

# Pessimistic Verification for Open Ended Math Questions

Yanxing Huang<sup>1</sup>, Zihan Tang<sup>1\*</sup>, Zejin Lin<sup>1\*</sup>, Peng Li<sup>3†</sup>, Yang Liu<sup>2,3†</sup>

<sup>1</sup>QiuZhen College, Tsinghua University

<sup>2</sup>Dept. of Comp. Sci. & Tech., Institute for AI, Tsinghua University

<sup>3</sup>Institute for AI Industry Research (AIR), Tsinghua University

[Code](#) [Dataset](#)

## Abstract

The key limitation of the verification performance lies in the ability of error detection. With this intuition we designed several variants of pessimistic verification, which are simple workflows that could significantly improve the verification of open-ended math questions. In pessimistic verification we construct multiple parallel verifications for the same proof, and the proof is deemed incorrect if any one of them reports an error. This simple technique significantly improves the performance across many math verification benchmarks without incurring substantial computational resources. Its token efficiency even surpassed extended long-CoT in test-time scaling. Our case studies further indicate that the majority of false negatives in stronger models are actually caused by annotation errors in the original dataset, so our method’s performance is in fact underestimated. Self-verification for mathematical problems can effectively improve the reliability and performance of language model outputs, and it also plays a critical role in enabling long-horizon mathematical tasks. We believe that research on pessimistic verification will help enhance the mathematical capabilities of language models across a wide range of tasks.

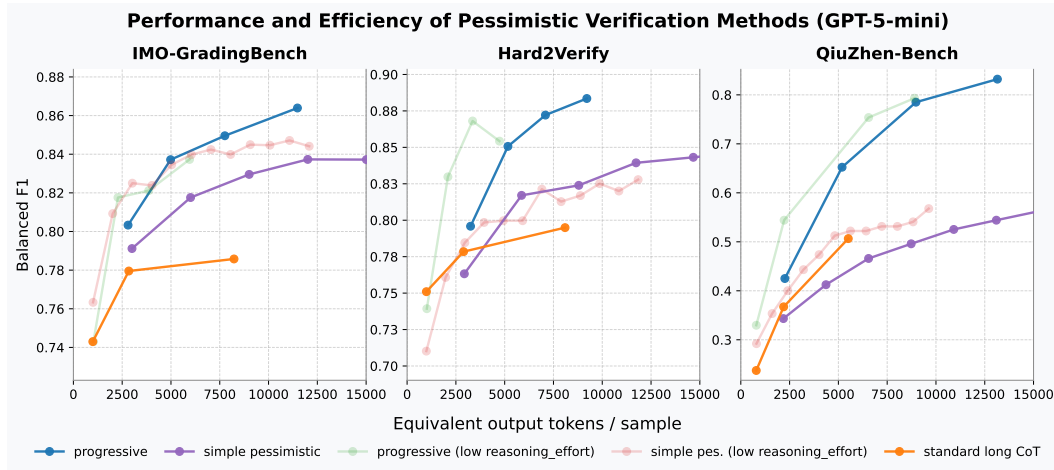


Figure 1: The response-level performance and efficiency of pessimistic verification methods on GPT-5-mini. The equivalent output tokens is calculated by weighting the model’s input and output token prices. For GPT-5-mini it is  $(\text{input\_tokens} / 8 + \text{output\_tokens})$ . These results indicate that pessimistic verification has stable and efficient scalability.

\*Indicates equal contribution.

†Corresponding authors: Peng Li (lipeng@air.tsinghua.edu.cn) and Yang Liu (liuyang2011@tsinghua.edu.cn).

---

## 1 Introduction

Since the release of OpenAI o1 (OpenAI et al., 2024) and DeepSeek-R1 (DeepSeek-AI et al., 2025), reasoning has become one of the most important topics in large language model (LLM) research within both academia and industry. Nevertheless, the scalability of the training recipe behind current large reasoning models (LRMs) is still limited by the requirement of verifiable reward. They typically require math problems or code test cases that come with ground-truth answers, along with feedback functions that can be directly computed through explicit rules to complete the training. Even in math, one of the most successful domains of LRM, the absence of a generic verifiable reward still introduces significant challenges to more advanced, open-ended and long-form reasoning tasks (Xu et al., 2025).

One possible solution to this problem is through formal theorem provers such as Lean (Chen et al., 2025; Varambally et al., 2025), which could provide completely reliable verification on math proofs. However, this approach would introduce notable external budget to the AI system while the performance of natural language provers can achieve 4x higher than formal provers on the same dataset (Dekoninck et al., 2025). Another line of work focuses on leveraging the internal capability of the LLM to achieve self evolution (Zuo et al., 2025; Yu et al., 2025; Xu et al., 2025). They typically rely on some form of self-verification to complete training, and we believe that such approaches have even greater potential. We contend that the importance of self-verification capabilities can be reflected in several key dimensions:

- Effective self-verification can substantially enhance the reliability of model outputs and significantly improve overall performance. The performance IMO level math problems can be notably enhanced via a verifier-guided workflow according to Huang & Yang (2025); Luong et al. (2025).
- Existing research indicates that the reliability of single-step task execution strongly influences the duration over which a system can operate dependably, thus introspective abilities may be particularly critical for long-horizon tasks (Kwa et al., 2025).
- We further argue that a general intelligent system should possess intrinsic mechanisms for self-validation, rather than relying exclusively on external ground-truth signals or verification modules.

The key limitation of verification lies in the ability of finding errors in a proof, while before the release of this work there is no well-known and scalable method that could enhance this ability other than long chain of thought. (Pandit et al., 2025). In this work, we conduct an in-depth investigation and extension of the previously existing but underexplored idea of pessimistic verification (Liu et al., 2025), and we develop several new methods that offer significantly improved performance and efficiency. Here we introduce three simple workflows which we call **simple pessimistic verification**, **vertical pessimistic verification**, and **progressive pessimistic verification**. These methods review a given solution for multiple times from different perspectives, and the whole proof will be considered false if any one review finds an error. More specifically, their differences lie in the way multi-round evaluations are constructed.

- **Simple**: directly perform multiple verification queries on the entire proof.
- **Vertical**: partition the proof into several segments and examine the details within each one more thoroughly.
- **Progressive**: combine the simple and vertical approaches to conduct multi-scale verification, ranging from the whole proof to fine-grained details, while filtering out easily detectable errors along the way.

We have conducted a series of experiments on three datasets, *Hard2Verify* (Pandit et al., 2025), *IMO-GradingBench* (Luong et al., 2025), and our *QiuZhen-Bench*. The former two benchmarks are both contest-level math grading benchmarks with expert annotations. *QiuZhenBench* was collected and curated from S.-T. Yau College Student Mathematics Contest and the doctoral qualifying exams from QiuZhen college, Tsinghua University. These exams covers a

wide range of topics in undergraduate-level mathematics and are well-known for their high difficulty. On all benchmarks our methods consistently show impressive improvements in error detection rate and overall balanced F1 score, indicating their effectiveness on proof verifications. Our case study further reveals that most of the errors made by stronger models arise from mislabeling in the original dataset or inconsistencies in rigor requirements. State-of-the-art models equipped with pessimistic verification already approach near human experts in verifying mathematical problems, yet this capability is underestimated by statistical metrics.

Our main contributions are:

- We proposed three variants of pessimistic verification methods that could significantly enhance the verification ability of large language models with a higher efficiency than scaling long chain of thought.
- We curated QiuZhen-Bench, a dataset containing 1,054 questions of high quality and difficulty in advanced mathematics.
- We conducted extensive and thorough experiments, which further confirm the effectiveness of our method.

## 2 Method

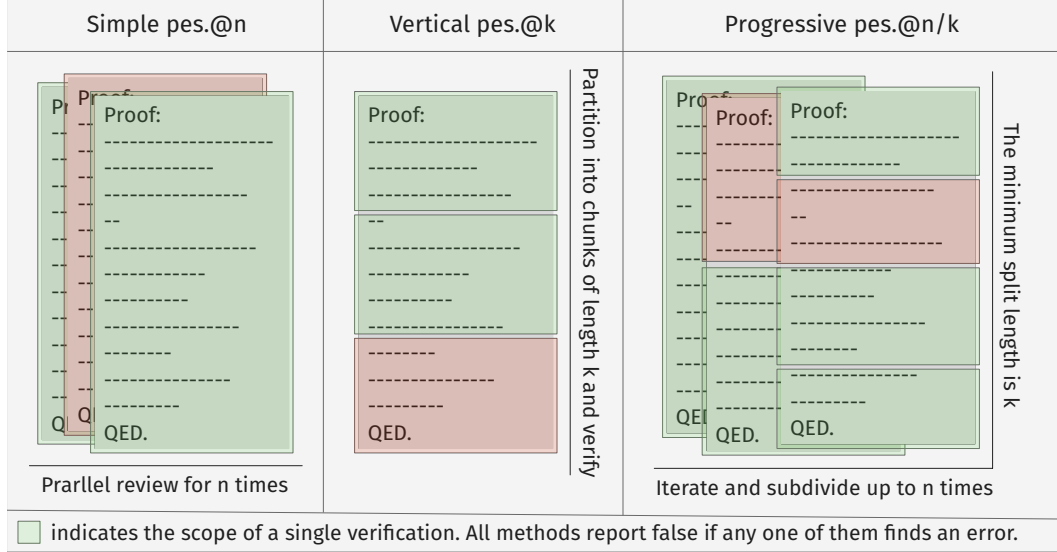


Figure 2: Three variants of pessimistic verification methods in this work. In following experiments they will separately be denoted as “pes@n”, “vp@l”, and “prog@n/l”

In Figure 2, we illustrate the three main pessimistic-verification approaches discussed in this paper. Among them, the progressive verification method combines the strengths of the other two and demonstrates the highest efficiency and performance ceiling in our experiments.

A common test-time scaling strategy of enhancing model capability is through majority voting. However, this mechanism does not work on verification tasks according to our experiments and some related researches (Pandit et al., 2025; Mahdavi et al., 2025b).

We can roughly understand this phenomenon as follows: Both our work and some recent studies (Pandit et al., 2025) indicate that identifying errors is the primary challenge in verifying mathematical proofs, it is likely that only a small number of evaluations can identify the critical mistakes. In this case, majority voting may actually restrict the model’s ability to uncover potential errors, while pessimistic verification further reinforces this ability. We also require the verifier to provide explanations of the error it detects, which

ensures the reliability and soundness of most verification results. Therefore, pessimistic verification does not significantly affect the verifier’s stability on positive examples.

## 2.1 Simple pessimistic verification

Simple pessimistic verification also conducts multiple reviews on a single proof as majority voting, but instead of using the majority as final answer, this method will constantly choose the first negative verification from these reviews if exists. As shown in Figure 3, this method drastically improves the overall performance of evaluation while majority voting has almost no effect.

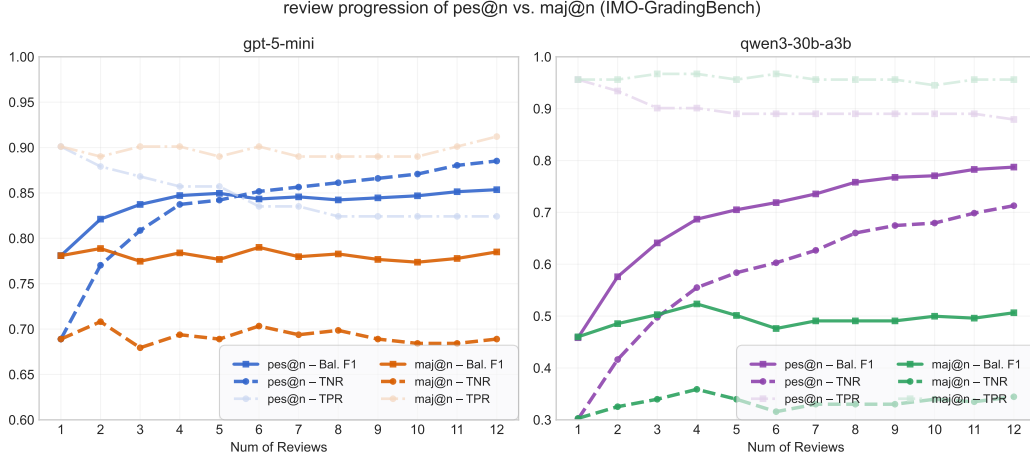


Figure 3: The comparison of simple pessimistic verification and majority voting. The former exhibits steady performance gains as sampling budget increases, whereas the latter shows almost no changes in performance.

## 2.2 Vertical pessimistic verification

In spite of simply applying multiple reviews on the whole proof, we also explored a pessimistic verification from another dimension. As shown in Figure 4, we adopted a special prompting method and require the LLM to focus on a certain part of the proof, and try looking deep into these contents to find errors. Vertical pessimistic verification adopts a hyperparameter  $l$ , it first splits the whole proof into chunks with  $l$  lines, and create a series of parallel review tasks for each chunk. Although this method only goes through the proof once, we also witnessed improved performance in error detection and even a higher scaling efficiency compared to simple pessimistic verification.

## 2.3 Progressive pessimistic verification

Progressive pessimistic verification combines the mechanism of both simple and vertical methods. It performs multi-scale verification, ranging from the level of the complete proof down to detailed steps similar to vertical pessimistic verification. This allows it to quickly filter out obvious errors in the proof, and then drill down into the detailed steps to carefully examine the correctness of each part.

Progressive pessimistic verification can also naturally support pruning operations, thereby further improving its execution efficiency. It begins with iterative verification on a single complete proof, and after each iteration, the proof is subdivided in half and each part is evaluated separately in the manner shown in Figure 4. After every iteration, proofs in which errors have been detected are filtered out, and verification continues on the remaining proofs. We use the parameter  $n$  to control the maximum number of iterations, in which case the maximum number of checks performed on a single proof is  $2^n - 1$ .

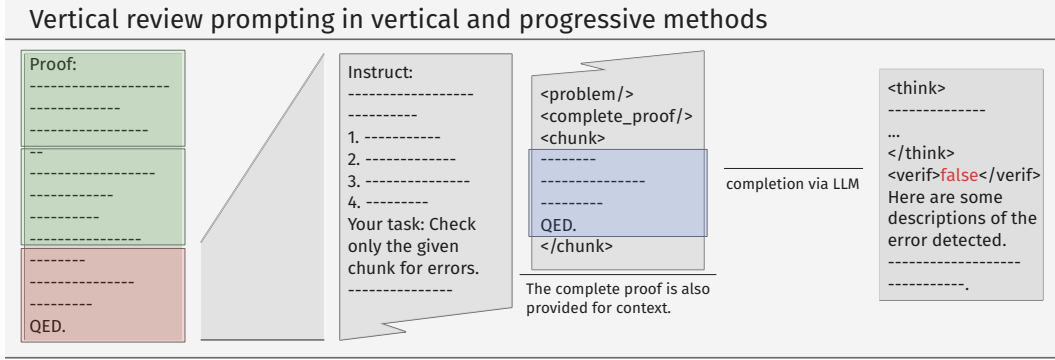


Figure 4: The vertical review prompting method used in vertical pessimistic verification and progressive pessimistic verification.

Pruning optimization in this method achieves the highest runtime efficiency when negative examples are prevalent in the data distribution, while the verification cost is greatest when positive examples dominate. However, in such cases, it is also better able to preserve the reliability of positive examples.

### 3 Experiments

#### 3.1 Datasets and settings

In our experiments we primarily use three datasets for evaluation, and we constantly use the same prompt and workflow setting across all these dataset. Here are some descriptions about them:

- **IMO-GradingBench** (Luong et al., 2025). This dataset contains 1,000 human-graded solutions to IMO-level proof problems from IMO-ProofBench (Luong et al., 2025), from which we selected a subset with 300 samples for evaluation. This is a challenging test of fine-grained mathematical proof evaluation.
- **Hard2Verify** (Pandit et al., 2025). Hard2Verify contains 200 challenging problems and solutions from recent math competitions such as IMO and Putnam. The solutions are generated by strong models such as GPT-5 (OpenAI, 2025) and Gemini 2.5 Pro (Google DeepMind, 2025) and annotated by humans.
- **QiuZhen-Bench**. This is a homemade collection of advanced math problems, with questions sourced from challenging university-level math competitions. It serves as a supplement to the previous two elementary math competition problem datasets. We randomly selected a subset of 300 problems, had them answered by GPT-5-mini, and used GPT-5 with progressive@3/6 for labeling. This subset can be used to evaluate the performance of weaker models. You can refer to Appendix A.1 for more details.

All evaluation in our experiments was conducted at the response level, where we require the verifier to decide whether the whole proof is correct instead of providing a score or step-level evaluation. For IMO-GradingBench, only the responses that obtained fully 7 points are considered correct, otherwise they will all be considered false. And aside from our experiments, we will simply use single pass verification with different reasoning effort setting as the baseline, since we already know that majority voting has almost no effect on evaluation. In all our experiments we set the temperature parameter to 1.0 to ensure diversity of the responses.

### 3.2 Metrics

In this work, we treat mathematical proof verification as a binary classification problem and focus on the following performance metrics:

- **True Negative Rate (TNR):** The proportion of detected errors among all erroneous proofs. This is the primary metric for evaluating the model’s ability to identify incorrect proofs.
- **True Positive Rate (TPR):** The proportion of proofs classified as correct among all truly correct proofs. This helps assess the model’s proof-verification capability.
- **Balanced F1 Score:** The harmonic mean of TPR and TNR, providing a balanced indicator of performance when both false positives and false negatives matter. This is also the primary indicator used in Pandit et al. (2025).

$$\text{Balanced F1} = \frac{2 \cdot \text{TPR} \cdot \text{TNR}}{\text{TPR} + \text{TNR}} \quad (1)$$

### 3.3 Main results

We begin by conducting some scaling experiments to test the effect of hyperparameter settings in pessimistic verification methods. In addition to the experiments on GPT-5-mini (OpenAI, 2025) shown in Figure 1, we also conducted corresponding experiments on the smaller models Qwen3-30B-A3B (Yang et al., 2025). These results are summarized in Figure 5.

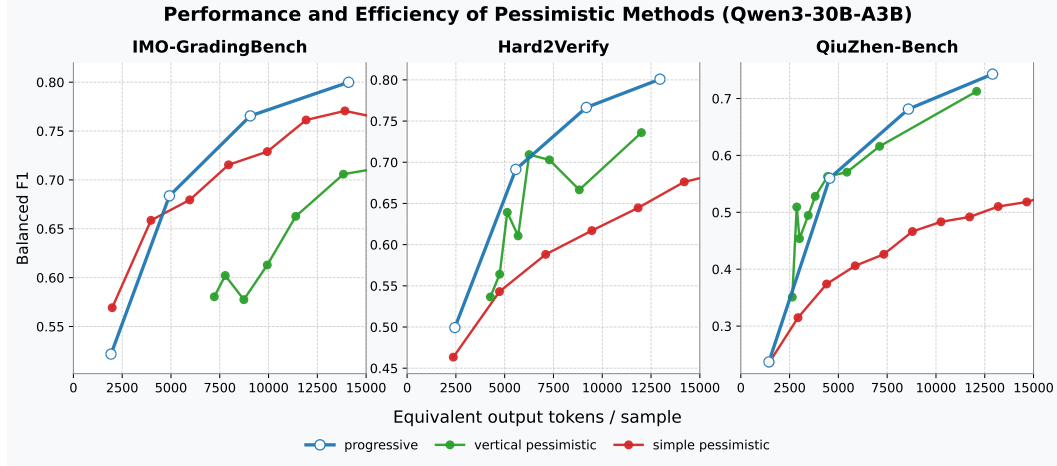


Figure 5: The response-level performance and efficiency of three pessimistic verification methods on Qwen3-30B-A3B.

These experiments showed similar trend of our methods. All three types of pessimistic verifications constantly benefit from scaling up computational budget. The performance of the vertical pessimistic method appears to be more influenced by the specific dataset, but when the segmentation is sufficiently fine-grained, it does indeed improve performance in situations where the proof has been read through only once. And as the combination of simple and vertical methods, progressive pessimistic verification constantly exhibits the highest performance and efficiency. Moreover, on models that the length of chain of thought is controllable (i.e. GPT-5-mini), simple and progressive methods constantly exhibits higher efficiency in test-time scaling than long chain of thought. Their parallel nature makes them even faster than long chains of reasoning.

Then we have also conducted vast experiments with standard and progressive pessimistic verification methods across different models on all three datasets, and the main results is

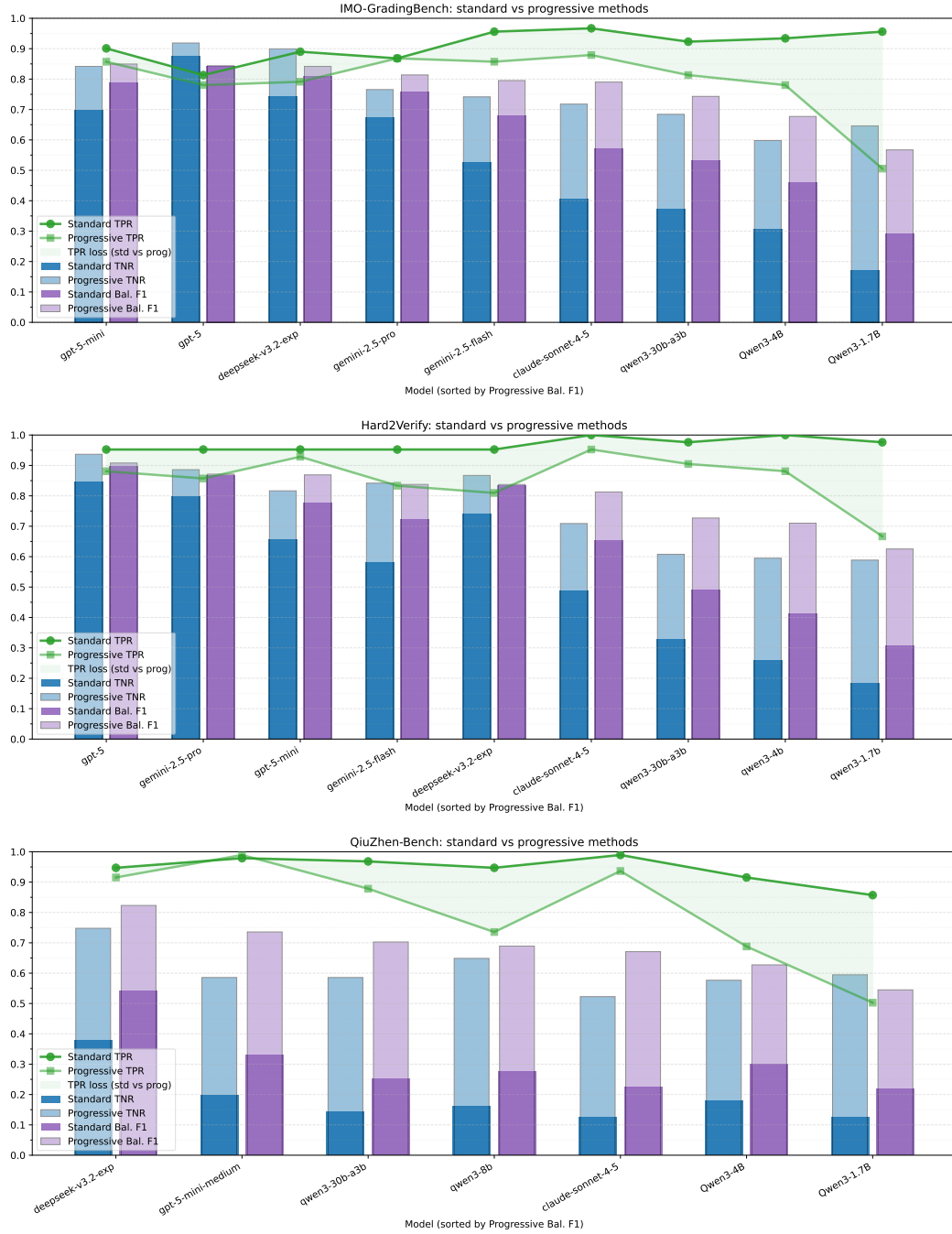


Figure 6: The main results on IMO-GradingBench, Hard2Verify and QiuZhen-Bench. Thinking mode is enabled for all models if possible, and the reasoning effort of gpt series model is set to medium. This method constantly improves the classification performance across all tested models.



summarized in Figure 6. In these experiments we use the hyperparameter setting `progressive@3/6` for balanced performance and efficiency. From these graphs we can read several key findings:

- Error detecting is the key ability that separates strong verifiers from the weak which is also discovered in some recent works (Pandit et al., 2025). On all datasets the TPR is almost the same across all models from strong to weak, while TNR varies a lot.
- Merely from statistics we can see that progressive pessimistic verification can significantly increase TNR and balanced F1 score for most models, with a small tradeoff of TPR.
- Weaker models tend to exhibit higher performance gains with a greater degression of TPR. The consistency on positive samples is also an important indicator of the verification ability of LLMs.
- In fact, based on more fine-grained case studies in Section 3.4, we found that for stronger models, the decrease in TPR is largely an illusion caused by dataset annotation, and the actual performance improvement is being underestimated.

### 3.4 Case study

To better analyze performance beyond aggregated statistics and reflect more realistic behavior, we also conducted a detailed case study on the results of the pessimistic verification. In this evaluation, we focus primarily on the False Negative cases, because TP and TN generally represent correct classifications, while most FP cases occur simply because our verifier failed to detect existing errors in the proof. FN cases are usually the most informative.

Surprisingly, we found that most of the FN cases produced by GPT-5-mini or stronger models under pessimistic verification did in fact identify real, critical errors in the proofs. These errors had been mistakenly labeled as correct by the dataset annotators. This indicates, on one hand, that our previous experiments substantially underestimated the verification capability of stronger models, and on the other hand, it supports the validity of using GPT-5 to annotate QiuZhen-Bench. Meanwhile, many smaller models do indeed produce a large number of unwarranted False Negative judgments.

The statistical results of our case study can be found in Table 1. In the Appendix A.3, we also provide several concrete examples of mislabeling in the original dataset, and the full set of cases we examined is available in our open-source code repository. Our classification criteria are based on whether the errors identified by these models in FN cases are:

- **Critical.** If our verifier does uncover a critical error that was overlooked in the original dataset annotation, or a difficult-to-resolve tricky skip, it directly affects the correctness of the entire proof.
- **Minor.** If our verifier identified a typo or minor errors that can be easily fixed. In such cases, whether it should be judged as an error depends entirely on subjective criteria.
- **Nonsense.** If our verifier misunderstands the problems and produces some unwarranted judgments. These are the truly harmful false negatives.

Model	Dataset	Critical	Minor	Nonsense	Total
GPT-5	IMO-GradingBench	9	9	2	20
GPT-5-mini	IMO-GradingBench	5	5	3	13
Qwen3-30B-A3B	IMO-GradingBench	1	1	13	15

Table 1: Manual review of FN cases in pessimistic verification. The classification is based on whether the judgments provided by the verifier in these FN cases are critical, minor, or nonsense.



---

We also encourage the community to conduct more experiments on pessimistic verification in order to better determine its true performance on state-of-the-art models.

## 4 Related work

Using LLM for evaluation or verification tasks is a natural idea. At the early development stage of LLM, some works have already tried this method on traditional tasks in natural language processing, such as text summarization (Liu et al., 2023), dialog generation (Liu et al., 2023), and machine translation (Zheng et al., 2023). This approach has achieved some results, but several problems still remain, such as scoring bias (Li et al., 2025) and self-inconsistency (Halder & Hockenmaier, 2025).

The open-ended math problem lies between the math answering problem and other evaluation problems. It lacks simple and direct means of verification, but its correctness is entirely objective. Existing works in this area primarily focus on the alignment of LLM grading and that of humans. Some of them proposed certain agentic workflows that could enhance this ability (Mahdavi et al., 2025a;b). Reinforcement learning on manually annotated data also exhibited effectiveness on the evaluation of mathematical proofs (Dekoning et al., 2025). However, these methods lack the scalability in further enhancing the performance, and they cannot distinguish performance improvements brought by subjective preference alignment from those resulting from objectively discovering new errors. Some work also highlights the importance of error detection, as this is the key ability that separates strong verifiers from weaker ones (Pandit et al., 2025).

## 5 Conclusion and discussion

In this work we proposed several variants of pessimistic verification method, which exhibits strong performance and even higher scaling potential than long chain of thought on the evaluation of open-ended math problems. These methods construct multiple different verification queries for a single mathematical proof in different ways, and deems the proof incorrect if any one of these queries determines it to be wrong.

Beyond the existing concrete implementations and results, we believe that the error-centered idea behind pessimistic verification is what truly deserves attention. This approach will naturally make the verification of mathematical problems increasingly stringent, which may also align with the field’s gradual trend toward greater formalization and rigor. It may likewise help guide large language models away from merely computing correct answers and toward generating fully rigorous proofs.

We can also envision several direct applications of pessimistic verification:

- Using pessimistic verification in math or code related workflows can further improve the reliability of the response, especially for long-form tasks.
- This method can further push the capability frontier of state-of-the-art large models, so there is an opportunity to incorporate it into the training pipeline to further raise the upper limit of large models’ abilities in executing verification and rigorous proof tasks.
- As the state-of-the-art performance of large language models approaching human experts, maybe we can also try to construct benchmarks for provers on open-ended math questions with these verification methods.
- In many software engineering applications, pessimistic verification may also be applicable, allowing us to use progressive verification to quickly identify potential issues in code without the need to design test cases or execute the code.

## References

Luoxin Chen, Jinming Gu, Liankai Huang, Wenhao Huang, Zhicheng Jiang, Allan Jie, Xiaoran Jin, Xing Jin, Chenggang Li, Kaijing Ma, Cheng Ren, Jiawei Shen, Wenlei Shi, Tong

- 
- Sun, He Sun, Jiahui Wang, Siran Wang, Zhihong Wang, Chenrui Wei, Shufa Wei, Yonghui Wu, Yuchen Wu, Yihang Xia, Huajian Xin, Fan Yang, Huaiyuan Ying, Hongyi Yuan, Zheng Yuan, Tianyang Zhan, Chi Zhang, Yue Zhang, Ge Zhang, Tianyun Zhao, Jianqiu Zhao, Yichi Zhou, and Thomas Hanwen Zhu. Seed-Prover: Deep and Broad Reasoning for Automated Theorem Proving, August 2025. URL <http://arxiv.org/abs/2507.23726>. arXiv:2507.23726 [cs].
- DeepSeek-AI, Daya Guo, Dejian Yang, Haowei Zhang, Junxiao Song, Ruoyu Zhang, Runxin Xu, Qihao Zhu, Shirong Ma, Peiyi Wang, Xiao Bi, Xiaokang Zhang, Xingkai Yu, Yu Wu, Z. F. Wu, Zhibin Gou, Zhihong Shao, Zhuoshu Li, Ziyi Gao, Aixin Liu, Bing Xue, Bingxuan Wang, Bochao Wu, Bei Feng, Chengda Lu, Chenggang Zhao, Chengqi Deng, Chenyu Zhang, Chong Ruan, Damai Dai, Deli Chen, Dongjie Ji, Erhang Li, Fangyun Lin, Fucong Dai, Fuli Luo, Guangbo Hao, Guanting Chen, Guowei Li, H. Zhang, Han Bao, Hanwei Xu, Haocheng Wang, Honghui Ding, Huajian Xin, Huazuo Gao, Hui Qu, Hui Li, Jianzhong Guo, Jiashi Li, Jiawei Wang, Jingchang Chen, Jingyang Yuan, Junjie Qiu, Junlong Li, J. L. Cai, Jiaqi Ni, Jian Liang, Jin Chen, Kai Dong, Kai Hu, Kaige Gao, Kang Guan, Kexin Huang, Kuai Yu, Lean Wang, Lecong Zhang, Liang Zhao, Litong Wang, Liyue Zhang, Lei Xu, Leyi Xia, Mingchuan Zhang, Minghua Zhang, Minghui Tang, Meng Li, Miaojuan Wang, Mingming Li, Ning Tian, Panpan Huang, Peng Zhang, Qiancheng Wang, Qinyu Chen, Qiushi Du, Ruiqi Ge, Ruisong Zhang, Ruizhe Pan, Runji Wang, R. J. Chen, R. L. Jin, Ruyi Chen, Shanghao Lu, Shangyan Zhou, Shanhuang Chen, Shengfeng Ye, Shiyu Wang, Shuiping Yu, Shunfeng Zhou, Shuting Pan, S. S. Li, Shuang Zhou, Shaoqing Wu, Shengfeng Ye, Tao Yun, Tian Pei, Tianyu Sun, T. Wang, Wangding Zeng, Wanbiao Zhao, Wen Liu, Wenfeng Liang, Wenjun Gao, Wenqin Yu, Wentao Zhang, W. L. Xiao, Wei An, Xiaodong Liu, Xiaohan Wang, Xiaokang Chen, Xiaotao Nie, Xin Cheng, Xin Liu, Xin Xie, Xingchao Liu, Xinyu Yang, Xinyuan Li, Xuecheng Su, Xuheng Lin, X. Q. Li, Xiangyue Jin, Xiaojin Shen, Xiaosha Chen, Xiaowen Sun, Xiaoxiang Wang, Xinnan Song, Xinyi Zhou, Xianzu Wang, Xinxia Shan, Y. K. Li, Y. Q. Wang, Y. X. Wei, Yang Zhang, Yanhong Xu, Yao Li, Yao Zhao, Yaofeng Sun, Yaohui Wang, Yi Yu, Yichao Zhang, Yifan Shi, Yiliang Xiong, Ying He, Yishi Piao, Yisong Wang, Yixuan Tan, Yiyang Ma, Yiyuan Liu, Yongqiang Guo, Yuan Ou, Yuduan Wang, Yue Gong, Yuheng Zou, Yujia He, Yunfan Xiong, Yuxiang Luo, Yuxiang You, Yuxuan Liu, Yuyang Zhou, Y. X. Zhu, Yanhong Xu, Yanping Huang, Yaohui Li, Yi Zheng, Yuchen Zhu, Yunxian Ma, Ying Tang, Yukun Zha, Yuting Yan, Z. Z. Ren, Zehui Ren, Zhangli Sha, Zhe Fu, Zhean Xu, Zhenda Xie, Zhengyan Zhang, Zhewen Hao, Zhicheng Ma, Zhigang Yan, Zhiyu Wu, Zihui Gu, Zijia Zhu, Zijun Liu, Zilin Li, Ziwei Xie, Ziyang Song, Zizheng Pan, Zhen Huang, Zhipeng Xu, Zhongyu Zhang, and Zhen Zhang. DeepSeek-R1: Incentivizing Reasoning Capability in LLMs via Reinforcement Learning, January 2025. URL <http://arxiv.org/abs/2501.12948> [cs].
- Jasper Dekoninck, Ivo Petrov, Kristian Minchev, Mislav Balunovic, Martin Vechev, Miroslav Marinov, Maria Drencheva, Lyuba Konova, Milen Shumanov, Kaloyan Tsvetkov, Nikolay Drenchev, Lazar Todorov, Kalina Nikolova, Nikolay Georgiev, Vanesa Kalinkova, and Margulan Ismoldayev. The Open Proof Corpus: A Large-Scale Study of LLM-Generated Mathematical Proofs, June 2025. URL <http://arxiv.org/abs/2506.21621>. arXiv:2506.21621 [cs].
- Google DeepMind. Gemini 2.5 pro. <https://ai.google.dev/gemini-api/docs/models>, 2025. Multimodal large language model.
- Rajarshi Haldar and Julia Hockenmaier. Rating Roulette: Self-Inconsistency in LLM-As-A-Judge Frameworks, October 2025. URL <http://arxiv.org/abs/2510.27106>. arXiv:2510.27106 [cs].
- Yichen Huang and Lin F. Yang. Gemini 2.5 Pro Capable of Winning Gold at IMO 2025, July 2025. URL <http://arxiv.org/abs/2507.15855>. arXiv:2507.15855 [cs].
- Thomas Kwa, Ben West, Joel Becker, Amy Deng, Katharyn Garcia, Max Hasin, Sami Jawhar, Megan Kinniment, Nate Rush, Sydney Von Arx, Ryan Bloom, Thomas Broadley, Haoxing Du, Brian Goodrich, Nikola Jurkovic, Luke Harold Miles, Seraphina Nix, Tao Lin, Neev Parikh, David Rein, Lucas Jun Koba Sato, Hjalmar Wijk, Daniel M. Ziegler, Elizabeth

- 
- Barnes, and Lawrence Chan. Measuring AI Ability to Complete Long Tasks, March 2025. URL <http://arxiv.org/abs/2503.14499>. arXiv:2503.14499 [cs].
- Qingquan Li, Shaoyu Dou, Kailai Shao, Chao Chen, and Haixiang Hu. Evaluating Scoring Bias in LLM-as-a-Judge, August 2025. URL <http://arxiv.org/abs/2506.22316>. arXiv:2506.22316 [cs].
- Yang Liu, Dan Iter, Yichong Xu, Shuohang Wang, Ruochen Xu, and Chenguang Zhu. G-Eval: NLG Evaluation using GPT-4 with Better Human Alignment, May 2023. URL <http://arxiv.org/abs/2303.16634>. arXiv:2303.16634 [cs].
- Yuanhang Liu, Yanxing Huang, Yanqiao Wang, Peng Li, and Yang Liu. AI Mathematician: Towards Fully Automated Frontier Mathematical Research, May 2025. URL <http://arxiv.org/abs/2505.22451>. arXiv:2505.22451 [cs].
- Thang Luong, Dawsen Hwang, Hoang H. Nguyen, Golnaz Ghiasi, Yuri Chervonyi, Insuk Seo, Junsu Kim, Garrett Bingham, Jonathan Lee, Swaroop Mishra, Alex Zhai, Clara Huiyi Hu, Henryk Michalewski, Jimin Kim, Jeonghyun Ahn, Junhwi Bae, Xingyou Song, Trieu H. Trinh, Quoc V. Le, and Junehyuk Jung. Towards Robust Mathematical Reasoning, November 2025. URL <http://arxiv.org/abs/2511.01846>. arXiv:2511.01846 [cs].
- Hamed Mahdavi, Pouria Mahdavinia, Samira Malek, Pegah Mohammadipour, Alireza Hashemi, Majid Daliri, Alireza Farhadi, Amir Khasahmadi, Niloofar Miresghallah, and Vasant Honavar. RefGrader: Automated Grading of Mathematical Competition Proofs using Agentic Workflows, October 2025a. URL <http://arxiv.org/abs/2510.09021>. arXiv:2510.09021 [cs].
- Sadeh Mahdavi, Branislav Kisacanin, Shubham Toshniwal, Wei Du, Ivan Moshkov, George Armstrong, Renjie Liao, Christos Thrampoulidis, and Igor Gitman. Scaling Generative Verifiers For Natural Language Mathematical Proof Verification And Selection, November 2025b. URL <http://arxiv.org/abs/2511.13027>. arXiv:2511.13027 [cs].
- OpenAI. Gpt-5 system card, August 2025. URL <https://cdn.openai.com/gpt-5-system-card.pdf>.
- OpenAI, Aaron Jaech, Adam Kalai, Adam Lerer, Adam Richardson, Ahmed El-Kishky, Aiden Low, Alec Helyar, Aleksander Madry, Alex Beutel, Alex Carney, Alex Iftimie, Alex Karpenko, Alex Tachard Passos, Alexander Neitz, Alexander Prokofiev, Alexander Wei, Allison Tam, Ally Bennett, Ananya Kumar, Andre Saraiva, Andrea Vallone, Andrew Duberstein, Andrew Kondrich, Andrey Mishchenko, Andy Applebaum, Angela Jiang, Ashvin Nair, Barret Zoph, Behrooz Ghorbani, Ben Rossen, Benjamin Sokolowsky, Boaz Barak, Bob McGrew, Borys Minaiev, Botao Hao, Bowen Baker, Brandon Houghton, Brandon McKinzie, Brydon Eastman, Camillo Lugaresi, Cary Bassin, Cary Hudson, Chak Ming Li, Charles de Bourcy, Chelsea Voss, Chen Shen, Chong Zhang, Chris Koch, Chris Orsinger, Christopher Hesse, Claudia Fischer, Clive Chan, Dan Roberts, Daniel Kappler, Daniel Levy, Daniel Selsam, David Dohan, David Farhi, David Mely, David Robinson, Dimitris Tsipras, Doug Li, Dragos Oprica, Eben Freeman, Eddie Zhang, Edmund Wong, Elizabeth Proehl, Enoch Cheung, Eric Mitchell, Eric Wallace, Erik Ritter, Evan Mays, Fan Wang, Felipe Petroski Such, Filippo Raso, Florencia Leoni, Foivos Tsimpourlas, Francis Song, Fred von Lohmann, Freddie Sulit, Geoff Salmon, Giambattista Parascandolo, Gildas Chabot, Grace Zhao, Greg Brockman, Guillaume Leclerc, Hadi Salman, Haiming Bao, Hao Sheng, Hart Andrin, Hessam Bagherinezhad, Hongyu Ren, Hunter Lightman, Hyung Won Chung, Ian Kivlichan, Ian O’Connell, Ian Osband, Ignasi Clavera Gilaberte, Ilge Akkaya, Ilya Kostrikov, Ilya Sutskever, Irina Kofman, Jakub Pachocki, James Lennon, Jason Wei, Jean Harb, Jerry Twore, Jiacheng Feng, Jiahui Yu, Jiayi Weng, Jie Tang, Jieqi Yu, Joaquin Quiñero Candela, Joe Palermo, Joel Parish, Johannes Heidecke, John Hallman, John Rizzo, Jonathan Gordon, Jonathan Uesato, Jonathan Ward, Joost Huizinga, Julie Wang, Kai Chen, Kai Xiao, Karan Singhal, Karina Nguyen, Karl Cobbe, Katy Shi, Kayla Wood, Kendra Rimbach, Keren Gu-Lemberg, Kevin Liu, Kevin Lu, Kevin Stone, Kevin Yu, Lama Ahmad, Lauren Yang, Leo Liu, Leon Maksin, Leyton Ho, Liam Fedus, Lilian Weng, Linden Li, Lindsay McCallum, Lindsey Held, Lorenz Kuhn, Lukas Kondraciuk, Lukasz

- 
- Kaiser, Luke Metz, Madelaine Boyd, Maja Trebacz, Manas Joglekar, Mark Chen, Marko Tintor, Mason Meyer, Matt Jones, Matt Kaufer, Max Schwarzer, Meghan Shah, Mehmet Yatbaz, Melody Y. Guan, Mengyuan Xu, Mengyuan Yan, Mia Glaese, Mianna Chen, Michael Lampe, Michael Malek, Michele Wang, Michelle Fradin, Mike McClay, Mikhail Pavlov, Miles Wang, Mingxuan Wang, Mira Murati, Mo Bavarian, Mostafa Rohaninejad, Nat McAleese, Neil Chowdhury, Neil Chowdhury, Nick Ryder, Nikolas Tezak, Noam Brown, Ofir Nachum, Oleg Boiko, Oleg Murk, Olivia Watkins, Patrick Chao, Paul Ashbourne, Pavel Izmailov, Peter Zhokhov, Rachel Dias, Rahul Arora, Randall Lin, Rapha Gontijo Lopes, Raz Gaon, Reah Miyara, Reimar Leike, Renny Hwang, Rhythm Garg, Robin Brown, Roshan James, Rui Shu, Ryan Cheu, Ryan Greene, Saachi Jain, Sam Altman, Sam Toizer, Sam Toyer, Samuel Miserendino, Sandhini Agarwal, Santiago Hernandez, Sasha Baker, Scott McKinney, Scottie Yan, Shengjia Zhao, Shengli Hu, Shibani Santurkar, Shraman Ray Chaudhuri, Shuyuan Zhang, Siyuan Fu, Spencer Papay, Steph Lin, Suchir Balaji, Suvansh Sanjeev, Szymon Sidor, Tal Broda, Aidan Clark, Tao Wang, Taylor Gordon, Ted Sanders, Tejal Patwardhan, Thibault Sottiaux, Thomas Degry, Thomas Dimson, Tianhao Zheng, Timur Garipov, Tom Stasi, Trapit Bansal, Trevor Creech, Troy Peterson, Tyna Eloundou, Valerie Qi, Vineet Kosaraju, Vinnie Monaco, Vitchyr Pong, Vlad Fomenko, Weiye Zheng, Wenda Zhou, Wes McCabe, Wojciech Zaremba, Yann Dubois, Yinghai Lu, Yining Chen, Young Cha, Yu Bai, Yuchen He, Yuchen Zhang, Yunyun Wang, Zheng Shao, and Zhuohan Li. OpenAI o1 System Card, December 2024. URL <http://arxiv.org/abs/2412.16720>. arXiv:2412.16720 [cs].
- Shrey Pandit, Austin Xu, Xuan-Phi Nguyen, Yifei Ming, Caiming Xiong, and Shafiq Joty. Hard2Verify: A Step-Level Verification Benchmark for Open-Ended Frontier Math, October 2025. URL <http://arxiv.org/abs/2510.13744>. arXiv:2510.13744 [cs].
- Sumanth Varambally, Thomas Voice, Yanchao Sun, Zhifeng Chen, Rose Yu, and Ke Ye. Hilbert: Recursively Building Formal Proofs with Informal Reasoning, September 2025. URL <http://arxiv.org/abs/2509.22819>. arXiv:2509.22819 [cs].
- Yifei Xu, Tusher Chakraborty, Srinagesh Sharma, Leonardo Nunes, Emre Kiciman, Songwu Lu, and Ranveer Chandra. Direct Reasoning Optimization: LLMs Can Reward And Refine Their Own Reasoning for Open-Ended Tasks, June 2025. URL <http://arxiv.org/abs/2506.13351>. arXiv:2506.13351 [cs].
- An Yang, Anfeng Li, Baosong Yang, Beichen Zhang, Binyuan Hui, Bo Zheng, Bowen Yu, Chang Gao, Chengen Huang, Chenxu Lv, Chujie Zheng, Dayiheng Liu, Fan Zhou, Fei Huang, Feng Hu, Hao Ge, Haoran Wei, Huan Lin, Jialong Tang, Jian Yang, Jianhong Tu, Jianwei Zhang, Jianxin Yang, Jiaxi Yang, Jing Zhou, Jingren Zhou, Junyang Lin, Kai Dang, Keqin Bao, Kexin Yang, Le Yu, Lianghao Deng, Mei Li, Mingfeng Xue, Mingze Li, Pei Zhang, Peng Wang, Qin Zhu, Rui Men, Ruize Gao, Shixuan Liu, Shuang Luo, Tianhao Li, Tianyi Tang, Wenbiao Yin, Xingzhang Ren, Xinyu Wang, Xinyu Zhang, Xuancheng Ren, Yang Fan, Yang Su, Yichang Zhang, Yinger Zhang, Yu Wan, Yuqiong Liu, Zekun Wang, Zeyu Cui, Zhenru Zhang, Zhipeng Zhou, and Zihan Qiu. Qwen3 Technical Report, May 2025. URL <http://arxiv.org/abs/2505.09388>. arXiv:2505.09388 [cs].
- Tianyu Yu, Bo Ji, Shouli Wang, Shu Yao, Zefan Wang, Ganqu Cui, Lifan Yuan, Ning Ding, Yuan Yao, Zhiyuan Liu, Maosong Sun, and Tat-Seng Chua. RLPR: Extrapolating RLVR to General Domains without Verifiers, June 2025. URL <http://arxiv.org/abs/2506.18254>. arXiv:2506.18254 [cs].
- Lianmin Zheng, Wei-Lin Chiang, Ying Sheng, Siyuan Zhuang, Zhanghao Wu, Yonghao Zhuang, Zi Lin, Zhuohan Li, Dacheng Li, Eric P. Xing, Hao Zhang, Joseph E. Gonzalez, and Ion Stoica. Judging LLM-as-a-Judge with MT-Bench and Chatbot Arena, December 2023. URL <http://arxiv.org/abs/2306.05685>. arXiv:2306.05685 [cs].
- Yuxin Zuo, Kaiyan Zhang, Shang Qu, Li Sheng, Xuekai Zhu, Biqing Qi, Youbang Sun, Ganqu Cui, Ning Ding, and Bowen Zhou. TTRL: Test-Time Reinforcement Learning, April 2025. URL <http://arxiv.org/abs/2504.16084>. arXiv:2504.16084 [cs].

---

## A Appendix

### A.1 Details in QiuZhen-Bench

QiuZhen-Bench is a dataset covering a wide range of topics in advanced math, including analysis, algebra, machine learning theory, geometry, topology, probability, statistics and theoretical physics. We have collected 143 exam papers from S.-T. Yau College Student Mathematics Contest since 2010, and the doctoral qualifying exams from QiuZhen college, Tsinghua University since 2023, and required GPT-5 to extract and reformat problems directly from the PDF files. Manual spot checks did not reveal any errors in these extracted contents, indicating the strong capability of GPT-5 in doing this task. We firstly divide the exam papers into pages, and for each page we used the following prompt for extraction:

#### system prompt

You are a meticulous exam parsing assistant.  
Extract ALL distinct problems from the provided text chunks.  
Return ONLY valid JSON (no commentary). Focus on problem statements; exclude answers and solutions.

#### user prompt

Task: Parse the following exam/contest page images to extract problem statements.  
Requirements:

- Identify each separate problem with its number or index if present.
- Preserve math using Markdown and LaTeX (inline  $\dots$ , display  $\dots$ ).
- Do not infer content; only use what appears in the images.
- Only include problems written in English; skip non-English (e.g., Chinese).
- If a problem is mixed-language, include only the English statement; otherwise skip.
- If a problem spans multiple pages in this chunk, include the full statement.

Output strictly as JSON array of objects with keys:

- problem\_number: string (e.g., '1', 'II-3', 'A.1'; use 'unknown' if missing)
- markdown\_statement: string (problem text in Markdown; no solution)
- section: string|null (e.g., 'Analysis', 'Geometry', or null)
- tags: array of strings (optional keywords, may be empty)
- source\_pages: array of integers (page numbers within this PDF chunk)

Return ONLY JSON.

fSource: {source.rel\_path}  
Pages: {chunk.start\_page}-{chunk.end\_page}

The problems that spans across two pages is extracted by part and combined in the following processing logic. This results in a dataset containing 1,054 problems in total. Here we provide some samples from this dataset:

#### Yau contest 2015, applied math

Let  $G$  be graph of a social network, where for each pair of members there is either no connection, or a positive or a negative one. An unbalanced cycle in  $G$  is a cycle which have odd number of negative edges. Traversing along such a cycle with social rules such as friend of enemy are enemy would result in having a negative relation of one with himself! A resigning in  $G$  at a vertex  $v$  of  $G$  is to switch the type (positive or negative) of all edges incident to  $v$ . Question: Show that one can switch all edge of  $G$  into positive edges using a sequence resigning if and only if there is no unbalanced cycle in  $G$ .



### QiuZhen qualifying 2024 spring, theoretical physics

$\Pi_{\mu\nu}(q)$  via dimensional regularization (35 points)

At one-loop, the photon propagator in QED receives the correction

$$i\Pi_{\mu\nu}(q) = - \int \frac{d^4 p}{(2\pi)^4} \text{Tr} \left( i\gamma^\nu \frac{i}{\not{p} + \not{q} - m} i\gamma^\mu \frac{i}{\not{p} - m} \right),$$

where  $\gamma_\mu$  is the gamma matrix and  $\not{p} = p^\mu \gamma_\mu$ .

i) 10' Show that this expression can be written as

$$i\Pi_{\mu\nu}(q) = -i \int \frac{d^4 p}{(2\pi)^4} \frac{N_{\mu\nu}}{D},$$

where

$$N_{\mu\nu} = \text{tr} [\gamma_\nu (\not{p} + \not{q} + m) \gamma_\mu (\not{p} - m)], \quad \frac{1}{D} = \int_0^1 d\alpha \frac{1}{D},$$

with

$$D = [l^2 + \alpha(1-\alpha)q^2 - m^2 + i\epsilon]^2, \quad \text{where } l = p + \alpha q.$$

ii) 10' Evaluate  $N_{\mu\nu}$  and shift momentum integration from  $p$  to  $l$  as above.

iii) 10' At the integration step over  $l$ , perform dimensional regularization by promoting to a  $d$ -dimensional integral. Derive the expression

$$\Pi_{\mu\nu}(q) = (q_\mu q_\nu - \eta_{\mu\nu} q^2) \Pi(q^2),$$

when taking the limit  $d \rightarrow 4$ .

iv) 5' Why does this result ensure gauge invariance?

### Yau contest 2019, analysis individual

Let  $\Omega \subset \mathbb{R}^2$  be a bounded domain with smooth boundary. Prove that, for all  $p > 1$  and  $1 \leq q < \infty$ , for all  $f \in L^p(\Omega)$ , there exists a unique  $u \in H_0^1(\Omega)$ , such that

$$\Delta u = |u|^{q-1}u + f \quad \text{in } \Omega.$$

### QiuZhen qualifying 2025 fall, algebra

Suppose  $r \leq n$  are positive integers. Let  $X$  be the subset of  $M(n \times n, \mathbb{C})$  consisting of complex  $n \times n$  matrices with rank at most  $r$ . Prove that  $X$  is Zariski closed. Find the number of irreducible components of  $X$ , and calculate the Krull dimension of each irreducible component of  $X$ .

## A.2 Prompt template

In our experiments, we use the same evaluation prompt template across all these datasets. For standard, majority voting, and simple pessimistic verification, we use this single-pass verification prompt:

### system prompt: single pass

You are an assistant highly proficient in mathematics.

The user will provide a math problem together with its proposed solution, and your task is to verify the correctness of that solution according to the given instruction.

### user prompt: single pass

Here is a math problem and a candidate solution of it, and you need to verify the correctness of this solution. Please check each of the following:

1. The provided content is indeed a math problem and its corresponding solution, rather than unrelated material supplied by mistake.
2. The solution actually derives the conclusion required by the original problem.
3. Every step of calculation and formula derivation in the solution is correct.
4. The hypotheses (conditions) and conclusions of any theorems used are correctly matched and applied.
5. The solution relies only on the conditions given in the problem and does not introduce any additional assumptions to obtain the conclusion.

Consistency and error-severity policy (important):

- If only minor, easily fixable issues exist (e.g., small algebraic slips later corrected, notational typos, superficial formatting), treat the solution as correct overall but briefly note such issues.
- If there is any critical error that undermines correctness (e.g., invalid step, wrong theorem usage without required conditions, uncorrected calculation error leading to a wrong result), treat the solution as incorrect.

Response requirements: If the solution is correct overall (possibly with minor issues), reply with ``<verification>true</verification>`` and briefly list minor issues if any.

If the solution is incorrect, reply with ``<verification>false</verification>`` followed by a concise description of the most harmful error.

Do not include any restatement of the entire solution or problem.

`<problem>{problem}</problem>`

`<answer>{solution}</answer>`

The first step of progressive pessimistic verifier also adopts the single pass prompt, and for the following verification of subdivisions, we will use the following chunk verification prompt. This prompt is also used in vertical pessimistic verification in our experiments.

### system prompt: chunk verification

You are an assistant highly proficient in mathematics.

The user will provide a math problem together with its proposed solution, and your task is to verify the correctness of that solution according to the given instruction.

### user prompt: chunk verification

We provide the original problem and the complete proposed solution for full context

Then we provide a specific chunk from the solution for focused checking.

Your task: Check ONLY the given chunk for errors while considering the overall context.

Checklist:

1. The chunk's reasoning and calculations adhere to mathematical correctness.
2. Any theorems used in the chunk match their hypotheses and conclusions.
3. The chunk does not rely on assumptions not justified by the problem or earlier proven steps.

Consistency and error-severity policy (important):



- If only minor, easily fixable issues exist (e.g., small algebraic slips later corrected, notational typos, superficial formatting), treat the chunk as correct overall but briefly note such issues.
- If there is any critical error that undermines correctness in this chunk (e.g., invalid step, wrong theorem usage without required conditions), treat the chunk as incorrect.

Response requirements: If the chunk is correct overall (possibly with minor issues), reply with ``<verification>true</verification>`` and briefly list minor issues if any.

If the chunk is incorrect, reply with ``<verification>false</verification>`` followed by a concise description of the most harmful error in the proof that you found in the chunk.

`<problem>{problem}</problem>`

`<full_answer>{full_proof}</full_answer>`

`<chunk_index>{idx}</chunk_index>`

`<chunk>{chunk}</chunk>`

### A.3 More detailed case studies

In this section we primarily focus on the false negative cases reported by different verifiers, these are the most informative cases in our experiments.

#### A.3.1 False negative examples on IMO-GradingBench

##### Case 1: problem

Let  $\angle XYZ$  be an acute angle with  $\angle XYZ \neq 60^\circ$ , and let  $A$  be a point inside  $\angle XYZ$ . Prove that there exists  $D \neq A$  inside  $\angle XYZ$  and  $\theta \in (0, 2\pi)$  satisfying the following condition:

For points  $B$  and  $C$  on the rays  $\overrightarrow{YX}$  and  $\overrightarrow{YZ}$  respectively, then

$$\angle BAC = \angle XYZ \implies \angle BDC = \theta.$$

##### Case 1: proof

**\*\*1. Summary\*\***

**\*\*a. Verdict:\*\*** I have successfully solved the problem. I have rigorously proven that the statement is true.

**\*\*b. Method Sketch:\*\*** The proof utilizes complex numbers to model the geometric configuration.

1. **\*\*Setup\*\***: We establish a coordinate system with  $Y$  at the origin. Let  $\alpha = \angle XYZ$ . The rays  $\overrightarrow{YX}$  and  $\overrightarrow{YZ}$  correspond to the positive real axis and the ray at angle  $\alpha$ , respectively.  $A = a$  is inside the angle.  $B = b$  and  $C = ce^{i\alpha}$  ( $b > 0, c > 0$ ). 2. **\*\*Analysis of Configurations ( $S_A$ )\*\***: We analyze the condition  $\angle BAC = \alpha$ . This geometric condition implies the directed angle from  $\vec{AB}$  to  $\vec{AC}$  is either  $\alpha$  (CCW) or  $-\alpha$  (CW). \* We prove that the set of CCW configurations is empty when  $A$  is strictly inside the angle. \* We prove that the set of CW configurations  $S_A$  is non-empty and infinite, corresponding to an interval of values for  $b$ . We rigorously derive the relationship between  $b$  and  $c$  (via an intermediate variable  $k$ ) as a rational function. 3. **\*\*Constant Angle Condition\*\***: We assume  $D = d$  is a point inside the angle such that  $\angle BDC = \theta$  is constant for all configurations in  $S_A$ . This implies the complex ratio  $X(b) = \frac{C(b)-d}{B(b)-d}$

has a constant argument as  $b$  varies. 4. **\*\*Analysis of the Rational Function\*\***:  $X(b)$  is shown to be a rational function  $P(b)/Q(b)$ , where  $\deg(P) = 1$  and  $\deg(Q) = 2$ . Since  $\arg(X(b))$  is constant,  $X(b)$  must be proportional to a real rational function. 5. **\*\*Poles Analysis\*\***: The poles of  $Q(b)$  are  $d$  (non-real) and  $b_0$  (real). For the corresponding real rational function to exist, its poles must be symmetric under conjugation. This forces a cancellation between  $P(b)$  and  $Q(b)$ . 6. **\*\*Case Analysis of Common Roots\*\***: We analyze the possibilities for the common root. \* If the common root is  $b_0$ , it implies  $A$  is on the ray  $YX$ , contradicting the premise. \* If the common root is  $d$ , it leads to two solutions for  $D$ :  $D = A$  and a unique other point  $D = D^*$ . 7. **\*\*Analysis of the Solution  $D^*$ \*\***: We explicitly calculate  $D^*$  and rigorously prove that  $D^*$  is always strictly inside the angle  $\angle XYZ$ . 8. **\*\*Exclusion of  $D^* = A^*$ \*\***: We determine the conditions under which  $D^* = A$ . This occurs if and only if  $\alpha = 60^\circ$  and  $A$  lies on the angle bisector. 9. **\*\*Conclusion\*\***: Since the problem stipulates that  $\alpha \neq 60^\circ$ , we must have  $D^* \neq A$ . Thus,  $D = D^*$  is the required point inside the angle, different from  $A$ . The constant angle is calculated to be  $\theta = \pi - 2\alpha$ .

**\*\*2. Detailed Solution\*\***

Step 1: Setup and Coordinate System. Let  $\alpha = \angle XYZ$ . We are given  $0 < \alpha < \pi/2$  and  $\alpha \neq \pi/3$ . We set up a coordinate system in the complex plane with  $Y$  at the origin  $0$ . The ray  $\vec{YX}$  is the positive real axis, and  $\vec{YZ}$  is the ray at angle  $\alpha$ . Let  $A$  be represented by the complex number  $a$ . Since  $A$  is strictly inside the angle,  $a = re^{i\phi_1}$  with  $r > 0$  and  $0 < \phi_1 < \alpha$ . Let  $B$  and  $C$  be represented by  $b$  and  $ce^{i\alpha}$  respectively, with  $b > 0, c > 0$ . Step 2: Analysis of the configurations  $\mathcal{S}_A$ . The condition is  $\angle BAC = \alpha$ . This geometric angle condition means the directed angle from  $\vec{AB}$  (represented by  $B - A$ ) to  $\vec{AC}$  (represented by  $C - A$ ) is either  $\alpha$  (CCW orientation) or  $-\alpha$  (CW orientation).

Case 2.1: CCW orientation. The directed angle is  $\alpha$ .

$$\frac{ce^{i\alpha} - a}{b - a} = ke^{i\alpha}, \quad \text{for some } k > 0.$$

$ce^{i\alpha} - a = k(b - a)e^{i\alpha}$ . Dividing by  $e^{i\alpha}$ :  $c - ae^{-i\alpha} = k(b - a)$ . Since  $c, k, b$  are real, we equate the imaginary parts:  $\text{Im}(c - ae^{-i\alpha}) = k\text{Im}(b - a)$ .  $\text{Im}(-ae^{-i\alpha}) = k\text{Im}(-a)$ .  $\text{Im}(-re^{i(\phi_1 - \alpha)}) = k\text{Im}(-re^{i\phi_1})$ .  $-r\sin(\phi_1 - \alpha) = k(-r\sin\phi_1)$ .  $\sin(\phi_1 - \alpha) = k\sin\phi_1$ . Since  $0 < \phi_1 < \alpha < \pi/2$ , we have  $\sin\phi_1 > 0$ . Also,  $-\pi/2 < -\alpha < \phi_1 - \alpha < 0$ . Thus  $\sin(\phi_1 - \alpha) < 0$ . This implies  $k = \frac{\sin(\phi_1 - \alpha)}{\sin\phi_1} < 0$ . This contradicts the requirement  $k > 0$ . Therefore, the set of CCW configurations is empty.

Case 2.2: CW orientation. The directed angle is  $-\alpha$ .

$$\frac{ce^{i\alpha} - a}{b - a} = ke^{-i\alpha}, \quad \text{for some } k > 0.$$

$ce^{i\alpha} - a = k(b - a)e^{-i\alpha}$ . Dividing by  $e^{i\alpha}$  is not helpful. Let's rearrange to isolate  $c$ .  $ce^{i\alpha} = a + k(b - a)e^{-i\alpha}$ .  $c = ae^{-i\alpha} + k(b - a)e^{-i2\alpha}$ . Since  $c$  is real, we equate the imaginary part to zero.  $\text{Im}(ae^{-i\alpha}) + k\text{Im}((b - a)e^{-i2\alpha}) = 0$ .  $\text{Im}(re^{i(\phi_1 - \alpha)}) + k\text{Im}(be^{-i2\alpha} - re^{i(\phi_1 - 2\alpha)}) = 0$ .  $r\sin(\phi_1 - \alpha) + k(b\sin(-2\alpha) - r\sin(\phi_1 - 2\alpha)) = 0$ .  $r\sin(\phi_1 - \alpha) = k(-b\sin(-2\alpha) + r\sin(\phi_1 - 2\alpha))$ .  $r\sin(\phi_1 - \alpha) = k(b\sin 2\alpha - r\sin(\phi_1 - 2\alpha))$ .

We rewrite this using  $\sin(x) = -\sin(-x)$ .  $-r\sin(\alpha - \phi_1) = k(b\sin 2\alpha + r\sin(2\alpha - \phi_1))$ .

Let  $C_1 = r\sin(\alpha - \phi_1)$ . Since  $0 < \alpha - \phi_1 < \pi/2$ ,  $C_1 > 0$ . Let  $C_2 = \sin 2\alpha$ . Since  $0 < 2\alpha < \pi$ ,  $C_2 > 0$ . Let  $C_3 = r\sin(2\alpha - \phi_1)$ . Since  $0 < \phi_1 < \alpha$  and  $\alpha < \pi/2$ , we have  $\alpha < 2\alpha - \phi_1 < 2\alpha < \pi$ . Thus  $C_3 > 0$ .

The equation is  $-C_1 = k(C_2b + C_3)$ . Since  $C_1 > 0$  and  $C_2, C_3, b > 0$ , the RHS is  $k(C_2b + C_3)$ . If  $k > 0$ , the RHS is positive, while the LHS is negative. This is a contradiction.

Let's recheck the derivation in the thought block, as I seem to have made a mistake again in the live derivation attempt.

Let's restart the derivation for Case 2.2 carefully.  $ce^{i\alpha} - a = k(b - a)e^{-i\alpha}$ . We want to eliminate  $c$ . We can multiply by  $e^{-i\alpha}$ .  $c - ae^{-i\alpha} = k(b - a)e^{-i2\alpha}$ . Since  $c$  is real, the

imaginary part of the LHS is  $Im(-ae^{-i\alpha})$ . The imaginary part of the RHS is  $kIm((b-a)e^{-i2\alpha})$ .  $Im(-re^{i(\phi_1-\alpha)}) = kIm((b-re^{i\phi_1})e^{-i2\alpha})$ .  $-r\sin(\phi_1-\alpha) = kIm(be^{-i2\alpha} - re^{i(\phi_1-2\alpha)})$ .  $-r\sin(\phi_1-\alpha) = k(b\sin(-2\alpha) - r\sin(\phi_1-2\alpha))$ .

Now we simplify. LHS:  $r\sin(\alpha-\phi_1)$ . RHS:  $k(-b\sin 2\alpha + r\sin(2\alpha-\phi_1))$ .

The equation is  $r\sin(\alpha-\phi_1) = k(r\sin(2\alpha-\phi_1) - b\sin 2\alpha)$ .

Using the definitions of  $C_1, C_2, C_3$ :  $C_1 = k(C_3 - C_2b)$ .

We require  $k > 0$ . Since  $C_1 > 0$ , we need  $C_3 - C_2b > 0$ .  $b < C_3/C_2$ . Let  $b_0 = C_3/C_2$ .

$b_0 > 0$ .  $k(b) = \frac{C_1}{C_3 - C_2b}$ .

We also require  $c(b) > 0$ .  $c(b) = Re(ae^{-i\alpha}) + k(b)Re((b-a)e^{-i2\alpha})$ .  $c(b) = \frac{Re(ae^{-i\alpha})(C_3 - C_2b) + C_1Re((b-a)e^{-i2\alpha})}{C_3 - C_2b}$ . The denominator is positive for  $b \in (0, b_0)$ . We

analyze the numerator  $N(b) = L_1b + L_0$ .  $L_1 = -C_2Re(ae^{-i\alpha}) + C_1Re(e^{-i2\alpha})$ .  $L_1 = -\sin 2\alpha \cdot r \cos(\phi_1 - \alpha) + r \sin(\alpha - \phi_1) \cos 2\alpha$ .  $L_1 = -r[\sin 2\alpha \cos(\alpha - \phi_1) - \cos 2\alpha \sin(\alpha - \phi_1)]$ .  $L_1 = -r \sin(2\alpha - (\alpha - \phi_1)) = -r \sin(\alpha + \phi_1)$ . Since  $0 < \alpha + \phi_1 < \pi$ ,  $L_1 < 0$ .

$L_0 = C_3Re(ae^{-i\alpha}) + C_1Re(-ae^{-i2\alpha})$ .  $L_0 = r \sin(2\alpha - \phi_1)r \cos(\alpha - \phi_1) - r \sin(\alpha - \phi_1)r \cos(2\alpha - \phi_1)$ .  $L_0 = r^2 \sin((2\alpha - \phi_1) - (\alpha - \phi_1)) = r^2 \sin \alpha$ . Since  $\alpha > 0$ ,  $L_0 > 0$ .

We need  $N(b) > 0$ .  $L_1b + L_0 > 0$ . Since  $L_1 < 0$ ,  $b < -L_0/L_1$ . Let  $b_1 = -L_0/L_1$ .  $b_1 > 0$ . The set of valid configurations  $\mathcal{S}_A$  corresponds to the interval  $I_A = (0, \min(b_0, b_1))$ .

This interval is non-empty, so  $\mathcal{S}_A$  is infinite.

Step 3: Analysis of the constant angle condition. Let  $D = d$  be a point inside the angle. This means  $d$  is not real and  $0 < \arg(d) < \alpha$ . We require  $D \neq A$ . We require  $\angle BDC = \theta$  to be constant for all  $(B, C) \in \mathcal{S}_A$ . Let  $C(b)$  denote the complex coordinate of  $C$  corresponding to  $B = b$ .  $C(b) = c(b)e^{i\alpha}$ . The condition means the argument of the complex ratio  $X(b) = \frac{C(b)-d}{b-d}$  is constant for  $b \in I_A$ .

From the definition of the CW configuration in Step 2.2, we have  $C(b) = a + k(b)(b-a)e^{-i\alpha}$ .  $X(b) = \frac{a-d+k(b)(b-a)e^{-i\alpha}}{b-d}$ . Substituting  $k(b) = \frac{C_1}{C_3 - C_2b}$ :  $X(b) = \frac{(a-d)(C_3 - C_2b) + C_1(b-a)e^{-i\alpha}}{(b-d)(C_3 - C_2b)} = \frac{P(b)}{Q(b)}$ .

$P(b)$  is a polynomial of degree at most 1 in  $b$ .  $Q(b)$  is a polynomial of degree 2 in  $b$ .  $Q(b) = -C_2(b-d)(b-b_0)$ , where  $b_0 = C_3/C_2$ . The roots are  $d$  (non-real) and  $b_0$  (real and positive).

Since  $\arg(X(b))$  is constant on the interval  $I_A$ ,  $X(b) = KR(b)$  where  $K$  is a complex constant and  $R(b)$  is a real rational function. The poles of a real rational function must be real or come in conjugate pairs.

If  $P(b)$  and  $Q(b)$  are coprime, the poles of  $X(b)$  (and thus  $R(b)$ ) are  $d$  and  $b_0$ . Since  $d$  is not real,  $\bar{d}$  must also be a pole. So  $\bar{d} \in \{d, b_0\}$ .  $\bar{d} = d$  is impossible as  $d$  is not real.  $\bar{d} = b_0$ . Since  $b_0$  is real,  $d = \bar{b}_0 = b_0$ . This implies  $d$  is real, which contradicts that  $D$  is inside the angle. Thus,  $P(b)$  and  $Q(b)$  must share a common root.

Step 4: Case analysis of the common root. Case 4.1: The common root is  $b_0$ .  $P(b_0) = 0$ .  $P(b_0) = (a-d)(C_3 - C_2b_0) + C_1(b_0 - a)e^{-i\alpha}$ . Since  $C_3 - C_2b_0 = 0$ ,  $P(b_0) = C_1(b_0 - a)e^{-i\alpha}$ .  $P(b_0) = 0$  implies  $C_1(b_0 - a)e^{-i\alpha} = 0$ . Since  $C_1 > 0$  and  $e^{-i\alpha} \neq 0$ , we must have  $b_0 - a = 0$ .  $a = b_0$ . This means  $A$  is real, so  $A$  is on the ray  $YX$ . This contradicts that  $A$  is strictly inside the angle ( $0 < \phi_1$ ).

Case 4.2: The common root is  $d$ .  $P(d) = 0$ .  $P(d) = (a-d)(C_3 - C_2d) + C_1(d-a)e^{-i\alpha} = 0$ .  $(a-d)[(C_3 - C_2d) - C_1e^{-i\alpha}] = 0$ . This yields two possibilities for  $D$ :  $D = A$  (i.e.,  $d = a$ ) or  $C_3 - C_2d - C_1e^{-i\alpha} = 0$ .

We are looking for  $D \neq A$ . Let  $D^*$  be the second solution, with coordinate  $d^*$ .  $C_2d^* = C_3 - C_1e^{-i\alpha}$ .  $d^* = \frac{C_3 - C_1e^{-i\alpha}}{C_2}$ .

If  $D = D^*$ , then  $(b-d^*)$  is a common factor of  $P(b)$  and  $Q(b)$ .  $P(b) = p_1(b-d^*)$ .  $Q(b) = -C_2(b-d^*)(b-b_0)$ .  $X(b) = \frac{p_1}{-C_2(b-b_0)}$ . For  $b \in I_A$ ,  $b < b_0$ . Since  $C_2 > 0$ , the denominator  $-C_2(b-b_0)$  is positive and real. Thus,  $\arg(X(b)) = \arg(p_1)$ , which is constant.

Step 5: Analysis of the location of  $D^*$ . We must verify that  $D^*$  is inside the angle, i.e.,  $0 < \arg(d^*) < \alpha$ .

$Im(d^*) = \frac{1}{C_2} Im(C_3 - C_1 e^{-i\alpha}) = \frac{1}{C_2} (-C_1 \sin(-\alpha)) = \frac{C_1 \sin \alpha}{C_2}$ . Since  $C_1 > 0, C_2 > 0, \sin \alpha > 0$ , we have  $Im(d^*) > 0$ . So  $\arg(d^*) > 0$ .

We check  $\arg(d^*) < \alpha$ . This is equivalent to  $Im(d^* e^{-i\alpha}) < 0$ .  $d^* e^{-i\alpha} = \frac{C_3 e^{-i\alpha} - C_1 e^{-i2\alpha}}{C_2}$ .

$Im(d^* e^{-i\alpha}) = \frac{C_3 \sin(-\alpha) - C_1 \sin(-2\alpha)}{C_2} = \frac{-C_3 \sin \alpha + C_1 \sin 2\alpha}{C_2}$ . Let  $H = C_1 \sin 2\alpha - C_3 \sin \alpha$ .

$H = r \sin(\alpha - \phi_1) \sin 2\alpha - r \sin(2\alpha - \phi_1) \sin \alpha$ .  $H = r \sin \alpha [2 \sin(\alpha - \phi_1) \cos \alpha - \sin(2\alpha - \phi_1)]$ . Using the identity  $2 \sin A \cos B = \sin(A + B) + \sin(A - B)$ :  $2 \sin(\alpha - \phi_1) \cos \alpha = \sin(2\alpha - \phi_1) + \sin(-\phi_1) = \sin(2\alpha - \phi_1) - \sin \phi_1$ .  $H = r \sin \alpha [\sin(2\alpha - \phi_1) - \sin \phi_1 - \sin(2\alpha - \phi_1)]$ .  $H = -r \sin \alpha \sin \phi_1$ . Since  $r > 0, \sin \alpha > 0, \sin \phi_1 > 0$  (as  $0 < \phi_1 < \alpha < \pi/2$ ), we have  $H < 0$ . Thus  $Im(d^* e^{-i\alpha}) < 0$ . So  $\arg(d^*) < \alpha$ .  $D^*$  is strictly inside the angle  $\angle XYZ$ .

Step 6: Analysis of the condition  $D^* = A$ . We check when  $d^* = a$ . This happens if  $a$  satisfies the equation for  $d^*$ .  $C_2 a = C_3 - C_1 e^{-i\alpha}$ .  $r e^{i\phi_1} \sin 2\alpha = r \sin(2\alpha - \phi_1) - r \sin(\alpha - \phi_1) e^{-i\alpha}$ . Dividing by  $r$  and examining the imaginary part:  $\sin \phi_1 \sin 2\alpha = Im(-\sin(\alpha - \phi_1) e^{-i\alpha}) = -\sin(\alpha - \phi_1) \sin(-\alpha) = \sin(\alpha - \phi_1) \sin \alpha$ .  $2 \sin \phi_1 \sin \alpha \cos \alpha = (\sin \alpha \cos \phi_1 - \cos \alpha \sin \phi_1) \sin \alpha$ . Since  $\sin \alpha \neq 0$ :  $2 \sin \phi_1 \cos \alpha = \sin \alpha \cos \phi_1 - \cos \alpha \sin \phi_1$ .  $3 \sin \phi_1 \cos \alpha = \sin \alpha \cos \phi_1$ .  $3 \tan \phi_1 = \tan \alpha$ . (Since  $\cos \alpha \neq 0, \cos \phi_1 \neq 0$ ).

Examining the real part:  $\cos \phi_1 \sin 2\alpha = \sin(2\alpha - \phi_1) - \sin(\alpha - \phi_1) \cos \alpha$ .  $\cos \phi_1 \sin 2\alpha = (\sin 2\alpha \cos \phi_1 - \cos 2\alpha \sin \phi_1) - (\sin \alpha \cos \phi_1 - \cos \alpha \sin \phi_1) \cos \alpha$ .  $0 = -\cos 2\alpha \sin \phi_1 - \sin \alpha \cos \alpha \cos \phi_1 + \cos^2 \alpha \sin \phi_1$ .  $0 = \sin \phi_1 (\cos^2 \alpha - \cos 2\alpha) - \sin \alpha \cos \alpha \cos \phi_1$ .  $0 = \sin \phi_1 (\cos^2 \alpha - (\cos^2 \alpha - \sin^2 \alpha)) - \frac{1}{2} \sin 2\alpha \cos \phi_1$ .  $0 = \sin \phi_1 \sin^2 \alpha - \sin \alpha \cos \alpha \cos \phi_1$ .  $0 = \sin \alpha (\sin \phi_1 \sin \alpha - \cos \alpha \cos \phi_1) = -\sin \alpha \cos(\alpha + \phi_1)$ . Since  $\sin \alpha \neq 0, \cos(\alpha + \phi_1) = 0$ . Since  $0 < \alpha + \phi_1 < \pi$ , we must have  $\alpha + \phi_1 = \pi/2$ .

If  $\alpha + \phi_1 = \pi/2$ , then  $\tan \alpha = \cot \phi_1 = 1/\tan \phi_1$ . Substituting into  $3 \tan \phi_1 = \tan \alpha$ :  $3 \tan \phi_1 = 1/\tan \phi_1$ .  $\tan^2 \phi_1 = 1/3$ . Since  $\phi_1 > 0, \tan \phi_1 = 1/\sqrt{3}$ .  $\phi_1 = \pi/6$  (30 degrees). Then  $\alpha = \pi/2 - \pi/6 = \pi/3$  (60 degrees). So  $D^* = A$  if and only if  $\alpha = 60^\circ$  and  $A$  is on the angle bisector ( $\phi_1 = \alpha/2$ ).

Step 7: Conclusion. The problem states that  $\angle XYZ \neq 60^\circ$  (i.e.,  $\alpha \neq \pi/3$ ). Therefore, the condition for  $D^* = A$  is not met, so  $D^* \neq A$ . We have shown that  $D^*$  is strictly inside the angle  $\angle XYZ$ . We have shown that for  $D = D^*$ , the angle  $\angle BDC$  is constant for all configurations  $(B, C)$  such that  $\angle BAC = \alpha$ .

We determine the constant angle  $\theta$ . The directed angle  $\Theta$  is  $\arg(X(b)) = \arg(p_1)$ .  $p_1$  is the leading coefficient of  $P(b)$ .  $p_1 = -C_2(a - d^*) + C_1 e^{-i\alpha}$ . Using the definition of  $d^*$ ,  $C_2 d^* = C_3 - C_1 e^{-i\alpha}$ , so  $C_1 e^{-i\alpha} = C_3 - C_2 d^*$ .  $p_1 = -C_2 a + C_2 d^* + (C_3 - C_2 d^*) = C_3 - C_2 a$ .  $p_1 = r \sin(2\alpha - \phi_1) - \sin 2\alpha \cdot r e^{i\phi_1}$ .  $p_1 = r[\sin(2\alpha - \phi_1) - (\cos \phi_1 + i \sin \phi_1) \sin 2\alpha]$ .  $p_1 = r[(\sin 2\alpha \cos \phi_1 - \cos 2\alpha \sin \phi_1) - \sin 2\alpha \cos \phi_1 - i \sin 2\alpha \sin \phi_1]$ .  $p_1 = r[-\cos 2\alpha \sin \phi_1 - i \sin 2\alpha \sin \phi_1]$ .  $p_1 = -r \sin \phi_1 (\cos 2\alpha + i \sin 2\alpha) = -r \sin \phi_1 e^{i2\alpha}$ . Since  $r > 0, \sin \phi_1 > 0, \arg(p_1) = \pi + 2\alpha$ . The directed angle is  $\Theta = \pi + 2\alpha \pmod{2\pi}$ . Since  $0 < \alpha < \pi/2, \pi < \Theta < 2\pi$ . The geometric angle  $\theta$  is  $2\pi - \Theta = 2\pi - (\pi + 2\alpha) = \pi - 2\alpha$ . Since  $0 < \alpha < \pi/2, 0 < \theta < \pi$ . Thus  $\theta \in (0, 2\pi)$ .

We have proven the existence of  $D = D^* \neq A$  inside  $\angle XYZ$  and  $\theta = \pi - 2\alpha \in (0, 2\pi)$  satisfying the condition.

#### Case 1: review from gpt-5

<verification>false</verification> The bilinear relation from  $Im\left(\frac{(c-a)e^{-i\alpha}}{be^{i\alpha}-a}\right) = 0$  is computed with a sign error. Expanding  $Im(N \cdot \overline{D}) = 0$  gives  $cb \sin(2\alpha) + rb \sin(2\alpha -$

$\phi) - rc \sin(\alpha + \phi) + r^2 \sin \alpha = 0$ , so the coefficient of  $b$  is  $A_{12} = r \sin(2\alpha - \phi)$ , not  $A_{12} = -r \sin(2\alpha - \phi)$  as stated. This sign error undermines the subsequent locus equation.

The manually annotated label in the dataset is true (7 points). However, our verification method found an algebraic error in the proof's procedure, suggesting the score should be lower (e.g., 5-6 points).

#### Case 1: our comment

The derivation of the bilinear relation coefficients ( $A_{11}$  through  $A_{22}$ ) contained a systematic sign error. For instance, the coefficient  $A_{12}$  was calculated with the wrong sign. This error did not propagate to the final construction of point  $D$  because the necessary values were derived from ratios of the coefficients (e.g.,  $A_{22}/A_{11}$ ), where the overall sign difference cancelled out.

#### Case 2: problem

Find all functions  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  such that the equation

$$f(x - f(xy)) = f(x)f(1 - y)$$

holds for all  $x, y \in \mathbb{Z}$ .

#### Case 2: proof

We are looking for all functions  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  such that the equation

$$f(x - f(xy)) = f(x)f(1 - y) \quad (*)$$

holds for all  $x, y \in \mathbb{Z}$ . Let  $P(x, y)$  be the assertion  $(*)$ .

**\*\*1. Constant solutions\*\*** If  $f(x) = c$  for all  $x \in \mathbb{Z}$ . Then  $c = c \cdot c$ , so  $c^2 = c$ . This implies  $c = 0$  or  $c = 1$ . Solution 1:  $f(x) = 0$ . (LHS=0, RHS=0). Solution 2:  $f(x) = 1$ . (LHS=1, RHS=1).

**\*\*2. Non-constant solutions\*\***  $P(0, y) \implies f(0 - f(0)) = f(0)f(1 - y)$ .  $f(-f(0)) = f(0)f(1 - y)$ . The LHS is constant. If  $f(0) \neq 0$ , then  $f(1 - y)$  is constant. As  $y$  spans  $\mathbb{Z}$ ,  $f(x)$  is constant. This contradicts the assumption that  $f$  is non-constant. Thus, we must have  $f(0) = 0$ .

$P(x, 0) \implies f(x - f(0)) = f(x)f(1)$ .  $f(x) = f(x)f(1)$ . Since  $f$  is not identically zero, there exists  $x_0$  such that  $f(x_0) \neq 0$ . Thus,  $f(1) = 1$ .

$P(x, 1) \implies f(x - f(x)) = f(x)f(1 - 1) = f(x)f(0) = 0$ . (E1)  $P(1, y) \implies f(1 - f(y)) = f(1)f(1 - y) = f(1 - y)$ . (E2)

Let  $Z = \{z \in \mathbb{Z} \mid f(z) = 0\}$  be the set of zeros of  $f$ . We know  $0 \in Z$ . (E1) implies that  $x - f(x) \in Z$  for all  $x \in \mathbb{Z}$ .

**\*\*3. Case  $Z = \{0\}$ \*\*** If 0 is the only zero, then  $x - f(x) = 0$  for all  $x$ . Solution 3:  $f(x) = x$ . Verification: LHS =  $f(x - xy) = x - xy$ . RHS =  $f(x)f(1 - y) = x(1 - y) = x - xy$ .

**\*\*4. Case  $Z \neq \{0\}$ \*\*** Let  $k$  be the smallest positive integer in  $Z$ . Since  $f(1) = 1$ ,  $k \geq 2$ . If  $z \in Z$ ,  $f(z) = 0$ . By (E2),  $f(1 - z) = f(1 - f(z)) = f(1 - 0) = f(1) = 1$ .

We will show that  $k$  must be 2 or 3.

**\*\*Step 4.1: Suppose  $k$  is even.\*\*** Let  $k = 2m$  for some integer  $m \geq 1$ .  $f(k) = f(2m) = 0$ .  $P(m, 2) \implies f(m - f(2m)) = f(m)f(1 - 2)$ .  $f(m - 0) = f(m)f(-1)$ .  $f(m) = f(m)f(-1)$ .

If  $m = 1$ , then  $k = 2$ . If  $m > 1$ , then  $1 < m < k$ . By the definition of  $k$ ,  $f(m) \neq 0$ . Thus  $f(-1) = 1$ .  $P(1, -1) \implies f(1 - f(-1)) = f(1)f(1 - (-1))$ .  $f(1 - 1) = 1 \cdot f(2)$ .  $f(0) = f(2)$ . So  $f(2) = 0$ . This means  $k \leq 2$ . Since  $k$  is even,  $k = 2$ . So, if  $k$  is even,  $k = 2$ .

**\*\*Step 4.2: Suppose  $k$  is odd.\*\***  $k \geq 3$ .  $f(k) = 0$ . Since  $k$  is odd,  $y = \frac{1-k}{2}$  is an integer.  $1 - y = \frac{1+k}{2}$ .  $P(2, y) \implies P(2, \frac{1-k}{2})$ .  $f(2 - f(2y)) = f(2)f(1 - y)$ .  $f(2 - f(1 -$

$k)) = f(2)f(\frac{1+k}{2})$ . We know  $f(1-k) = 1$  (since  $k \in \mathbb{Z}$ ).  $f(2-1) = f(1) = 1$ . So  $1 = f(2)f(\frac{1+k}{2})$ .

Since  $f(x) \in \mathbb{Z}$ ,  $f(2)$  must be 1 or -1.

If  $f(2) = 1$ . By (E1),  $f(2-f(2)) = f(2-1) = f(1) = 0$ . But  $f(1) = 1$ . Contradiction ( $1 = 0$ ). If  $f(2) = -1$ . By (E1),  $f(2-f(2)) = f(2-(-1)) = f(3) = 0$ . This implies  $k \leq 3$ . Since  $k$  is odd and  $k \geq 3$ , we must have  $k = 3$ .

So, the smallest positive zero  $k$  can only be 2 or 3.

**\*\*Step 4.3: Case  $k=2$ .**  $f(2) = 0$ .  $f(1) = 1$ . We look for a solution where  $Z = 2\mathbb{Z}$  (the set of even integers). If  $Z = 2\mathbb{Z}$ ,  $f(x) = 0$  iff  $x$  is even. Let  $x$  be odd (so  $f(x) \neq 0$ ) and  $y$  be even.  $xy$  is even, so  $f(xy) = 0$ .  $1-y$  is odd.  $P(x, y) \implies f(x-f(xy)) = f(x)f(1-y)$ .  $f(x-0) = f(x)f(1-y)$ .  $f(x) = f(x)f(1-y)$ . Since  $f(x) \neq 0$ , we must have  $f(1-y) = 1$ . As  $y$  ranges over all even integers,  $1-y$  ranges over all odd integers. So  $f(z) = 1$  for all odd  $z$ .

Solution 4:  $f(x) = \begin{cases} 0 & \text{if } x \text{ is even} \\ 1 & \text{if } x \text{ is odd} \end{cases}$  (i.e.,  $f(x) = x \pmod{2}$ ). Verification: Case 1:  $x$  is even.  $f(x) = 0$ . RHS=0.  $f(xy)$  is 0 (if  $y$  is even) or  $f(x) = 0$  (if  $y$  is odd). In both cases  $f(xy) = 0$ . LHS= $f(x-0) = 0$ . Case 2:  $x$  is odd.  $f(x) = 1$ . RHS= $f(1-y)$ . If  $y$  is even,  $1-y$  is odd (RHS=1).  $xy$  is even,  $f(xy) = 0$ . LHS= $f(x-0) = f(x) = 1$ . If  $y$  is odd,  $1-y$  is even (RHS=0).  $xy$  is odd,  $f(xy) = 1$ . LHS= $f(x-1)$ . Since  $x$  is odd,  $x-1$  is even, LHS=0.

**\*\*Step 4.4: Case  $k=3$ .**  $f(3) = 0$ .  $f(1) = 1$ . We found  $f(2) = -1$ . We look for a solution where  $Z = 3\mathbb{Z}$ .  $x-f(x) \in Z = 3\mathbb{Z}$ , so  $f(x) \equiv x \pmod{3}$ . If  $f(x) \in \{-1, 0, 1\}$ , then  $f(x)$  is uniquely determined by  $x \pmod{3}$ .

Solution 5:  $f(x) = \begin{cases} 0 & \text{if } x \equiv 0 \pmod{3} \\ 1 & \text{if } x \equiv 1 \pmod{3} \\ -1 & \text{if } x \equiv 2 \pmod{3} \end{cases}$ . (This is the Legendre symbol  $(x/3)$ ).

Verification: This function  $f$  is completely multiplicative,  $f(xy) = f(x)f(y)$ , and  $f(x) \in \{-1, 0, 1\}$ . The equation is  $f(x-f(x)f(y)) = f(x)f(1-y)$ . If  $f(x) = 0$  (i.e.,  $3|x$ ), LHS=0, RHS=0. If  $f(x) \neq 0$ , then  $f(x) \in \{1, -1\}$ . LHS =  $f(x(1-f(x)^{-1}f(y)))$ . Since  $f(x)^{-1} = f(x)$ , LHS =  $f(x)f(1-f(x)f(y))$ . We need  $f(x)f(1-f(x)f(y)) = f(x)f(1-y)$ . Since  $f(x) \neq 0$ , we need  $f(1-f(x)f(y)) = f(1-y)$ .

If  $f(x) = 1$ .  $f(1-f(y)) = f(1-y)$ . This holds by (E2). If  $f(x) = -1$ .  $f(1-(-1)f(y)) = f(1+f(y))$ . We need  $f(1+f(y)) = f(1-y)$ . If  $f(y) = 0$  ( $y \equiv 0 \pmod{3}$ ).  $f(1) = 1$ .  $1-y \equiv 1 \pmod{3}$ ,  $f(1-y) = 1$ . If  $f(y) = 1$  ( $y \equiv 1 \pmod{3}$ ).  $f(1+1) = f(2) = -1$ .  $1-y \equiv 0 \pmod{3}$ ,  $f(1-y) = 0$ .  $-1 = 0$ .

Let's re-verify Solution 5 using the congruence property  $f(x) \equiv x \pmod{3}$ . LHS =  $f(x-f(xy))$ . RHS =  $f(x)f(1-y)$ . RHS  $\equiv x(1-y) = x-xy \pmod{3}$ . LHS  $\equiv x-f(xy) \pmod{3}$ . Since  $f(xy) \equiv xy \pmod{3}$ . LHS  $\equiv x-xy \pmod{3}$ . Since LHS and RHS are both in  $\{-1, 0, 1\}$  and congruent modulo 3, they must be equal.

Wait, there was an error in the verification attempt using multiplicativity above: If  $f(x) = -1$  and  $f(y) = 1$ .  $f(1+f(y)) = f(2) = -1$ .  $f(1-y) = 0$ .  $-1 = 0$ . This derivation must be flawed.

Let's check the specific case:  $x = 2$  ( $f(x) = -1$ ),  $y = 1$  ( $f(y) = 1$ ). LHS:  $f(x-f(xy)) = f(2-f(2)) = f(2-(-1)) = f(3) = 0$ . RHS:  $f(x)f(1-y) = f(2)f(0) = (-1)(0) = 0$ . The specific case holds.

The error in the multiplicative derivation was assuming  $f(x-f(x)f(y)) = f(x)f(1-f(x)f(y))$  when  $f(x) \neq 0$ . This requires  $f(a-b) = f(a)f(1-b/a)$  which is generally not true. Multiplicativity  $f(ab) = f(a)f(b)$  works, but we cannot factor  $x$  out if  $x$  is not coprime to 3 (and even if it is,  $f(x-y) \neq f(x)f(1-y/x)$ ).

The verification using  $f(x) \equiv x \pmod{3}$  is correct.

We have found five solutions.



Final Answer: The functions satisfying the equation are: 1.  $f(x) = 0$  for all  $x \in \mathbb{Z}$ . 2.  $f(x) = 1$  for all  $x \in \mathbb{Z}$ . 3.  $f(x) = x$  for all  $x \in \mathbb{Z}$ . 4.  $f(x) = \begin{cases} 0 & \text{if } x \text{ is even} \\ 1 & \text{if } x \text{ is odd} \end{cases}$ . 5.  $f(x) = \begin{cases} 0 & \text{if } x \equiv 0 \pmod{3} \\ 1 & \text{if } x \equiv 1 \pmod{3} \\ -1 & \text{if } x \equiv 2 \pmod{3} \end{cases}$ .

#### Case 2: review from gpt-5

The chunk makes critical, unjustified assumptions when classifying the non-constant solutions. In particular:

- In Step 4.3 ( $k = 2$ ), it assumes  $Z = 2\mathbb{Z}$  (all even numbers are zeros) and then derives the parity solution, but it never proves from the functional equation that  $Z$  must equal  $2\mathbb{Z}$  or that  $f(\text{odd}) = 1$ .
- In Step 4.4 ( $k = 3$ ), it assumes  $Z = 3\mathbb{Z}$  and that  $f(x) \in \{-1, 0, 1\}$ , using this to conclude the “Legendre symbol” form via a congruence argument. These constraints are not derived from  $k = 3$  (only  $f(x) \equiv x \pmod{3}$  is shown).

Thus the claim that the five listed functions are all solutions is not established.

The manually annotated label in the dataset is true (7 points), while our method found a critical error in it:

#### Case 2: our comment

The proof is incomplete in steps 4.3 and 4.4 due to unjustified assumptions. For the  $k = 2$  and  $k = 3$  cases, the author assumes the set of zeros is  $Z = k\mathbb{Z}$  and, for  $k = 3$ , further assumes  $f(x) \in \{-1, 0, 1\}$  to match the Legendre symbol form. While these specific functions are indeed solutions, the proof fails to derive that *all* solutions satisfying  $f(k) = 0$  must necessarily possess this periodicity and range structure.

#### Case 3: problem

For a positive integer  $n$ , let  $A_n$  be the number of perfect powers less than or equal to  $n$ . Here, a perfect power is a number that can be expressed in the form  $a^b$ , where  $a$  is a positive integer and  $b$  is an integer greater than or equal to 2. Prove that there are infinitely many  $n$  such that  $A_n$  divides  $n + 2024$ .

#### Case 3: proof

Let  $S$  be the set of perfect powers. A perfect power is a positive integer that can be expressed in the form  $a^b$ , where  $a$  is a positive integer and  $b$  is an integer greater than or equal to 2. The set of perfect powers, sorted in increasing order, is  $\{1, 4, 8, 9, 16, 25, 27, \dots\}$ . For a positive integer  $n$ ,  $A_n$  is the number of perfect powers less than or equal to  $n$ . So,  $A_n = |S \cap \{1, 2, \dots, n\}|$ . The function  $A_n$  is a non-decreasing step function. It is constant between consecutive perfect powers. Let  $p_k$  denote the  $k$ -th perfect power. For any integer  $n$  in the range  $p_k \leq n < p_{k+1}$ , the value of  $A_n$  is constant and equal to  $k$ .

We want to prove that there are infinitely many positive integers  $n$  such that  $A_n$  divides  $n + 2024$ .

Our strategy is to find an infinite sequence of disjoint integer intervals, and in each interval, find one such integer  $n$ . Let's consider an interval  $[p_k, p_{k+1} - 1]$  where  $A_n$  is constant, equal to  $k$ . The condition we want to satisfy is  $k \mid (n + 2024)$ . This is equivalent to the congruence relation  $n \equiv -2024 \pmod{k}$ . The interval  $[p_k, p_{k+1} - 1]$



contains  $p_{k+1} - p_k$  consecutive integers. If the length of this interval is at least  $k$ , the integers it contains form a complete residue system modulo  $k$ . In such a case, there is guaranteed to be an integer  $n$  in the interval satisfying  $n \equiv -2024 \pmod{k}$ .

So, the problem reduces to showing that there are infinitely many values of  $k$  for which the gap between consecutive perfect powers,  $p_{k+1} - p_k$ , is greater than or equal to  $k$ . We will focus on a special subsequence of perfect powers: the squares. Consider the gaps between consecutive squares. Let  $m \geq 2$  be an integer. We will analyze the interval of integers  $I_m = [(m-1)^2, m^2 - 1]$ . We will show that for all sufficiently large  $m$ : 1.  $A_n$  is constant for all  $n \in I_m$ . To do this, we will prove that there are no perfect powers in the open interval  $((m-1)^2, m^2)$ . 2. The length of the interval  $I_m$  is greater than the value of  $A_n$  for  $n \in I_m$ .

If these two conditions hold for some large  $m$ , let  $k_m$  be the constant value of  $A_n$  on  $I_m$ . The length of  $I_m$  is  $(m^2 - 1) - (m-1)^2 + 1 = 2m - 1$ . Condition 2 means  $2m - 1 > k_m$ . As argued above, since the interval  $I_m$  contains  $2m - 1 > k_m$  consecutive integers, there must exist an integer  $n_m \in I_m$  such that  $n_m \equiv -2024 \pmod{k_m}$ . For this  $n_m$ , we have  $A_{n_m} = k_m$ , and thus  $A_{n_m} \mid (n_m + 2024)$ .

If we can show that these conditions hold for all  $m$  greater than some integer  $M$ , we can find a solution  $n_m$  for each  $m \geq M$ . Since the intervals  $I_m$  are disjoint for different  $m$ , the solutions  $n_m$  will be distinct, proving that there are infinitely many such integers. We now formalize and prove these two main claims.

**\*\*Claim 1:\*\*** For all sufficiently large integers  $m$ , there are no perfect powers in the open interval  $((m-1)^2, m^2)$ .

**\*\*Proof:\*\*** A perfect power in the interval  $((m-1)^2, m^2)$  cannot be a perfect square, as there are no integers between  $m-1$  and  $m$ . Therefore, any such perfect power must be of the form  $x^j$  where  $x \geq 2$  and  $j \geq 3$  are integers. For  $x^j$  to be in the interval, we must have  $(m-1)^2 < x^j < m^2$ . Taking the  $j$ -th root of this inequality gives  $(m-1)^{2/j} < x < m^{2/j}$ . For an integer  $x$  to exist in this range, the length of the interval  $((m-1)^{2/j}, m^{2/j})$  must be greater than 1 for some  $j$ . The possible values for the exponent  $j$  are limited by the condition  $2^j \leq x^j < m^2$ , which implies  $j < 2 \log_2 m$ . Let  $N(a, b)$  be the number of perfect powers in the open interval  $(a, b)$ . The total number of perfect powers in  $((m-1)^2, m^2)$  is bounded by the sum of the number of integer candidates for the base  $x$  over all possible prime exponents  $j$ . For simplicity, we sum over all integers  $j \geq 3$ :

$$N((m-1)^2, m^2) \leq \sum_{j=3}^{\lfloor 2 \log_2 m \rfloor} \left( \lfloor m^{2/j} \rfloor - \lceil (m-1)^{2/j} \rceil + 1 \right)$$

A simpler bound is the sum of the lengths of the intervals for  $x$ :

$$N((m-1)^2, m^2) \leq \sum_{j=3}^{\lfloor 2 \log_2 m \rfloor} \left( m^{2/j} - (m-1)^{2/j} \right)$$

Let's analyze the term  $m^{2/j} - (m-1)^{2/j}$ . By the Mean Value Theorem applied to the function  $f(t) = t^{2/j}$  on the interval  $[m-1, m]$ , for each  $j$  there exists some  $c_j \in (m-1, m)$  such that:

$$m^{2/j} - (m-1)^{2/j} = f'(c_j) \cdot (m - (m-1)) = \frac{2}{j} c_j^{\frac{2}{j}-1}$$

Since  $j \geq 3$ , the exponent  $\frac{2}{j} - 1$  is negative. As  $c_j > m-1$ , we have  $c_j^{\frac{2}{j}-1} < (m-1)^{\frac{2}{j}-1}$ . So,  $m^{2/j} - (m-1)^{2/j} < \frac{2}{j} (m-1)^{\frac{2}{j}-1}$ .

Now we bound the sum. For  $j \geq 3$ , the exponent  $\frac{2}{j} - 1 \leq \frac{2}{3} - 1 = -\frac{1}{3}$ . Assuming  $m - 1 \geq 1$ , we have  $(m - 1)^{\frac{2}{j} - 1} \leq (m - 1)^{-1/3}$ .

$$N((m - 1)^2, m^2) < \sum_{j=3}^{\lfloor 2 \log_2 m \rfloor} \frac{2}{j} (m - 1)^{\frac{2}{j} - 1} \leq \sum_{j=3}^{\lfloor 2 \log_2 m \rfloor} \frac{2}{3} (m - 1)^{-1/3}$$

The number of terms in the summation is  $\lfloor 2 \log_2 m \rfloor - 2 < 2 \log_2 m$ .

$$N((m - 1)^2, m^2) < (2 \log_2 m) \cdot \frac{2}{3} (m - 1)^{-1/3} = \frac{4 \log_2 m}{3(m - 1)^{1/3}}$$

As  $m \rightarrow \infty$ , this upper bound tends to 0. To see this formally:

$$\lim_{m \rightarrow \infty} \frac{4 \log_2 m}{3(m - 1)^{1/3}} = \lim_{m \rightarrow \infty} \frac{4 \ln m}{3 \ln 2 \cdot (m - 1)^{1/3}} = 0$$

since any positive power of  $m$  grows faster than any power of  $\ln m$ . Since  $N((m - 1)^2, m^2)$  must be a non-negative integer, there must exist an integer  $M_1$  such that for all  $m \geq M_1$ ,  $N((m - 1)^2, m^2) = 0$ . This proves Claim 1.

**\*\*Claim 2:\*\*** For all sufficiently large integers  $m$ ,  $2m - 1 > A_{(m-1)^2}$ .

**\*\*Proof:\*\*** We need an upper bound for  $A_n$ . The set of perfect powers is the union of the sets of squares, cubes, 5th powers, and so on. A simple upper bound for  $A_n$  is obtained by summing the number of  $j$ -th powers up to  $n$  for all  $j \geq 2$ , ignoring overlaps which leads to overcounting. The number of  $j$ -th powers up to  $n$  is  $\lfloor n^{1/j} \rfloor$ .

$$A_n \leq \sum_{j=2}^{\lfloor \log_2 n \rfloor} \lfloor n^{1/j} \rfloor < \sum_{j=2}^{\lfloor \log_2 n \rfloor} n^{1/j}$$

We can write this as:

$$A_n < n^{1/2} + n^{1/3} + n^{1/4} + \dots + n^{1/\lfloor \log_2 n \rfloor}$$

The number of terms in the sum after  $n^{1/2}$  is  $\lfloor \log_2 n \rfloor - 2$ . For  $j \geq 3$ , we have  $n^{1/j} \leq n^{1/3}$  (for  $n \geq 1$ ). So, we can bound the sum as:

$$A_n < n^{1/2} + (\lfloor \log_2 n \rfloor - 2)n^{1/3} < n^{1/2} + (\log_2 n)n^{1/3}$$

Now, let's apply this bound to  $A_{(m-1)^2}$ . Let  $n = (m - 1)^2$ .

$$A_{(m-1)^2} < \sqrt{(m - 1)^2} + (\log_2((m - 1)^2)) \cdot ((m - 1)^2)^{1/3}$$

$$A_{(m-1)^2} < m - 1 + (2 \log_2(m - 1)) \cdot (m - 1)^{2/3}$$

We want to prove that  $2m - 1 > A_{(m-1)^2}$  for sufficiently large  $m$ . It is sufficient to show that:

$$2m - 1 > m - 1 + 2 \log_2(m - 1) \cdot (m - 1)^{2/3}$$

This inequality simplifies to:

$$m > 2 \log_2(m - 1) \cdot (m - 1)^{2/3}$$

For  $m > 1$ , we can divide by  $m^{2/3}$ :

$$m^{1/3} > 2 \log_2(m - 1) \cdot \left( \frac{m - 1}{m} \right)^{2/3} = 2 \log_2(m - 1) \cdot \left( 1 - \frac{1}{m} \right)^{2/3}$$

As  $m \rightarrow \infty$ , the term  $(1 - 1/m)^{2/3}$  approaches 1. The inequality will hold if  $m^{1/3}$  grows faster than  $2 \log_2(m - 1)$ . To confirm this, we examine the limit of their ratio:

$$\lim_{x \rightarrow \infty} \frac{x^{1/3}}{2 \log_2(x - 1)} = \lim_{x \rightarrow \infty} \frac{x^{1/3} \ln 2}{2 \ln(x - 1)}$$

Using L'Hôpital's rule on this  $\frac{\infty}{\infty}$  form:

$$= \lim_{x \rightarrow \infty} \frac{\frac{1}{3}x^{-2/3} \ln 2}{2 \cdot \frac{1}{x-1}} = \lim_{x \rightarrow \infty} \frac{\ln 2}{6} \frac{x-1}{x^{2/3}} = \lim_{x \rightarrow \infty} \frac{\ln 2}{6} (x^{1/3} - x^{-2/3}) = \infty$$

Since this limit is infinite,  $x^{1/3}$  grows faster than any constant multiple of  $\log_2(x-1)$ . Therefore, there exists an integer  $M_2$  such that for all  $m \geq M_2$ , the inequality  $m^{1/3} > 2 \log_2(m-1) \cdot (1-1/m)^{2/3}$  holds. This proves Claim 2.

**\*\*Conclusion\*\***

Let  $M = \max(M_1, M_2)$ . For every integer  $m \geq M$ , we have established the following:

1. From Claim 1, the interval  $((m-1)^2, m^2)$  contains no perfect powers. This means that for any integer  $n$  in the interval  $I_m = [(m-1)^2, m^2 - 1]$ , the set of perfect powers less than or equal to  $n$  is the same as the set of perfect powers less than or equal to  $(m-1)^2$ . Thus,  $A_n$  has a constant value for all  $n \in I_m$ . Let this value be  $k_m = A_{(m-1)^2}$ .
2. From Claim 2, the inequality  $2m-1 > A_{(m-1)^2} = k_m$  holds. The length of the interval  $I_m$  is  $(m^2 - 1) - (m-1)^2 + 1 = 2m - 1$ . So, for  $m \geq M$ , the length of the interval  $I_m$  is strictly greater than  $k_m$ .

We are looking for an integer  $n$  such that  $A_n \mid (n + 2024)$ . For any  $m \geq M$ , we seek an  $n \in I_m$  satisfying  $k_m \mid (n + 2024)$ . This is equivalent to finding an  $n$  in the range  $(m-1)^2 \leq n \leq m^2 - 1$  that satisfies the congruence:

$$n \equiv -2024 \pmod{k_m}$$

The interval  $I_m$  contains the sequence of consecutive integers  $(m-1)^2, (m-1)^2 + 1, \dots, m^2 - 1$ . The length of this sequence is  $2m - 1$ . Since  $m \geq M$ , we have  $2m - 1 > k_m$ . Because the sequence of integers in  $I_m$  is longer than a full cycle of residues modulo  $k_m$ , it is guaranteed to contain at least one integer from each residue class modulo  $k_m$ . In particular, there must exist an integer  $n_m \in I_m$  such that  $n_m \equiv -2024 \pmod{k_m}$ . For this  $n_m$ , we have  $A_{n_m} = k_m$ , and so  $A_{n_m} \mid (n_m + 2024)$ .

For each integer  $m \geq M$ , we have successfully found an integer  $n_m$  that satisfies the given property. To complete the proof, we must show that this process generates an infinite number of distinct solutions.

Let  $m$  and  $m'$  be two integers such that  $M \leq m < m'$ . The solution  $n_m$  lies in the interval  $I_m = [(m-1)^2, m^2 - 1]$ . The solution  $n_{m'}$  lies in the interval  $I_{m'} = [(m'-1)^2, (m')^2 - 1]$ . Since  $m' > m$ , we must have  $m' \geq m + 1$ . The smallest value in the interval  $I_{m'}$  is  $(m'-1)^2 \geq (m+1-1)^2 = m^2$ . The largest value in the interval  $I_m$  is  $m^2 - 1$ . So, for any  $n \in I_m$  and  $n' \in I_{m'}$ , we have  $n \leq m^2 - 1 < m^2 \leq n'$ . This shows that the intervals  $I_m$  and  $I_{m'}$  are disjoint. Consequently, the solutions  $n_m \in I_m$  and  $n_{m'} \in I_{m'}$  are distinct. Since we can choose any integer  $m \geq M$ , we have constructed an infinite sequence of distinct solutions  $\{n_m\}_{m \geq M}$ . Therefore, there are infinitely many integers  $n$  such that  $A_n$  divides  $n + 2024$ .

The manually annotated label in the dataset is true (7 points), while our method found a critical error in it:

#### Case 3: review from GPT-5-mini

<verification>false</verification>

The main (fatal) error is in Claim 1: the author replaces the integer count

$$\lfloor m^{2/j} \rfloor - \lceil (m-1)^{2/j} \rceil + 1$$

by the real length  $m^{2/j} - (m-1)^{2/j}$ . That inequality is not valid in general (e.g. an interval of real length  $< 1$  can still contain an integer). Thus the bound

$$N((m-1)^2, m^2) \leq \sum_j (m^{2/j} - (m-1)^{2/j})$$

is unjustified, and the conclusion “this sum  $\rightarrow 0 \Rightarrow N = 0$  for large  $m$ ” does not follow. Because of this mistake the proof does not establish that the open interval  $((m-1)^2, m^2)$  contains no perfect powers for all large  $m$ , and the rest of the argument (which depends critically on that claim) fails.

#### Case 3: our comment

The `gt_label` is wrong, the `pred_label` is correct.

Fatal error in Claim 1: It is invalid to use the length of a real interval to upper-bound the number of integer points, because when the interval length is  $\geq 1$  it may still contain integers. Therefore, the conclusion “limit = 0  $\Rightarrow$  final result = 0” does not hold; Moreover, the proof has a totally wrong derivation that

$$N((m-1)^2, m^2) \leq \sum_{j=3}^{\lfloor 2 \log_2 m \rfloor} \left( \lfloor m^{2/j} \rfloor - \lceil (m-1)^{2/j} \rceil + 1 \right)$$

$$\Rightarrow N((m-1)^2, m^2) \leq \sum_{j=3}^{\lfloor 2 \log_2 m \rfloor} \left( m^{2/j} - (m-1)^{2/j} \right)$$

which is actually an unacceptable gap.

#### Case 4: problem

Prove that there exist two real-coefficient polynomials  $P(x)$  and  $Q(x)$  satisfying the following condition:

(Condition) The degree of the polynomial  $P(x)$  is at least 2024, the degree of  $Q(x)$  is at least 2, and for any real number  $x$ , the following holds:

$$P(Q(x) - x - 1) = Q(P(x))$$

#### Case 4: proof

We will prove the existence of such polynomials  $P(x)$  and  $Q(x)$  by constructing them. Our construction will follow three main steps: 1. Find a pair of polynomials of low degree, let's call them  $(P_0(x), Q_0(x))$ , that satisfies the given functional equation. 2. Construct a high-degree polynomial, let's call it  $R(x)$ , that commutes with  $Q_0(x)$ . 3. Combine these polynomials to form the final pair  $(P(x), Q(x))$  that satisfies all the given conditions.

Let the given equation be

$$P(Q(x) - x - 1) = Q(P(x)) \quad (*).$$

##### Step 1: Finding a base solution pair

We look for a simple solution where  $P_0(x)$  and  $Q_0(x)$  have degree 2. Let's try to relate them. A simple assumption is that  $Q_0(x)$  is a linear perturbation of  $P_0(x)$ . Let's try the ansatz:

$$Q_0(x) = P_0(x) + x + c$$

for some real constant  $c$ . Substituting this into the equation (\*): The left side's argument becomes  $Q_0(x) - x - 1 = (P_0(x) + x + c) - x - 1 = P_0(x) + c - 1$ . So the equation becomes  $P_0(P_0(x) + c - 1) = Q_0(P_0(x))$ .

Let  $y = P_0(x)$ . For any value  $y$  in the range of  $P_0(x)$ , the equation must hold:

$$P_0(y + c - 1) = Q_0(y).$$

Using our ansatz for  $Q_0(y)$ , we have  $Q_0(y) = P_0(y) + y + c$ . So, we obtain a functional equation for  $P_0(y)$ :

$$P_0(y + c - 1) = P_0(y) + y + c.$$

Let  $b = c - 1$ , so  $c = b + 1$ . The equation is

$$P_0(y + b) - P_0(y) = y + b + 1.$$

Let  $m = \deg(P_0)$ . If  $b \neq 0$ , the degree of the polynomial on the left,  $P_0(y + b) - P_0(y)$ , is  $m - 1$ . The degree of the polynomial on the right is 1. For the identity to hold, the degrees must be equal, so  $m - 1 = 1$ , which implies  $m = 2$ .

Let  $P_0(y) = Ay^2 + By + C$  for some constants  $A, B, C \in \mathbb{R}$  with  $A \neq 0$ . Substituting this into the difference equation:  $A(y + b)^2 + B(y + b) + C - (Ay^2 + By + C) = y + b + 1$   
 $A(y^2 + 2by + b^2) + B(y + b) + C - Ay^2 - By - C = y + b + 1$   
 $2Aby + Ab^2 + Bb = y + b + 1$ . This must be a polynomial identity in  $y$ . We can equate the coefficients of the powers of  $y$ : 1. Coefficient of  $y$ :  $2Ab = 1$ . 2. Constant term:  $Ab^2 + Bb = b + 1$ .

This is a system of two equations with three unknowns ( $A, B, b$ ). We can choose a value for one of them. Let's choose  $b = 2$ . From (1):  $2A(2) = 1 \implies 4A = 1 \implies A = 1/4$ . Substitute  $A = 1/4$  and  $b = 2$  into (2):  $(1/4)(2^2) + B(2) = 2 + 1 \implies 1 + 2B = 3 \implies 2B = 2 \implies B = 1$ . The constant  $C$  can be any real number. For simplicity, we choose  $C = 0$ . So, we have found a polynomial  $P_0(x) = \frac{1}{4}x^2 + x$ . We have  $b = 2$ , so  $c = b + 1 = 3$ . Now we find the corresponding  $Q_0(x)$ :  $Q_0(x) = P_0(x) + x + c = (\frac{1}{4}x^2 + x) + x + 3 = \frac{1}{4}x^2 + 2x + 3$ .

So, we have found a base solution pair:  $P_0(x) = \frac{1}{4}x^2 + x$   $Q_0(x) = \frac{1}{4}x^2 + 2x + 3$  Both have degree 2, satisfying the degree condition for  $Q_0(x)$ . Let's verify that they satisfy (\*).  $Q_0(x) - x - 1 = (\frac{1}{4}x^2 + 2x + 3) - x - 1 = \frac{1}{4}x^2 + x + 2$ . LHS:  $P_0(Q_0(x) - x - 1) = P_0(\frac{1}{4}x^2 + x + 2) = \frac{1}{4}(\frac{1}{4}x^2 + x + 2)^2 + (\frac{1}{4}x^2 + x + 2) = \frac{1}{64}x^4 + \frac{1}{8}x^3 + \frac{3}{4}x^2 + 2x + 3$ . RHS:  $Q_0(P_0(x)) = \frac{1}{4}(P_0(x))^2 + 2P_0(x) + 3 = \frac{1}{4}(\frac{1}{4}x^2 + x)^2 + 2(\frac{1}{4}x^2 + x) + 3 = \frac{1}{64}x^4 + \frac{1}{8}x^3 + \frac{3}{4}x^2 + 2x + 3$ . LHS=RHS, so the pair  $(P_0, Q_0)$  is a solution.

### Step 2: Construction of a high-degree commuting polynomial

Our plan is to set  $Q(x) = Q_0(x)$  and  $P(x) = R(P_0(x))$  for some polynomial  $R(x)$  with high degree. Let's see what condition  $R(x)$  must satisfy. Substitute  $P(x) = R(P_0(x))$  and  $Q(x) = Q_0(x)$  into (\*): LHS:  $P(Q(x) - x - 1) = R(P_0(Q_0(x) - x - 1))$ . Using the property of our base solution,  $P_0(Q_0(x) - x - 1) = Q_0(P_0(x))$ , we get: LHS =  $R(Q_0(P_0(x)))$ . RHS:  $Q(P(x)) = Q_0(R(P_0(x)))$ . So, the equation becomes  $R(Q_0(P_0(x))) = Q_0(R(P_0(x)))$ . Let  $y = P_0(x)$ . This identity must hold for all values  $y$  in the range of  $P_0$ . If we find a polynomial  $R$  that commutes with  $Q_0$ , i.e.,  $R(Q_0(y)) = Q_0(R(y))$  for all  $y$ , then our equation will be satisfied. Since this is a polynomial identity, if it holds for all  $y$  in the range of  $P_0$  (an infinite set), it holds for all  $y$ .

We need to find a polynomial  $R(x)$  that commutes with  $Q_0(x)$ . A standard method for finding commuting polynomials is to simplify  $Q_0(x)$  by a linear change of variables (conjugation).  $Q_0(x) = \frac{1}{4}x^2 + 2x + 3 = \frac{1}{4}(x^2 + 8x) + 3 = \frac{1}{4}(x^2 + 8x + 16) - 4 + 3 = \frac{1}{4}(x + 4)^2 - 1$ . Let  $L(x) = x + 4$ . Its inverse is  $L^{-1}(x) = x - 4$ . Let  $S(y) = \frac{1}{4}y^2 - 1$ . Then  $Q_0(x) = L^{-1}(S(L(x)))$ .

If we find a polynomial  $\tilde{R}(y)$  that commutes with  $S(y)$ , then  $R(x) = L^{-1}(\tilde{R}(L(x)))$  will commute with  $Q_0(x)$ . This is because:  $R(Q_0(x)) = L^{-1}(\tilde{R}(L(L^{-1}(S(L(x))))) = L^{-1}(\tilde{R}(S(L(x))))$ .  $Q_0(R(x)) = L^{-1}(S(L(L^{-1}(\tilde{R}(L(x))))) = L^{-1}(S(\tilde{R}(L(x))))$ . Since  $\tilde{R}$  and  $S$  commute,  $\tilde{R}(S(y)) = S(\tilde{R}(y))$ , so  $R$  and  $Q_0$  commute.

A simple way to find a polynomial that commutes with  $S(y)$  is to take iterates of  $S(y)$ . Let  $S_1(y) = S(y)$  and  $S_k(y) = S(S_{k-1}(y))$  for  $k \geq 2$ . Then  $S_k(y)$  commutes with  $S(y)$  for any  $k \geq 1$ . Let's choose  $\tilde{R}(y) = S_k(y)$  for a sufficiently large  $k$ . The degree of  $S(y)$  is 2. The degree of  $\tilde{R}(y) = S_k(y)$  is  $2^k$ . We define  $R(x) = L^{-1}(S_k(L(x))) = S_k(x + 4) - 4$ . The degree of  $R(x)$  is  $2^k$ .

### Step 3: Assembling the final solution

We are now ready to define  $P(x)$  and  $Q(x)$ . - Let  $Q(x) = Q_0(x) = \frac{1}{4}x^2 + 2x + 3$ . - Let  $R(x)$  be constructed with  $k = 10$ , so  $\tilde{R}(y) = S_{10}(y) = S^{\circ 10}(y)$ , and  $R(x) = S_{10}(x + 4) - 4$ . - Let  $P(x) = R(P_0(x)) = R(\frac{1}{4}x^2 + x)$ .

Let's check the conditions: 1. **Coefficients**:  $P_0, Q_0, L, L^{-1}, S$  are all real-coefficient polynomials. Any composition or iteration will also result in polynomials with real coefficients. So  $P(x)$  and  $Q(x)$  have real coefficients.

2. **Degree of  $Q(x)$** :  $\deg(Q) = \deg(Q_0) = 2$ . This satisfies  $\deg(Q) \geq 2$ .

3. **Degree of  $P(x)$** : The degree of  $P(x)$  is the product of the degrees of  $R(x)$  and  $P_0(x)$ .  $\deg(P_0) = 2$ .  $\deg(R) = \deg(S_{10}) = 2^{10} = 1024$ .  $\deg(P) = \deg(R) \cdot \deg(P_0) = 1024 \cdot 2 = 2048$ . We need  $\deg(P) \geq 2024$ . Our value 2048 satisfies this condition. (If we needed an even higher degree, we could just choose a larger  $k$ . For instance,  $k = 11$  would give  $\deg(P) = 4096$ .)

4. **The functional equation**: As shown in Step 2, the choice  $P(x) = R(P_0(x))$  and  $Q(x) = Q_0(x)$  reduces the equation (\*) to  $R(Q_0(P_0(x))) = Q_0(R(P_0(x)))$ . Our construction of  $R(x)$  ensures it commutes with  $Q_0(x)$ , so this identity holds for all inputs, and thus for all real numbers  $x$ .

We have successfully constructed polynomials  $P(x)$  and  $Q(x)$  that satisfy all the required conditions. For completeness, the polynomials are: -  $Q(x) = \frac{1}{4}x^2 + 2x + 3$ . -  $P(x) = S_{10}(\frac{1}{4}x^2 + x + 4) - 4$ , where  $S(y) = \frac{1}{4}y^2 - 1$  and  $S_{10}$  is the 10th iterate of  $S(y)$ . This completes the proof that such polynomials exist.

#### Case 4: review from GPT-5-mini

<verification>false</verification> The fatal error is in Step 2: the author claims  $Q_0(x) = L^{-1}(S(L(x)))$  with  $L(x) = x + 4$  and then uses this to conclude that  $R(x) = L^{-1}(S_k(L(x)))$  commutes with  $Q_0$  whenever  $S_k$  commutes with  $S$ . But  $Q_0(x) = 1/4(x+4)^2 - 1$  equals  $S(L(x))$ , not  $L^{-1}(S(L(x)))$ . That mistaken conjugation breaks the commuting argument: the equality  $R(Q_0(x)) = Q_0(R(x))$  does not follow from  $S_k \circ S = S \circ S_k$  under the provided definitions, and the displayed  $R(x) = S_k(x+4) - 4$  need not commute with  $Q_0$ . (Thus the key commuting-polynomial construction is invalid as written.)

The manually annotated label in the dataset is true (7 points), while our method found a typo in it:

#### Case 4: our comment

The review from GPT-5-mini actually points out a typo that  $Q_0(x) \neq L^{-1}(S(L(x)))$ . However, the choice of  $S$  does not really matter so we can fix the proof easily.

#### Case 5: problem

Prove that there exist two real-coefficient polynomials  $P(x)$  and  $Q(x)$  satisfying the following condition:

(Condition) The degree of the polynomial  $P(x)$  is at least 2024, the degree of  $Q(x)$  is at least 2, and for any real number  $x$ , the following holds:

$$P(Q(x) - x - 1) = Q(P(x))$$

#### Case 5: proof

Let  $n = 2024$ . We are looking for real-coefficient polynomials  $P(x)$  and  $Q(x)$  such that  $\deg(P) \geq n$  and  $P(Q(x) - x - 1) = Q(P(x))$  for all real  $x$ .

Let  $\deg(P) = d_P$  and  $\deg(Q) = d_Q$ . Comparing the degrees of both sides of the equation  $P(Q(x) - x - 1) = Q(P(x))$ , we have  $\deg(Q(P(x))) = d_Q d_P$ .  $\deg(P(Q(x) - x - 1)) = d_P \deg(Q(x) - x - 1)$ . If  $d_Q > 1$ ,  $\deg(Q(x) - x - 1) = d_Q$ , so  $\deg(P(Q(x) - x - 1)) = d_P d_Q$ . If  $d_Q = 1$ , let  $Q(x) = cx + d$  with  $c \neq 0$ . Then  $Q(x) - x - 1 = (c - 1)x +$



$d - 1$ . If  $c \neq 1$ ,  $\deg(Q(x) - x - 1) = 1 = d_Q$ , so  $\deg(P(Q(x) - x - 1)) = d_P \cdot 1 = d_P$ . The degree of  $Q(P(x))$  is  $1 \cdot d_P = d_P$ . If  $c = 1$ ,  $Q(x) = x + d$ .  $Q(x) - x - 1 = d - 1$ .  $\deg(P(Q(x) - x - 1)) = \deg(P(d - 1)) = 0$ . The degree of  $Q(P(x)) = P(x) + d$  is  $d_P$ . For  $0 = d_P$ ,  $P(x)$  must be a constant. Let  $P(x) = k$ . Then  $P(Q(x) - x - 1) = k$  and  $Q(P(x)) = Q(k) = k + d$ . So  $k = k + d \implies d = 0$ .  $Q(x) = x$ . In this case  $P(Q(x) - x - 1) = P(-1)$  and  $Q(P(x)) = P(x)$ . So  $P(x)$  must be a constant  $k$  such that  $P(-1) = k$ . This works, but we need  $d_P \geq 2024$ . So we must have  $d_Q \geq 1$  and  $c \neq 1$  if  $d_Q = 1$ .

Let's try to find a solution with  $d_Q = 1$ . Let  $Q(x) = cx + d$  with  $c \neq 1$ . The condition is  $P((c - 1)x + d - 1) = cP(x) + d$ . Let  $d_P = n = 2024$ . Let  $P(x) = a_n x^n + \dots + a_0$  with  $a_n \neq 0$ . Comparing the leading coefficients,  $a_n((c - 1)x)^n + \dots = c(a_n x^n) + \dots$ .  $a_n(c - 1)^n x^n + \dots = ca_n x^n + \dots$ . Since  $a_n \neq 0$ , we must have  $(c - 1)^n = c$ . Let  $n = 2024$ . We need to show that there exists a real number  $c \neq 1$  such that  $(c - 1)^{2024} = c$ . Consider the function  $f(t) = (t - 1)^{2024} - t$  for  $t \in \mathbb{R}$ .  $f(t)$  is continuous.  $f(0) = (0 - 1)^{2024} - 0 = 1 > 0$ .  $f(1) = (1 - 1)^{2024} - 1 = -1 < 0$ . By the Intermediate Value Theorem, there exists  $c_1 \in (0, 1)$  such that  $f(c_1) = 0$ . Since  $c_1 \in (0, 1)$ ,  $c_1 \neq 1$ .  $f(3) = (3 - 1)^{2024} - 3 = 2^{2024} - 3 > 0$ . By the Intermediate Value Theorem, there exists  $c_2 \in (1, 3)$  such that  $f(c_2) = 0$ . Since  $c_2 \in (1, 3)$ ,  $c_2 \neq 1$ . Let  $c$  be any such real root. For example,  $c = c_1$  or  $c = c_2$ . Now we need to find a polynomial  $P(x)$  of degree  $n = 2024$  satisfying  $P((c - 1)x + d - 1) = cP(x) + d$  for some  $d$ . Let's try  $d = 1$ . We need  $P((c - 1)x) = cP(x) + 1$ . Let  $P(x) = ax^n + b$ .  $a((c - 1)x)^n + b = c(ax^n + b) + 1$ .  $a(c - 1)^n x^n + b = acx^n + cb + 1$ .  $acx^n + b = acx^n + cb + 1$ . This requires  $b = cb + 1$ , so  $b(1 - c) = 1$ . Since  $c \neq 1$ , we can choose  $b = \frac{1}{1 - c}$ . Let  $a = 1$ . Then  $P(x) = x^n + \frac{1}{1 - c}$  is a polynomial of degree  $n = 2024$ . Let  $P(x) = x^{2024} + \frac{1}{1 - c}$  and  $Q(x) = cx + 1$ .  $P(x)$  has degree  $2024 \geq 2024$ .  $P(x)$  and  $Q(x)$  have real coefficients since  $c$  is real. Let's verify the condition:  $P(Q(x) - x - 1) = P(cx + 1 - x - 1) = P((c - 1)x) = ((c - 1)x)^{2024} + \frac{1}{1 - c} = (c - 1)^{2024} x^{2024} + \frac{1}{1 - c} = cx^{2024} + \frac{1}{1 - c}$ .  $Q(P(x)) = cP(x) + 1 = c(x^{2024} + \frac{1}{1 - c}) + 1 = cx^{2024} + \frac{c}{1 - c} + 1 = cx^{2024} + \frac{c + 1 - c}{1 - c} = cx^{2024} + \frac{1}{1 - c}$ . The condition  $P(Q(x) - x - 1) = Q(P(x))$  is satisfied. Thus, such polynomials  $P(x)$  and  $Q(x)$  exist. Final Answer: The final answer is  $P(x)$  and  $Q(x)$  exist

#### Case 5: review from qwen3-30b-a3b

<verification>false</verification> The solution incorrectly constructs  $Q(x)$  as a linear polynomial (degree 1), violating the problem's requirement that  $\deg(Q) \geq 2$ . This critical error undermines the correctness of the solution.

The manually annotated label in the dataset is true (7 points), while our method found a critical error in it:

#### Case 5: our comment

$\deg(Q)$  has to be at least 2 so this construction cannot be accepted.

#### Case 6: problem

There are 120 students in DeepMind highschool, and each student 'loves' some of the students. Here 'love' is not always a mutual relationship; a student A may love B while B may not love A, and it is also possible that a student loves oneself. For the set  $F$  of all  $2^{120}$  subsets of the students, define the function  $f : F \rightarrow F$  as the function that maps each  $X \in F$  to the set of students loved by one or more students in  $X$ . For  $A, B \in F$ ,  $A$  and  $B$  are said to have a 'lovely relationship' iff there exists a natural number  $k$  such that  $f^k(A) = B$ . Now we want to select distinct elements  $A_1, A_2, \dots, A_t$  from  $F$  such



that for any  $1 \leq i < j \leq t$ ,  $A_i$  and  $A_j$  have a lovely relationship. Let  $M(f)$  be the maximum possible value of  $t$ . Show that  $M(f) \leq 2^{70}$ . ", "proof": "Let  $S$  be the set of 120 students, and  $F = 2^S$  be the power set of  $S$ . The function  $f : F \rightarrow F$  is defined as  $f(X) = \bigcup_{u \in X} N(u)$ , where  $N(u)$  is the set of students loved by student  $u$ . This defines a directed graph  $G = (S, E)$  where  $(u, v) \in E$  if  $u$  loves  $v$ .

#### Case 6: proof

The function  $f$  is a linear map over the vector space  $\mathbb{F}_2^{120}$ , where subsets are identified with their characteristic vectors. The union operation corresponds to vector addition. A 'lovely relationship' between two sets  $A, B \in F$  means that one can be reached from the other by repeated application of  $f$ . A collection of distinct sets  $\{A_1, \dots, A_t\}$  where any pair has a lovely relationship must form a chain under this reachability relation. We can reorder these sets to form a simple path in the functional graph of  $f$ . The maximum possible value of  $t$ , denoted  $M(f)$ , is the length of the longest such path. A maximal path can be written as  $A_0, A_1, \dots, A_{t-1}$ , where  $A_{i+1} = f(A_i)$  for  $i = 0, \dots, t-2$ , and all  $A_i$  are distinct.

Our proof strategy is to analyze the structure of the graph  $G$  by decomposing it into a core part and an acyclic part. We will use the properties of the iterates of  $f$  on the whole set  $S$ .

Let  $L_0 = S$ , and define a sequence of subsets  $L_k = f^k(S)$  for  $k \geq 1$ . The function  $f$  is monotone, i.e., if  $X \subseteq Y$ , then  $f(X) \subseteq f(Y)$ . Since  $L_1 = f(S) \subseteq S = L_0$ , monotonicity implies  $f(L_1) \subseteq f(L_0)$ , so  $L_2 \subseteq L_1$ . This gives a descending chain of subsets:  $S = L_0 \supseteq L_1 \supseteq L_2 \supseteq \dots$ .

Since  $S$  is finite, this sequence must stabilize. Let  $p$  be the first integer such that  $L_p = L_{p+1}$ . Let  $C_f = L_p$ . This is the image core of the graph under  $f$ . From  $L_{p+1} = f(L_p)$ , the stability condition  $L_p = L_{p+1}$  means  $f(C_f) = C_f$ . This implies that the function  $f$ , when restricted to the set  $C_f$ , acts as a permutation on the elements of  $C_f$ . Let's call this permutation  $\pi$ . Let  $x \in C_f$ . Since  $f(C_f) = C_f$ , every element in  $C_f$  must have at least one preimage in  $C_f$ . Since  $C_f$  is finite, this means  $f$  restricted to the elements of  $C_f$  is a bijection, i.e., a permutation  $\pi$ . This implies that for each  $x \in C_f$ , its neighborhood  $N(x)$  must be a singleton set  $\{\pi(x)\}$ , and  $\pi(x) \in C_f$ .

Now we analyze the structure of any trajectory  $A_0, A_1, \dots, A_{t-1}$ . Let  $m$  be the first index in the sequence such that  $A_m$  is a subset of  $C_f$ . So  $A_i \not\subseteq C_f$  for  $i < m$ , and  $A_m \subseteq C_f$ .

The property  $f(C_f) = C_f$  implies that for any  $u \in C_f$ ,  $N(u) \subseteq C_f$ . Let  $X \subseteq C_f$ . Then  $f(X) = \bigcup_{u \in X} N(u) \subseteq C_f$ . This means that once a set in the trajectory is a subset of  $C_f$ , all subsequent sets in the trajectory are also subsets of  $C_f$ . So  $A_i \subseteq C_f$  for all  $i \geq m$ . The total length of the trajectory is  $t = m + (t - m)$ . The first part of the trajectory has length  $m$ , and the second part has length  $t - m$ . The two parts are disjoint because sets in the first part are not subsets of  $C_f$ , while sets in the second part are.

Let's bound the length of the pre-core part of the trajectory,  $m$ . Let  $U = S \setminus C_f$ . For  $i < m$ ,  $A_i \cap U \neq \emptyset$ . Let  $A_i^U = A_i \cap U$  and  $A_i^C = A_i \cap C_f$ .  $A_{i+1} = f(A_i) = f(A_i^C \cup A_i^U) = f(A_i^C) \cup f(A_i^U)$ . Since  $N(C_f) \subseteq C_f$ , we have  $f(A_i^C) \subseteq C_f$ . Therefore, the part of  $A_{i+1}$  outside the core is determined solely by the part of  $A_i$  outside the core:  $A_{i+1}^U = A_{i+1} \cap U = (f(A_i^C) \cup f(A_i^U)) \cap U = f(A_i^U) \cap U$ . The sequence of non-core parts  $(A_i^U)_{i \geq 0}$  evolves autonomously.

The set  $C_f = L_p = \bigcap_{k \geq 0} L_k$  consists of all vertices that can be reached by paths of arbitrary length from  $S$ . This means any vertex not in  $C_f$  cannot reach a cycle. The subgraph  $G[U]$  induced by  $U = S \setminus C_f$  must be a Directed Acyclic Graph (DAG).

Let  $v_1, v_2, \dots, v_{|U|}$  be a topological sort of the vertices in  $U$ . For any edge  $(v_i, v_j)$  in  $G[U]$ , we have  $i < j$ . For any non-empty subset  $Y \subseteq U$ , define its rank as  $\min_r(Y) =$

$\min\{i \mid v_i \in Y\}$ . Let  $Y_i = A_i^U$ . Suppose  $Y_i \neq \emptyset$ . Let  $j = \min_r(Y_i)$ .  $Y_{i+1} = f(Y_i) \cap U = \bigcup_{u \in Y_i} (N(u) \cap U)$ . Any vertex  $w \in N(v_j) \cap U$  must have an index greater than  $j$ . The same applies to any other  $u \in Y_i$  with index greater than  $j$ . Thus, for any vertex in  $Y_{i+1}$ , its index in the topological sort is greater than  $j$ . This means  $\min_r(Y_{i+1}) > \min_r(Y_i)$ . The sequence of ranks  $\min_r(A_0^U), \min_r(A_1^U), \dots$  is strictly increasing as long as the sets are non-empty. Since the rank is at most  $|U|$ , the sequence of non-empty sets  $(A_i^U)$  can have length at most  $|U|$ .  $A_{m-1}^U \neq \emptyset$  and  $A_m^U = \emptyset$ . So  $m \leq |U|$ . Let  $k = |C_f|$ . Then  $|U| = 120 - k$ . We have  $m \leq 120 - k$ .

Now let's bound the length of the core part of the trajectory,  $t - m$ . For  $i \geq m$ , the sets  $A_i$  are subsets of  $C_f$ . As we established, for any  $x \in C_f$ ,  $N(x) = \{\pi(x)\}$  where  $\pi$  is a permutation of  $C_f$ . So, for any  $X \subseteq C_f$ ,  $f(X) = \bigcup_{x \in X} \{\pi(x)\} = \pi(X)$ . The trajectory for  $i \geq m$  is given by  $A_{i+1} = \pi(A_i)$ . The sequence is  $A_m, \pi(A_m), \pi^2(A_m), \dots, A_{t-1}$ . Since these sets must be distinct, the length  $t - m$  is bounded by the order of the permutation  $\pi$  in the symmetric group on  $2^{C_f}$ . This is the order of the element  $A_m$ . The maximum length is the maximum cycle length in the permutation group induced by  $\pi$  on  $2^{C_f}$ . This is bounded by the order of  $\pi$  acting on  $C_f$ . The order of  $\pi \in S_k$  is the least common multiple (lcm) of the lengths of its disjoint cycles. The maximum possible order for a permutation of  $k$  elements is given by Landau's function  $g(k)$ , defined as  $g(k) = \max_{n_1 + \dots + n_r = k} \text{lcm}(n_1, \dots, n_r)$ . So,  $t - m \leq g(k)$ .

Combining the bounds for both parts, we get:  $t = m + (t - m) \leq (120 - k) + g(k)$ . We need to find an upper bound for  $h(k) = 120 - k + g(k)$  for  $k \in \{0, 1, \dots, 120\}$ . Landau's function  $g(k)$  grows subexponentially. The asymptotic growth is  $\ln g(k) \sim \sqrt{k} \ln k$ . Let's check some values:  $h(0) = 120 - 0 + g(0) = 120 + 1 = 121$  (a trajectory on  $\emptyset$  has length 1).  $h(1) = 120 - 1 + g(1) = 119 + 1 = 120$ .  $h(10) = 110 + g(10) = 110 + \text{lcm}(2, 3, 5) = 110 + 30 = 140$ .  $h(20) = 100 + g(20) = 100 + \text{lcm}(3, 4, 5, 7) = 100 + 420 = 520$ .  $h(40) = 80 + g(40) = 80 + \text{lcm}(3, 5, 7, 8, 11) = 80 + 9240 = 9320$ .  $h(70) = 50 + g(70)$ .  $g(70) = \text{lcm}(3, 4, 5, 7, 11, 13, 17) = 1161160$ .  $h(70) = 1161210$ . For  $k = 120$ ,  $h(120) = g(120)$ . A partition of 120 into prime powers that gives a large lcm is 3, 4, 5, 7, 11, 13, 17, 19, 23. The sum is 102. Remaining 18. Add 1.  $g(102)$  is huge. For  $k = 120$ ,  $g(120)$  is known to be  $\approx 3.2 \times 10^{10}$ .

In all cases, the value of  $h(k) = 120 - k + g(k)$  is significantly smaller than  $2^{70}$ .  $2^{10} \approx 10^3$ .  $2^{70} = (2^{10})^7 \approx (10^3)^7 = 10^{21}$ . The maximum value of  $h(k)$  for  $k \in [0, 120]$  is many orders of magnitude smaller than  $2^{70}$ . Therefore, we have shown that for any graph  $G$ ,  $M(f) \leq \max_{0 \leq k \leq 120} (120 - k + g(k)) \ll 2^{70}$ .

This shows that  $M(f) \leq 2^{70}$ .

#### Case 6: review from qwen3-30b-a3b

<verification>false</verification>

The solution incorrectly claims that the function  $f$  corresponds to vector addition in  $\mathbb{F}_2^{120}$ , which is invalid because  $f$  is defined via union (not symmetric difference). It also wrongly asserts that  $f$  restricted to the core  $C_f$  is a permutation, which fails to account for possible non-injective behavior of  $f$ . Additionally, the analysis of the core assumes each  $N(x)$  is a singleton, which is not guaranteed by the problem's conditions. These critical errors undermine the proof's validity."

The case shows that model may misunderstand definitions if the explanation is unclear in some sense. Actually, the human expert misunderstands it as well at first.

#### Case 6: our comment

The proof is true. The model misunderstands the definition of the function  $f$ . However, It can easily be misunderstood due to the unclear explanation.