

# Multiple change-points detection based on $U$ -Statistics under weak dependence

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## Abstract

We study multiple change-points detection using multi-samples tests based on  $U$ -statistics for absolutely regular observations. Our results extend those of [Ngatchou-Wandji et al. \(2022\)](#) concerned with the study of one single change-point. The asymptotic distributions of the test statistics under the null hypothesis and under a sequence of local alternatives are given explicitly, and the tests are shown to be consistent. A small set of simulations is done for evaluating the performance of the tests in detecting multiple changes in the mean, variance and autocorrelation of some simple times series models.

**Keywords:** Multiple change-points,  $U$ -statistics, Testing, Weak convergence, Weak dependance

**MSC Classification:** 60F17 , 62F03 , 62M10

# 1 Introduction

Testing for changes in the structure of data is known in statistics as a change-point problem. This is of great importance in a wide range of disciplines, such as industrial quality control, financial market, epidemic, medical diagnostics, hydrology and climatology, to name a few.

Figure 1 presents the chronograms of series with two changes, simulated from the model (8) in Section 4. Both changes occurred either in the mean, or in the variance, or in the autocorrelation, or in the mean and the autocorrelation, or in the mean and the variance, or in the variance and the autocorrelation. Given such series, the change-point theory provides methods for their detection and for estimating their locations.

The literature on change-point is vast. Parametric as well as nonparametric methods are used for both independent and dependent observations. The major part of the earliest works was done in the independent context. A small sample of these are Page (1954) who first, studied change-point problem, Chernoff and Zacks (1964) who propose test statistics for shifts detection in the mean of a normal distribution function. This work was extended to exponential family by Kander and Zacks (1966), Gardner (1969) and MacNeill (1974). Matthews et al. (1985) study maximal score statistics to test for constant hazard against a change-point alternative. Haccou et al. (1988) propose a likelihood ratio test for a change-point in a sequence of independent exponentially distributed random variables, and prove its optimality in the sense of Bahadur.

Most of the recent papers on change-points are in the dependence context. Among them are Vogelsang (1997) who constructs Wald-type tests for breaks detection in the trend of a dynamic time series. Vogelsang (1999) who studies tests for change detection in the mean of various time series models. Berkes et al. (2004), Gombay (2008), Gombay and Serban (2009), Bardet et al. (2012), Kengne (2012), Bardet and Kengne (2014) study change detection in the parameters of various sub-classes of nonlinear heteroscedastic time series models. More recently, Ma et al. (2020), Mohr and Selk (2020) and Ngatchou-Wandji and Ltaifa (2023b,a) have also studied change detection within sub-classes of nonlinear heteroscedastic time series models.

However, a substantial part of the literature is devoted to the case of one single change-point. In this paper, we stress on multiple change-points detection for weakly dependent observations containing a given number of changes. This important problem has attracted more and more attention recently. Our approach, based on  $U$ -statistics, is an extension of Ngatchou-Wandji et al. (2022) who study the case of one single change.

There are various approaches in the literature dealing with  $U$ -statistics in change point analysis. Döring (2010) proposes an estimation of change-points by using  $U$ -statistics, while Döring (2011) studies the convergence of such estimators. Ferger (2001) estimates change location in the independence context. Horváth and Husková (2005) compute critical values for various tests based on  $U$ -statistics to detect a possible change. Kirch and Stoehr (2022) propose a general framework of sequential change-point testing procedures. Dehling et al. (2022) study the large-sample behavior

of change-point tests in the case of short-range dependent data. Additional change-point studies based on  $U$ -statistics are, among others [Aly and Subhash \(1997\)](#), [Gombay \(2000, 2001\)](#), [Orasch \(2004\)](#), [Horváth and Husková \(2005\)](#), [Dehling et al. \(2013\)](#) and [Dehling et al. \(2017\)](#), [Račkauskas and Wendler \(2020\)](#).

The local power of existing change-points tests are generally not studied, in particular those of multichange-points tests. For the tests studied in this paper, not only their null distributions and consistency are studied, but also their local powers. These tests are derived from a basic process of the following form

$$\mathcal{Z}_n^*(t_1, t_2, \dots, t_k) = n^{-3/2} \sum_{l=1}^k \sum_{i=[nt_{l-1}]+1}^{[nt_l]} \sum_{j=[nt_l]+1}^{[nt_{l+1}]} h(X_i, X_j),$$

for  $0 \leq t_{l-1} < t_l \leq 1$ ,  $1 \leq l \leq k+1$ , where  $k$  stands for the number of changes in the series  $X_1, \dots, X_n$ ,  $n$  is the sample length,  $[nt_0] = 1$ ,  $[nt_{k+1}] = n$  (by convention) and  $h : \mathbb{R}^2 \rightarrow \mathbb{R}$  is a kernel function. This process is a generalization of that defined in [Ngatchou-Wandji et al. \(2022\)](#) from the one single change-point to the multiple change-points. The study of the asymptotic properties of such processes is simplified by the use of  $U$ -statistics and the Hoeffding decomposition (see, eg, [Hoeffding \(1948\)](#)).

The paper is organized as follows. In Section 2, we give some useful definitions and list the main assumptions. In Section 3 we study the asymptotic properties of our test statistics under the null hypothesis, under a sequence of local alternatives and under fixed alternatives. Practical considerations are presented and discussed in Section 4. Section 5 concludes our work, while the last section contains the proofs of the main results.

## 2 Definitions and assumptions

In this section, we define our test statistics and give the main assumptions needed for the study of their behavior under the hypotheses considered.

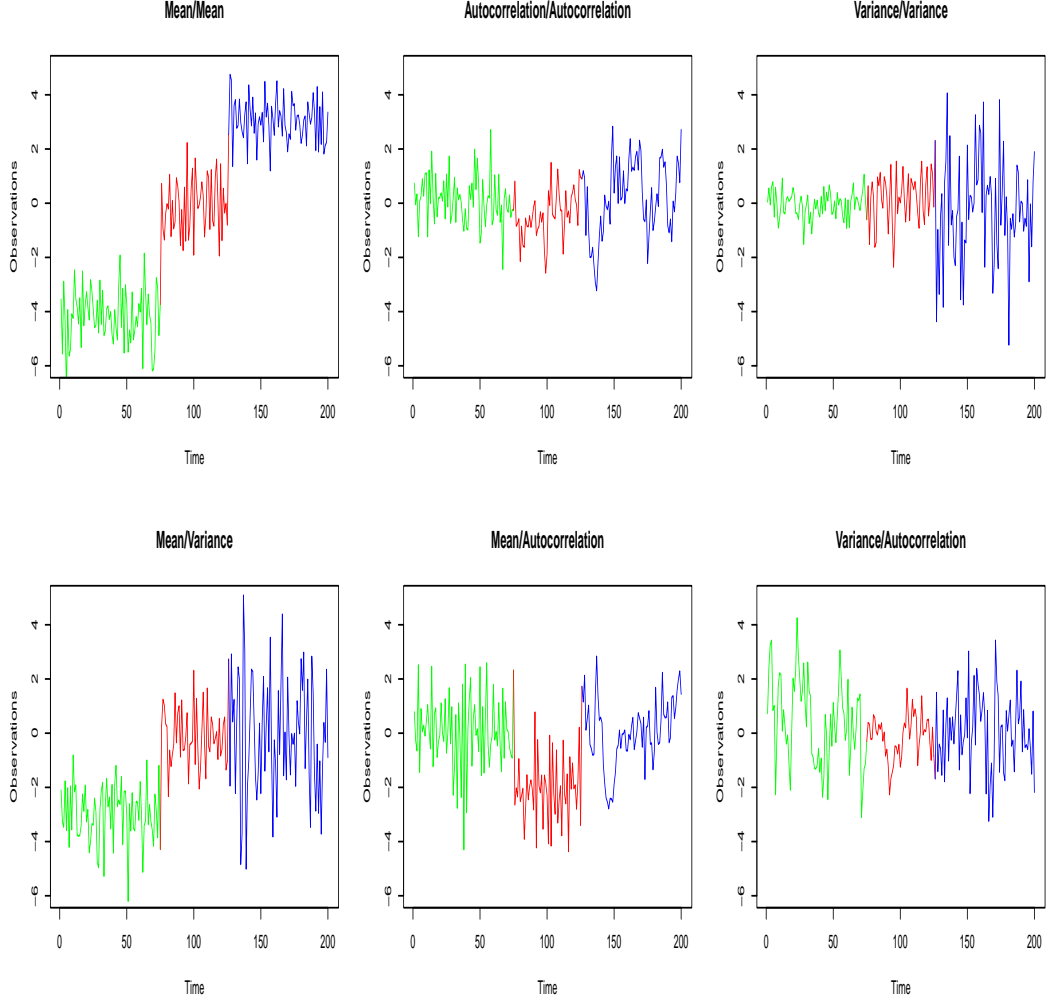
Let  $Q$  and  $R$  be cumulative distribution functions. Let  $h$  be a kernel defined on  $\mathbb{R}^2$ . Define by  $\theta_h(Q, R)$  the following real number

$$\theta_h(Q, R) = \int \int h(x, y) dQ(x) dR(y).$$

For any  $i = 1, 2, \dots, n$ , denote by  $F_i$  be the cumulative distribution function of  $X_i$ . Our aim is to check the equality of the  $F_i$ 's. For doing this, we construct tests for testing the hypothesis  $\mathcal{H}_0$  against the fixed alternative  $\mathcal{H}_1^k$ , or against the sequence of local alternatives  $\mathcal{H}_{1,n}^k$  defined respectively by

$\mathcal{H}_0$ : The distribution function of  $X_i$  is  $F_1$  for all  $i \in \{1, 2, \dots, n\}$

$\mathcal{H}_1^k$ : There exist  $k+2$  real numbers  $0 = t_{0,0} < t_{0,1} < t_{0,2} < \dots < t_{0,k} < t_{0,k+1} = 1$  such that the cumulative distribution function of  $X_i$  is  $\dot{F}_l$  if  $[nt_{0,l-1}] \leq i < [nt_{0,l}]$ ,  $1 \leq l \leq k+1$  and the cumulative distribution function of  $X_n$  is  $\dot{F}_{k+1}$ ,



**Fig. 1** Chronograms of series with two changes. On the first row, the first graphic is that of a series with two changes in the mean. The second is that of a series with two changes in the autocorrelation, the last is that of a series with two changes in the variance. On the second row, the first chronogram is that of a series with one change in the mean and one in the autocorrelation. The next is that of a series with one changes in the mean and one in the variance, and the last is that of a series with one change in the variance and another one in the autocorrelation.

there exist some integers  $l$  and  $l'$ ,  $1 \leq l < l' \leq k + 1$  such that  $\dot{F}_l \neq \dot{F}_{l'}$  with  $\theta_h(\dot{F}_l, \dot{F}_l) \neq \theta_h(\dot{F}_l, \dot{F}_{l'})$ .

$\mathcal{H}_{1,n}^k$ : There exist  $k + 2$  real values :  $0 = t_{0,0} < t_{0,1} < t_{0,2} < \dots < t_{0,k} < t_{0,k+1} = 1$  such that  $F_{[nt_l]+1} = F_{[nt_l]+2} = \dots = F_{[nt_{l+1}]} = F_{l+1}^{(n)}$ ,  $0 \leq l \leq k$ , the distribution

function of  $X_n$  is  $F_{k+1}^{(n)}$ , there exists some  $l$ ,  $1 \leq l \leq k+1$  such that  $\theta_h(F_1, F_l^{(n)}) = \theta_h(F_1, F_1) + n^{-1/2}[A_l + o(1)]$ , where  $A_l$  is some constant.

**Remark 1.** As example of local alternatives  $\mathcal{H}_{1,n}^k$ , we can take those defined by  $F_l^{(n)}(x) = F_1(x + n^{-1/2}\gamma_l)$ , for some  $\gamma_l \in \mathbb{R}$ , together with a twice differentiable kernel function  $h$  satisfying  $\int \int (\partial h(x, y)/\partial y) dF_1(x) dF_1(y) < \infty$  and  $|\partial^2 h(x, y)/\partial^2 y| < C$  for some  $C > 0$ . These can be checked easily by an application of the Taylor-Young formula. The computations with this example, yield, for any  $l$ ,

$$A_l = -\gamma_l \int \int \frac{\partial h}{\partial u}(x, u) dF_1(x) dF_1(u).$$

For solving our testing problem we can construct Kolmogorov-Smirnov (KS) or Cramér-von Mises (CV) type-tests based respectively on the following KS and CM statistics

$$\mathcal{T}_{1,n} = \max_{1 < m_1 < m_2 < \dots < m_k < n} \left| n^{-3/2} \sum_{l=1}^k \sum_{i=m_{l-1}}^{m_l} \sum_{j=m_l+1}^{m_{l+1}} \left[ h(X_i, X_j) - \theta_h(\hat{F}_1, \hat{F}_1) \right] \right| \quad (1)$$

$$\mathcal{T}_{2,n} = \frac{1}{n^k} \sum_{1 < m_1 < m_2 < \dots < m_k < n} \left\{ n^{-3/2} \sum_{l=1}^k \sum_{i=m_{l-1}}^{m_l} \sum_{j=m_l+1}^{m_{l+1}} \left[ h(X_i, X_j) - \theta_h(\hat{F}_1, \hat{F}_1) \right] \right\}^2 \quad (2)$$

where by convention  $m_0 = 1$  and  $m_{k+1} = n$  and  $\hat{F}_1$  stands for any consistent estimator of  $F_1$ , a simple example being the associated empirical cumulative distribution function.

Denote by  $[x]$  the integer part of any real number  $x$  and by  $\Theta_k$  the subset of  $[0, 1]^k$  defined by

$$\Theta_k = \{t = (t_1, \dots, t_k) \in [0, 1]^k; 0 < t_1 < \dots < t_k < 1\}.$$

Noting that for any  $k \in \{1, \dots, n-1\}$ , there exists  $t_l^* \in [0, 1]$  ( $1 \leq l \leq k$ ) such that  $k = [nt_l^*]$ , one can write, asymptotically,

$$\begin{aligned} \mathcal{T}_{1,n} &= \sup_{t_1, \dots, t_k \in [0, 1]} |\mathcal{Z}_n(t_1, \dots, t_k)| \\ \mathcal{T}_{2,n} &= \int_0^1 \dots \int_0^1 \mathcal{Z}_n^2(t_1, \dots, t_k) dt_1 \dots dt_k, \end{aligned}$$

where  $(\mathcal{Z}_n(\cdot))$  stands for the sequence of stochastic processes defined for all  $t = (t_1, \dots, t_k) \in [0, 1]^k$ , by

$$\mathcal{Z}_n(t_1, \dots, t_k) = \begin{cases} n^{-3/2} \sum_{l=1}^k \sum_{i=[nt_{l-1}]+1}^{[nt_l]} \sum_{j=[nt_l]+1}^{[nt_{l+1}]} \{h(X_i, X_j) - \theta_h(F_1, F_1)\}, & t \in \Theta_k \\ 0 & t \notin \Theta_k, \end{cases} \quad (3)$$

with the convention that  $[nt_0] = 1$  and  $[nt_{k+1}] = n$ .

For any kernel  $h$ , noting that  $\dot{F}_1 = F_1$ , define the following functions used later in the Hoeffding decomposition :

$$\begin{aligned} h_{F_1,1}(x) &= \int h(x, y) dF_1(y) - \theta_h(F_1, F_1), \quad x \in \mathbb{R} \\ h_{F_1,2}(y) &= \int h(x, y) dF_1(x) - \theta_h(F_1, F_1), \quad y \in \mathbb{R} \\ h_{\dot{F}_l,1}(x) &= \int h(x, y) d\dot{F}_l(y) - \theta_h(\dot{F}_{l-1}, \dot{F}_l), \quad l = 2, \dots, k, \quad x \in \mathbb{R} \\ h_{\dot{F}_l,2}(y) &= \int h(x, y) d\dot{F}_l(x) - \theta_h(\dot{F}_{l-1}, \dot{F}_l), \quad l = 2, \dots, k, \quad y \in \mathbb{R} \\ g_{F_1, F_1}(x, y) &= h(x, y) - h_{F_1,1}(x) - h_{F_1,2}(y) + \theta_h(F_1, F_1), \quad x, y \in \mathbb{R} \end{aligned}$$

$$g_{\dot{F}_{l-1}, \dot{F}_l}(x, y) = h(x, y) - h_{\dot{F}_l,1}(x) - h_{\dot{F}_l,2}(y) + \theta_h(\dot{F}_{l-1}, \dot{F}_l), \quad l = 2, \dots, k, \quad x, y \in \mathbb{R}.$$

Also, we define the following real numbers

$$\sigma_{p,r} = \mathbb{E}[h_{F_1,p}(X_1)h_{F_1,r}(X_1)] + 2 \sum_{i=1}^{\infty} \mathbb{Cov}(h_{F_1,p}(X_1), h_{F_1,r}(X_{i+1})), \quad p, r = 1, 2.$$

Now, we recall from [Harel and Puri \(1994\)](#) that a non-necessarily stationary triangular sequence  $\{\mathcal{V}_{ni}, 1 \leq i \leq n, n \geq 1\}$  is absolutely regular if, as  $m \rightarrow \infty$ ,

$$\beta(m) = \sup_{n \in \mathbb{N}} \sup_{m \leq n} \max_{1 \leq j \leq n-m} \mathbb{E} \left\{ \sup_{A \in \mathcal{A}_{n,j+m}^{\infty}} \left| \mathbb{P}(A \mid \mathcal{A}_{n,0}^j) - \mathbb{P}(A) \right| \right\} \longrightarrow 0,$$

with  $\mathcal{A}_{n,i}^j$  standing for the  $\sigma$ -algebra generated by  $\mathcal{V}_{ni}, \dots, \mathcal{V}_{nj}$ ,  $i < j$ ,  $i, j \in \mathbb{N} \cup \{\infty\}$ .

In the remaining, we make the following main assumptions :

(A1) The sequence  $(X_i)_{i \in \mathbb{N}}$  is stationary and absolutely regular with rate

$$\beta(n) = \mathcal{O}(\tau^n), \quad 0 < \tau < 1. \quad (4)$$

- (A2) For any  $l = 1, \dots, k$ , consider  $(Y_i^l)_{1 \leq i \leq n}$  a sequence of stationary and absolute regular random variables with rate (4) and cumulative distribution function  $\dot{F}_l$ . For any  $i, j \in \mathbb{N}$ , the absolute regular dependence between  $Y_i^l$  and  $Y_j^l$  is the same as the dependence between  $X_i$  and  $X_j$ . The random variables  $Y_i^1$ 's have the same law as the random variables  $X_i$ .
- (A3) For any  $l = 1, \dots, k$  consider  $(Y_{ni}^l)_{1 \leq i \leq n, n \geq 1}$  an array of stationary and absolute regular random variables with cumulative distribution function  $F_l^{(n)}(x) = F_1(x + n^{-1/2}\gamma_l)$ . We assume the cumulative distribution functions  $F_{l,ij}^{(n)}$  and  $F_{l,ij}^{*(n)}$  of the  $(Y_{ni}^l, Y_{nj}^l)$ 's and  $(X_i, Y_{nj}^l)$ 's respectively satisfy

$$\lim_{n \rightarrow \infty} F_{l,ij}^{(n)}(x, y) = F_{ij}(x, y) \text{ and } \lim_{n \rightarrow \infty} F_{l,ij}^{*(n)}(x, y) = F_{ij}(x, y), \quad 1 \leq i < j \leq n,$$

where  $F_{ij}$  is the cumulative distribution function of  $(X_i, X_j)$ .

**Remark 2.** Assumption (A1) is met by ARMA, GARCH and some other usual nonlinear time series as indicated for instance in Example 2.2 of [Schmidt \(2024\)](#). Assumptions (A1) and (A2) allow for the study of the behavior of the test statistic under the alternatives, while assumptions (A1) and (A3) allow for the study under the local alternatives defined by the sequence  $(\mathcal{H}_{1,n}^k)$ . In particular (A3) implies that  $\mathcal{H}_0$  and  $(\mathcal{H}_{1,n}^k)$  are contiguous (see, e.g. [Le Cam \(1986\)](#)).

### 3 Asymptotics

This section is devoted to the statement of our theoretical results. The main proofs are postponed to the last section.

#### 3.1 Results for general kernels

In this subsection, we state the results for general kernels  $h$ . These results show that, in general, the test statistics considered are not asymptotic distribution free. However, we explain how to approximate the limiting distribution of the Cramér-type test statistic presented later.

**Theorem 1.** Assume that (A1)-(A2) hold. Then, under  $\mathcal{H}_0$ , if for some  $\delta > 0$

$$\max \left\{ \sup_{ij} \mathbb{E} \left( |h(X_i, X_j)|^{2+\delta} \right), \int \int_{\mathbb{R}^2} |h(x, y)|^{2+\delta} dF_1(x) dF_1(y) \right\} < \infty,$$

then for any  $p, r = 1, 2$ ,  $\sigma_{pr} < \infty$ .

If in addition  $\sigma_{pr} > 0$ ,  $1 \leq p, r \leq 2$ , then  $\{\mathcal{Z}_n(t_1, t_2, \dots, t_k); t_1, t_2, \dots, t_k \in [0, 1]\}_{n \in \mathbb{N}}$  converges in distribution towards the process  $\mathcal{Z}(\cdot)$  defined for any  $t = (t_1, \dots, t_k) \in [0, 1]^k$  by

$$\mathcal{Z}(t_1, t_2, \dots, t_k) =$$

$$\begin{cases} \sum_{l=1}^k \{(t_{l+1} - t_l) [W_1(t_l) - W_1(t_{l-1})] + (t_l - t_{l-1}) [W_2(t_{l+1}) - W_2(t_l)]\}, & t \in \Theta_k, \\ 0 & t \notin \Theta_k, \end{cases}$$

where by convention  $t_0 = 0$ ,  $t_{k+1} = 1$ ,  $\{(W_1(t), W_2(t))\}_{0 \leq t \leq 1}$ , is a two-dimensional zero-mean Brownian motion with covariance kernel matrix with entries  $\text{Cov}(W_p(s), W_r(t)) = \min(s, t) \sigma_{pr}$ ,  $p, r = 1, 2$ ,  $s, t \in [0, 1]$ .

**Proof** - See Appendix.

**Remark 3.** The covariance kernel  $\Gamma$  of the limiting process  $\mathcal{Z}(\cdot)$  is given for any  $s = (s_1, \dots, s_k)$ ,  $t = (t_1, \dots, t_k) \in \mathbb{R}^k$  by  $\Gamma(s, t) = 0$  if  $s = (s_1, \dots, s_k)$ ,  $t = (t_1, \dots, t_k) \notin \Theta_k$  and for  $s = (s_1, \dots, s_k)$ ,  $t = (t_1, \dots, t_k) \in \Theta_k$ , by

$$\begin{aligned} \Gamma(s, t) = & \sigma_{11} \sum_{l=1}^k (t_{l+1} - t_l)(s_{l+1} - s_l)(t_l \wedge s_l - t_l \wedge s_{l-1} - t_{l-1} \wedge s_l + t_{l-1} \wedge s_{l-1}) \\ & + \sigma_{12} \sum_{l=1}^k (t_{l+1} - t_l)(s_l - s_{l-1})(t_l \wedge s_{l+1} - t_l \wedge s_l - t_{l-1} \wedge s_{l+1} + t_{l-1} \wedge s_l) \\ & + \sigma_{12} \sum_{l=1}^k (t_l - t_{l-1})(s_{l+1} - s_l)(t_{l+1} \wedge s_l - t_{l+1} \wedge s_{l-1} - t_l \wedge s_l + t_l \wedge s_{l-1}) \\ & + \sigma_{22} \sum_{l=1}^k (t_l - t_{l-1})(s_l - s_{l-1})(t_{l+1} \wedge s_{l+1} - t_{l+1} \wedge s_l - t_l \wedge s_{l+1} + t_l \wedge s_l) \\ & + 2\sigma_{11} \sum_{l=1}^k \sum_{j \leq l} (t_{l+1} - t_l)(s_{j+1} - s_j)(t_l \wedge s_j - t_l \wedge s_{j-1} - t_{l-1} \wedge s_j + t_{l-1} \wedge s_{j-1}) \\ & + 2\sigma_{12} \sum_{l=1}^k \sum_{j \leq l} (t_{l+1} - t_l)(s_j - s_{j-1})(t_l \wedge s_{j+1} - t_l \wedge s_j - t_{l-1} \wedge s_{j+1} + t_{l-1} \wedge s_j) \\ & + 2\sigma_{12} \sum_{l=1}^k \sum_{j \leq l} (t_l - t_{l-1})(s_{j+1} - s_j)(t_{l+1} \wedge s_j - t_{l+1} \wedge s_{j-1} - t_l \wedge s_j + t_l \wedge s_{j-1}) \\ & + 2\sigma_{22} \sum_{l=1}^k \sum_{j \leq l} (t_l - t_{l-1})(s_j - s_{j-1})(t_{l+1} \wedge s_{j+1} - t_{l+1} \wedge s_j - t_l \wedge s_{j+1} + t_l \wedge s_j) \end{aligned} \quad (5)$$

**Theorem 2.** Assume (A1) and (A3) hold,  $h$  is twice differentiable with bounded second-order derivatives  $\partial^2 h(x, y)/\partial x \partial y$ , and the integral  $\int \int (\partial h(x, y)/\partial y) dF_1(x) dF_1(y)$  is finite. Then, under  $\mathcal{H}_{1,n}^k$ , if there exists  $\delta > 0$  such



that for any  $1 \leq l \leq k+1$ ,

$$\begin{aligned} & \sup_{1 \leq i, j \leq n} \mathbb{E} \left( |h(X_i, X_j)|^{2+\delta} \right), \quad \sup_{n \geq 1} \sup_{i, j} \mathbb{E} \left( |h(Y_{ni}^l, Y_{nj}^l)|^{2+\delta} \right), \\ & \sup_{n \geq 1} \sup_{1 \leq i, j \leq n} \mathbb{E} \left( |h(X_i, Y_{nj}^l)|^{2+\delta} \right), \quad \int \int_{\mathbb{R}^2} |h(x, y)|^{2+\delta} dF_1(x) dF_1(y), \\ & \sup_{n \geq 1} \int \int_{\mathbb{R}^2} |h(x, y)|^{2+\delta} dF_l^{(n)}(x) dF_l^{(n)}(y), \quad \sup_{n \geq 1} \int \int_{\mathbb{R}^2} |h(x, y)|^{2+\delta} dF_1(x) dF_l^{(n)}(y) \end{aligned}$$

are finite, if for any  $p, r = 1, 2$ ,  $\sigma_{pr} > 0$ , then the sequence of processes  $\{\mathcal{Z}_n(t_1, t_2, \dots, t_k); t_1, t_2, \dots, t_k \in [0, 1]\}_{n \in \mathbb{N}}$  converges in distribution towards a Gaussian process  $\tilde{\mathcal{Z}}(\cdot)$  with representation given for any  $t = (t_1, \dots, t_k) \in [0, 1]^k$  by

$$\tilde{\mathcal{Z}}(t_1, t_2, \dots, t_k) = \begin{cases} \mathcal{Z}(t_1, t_2, \dots, t_k) + \sum_{l=1}^k (t_{l+1} - t_l)(t_l - t_{l-1})A_l, & t \in \Theta_k \\ 0 & t \notin \Theta_k, \end{cases}$$

where by convention  $t_0 = 0$ ,  $t_{k+1} = 1$  and  $\{\mathcal{Z}(t_1, t_2, \dots, t_k), t_1, t_2, \dots, t_k \in [0, 1]\}$  is the zero-mean Gaussian process defined in Theorem 1 and the constants  $A_l$  are invoked in  $\mathcal{H}_{1,n}^k$ .

**Proof** - See Appendix.

**Corollary 3.** Assume that the cumulative distribution function of  $X_i$  is  $\dot{F}_l$  for  $[nt_{0,l-1}] \leq i < [nt_{0,l}]$ ,  $1 \leq l \leq k+1$  and that of  $X_n$  is  $\dot{F}_{k+1}$ . Assume there exist constants  $B_l$ ,  $1 \leq l \leq k+1$  such that  $F_l^{(n)}(x) = F_1(x + n^{-1/2}B_l)$  for all  $x \in \mathbb{R}$ , and that the kernel function  $h$  is twice differentiable with  $\int \int (\partial h(x, y)/\partial y) dF_1(x) d\dot{F}_l(y) < \infty$  and  $\partial^2 h(x, y)/\partial^2 y$  bounded. Then Theorem 2 holds.

**Proof** - It suffices to check the assumptions of Theorem 2.

**Theorem 4.** Assume (A1)-(A3) hold and that under  $\mathcal{H}_1^k$ , the integrability conditions in Theorem 2 hold. Then, as  $n$  tends to infinity, the sequence of processes  $n^{-1/2}\mathcal{Z}_n^*(t_1, t_2, \dots, t_k)$  converges almost surely in  $D([0, 1]^k)$  to either of the following limits

$$\begin{aligned} (i) & \sum_{l=1}^k \left[ \theta_h(\dot{F}_l, \dot{F}_l)(t_l - t_{0,l-1})(t_{0,l} - t_l) + \theta_h(\dot{F}_l, \dot{F}_{l+1})(t_l - t_{0,l-1})(t_{0,l+1} - t_{0,l}) \right], \\ & \text{if } t_{0,l-1} \leq t_l \leq t_{0,l}, \quad l = 1, \dots, k, (t_1, t_2, \dots, t_k) \in \Theta_k; \\ (ii) & \sum_{l=1}^k \left[ \theta_h(\dot{F}_{l+1}, \dot{F}_{l+1})(t_l - t_{0,l-1})(t_{0,l+1} - t_l) + \theta_h(\dot{F}_l, \dot{F}_{l+1})t_{0,l}(t_{0,l+1} - t_l) \right], \\ & \text{if } t_{0,l} \leq t_l \leq t_{0,l+1}, \quad l = 1, \dots, k, (t_1, t_2, \dots, t_k) \in \Theta_k; \\ (iii) & 0, \text{ if } (t_1, t_2, \dots, t_k) \notin \Theta_k \end{aligned} \tag{6}$$

where by convention  $t_{0,0} = 0$ .

**Proof** - See Appendix.

From Theorem 4, we easily deduce the following corollary.

**Corollary 5.** *Under  $\mathcal{H}_0$  and the conditions of Theorem 1, as  $n$  tends to infinity, almost surely,*

$$n^{-1/2} \mathcal{Z}_n^*(t_1, t_2, \dots, t_k) \longrightarrow \theta_h(F_1, F_1) \sum_{l=1}^k t_l(t_{0,l+1} - t_l), \quad (t_1, \dots, t_k) \in \Theta_k.$$

Let  $\lambda$  be any square integrable function on  $[0, 1]^k$ . Define the integral operator  $\mathcal{K}_\Gamma$  by

$$\mathcal{K}_\Gamma[\lambda(\cdot)] = \int_{[0,1]^k} \Gamma(\cdot, s) \lambda(s) ds, \quad (7)$$

where we recall that  $\Gamma$  is the covariance kernel of the process  $\mathcal{Z}$  defined in Theorem 1.

**Theorem 6.** *Assume that the assumptions of Theorem 2 hold. Let  $(\mathcal{Z}(s) : s \in [0, 1]^k)$  be the limiting process defined in Theorems 1 and 2. Then*

i- *Under  $\mathcal{H}_0$ , as  $n$  tends to infinity, one has the following convergence in distribution,*

$$\mathcal{T}_{1,n} \longrightarrow \sup_{s \in [0,1]^k} |\mathcal{Z}(s)| \quad \text{and} \quad \mathcal{T}_{2,n} \longrightarrow \int_{[0,1]^k} \mathcal{Z}^2(s) ds = \sum_{j \geq 1} \zeta_j \chi_j^2,$$

where the  $\chi_j^2$ 's are iid chi-square random variables with one degree of freedom and the  $\zeta_j$ 's are standing for the eigenvalues of the operator  $\mathcal{K}_\Gamma$ .

ii- *Under  $\mathcal{H}_{1,n}^k$ , as  $n$  tends to infinity, one has the following convergence in distribution,*

$$\mathcal{T}_{1,n} \longrightarrow \sup_{s \in [0,1]^k} |\tilde{\mathcal{Z}}(s)| \quad \text{and} \quad \mathcal{T}_{2,n} \longrightarrow \int_{[0,1]^k} \tilde{\mathcal{Z}}^2(s) ds = \sum_{j \geq 1} \zeta_j \chi_j^{*2},$$

where the  $\chi_j^{*2}$ 's are iid non-central chi-square random variables with one degree of freedom and non-centrality parameters  $\rho_j^2 \zeta_j^{-1}$  and

$$\rho_j = \sum_{l=1}^k A_l \int_0^1 \dots \int_0^1 (s_{l+1} - s_l)(s_l - s_{l-1}) g_j(s_1, \dots, s_k) \mathbb{1}((s_1, \dots, s_k) \in \Theta_k) ds_1 \dots ds_k,$$

in which the  $g_j$ 's stand for the eigenfunctions of the integral operator  $\mathcal{K}_\Gamma$  associated with the eigen-value  $\zeta_j$ 's.

iii- *Under  $\mathcal{H}_1^k$ , as  $n$  tends to infinity, one has the following convergence in probability,*

$$\mathcal{T}_{1,n} \longrightarrow \infty, \quad \mathcal{T}_{2,n} \longrightarrow \infty.$$

**Proof - i-** It is immediate from Theorem 1 that  $\mathcal{T}_{1,n}$  and  $\mathcal{T}_{2,n}$  converge in distribution respectively to  $\sup_{t \in \Theta_k} |\mathcal{Z}(t)|$  and  $\int_{\Theta_k} \mathcal{Z}^2(t) dt$ .

We prove now that the distribution of  $\int_{\Theta_k} \mathcal{Z}^2(t) dt$  is that of a sum of weighted iid chi-square distribution with one degree of freedom.

From usual arguments, it is easy to see that the integral operator defined by (7) admits eigenvalues  $\zeta_1 \geq \zeta_2 \geq \dots \geq 0$  with associated eigenfunctions  $g_1, g_2, \dots$  forming an orthonormal basis of  $L^2(\Theta_k)$ , the set of square integrable functions on  $\Theta_k$ . Then the zero-mean Gaussian process  $\mathcal{Z}$  as a function in  $L^2(\Theta_k)$ , has the Karhunen-Loève representation

$$\mathcal{Z}(t) = \sum_{j \geq 1} G_j g_j(t), t \in \Theta_k$$

with the independent random variables  $G_j$ 's defined as  $G_j = \int_{\Theta_k} \mathcal{Z}(t) g_j(t) dt \sim \mathcal{N}(0, \zeta_j)$ . It results from this that in distribution

$$\int_{\Theta_k} \mathcal{Z}^2(t) dt = \sum_{j \geq 1} \zeta_j \chi_j^2,$$

where the  $\chi_j^2$ 's are iid chi-square random variables with one degree of freedom.

*ii-* Here also, it is immediate from Theorem 2 that  $\mathcal{T}_{1,n}$  and  $\mathcal{T}_{2,n}$  converge in distribution respectively to

$$\sup_{(t_1, \dots, t_k) \in \Theta_k} |\mathcal{Z}(t_1, \dots, t_k) + \sum_{l=1}^k (t_{l+1} - t_l)(t_l - t_{l-1}) A_l| \quad \text{and}$$

$$\int_{[0,1]^k} \mathbb{1}((t_1, \dots, t_k) \in \Theta_k) |\mathcal{Z}(t_1, \dots, t_k) + \sum_{l=1}^k (t_{l+1} - t_l)(t_l - t_{l-1}) A_l|^2 dt_1 \dots dt_k.$$

For the same reasons as above, one has the decomposition

$$\mathcal{Z}(t) + \sum_{l=1}^k (t_{l+1} - t_l)(t_l - t_{l-1}) A_l = \sum_{j \geq 1} \tilde{G}_j g_j(t), t = (t_1, \dots, t_k) \in \Theta_k,$$

with the independent random variables  $\tilde{G}_j$ 's defined as  $\tilde{G}_j = \int_{\Theta_k} \mathcal{Z}^2(t) dt \sim \mathcal{N}(\rho_j, \zeta_j)$ , where

$$\rho_j = \sum_{l=1}^k A_l \int_0^1 \dots \int_0^1 (s_{l+1} - s_l)(s_l - s_{l-1}) g_j(s_1, \dots, s_k) \mathbb{1}((s_1, \dots, s_k) \in \Theta_k) ds_1 \dots ds_k.$$

It follows from this that, in distribution,

$$\begin{aligned} & \int_0^1 \dots \int_0^1 \mathbb{1}((t_1, \dots, t_k) \in \Theta_k) |\mathcal{Z}(t_1, \dots, t_k) + \sum_{l=1}^k (t_{l+1} - t_l)(t_l - t_{l-1}) A_l|^2 dt_1 \dots dt_k \\ &= \sum_{j \geq 1} \zeta_j \chi_j^{*2}, \end{aligned}$$

where the  $\chi_j^{*2}$ 's are non-central iid chi-square random variables with one degree of freedom and non-centrality parameter  $\rho_j^2 \zeta_j^{-1}$ .

iii- The last part follows easily from Theorem 4.

### 3.2 Results for some particular kernels

For some particular kernels, the results of the above subsection simplify, and, can lead to the construction of asymptotic distribution free test statistics whose distributions can be approximated more easily than those of the preceding more general test statistics.

Denote by  $\sigma^2$  the long-run variance of the series  $(h_{F_1,1}(X_i))$  :

$$\sigma^2 = \mathbb{E}\{[h_{F_1,1}(X_1)]^2\} + 2 \sum_{i=1}^{\infty} \mathbb{Cov}(h_{F_1,1}(X_1), h_{F_1,1}(X_{i+1})).$$

We have the following result.

**Theorem 7.** *If the kernel  $h$  is such that for all  $l = 1, \dots, k$  its associated  $h_{\dot{F}_l,1}$  and  $h_{\dot{F}_l,2}$  satisfy  $h_{\dot{F}_l,1}(x) = -h_{\dot{F}_l,2}(x)$ , then*

- (i) - *Under the assumptions of Theorem 1, under  $H_0$ ,  $\mathcal{Z}_n(t_1, t_2, \dots, t_k)$  converges weakly to  $\mathcal{Z}_0(t_1, t_2, \dots, t_k)$  with representation*

$$\mathcal{Z}_0(t_1, t_2, \dots, t_k) = \begin{cases} \sigma \mathcal{B}(t_1, t_2, \dots, t_k), & (t_1, t_2, \dots, t_k) \in \Theta_k \\ 0, & (t_1, t_2, \dots, t_k) \notin \Theta_k, \end{cases}$$

where for all  $(t_1, t_2, \dots, t_k) \in \Theta_k$ ,

$$\mathcal{B}(t_1, t_2, \dots, t_k) = \sum_{l=1}^k \{(t_{l+1} - t_l) [W_0(t_l) - W_0(t_{l-1})] - (t_l - t_{l-1}) [W_0(t_{l+1}) - W_0(t_l)]\},$$

with  $W_0$  standing for the standard Brownian bridge.

- (ii) - Under the assumptions of Theorem 2, under  $H_{1,n}^k$ ,  $\mathcal{Z}_n(t_1, t_2, \dots, t_k)$  converges weakly to  $\tilde{\mathcal{Z}}_0(t_1, t_2, \dots, t_k)$  with representation

$$\tilde{\mathcal{Z}}_0(t_1, t_2, \dots, t_k) = \mathcal{Z}_0(t_1, t_2, \dots, t_k) + \sum_{l=1}^k A_l(t_{l+1} - t_l)(t_l - t_{l-1}).$$

- (iii) - Under the assumptions of Theorem 4, under  $H_1^k$ ,  $\mathcal{Z}_n(\cdot)$  diverges in probability to  $\infty$ .

**Proof** - Theorem 1 holds for any kernel  $h$ . In the particular case where  $h$  is such that its associated  $h_{\hat{F}_{l,1}}$  and  $h_{\hat{F}_{l,2}}$  satisfy  $h_{\hat{F}_{l,1}}(x) = -h_{\hat{F}_{l,2}}(x)$ , one has that  $W_1(t) = -W_2(t)$ . Theorem 7 then follows by taking into account this equality in Theorem 6.

**Remark 4.** The common covariance kernel of  $\mathcal{Z}_0$  and  $\tilde{\mathcal{Z}}_0$  is defined by  $\Gamma(s, t)$ ,  $s, t \in \mathbb{R}^k$  with  $\sigma_{11} = \sigma_{22} = \sigma^2$  and  $\sigma_{12} = \sigma_{21} = -\sigma^2$ .

**Remark 5.** It is easy to check that anti-symmetric kernels  $h$  are such that their associated  $h_{\hat{F}_{l,1}}$  and  $h_{\hat{F}_{l,2}}$  satisfy the property  $h_{\hat{F}_{l,1}}(x) = -h_{\hat{F}_{l,2}}(x)$ .

There are various techniques for estimating long-run variances. The subsampling approach introduced by Carlstein (1986) (see, e.g., Schmidt et al. (2021)) is based on the idea that, given a stationary time series  $(\vartheta_i)$ , its associated long-run variance  $\sigma_\vartheta^2$  is defined as  $\lim_{n \rightarrow \infty} \text{Var}(\sqrt{n}\bar{\vartheta}_n) = \sum_{j=-\infty}^{\infty} \gamma_\vartheta(j) = \gamma_\vartheta(0) + 2 \sum_{j=1}^{\infty} \gamma_\vartheta(j)$ , where  $\gamma_\vartheta$  is the autocorrelation function of  $(\vartheta_i)$  and  $\bar{\vartheta}_n$  is the sample mean of  $\vartheta_1, \dots, \vartheta_n$ . Then, for some suitable integer  $b_n$  ( $b_n < n$ ), subsamples of lengths, say  $l_{n_j} < n$ ,  $1 \leq j \leq b_n$  are obtained from  $\vartheta_1, \dots, \vartheta_n$ . From these sample,  $b_n$  copies  $\sqrt{l_{n_j}}\bar{\vartheta}_{l_{n_j}}$ ,  $j = 1, \dots, b_n$  of  $\sqrt{n}\bar{\vartheta}_n$  are computed. The sample variances of these copies is taken as an estimator of  $\sigma_\vartheta^2$ . From the latter expression of  $\sigma_\vartheta^2$ , another class of estimators proposed by Newey and West (1987) is given as

$$\hat{\sigma}_\vartheta = \hat{\gamma}_\vartheta(0) + 2 \sum_{j=1}^{\infty} K\left(\frac{j}{\ell_n}\right) \hat{\gamma}_\vartheta(j),$$

where  $\ell_n$  is a bandwidth and  $K$  is a symmetric kernel such that  $K(0) = 1$ . The last approach that we present is as follows. Denoting the spectral density of  $(\vartheta_i)$  by  $f_\vartheta$ , and recalling that  $\sigma_\vartheta^2 = 2\pi f_\vartheta(0)$ , an estimator of  $\sigma_\vartheta^2$  is obtained by plugging an estimator of  $f_\vartheta(0)$  in the preceding equality. A candidate estimator of  $f_\vartheta(0)$  is

$$\hat{f}_\vartheta(0) = \frac{1}{2\pi} \sum_{|j| \leq n-1} w_n(j) \hat{\gamma}_\vartheta(j),$$

where the  $w_n(j)$  are weights such that  $w_n(j) \rightarrow 0$ , as  $|j| \rightarrow \infty$ , and the  $\hat{\gamma}_\vartheta(j)$  are the usual sample autocovariances. We can also note that if  $\sigma_\vartheta^2$  is expressed as a function of unknown parameters whose estimators can be found, based on these estimators, an estimator of  $\sigma_\vartheta^2$  can be obtained. This is the case for the  $\vartheta_i$ 's generated by an AR(1) model, as will be seen in the next section.

The consistency of the above classes of estimators is established for a broad class of stationary time series. Let  $\hat{\sigma}^2$  be any such consistent estimator of  $\sigma^2$  under  $\mathcal{H}_0$ , converging in probability to a positive number  $\sigma^*$  under  $\mathcal{H}_1^k$ . Consider the statistics

$$\tilde{\mathcal{T}}_{1,n} = \frac{\mathcal{T}_{1,n}}{\hat{\sigma}} \quad \text{and} \quad \tilde{\mathcal{T}}_{2,n} = \frac{\mathcal{T}_{2,n}}{\hat{\sigma}^2}.$$

**Corollary 8.** *Assume that the assumptions of Theorem 2 hold. Let  $(\mathcal{B}(s) : s \in [0, 1]^k)$  be the process defined in Theorem 7. Then*

i- *Under  $\mathcal{H}_0$ , as  $n$  tends to infinity, one has the following convergence in distribution,*

$$\tilde{\mathcal{T}}_{1,n} \longrightarrow \sup_{s \in [0, 1]^k} |\mathcal{B}(s)| \quad \text{and} \quad \tilde{\mathcal{T}}_{2,n} \longrightarrow \int_{[0, 1]^k} \mathcal{B}^2(s) ds = \sum_{j \geq 1} \tilde{\zeta}_j \chi_j^2,$$

*where the  $\chi_j^2$ 's are iid chi-square random variables with one degree of freedom and the  $\tilde{\zeta}_j$ 's are the eigenvalues of the operator  $\mathcal{K}_\Gamma/\sigma^2$ .*

ii- *Under  $\mathcal{H}_{1,n}^k$ , as  $n$  tends to infinity, one has the following convergence in distribution,*

$$\tilde{\mathcal{T}}_{1,n} \longrightarrow \sup_{s \in [0, 1]^k} |\tilde{\mathcal{Z}}_0(s)| \quad \text{and} \quad \tilde{\mathcal{T}}_{2,n} \longrightarrow \int_{[0, 1]^k} \tilde{\mathcal{Z}}_0^2(s) ds = \sum_{j \geq 1} \tilde{\zeta}_j \chi_j^{*2},$$

*where the  $\chi_j^{*2}$ 's are iid non-central chi-square random variables with one degree of freedom and non-centrality parameters  $\rho_j^2 \tilde{\zeta}_j^{-1} \sigma^{-2}$  with the  $\rho_j$ 's defined in Theorem 6 and the  $g_j$ 's appearing in their expressions being the eigenfunctions of the integral operator  $\mathcal{K}_\Gamma/\sigma^2$ , associated with the eigen-value  $\tilde{\zeta}_j$ .*

iii- *Under  $\mathcal{H}_1^k$ , as  $n$  tends to infinity, one has the following convergence in probability,*

$$\tilde{\mathcal{T}}_{1,n} \longrightarrow \infty, \quad \tilde{\mathcal{T}}_{2,n} \longrightarrow \infty.$$

**Proof** - Using the consistency of  $\hat{\sigma}^2$  to  $\sigma^2$  and Theorem 7, proceeding as in the proof of this theorem with  $\mathcal{Z}_n/\hat{\sigma}$  substituted for  $\mathcal{Z}_n$ , one easily establishes this corollary.

**Remark 6.** *Using Theorem 6, the asymptotic distributions of  $\mathcal{T}_{2,n}$  and  $\tilde{\mathcal{T}}_{2,n}$  under  $\mathcal{H}_0$  and  $\mathcal{H}_{1,n}^k$  can be approximated, even for more general kernel  $h$ . This is an advantage over  $\mathcal{T}_{1,n}$  and  $\tilde{\mathcal{T}}_{1,n}$  whose asymptotic distributions are more difficult to approximate. However, since  $\tilde{\mathcal{T}}_{1,n}$  and  $\tilde{\mathcal{T}}_{2,n}$  are asymptotic distribution-free under the null hypothesis, in the next section, we only consider kernels leading to these statistics. That is, the tests studied in that section are based on  $\tilde{\mathcal{T}}_{1,n}$  and  $\tilde{\mathcal{T}}_{2,n}$  whose asymptotic null distributions are free. Then, the quantiles of their limiting distributions are approximated by their simulated empirical counter part.*

## 4 Simulation study

For the simulations, we use the version 4.4.2 of the software R. As mentioned in Remark 6 above, we study only the tests based on  $\tilde{T}_{1,n}$  and  $\tilde{T}_{2,n}$ . Potential kernels are anti-symmetric kernels, in particular  $h(x, y) = \mathbb{1}(x < y)$  and  $h(x, y) = x - y$ . For the first, it is straightforward that  $\theta(F_1, F_1) = 1/2$  and  $h_{F_1,1}(x) = 1/2 - F_1(x) = -h_{F_1,2}(x)$ . For the latter, it is a trivial matter that  $\theta(F_1, F_1) = 0$  and that  $h_{F_1,1}(x) = -h_{F_1,2}(x)$  as  $h$  is anti-symmetric. The corresponding  $\sigma^2$  are respectively

$$\sigma_1^2 = \sigma^2 = \text{Var}[F_1(X_1)] + 2 \sum_{j \geq 1} \text{Cov}(F_1(X_1), F_1(X_{1+j}))$$

and

$$\sigma_2^2 = \sigma^2 = \text{Var}(X_1) + 2 \sum_{j \geq 1} \text{Cov}(X_1, X_{1+j}).$$

We restrict to  $h(x, y) = x - y$  and  $k = 2$ , the case  $k = 1$  being treated in [Ngatchou-Wandji et al. \(2022\)](#). Our results are applied to simple models for detecting : (i) two changes in the mean of a shifted white noise, (ii) two changes in the mean of an AR(1), (iii) one change in the mean and one change in the autocorrelation of an AR(1), and (iv) one change in the mean and one change in the variance of shifted white noise.

To obtain the critical values of the tests, we simulated 5000 observations of the stochastic process  $\mathcal{B}(t_1, t_2)$  defined in Theorem 7 for  $k = 2$ , and considered empirical quantiles of the limiting distribution as critical values. To simulate the Brownian bridge, following Donsker's theorem, we approximated the Brownian motion by the normalized partial sum process

$$S_m(t) = \frac{1}{\sqrt{m}} \sum_{i=1}^{\lfloor mt \rfloor} \eta_i, \quad t \in [0, 1], \quad m = 2000,$$

with the  $\eta_i$ 's simulated from a standard Gaussian distribution. With this, the Brownian bridge was approximated by  $S_m(t) - tS_m(1)$ ,  $t \in [0, 1]$ . The process  $\mathcal{B}(t_1, t_2)$  was then approximated by substituting  $S_m(t) - tS_m(1)$  for  $W_0(t)$ . For computing the values of  $\sup_{0 \leq t_1 < t_2 \leq 1} |\mathcal{B}(t_1, t_2)|$ , we considered the couples  $(i/m, j/m)$ ,  $j = 2, \dots, m-2$ ,  $i = 1, \dots, j-1$  and took it as the random variable  $\max_{2 \leq j \leq m-2, 1 \leq i \leq j-1} |\mathcal{B}(i/m, j/m)|$ . For computing the values of  $\int_{\Theta_2} \mathcal{B}^2(t_1, t_2) dt_1 dt_2$ , we approximated it by the random variable  $\sum_{2 \leq j \leq m-2, 1 \leq i \leq j-1} \mathcal{B}^2(i/m, j/m)/m^2$ . Using all these approximations, and based on 5000 replications, the quantiles of  $\sup_{0 \leq t_1 < t_2 \leq 1} |\mathcal{B}(t_1, t_2)|$  and  $\int_{\Theta_2} \mathcal{B}^2(t_1, t_2) dt_1 dt_2$  are approximated by the values in Table 1.

For evaluating the performances of our tests, we sampled 1000 sets of  $n = 200$  data  $X_1, X_2, \dots, X_n$  with  $X_i = Y_{1,i}$ ,  $i = 25, \dots, 100$ ,  $X_{75+i} = Y_{2,25+i}$ ,  $i = 1, \dots, 75$ ,  $X_{150+i} = Y_{3,25+i}$ ,  $i = 1, \dots, 50$ , where the  $Y_{i,j}$ 's are from the piece-wise stationary

**Table 1** Empirical quantiles of the limiting distributions of the test statistics.

Level	Quantile	KS	CV
<b>0.01</b>		<b>1.66</b>	<b>0.249</b>
0.02		1.55	0.206
0.03		1.45	0.182
0.04		1.43	0.162
<b>0.05</b>		<b>1.38</b>	<b>0.145</b>
0.06		1.34	0.134
0.07		1.32	0.126
0.08		1.29	0.119
0.09		1.27	0.113
<b>0.10</b>		<b>1.26</b>	<b>0.107</b>

models

$$\begin{aligned}
Y_{1,j} &= \mu_1 + \rho_1 Y_{1,j-1} + \omega_1 \varepsilon_j \quad j = 1, \dots, 100, \\
Y_{2,j} &= \mu_2 + \rho_2 X_{j-1} + \omega_2 \varepsilon_j \quad j = 1, \dots, 100, \\
Y_{3,j} &= \mu_3 + \rho_3 X_{j-1} + \omega_3 \varepsilon_j \quad j = 1, \dots, 100,
\end{aligned} \tag{8}$$

with the  $\mu_j$ 's,  $\rho_j$ 's and the  $\omega_j$ 's being real numbers, the  $\varepsilon_i$ 's iid random variables and for all  $i = 1, \dots, 200$ ,  $\varepsilon_i \sim \mathcal{N}(0, 1)$ . These models are studied in [Yau and Zhao \(2016\)](#) and their stationarity and ergodicity are studied in [Ngatchou-Wandji and Ltaifa \(2023b\)](#).

We considered the following scenarios :

- (i) Two changes in the mean :  $\mu_1 = 0, \mu_2, \mu_3 \neq 0, \mu_2 \neq \mu_3, \rho_i = 0$  and  $\omega_i = 1, i = 1, 2, 3$ .
- (ii) Two changes in the mean :  $\mu_1 = 0, \mu_2, \mu_3 \neq 0, \mu_2 \neq \mu_3, \rho_i = 0.2$  and  $\omega_i = 1, i = 1, 2, 3$ .
- (iii) One change in the mean and one change in the autocorrelation :  $\mu_1 = 0, \mu_2 = \mu_3 \neq 0, \rho_1 = \rho_2 = 0, \rho_3 \neq 0$  and  $\omega_i = 1, i = 1, 2, 3$ .
- (iv) One change in the mean and one change in the variance :  $\mu_1 = 0, \mu_2 = \mu_3 \neq 0, \rho_i = 0, i = 1, 2, 3, \omega_1 = \omega_2 = 1$  and  $\omega_3 \neq 1$ .

Here, the null model is an AR(1) of the form

$$X_i = \mu + \rho X_{i-1} + \omega \varepsilon_i.$$

From simple computations, the corresponding long-run variance is

$$\sigma^2 = \frac{\omega^2}{(1 - \rho)^2}.$$

Let  $\hat{\omega}^2$  and  $\hat{\rho}$  be consistent estimators of  $\omega^2$  and  $\rho$ . An estimator  $\hat{\sigma}^2$  of  $\sigma^2$  can be obtained by substituting these quantities in the above expression. Examples of  $\hat{\omega}^2$  and



$\hat{\rho}$  are the Yule-Walker estimators (see, eg, [Brockwell and Davis \(2002\)](#)) given by

$$\begin{cases} \hat{\rho} = \frac{\sum_{t=1}^{n-1} (X_{t-1} - \bar{X})(X_t - \bar{X})}{\sum_{j=1}^n (X_j - \bar{X})^2} \\ \hat{\mu} = \bar{X}(1 - \hat{\rho}) \\ \hat{\omega}^2 = \frac{1}{n} \sum_{t=1}^n (X_t - \hat{\mu} - \hat{\rho}X_{t-1})^2. \end{cases}$$

In what follows, we use the estimator of  $\sigma^2$  obtained this way, because it is less subjective than the ones described in Subsection 3.2. With it, the KS-type statistic  $\mathcal{T}_{1,n}$  and the CV-type statistic  $\mathcal{T}_{2,n}$  were computed by taking respectively, the maximum and the sum of the  $X_i - X_j$ 's over the couples  $(i, j)$ ,  $j = 2, \dots, n-2$ ,  $i = 1, \dots, j-1$ . Then the distribution-free test statistics  $\tilde{\mathcal{T}}_{1,n}$  and  $\tilde{\mathcal{T}}_{2,n}$  were obtained by dividing the resulting quantities, respectively by  $\hat{\sigma}$  and  $\hat{\sigma}^2$ .

For each of the 1000 samples generated from (8), the values of these distribution-free statistics were respectively compared to the 0.95-quantiles of the limiting distribution of the KS and CV tests. The ratios of the number of times they were larger than these numbers, over 1000 are their empirical powers. These were compared to those of the CUSUM test, the Rényi-type test and the Darling-Erdős test presented in [Horváth et al. \(2020\)](#), and whose codes are available in the package CPAT of the software R. Table 2 displays the empirical levels of the tests for some particular null models, while Tables 3 and 4 list their empirical powers for all the scenarios studied. From Table 2, it can be seen that the empirical level of the RC test ranges from 0.071 to 0.734. That of the DE test is nil for almost all the models except for the zero-mean AR(1) models (lines 9-13). The empirical level of the remaining tests are more closer to the nominal level of 5% except for the pure AR(1) (lines 7-13). While those of the CU test are too large besides the nominal level, those of the KS test are rather too small.

For changes in the mean (see Table 3), all the tests have nice powers, except the DE test for which the local power is generally too poor. For changes in the mean and correlation or variance (see Table 4), the power of the RC test is generally smaller than the others; that of the DE test is unstable, while those of the remaining tests are nice and competitive.

## 5 Concluding remarks

The current work is an extension of [Ngatchou-Wandji et al. \(2022\)](#) to multiple change-points detection. The results are applied to testing for two changes of possibly different natures in a series simulated from a piece-wise stationary AR(1) model. The critical values were obtained as simulated empirical quantiles of the limiting distributions. For the kernel  $h(x, y) = x - y$  that we used, we observed that our tests have nice empirical levels, and that their empirical powers compared to the CUSUM tests, the Rényi-type test and the Darling-Erdős test were very competitive.

The powers of all the tests were generally too poor for the series with the two changes either in the autocorrelation or in the variance. For our tests, this can be

**Table 2** Empirical level of the tests for scenarios from model (8) :  $\mu_1 = \mu_2 = \mu_3$ ,  $\rho_1 = \rho_2 = \rho_3$  and  $\omega_1 = \omega_2 = \omega_3$  (no change neither in the mean nor in the autocorrelation and in the variance).

$(\mu_1, \rho_1, \omega_1)$	KS	CV	CU	RC	DE
(0.1, 0, 1)	0.042	0.052	0.051	<b>0.098</b>	0.000
(0.4, 0, 1)	0.047	0.048	0.055	<b>0.074</b>	0.000
(-0.8, 0, 1)	0.042	0.046	0.041	<b>0.081</b>	0.000
(1.2, 0, 1)	0.049	0.057	0.055	<b>0.109</b>	0.000
(-1.6, 0, 1)	0.046	0.052	0.053	<b>0.121</b>	0.000
(2, 0, 1)	0.046	0.052	0.043	<b>0.152</b>	0.000
(0, 0.1, 1)	0.030	0.035	0.035	<b>0.111</b>	0.003
(0, 0.3, 1)	0.032	0.038	0.062	<b>0.156</b>	0.002
(0, 0.5, 1)	<b>0.024</b>	0.034	<b>0.139</b>	<b>0.244</b>	<b>0.018</b>
(0, 0.7, 1)	<b>0.010</b>	0.032	<b>0.287</b>	<b>0.483</b>	<b>0.087</b>
(0, 0.9, 1)	<b>0.010</b>	0.039	<b>0.719</b>	<b>0.734</b>	<b>0.486</b>
(0.7, 0.4, 1)	<b>0.020</b>	0.032	<b>0.074</b>	<b>0.224</b>	0.002
(-1, 0.6, 1)	<b>0.019</b>	0.029	<b>0.174</b>	<b>0.380</b>	0.031
(0, 0, 0.2)	0.042	0.052	0.050	<b>0.078</b>	0.000
(0, 0, 0.4)	0.040	0.054	0.043	<b>0.083</b>	0.000
(0, 0, 0.6)	0.045	0.047	0.042	<b>0.072</b>	0.000
(0, 0, 0.8)	0.050	0.052	0.051	<b>0.082</b>	0.000
(0, 0, 1)	0.045	0.049	0.047	<b>0.071</b>	0.000
(0, 0, 1.5)	0.043	0.049	0.051	<b>0.081</b>	0.000
(0, 0, 2)	0.041	0.045	0.049	<b>0.082</b>	0.001

**Table 3** Power of the tests for scenarios from model (8) :  $\mu_1 = 0$ ,  $\mu_2, \mu_3 \neq 0$ ,  $\mu_2 \neq \mu_3$ ,  $\rho_i = 0$  and  $\omega_i = 1$ ,  $i = 1, 2, 3$  (left side of the table) ;  $\mu_1 = 0$ ,  $\mu_2, \mu_3 \neq 0$ ,  $\mu_2 \neq \mu_3$ ,  $\rho_i = 0.2$  and  $\omega_i = 1$ ,  $i = 1, 2, 3$  (right side of the table).

$(\mu_2, \mu_3)$	KS	CV	CU	RC	DE	KS	CV	CU	RC	DE
(0.00, 0.00)	0.042	0.051	0.048	0.091	0.002	0.039	0.052	0.051	0.133	0.000
(0.01, -0.05)	0.046	0.053	0.043	0.071	0.000	0.038	0.045	0.050	0.129	0.000
(-0.03, 0.05)	0.049	0.056	0.051	0.081	0.000	0.037	0.046	0.047	0.107	0.000
(0.06, -0.05)	0.050	0.058	0.054	0.077	0.000	0.034	0.047	0.067	0.135	0.001
(0.06, -0.07)	0.056	0.058	0.060	0.082	0.000	0.037	0.048	0.065	0.135	0.001
(0.08, -0.09)	0.066	0.063	0.072	0.085	0.001	0.042	0.058	0.064	0.128	0.001
(0.10, 0.14)	0.080	0.105	0.089	0.086	0.001	0.074	0.100	0.128	0.135	0.001
(0.15, 0.20)	0.158	0.193	0.178	0.083	0.001	0.096	0.132	0.135	0.144	0.002
(-0.50, 0.5)	0.854	0.376	0.945	0.203	0.444	0.722	0.274	0.948	0.229	0.358
(0.25, 0.20)	0.057	0.057	0.054	0.081	0.000	0.158	0.183	0.253	0.152	0.003
(0.50, -0.30)	0.223	0.230	0.245	0.093	0.004	0.365	0.088	0.689	0.175	0.079
(-0.80, 0.40)	0.549	0.132	0.712	0.106	0.081	0.747	0.095	0.986	0.223	0.489
(-0.45, -0.60)	0.870	0.182	0.987	0.223	0.482	0.786	0.876	0.920	0.195	0.197
(-0.90, 0.90)	0.870	0.917	0.901	0.143	0.179	1.000	0.770	1.000	0.916	0.998
(1.00, -0.80)	1.000	0.840	1.000	0.928	0.998	1.000	0.560	1.000	0.840	0.995
(1.00, 1.50)	1.000	0.659	1.000	0.874	0.994	1.000	1.000	1.000	0.875	1.000
(-1.50, 1.00)	1.000	1.000	1.000	0.882	0.999	1.000	1.000	0.356	1.000	1.000

**Table 4** Power of the tests for scenarios from model (8) :  $\mu_1 = 0, \mu_2 = \mu_3 \neq 0, \rho_1 = \rho_2 = 0, \rho_3 \neq 0$  and  $\omega_i = 1, i = 1, 2, 3$  (second and third columns);  $\mu_1 = 0, \mu_2 = \mu_3 \neq 0, \rho_i = 0, i = 1, 2, 3, \omega_1 = \omega_2 = 1$  and  $\omega_3 \neq 1$  (last two columns).

$(\mu_2, \rho_3)$	KS	CV	CU	RC	DE	$(\mu_2, \omega_3)$	KS	CV	CU	RC	DE
(0.0,0.0)	0.047	0.052	0.047	0.075	0.000	(0.0,0.0)	0.046	0.056	0.044	0.083	0.001
(0.1,0.1)	0.062	0.075	0.073	0.083	0.001	(1.0,0.5)	0.097	0.089	0.108	0.106	0.000
(-0.1,-0.3)	0.109	0.126	0.123	0.094	0.002	(-0.2,0.5)	0.238	0.230	0.269	0.114	0.011
(0.1,-0.5)	0.096	0.114	0.112	0.088	0.001	(-0.3,0.5)	0.469	0.476	0.509	0.124	0.022
(-0.1,0.7)	0.038	0.042	0.036	0.085	0.001	(0.5,0.5)	0.899	0.914	0.923	0.169	0.277
(0.1,0.9)	0.036	0.043	0.032	0.102	0.001	(0.1,1.5)	0.076	0.110	0.082	0.106	0.006
(-0.5,-0.2)	0.779	0.797	0.836	0.111	0.129	(-0.3,1.5)	0.262	0.327	0.293	0.111	0.008
(0.5,0.4)	0.850	0.868	0.893	0.139	0.108	(0.5,1.5)	0.643	0.697	0.697	0.013	0.066
(-0.5,0.6)	0.787	0.817	0.834	0.122	0.078	(0.8,1.5)	0.990	0.972	0.991	0.208	0.587
(0.5,-0.8)	0.716	0.746	0.756	0.145	0.127	(1.0,1.5)	0.999	0.998	1.000	0.319	0.908
(-0.5,-0.9)	0.682	0.738	0.740	0.119	0.101	(-0.5,-1.5)	0.649	0.678	0.701	0.127	0.053
(1.0,0.1)	1.000	1.000	1.000	0.495	0.987	(0.4,-0.5)	0.701	0.709	0.731	0.142	0.100
(-1.0,-0.5)	1.000	1.000	1.000	0.482	0.957	(-0.7,-0.5)	0.997	0.998	1.000	0.253	0.777
(0.9,-0.7)	0.954	0.968	0.964	0.159	0.219	(1.0,-2.0)	0.998	0.994	0.998	0.223	0.707
(1.0,-0.9)	0.999	0.999	0.999	0.307	0.857	(-1.0,-0.2)	1.000	1.000	1.000	0.782	0.999
(-0.6,0.7)	1.000	0.997	1.000	0.376	0.924	(-0.2,0.1)	0.246	0.240	0.285	0.119	0.004
(0.8,0.8)	1.000	0.999	0.999	0.201	0.714	(0.9,2.0)	0.979	0.967	0.985	0.200	0.492

explained by the fact that the kernel used is more adapted to testing changes in the location parameters. This argument can also be invoked for the other tests.

## 6 Appendix : Proofs of the results

### 6.1 Some useful results

We need the following result proved by [Oodaira and Yoshihara \(1972\)](#).

Let  $\xi_1, \xi_2, \dots, \xi_n, \dots$  be a strictly stationary sequence of zero-mean random variables, and let

$$\sigma_*^2 = \mathbb{E}(\xi_1^2) + 2 \sum_{i=1}^{\infty} \mathbb{E}(\xi_1 \xi_{i+1}).$$

**Proposition 9.** Assume  $\mathbb{E}(|\xi_i|^{2+\delta}) < \infty$  for some positive  $\delta$  and  $\xi_1, \xi_2, \dots, \xi_n, \dots$  is  $\alpha$ -mixing with  $\alpha$ -rate satisfying

$$\sum_{i=1}^{\infty} [\alpha(i)]^{\frac{\delta}{2+\delta}} < \infty.$$

Then  $\sigma_*^2 < \infty$ .

If  $\sigma_* > 0$ , then the sequence of processes

$$S_n(t) = \frac{1}{\sigma_* \sqrt{n}} \sum_{i=1}^{[nt]} \xi_i, \quad t \in [0, 1]$$

converges weakly to a Wiener measure on  $(D, \mathcal{D})$ , where  $\mathcal{D}$  is the  $\sigma$ -fields of Borel sets for the Skorohod topology.

**Lemma 10.** (*Harel and Puri (1989)*) Let  $\{\mathcal{X}_{ni}^*\}$  be a sequence of zero-mean absolutely regular random variables (rv)'s with rates satisfying

$$\sum_{n \geq 1} [\beta(n)]^{\delta/(2+\delta)} < \infty \text{ for some } \delta > 0. \quad (9)$$

Suppose that for any  $\kappa$ , there exists a sequence  $\{\mathcal{Y}_{ni}^\kappa\}$  of rv's satisfying (9) such that

$$\sup_{n \in \mathbb{N}} \max_{0 \leq i \leq n} |\mathcal{Y}_{ni}^\kappa| \leq B_\kappa < \infty, \quad (10)$$

where  $B_\kappa$  is some positive constant

$$\sup_{n \in \mathbb{N}} \max_{0 \leq i \leq n} \mathbb{E}(|\mathcal{X}_{ni}^* - \mathcal{Y}_{ni}^\kappa|^{2+\delta}) \longrightarrow 0 \text{ as } \kappa \rightarrow \infty \quad (11)$$

$$\frac{1}{n} \mathbb{E} \left[ \left( \sum_{i=1}^n \mathcal{X}_{ni}^* \right)^2 \right] \longrightarrow c \text{ as } n \rightarrow \infty, \quad (12)$$

where  $c$  is some positive constant

$$\frac{1}{n} \mathbb{E} \left[ \left( \sum_{i=1}^n \mathcal{Y}_{ni}^\kappa - \mathbb{E}(\mathcal{Y}_{ni}^\kappa) \right)^2 \right] \longrightarrow c_\kappa \text{ as } n \rightarrow \infty, \quad (13)$$

where  $c_\kappa$  is some constant  $> 0$

$$c_\kappa \longrightarrow c \text{ as } \kappa \rightarrow \infty. \quad (14)$$

Then

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n \mathcal{X}_{ni}^*$$

converges in distribution to the normal distribution with mean 0 and variance  $c$ .

**Lemma 11.** (*Phillips and Durlauf (1986)*) Probability measures on a product space are tight iff all the marginal probability measures are tight on the component spaces.

## 6.2 Preliminaries

In this subsection, we prove some preliminary results necessary to the proofs of Theorems 1 and 2.

**Proposition 12.** *Under the conditions of Theorem 1, as  $n$  tends to infinity, in probability, we have*

$$n^{-3/2} \sup_{(t_1, t_2, \dots, t_k) \in \Theta_k} \left| \sum_{l=1}^k \sum_{i=[nt_{l-1}]+1}^{[nt_l]} \sum_{j=[nt_l]+1}^{[nt_{l+1}]} g_{F_1, F_1}(X_i, X_j) \right| \longrightarrow 0.$$

*Under the conditions of Theorem 2, as  $n$  tends to infinity, in probability, we have*

$$n^{-3/2} \sup_{(t_1, t_2, \dots, t_k) \in \Theta_k} \left| \sum_{l=1}^k \sum_{i=[nt_{l-1}]+1}^{[nt_l]} \sum_{j=[nt_l]+1}^{[nt_{l+1}]} g_{F_{l-1}^{(n)}, F_l^{(n)}}(Y_{ni}^{l-1}, Y_{nj}^l) \right| \longrightarrow 0$$

where by convention  $[nt_0] = 1$  and  $[nt_{k+1}] = n$ .

**Proof** - We only prove the first part. This needs two lemmas that we first state and prove.

**Lemma 13.** *Under the conditions of Theorem 1, there exists a constant  $Cst > 0$  such that for any  $(t_1, t_2, \dots, t_k) \in \Theta_k$ ,*

$$\begin{aligned} & \mathbb{E} \left\{ \left[ \sum_{l=1}^k \sum_{i=[nt_{l-1}]+1}^{[nt_l]} \sum_{j=[nt_l]+1}^{[nt_{l+1}]} g_{F_1, F_1}(X_i, X_j) \right]^2 \right\} \\ & \leq Cst \left\{ \sum_{l=1}^k ([nt_l] - [nt_{l+1}])([nt_{l+1}] - [nt_l]) \right. \\ & \quad \left. + 2 \sum_{1 \leq l < l' \leq k} ([nt_l] - [nt_{l+1}])([nt_{l'}] - [nt_{l'+1}]) \right\}, \end{aligned}$$

with the convention that convention  $[nt_0] = 1$  and  $[nt_{k+1}] = n$ .

**Proof** - For any  $(t_1, t_2, \dots, t_k) \in \Theta_k$ , we can write

$$\begin{aligned} & \mathbb{E} \left\{ \left[ \sum_{l=1}^k \sum_{i=[nt_{l-1}]+1}^{[nt_l]} \sum_{j=[nt_l]+1}^{[nt_{l+1}]} g_{F_1, F_1}(X_i, X_j) \right]^2 \right\} \\ & \leq \sum_{l=1}^k \sum_{i=[nt_{l-1}]+1}^{[nt_l]} \sum_{j=[nt_l]+1}^{[nt_{l+1}]} \mathbb{E} \{ [g_{F_1, F_1}(X_i, X_j)]^2 \} \\ & \quad + 2 \sum_{1 \leq l < l' \leq k} \sum_{1 \leq i_1 < i_2 \leq [nt_l]} \sum_{[nt_{l'}]+1 \leq j_1 < j_2 \leq [nt_{l'+1}]} H((i_1, j_1), (i_2, j_2)) \end{aligned}$$

$$= \sum_{l=1}^k A_n(t_l) + 2 \sum_{1 \leq l < l' \leq k} B_n(t_l, t_{l'}),$$

where

$$\begin{aligned} H((i_1, j_1), (i_2, j_2)) &= \mathbb{E}\{[g_{F_1, F_1}(X_{i_1}, X_{j_1}) - h_{F_1, 1}(X_{i_1}) - h_{F_1, 2}(X_{j_1}) + \theta_h(F_1, F_1)] \\ &\quad \times [g_{F_1, F_1}(X_{i_2}, X_{j_2}) - h_{F_1, 1}(X_{i_2}) - h_{F_1, 2}(X_{j_2}) + \theta_h(F_1, F_1)]\}. \end{aligned}$$

From the integrability condition, we have

$$\sup_{i, j \in \mathbb{N}} \mathbb{E}\{[g_{F_1, F_1}(X_1, X_2)]^2\} \leq Cst.$$

Then, for any  $1 \leq l \leq k$ ,

$$A_n(t_l) \leq Cst([nt_l] - [nt_{l-1}])([nt_{l+1}] - [nt_l]).$$

Similarly as in [Ngatchou-Wandji et al. \(2022\)](#), we prove that

$$B_n(t_l, t_{l'}) \leq Cst([nt_l] - [nt_{l+1}])([nt_{l'}] - [nt_{l'-1}])$$

and Lemma 13 is proved.

For any  $(t_1, t_2, \dots, t_k) \in \Theta_k$ , we now define the process  $\mathcal{G}_n(t_1, \dots, t_k)$  by

$$\mathcal{G}_n(t_1, \dots, t_k) = n^{-3/2} \sum_{l=1}^k \sum_{i=[nt_{l-1}]+1}^{[nt_l]} \sum_{j=[nt_l]+1}^{[nt_{l+1}]} g_{F_1, F_1}(X_i, X_j).$$

**Lemma 14.** *Under the conditions of Theorem 1, we have*

$$\mathbb{E} \left[ \left( \mathcal{G}_n(t_1^1, \dots, t_1^k) - \mathcal{G}_n(t_2^1, \dots, t_2^k) \right)^2 \right] \leq \frac{Cst}{n} \sum_{l=1}^k (t_1^l - t_2^l), \quad (15)$$

for all  $0 < t_1^1 < t_2^1 < t_1^2 < t_2^2 < \dots < t_1^k < t_2^k < 1$ .

**Proof** - For any  $0 < t_1^1 < t_2^1 < t_1^2 < t_2^2 < \dots < t_1^k < t_2^k < 1$ , we can write

$$\begin{aligned} &\mathbb{E} \left[ \left( \mathcal{G}_n(t_1^1, \dots, t_1^k) - \mathcal{G}_n(t_2^1, \dots, t_2^k) \right)^2 \right] \\ &= \mathbb{E} \left[ \left( n^{-3/2} \sum_{l=1}^k \sum_{i=[nt_1^{l-1}]+1}^{[t_1^l]} \sum_{j=[nt_l]+1}^{[nt_1^{l+1}]} g_{F_1, F_1}(X_i, X_j) \right. \right. \end{aligned}$$

$$\begin{aligned}
& -n^{-3/2} \sum_{l=1}^k \sum_{i=[nt_2^{l-1}]+1}^{[nt_2^l]} \sum_{j=[nt_2^l]+1}^{[nt_2^{l+1}]} g_{F_1, F_1}(X_i, X_j) \Big)^2 \Big] \\
& \leq 2n^{-3} \mathbb{E} \left[ \left( \sum_{l=1}^k \sum_{i=[nt_2^{l-1}]+1}^{[nt_2^l]} \sum_{j=[nt_2^l]+1}^{[nt_2^{l+1}]} g_{F_1, F_1}(X_i, X_j) \right)^2 \right] \\
& \quad + 2n^{-3} \mathbb{E} \left[ \left( \sum_{l=1}^k \sum_{i=[nt_1^{l-1}]+1}^{[nt_1^l]} \sum_{j=[nt_1^l]+1}^{[nt_1^{l+1}]} g_{F_1, F_1}(X_i, X_j) \right)^2 \right].
\end{aligned}$$

From Lemma 13, we deduce (15) and Lemma 14 is proved.

For the simplification of the proof, we fix  $k = 2$ . The case  $k > 2$  is lengthy be easy.

From Lemma 14, we deduce that for all  $\epsilon > 0$  and all  $0 \leq t_1^1 < t_2^1 < t_1^2 < t_2^2 \leq 1$ ,

$$\mathbb{P}(|\mathcal{G}_n(t_1^1, t_2^1) - \mathcal{G}_n(t_1^2, t_2^2)| \geq \epsilon) \leq \frac{Cst}{\epsilon^2 n} [(t_2^1 - t_1^1) + (t_2^2 - t_1^2)].$$

It implies that for all  $0 \leq s_1^1 < s_2^1 < s_1^2 < s_2^2 \leq n$  with  $0 \leq s_1^1 \ll s_2^1 \ll s_1^2 \ll s_2^2 \leq n$

$$\begin{aligned}
\mathbb{P} \left[ \left| \mathcal{G}_n \left( \frac{s_1^1}{n}, \frac{s_2^1}{n} \right) - \mathcal{G}_n \left( \frac{s_1^2}{n}, \frac{s_2^2}{n} \right) \right| \geq \epsilon \right] & \leq \frac{Cst}{\epsilon^2 n^2} [(s_2^1 - s_1^1) + (s_2^2 - s_1^2)] \\
& \leq \frac{Cst}{\epsilon^2} \left[ \frac{(s_2^1 - s_1^1) + (s_2^2 - s_1^2)}{n^{5/4}} \right]^{4/3}.
\end{aligned}$$

Now consider the partial sum process defined by

$$\mathcal{S}_{s,t} = \sum_{2 \leq j \leq t} \sum_{1 \leq i \leq \min\{j-1, s\}} \mathcal{A}(i, j),$$

where

$$\mathcal{A}(i, j) = \begin{cases} \mathcal{G}_n \left( \frac{i}{n}, \frac{j}{n} \right) - \mathcal{G}_n \left( \frac{i-1}{n}, \frac{j}{n} \right) - \mathcal{G}_n \left( \frac{i}{n}, \frac{j-1}{n} \right) + \mathcal{G}_n \left( \frac{i-1}{n}, \frac{j-1}{n} \right) & \text{if } i < j \\ \mathcal{G}_n \left( \frac{i}{n}, \frac{j}{n} \right) = 0 & \text{if } i \geq j. \end{cases}$$

It results that

$$\mathcal{S}_{s,t} = \mathcal{G}_n \left( \frac{s}{n}, \frac{t}{n} \right).$$

The last inequality is equivalent to

$$\begin{aligned}
\mathbb{P}(|\mathcal{S}_{s,t} - \mathcal{S}_{s',t'}| \geq \epsilon) & \leq \frac{Cst}{\epsilon^2} \left\{ \frac{[(s' - s) + (t' - t)]}{n^{5/4}} \right\}^{4/3} \\
& \leq \frac{Cst}{\epsilon^2 n^{1/3}}.
\end{aligned}$$

From Theorem 10.2 of Billingsley (1968), we easily deduce

$$\mathbb{P} \left( \max_{1 \leq s < t \leq n-1} |\mathcal{S}_{s,t}| \geq \epsilon \right) \leq \frac{Cst}{\epsilon^2 n^{1/3}},$$

which implies that, as  $n$  tends to infinity, in probability

$$n^{-3/2} \sup_{0 < t_1 < t_2 < 1} \left| \sum_{l=1}^2 \sum_{i=[nt_{l-1}]+1}^{[nt_l]} \sum_{j=[nt_l]+1}^{[nt_{l+1}]} g_{F_1, F_1}(X_i, X_j) \right| \longrightarrow 0.$$

This completes the proof of Proposition 12.

**Proposition 15.** *Under the conditions of Theorem 1, as  $n$  tends to infinity, we have the weak convergence result*

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} \begin{pmatrix} h_{F_1,1}(X_i) \\ h_{F_1,2}(X_i) \end{pmatrix} \longrightarrow \begin{pmatrix} W_1(t) \\ W_2(t) \end{pmatrix}, \quad (16)$$

in the space  $(D[0,1])^2$ .

Under the conditions of Theorem 2, for any  $l = 1, \dots, k$ , as  $n$  tends to infinity, we have the weak convergence result

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} \begin{pmatrix} h_{F_l^{(n)},1}(Y_{ni}^l) \\ h_{F_l^{(n)},2}(Y_{ni}^{l+1}) \end{pmatrix} \longrightarrow \begin{pmatrix} W_1(t) \\ W_2(t) \end{pmatrix}, \quad (17)$$

in the space  $(D[0,1])^2$ .

**Proof** - We start with the proof of (16). Under the assumptions of Theorem 2, the sequences  $(h_{F_1,j}(X_i))_{i \in \mathbb{Z}}$ ,  $j = 1, 2$  and  $\left( \begin{pmatrix} h_{F_1,1}(X_i) \\ h_{F_1,2}(X_i) \end{pmatrix} \right)_{i \in \mathbb{Z}}$  are strictly stationary. For any  $a_1, a_2 \in \mathbb{R}$ , one has by Proposition 1,

$$\begin{cases} \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} h_{F_1,1}(X_i) \longrightarrow \sigma_{11} W_1(t) \\ \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} h_{F_1,2}(X_i) \longrightarrow \sigma_{22} W_2(t) \\ \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} [a_1 h_{F_1,1}(X_i) + a_2 h_{F_1,2}(X_i)] \longrightarrow \tilde{\sigma} \widetilde{W}(t), \end{cases}$$

where  $W_1, W_2$  and  $\widetilde{W}$  are Wiener processes and  $\tilde{\sigma}$  equals  $\sigma_{rp}$  with  $a_1 h_{F_1,1}(X_i) + a_2 h_{F_1,2}(X_i)$  substituted for  $h_{F_1,1}(X_i)$  and  $h_{F_1,2}(X_i)$ .



It is easy to see that in distribution  $\widetilde{\sigma W}(t)$  equals  $\sigma_{11}W_1(t) + \sigma_{22}W_2(t)$ , which establishes (16).

For establishing (17) we cannot proceed directly as for (16) since the bidimensional series  $\left( \begin{pmatrix} h_{F_l^{(n)},1}(Y_{ni}^l) \\ h_{F_l^{(n)},2}(Y_{ni}^{l+1}) \end{pmatrix} \right)_{i \in \mathbb{Z}}$  is not strictly stationary. We need to study the tightness and finite-dimensional distributions of the bi-dimensional partial sum in the left-hand side of (17).

For the study of the finite-dimensional distributions, by the Cramér-Wold device it suffices to show that for any  $\ell \in \mathbb{N}^*$ , any  $a_j, b_j, t_j \in \mathbb{R}$ ,  $0 = t_0 < t_1 < \dots < t_\ell = 1$

$$\sum_{j=1}^{\ell} \frac{1}{\sqrt{n}} \sum_{i=[nt_{j-1}]+1}^{[nt_j]} \left[ a_j h_{F_l^{(n)},1}(Y_{ni}^l) + b_j h_{F_l^{(n)},2}(Y_{ni}^{l+1}) \right]$$

converges in distribution to a Gaussian random variable.

For the simplification of the presentation of the proof, we only treat the case  $\ell = 2$  and  $0 = t_0 < t_1 < t_2 = 1$ .

The assumption (9) readily holds from (4).

Define, for  $j = 1, 2$ ,

$$\vartheta_{ni}^{(j)} = a_j h_{F_l^{(n)},1}(Y_{ni}^l) + b_j h_{F_l^{(n)},2}(Y_{ni}^{l+1}).$$

For establishing (12), we need proving that, as  $n$  tends to infinity,

$$\mathbb{E} \left\{ \left[ \frac{1}{\sqrt{n}} \left( \sum_{i=1}^{[nt_1]} \vartheta_{ni}^{(1)} + \sum_{i=[nt_1]+1}^n \vartheta_{ni}^{(2)} \right) \right]^2 \right\}$$

tends to some positive constant  $c$ .

We have

$$\begin{aligned} \mathbb{E} \left\{ \left[ \frac{1}{\sqrt{n}} \left( \sum_{i=1}^{[nt_1]} \vartheta_{ni}^{(1)} + \sum_{i=[nt_1]+1}^n \vartheta_{ni}^{(2)} \right) \right]^2 \right\} = \\ \frac{1}{n} \left\{ \mathbb{E} \left[ \left( \sum_{i=1}^{[nt_1]} \vartheta_{ni}^{(1)} \right)^2 \right] + 2 \mathbb{E} \left[ \left( \sum_{i=1}^{[nt_1]} \vartheta_{ni}^{(1)} \right) \left( \sum_{i=[nt_1]+1}^n \vartheta_{ni}^{(2)} \right) \right] + \mathbb{E} \left[ \left( \sum_{i=[nt_1]+1}^n \vartheta_{ni}^{(2)} \right)^2 \right] \right\}. \end{aligned}$$

Since the random variables  $\vartheta_{ni}^{(1)}$  and  $\vartheta_{ni}^{(2)}$  are centered, we obtain

$$\mathbb{E} \left[ \left( \sum_{i=1}^{[nt_1]} \vartheta_{ni}^{(1)} \right)^2 \right] = [nt_1] \mathbb{E} \left[ \left( \vartheta_{n1}^{(1)} \right)^2 \right] + 2 \sum_{i=1}^{[nt_1]} \sum_{j=1}^{[nt_1]-i} \mathbb{E} \left( \vartheta_{ni}^{(1)} \vartheta_{n,i+j}^{(1)} \right).$$

From the condition of Theorem 2, we deduce that  $\mathbb{E} \left[ \left( \vartheta_{n1}^{(1)} \right)^{2+\delta} \right] < \infty$ , which implies that

$$\sup_{n,i,j \geq 1} \left| \mathbb{E} \left( \vartheta_{ni}^{(1)} \vartheta_{n,i+j}^{(1)} \right) \right| \leq \beta^{\frac{\delta}{2+\delta}}(j) \left\{ \mathbb{E} \left( \left| \vartheta_{ni}^{(1)} \right|^{2+\delta} \right) \right\}^{\frac{1}{2+\delta}} \left\{ \mathbb{E} \left( \left| \vartheta_{n,i+j}^{(1)} \right|^{2+\delta} \right) \right\}^{\frac{1}{2+\delta}}.$$

We get

$$\mathbb{E} \left[ \left( \sum_{i=1}^{[nt_1]} \vartheta_{ni}^{(1)} \right)^2 \right] \leq [nt_1] \mathbb{E} \left[ \left( \vartheta_{n1}^{(1)} \right)^2 \right] + 2[nt_1] \sum_{j=1}^{[nt_1]} \beta^{\frac{\delta}{2+\delta}}(j) M^2,$$

where  $M = \sup_{n \geq 1} \left\{ \mathbb{E} \left[ \left( \vartheta_{n1}^{(1)} \right)^{2+\delta} \right] \right\}^{\frac{1}{2+\delta}}$ .

It results that

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{1}{n} \left\{ [nt_1] \mathbb{E} \left[ \left( \vartheta_{n1}^{(1)} \right)^2 \right] + 2 \sum_{i=1}^{[nt_1]} \sum_{j=1}^{[nt_1]-i} \left| \mathbb{E} \left( \vartheta_{ni}^{(1)} \vartheta_{n,i+j}^{(1)} \right) \right| \right\} \\ & \leq t_1 \left\{ \mathbb{E} \left[ \left( \vartheta_{n1}^{(1)} \right)^2 \right] + 2 \sum_{j=1}^{\infty} \beta^{\frac{\delta}{2+\delta}}(j) M^2 \right\}. \end{aligned} \quad (18)$$

We also have

$$\mathbb{E} \left[ \left( \sum_{i=1}^{[nt_1]} \vartheta_{ni}^{(1)} \right) \left( \sum_{i=[nt_1]+1}^n \vartheta_{ni}^{(2)} \right) \right] = \sum_{i=1}^{[nt_1]} \sum_{j=[nt_1]+1}^n \mathbb{E} \left( \vartheta_{ni}^{(1)} \vartheta_{nj}^{(2)} \right).$$

From

$$\sup_{n \geq 1} \sup_{i,j \geq 1} \left| \mathbb{E} \left( \vartheta_{ni}^{(1)} \vartheta_{nj}^{(2)} \right) \right| \leq \beta^{\frac{\delta}{2+\delta}}(j-i) M M^*,$$

where  $M^* = \sup_{n \geq 1} \left\{ \mathbb{E} \left[ \left( \vartheta_{n1}^{(2)} \right)^{2+\delta} \right] \right\}^{\frac{1}{2+\delta}}$ , it results that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^{[nt_1]} \sum_{j=[nt_1]+1}^n \left| \mathbb{E} \left( \vartheta_{ni}^{(1)} \vartheta_{nj}^{(2)} \right) \right| \leq t_1 \sum_{j=1}^{\infty} \beta^{\frac{\delta}{2+\delta}}(j) M M^*. \quad (19)$$

Similarly, we get

$$\lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E} \left[ \left( \sum_{i=[nt_1]+1}^n \vartheta_{ni}^{(2)} \right)^2 \right] \leq (1-t_1) \left\{ \mathbb{E} \left[ (\vartheta_{n1}^{(1)})^2 \right] + 2 \sum_{j=1}^{\infty} \beta^{\frac{\delta}{2+\delta}}(j) (M^*)^2 \right\}. \quad (20)$$

From (18)-(20), we deduce (12).

Now, we turn to proving (11). For all  $i \geq 1$ , and for any  $\kappa > 0$ , define

$$\vartheta_{ni}^{(j),\kappa} = \begin{cases} \vartheta_{ni}^{(j)} & \text{if } |\vartheta_{ni}^{(j)}| \leq \kappa \\ 0 & \text{if } |\vartheta_{ni}^{(j)}| \geq \kappa, \quad j = 1, 2. \end{cases}$$

It is immediate that

$$\sup_{n \geq 1} \sup_{i \geq 1} |\vartheta_{ni}^{(j)}| \leq \kappa < \infty.$$

It results from the integrability condition in Theorem 2 that the sequences  $\{\vartheta_{ni}^{(j)}; i \geq 1, j = 1, 2\}$  are uniformly integrable.

Whence

$$\sup_{i \geq 1} \mathbb{E} \left( \left| \vartheta_{ni}^{(j)} - \vartheta_{ni}^{(j),\kappa} \right|^{2+\delta} \right) \longrightarrow 0 \quad \text{as } \kappa \rightarrow \infty, \quad j = 1, 2$$

and (11) is proved.

The proof of (13), that is

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[ \left( \frac{1}{\sqrt{n}} \left\{ \sum_{i=1}^{[nt_1]} \left[ \vartheta_{ni}^{(1),\kappa} - \mathbb{E}(\vartheta_{ni}^{(1),\kappa}) \right] + \sum_{i=[nt_1]+1}^n \left[ \vartheta_{ni}^{(2),\kappa} - \mathbb{E}(\vartheta_{ni}^{(2),\kappa}) \right] \right\} \right)^2 \right] = c_\kappa,$$

where  $c_\kappa$  is some positive constant, is similar to that of (12).

It remains to prove (14).

For any  $i \geq 1$  and any  $j = 1, 2$ , denote by  $\vartheta_i^{(j),\kappa}$  the counterpart of  $\vartheta_{ni}^{(j),\kappa}$  obtained by substituting the  $X_i$ 's for the  $Y_{ni}^l$ 's.

We have

$$\begin{aligned} c_\kappa &= t_1 \mathbb{E} \left\{ \left[ \vartheta_1^{(1),\kappa} - \mathbb{E}(\vartheta_1^{(1),\kappa}) \right]^2 \right\} \\ &\quad + 2t_1 \sum_{i=1}^{\infty} \mathbb{E} \left\{ \left[ \vartheta_1^{(1),\kappa} - \mathbb{E}(\vartheta_1^{(1),\kappa}) \right] \left[ \vartheta_{i+1}^{(1),\kappa} - \mathbb{E}(\vartheta_{i+1}^{(1),\kappa}) \right] \right\} \\ &\quad + t_1 \sum_{i=1}^{\infty} \mathbb{E} \left\{ \left[ \vartheta_1^{(1),\kappa} - \mathbb{E}(\vartheta_1^{(1),\kappa}) \right] \left[ \vartheta_{i+1}^{(2),\kappa} - \mathbb{E}(\vartheta_{i+1}^{(2),\kappa}) \right] \right\} \end{aligned}$$

$$\begin{aligned}
& + (1 - t_1) \mathbb{E} \left\{ \left[ \vartheta_1^{(2),\kappa} - \mathbb{E}(\vartheta_1^{(2),\kappa}) \right]^2 \right\} \\
& + 2(1 - t_1) \sum_{i=1}^{\infty} \mathbb{E} \left\{ \left[ \vartheta_1^{(2),\kappa} - \mathbb{E}(\vartheta_1^{(2),\kappa}) \right] \left[ \vartheta_{i+1}^{(1),\kappa} - \mathbb{E}(\vartheta_{i+1}^{(2),\kappa}) \right] \right\}.
\end{aligned}$$

By the Lebesgue dominated convergence theorem, one obtains

$$\begin{aligned}
& \mathbb{E} \left\{ \left[ \vartheta_1^{(1),\kappa} - \mathbb{E}(\vartheta_1^{(1),\kappa}) \right]^2 \right\} \longrightarrow \mathbb{E} \left[ \left( \vartheta_1^{(1)} \right)^2 \right] \quad \text{as } \kappa \rightarrow \infty, \\
& \mathbb{E} \left\{ \left[ \vartheta_1^{(1),\kappa} - \mathbb{E}(\vartheta_1^{(1),\kappa}) \right] \left[ \vartheta_{i+1}^{(1),\kappa} - \mathbb{E}(\vartheta_{i+1}^{(1),\kappa}) \right] \right\} \longrightarrow \mathbb{E} \left( \vartheta_1^{(1)} \vartheta_{i+1}^{(1)} \right) \quad \text{as } \kappa \rightarrow \infty, \\
& \mathbb{E} \left\{ \left[ \vartheta_1^{(1),\kappa} - \mathbb{E}(\vartheta_1^{(1),\kappa}) \right] \left[ \vartheta_{i+1}^{(2),\kappa} - \mathbb{E}(\vartheta_{i+1}^{(2),\kappa}) \right] \right\} \longrightarrow \mathbb{E} \left( \vartheta_1^{(2)} \vartheta_{i+1}^{(2)} \right) \quad \text{as } \kappa \rightarrow \infty, \\
& \mathbb{E} \left\{ \left[ \vartheta_1^{(2),\kappa} - \mathbb{E}(\vartheta_1^{(2),\kappa}) \right]^2 \right\} \longrightarrow \mathbb{E} \left[ \left( \vartheta_1^{(2)} \right)^2 \right] \quad \text{as } \kappa \rightarrow \infty
\end{aligned}$$

and

$$\mathbb{E} \left\{ \left[ \vartheta_1^{(2),\kappa} - \mathbb{E}(\vartheta_1^{(2),\kappa}) \right] \left[ \vartheta_{i+1}^{(2),\kappa} - \mathbb{E}(\vartheta_{i+1}^{(2),\kappa}) \right] \right\} \longrightarrow \mathbb{E} \left( \vartheta_1^{(2)} \vartheta_{i+1}^{(2)} \right) \quad \text{as } \kappa \rightarrow \infty.$$

Therefore

$$\lim_{\kappa \rightarrow \infty} c_\kappa = c$$

and (14) is proved. Whence, the finite dimensional convergence is established.

For proving the tightness define

$$\mathcal{Q}_n(t^l) = \sigma_{11}^{-1/2} \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt^l]} h_{F_l^{(n)}, 1}(Y_{ni}^l).$$

If  $t_1^l \leq t^l \leq t_2^l$ , from the integral conditions and condition (4), there exists a constant  $C$  such that

$$\begin{aligned}
\mathbb{E} \left( \left| \mathcal{Q}_n(t^l) - \mathcal{Q}_n(t_1^l) \right|^2 \mid \mathcal{Q}_n(t_2^l) - \mathcal{Q}_n(t^l) \right)^2 & \leq C \frac{1}{n^2} ([nt^l] - [nt_1^l]) ([nt_2^l] - [nt^l]) \\
& \leq C \frac{1}{n^2} ([nt_1^l] - [nt^l]) ([nt^l] - [nt_2^l]) \\
& \leq C \frac{1}{n^2} ([nt_2^l] - [nt_1^l])^2 \\
& \leq C (t_2^l - t_1^l)^2.
\end{aligned}$$

If  $t_2^l - t_1^l \geq 1/n$  the last inequality follows and if  $t_2^l - t_1^l < 1/n$ , then either  $t_1^l$  and  $t^l$  lie in the same subinterval  $[(i-1)/n, i/n]$  or else  $t^l$  and  $t_2^l$  do. In either of these cases the left hand of last inequality vanishes. From Theorem 13.5 of Billingsley (1968), the process  $\mathcal{Q}_n$  is tight.

With similar methods, we prove that any linear combinations of the components converges to a Gaussian random variable. Therefore Proposition 12 is proved by Lemma 11.

### 6.3 Proof of Theorem 1

Using the Hoeffding decomposition, for any  $(t_1, \dots, t_k) \in \Theta_k$ , we can write  $\mathcal{Z}_n(t_1, \dots, t_k)$  as

$$\begin{aligned} \mathcal{Z}_n(t_1, \dots, t_k) &= n^{-3/2} \sum_{l=1}^k \sum_{i=[nt_{l-1}]+1}^{[nt_l]} \sum_{j=[nt_l]+1}^{[nt_{l+1}]} [h_{F_1,1}(X_i) + h_{F_1,2}(X_j) + g_{F_1,F_1}(X_i, X_j)] \\ &= n^{-3/2} \sum_{l=1}^k \left[ ([nt_{l+1}] - [nt_l]) \sum_{i=[nt_{l-1}]+1}^{[nt_l]} h_{F_1,1}(X_i) + ([nt_l] - [nt_{l-1}]) \sum_{j=[nt_l]+1}^{[nt_{l+1}]} h_{F_1,2}(X_j) \right] \\ &\quad + n^{-3/2} \sum_{l=1}^k \sum_{i=[nt_{l-1}]+1}^{[nt_l]} \sum_{j=[nt_l]+1}^{[nt_{l+1}]} g_{F_1,F_1}(X_i, X_j), \end{aligned} \quad (21)$$

where by convention  $[nt_0] = 1$  and  $[nt_{k+1}] = n$ .

From Proposition 12, as  $n$  tends to infinity, in probability, we have,

$$n^{-3/2} \sup_{(t_1, \dots, t_k) \in \Theta_k} \left| \sum_{l=1}^k \sum_{i=[nt_{l-1}]+1}^{[nt_l]} \sum_{j=[nt_l]+1}^{[nt_{l+1}]} g_{F_1,F_1}(X_i, X_j) \right| \rightarrow 0.$$

Thus, by Slutsky's lemma, it suffices to show that the sequence of the following processes indexed by  $(t_1, \dots, t_k) \in \Theta_k$

$$n^{-3/2} \sum_{l=1}^k \left[ ([nt_{l+1}] - [nt_l]) \sum_{i=[nt_{l-1}]+1}^{[nt_l]} h_{F_1,1}(X_i) + ([nt_l] - [nt_{l-1}]) \sum_{j=[nt_l]+1}^{[nt_{l+1}]} h_{F_1,2}(X_j) \right]$$

converges weakly to the desired limit process.

Now, observing that the following mapping

$$\begin{pmatrix} x_1(t_1) \\ x_2(t_1) \\ x_3(t_2) \\ x_4(t_2) \\ \vdots \\ x_l(t_l) \\ x_{l+1}(t_l) \\ \vdots \\ x_k(t_k) \\ x_{k+1}(t_k) \end{pmatrix} \mapsto \sum_{l=1}^k \{ (t_{l+1} - t_l)[x_l(t_l) - x_l(t_{l-1})] + (t_l - t_{l-1})[x_{l+1}(t_{l+1}) - x_l(t_l)] \}$$

is continuous from  $(D[0, 1])^{2k}$  to  $D[0, 1]$ , one has by the continuous mapping theorem and Proposition 15, the following weak convergence

$$\begin{aligned} & n^{-3/2} \sum_{l=1}^k ([nt_{l+1}] - [nt_l]) \sum_{i=[nt_{l-1}]+1}^{[nt_l]} h_{F_1,1}(X_i) \\ & + n^{-3/2} \sum_{l=1}^k ([nt_l] - [nt_{l-1}]) \sum_{j=[nt_l]+1}^{[nt_{l+1}]} h_{F_1,2}(X_j) \\ & \longrightarrow \mathcal{Z}(t_1, \dots, t_k) \end{aligned}$$

and Theorem 1 is proved.  $\square$

## 6.4 Proof of Theorem 2

For the proof of Theorem 2, we observe that under its assumptions, for the random variables  $V_{n, \dot{F}_l}$  defined by

$$V_{n, \dot{F}_l} = \sum_{i=[nt_{l-1}]+1}^{[nt_l]} \sum_{j=[nt_l]+1}^{[nt_{l+1}]} h(X_i, X_j),$$

one can check easily that

$$\begin{aligned} & V_{n, \dot{F}_l} - ([nt_l] - [nt_{l-1}])([nt_{l+1}] - [nt_l])\theta_h(\dot{F}_{l-1}, \dot{F}_{l-1}) \\ & = ([nt_l] - [nt_{l-1}])([nt_{l+1}] - [nt_l])\theta_h(\dot{F}_{l-1}, \dot{F}_l) - \theta_h(\dot{F}_{l-1}, \dot{F}_{l-1}) \\ & + ([nt_{l+1}] - [nt_l]) \sum_{i=[nt_{l-1}]+1}^{[nt_l]} h_{\dot{F}_l,1}(X_i) + ([nt_l] - [nt_{l-1}]) \sum_{i=[nt_l]+1}^{[nt_{l+1}]} h_{\dot{F}_l,2}(X_i) \end{aligned}$$

$$+ \sum_{i=[nt_{l-1}]+1}^{[nt_l]} \sum_{j=[nt_l]+1}^{[nt_{l+1}]} g_{\dot{F}_{l-1}, \dot{F}_l}(X_i, X_j). \quad (22)$$

Thus, it results from (22), that for any  $(t_1, \dots, t_k) \in \Theta_k$ , the following equalities hold

$$\begin{aligned} \mathcal{Z}_n(t_1, \dots, t_k) &= n^{-3/2} \sum_{l=1}^k V_{n, F_l^{(n)}} \\ &\quad - n^{-3/2} \sum_{l=1}^k ([nt_l] - [nt_{l-1}])([nt_{l+1}] - [nt_l]) \left[ \theta_h(F_{l-1}^{(n)}, F_l^{(n)}) - \theta_h(F_{l-1}^{(n)}, F_{l-1}^{(n)}) \right] \\ &= n^{-3/2} \sum_{l=1}^k \left\{ ([nt_{l+1}] - [nt_l]) \sum_{i=[nt_{l-1}]+1}^{[nt_l]} h_{F_l^{(n)}, 1}(Y_{ni}^{l-1}) \right. \\ &\quad \left. + ([nt_l] - [nt_{l-1}]) \sum_{j=[nt_l]+1}^{[nt_{l+1}]} h_{F_l^{(n)}, 2}(Y_{nj}^l) \right. \\ &\quad \left. + \sum_{i=[nt_{l-1}]+1}^{[nt_l]} \sum_{j=[nt_l]+1}^{[nt_{l+1}]} g_{F_{l-1}^{(n)}, F_l^{(n)}}(Y_{ni}^{l-1}, Y_{nj}^l) \right\} \\ &\quad - n^{-3/2} \sum_{l=1}^k ([nt_l] - [nt_{l-1}])([nt_{l+1}] - [nt_l]) \left[ \theta_h(F_{l-1}^{(n)}, F_l^{(n)}) - \theta_h(F_{l-1}^{(n)}, F_{l-1}^{(n)}) \right]. \end{aligned}$$

From Proposition 15, we deduce that as  $n$  tends to infinity, in probability,

$$n^{-3/2} \sup_{(t_1, \dots, t_k) \in \Theta_k} \left| \sum_{l=1}^k \sum_{i=[nt_{l-1}]+1}^{[nt_l]} \sum_{j=[nt_l]+1}^{[nt_{l+1}]} g_{F_{l-1}^{(n)}, F_l^{(n)}}(Y_{ni}^{l-1}, Y_{nj}^l) \right| \longrightarrow 0.$$

By Proposition 15, it is easy to see that, for any  $l = 1, \dots, k$ , the sequence of processes

$$n^{-1/2} \sum_{i=1}^{[nt_l]} h_{F_l^{(n)}, 1}(Y_{ni}^{l-1})$$

converges weakly to the Brownian process  $\{W_1(t_l)\}_{0 \leq t_l \leq 1}$  and that the sequence of processes

$$n^{-1/2} \sum_{j=1}^{[nt_l]} h_{F_l^{(n)}, 2}(Y_{nj}^l)$$

converges weakly to the Brownian process  $\{W_2(t_l)\}_{0 \leq t_l \leq 1}$ .

As in the proof of Theorem 1, there is a continuous mapping from  $(D[0, 1])^{2k}$  to  $D[0, 1]$  such that, as  $n$  tends to infinity, the following weak convergence holds

$$n^{-3/2} \sum_{l=1}^k \left\{ ([nt_{l+1}] - [nt_l]) \sum_{i=[nt_{l-1}]+1}^{[nt_l]} h_{F_l^{(n)}, 1}(Y_{ni}^{l-1}) + ([nt_l] - [nt_{l-1}]) \sum_{j=[nt_l]+1}^{[nt_{l+1}]} h_{F_l^{(n)}, 2}(Y_{nj}^l) \right\} \\ \longrightarrow \mathcal{Z}(t_1, \dots, t_k).$$

Also, under  $H_{1,n}^k$ , as  $n$  tends to infinity,

$$n^{-3/2} \sum_{l=1}^k ([nt_l] - [nt_{l-1}]) ([nt_{l+1}] - [nt_l]) \left[ \theta_h(F_{l-1}^{(n)}, F_l^{(n)}) - \theta_h(F_{l-1}^{(n)}, F_{l-1}^{(n)}) \right] \\ \longrightarrow \sum_{l=1}^k (t_{l+1} - t_l)(t_l - t_{l+1}) A_l$$

and Theorem 2 is proved.  $\square$

## 6.5 Proof of Theorem 4

Let  $[nt_{0,l-1}] \leq [nt_l] \leq [nt_{0,l}]$ ,  $1 \leq l \leq k$ , then

$$\begin{aligned} \mathcal{Z}_n^*(t_1, \dots, t_k) &= n^{-3/2} \sum_{l=1}^k \sum_{[nt_{0,l-1}] \leq i < j \leq [nt_{0,l}]} h(X_i, X_j) \\ &\quad + n^{-3/2} \sum_{l=1}^k \sum_{i=[nt_{l-1}]+1}^{[nt_{0,l}]} \sum_{j=[nt_{0,l}]+1}^{[nt_{0,l+1}]} h(X_i, X_j) \\ &\quad - \left\{ n^{-3/2} \sum_{l=1}^k \sum_{[nt_{0,l-1}] \leq i < j \leq [nt_l]} h(X_i, X_j) \right. \\ &\quad + n^{-3/2} \sum_{l=1}^k \sum_{[nt_l]+1 \leq i < j \leq [nt_{0,l}]} h(X_i, X_j) \\ &\quad \left. + n^{-3/2} \sum_{l=1}^k \sum_{[nt_l]+1 \leq i \leq [nt_{0,l}]} \sum_{[nt_{0,l}]+1 \leq j \leq [nt_{0,l+1}]} h(X_i, X_j) \right\} \\ &= R_n^{(1)} + R_n^{(2)} - \left\{ R_n^{(3)} + R_n^{(4)} + R_n^{(5)} \right\}. \end{aligned}$$



Similar to the proof of Theorem 3 (see [Ngatchou-Wandji et al. \(2022\)](#)). First we prove that as  $n$  tends to infinity, almost surely,

$$n^{-1/2}R_n^{(1)} \longrightarrow \sum_{l=1}^k t_{0,l}^2 \theta_h(\dot{F}_l, \dot{F}_l)/2.$$

Similarly, we prove that as  $n$  tends to infinity, almost surely,

$$n^{-1/2}R_n^{(3)} \longrightarrow \sum_{l=1}^k t_l^2 \theta_h(\dot{F}_l, \dot{F}_l)/2,$$

$$n^{-1/2}R_n^{(4)} \stackrel{\mathcal{D}}{=} \sum_{l=1}^k \sum_{1 \leq i < j \leq [nt_{0,l}] - [(n+1)t_l]} h(X_i, X_j) \longrightarrow \sum_{l=1}^k (t_l - t_{0,l})^2 \theta_h(\dot{F}_l, \dot{F}_l)/2.$$

and, we establish that, as  $n$  tends to infinity, almost surely,

$$n^{-1/2}R_n^{(2)} \longrightarrow \sum_{l=1}^k t_{0,l}(t_{0,l+1} - t_{0,l}) \theta_h(\dot{F}_l, \dot{F}_{l+1}).$$

Similarly, we prove that, as  $n$  tends to infinity, in probability,

$$n^{-1/2}R_n^{(5)} \longrightarrow \sum_{l=1}^k (t_{0,l} - t_l)(t_{0,l+1} - t_{0,l}) \theta_h(\dot{F}_l, \dot{F}_{l+1}).$$

The first part of (6) clearly follows from these results. The proof of its second part can be handled in similar lines.  $\square$

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