, $\forall i \in C_{\sigma\sim\mu}$, $\mathcal{I}_i(\mu(i)) = \mathcal{I}_i(\mu'(i)) \subseteq \omega(C)$.

(2) By Proposition 3, the rectified exclusion core is a subset of the rectified strong core, which is a subset of the weak core. So, the rectified exclusion core is a subset of the weak core. Below, we prove that it is a superset of the exclusion core.

Consider any allocation μ that is rectification exclusion blocked by a coalition C via another σ. If μ is not Pareto efficient, then it is evident that μ is also exclusion blocked.

If μ is Pareto efficient, there must exist $j \in I \backslash C$ such that $\mu(j) \succ_j \sigma(j)$, and for every such j, $\mu(j) \in \Omega^*(C|\omega,\mu)$. In the following, we prove that, for every such j, $\mu(j) \in \Omega(C_{\sigma \succ \mu}|\omega,\mu)$. This means that μ is exclusion blocked by $C_{\sigma \succ \mu}$ via σ.

It is evident that $\Omega^*(C|\omega,\mu) \subseteq \Omega(C|\omega,\mu)$ and $\Omega(C|\omega,\mu) = \Omega(C_{\sigma\sim\mu}|\omega,\mu) \cup \Omega(C_{\sigma\succ\mu}|\omega,\mu)$. So, for every $j \in I\backslash C$ such that $\mu(j) \succ_j \sigma(j)$, either $\mu(j) \in \Omega(C_{\sigma\sim\mu}|\omega,\mu)$ or $\mu(j) \in \Omega(C_{\sigma\succ\mu}|\omega,\mu)$. In the latter case, we are done. In the former case, there must exist a chain $j \to j_1 \to j_2 \to \cdots \to j_m \to i$, in which every $x \to y$ means that agent x obtains agent y's endowment in μ, and $i \in C_{\sigma\sim\mu}$, yet every $j_k \notin C_{\sigma\sim\mu}$ $(k = 1,\ldots,m)$. We prove that $\mu(i) \in \Omega(C_{\sigma\succ\mu}|\omega,\mu)$. This implies that $\mu(j) \in \Omega(C_{\sigma\succ\mu}|\omega,\mu)$

Because $i \in C_{\sigma\sim\mu}$, $\mu(i) \in \omega(C)$. Then, either $\mu(i) \in \omega(C_{\sigma\succ\mu})$ or $\mu(i) \in \omega(C_{\sigma\sim\mu})$. In the former case, we are done. In the latter case, $\mu(i) = \omega(i_1)$ for some $i_1 \in C_{\sigma\sim\mu}$, and thus $\mu(i_1) \in \omega(C)$. If $\mu(i_1) \in \omega(C_{\sigma\succ\mu})$, then $\mu(i) \in \Omega(C_{\sigma\succ\mu}|\omega,\mu)$, and we are done. If $\mu(i_1) \in \omega(C_{\sigma\sim\mu})$, then $\mu(i_1) = \omega(i_2)$ for some $i_2 \in C_{\sigma\sim\mu}$, and thus $\mu(i_2) \in \omega(C)$. Continuing this process, we will find a chain $i \to i_1 \to i_2 \to \ldots \to i_\ell$ in which all agents belong to C and every agent obtains the next agent's endowment. It is impossible that all of these agents belong to $C_{\sigma\sim\mu}$, because otherwise they would form a cycle, which contradicts that $j_m \notin C_{\sigma\sim\mu}$ obtains i's endowment. Therefore, the last agent i_ℓ in the chain must belong to $C_{\sigma\succ\mu}$, which means that $\mu(i) \in \Omega(C_{\sigma\succ\mu}|\omega,\mu)$.

(3) When the strong core is nonempty, by Proposition 2 and Proposition 3, the exclusion core and the rectified strong core both coincide with the strong core. Since the rectified exclusion core lies between the exclusion core and the rectified strong core, it must coincide with the strong core and with the exclusion core. ■