# A Local Parametrization of the State-Feedback Matrices in the Pole Assignment Problem

I. Baragaña [*] [†] F. Puerta [‡] [§]

I. Zaballa [¶] [‖]

[*] Departamento de Ciencia de la Computación e IA
University of the Basque Country UPV/EHU
Apdo. 649. 20080 Donostia-San Sebastián. Spain

[‡] Departament de Matemàtica Aplicada I.
E.T.S. Enginyeria Industrial de Barcelona. UPC
Institut d'Organització i Control(IOC) UPC
Diagonal 647. 08028 Barcelona. Spain

[¶] Departamento de Matemáticas.
University of the Basque Country UPV/EHU
Apdo. 644. 48080 Bilbao. Spain

November 26, 2025

**Abstract**

Given a controllable system $(F, G)$, a local parametrization is obtained for the set of the feedback gain matrices $K$ such that the state matrix, $F + GK$, of the closed-loop system is in a prescribed similarity class. It is shown that this set can be endowed with the structure of a differentiable manifold whose dimension is also computed. Then a local parametrization and a local system of coordinates are obtained using a diffeomorphism between this set of state-feedback matrices and the orbit space of a set of truncated observability matrices via the action of a Lie group.

*Keywords:* Linear systems, pole assignment, controllability indices, Brunovsky indices, differentiable manifold, local parametrization, local coordinate system.

*MSC:* 93B27, 93B55, 93C05

[†]e-mail: itziar.baragana@ehu.eus. Supported by the Agencia Estatal de Investigación of Spain MCIN/AEI/10.13039/501100011033/ and by "ERDF A way of making Europe" of the "European Union" through grant PID2021-124827NB-I00.

[§]e-mail: ferran.puerta@upc.edu.

[‖]Corresponding author, e-mail: ion.zaballa@ehu.eus. Supported by the Agencia Estatal de Investigación of Spain MCIN/AEI/10.13039/501100011033/ and by "ERDF A way of making Europe" of the "European Union" through grant PID2021-124827NB-I00.

# 1 Introduction

One of the most classic problems in the literature on linear control systems is the *pole assignment problem* (also known as the *pole placement problem*). Assume that we are given a linear, time-invariant control system

$$\dot{x}(t) = Fx(t) + Gu(t), \quad F \in \mathbb{R}^{n\times n}, G \in \mathbb{R}^{n\times m}$$

where $\mathbb{R}$ is the field of real numbers and $\mathbb{R}^{p\times q}$ the set of $p \times q$ matrices over $\mathbb{R}$. The system above will be identified with the pair of matrices $(F, G)$. Assume that we are also given a self-conjugate sequence $\Lambda$ of $n$ complex numbers (i.e., such that $\overline{\Lambda} = \Lambda$), the pole placement problem is to find a state-feedbak matrix $K$ such that the eigenvalues of $F + GK$ are the complex numbers of $\Lambda$. If the given system $(F, G)$ is controllable, it is well known that such a state-feedback matrix always exists.

A more general and difficult problem, sometimes called *the general pole assignment problem*, consists of assigning not only the eigenvalues but the complete similarity class for $F + GK$. Recall that two matrices $A, A' \in \mathbb{R}^{n\times n}$ are said to be *similar* if $A' = T^{-1}AT$ with $T \in \mathrm{Gl}(n)$, the general linear group of all invertible matrices in $\mathbb{R}^{n\times n}$. A complete system of invariants for matrix similarity is given by the invariant polynomials (see, for example, [9, Ch. 6]). Thus, the orbit of $A$ under the similarity action only depends on its invariant polynomials $\underline{\alpha} : \alpha_1(s) \mid \cdots \mid \alpha_n(s)$ and will be denoted by $\mathcal{O}(\underline{\alpha})$. Therefore, given the system $(F, G)$, the general pole assignment problem is to find $K \in \mathbb{R}^{m\times n}$ such that $F+GK \in \mathcal{O}(\underline{\alpha})$ for a given sequence of monic polynomials $\underline{\alpha} : \alpha_1(s) \mid \cdots \mid \alpha_n(s)$ such that $\sum_{i=1}^{n} \deg(\alpha_i(s)) = n$.

Necessary and sufficient conditions for such a matrix $K$ to exist were given by Rosenbrock ([21]) when the system $(F, G)$ is controllable and by Zaballa ([23]) when this controllability restriction is removed (see Proposition 2.9 below). In both cases the proofs are constructive (see also [7, 8] for alternative geometric proofs of Rosenbrock's result). In general, if there is a matrix $K$ such that $F + GK$ is in a prescribed similarity class, it is not unique. We aim to *locally parameterize* this set of feedback gain matrices. To achieve this, we first study the geometry of the set of state-feedback matrices for the given controllable system $(F, G)$:

$$\mathcal{H}_{(F,G)} = \{K \in \mathbb{R}^{m\times n} \ : \ F + GK \in \mathcal{O}(\underline{\alpha})\}. \tag{1}$$

Specifically, we aim to show that $\mathcal{H}_{(F,G)}$ can be endowed with the structure of a differentiable manifold making it is an (immersed) submanifold of $\mathbb{R}^{m\times n}$. In addition, its dimension will be computed. It is a well-known general result that immersed submanifolds admit *local parametrizations* and *local systems of coordinates* (see, for example[15, Lemma 8.18]). In this paper, a local parametrization for $\mathcal{H}_{(F,G)}$ is obtained using a diffeomorphism of this set and an orbit space of matrices via the action of a Lie group.

Having a local parametrization defined in $\mathcal{H}_{(F,G)}$ may be interesting in several respects. For example, if $K \in \mathcal{H}_{(F,G)}$ and $K'$ is a small, arbitrary perturbation of $K$, then $K'$ may not be in $\mathcal{H}_{(F,G)}$. A differentiable structure and a local parametrization in $\mathcal{H}_{(F,G)}$ can be used to determine the possible perturbations of $K$ that remain in $\mathcal{H}_{(F,G)}$; i.e., those such that $F + GK$ and $F + GK'$ have the same invariant polynomials. On the other hand, a local coordinate system provides the minimum number of parameters required to fully describe the perturbed feedback matrices $K' \in \mathcal{H}_{(F,G)}$. A local parametrization may also be useful to determine the optimal, in some sense, feedback gain matrix $K$. This idea is explored in [17] in relation to the *optimal pole placement problem* (see also the references therein). When tackling this problem the starting point is usually to parametrize the set of allowable matrices $K$.

The rest of the paper is organized as follows. Section 2 presents the notation and the necessary preliminary results. In particular, it reviews the complex and real Jordan and Weyr canonical forms and their centralizers, recalls the feedback equivalence of linear control systems and the statement of Rosenbrock's theorem on pole assignment, and introduces a new Brunovsky canonical form. In Section 3 we study the geometry of $\mathcal{H}_{(F,G)}$ in (1) for a given controllable system $(F,G)$. It is shown that this set is a differentiable submanifold of $\mathbb{R}^{m\times n}$ and we also compute its dimension. The final three sections are dedicated to obtaining a local parametrization and a local system of coordinates for $\mathcal{H}_{(F,G)}$. First, we will show in Section 4 that $\mathcal{H}_{(F,G)}$ is diffeomorphic to an orbit space of truncated observability matrices via the action of a Lie group. This Lie group consists of the invertible matrices that commute with the state matrix of the observability system from which the truncated observability matrix is derived. Using this action a unique local reduced form is found in Section 5 for each orbit. This reduced form provides a local parametrization of the orbit space of truncated observability matrices that is translated in Section 6 to a local parametrization and a local system of coordinates of $\mathcal{H}_{(F,G)}$ by means of the diffeomorphism found in Section 4. An example that illustrates the whole process is provided in Section 6.

# 2 Notation and Preliminary results

## 2.1 Partitions

If $s$ and $p$ are positive integers ($0 < s \leq p$), $Q_{s,p} := \{(i_1,\ldots,i_s) : 1 \leq i_1 < \cdots < i_s \leq p\}$ and $Q_{0,p} := \{\emptyset\}$. If $A \in \mathbb{R}^{n\times m}$, $I \in Q_{s,n}$ and $J \in Q_{r,m}$ then $A(I,J)$ will denote the $s \times r$ submatrix of $A$ formed by the rows in $I$ and the columns in $J$; that is, if $I = (i_1,\ldots,i_s)$, $J = (j_1,\ldots,j_r)$ and $B = A(I,J) \in \mathbb{R}^{s\times r}$ then $b_{k\ell} = a_{i_k j_\ell}$, $k = 1,\ldots,s$ and $\ell = 1,\ldots,r$. Similarly, $A(I,:) \in \mathbb{R}^{s\times m}$ and $A(:,J) \in \mathbb{R}^{n\times r}$ are the submatrices of $A$ formed by the rows in $I$ (and all columns) and the columns in $J$ (and all rows), respectively. If $p \leq q$ are integers the symbol $p : q$ denotes the sequence $(p, p+1,\ldots,q)$.

It is well known that a *partition* is a finite or infinite sequence $\underline{a} = (a_1, a_2, \ldots,)$ of nonnegative integers almost all zero. In this manuscript we will only use partitions whose components are arranged in nonincreasing order. The sequence $\underline{a}$ is said to be a partition of $n$ if $n = \sum_{i\geq 1} a_i$. If $\underline{a}$ and $\underline{b}$ are partitions then $\underline{a} + \underline{b} = (a_1 + b_1, a_2 + b_2, \ldots)$ and $\underline{a} \cup \underline{b}$ is the partition whose elements are those of $\underline{a}$ and those of $\underline{b}$ (with possible repetitions) reordered so that they do not increase. If $\underline{a}$ is a partition of $n$ and we define $b_i = \#\{j : a_j \geq i\}$, where # stands for cardinality, then $\underline{b} = (b_1, b_2, \ldots)$ is said to be the *conjugate* or *dual* partition of $\underline{a}$ and $\sum_{i\geq 1} b_i = n$. It is well-known (see, for example, [18, Section 7.B]), that the conjugation of non-increasingly ordered partitions is an involution; i.e., if $\underline{b}$ is the conjugate partition of $\underline{a}$ then the latter is the conjugate partition of the former. On the other hand, if $\underline{a} = (a_1, a_2, \ldots, a_n)$ and $\underline{b} = (b_1, b_2, \ldots, b_n)$ are finite partitions with $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$ and $b_1 \geq b_2 \geq \cdots \geq b_n \geq 0$, following [12] (see also [18, Chapter 1]), we say that $\underline{a}$ is majorized by $\underline{b}$, and we write $\underline{a} \prec \underline{b}$, if

$$\sum_{j=1}^{k} a_j \leq \sum_{j=1}^{k} b_j \quad 1 \leq k \leq m, \qquad \text{and} \qquad \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i. \tag{2}$$

It should be noted that we can always assume that any two finite partitions have the same number of components by adding or deleting zeros. The following result is well-known (see, for example, [16, p. 6,7]):

**Proposition 2.1** *If $\underline{a}$ and $\underline{b}$ are partitions then*

*(i)* $(\underline{a} + \underline{b})^* = \underline{a}^* \cup \underline{b}^*$

*(ii)* $\underline{a} \prec \underline{b} \quad \Leftrightarrow \quad \underline{b}^* \prec \underline{a}^*$

*where $\underline{a}^*$, $\underline{b}^*$ and $(\underline{a} + \underline{b})^*$ are the conjugate partitions of $\underline{a}^*$, $\underline{b}^*$ and $(\underline{a} + \underline{b})^*$, respectively.*

A proof for item (ii) above can also be found in[18, Section 7.B].

## 2.2 Real Jordan and Weyr Canonical Forms

A canonical form for the similarity of real square matrices is the *real Jordan canonical form* (see, for instance, [13, Theorem 6.7.1] or [10, Theorem[12.2.2]). Let $A \in \mathbb{R}^{n\times n}$ be a matrix with invariant polynomials $\underline{\alpha} : \alpha_1(s) \mid \cdots \mid \alpha_n(s)$, let $\lambda_1, \ldots, \lambda_t \in \mathbb{C}$ be the distinct eigenvalues of $A$ and split $\alpha_i(s)$ as a product of powers of irreducible polynomials

$$\alpha_{n-i+1}(s) = (s-\lambda_1)^{m_{1i}}(s-\lambda_2)^{m_{2i}} \cdots (s-\lambda_t)^{m_{t\,i}}, \quad 1 \leq i \leq n.$$

Then $m_{i1} \geq m_{i2} \geq \cdots \geq m_{i\,w_i} > 0 = m_{i\,w_{i+1}} = \cdots = m_{i\,n}$ for $i = 1, \ldots, t$ and the sequence $\big((m_{11}, \ldots, m_{1w_1}), \ldots, (m_{t1}, \ldots, m_{t\,w_t})\big)$ is known as *the Segre characteristic* of $A$. If $n_i$ is the algebraic multiplicity of $\lambda_i$ then the Segre characteristic of $A$, $(m_{i1}, \ldots, m_{i\,w_i})$, is a partition of $n_i$. Its conjugate partition will be denoted by $(w_{i1}, \ldots, w_{i\,m_i})$ (observe that $w_{i1} = w_i$ and $m_{i1} = m_i$).The sequence of partitions $\big((w_{11}, \ldots, w_{1m_1}), \ldots, (w_{t1}, \ldots, w_{t\,m_t})\big)$ is called *the Weyr characteristic* of $A$ (see, for example, [22]). Each of these characteristics has an associated canonical form: the Jordan canonical form for the Segre characteristic and the Weyr canonical form for the Weyr characteristic. If the eigenvalues are all real; i. e., $\lambda_1, \ldots, \lambda_t \in \mathbb{R}$, then the well-known Jordan canonical form of $A$ is $J = \operatorname{diag}\big(J(\lambda_1), \ldots, J(\lambda_t)\big)$ where

$$\begin{aligned} J(\lambda_i) &= \operatorname{diag}\big(J_1(\lambda_i), \ldots, J_{w_i}(\lambda_i)\big) \in \mathbb{R}^{n_i\times n_i}, \quad 1 \leq i \leq t \\ J_k(\lambda_i) &= \begin{bmatrix} \lambda_i & 1 & \cdots & 0 & 0 \\ 0 & \lambda_i & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_i & 1 \\ 0 & 0 & \ldots & 0 & \lambda_i \end{bmatrix} \in \mathbb{R}^{m_{ik}\times m_{ik}}, \quad 1 \leq k \leq w_i. \end{aligned} \tag{3}$$

The Weyr canonical form is, perhaps, less known but it is more convenient for our developments (see [20, Chapter 2] or [22]; we willl use the notation of the latter): $W = \operatorname{diag}\big(W(\lambda_1), \ldots, W(\lambda_t)\big)$ where, for $1 \le i \le t$,

$$W(\lambda_i) = \begin{bmatrix} \lambda_i I_{w_{i1}} & I_{w_{i1},w_{i2}} & \cdots & 0 & 0 \\ 0 & \lambda_i I_{w_{i2}} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_i I_{w_{i\,m_i-1}} & I_{w_{i\,m_i-1},w_{i\,m_i}} \\ 0 & 0 & \cdots & 0 & \lambda_i I_{w_{i\,m_i}} \end{bmatrix} \in \mathbb{R}^{n_i \times n_i}, \quad (4)$$

and for $p \ge q$

$$I_{p,q} = \begin{bmatrix} I_q \\ 0 \end{bmatrix} \in \mathbb{R}^{p \times q}. \quad (5)$$

The Jordan and Weyr canonical forms of $A$ are closely related. The following lemma shows that they can be obtained from each other by a permutation similarity. This is a well-known result (see, for example, [20, Chapter 2] or [19] for a generalization to arbitrary fields). We offer a simple proof that will be used to define the real Weyr canonical form.

**Lemma 2.2** *([20, Chapter 2]) Let $H \in \mathbb{R}^{n\times n}$ be a matrix with $\lambda_0 \in \mathbb{R}$ as its only eigenvalue and let $(m_1, \ldots, m_w)$ and $(w_1, \ldots, w_m)$ be their Segre and Weyr characteristics, respectively, where $m = m_1$ and $w = w_1$. Let $J$ and $W$ be the Jordan and Weyr canonical forms of $H$ and define*

$$s_i = w_1 + \ldots + w_i, \quad i = 1, \ldots, m. \quad (6)$$

*Let $e_k$ be the $k$-th column of $I_n$, $k = 1, \ldots, n$, and*

$$\begin{array}{c} Q = \begin{bmatrix} Q_1^T & Q_2^T & \cdots & Q_w^T \end{bmatrix}^T, \\ Q_i = \begin{bmatrix} e_i & e_{s_1+i} & \cdots & e_{s_{m_i-1}+i} \end{bmatrix}^T \in \mathbb{R}^{m_i \times n}, \ \ 1 \le i \le w \end{array} \quad (7)$$

*where $^T$ stands for transpose. Then $W = Q^T J Q$ .*

**Proof.** Since $J$ and $W$ are the Jordan and Weyr canonical forms of $H$, they are similar. Put $\widetilde{J} = \lambda_0 I_n - J$ and $\widetilde{W} = \lambda_0 I_n - W$. The matrices $T \in \mathrm{Gl}(n)$ such that $\widetilde{J}T = T\widetilde{W}$ form an open set of the following linear subspace of dimension $wn$:

$$\mathcal{T}(\widetilde{W}, \underline{m}) = \left\{ T = \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_w \end{bmatrix}, T_i = \begin{bmatrix} t_i \\ t_i\widetilde{W} \\ \vdots \\ t_i \widetilde{W}^{m_i-1} \end{bmatrix}, t_i \in \mathbb{R}^{1\times n}, 1 \le i \le w \right\} \quad (8)$$

It follows from the definition of $\widetilde{W}$ that, for $1 \le i \le w$, $1 \le j \le m_i - 1$, $e_{s_{j-1}+i}^T \widetilde{W} = e_{s_j+i}^T$ ($s_0 = 0$) and so $e_{s_j+i}^T = e_i^T \widetilde{W}^j$. Hence, $Q_i = \begin{bmatrix} e_i^T \\ e_i^T \widetilde{W} \\ \vdots \\ e_i^T \widetilde{W}^{m_i-1} \end{bmatrix}$, $1 \le i \le w$ and $Q \in \mathcal{T}(\widetilde{W}, \underline{m})$. Therefore $Q^T J Q = W$ as desired. □

Assume now that $A \in \mathbb{R}^{n\times n}$ has real and nonreal eigenvalues: $\lambda_1, \ldots, \lambda_p \in \mathbb{R}$ and $\lambda_{p+1}, \ldots, \lambda_{p+q}, \overline{\lambda}_{p+1}, \ldots, \overline{\lambda}_{p+q} \in \mathbb{C} \setminus \mathbb{R}$ where $\overline{\lambda}_i$ stands for the complex conjugate of $\lambda_i$ and put $t = p + 2q$. In this case the prime factorization of $\alpha_{n-i+1}(s)$, $i = 1, \ldots, n$, would be

$$\alpha_{n-i+1}(s) = (s-\lambda_1)^{m_{1i}} \cdots (s-\lambda_p)^{m_{pi}} (s^2+c_1s+c_2)^{m_{p+1\,i}} \cdots (s^2+c_{2q-1}s+c_{2q})^{m_{p+q\,i}}, \tag{9}$$

where we can assume without loss of generality that $s^2 + c_{2k-1}s + c_{2k} = (s - \lambda_{p+k})(s - \overline{\lambda}_{p+k})$, $k = 1, \ldots, q$. Then $m_{i1} \geq m_{i2} \geq \cdots \geq m_{i\,w_i} > 0 = m_{i\,w_{i+1}} = \cdots = m_{in}$ for $i = 1, \ldots, t = p + 2q$ and $\big((m_{11}, \ldots, m_{1w_1}), \ldots, (m_{t1}, \ldots, m_{t\,w_t})\big)$ is the Segre characteristic of $A$. Note that for $k = p+1, \ldots, t = p+2q$, the Segre characteristics of $A$ for the eigenvalues $\lambda_k$ and $\overline{\lambda}_k$ coincide; i.e., $m_{kj} = m_{k+q\,j}$ for $j = 1, \ldots, w_k$. Then the Real Jordan canonical form of $A$ is (see [13, Theorem 6.7.1] or [10, Theorem 12.2.2]):

$$J_R = \operatorname{diag}\Big(J(\lambda_1), \ldots, J(\lambda_p), \widehat{J}(\lambda_{p+1}, \overline{\lambda}_{p+1}), \ldots, \widehat{J}(\lambda_{p+q}, \overline{\lambda}_{p+q})\Big), \tag{10}$$

where $J(\lambda_i)$ are the matrices of (3),

$$\widehat{J}(\lambda_j, \overline{\lambda}_j)) = \operatorname{diag}\Big(\widehat{J}_1(\lambda_j, \overline{\lambda}_j), \ldots, \widehat{J}_{w_j}(\lambda_j, \overline{\lambda}_j)\Big), \quad p+1 \leq j \leq p+q,$$

and, if $\lambda_j = a_j + b_j i \in \mathbb{C} \setminus \mathbb{R}$ then

$$\widehat{J}_k(\lambda_j, \overline{\lambda}_j) = \begin{bmatrix} B_j & I_2 & 0 & \ldots & 0 & 0 \\ 0 & B_j & I_2 & \ldots & 0 & 0 \\ 0 & 0 & B_j & \ldots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & B_j & I_2 \\ 0 & 0 & 0 & \ldots & 0 & B_j \end{bmatrix} \in \mathbb{R}^{2m_{jk} \times 2m_{jk}}, \quad 1 \leq k \leq w_i, \tag{11}$$

with $B_j = \left[\begin{smallmatrix} a_j & b_j \\ -b_j & a_j \end{smallmatrix}\right]$.

Now, for $i = 1, \ldots, t = p+2q$, let $(w_{i1}, \ldots, w_{i\,m_i})$ be the conjugate partition of $(m_{i1}, \ldots, m_{i\,w_i})$ ($m_i = m_{i1}$ and $w_i = w_{i1}$). Then, as in the case when all eigenvalues are real, $\big((w_{11}, \ldots, w_{1m_1}), \ldots, (w_{t1}, \ldots, w_{t\,m_t})\big)$ is the Weyr characteristic of $A$. Weyr canonical forms for matrices over arbitrary fields where studied in [19]. However an explicit definition of a real Weyr canonical form is not provided. It can be obtained, after some manipulations, by applying the technique of Section 3 in that paper to the *Generalized Jordan canonical form of the first kind* in [19, Theorem 2.7] for matrices with real entries . We take a more direct approach generalizing Lemma 2.2 to the case of nonreal eigenvalues.

Recall that if $X \in \mathbb{R}^{m \times n}$ the Kronecker product $I_p \otimes X = \operatorname{diag}(\overbrace{X, \ldots, X}^{p})$. For notational simplicity we will use the notation

$$X^{(p)} = I_p \otimes X = \operatorname{diag}(\overbrace{X, \ldots, X}^{p}). \tag{12}$$

**Lemma 2.3** *Let $H \in \mathbb{R}^{2n \times 2n}$ be a matrix with $\lambda_0, \overline{\lambda}_0 \in \mathbb{C} \setminus \mathbb{R}$ as its only eigenvalues and let $(m_1, \ldots, m_w)$ and $(w_1, \ldots, w_m)$ be the common Segre and Weyr characteristics of $H$ for both $\lambda_0$ and $\overline{\lambda}_0$. Assume that $\lambda_0 = a_0 + b_0 i$ and let $B_0 = \left[\begin{smallmatrix} a_0 & b_0 \\ -b_0 & a_0 \end{smallmatrix}\right]$. Let the real Jordan canonical form of $H$ be $\widehat{J}(\lambda_0, \overline{\lambda}_0) =$*

diag $\left(\widehat{J}_1(\lambda_0,\overline{\lambda}_0),\ldots,\widehat{J}_w(\lambda_0,\overline{\lambda}_0)\right)$ *where* $\widehat{J}_k(\lambda_0,\overline{\lambda}_0)$ *is the matrix of* (11) *with* $j=0$ *and of size* $2m_k\times 2m_k$, $k=1,\ldots,w$. *Define*

$$\widehat{W}(\lambda_0,\overline{\lambda}_0)=\begin{bmatrix} B_0^{(w_1)} & I_{2w_1,2w_2} & \ldots & 0 & 0\\ 0 & B_0^{(w_2)} & \ldots & 0 & 0\\ \vdots & \vdots & \ddots & \ddots & \vdots\\ 0 & 0 & \ldots & B_0^{(w_{m-1})} & I_{2w_{m-1},2w_m}\\ 0 & 0 & \ldots & 0 & B_0^{(w_m)}\end{bmatrix}, \tag{13}$$

*For* $i=1,\ldots,m$, *let* $s_i$ *be the positive integer of* (6). *Let* $E_k=\begin{bmatrix}e_{2k-1} & e_{2k}\end{bmatrix}\in\mathbb{R}^{2n\times 2}$ *where* $e_k$ *is the* $k$*-th column of* $I_{2n}$, $k=1,\ldots,n$, *and*

$$\begin{gathered} Q=\begin{bmatrix}Q_1^T & Q_2^T & \cdots & Q_w^T\end{bmatrix}^T,\\ Q_i=\begin{bmatrix}E_i & E_{s_1+i} & \cdots & E_{s_{m_i-1}+i}\end{bmatrix}^T\in\mathbb{R}^{2m_i\times 2n},\ \ 1\le i\le w.\end{gathered} \tag{14}$$

*Then* $\widehat{W}(\lambda_0,\overline{\lambda}_0)=Q^T\widehat{J}(\lambda_0,\overline{\lambda}_0)Q$.

**Proof.** Let $\widetilde{J}=\widehat{J}(\lambda_0,\overline{\lambda}_0)-B_0^{(n)}$ and $\widetilde{W}=\widehat{W}(\lambda_0,\overline{\lambda}_0)-B_0^{(n)}$. Then the only eigenvalue of $\widetilde{J}$ and $\widetilde{W}$ is 0, the Segre characteristic of $\widetilde{J}$ is $\underline{m}\cup\underline{m}=(m_1,m_1,m_2,m_2,\ldots,m_w,m_w)$ and the Weyr characteristic of $\widetilde{W}$ is $\underline{w}+\underline{w}=(2w_1,2w_2,\ldots,2w_m)$. Since $\underline{m}\cup\underline{m}$ and $\underline{w}+\underline{w}$ are conjugate partitions, $\widetilde{J}$ and $\widetilde{W}$ are similar matrices. As in Lemma 2.2, the set of matrices $T\in\mathrm{Gl}(2n)$ such that $\widetilde{J}T=T\widetilde{W}$ form an open set of a linear subspace of dimension $4wn$:

$$\mathcal{TR}(\widetilde{W},\underline{m}+\underline{m})=\left\{T=\begin{bmatrix}T_1\\T_2\\ \vdots\\ T_w\end{bmatrix}, T_i=\begin{bmatrix}X_i\\ X_i\widetilde{W}\\ \vdots\\ X_i\widetilde{W}^{m_i-1}\end{bmatrix}, X_i\in\mathbb{R}^{2\times 2n}, 1\le i\le w\right\}.$$

Now, as in the proof of Lemma 2.2, for $1\le i\le w$, $1\le j\le m_i-1$, $E_{s_{j-1}+i}^T\widetilde{W}=E_{s_j+i}^T$ $(s_0=0)$, $E_{s_j+i}^T=E_i^T\widetilde{W}^j$, $Q_i=\begin{bmatrix}E_i^T\\ E_i^T\widetilde{W}\\ \vdots\\ E_i^T\widetilde{W}^{m_i-1}\end{bmatrix}$, $1\le i\le w$, $Q\in\mathcal{T}(\widetilde{W},\underline{m})$ and so $Q^T\widetilde{J}Q=\widetilde{W}$. Since $B_0^{(n)}$ is a block diagonal matrix with repeated $2\times 2$ diagonal blocks, it is not difficult to see that $Q^TB_0^{(n)}Q=B_0^{(n)}$ and the Lemma follows. $\square$

In general, if $A\in\mathbb{R}^{n\times n}$, there is permutation matrix $Q\in\mathbb{R}^{n\times n}$such that

$$W_R=Q^TJ_RQ=\mathrm{diag}(W(\lambda_1),\ldots,W(\lambda_p),\widehat{W}(\lambda_{p+1},\overline{\lambda}_{p+1}),\ldots,\widehat{W}(\lambda_{p+q},\overline{\lambda}_{p+q})), \tag{15}$$

$$\widehat{W}(\lambda_j,\overline{\lambda}_j)=\begin{bmatrix} B_j^{(w_{j1})} & I_{2w_{j1},2w_{j2}} & \ldots & 0 & 0\\ 0 & B_j^{(w_{j2})} & \ldots & 0 & 0\\ \vdots & \vdots & \ddots & \ddots & \vdots\\ 0 & 0 & \ldots & B_j^{(w_{j\,m_j-1})} & I_{2w_{j\,m_j-1},2w_{j\,m_j}}\\ 0 & 0 & \ldots & 0 & B_j^{(w_{j\,m_j})}\end{bmatrix}.$$

The matrix $W_R$ will be called the *real Weyr canonical form* of $A$.

## 2.3 The centralizer

Given a matrix $A \in \mathbb{R}^{n\times n}$, we denote by $C_A$ the *centralizer* of $A$, i.e.

$$C_A = \{X \in \mathbb{R}^{n\times n} \,:\, XA = AX\}.$$

It is well-known (see, for example, [9, Ch. 8]) that if $\alpha_1(s) \mid \cdots \mid \alpha_n(s)$ are the invariant polynomials of $A$ then $C_A$ is a subspace of $\mathbb{R}^{n\times n}$ of $\dim C_A = N$, where

$$N = \deg(\alpha_n) + 3\deg(\alpha_{n-1}) + 5\deg(\alpha_{n-2}) \cdots = \sum_{k=1}^{n} (2k-1)\deg(\alpha_{n-k+1}). \quad (16)$$

Let $\widetilde{C}_A$ be the subgroup of $\mathrm{Gl}(n)$ formed by the invertible matrices of $C_A$, $\widetilde{C}_A = \{X \in \mathrm{Gl}(n) \,:\, X \in C_A\}$. Then $\widetilde{C}_A$ is an open subset of a linear manifold and $\dim \widetilde{C}_A = N$. It turns out that (see, for instance, [2, Proposition 3.2], [6, Theorem 2.1] or the proof of Theorem 9.16 in [15]), also $\mathcal{O}(\underline{\alpha})$ is a differentiable manifold of codimension $N$.

Let $A = J_R$ be the matrix in (10). Then it is easily seen that $X \in C_A$ if and only if $X = \mathrm{diag}(X_1, \ldots, X_p, \widehat{X}_{p+1}, \ldots, \widehat{X}_{p+q})$ where $X_i \in C_{J(\lambda_i)}$, $1 \le i \le p$ and $\widehat{X}_i \in C_{\widehat{J}(\lambda_i,\overline{\lambda}_i)}$, $p+1 \le i \le p+q$. Similarly, if $A = W_R$ is the matrix of (15) then $X \in C_A$ if and only if $X = \mathrm{diag}(X_1, \ldots, X_p, \widehat{X}_{p+1}, \ldots, \widehat{X}_{p+q})$ where $X_i \in C_{W(\lambda_i)}$, $1 \le i \le p$ and $\widehat{X}_i \in C_{\widehat{W}(\lambda_i,\overline{\lambda}_i)}$, $p+1 \le i \le p+q$.

Let $\lambda_0 \in \mathbb{R}$ be an eigenvalue of $A$ and let $(m_1, \ldots, m_w)$ be its Segre Characteristic. If $J(\lambda_0) = \mathrm{diag}\,(J_1(\lambda_0), \ldots, J_w(\lambda_0))$ is the block associated to $\lambda_0$ in the Jordan canonical form of $A$, then the characterization of the centralizer of $J(\lambda_0)$, $\mathcal{C}_{J(\lambda_0)}$, can be found in many books (for example in [9, Ch. 8], [10, Theorem 12.4.2] or [13, Ch. 12]). We are interested in the less known characterization of the centralizer of $W(\lambda_0)$, the block associated to $\lambda_0$ in the Weyr canonical form of $A$. Taking into account that $Q^T J(\lambda_0) Q = W(\lambda_0)$ where $Q$ is the matrix of (7), $X \in C_{J(\lambda_0)}$ if and only if $Q^T X Q \in C_{W(\lambda_0)}$. Using this property we get

**Lemma 2.4** *Let $\lambda_0 \in \mathbb{R}$ and $W(\lambda_0)$ be the matrix of (4) with $i = 0$ and Weyr characteristic $(w_1, w_2, \ldots, w_m)$. Then $Y \in C_{W(\lambda_0)}$ if and only if*

$$Y = \begin{bmatrix} Y_{11} & Y_{12} & \cdots & Y_{1,m} \\ 0 & Y_{22} & \cdots & Y_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & Y_{mm} \end{bmatrix} \quad (17)$$

*where*

*(i)*

$$Y_{1j} = \begin{bmatrix} D^{(j)}_{11} & D^{(j)}_{12} & \cdots & D^{(j)}_{1\,m-j+1} \\ \vdots & \vdots & & \vdots \\ D^{(j)}_{j1} & D^{(j)}_{j2} & \cdots & D^{(j)}_{j\,m-j+1} \\ 0 & D^{(j)}_{j+1\,2} & \cdots & D^{(j)}_{j+1\,m-j+1} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & D^{(j)}_{m\,m-j+1} \end{bmatrix}, \quad (18)$$

*and*

$$D_{i,k}^{(j)} \in \mathbb{R}^{(\tau_i-\tau_{i-1})\times(\tau_k-\tau_{k-1})}, \quad 1 \le i,j \le m,\ \max\{i-j+1,1\} \le k \le m-j+1, \tag{19}$$

*with* $\tau_i = w_{m-i+1}$, $0 \le i \le m$ $(w_{m+1} = 0)$.

*(ii) For* $1 \le i \le j \le m-1$

$$Y_{i+1\,j+1} = I_{w_i,w_{i+1}}^T Y_{i\,j} I_{w_j,w_{j+1}}. \tag{20}$$

**Remark 2.5** Condition (20) means that $Y_{ij}$ has de form $Y_{ij} = \left[\begin{smallmatrix} Y_{i+1\,j+1} & * \\ 0 & * \end{smallmatrix}\right]$. So, all distinct parameters of $Y$ are concentrated in $Y_{1j}$, $1 \le j \le m$. The number of parameters in $Y_{1j}$ is $(w_j - w_{j+1})w_1 + (w_{j+1} - w_{j+2})w_2 + \ldots + (w_{m-1} - w_m)w_{m-j} + w_m w_{m-j+1}$. Thus the number of distinct parameters in $Y$ is

$$\sum_{j=1}^{m}(w_j - w_{j+1})w_1 + \sum_{j=1}^{m}(w_{j+1} - w_{j+2})w_2 + \ldots + \sum_{j=1}^{m} w_m w_{m-j+1} = w_1^2 + w_2^2 + \ldots + w_m^2.$$

This is, actually, the value of $N$ in (16) when $A$ has only one eigenvalue. In fact, in that case, if $\underline{m} = (m_1, \ldots, m_w)$ is the Segre characteristic of $A$ then $N = \sum_{j=1}^{n}(2j-1)m_j$. Now, $w_i - w_{i+1} = \#\{j : m_j = i\}$; that is, there are $w_m$ numbers in $\underline{m}$ equal to $m_1 = m$, $w_{m-1} - w_m$ equal to $m-1$, $w_{m-2} - w_{m-1}$ equal to $m-2$ , ..., $w_2 - w_1$ equal to 1 (of course $w_j - w_{j+1}$ can be 0 for some $j$). Hence, with the agreement $\sum_{i=p+1}^{p} := 0$ $(p \ge 0)$, we get

$$\begin{aligned} N &= \sum_{j=1}^{n}(2j-1)m_j = \sum_{j=1}^{w_m}(2j-1)m + \sum_{j=w_m+1}^{w_{m-1}}(2j-1)(m-1) \\ &\quad + \sum_{j=w_{m-1}+1}^{w_{m-2}}(2j-1)(m-2) + \ldots + \sum_{j=w_2+1}^{w_1}(2j-1)1 \\ &= w_m^2 m + (w_{m-1}^2 - w_m^2)(m-1) + (w_{m-2}^2 - w_{m-1}^2)(m-2) \\ &\quad + \ldots + (w_1^2 - w_2^2)1 = w_m^2 + w_{m-1}^2 + w_{m-2}^2 + \ldots + w_1^2. \end{aligned}$$

□

**Example 2.6** Assume that $A \in \mathbb{R}^{12\times12}$ has $\lambda_0 \in \mathbb{R}$ as its only eigenvalue and let $\underline{m} = (4,2,2,2,1,1)$ and $\underline{w} = (6,4,1,1)$ be its Segre and Weyr characteristics, respectively. Then $N = 4 + 3\cdot2 + 5\cdot2 + 7\cdot2 + 9\cdot1 + 11\cdot1 = 6^2 + 4^2 + 1 + 1 = 54$ and the matrices of $C_{W(\lambda_0)}$ have the following form (recall that $\tau_i = w_{m-i+1}$ and so $\tau_1 = \tau_2 = 1$, $\tau_3 = 4$ and $\tau_4 = 6$ and note that $\tau_2 - \tau_1 = 0$):

$$\begin{array}{cc} & \begin{array}{ccccccc} \overset{\tau_1}{1} & \overset{\tau_3-\tau_2}{3} & \overset{\tau_4-\tau_3}{2} & \overset{\tau_1}{1} & \overset{\tau_3-\tau_2}{3} & \overset{\tau_1}{1} & \overset{\tau_1}{1} \end{array} \\ Y = & \left[\begin{array}{ccc|cc|c|c} d_{11}^{(1)} & D_{13}^{(1)} & D_{14}^{(1)} & d_{11}^{(2)} & D_{13}^{(2)} & d_{11}^{(3)} & d_{11}^{(4)} \\ 0 & D_{33}^{(1)} & D_{34}^{(1)} & 0 & D_{33}^{(2)} & D_{31}^{(3)} & D_{31}^{(4)} \\ 0 & 0 & D_{44}^{(1)} & 0 & D_{43}^{(2)} & 0 & D_{41}^{(4)} \\ \hline 0 & 0 & 0 & d_{11}^{(1)} & D_{13}^{(1)} & d_{11}^{(2)} & d_{11}^{(3)} \\ 0 & 0 & 0 & 0 & D_{33}^{(1)} & 0 & D_{31}^{(3)} \\ \hline 0 & 0 & 0 & 0 & 0 & d_{11}^{(1)} & d_{11}^{(2)} \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & d_{11}^{(1)} \end{array}\right] \begin{array}{l} \tau_1 = 1 \\ \tau_3 - \tau_2 = 3 \\ \tau_4 - \tau_3 = 2 \\ \tau_1 = 1 \\ \tau_3 - \tau_2 = 3 \\ \tau_1 = 1 \\ \tau_1 = 1. \end{array} \end{array} \tag{21}$$

□

Let $\lambda_0, \overline{\lambda}_0 \in \mathbb{C} \setminus \mathbb{R}$ be eigenvalues of $A \in \mathbb{R}^{n \times n}$. Let $(m_1, \ldots, m_w)$ and $(w_1, \ldots, w_m)$ be the Segre and Weyr characteristics for both $\lambda_0$ and $\overline{\lambda}_0$. Let $\widehat{J}(\lambda_0, \overline{\lambda}_0) = \operatorname{diag}\left(\widehat{J}_1(\lambda_0, \overline{\lambda}_0), \ldots, \widehat{J}_w(\lambda_0, \overline{\lambda}_0)\right)$ be the block associated to $\lambda_0$ and $\overline{\lambda}_0$ in the real Jordan canonical form of $A$. A characterization of $C_{\widehat{J}(\lambda_0, \overline{\lambda}_0)}$ can be found in several publications (see, for example, [10, Theorem 12.4.2], [19, Theorem 5.6] or [14, Section 3]). As in the case when all eigenvalues are real, we are interested in the centralizer of $\widehat{W}(\lambda_0, \overline{\lambda}_0)$, the block associated to the pair of eigenvalues $\lambda_0$ and $\overline{\lambda}_0$ in the real Weyr canonical form of $A$. Since $Q^T \widehat{J}(\lambda_0, \overline{\lambda}_0) Q = \widehat{W}(\lambda_0, \overline{\lambda}_0)$ where $Q$ is the matrix of (14), $X \in C_{\widehat{J}(\lambda_0, \overline{\lambda}_0)}$ if and only if $Q^T X Q \in C_{\widehat{W}(\lambda_0, \overline{\lambda}_0)}$. Using this property we get

**Lemma 2.7** *Let $\lambda_0, \overline{\lambda}_0 \in \mathbb{C} \setminus \mathbb{R}$ and $\widehat{W}(\lambda_0, \overline{\lambda}_0)$ be the matrix of* (13) *with Weyr characteristic $(w_1, w_2, \ldots, w_m)$ for each eigenvalue $\lambda_0$ and $\overline{\lambda}_0$. Then $Y \in C_{\widehat{W}(\lambda_0, \overline{\lambda}_0)}$ if and only if $Y$ has the structure of* (17) *satisfying the properties* (18)*, for $1 \leq i \leq j \leq m-1$,*

$$Y_{i+1\, j+1} = I^T_{2w_i, 2w_{i+1}} Y_{i\, j} I_{2w_j, 2w_{j+1}},$$

*and for $1 \leq i, j \leq m$ and $\max\{i - j + 1, 1\} \leq k \leq m - j + 1$*

$$D^{(j)}_{i,k} = \left[T^{(j)}_{\alpha\, \beta}\right]_{\substack{\tau_{i-1}+1 \leq \alpha \leq \tau_i \\ \tau_{k-1}+1 \leq \beta \leq \tau_k}} \in \mathbb{R}^{2(\tau_i - \tau_{i-1}) \times 2(\tau_k - \tau_{k-1})}, \tag{22}$$

*where $\tau_0 = 0$, $\tau_i = w_{m-i+1}$, $1 \leq i \leq m$, and $T^{(j)}_{\alpha, \beta} = \begin{bmatrix} x^{(j)}_{\alpha\, \beta} & y^{(j)}_{\alpha\, \beta} \\ -y^{(j)}_{\alpha\, \beta} & x^{(j)}_{\alpha\, \beta} \end{bmatrix} \in \mathbb{R}^{2 \times 2}$.*

Note that if $B = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \in \mathbb{R}^{2\times 2}$ with $b \neq 0$ then $X \in \mathbb{R}^{2 \times 2}$ commutes with $B$ if and only if $X = \begin{bmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{bmatrix}$ for any $x_1, x_2 \in \mathbb{R}$.

On the other hand, because of (16), the dimension of $C_A$ is the same whether it is computed on $\mathbb{C}$ or $\mathbb{R}$. Hence, since $\lambda_0$ and $\overline{\lambda}_0$ have the same associated Segre and Weyr characteristics, the number of parameters in $Y$ when $\lambda_0$ and $\overline{\lambda}_0$ are the only eigenvalues of $A$ is

$$N = 2\sum_{j=1}^{n} (2j-1) m_j = 2(w_m^2 + w_{m-1}^2 + \ldots + w_1^2) = 2(\tau_1^2 + \tau_2^2 + \ldots + \tau_m^2).$$

**Example 2.8** As in Example 2.6, assume that $A \in \mathbb{R}^{12 \times 12}$ has $\lambda_0, \overline{\lambda}_0 \in \mathbb{C} \setminus \mathbb{R}$ as its only eigenvalues and let $\underline{m} = (4, 2, 2, 2, 1, 1)$ and $\underline{w} = (6, 4, 1, 1)$ be the Segre and Weyr characteristics, respectively, for both $\lambda_0$ and $\overline{\lambda}_0$. Then $N = 2(4 + 3\cdot 2 + 5 \cdot 2 + 7 \cdot 2 + 9 \cdot 1 + 11 \cdot 1) = 2(6^2 + 4^2 + 1 + 1) = 108$ and the matrices of $C_{W(\lambda_0)}$ have the following form :

$$\begin{array}{ccc}
 & \begin{array}{ccccccc} \substack{2\tau_1 \\ 2} & \substack{2(\tau_3 - \tau_2) \\ 6} & \substack{2(\tau_4 - \tau_3) \\ 4} & \substack{2\tau_1 \\ 2} & \substack{2(\tau_3 - \tau_2) \\ 6} & \substack{2\tau_1 \\ 2} & \substack{2\tau_1 \\ 2} \end{array} & \\
Y = & \left[\begin{array}{ccc|cc|c|c}
D^{(1)}_{11} & D^{(1)}_{13} & D^{(1)}_{14} & D^{(2)}_{11} & D^{(2)}_{13} & D^{(3)}_{11} & D^{(4)}_{11} \\
0 & D^{(1)}_{33} & D^{(1)}_{34} & 0 & D^{(2)}_{33} & D^{(3)}_{31} & D^{(4)}_{31} \\
0 & 0 & D^{(1)}_{44} & 0 & D^{(2)}_{43} & 0 & D^{(4)}_{41} \\
\hline
0 & 0 & 0 & D^{(1)}_{11} & D^{(1)}_{13} & d^{(2)}_{11} & D^{(3)}_{11} \\
0 & 0 & 0 & 0 & D^{(1)}_{33} & 0 & D^{(3)}_{31} \\
\hline
0 & 0 & 0 & 0 & 0 & D^{(1)}_{11} & D^{(2)}_{11} \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & D^{(1)}_{11}
\end{array}\right] & \begin{array}{c} 2\tau_1 = 2 \\ 2(\tau_3 - \tau_2) = 6 \\ 2(\tau_4 - \tau_3) = 4 \\ 2\tau_1 = 2 \\ 2(\tau_3 - \tau_2) = 6 \\ 2\tau_1 = 2 \\ 2\tau_1 = 2, \end{array}
\end{array}$$

with

$$D_{11}^{(i)} = T_{11}^{(i)}, 1 \le i \le 4, \quad D_{13}^{(i)} = \begin{bmatrix} T_{12}^{(i)} & T_{13}^{(i)} & T_{14}^{(i)} \end{bmatrix}, i = 1, 3, \quad D_{14}^{(1)} = \begin{bmatrix} T_{15}^{(1)} & T_{16}^{(1)} \end{bmatrix}$$

$$D_{33}^{(i)} = \begin{bmatrix} T_{22}^{(i)} & T_{23}^{(i)} & T_{24}^{(i)} \\ T_{32}^{(i)} & T_{33}^{(i)} & T_{34}^{(i)} \\ T_{42}^{(i)} & T_{43}^{(i)} & T_{44}^{(i)} \end{bmatrix}, i = 1, 2, \quad D_{34}^{(1)} = \begin{bmatrix} T_{25}^{(1)} & T_{26}^{(1)} \\ T_{35}^{(1)} & T_{36}^{(1)} \\ T_{45}^{(1)} & T_{66}^{(1)} \end{bmatrix}, \quad D_{31}^{(i)} = \begin{bmatrix} T_{21}^{(i)} \\ T_{31}^{(i)} \\ T_{41}^{(i)} \end{bmatrix}, i = 3, 4$$

$$D_{44}^{(1)} = \begin{bmatrix} T_{55}^{(1)} & T_{56}^{(1)} \\ T_{65}^{(1)} & T_{66}^{(1)} \end{bmatrix}, D_{43}^{(2)} = \begin{bmatrix} T_{52}^{(2)} & T_{53}^{(2)} & T_{54}^{(2)} \\ T_{62}^{(2)} & T_{63}^{(2)} & T_{64}^{(2)} \end{bmatrix}, D_{41}^{(4)} = \begin{bmatrix} T_{51}^{(4)} \\ T_{61}^{(i4)} \end{bmatrix}$$

and

$$T_{\alpha,\beta}^{(i)} = \begin{bmatrix} x_{\alpha,\beta}^{(i)} & y_{\alpha,\beta}^{(i)} \\ -y_{\alpha,\beta}^{(i)} & x_{\alpha,\beta}^{(i)} \end{bmatrix} \in \mathbb{R}^{2\times 2}, \quad 1 \le \alpha, \beta \le 6, \quad 1 \le i \le 4.$$

□

## 2.4 Feedback Equivalence and the Pole Assignment Problem by State-Feedback

Let $\Sigma^c$ be the open subset of $\mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$ formed by the controllable pairs of matrices. That is to say,

$$\Sigma^c = \{(F, G) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \ : \ \text{rank}\begin{bmatrix} G & FG \dots & F^{n-1}G \end{bmatrix} = n\}.$$

Let $\mathcal{G}_c = \left\{ \left[\begin{smallmatrix} P & 0 \\ R & Q \end{smallmatrix}\right] : P \in \text{Gl}(n), Q \in \text{Gl}(m), R \in \mathbb{R}^{m\times n} \right\}$ denote the feedback group. Two pairs of matrices $(F, G), (F', G') \in \Sigma^c$ are said to be *feedback equivalent* if $\left[\begin{smallmatrix} F' & G' \end{smallmatrix}\right] = P^{-1}\left[\begin{smallmatrix} F & G \end{smallmatrix}\right]\left[\begin{smallmatrix} P & 0 \\ R & Q \end{smallmatrix}\right]$ with $\left[\begin{smallmatrix} P & 0 \\ R & Q \end{smallmatrix}\right] \in \mathcal{G}_c$.

It is well-known that the controllability indices form a complete system of invariants for the feedback equivalence relation. However also *the Brunovsky indices* form a complete system of invariants. Both indices are closely related. Let us briefly recall their definitions.

Assume that we are given a controllable system $(F, G) \in \Sigma^c$ and $\text{rank}\, G = r$. For $i = 1, \dots n$ let $r_1 + \dots + r_i = \text{rank}\begin{bmatrix} G & FG & \cdots & F^{i-1}G \end{bmatrix}$ (see [5]). Then there is a positive integer $k$ such that $r = r_1 \ge r_2 \ge \cdots \ge r_k > 0 = r_{k+1} = \dots = r_n$. The nonnegative integers $r_1 \ge r_2 \ge \cdots \ge r_n \ge 0$ are called the Brunovsky indices of $(F, G)$ (they were called $r$-numbers in [4]). As $r_1 + r_2 + \cdots + r_n = n$, $\underline{r} = (r_1, r_2, \dots, r_n)$ is a partition of $n$. The controllability indices of $(F, G)$ are the components of its *conjugate partition*. That is to say, for $i = 1, \dots, m$, $k_i$ is the number of elements of $\underline{r}$ that are not smaller than $i$: $k_i = \#\{j : r_j \ge i\}$. Hence, bearing in mind that $r = r_1$, $k = k_1 \ge k_2 \ge \cdots \ge k_r > 0 = k_{r+1} = \cdots = k_m$.

There are canonical representatives in each feedback equivalence class associated to either the controllability indices or the Brunovsky indices. The so-called and well-known *Brunovsky canonical form* is associated to the controllability indices (see, for instance, [10, Theorem 6.2.5]): Let $(F, G) \in \Sigma^c$ be a controllable pair with controllability indices $\underline{k} : k_1 \ge \cdots \ge k_r > 0 = k_{r+1} = \cdots = k_m$. Then $(F, G)$ is feedback equivalent to $(F_c, G_c)$, where

$$F_c = \text{diag}(J_1(0), \dots, J_r(0)) \in \mathbb{R}^{n\times n}, \quad G_c = \begin{bmatrix} G_1 & 0 \end{bmatrix} \in \mathbb{R}^{n\times(r+(m-r))},$$

$$J_i(0) = \left[\begin{smallmatrix} 0 & I_{k_i-1} \\ 0 & 0 \end{smallmatrix}\right] \in \mathbb{R}^{k_i\times k_i}, \ G_1 = \begin{bmatrix} E_1 \\ \vdots \\ E_r \end{bmatrix} \in \mathbb{R}^{n\times r}, \ E_i = \begin{bmatrix} 0 \\ e_i^T \end{bmatrix} \in \mathbb{R}^{((k_i-1)+1)\times r},$$

and $e_i$ is the $i$-th column of the identity matrix $I_r$, $1 \leq i \leq r$.

Using the permutation matrix of (7) with $m_i = k_i$, we get (compare with [4, Theorem 3.3] that it is sometimes called the *dual Brunovsky canonical form*):

$$
\begin{array}{l}
F_p = Q^T F_c Q = \begin{bmatrix} 0 & I_{r_1,r_2} & 0 & \cdots & 0 & 0 \\ 0 & 0 & I_{r_2,r_3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & I_{r_{k-1},r_k} \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \\
G_p = Q^T G_c = \begin{bmatrix} 0 & 0 & \cdots & 0 & E_{r_1-r_2} & 0 \\ 0 & 0 & \cdots & E_{r_2-r_3} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & E_{r_{k-1}-r_k} & \cdots & 0 & 0 & 0 \\ I_{r_k} & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}
\end{array} \tag{23}
$$

where $I_{r_i,r_{i+1}}$ is defined in (5) and

$$
E_{r_i-r_{i+1}} = \begin{bmatrix} 0 \\ I_{r_i-r_{i+1}} \end{bmatrix} \in \mathbb{R}^{r_i \times (r_i - r_{i+1})}, i = 1, 2, \ldots, k-1.
$$

Note that $F_p$ is the Weyr canonical form of $F_c$. The pair $(F_p, G_p)$ will be called the *permuted dual Brunovsky canonical form* or, for short, the *p-Brunovsky canonical form* of $(F, G)$.

Recall that for a given controllable system $(F, G) \in \Sigma^c$ and a given sequence of monic polynomials $\underline{\alpha} : \alpha_1(s) \mid \cdots \mid \alpha_n(s)$ with $\sum_{i=1}^n \deg(\alpha_i(s)) = n$, we aim to parametrize the set of feedback matrices $K \in \mathbb{R}^{m \times n}$ such that $F + GK$ has the polynomials in $\underline{\alpha}$ as invariant polynomials; i.e., such that $F + GK \in \mathcal{O}(\underline{\alpha})$. Necessary and sufficient conditions for such a set not to be empty were obtained in [21] when $(F, G)$ is controllable and in [23] in the general case. We state the result for the controllable case.

**Proposition 2.9** [21, Ch. 5, Sec. 4], [23, Theorem 2.6] Let $(F, G) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m}$ be a controllable pair and let $k_1 \geq \cdots \geq k_m$ be its controllability indices. Let $\underline{\alpha} : \alpha_1(s) \mid \cdots \mid \alpha_n(s)$ be monic polynomials. There exists $K \in \mathbb{R}^{m \times n}$ such that $F + GK \in \mathcal{O}(\underline{\alpha})$ if and only if (see (2))

$$
(k_1, k_2, \ldots, k_m) \prec (\deg(\alpha_n(s)), \deg(\alpha_{n-1}(s)), \ldots, \deg(\alpha_1(s))). \tag{24}
$$

**Remark 2.10** Assume that the prime factorization of $\alpha_{n-i+1}(s)$ is given by (9) and $\alpha_1(s) = \cdots = \alpha_h(s) = 1 \neq \alpha_{h+1}(s)$.

- If $\underline{m}_i = (m_{i1}, \ldots, m_{iw_i})$ is the Segre characteristic for $\lambda_i$, $1 \leq i \leq t$, then

  $$
  (\deg(\alpha_n(s)), \deg(\alpha_{n-1}(s)), \ldots, \deg(\alpha_{h+1}(s))) = \underline{m}_1 + \underline{m}_2 + \cdots + \underline{m}_{p+2q}.
  $$

  Therefore, (24) is equivalent to

  $$
  (k_1, k_2, \ldots, k_m) \prec (m_{11}, \ldots, m_{1w_1}) + \cdots + (m_{p+2q\,1}, \ldots, m_{p+2q\,w_{p+2q}}) \tag{25}
  $$

- For $i = 1, \ldots, t$, let $\underline{w}_i = (w_{i1}, \ldots, m_{iw_i})$ be the Weyr characteristic of $\lambda_i$ and let $\underline{r} = (r_1, \ldots, r_n)$ be the Brunovsky indices of $(F, G)$. Then, by definition, $\underline{w}_i$ and $\underline{r}$ are the conjugate partitions of $\underline{m}_i$, $1 \leq i \leq p+2q$, and $\underline{k}$, respectively. It follows from (25) and Proposition 2.1 that condition (24) is equivalent to

$$(w_{11}, \ldots, w_{1m_1}) \cup \cdots \cup (w_{p+2q\,1}, \ldots, w_{p+2q\,m_{p+2q}}) \prec (r_1, r_2, \ldots, r_n). \quad (26)$$

Note that $\underline{w}_1 \cup \underline{w}_2 \cup \cdots \cup \underline{w}_t = (\deg(\alpha_n(s)), \deg(\alpha_{n-1}(s)), \ldots, \deg(\alpha_1(s)))^*$, the conjugate partition of $(\deg(\alpha_n(s)), \deg(\alpha_{n-1}(s)), \ldots, \deg(\alpha_1(s)))$.

□

# 3 Geometric structure of $\mathcal{H}_{(F,G)}$

Let $(F, G) \in \Sigma^c$ be a controllable system with $k_1 \geq k_2 \geq \cdots \geq k_r > 0 = k_{r+1} = \cdots = k_m$ as controllability indices and let $\underline{\alpha} : \alpha_1(s) \mid \cdots \mid \alpha_n(s)$ be monic polynomials. Let $\mathcal{H}_{(F,G)}$ is the set of (1) and assume that it is not empty; i.e., condition (24) holds true. In this section we will prove that $\mathcal{H}_{(F,G)}$ is a submanifold of $\mathbb{R}^{m\times n}$ whose dimension is $\dim \mathcal{H}_{(F,G)} = nm - N$, where $N$ is given in (16).

**Lemma 3.1** *Let $\underline{\alpha} : \alpha_1(s) \mid \cdots \mid \alpha_n(s)$ be monic polynomials such that $\sum_{i=1}^n \deg(\alpha_n) = n$ and let $A \in \mathcal{O}(\underline{\alpha})$. Then the tangent space of $\mathcal{O}(\underline{\alpha})$ at $A$ is*

$$T_A\mathcal{O}(\underline{\alpha}) = \{[A, X] \; : \; X \in \mathbb{R}^{n\times n}\},$$

*where $[A, X] = AX - XA$ is the commutator of $A$ and $X$.*

**Proof.** Let $\gamma_A : \mathrm{Gl}(n) \longrightarrow \mathbb{R}^{n\times n}$ be the map defined by $\gamma_A(P) = P^{-1}AP$. It follows from the proof of Theorem 9.16 in [15] (see also [2, Proposition 3.2]) that $T_A\mathcal{O}(\underline{\alpha}) = \operatorname{Im} d\gamma_{A,I_n}$. For $X \in \mathbb{R}^{n\times n}$,

$$\gamma_A(I_n + \epsilon X) = (I_n + \epsilon X)^{-1}A(I_n + \epsilon X) = (I_n - \epsilon X + \epsilon^2 X^2 - \ldots)A(I_n + \epsilon X).$$

Then

$$\gamma_A(I_n + \epsilon X) - \gamma_A(I_n) = \epsilon(AX - XA) + \epsilon^2 P(\epsilon)$$

where $P(\epsilon)$ is a polynomial matrix whose coefficients depend on $A$ and $X$. Therefore $d\gamma_{A,I_n}(X) = [A, X]$. □

In the proof of the following theorem we will use the Frobenius inner product in $\mathbb{R}^{n\times n}$: if $A, B \in \mathbb{R}^{n\times n}$, $< A, B >= \operatorname{tr}(A^T B)$, where tr stands for trace.

**Theorem 3.2** *Let $(F, G) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$ be a controllabe pair with controllability indices $k_1 \geq \cdots \geq k_m$ and let $\underline{\alpha} : \alpha_1(s) \mid \cdots \mid \alpha_n(s)$ be monic polynomials satisfying (24). Then the set $\mathcal{H}_{(F,G)}$ defined in (1) is a submanifold of $\mathbb{R}^{m\times n}$ and $\dim \mathcal{H}_{(F,G)} = nm - N$, where $N$ is given in (16).*

**Proof.** Let $\varphi : \mathbb{R}^{m\times n} \longrightarrow \mathbb{R}^{n\times n}$ be the differentiable map defined by $\varphi(K) = F + GK$. Then $\varphi^{-1}(\mathcal{O}(\underline{\alpha})) = \mathcal{H}_{(F,G)}$ and $d\varphi_K(V) = GV$ for all $V \in \mathbb{R}^{m\times n}$.

If we prove that $\varphi$ is transversal to $\mathcal{O}(\underline{\alpha})$ then ([11, p. 28]) $\varphi^{-1}(\mathcal{O}(\underline{\alpha})) = \mathcal{H}_{(F,G)}$ would be a submanifold of $\mathbb{R}^{m\times n}$ of dimension $\dim \mathcal{H}_{(F,G)} = mn - N$, as desired.

We take $K \in \varphi^{-1}(\mathcal{O}(\underline{\alpha}))$ and we are to prove that $\operatorname{Im} d\varphi_k + T_{\varphi(K)}\mathcal{O}(\underline{\alpha}) = T_{\varphi(K)}\mathbb{R}^{n\times n}$. Equivalently, bearing in mind that $T_{\varphi(K)}\mathbb{R}^{n\times n} = \mathbb{R}^{n\times n}$,

$$(\operatorname{Im} d\varphi_k)^{\perp} \cap (T_{\varphi(K)}\mathcal{O}(\underline{\alpha}))^{\perp} = \{0\}.$$

Let $U \in \mathbb{R}^{n\times n}$. On one hand, by Lemma 3.1, $U \in (T_{\varphi(K)}\mathcal{O}(\underline{\alpha}))^{\perp}$ if and only if $0 =< U, [F+GK, X] >= \operatorname{tr}(U^T(F+GK)X - U^TX(F+GK)) = \operatorname{tr}(U^T(F+GK)X - (F+GK)U^TX) = \operatorname{tr}((U^T(F+GK) - (F+GK)U^T)X)$ for all $X \in \mathbb{R}^{n\times n}$, i.e., if and only if $[F+GK, U^T] = 0$. On the other hand, $U \in (\operatorname{Im} d\varphi_k)^{\perp}$ if and only if $\operatorname{tr}(U^TGV) = 0$ for all $V \in \mathbb{R}^{m\times n}$, i.e., if and only if $U^TG = 0$.

Therefore, if $U \in (\operatorname{Im} d\varphi_k)^{\perp} \cap (T_{\varphi(K)}\mathcal{O}(\underline{\alpha}))^{\perp}$ then

$$\begin{aligned} 0 = &\begin{bmatrix} U^TG & (F+GK)U^TG & \dots & (F+GK)^{n-1}U^TG \end{bmatrix} \\ = &\begin{bmatrix} U^TG & U^T(F+GK)G & \dots & U^T(F+GK)^{n-1}G \end{bmatrix} \\ = &\; U^T\begin{bmatrix} G & (F+GK)G & \dots & (F+GK)^{n-1}G \end{bmatrix}. \end{aligned}$$

Taking into account that $(F+GK, G)$ is controllable, the matrix

$$\begin{bmatrix} G & (F+GK)G & \dots & (F+GK)^{n-1}G \end{bmatrix}$$

is right invertible and so $U = 0$ as desired. □

The next proposition shows that in order to study the geometry of the set $\mathcal{H}_{(F,G)}$, the pair $(F, G)$ can be replaced by any other pair in its orbit of feedback equivalence.

**Proposition 3.3** *Let $(F, G), (F', G') \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$ be controllabe pairs. If $(F, G), (F', G')$ are feedback equivalent then $\mathcal{H}_{(F,G)}$ and $\mathcal{H}_{(F',G')}$ are diffeomorphic.*

**Proof.** There exist $P \in \mathrm{Gl}(n)$, $Q \in \mathrm{Gl}(m)$ and $R \in \mathbb{R}^{m\times n}$ such that $P^{-1}FP + P^{-1}GR = F'$ and $P^{-1}GQ = G'$.

Let $K \in \mathbb{R}^{m\times n}$. Then $F' + G'Q^{-1}(KP - R) = P^{-1}(F+GK)P$. Therefore, the map $\psi : \mathcal{H}_{(F,G)} \longrightarrow \mathcal{H}_{(F',G')}$ defined by $\psi(K) = Q^{-1}(KP - R)$ is well defined and bijective. It is easily seen that it is a diffeomorphism. □

# 4 The manifold $\mathcal{P}_{(A;\underline{r})}/\widetilde{C}_A$

In order to obtain a parameterization of the manifold $\mathcal{H}_{(F,G)}$ defined in Section 3, we will prove that it is diffeomorphic to an orbit space by the action of a Lie group. We are led by the following idea taken from [1, Section 2.2] (see also [3, Section 2.3]): Assume that we are given a controllable system $(F_c, G_c) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$ in Brunovsky canonical form with $\underline{k} : k_1 \geq \cdots \geq k_r > 0 = k_{r+1} = \cdots = k_m$ as controllability indices. Then, by [1, Theorem 2.15], a subspace $\mathcal{V} \subset \mathbb{R}^n$ of dimension $d$ is $(F_c, G_c)-$*invariant* (*i.e.*; $F_c\mathcal{V} \subset \mathcal{V} + \operatorname{Im}\, G_c$) if and only if there exists a pair of matrices $(\widehat{H}, \widehat{F}) \in \mathbb{R}^{r\times d} \times \mathbb{R}^{d\times d}$ such that $\mathcal{V} = \operatorname{Im} O_\pi(\widehat{H}, \widehat{F})$ where $\widehat{H} = \begin{bmatrix} h_1^T & \cdots & h_r^T \end{bmatrix}^T$ and

$$O_\pi(\widehat{H}, \widehat{F}) = \begin{bmatrix} O_1^T & \cdots & O_r^T \end{bmatrix}^T, \; O_i = \begin{bmatrix} h_i^T & \widehat{F}^Th_i^T \cdots (\widehat{F}^T)^{k_i-1}h_i^T \end{bmatrix}^T, \; 1 \leq i \leq r.$$

Using Antoulas' notation, $O_\pi(\widehat{H},\widehat{F})$ is a *permuted and truncated observability matrix* of $(\widehat{H},\widehat{F})$. Note that $O_\pi(\widehat{H},\widehat{F}) \in \mathcal{T}(\widehat{F},\underline{k})$ (cf. (8)). It is then shown ([1, Corollary 2.18]) that $\widehat{F} = (F_c + G_cK)_{|\mathcal{V}}$, i.e., $\widehat{F}$ is the restriction of $F_c + G_cK$ to $\mathcal{V}$ for some state-feedback matrix $K$. In other words; if $R = \begin{bmatrix} O_\pi(\widehat{H},\widehat{F}) & X\end{bmatrix} \in \mathrm{Gl}(n)$ then $(F_c + G_cK)R = O_\pi(\widehat{H},\widehat{F})\widehat{F}$ for some feedback transformation $K$. In particular, if $\mathcal{V} = \mathbb{R}^n$ then there is a pair of matrices $(\widehat{H},\widehat{F}) \in \mathbb{R}^{r\times n}\times\mathbb{R}^{n\times n}$ such that $O_\pi(\widehat{H},\widehat{F})$ is invertible and

$$O_\pi(\widehat{H},\widehat{F})\widehat{F} = (F_c + G_cK)O_\pi(\widehat{H},\widehat{F}). \tag{27}$$

This result establishes a close relationship between the Antoulas' permuted and truncated observability matrices with fixed state matrix $A \in \mathcal{O}(\underline{\alpha})$ and the set $\mathcal{H}_{(F_c,G_c)}$ and, by Proposition 3.3, with the set $\mathcal{H}_{(F,G)}$ provided that $(F,G)$ and $(F_c,G_c)$ are feedback equivalent. As Antoulas himself remarks if $(F,G) = (T^{-1}F_cT, T^{-1}G_c)$ for some invertible matrix $T$, then a subspace is $(F,G)$-invariant if and only if it is spanned by $T^{-1}O_\pi(\widehat{H},\widehat{F})$. In order to simplify the computations in Section 5, it is most convenient for us to work with some matrices whose rows are obtained by permuting in a precise way the rows of Antoulas' permuted and truncated observability matrices $O_\pi(\widehat{H},\widehat{F})$. Specifically if $Q$ is the permutation matrix of (7) then $(F_p,G_p) = (Q^TF_cQ, Q^TG_c)$ is the $p$-Brunovsky canonical form of (23) and so a subspace is $(F_p,G_p)$-invariant if and only if it is spanned by $P = Q^TO_\pi(\widehat{H},\widehat{F})$. A direct computation shows that $P$ has the following form: If $\underline{r} = (r_1,r_2,\dots,r_k)$ $(r_{k+1}=0)$ is the conjugate partition of $\underline{k} = (k_1,k_2,\dots,k_r)$ then

$$P = \begin{bmatrix} P_1\\ P_2\\ \vdots\\ P_k\end{bmatrix},\quad P_1 = \begin{bmatrix} P_{11}\\ P_{12}\\ \vdots\\ P_{1k}\end{bmatrix} \in \mathbb{R}^{r_1\times d}, P_{1i} = \begin{bmatrix} h_{r_{k-i+2}+1}\\ h_{r_{k-i+2}+2}\\ \vdots\\ h_{r_{k-i+1}}\end{bmatrix} \in \begin{array}{l}\mathbb{R}^{(r_{k-i+1}-r_{k-i+2})\times d},\\ 1\le i\le k,\end{array}$$

and for $i = 1,\dots,k-1$

$$P_{i+1} = I^T_{r_i,r_{i+1}}P_i\widehat{F} = I^T_{r_1,r_{i+1}}P_1\widehat{F}^i = \begin{bmatrix} P_{11}\widehat{F}^i\\ P_{12}\widehat{F}^i\\ \vdots\\ P_{1k-i}\widehat{F}^i\end{bmatrix} \in \mathbb{R}^{r_{i+1}\times d},$$

where $I_{p,q}$ is the matrix of (5).

Note that $P$ is a truncated observability matrix of $(\widehat{H},\widehat{F})$ but it is obtained from that matrix without permuting its rows.

We define formally the set of matrices $P$ introduced above. Given an arbitrary matrix $A \in \mathbb{R}^{d\times d}$ and arbitrary nonnegative integers $\underline{r} : r = r_1 \ge r_2 \ge \cdots \ge r_k > 0 = r_{k+1} = \cdots = r_d$ such that $n = \sum_{i=1}^d r_i \ge d$, we define

$$\mathcal{P}_{(A;\underline{r})} := \left\{ P = \begin{bmatrix} P_1\\ P_2\\ \vdots\\ P_k\end{bmatrix},\ P_i = \begin{bmatrix} P_{11}A^{i-1}\\ P_{12}A^{i-1}\\ \vdots\\ P_{1k-i+1}A^{i-1}\end{bmatrix} \in \mathbb{R}^{r_i\times d}, i=1,\dots,k, \operatorname{rank} P = d \right\}.$$

Note that if $\mathcal{P}_{(A;\underline{r})} \neq \emptyset$ then it is an open set of a linear subspace or dimension $rd$. Hence $\mathcal{P}_{(A;\underline{r})}$ is a linear manifold of dimension $rd$.

**Remark 4.1** (i) We have seen that if $P = \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_k \end{bmatrix} \in \mathcal{P}(A, \underline{r})$ with $P_1 = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_r \end{bmatrix}$ then $P_{1i} = \begin{bmatrix} p_{r_{k-i+2}+1} \\ p_{r_{k-i+2}+2} \\ \vdots \\ p_{r_{k-i+1}} \end{bmatrix}$, $i = 1, \ldots, k$. It follows from this and $r_i - r_{i+1} = \#\{j : k_j = i\}$ that

$$\begin{bmatrix} P_{11}A^{k_1-1} \\ P_{12}A^{k_1-2} \\ \vdots \\ P_{1\,k-1}A \\ P_{1k} \end{bmatrix} = \begin{bmatrix} p_1A^{k_1-1} \\ p_2A^{k_2-1} \\ \vdots \\ p_{r-1}A^{k_{r-1}-1} \\ p_rA^{k_r-1} \end{bmatrix}. \tag{28}$$

(ii) It is worth-noticing that $\mathcal{P}_{(A;\underline{r})}$ can be empty for some matrices $A$ and some sequences $\underline{r}$. For example, if $A = I_4$ and $\underline{r} = (2,2)$ then $\operatorname{rank}\begin{bmatrix} p_1 \\ p_2 \\ p_1A \\ p_2A \end{bmatrix} < 4$ for all vectors $p_1, p_2 \in \mathbb{R}^{1\times 4}$.

$\square$

The following proposition provides a necessary and sufficient condition for $\mathcal{P}_{(A;\underline{r})} \neq \emptyset$. As it is lengthy and does not significantly contribute to the aim of the paper, its proof is deferred to the Appendix.

**Proposition 4.2** *With the above notation, if $\alpha_1(s) \mid \cdots \mid \alpha_d(s)$ are the invariant polynomials of $A$ and $(w_1, \ldots, w_d)$ is the conjugate partition of* $(\deg(\alpha_d(s)), \ldots, \deg(\alpha_1(s)))$ *then $\mathcal{P}_{(A;\underline{r})} \neq \emptyset$ if and only if each of the following equivalent conditions holds:*

$$\sum_{j=i+1}^{r} k_j \geq \sum_{j=1}^{d-i} \deg(\alpha_j), \quad i \geq 1 \tag{29}$$

$$\sum_{j=1}^{i} w_j \leq \sum_{j=1}^{i} r_j, \quad 1 \leq i \leq d. \tag{30}$$

**Remark 4.3** It should be noted that when $n = \sum_{i=1}^d r_i = d$ then, since $\sum_{i=1}^d \deg(\alpha_i(s)) = d$, $\sum_{j=1}^d w_j = \sum_{j=1}^d r_j$. This and (30) implies $(w_1, \ldots, w_d) \prec (r_1, \ldots, r_d)$. Taking into account the second item of Remark 2.10, if $\sum_{i=1}^d r_i = d$ then $\mathcal{H}_{(F,G)} \neq \emptyset$ if and only if $\mathcal{P}_{(A;\underline{r})} \neq \emptyset$. In addition, in this case, all matrices in $\mathcal{P}_{(A;\underline{r})}$ are square and invertible.

Our goal in this section is to show that, for any $A \in \mathbb{R}^{d\times d}$, $\mathcal{P}_{(A;\underline{r})}/\widetilde{C}_A$ is a differentiable manifold and that there is a diffeomorphism between $\mathcal{H}_{(F,G)}$ and $\mathcal{P}_{(A;\underline{r})}/\widetilde{C}_A$ when $d = n$ and $A \in \mathcal{O}(\underline{\alpha})$.

**Proposition 4.4** *The action $\sigma : \widetilde{C}_A \times \mathcal{P}_{(A;\underline{r})} \longrightarrow \mathcal{P}_{(A;\underline{r})}$ of $\widetilde{C}_A$ on $\mathcal{P}_{(A;\underline{r})}$ defined by $\sigma(X, P) = PX$ is free and proper.*

**Proof.** If $P \in \mathcal{P}_{(A;\underline{r})}$ then $P$ is left invertible and thus $\sigma$ is free.

Let $\{P_i\}$ be a convergent sequence in $\mathcal{P}_{(A;\underline{r})}$ and $\{X_i\}$ a sequence in $\widetilde{C}_A$ such that $\{P_i X_i\}$ converges. Then $\{(P_i^T P_i)^{-1} P_i^T P_i X_i\} = \{X_i\}$ converges. By [15, Proposition 9.13], the action $\sigma$ is proper. □

As a consequence we can apply the quotient manifold theorem (see, for example, [15, Theorem 9.16]).

**Corollary 4.5** *The space of orbits $\mathcal{P}_{(A;\underline{r})}/\widetilde{C}_A$ is a differentiable manifold, the natural projection $\pi : \mathcal{P}_{(A;\underline{r})} \longrightarrow \mathcal{P}_{(A;\underline{r})}/\widetilde{C}_A$ is a submersion and $\dim \mathcal{P}_{(A;\underline{r})}/\widetilde{C}_A = rd - \dim \widetilde{C}_A$.*

In what follows $\underline{\alpha} : \alpha_1(s) \mid \cdots \mid \alpha_n(s)$ will be assumed to be monic polynomials satisfying $\sum_{i=1}^{n} \deg(\alpha_i(s)) = n$ and $(F, G) \in \Sigma^c$ a given controllable pair with controllability indices $\underline{k} : k_1 \geq \cdots \geq k_r > 0 = k_{r+1} = \cdots = k_m$ satisfying (24) and Brunovsky indices $r_1 \geq \cdots r_k > 0 = r_{k+1} = \cdots = r_n$ ($k = k_1$ and $r = r_1$). We aim to obtain a parameterization of $\mathcal{H}_{(F,G)}$. This will be achieved in Section 6 throughout a diffeomorphism between $\mathcal{H}_{(F,G)}$ and $\mathcal{P}_{(A;\underline{r})}/\widetilde{C}_A$ where $A \in \mathbb{R}^{n\times n}$ is a matrix with $\alpha_1(s) \mid \cdots \mid \alpha_n(s)$ as invariant polynomials. That $\mathcal{H}_{(F,G)}$ and $\mathcal{P}_{(A;\underline{r})}/\widetilde{C}_A$ are diffeomorphic is proved in Theorem 4.8 below.

Since $\underline{\alpha}$ and $\underline{k}$ satisfy (24), $\mathcal{H}_{(F,G)} \neq \emptyset$ and, by Remark 4.3, $\mathcal{P}_{(A;\underline{r})} \neq \emptyset$ and $\mathcal{P}_{(A;\underline{r})} \subset \mathrm{Gl}(n)$. Also, it follows from $k_r > 0 = k_{r+1}$ that $\operatorname{rank} G = r$.

**Remark 4.6** By Proposition 3.3, we can assume that $(F, G) = (F_p, G_p)$ where $(F_p, G_p)$ is the $p$-Brunovsky canonical form given in (23). Let $G = \begin{bmatrix} G_1 & O \end{bmatrix}$, with $G_1 \in \mathbb{R}^{n\times r}$, $\operatorname{rank} G_1 = r$. If $K = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} \in \mathbb{R}^{(r+(m-r))\times n}$, then $F + GK = F + G_1 K_1$ and therefore

$$\mathcal{H}_{(F,G)} = \mathcal{H}_{(F,G_1)} \times \mathbb{R}^{(m-r)\times n}.$$

Thus, it is enough to obtain a parameterization of $\mathcal{H}_{(F,G_1)}$.

The following lemma gives the counterpart of (27) when the matrices of $\mathcal{P}_{(A;\underline{r})}$ are used.

**Lemma 4.7** *Let $A \in \mathbb{R}^{n\times n}$ and let $(F, G)$ be in p-Brunovsky canonical form with $G = \begin{bmatrix} G_1 & 0 \end{bmatrix}$, $G_1 \in \mathbb{R}^{n\times r}$, $\operatorname{rank} G_1 = r$. Let $k_1 \geq k_2 \geq \cdots \geq k_r > 0$ and $r_1 \geq r_2 \geq \cdots \geq r_k > 0$ be the nonzero controllability and Brunovsky indices of $(F, G)$. Then:*

*(i) For each $P \in \mathcal{P}_{(A;\underline{r})}$ the matrix $K = \begin{bmatrix} p_1 A^{k_1} \\ \vdots \\ p_r A^{k_r} \end{bmatrix} P^{-1}$, where $p_1, \ldots, p_r$ are the first $r$ rows of $P$, is in $\mathcal{H}_{(F,G_1)}$.*

*(ii) For each $K \in \mathcal{H}_{(F,G_1)}$ there is $P \in \mathcal{P}_{(A;\underline{r})}$ such that $PA = (F + G_1 K)P$ and $KP = \begin{bmatrix} p_1 A^{k_1} \\ \vdots \\ p_r A^{k_r} \end{bmatrix}$ where $p_1, \ldots, p_r$ are the first $r$ rows of $P$.*

**Proof.** Assume that $P \in \mathcal{P}_{(A;\underline{r})}$. Then $P = \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_k \end{bmatrix}$ with $P_i = \begin{bmatrix} P_{i1} \\ P_{i2} \\ \vdots \\ P_{i\,k-i+1} \end{bmatrix} \in \mathbb{R}^{r_i \times d}$ and $P_{ij} = P_{1j}A^{i-1} \in \mathbb{R}^{(r_{k-j+1}-r_{k-j+2})\times d}$, $i = 1, \ldots, k$, $j = 1, \ldots, k-i+1$. Let $K = \begin{bmatrix} p_1 A^{k_1} \\ \vdots \\ p_r A^{k_r} \end{bmatrix} P^{-1}$ where $p_1, \ldots, p_r$ are the rows of $P_1$ (recall that $r_1 = r$). We aim to show that $PA = (F + G_1K)P$. As $\operatorname{rank} P = n$, this implies that $F + G_1K = PAP^{-1} \in \mathcal{O}(\underline{\alpha})$ and so $K \in \mathcal{H}_{(F,G_1)}$. In fact, define $Z = KP$. Then $Z = \begin{bmatrix} P_{11}A^{k_1} \\ P_{12}A^{k_1-1} \\ \vdots \\ P_{1\,k-1}A^2 \\ P_{1k}A \end{bmatrix}$. Put $Z_i = P_{1i}A^{k_1-i+1}$, $i = 1, \ldots, k = k_1$. Now, $PA = \begin{bmatrix} P_1A \\ P_2A \\ \vdots \\ P_kA \end{bmatrix}$ and it follows from (23) that

$$FP = \begin{array}{cc} \left[\begin{array}{c} P_2 \\ 0 \\ \hline P_3 \\ 0 \\ \hline \vdots \\ \hline P_k \\ 0 \\ \hline 0 \end{array}\right] & \begin{array}{c} r_2 \\ r_1 - r_2 \\ r_3 \\ r_2 - r_3 \\ \vdots \\ r_k \\ r_{k-1} - r_k \\ r_k \end{array} \end{array} \quad \text{and} \quad G_1Z = \begin{array}{cc} \left[\begin{array}{c} 0 \\ Z_k \\ \hline 0 \\ Z_{k-1} \\ \hline \vdots \\ \hline 0 \\ Z_2 \\ \hline Z_1 \end{array}\right] & \begin{array}{c} r_2 \\ r_1 - r_2 \\ r_3 \\ r_2 - r_3 \\ \vdots \\ r_k \\ r_{k-1} - r_k \\ r_k \end{array} \end{array}. \tag{31}$$

Bearing in mind that

$$P_2 = \begin{bmatrix} P_{11}A \\ P_{12}A \\ \vdots \\ P_{1\,k-1}A \end{bmatrix}, \; P_3 = \begin{bmatrix} P_{11}A^2 \\ P_{12}A^2 \\ \vdots \\ P_{1\,k-2}A^2 \end{bmatrix}, \; \ldots, \; P_k = P_{11}A^{k_1-1},$$

and

$$Z_k = P_{1k}A, \; Z_{k-1} = P_{1\,k-1}A^2, \; \ldots, \; Z_2 = P_{12}A^{k_1-1}, \; Z_1 = P_{11}A^{k_1},$$

we get $PA = FP + G_1Z = (F + G_1K)P$, as claimed. Finally, by (28), $Z = \begin{bmatrix} P_{11}A^{k_1} \\ P_{12}A^{k_1-1} \\ \vdots \\ P_{1\,k-1}A^2 \\ P_{1k}A \end{bmatrix} = \begin{bmatrix} p_1A^{k_1} \\ p_2A^{k_2} \\ \vdots \\ p_{r-1}A^{k_{r-1}} \\ p_rA^{k_r} \end{bmatrix}$.

Conversely, if $K \in \mathcal{H}_{(F,G_1)}$ then there is $P \in \mathrm{Gl}(n)$ such that $PA = (F + G_1K)P$. Split $P = \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_k \end{bmatrix}$ with $P_i = \begin{bmatrix} P_{i1} \\ P_{i2} \\ \vdots \\ P_{i\,k-i+1} \end{bmatrix} \in \mathbb{R}^{r_i \times d}$ and $P_{ij} \in \mathbb{R}^{(r_{k-j+1}-r_{k-j+2})\times d}$, $i = 1, \ldots, k$, $j = 1, \ldots, k-i+1$. Put $Z = KP = \begin{bmatrix} Z_1 \\ \vdots \\ Z_k \end{bmatrix}$

with $Z_i \in \mathbb{R}^{(r_{k-i+1}-r_{k-i+2})\times n}$, $i = 1,\dots,k$. By using that $PA = \begin{bmatrix} P_1A \\ P_2A \\ \vdots \\ P_kA \end{bmatrix}$ and (31) we get $P_{ij} = P_{1j}A^{i-1}$ for $i = 1,\dots,k$, $j = 1,\dots,k-i+1$, and $Z_i = P_{1i}A^{k_1-i+1}$, $i = 1,\dots,k$. Therefore $P \in \mathcal{P}_{(A;\underline{r})}$ and by (28), $KP = Z = \begin{bmatrix} p_1A^{k_1} \\ \vdots \\ p_rA^{k_r} \end{bmatrix}$ where $p_1,\dots,\ p_r$ are the rows of $P_1$. $\square$

We are ready to prove that $\mathcal{H}_{(F,G_1)}$ and $\mathcal{P}_{(A;\underline{r})}/\widetilde{C}_A$ are diffeomorphic manifolds.

**Theorem 4.8** *Let $(F,G) \in \Sigma^c$ be in p-Brunovsky canonical form, with $G = \begin{bmatrix} G_1 & 0 \end{bmatrix}$, $G_1 \in \mathbb{R}^{n\times r}$, $\operatorname{rank} G_1 = r$ and $k_1 \geq \cdots \geq k_r > 0 = k_{r+1} = \cdots = k_m$ as controllability indices, and let $A \in \mathbb{R}^{n\times n}$ be a matrix in $\mathcal{O}(\underline{\alpha})$. Then the map*

$$\begin{array}{rccc} \phi: & \mathcal{P}_{(A;\underline{r})}/\widetilde{C}_A & \longrightarrow & \mathcal{H}_{(F,G_1)} \\ & \widetilde{P} & \mapsto & \begin{bmatrix} p_1A^{k_1} \\ p_2A^{k_2} \\ \vdots \\ p_rA^{k_r} \end{bmatrix} P^{-1}, \end{array} \tag{32}$$

*where $P \in \mathcal{P}_{(A;\underline{r})}$ is any matrix in the orbit $\widetilde{P}$ and $p_1,\dots,p_r$ are its first $r$ rows, is a diffeomorphism.*

**Proof.** Let us see first that $\phi$ is well-defined. If $\widetilde{P}_1 = \widetilde{P}_2$ then for any $P_1 \in \widetilde{P}_1$ and $P_2 \in \widetilde{P}_2$, $P_1 = P_2X$ for some $X \in \widetilde{C}_A$. Then if $\phi(\widetilde{P}_i) = \begin{bmatrix} p_{i1}A^{k_1} \\ p_{i2}A^{k_2} \\ \vdots \\ p_{ir}A^{r} \end{bmatrix} P_i^{-1}$, $i = 1,2$, then

$$\phi(\widetilde{P}_1) = \begin{bmatrix} p_{21}XA^{k_1} \\ \vdots \\ p_{2r}XA^{k_r} \end{bmatrix} X^{-1}P_2^{-1} = \begin{bmatrix} p_{21}A^{k_1} \\ \vdots \\ p_{2r}A^{k_r} \end{bmatrix} XX^{-1}P_2^{-1} = \phi(\widetilde{P}_2).$$

Next, let $P \in \widetilde{P}$. By item (i) of Lemma 4.7, the matrix $K = \begin{bmatrix} p_1A^{k_1} \\ \vdots \\ p_rA^{k_r} \end{bmatrix} P^{-1}$ is in $\mathcal{H}_{(F,G_1)}$. Thus $\phi$ is well-defined.

Conversely, if $K \in \mathcal{H}_{F,G_1)}$, by item (ii) of Lemma 4.7, there is $P \in \mathcal{P}(A,\underline{k})$ such that $KP = \begin{bmatrix} p_1A^{k_1} \\ \vdots \\ p_rA^{k_r} \end{bmatrix}$. Therefore $\phi$ is surjective.

Assume now that $K_1, K_2 \in \mathcal{H}_{(F,G_1)}$ and $K_1 = K_2$. By Lemma 4.7 there are $P_1, P_2 \in \mathcal{P}(A,\underline{k})$ such that $P_1AP_1^{-1} = F + G_1K_1 = F + G_1K_2 = P_2AP_2^{-1}$. Then $X = P_1^{-1}P_2 \in \widetilde{C}_A$ and $P_2 = P_1X$. Hence $\widetilde{P}_1 = \widetilde{P}_2$ and $\phi$ is injective.

In order to prove that $\phi$ is a diffeomorphism, we introduce the set

$$\widehat{\mathcal{H}}_{(F,G_1)} = \{F + G_1K \ :\ K \in \mathcal{H}_{(F,G_1)}\}$$

and the map $\widehat{\theta} : \mathbb{R}^{r\times n} \longrightarrow \mathbb{R}^{n\times n}$ defined by $\widehat{\theta}(K) = F + G_1K$. This is a linear map whose differential has constant rank. Then it is an embedding and

$\widehat{\theta}(\mathcal{H}_{(F,G_1)}) = \widehat{\mathcal{H}}_{(F,G_1)}$. If $\widehat{\theta}_{\mathcal{H}_{(F,G_1)}}$ is the restriction of $\widehat{\theta}$ to $\mathcal{H}_{(F,G_1)}$, since $\mathcal{H}_{(F,G_1)}$ is a smooth manifold (Theorem 3.2), we can provide $\widehat{\mathcal{H}}_{(F,G_1)}$ with a smooth structure for which $\widehat{\theta}_{\mathcal{H}_{(F,G_1)}}$ is a diffeomorphism.

Let $\psi : \mathcal{P}_{(A;\underline{r})}/\widetilde{C}_A \longrightarrow \widehat{\mathcal{H}}_{(F,G_1)}$ be the map defined by $\psi(\widetilde{P}) = PAP^{-1}$, where $P \in \mathcal{P}_{(A,\underline{r})}$ is any matrix in $\widetilde{P}$. This is a well-defined and bijective map because $\psi = \widehat{\theta}_{\mathcal{H}_{(F,G_1)}} \circ \phi$. In fact,

$$(\widehat{\theta}_{\mathcal{H}_{(F,G_1)}} \circ \phi)(\widetilde{P}) = \widehat{\theta}_{\mathcal{H}_{(F,G_1)}}\left(\begin{bmatrix} p_1A^{k_1} \\ \vdots \\ p_rA^{k_r} \end{bmatrix} P^{-1}\right) = F + G_1K$$

where $P \in \mathcal{P}_{(A,\underline{r})}$ is any representative of $\widetilde{P}$ and $K = \begin{bmatrix} p_1A^{k_1} \\ \vdots \\ p_rA^{k_r} \end{bmatrix} P^{-1}$. But $PA = (F + G_1K)P$ so that $(\widehat{\theta}_{\mathcal{H}_{(F,G_1)}} \circ \phi)(\widetilde{P}) = PAP^{-1}$. Taking into account that $\widehat{\theta}_{\mathcal{H}_{(F,G_1)}}$ is a diffeomorphism, we are going to prove that $\psi$ and $\psi^{-1}$ are differentiable. This proves that $\phi$ is a diffeomorphism.

We prove first that $\psi$ is differentiable. Let $\pi : \mathcal{P}_{(A;\underline{r})} \longrightarrow \mathcal{P}_{(A;\underline{r})}/\widetilde{C}_A$ be the natural projection, then $f := \psi\circ\pi$ is the restriction to $\mathcal{P}_{(A;\underline{r})}$ of the differentiable map $\widehat{f} : \mathrm{Gl}(n) \longrightarrow \mathbb{R}^{n\times n}$ defined by $\widehat{f}(P) = PAP^{-1}$. That $f$ is differentiable follows from the fact that $\widehat{f}$ is differentiable. Since $f$ is differentiable and $\pi$ is, by Corollary 4.5, a submersion, using [15, Proposition 7.17], we can conclude that $\psi$ is differentiable.

Let us see now that $\psi^{-1}$ is also differentiable. First $f = \psi \circ \pi$ and so $f$ is surjective because $\pi$ is surjective and $\psi$ is bijective. Now, for $P \in \mathcal{P}_{(A,\underline{r})}$ and $U \in \mathbb{R}^{n\times n}$, a direct computation shows that $d\widehat{f}_P(U) = UAP^{-1} - PAP^{-1}UP^{-1}$. Let $\widetilde{\mathcal{P}}_{(A,\underline{r})} = T_P\mathcal{P}_{(A,\underline{r})}$ be the tangent space of $\mathcal{P}_{(A,\underline{r})}$ at $P$. Then

$$\widetilde{\mathcal{P}}_{(A,\underline{r})} = \left\{ \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_k \end{bmatrix} \in \mathbb{R}^{n\times n} \ : \ P_i = \begin{bmatrix} P_{11}A^{i-1} \\ \vdots \\ P_{1\,k-i+1}A^{i-1} \end{bmatrix} \in \mathbb{R}^{r_i\times n}, 1 \le i \le k \right\}$$

and $\dim \widetilde{\mathcal{P}}_{(A,\underline{r})} = rn$. In addition, $df_P = d\widehat{f}_P \mid_{\widetilde{\mathcal{P}}_{(A,\underline{r})}}$ and so $\operatorname{Ker} df_P = \{U \in \widetilde{\mathcal{P}}_{(A,\underline{r})} \ : \ P^{-1}UA = AP^{-1}U\} = \widetilde{\mathcal{P}}_{(A,\underline{r})} \cap PC_A$. We claim that $\dim \operatorname{Ker} df_P = \dim C_A$. In fact, the map $\alpha : C_A \longrightarrow \operatorname{Ker} df_P$, defined by $\alpha(X) = PX$ is well-defined because $PX \in PC_A$ and $P_iX = \begin{bmatrix} P_{11}A^{i-1} \\ \vdots \\ P_{1\,k-i+1}A^{i-1} \end{bmatrix} X = \begin{bmatrix} P_{11}XA^{i-1} \\ \vdots \\ P_{1\,k-i+1}A^{i-1}X \end{bmatrix}$ so that $PX \in \widetilde{\mathcal{P}}_{(A,\underline{r})}$. It is easy to see that $\alpha$ is bijective. Thus, $\alpha$ is an isomorphism of linear spaces. As a conclusion we get $\dim \operatorname{Im} df_P = rn - N = \dim \widehat{\mathcal{H}}_{(F,G_1)}$. Therefore $f$ is a surjective submersion. Using again [15, Proposition 7.17] with $\psi^{-1} \circ f = \pi$ we conclude that $\psi^{-1}$ is differentiable, as claimed. $\square$

# 5 Parameterization of $\mathcal{P}_{(A;\underline{r})}/\widetilde{C}_A$

Let $\underline{\alpha} : \alpha_1(s) \mid \cdots \mid \alpha_n(s)$ be a sequence of monic polynomials such that $\sum_{i=1}^n \deg(\alpha_i(s)) = n$ and let $(F, G) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$ be a controllable pair with controllability indices $\underline{k} : k = k_1 \geq \cdots \geq k_r > 0 = k_{r+1} = \cdots = k_m$ satisfying (24), and Brunovsky indices $\underline{r} : r = r_1 \geq \cdots \geq r_k > 0 = r_{k+1} = \cdots = r_n$. By Remark 4.6 and Theorem 4.8, obtaining a parameterization of $\mathcal{H}_{(F,G)}$ is equivalent to obtaining a parameterization of $\mathcal{P}_{(A;\underline{r})}/\widetilde{C}_A$ for any matrix $A \in \mathcal{O}(\underline{\alpha})$. So, we can assume that $\alpha_{n-i+1}(s)$ factorizes as in (9) and that $A$ is the associated real Weyr canonical form:

$$A = \mathrm{diag}(W_1, \ldots, W_p, \widehat{W}_{p+1}, \ldots, \widehat{W}_{p+q}), \tag{33}$$

where $W_i = W(\lambda_i)$, $1 \leq i \leq p$ and $\widehat{W}_{p+i} = \widehat{W}(\lambda_{p+i}, \overline{\lambda_{p+i}})$, $1 \leq i \leq q$ are the matrices of (4) and (15), respectively. Let $s_i = \sum_{j=1}^{m_i} w_{i,j}$, $1 \leq i \leq p+q$ and $P = \begin{bmatrix} P_1 \\ \vdots \\ P_k \end{bmatrix} \in \mathcal{P}_{(A;\underline{r})}$. Put

$$P_i = \begin{bmatrix} P_i^{(1)} & \cdots & P_i^{(p)} & P_i^{(p+1)} & \cdots & P_i^{(p+q)} \end{bmatrix}, \quad 1 \leq i \leq k,$$

and

$$P^{(j)} = \begin{bmatrix} P_1^{(j)} \\ \vdots \\ P_k^{(j)} \end{bmatrix}, \quad 1 \leq j \leq p+q.$$

where for $1 \leq i \leq k$,

$$P_i^{(j)} \in \mathbb{R}^{r_i \times s_j} \; 1 \leq i \leq p, \quad P_i^{(j)} \in \mathbb{R}^{r_i \times 2s_j} \; p+1 \leq j \leq p+q.$$

For $1 \leq i \leq k-1$,

$$\begin{aligned} P_{i+1} &= I_{r_i,r_{i+1}}^T P_i A \\ &= I_{r_i,r_{i+1}}^T \begin{bmatrix} P_i^{(1)} W_1 & \cdots & P_i^{(p)} W_p & P_i^{(p+1)} \widehat{W}_{p+1} & \cdots & P_i^{(p+q)} \widehat{W}_{p+q} \end{bmatrix}. \end{aligned} \tag{34}$$

Hence

$$P_{i+1}^{(j)} = I_{r_i,r_{i+1}}^T P_i^{(j)} W_i, \; 1 \leq i \leq p \text{ and } P_{i+1}^{(j)} = I_{r_i,r_{i+1}}^T P_i^{(j)} \widehat{W}_i, \; p+1 \leq j \leq p+q.$$

As $P^{(j)}$ are full column rank matrices, $1 \leq j \leq p+q$, $P_i \in \mathcal{P}_{(W_i;\underline{r})}$, $1 \leq i \leq p$, and $P_i \in \mathcal{P}_{(\widehat{W}_i;\underline{r})}$, $p+1 \leq i \leq p+q$. Let

$$\mathcal{P} = \mathcal{P}_{(W_1;\underline{r})} \times \cdots \times \mathcal{P}_{(W_p;\underline{r})} \times \mathcal{P}_{(\widehat{W}_{p+1};\underline{r})} \times \cdots \times \mathcal{P}_{(\widehat{W}_{p+q};\underline{r})}.$$

and note that $\mathcal{P}_{(A;\underline{r})}$ can be identified with the subset of $\mathcal{P}$ formed by their invertible matrices . Thus we can think of $\mathcal{P}_{(A;\underline{r})}$ as an open subset of $\mathcal{P}$.

Recall that $X \in C_A$ if and only if $X = \mathrm{diag}(X_1, \ldots, X_p, \widehat{X}_{p+1}, \ldots, \widehat{X}_{p+q})$ with $X_i \in C_{W_i}$, $1 \leq i \leq p$ and $\widehat{X}_i \in C_{\widehat{W}_i}$, $p+1 \leq i \leq p+q$. Then, $PX = \left[\begin{smallmatrix} P^{(1)}X_1 & \ldots & P^{(p)}X_p & P^{(p+1)}X_{p+1} & \ldots & P^{(p+q)}X_{p+q} \end{smallmatrix}\right]$. As a consequence (see Corollary 4.5) we can identify $\mathcal{P}_{(A;\underline{r})}/\widetilde{C}_A$ with an open subset of

$$\mathcal{P}_{(W_1;\underline{r})}/\widetilde{C}_{W_1} \times \cdots \times \mathcal{P}_{(W_p;\underline{r})}/\widetilde{C}_{W_p} \times \mathcal{P}_{(\widehat{W}_{p+1};\underline{r})}/\widetilde{C}_{\widehat{W}_{p+1}} \times \cdots \times \mathcal{P}_{(\widehat{W}_{p+q};\underline{r})}/\widetilde{C}_{\widehat{W}_{p+q}},$$

and we can parametrize $\mathcal{P}_{(A;\underline{r})}/\widetilde{C}_A$ from a parametrization of $\mathcal{P}_{(W_i;\underline{r})}/\widetilde{C}_{W_i}$, $1 \leq i \leq p$ and $\mathcal{P}_{(\widehat{W}_i;\underline{r})}/\widetilde{C}_{\widehat{W}_i}$, $p+1 \leq i \leq p+q$.

Then, we aim to parameterize the manifold $\mathcal{P}_{(A;\underline{r})}/\widetilde{C}_A$ in two cases, when $A$ has only one real eigenvalue, $A = W(\lambda)$, and when $A$ has two conjugate complex eigenvalues, $A = W(\lambda, \overline{\lambda})$. To do this, we will obtain in both cases a local reduced form for the equivalence relation associated to the action of $\widetilde{C}_A$ on $\mathcal{P}_{(A;\underline{r})}$. This equivalence relation will be denoted by $\overset{\widetilde{C}_A}{\sim}$. That is to say, given $P, \widehat{P} \in \mathcal{P}_{(A;\underline{r})}$, we will write $P \overset{\widetilde{C}_A}{\sim} \widehat{P}$ if there exists $X \in \widetilde{C}_A$ such that $\widehat{P} = PX$. If there is no risk of confusion we will write $P{\sim}\widehat{P}$ instead of $P \overset{\widetilde{C}_A}{\sim} \widehat{P}$.

## 5.1 Reduced form when there is only a real eigenvalue

Let $W = W(\lambda)$, $\lambda \in \mathbb{R}$, with Weyr characteristic $(w_1, \dots, w_m)$ and assume that $\mathcal{P}_{(W;\underline{r})} \neq \emptyset$. The procedure to bring a matrix $P \in \mathcal{P}_{(W;\underline{r})}$ to a reduced form is based on a sequence of elementary transformations defined by some subgroups of $\widetilde{C}_W$. It is worth-recalling at this point the structure of the matrices in $C_W$ (Lemma 2.4) and that $\tau_i = w_{m-i+1}$, $1 \leq i \leq m$ and $\tau_0 = 0$ (see (19)).

**Definition 5.1**

1. *Let* $T_i \in \mathrm{Gl}(\tau_i - \tau_{i-1})$ , $1 \leq i \leq m$ *and* $Y_I = \mathrm{diag}(Y_{11}, \dots, Y_{mm})$ *with* $Y_{ii} = \mathrm{diag}(T_1, \dots, T_{m-i+1})$, $1 \leq i \leq m$. *The matrices of this type will be called* elementary matrices of type I *and they form a subgroup of* $\widetilde{C}_W$.
2. *For* $j = 1$, $1 \leq i < k \leq m$, *and for* $2 \leq j \leq m$, $1 \leq k \leq m - j + 1$ $1 \leq i \leq k + j - 1$ *let* $Y^{(j)}_{II,i,k}$ *be the matrix of* (17), *with, perhaps,* $D^{(j)}_{ik} \neq 0$,
$$D^{(1)}_{ii} = I_{\tau_i - \tau_{i-1}}, \quad 1 \leq i \leq m,$$
*and all the other blocks zero. This type of matrices will be called* elementary matrices of type II *and they form a subgroup of* $\widetilde{C}_W$.

In addition to these elementary matrices we will use some auxiliary results.

**Proposition 5.2** *Let* $P = \begin{bmatrix} P_1 \\ \vdots \\ P_k \end{bmatrix} \in \mathcal{P}_{(W,\underline{r})}$ *and partition* $P_i = \begin{bmatrix} P_{i1} & P_{i2} & \cdots & P_{im} \end{bmatrix}$ *with* $P_{ij} \in \mathbb{R}^{r_i \times w_j}$, $1 \leq i \leq k$, $1 \leq j \leq m$. *Then* $\operatorname{rank} P_{11} = w_1$.

**Proof.** Since $\operatorname{rank} P = \sum_{j=1}^m w_j$, $\operatorname{rank} \begin{bmatrix} P_{1j} \\ \vdots \\ P_{kj} \end{bmatrix} = w_j$, $1 \leq j \leq m$. On the other hand (see (34)), $P_{i+1} = I^T_{r_i, r_{i+1}} P_i W(\lambda) = I^T_{r_1, r_{i+1}} P_1 W(\lambda)^i$, $1 \leq i \leq k-1$. Thus for $i = 1, \dots, k-1$,

$$P_{i+1\,1} = I^T_{r_1, r_{i+1}} P_1 W(\lambda)^i \begin{bmatrix} I_{w_1} \\ 0 \end{bmatrix} = I^T_{r_1, r_{i+1}} P_1 \begin{bmatrix} \lambda^i I_{w_1} \\ 0 \end{bmatrix} = I^T_{r_1, r_{i+1}} P_{11} \lambda^i.$$

Then

$$\begin{bmatrix} P_{11} \\ \vdots \\ P_{k1} \end{bmatrix} = \begin{bmatrix} P_{11} \\ I^T_{r_1, r_2} \lambda P_{11} \\ \vdots \\ I^T_{r_1, r_k} \lambda^{k-1} P_{11} \end{bmatrix} = \mathrm{diag}(I_{r_1}, I^T_{r_1, r_2} \lambda, \dots I^T_{r_1, r_k} \lambda^{k-1}) P_{11}.$$

Therefore

$$w_1 = \operatorname{rank}\begin{bmatrix} P_{11} \\ \vdots \\ P_{k1} \end{bmatrix} = \operatorname{rank} P_{11}.$$

□

Recall (see Section 2) that if $s$ and $p$ are positive integers ($0 < s \le p$) then $Q_{s,p} := \{(i_1, \dots, i_s) \,:\, 1 \le i_1 < \cdots < i_s \le p\}$ and $Q_{0,p} := \{\emptyset\}$.

**Corollary 5.3** *Let $P \in \mathcal{P}_{(W;\underline{r})}$. Then, for each $j = 1, \dots, m$, there is a sequence of $\tau_j$ indices $\mathcal{I}_j \subseteq \{1, \dots, r\}$ such that*

$$\mathcal{I}_j \subseteq \mathcal{I}_{j+1}, \quad 1 \le j \le m-1, \tag{35}$$

$$\mathcal{I}_j \setminus \mathcal{I}_{j-1} \in Q_{\tau_j - \tau_{j-1}, r}, \quad 1 \le j \le m, \quad (\mathcal{I}_0 = \emptyset), \tag{36}$$

$$P(\mathcal{I}_j, 1 : \tau_j) \in \mathrm{Gl}(\tau_j), \quad 1 \le j \le m, \tag{37}$$

**Proof**. With the same notation as in Proposition 5.2, partition $P_{11} = \begin{bmatrix} P_{11}^{(1)} & P_{11}^{(2)} & \cdots & P_{11}^{(m)} \end{bmatrix}$ with $P_{11}^{(j)} \in \mathbb{R}^{r_1 \times (\tau_j - \tau_{j-1})}$, $1 \le j \le m$. By Proposition 5.2, $\operatorname{rank} P_{11} = w_1 = \tau_m$. Thus, $\operatorname{rank} \begin{bmatrix} P_{11}^{(1)} & \cdots & P_{11}^{(j)} \end{bmatrix} = \tau_j$, $1 \le j \le m$.

Since $\operatorname{rank} P_{11}^{(1)} = \tau_1$, in $P_{11}^{(1)}$ there must be $\tau_1$ linearly independent rows $i_1 < \cdots < i_{\tau_1}$. Then $\mathcal{I}_1 = (i_1, \dots, i_{\tau_1}) \in Q_{\tau_1, r} = Q_{\tau_1 - \tau_0, r}$ and $P(\mathcal{I}_1, 1 : \tau_1) = P_{11}^{(1)}(\mathcal{I}_1, :) \in \mathrm{Gl}(\tau_1)$. Now, $\operatorname{rank} \begin{bmatrix} P_{11}^{(1)} & P_{11}^{(2)} \end{bmatrix} = \tau_2$. Thus, in $P_{11}^{(2)}$ there must be $\tau_2 - \tau_1$ rows $i_{\tau_1+1} < i_{\tau_1+2} < \cdots < i_{\tau_2}$ such that the rows $i_1 < \cdots < i_{\tau_1}, i_{\tau_1+1} < \cdots < i_{\tau_2}$ of $\begin{bmatrix} P_{11}^{(1)} & P_{11}^{(2)} \end{bmatrix}$ are linearly independet. Put $\mathcal{I}_2 = (i_1, \dots, i_{\tau_2})$. Then $I_1 \subseteq I_2$, $I_2 \setminus I_1 = (i_{\tau_1+1}, \dots, i_{\tau_2}) \in Q_{\tau_2 - \tau_1, r}$, and $P(\mathcal{I}_2, 1 : \tau_2) = \begin{bmatrix} P_{11}^{(1)} & P_{11}^{(2)} \end{bmatrix}(\mathcal{I}_2, :) \in \mathrm{Gl}(\tau_2)$. Continuing the process, we can obtain $m$ sequences of indices, $\mathcal{I}_1, \dots, \mathcal{I}_m$ satisfying (35)– (37). □

**Definition 5.4** *Given $P \in \mathcal{P}_{(W;\underline{r})}$, let $\mathcal{I}_i$, $1 \le i \le m$, be sequences of indices satisfying (35)-(37). Then $\underline{\mathcal{I}} = (\mathcal{I}_1, \dots, \mathcal{I}_m)$ will be called an* admissible sequence of indices *for $P$.*

**Proposition 5.5** *Let $P, \widehat{P} \in \mathcal{P}_{(W;\underline{r})}$ be matrices such that $\widehat{P} \sim P$ and let $\underline{\mathcal{I}} = (\mathcal{I}_1, \dots, \mathcal{I}_m)$ be an admissible sequence of indices for $P$. Then $\underline{\mathcal{I}}$ is also an admissible sequence of indices for $\widehat{P}$.*

**Proof.** First of all, since $\underline{\mathcal{I}} = (\mathcal{I}_1, \dots, \mathcal{I}_m)$ is an admissible sequence of indices for $P$, it satisfies (35) and (36). So, it only remains to prove that $\widehat{P}(\mathcal{I}_j, 1 : \tau_j) \in \mathrm{Gl}(\tau_j)$ for $1 \le j \le m$.

Since $\widehat{P} \sim P$, there exists $Y \in \widetilde{C}_W$ such that $\widehat{P} = PY$ and so $\widehat{P}(\mathcal{I}_j, 1 : \tau_j) = P(\mathcal{I}_j, :)Y(:, 1 : \tau_j)$. By (18),

$$Y(:, 1 : \tau_j) = \left[\begin{array}{cccc} D_{11}^{(1)} & D_{12}^{(1)} & \dots & D_{1j}^{(1)} \\ 0 & D_{22}^{(1)} & \dots & D_{2j}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & D_{jj}^{(1)} \\ \hline 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{array}\right]$$

where $D_{ii}^{(1)} \in \mathrm{Gl}(\tau_i - \tau_{i-1})$, $1 \leq i \leq j$. On the other hand, it follows from (37) that for $1 \leq j \leq m$, $P(\mathcal{I}_j, 1:\tau_j) \in \mathrm{Gl}(\tau_j)$. Henceforth

$$\widehat{P}(\mathcal{I}_j, 1:\tau_j) = P(\mathcal{I}_j, :)Y(:, 1:\tau_j) = P(\mathcal{I}_j, 1:\tau_j) \begin{bmatrix} D_{11}^{(1)} & D_{12}^{(1)} & \dots & D_{1j}^{(1)} \\ 0 & D_{22}^{(1)} & \dots & D_{2,j}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & D_{jj}^{(1)} \end{bmatrix} \in \mathrm{Gl}(\tau_j)$$

and the Proposition follows. □

Let

$$\mathcal{A}_W = \{\underline{\mathcal{I}} = (\mathcal{I}_1, \dots, \mathcal{I}_m) \;:\; \mathcal{I}_j \text{ satisfies (35) and (36)}, \, 1 \leq j \leq m\}. \tag{38}$$

Given $\underline{\mathcal{I}} \in \mathcal{A}_W$, we denote by $\mathcal{U}_{\underline{\mathcal{I}}}$ the open subset of $\mathcal{P}_{(W;\underline{r})}$ formed by their matrices with $\underline{\mathcal{I}}$ as an admissible sequence of indices.

We are ready to show a procedure to bring any $P \in \mathcal{P}_{(W,\underline{r})}$ to a reduced form. First we illustrate this procedure with an example.

**Example 5.6** Consider the invariant polynomials of the matrix $A \in \mathbb{R}^{12\times 12}$ of Example 2.6. Let $W \in \mathbb{R}^{12\times 12}$ be its Weyr canonical form. Then its Weyr characteristic is $(w_1, \dots, w_4) = (6, 4, 1, 1)$, $s = \sum_{i=1}^{m} w_i = 12$, $\tau_1 - \tau_0 = 1$ (recall that $\tau_0 = 0$), $\tau_2 - \tau_1 = 0$, $\tau_3 - \tau_2 = 3$, $\tau_4 - \tau_3 = 2$ and $Y \in \widetilde{C}_W$ is the matrix of (21) with $d_{11}^{(1)} \in \mathrm{Gl}(1)$, $D_{33}^{(1)} \in \mathrm{Gl}(3)$ and $D_{44}^{(1)} \in \mathrm{Gl}(2)$. Let $\underline{r} = (7, 4, 2, 1)$ and note that $\underline{r}$ and $\underline{w}$ satisfies the conditions of Proposition 4.2 so that $\mathcal{P}_{(W,\underline{r})} \neq \emptyset$. Let

$$P = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix} \in \mathcal{P}_{(W;\underline{r})}$$

Put $P_1 = \begin{bmatrix} p_1 \\ \vdots \\ p_7 \end{bmatrix}$ and recall that $P_j = I_{r_1,r_j}^T P_1 W^{j-1}$, $2 \leq j \leq 4$. Thus

$$P_2 = \begin{bmatrix} p_1 W \\ \vdots \\ p_4 W \end{bmatrix}, \quad P_3 = \begin{bmatrix} p_1 W^2 \\ p_2 W^2 \end{bmatrix}, \quad P_4 = p_1 W^3.$$

Since $P_i$, $2 \leq i \leq 4$, can be obtained from $P_1$, we only need to reduce $P_1$. Put $P_1 = \begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} \end{bmatrix}$, $P_{1j} \in \mathbb{R}^{r_1 \times w_j}$, $1 \leq j \leq 4$, and partition $P_{1j}$ as follows:

$$P_{1j} = \begin{bmatrix} P_{11}^{(j)} & P_{12}^{(j)} & \cdots & P_{1\,5-j}^{(j)} \end{bmatrix}, \; P_{1k}^{(j)} \in \mathbb{R}^{r_1 \times (\tau_k - \tau_{k-1})}, \; 1 \leq j \leq 4, \, 1 \leq k \leq 5-j.$$

Specifically,

$$P_1 = \left[\begin{array}{c|ccc|cc||c|ccc||c||c} p_{11} & p_{12} & p_{13} & p_{14} & p_{15} & p_{16} & p_{17} & p_{18} & p_{19} & p_{1\,10} & p_{1\,11} & p_{1\,12} \\ p_{21} & p_{22} & p_{23} & p_{24} & p_{25} & p_{26} & p_{27} & p_{28} & p_{29} & p_{2\,10} & p_{2\,11} & p_{2\,12} \\ p_{31} & p_{32} & p_{33} & p_{34} & p_{35} & p_{36} & p_{37} & p_{38} & p_{39} & p_{3\,10} & p_{3\,11} & p_{3\,12} \\ p_{41} & p_{42} & p_{43} & p_{44} & p_{45} & p_{46} & p_{47} & p_{48} & p_{49} & p_{4\,10} & p_{4\,11} & p_{4\,12} \\ p_{51} & p_{52} & p_{53} & p_{54} & p_{55} & p_{56} & p_{57} & p_{58} & p_{59} & p_{5\,10} & p_{5\,11} & p_{5\,12} \\ p_{61} & p_{62} & p_{63} & p_{64} & p_{65} & p_{66} & p_{67} & p_{68} & p_{69} & p_{6\,10} & p_{6\,11} & p_{6\,12} \\ p_{71} & p_{72} & p_{73} & p_{74} & p_{75} & p_{76} & p_{77} & p_{78} & p_{79} & p_{7\,10} & p_{7\,11} & p_{7\,12} \end{array}\right].$$

Recall now that, by Proposition 5.2, $\operatorname{rank} P_{11} = w_1 = 6$. This means that $\operatorname{rank} P_{11}^{(1)} = \tau_1 = 1$, $\operatorname{rank}[\, P_{11}^{(1)} \; P_{12}^{(1)} \,] = \tau_2 = 1$, $\operatorname{rank}[\, P_{11}^{(1)} \; P_{12}^{(1)} \; P_{13}^{(1)} \,] = \tau_3 = 4$ and rank$\left[\, P^{(1)}_{11}\ P^{(1)}_{12}\ P^{(1)}_{13}\ P^{(1)}_{14} \,\right] = \tau_4 = 6$. Then $P^{(1)}_{12}$ is an empty matrix. Let us assume that, for example,

$$P^{(1)}_{11}(3,:) = p_{31} \neq 0, \qquad \det\begin{bmatrix} P^{(1)}_{11} & P^{(1)}_{13}\end{bmatrix}((3,1,4,7),:) \neq 0,$$

$$\det\begin{bmatrix} P^{(1)}_{11} & P^{(1)}_{12} & P^{(1)}_{13} & P^{(1)}_{14}\end{bmatrix}((3,1,4,7,5,6),:) \neq 0.$$

Then $\underline{\mathcal{I}} = ((3),(3,1,4,7),(3,1,4,7,5,6))$ is an admissible sequence of indices for $P$. For this admissible sequence of indices we define $Y^a_{11} = \operatorname{diag}(\left[p^{-1}_{31}\right], I_3, I_2)$, $Y^a_{22} = \operatorname{diag}(\left[p^{-1}_{31}\right], I_3)$, $Y^a_{33} = Y^a_{44} = \left[p^{-1}_{31}\right]$ and $Y^a_I = \operatorname{diag}(Y^a_{11}, Y^a_{22}, Y^a_{33}, Y^a_{44})$. Then $Y^a_I$ is an elementary matrix of type I (see Definition 5.1) and

$$P_1 Y^a_I = \left[\begin{array}{cccccc|cccc|c|c}
p^a_{11} & p_{12} & p_{13} & p_{14} & p_{15} & p_{16} & p^a_{17} & p_{18} & p_{19} & p_{1\,10} & p^a_{1\,11} & p^a_{1\,12} \\
p^a_{21} & p_{22} & p_{23} & p_{24} & p_{25} & p_{26} & p^a_{27} & p_{28} & p_{29} & p_{2\,10} & p^a_{2\,11} & p^a_{2\,12} \\
1 & p_{32} & p_{33} & p_{34} & p_{35} & p_{36} & p^a_{37} & p_{38} & p_{39} & p_{3\,10} & p^a_{3\,11} & p^a_{3\,12} \\
p^a_{41} & p_{42} & p_{43} & p_{44} & p_{45} & p_{46} & p^a_{47} & p_{48} & p_{49} & p_{4\,10} & p^a_{4\,11} & p^a_{4\,12} \\
p^a_{51} & p_{52} & p_{53} & p_{54} & p_{55} & p_{56} & p^a_{57} & p_{58} & p_{59} & p_{5\,10} & p^a_{5\,11} & p^a_{5\,12} \\
p^a_{61} & p_{62} & p_{63} & p_{64} & p_{65} & p_{66} & p^a_{67} & p_{68} & p_{69} & p_{6\,10} & p^a_{6\,11} & p^a_{6\,12} \\
p^a_{71} & p_{72} & p_{73} & p_{74} & p_{75} & p_{76} & p^a_{77} & p_{78} & p_{79} & p_{7\,10} & p^a_{7\,11} & p^a_{7\,12}
\end{array}\right]$$

where $p^a_{ij} = p_{ij}p^{-1}_{31}$ for $1 \leq i \leq 7$ and $j = 1, 7, 11, 12$.

Now, take the elementary matrix $Y^{(1)}_{II,1,3}$ of Definition 5.1 with $D^{(1)}_{13} = -\left[\, p_{32}\ p_{33}\ p_{34} \,\right]$. This is an elementary matrix of type II and

$$P_1 Y^a_I Y^{(1)}_{II,1,3} = \left[\begin{array}{cccccc|cccc|c|c}
p^a_{11} & p^a_{12} & p^a_{13} & p^a_{14} & p_{15} & p_{16} & p^a_{17} & p^a_{18} & p^a_{19} & p^a_{1\,10} & p^a_{1\,11} & p^a_{1\,12} \\
p^a_{21} & p^a_{22} & p^a_{23} & p^a_{24} & p_{25} & p_{26} & p^a_{27} & p^a_{28} & p^a_{29} & p^a_{2\,10} & p^a_{2\,11} & p^a_{2\,12} \\
1 & 0 & 0 & 0 & p_{35} & p_{36} & p^a_{37} & p^a_{38} & p^a_{39} & p^a_{3\,10} & p^a_{3\,11} & p^a_{3\,12} \\
p^a_{41} & p^a_{42} & p^a_{43} & p^a_{44} & p_{45} & p_{46} & p^a_{47} & p^a_{48} & p^a_{49} & p^a_{4\,10} & p^a_{4\,11} & p^a_{4\,12} \\
p^a_{51} & p^a_{52} & p^a_{53} & p^a_{54} & p_{55} & p_{56} & p^a_{57} & p^a_{58} & p^a_{59} & p^a_{5\,10} & p^a_{5\,11} & p^a_{5\,12} \\
p^a_{61} & p^a_{62} & p^a_{63} & p^a_{64} & p_{65} & p_{66} & p^a_{67} & p^a_{68} & p^a_{69} & p^a_{6\,10} & p^a_{6\,11} & p^a_{6\,12} \\
p^a_{71} & p^a_{72} & p^a_{73} & p^a_{74} & p_{75} & p_{76} & p^a_{77} & p^a_{78} & p^a_{79} & p^a_{7\,10} & p^a_{7\,11} & p^a_{7\,12}
\end{array}\right].$$

Defining elementary matrices $Y^{(1)}_{II,1,4}$, $Y^{(2)}_{II,1,1}$, $Y^{(2)}_{II,1,3}$, $Y^{(3)}_{II,1,1}$ and $Y^{(4)}_{II,1,1}$ of type II in a similar way we can zero out the remaining elements of the third row:

$$P^a_1 = \left[\begin{array}{cccccc|cccc|c|c}
p^a_{11} & p^a_{12} & p^a_{13} & p^a_{14} & p^a_{15} & p^a_{16} & p^{a'}_{17} & p^{a'}_{18} & p^{a'}_{19} & p^{a'}_{1\,10} & p^{a'}_{1\,11} & p^{a'}_{1\,12} \\
p^a_{21} & p^a_{22} & p^a_{23} & p^a_{24} & p^a_{25} & p^a_{26} & p^{a'}_{27} & p^{a'}_{28} & p^{a'}_{29} & p^{a'}_{2\,10} & p^{a'}_{2\,11} & p^{a'}_{2\,12} \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
p^a_{41} & p^a_{42} & p^a_{43} & p^a_{44} & p^a_{4,} & p^a_{46} & p^{a'}_{47} & p^{a'}_{48} & p^{a'}_{4,9} & p^{a'}_{4\,10} & p^{a'}_{4\,11} & p^{a'}_{4\,12} \\
p^a_{51} & p^a_{52} & p^a_{53} & p^a_{54} & p^a_{55} & p^a_{56} & p^{a'}_{57} & p^{a'}_{58} & p^{a'}_{5,9} & p^{a'}_{5\,10} & p^{a'}_{5\,11} & p^{a'}_{5\,12} \\
p^a_{61} & p^a_{62} & p^a_{63} & p^a_{64} & p^a_{65} & p^a_{66} & p^{a'}_{67} & p^{a'}_{68} & p^{a'}_{6,9} & p^{a'}_{6\,10} & p^{a'}_{6\,11} & p^{a'}_{6\,12} \\
p^a_{71} & p^a_{72} & p^a_{73} & p^a_{74} & p^a_{75} & p^a_{76} & p^{a'}_{77} & p^{a'}_{78} & p^{a'}_{7,9} & p^{a'}_{7\,10} & p^{a'}_{7\,11} & p^a_{7\,12}
\end{array}\right].$$

Let $P^a = \begin{bmatrix} P^a_1 \\ P^a_2 \\ P^a_3 \\ P^a_4 \end{bmatrix} \in \mathcal{P}_{(W;\underline{r})}$. Then $P^a \sim P$. By Proposition 5.5, $\underline{\mathcal{I}}$ is an admissible sequence of indices for $P^a$. Therefore $\det P^a_1((3,1,4,7), 1:4) \neq 0$ and $T_3 = \begin{bmatrix} p^a_{12} & p^a_{13} & p^a_{14} \\ p^a_{42} & p^a_{43} & p^a_{44} \\ p^a_{72} & p^a_{73} & p^a_{74} \end{bmatrix} \in \mathrm{Gl}(3)$. Put $Y^b_{11} = \operatorname{diag}(1, T^{-1}_3, I_2)$, $Y^b_{22} = \operatorname{diag}(1, T^{-1}_3)$, $Y^b_{33} = Y^b_{44} = 1$ and $Y^b_I = \operatorname{diag}(Y^b_{11}, Y^b_{22}, Y^b_{33}, Y^b_{44})$. Then

$$P^a_1 Y^b_I = \left[\begin{array}{cccccc|cccc|c|c}
p^a_{11} & 1 & 0 & 0 & p^a_{15} & p^a_{16} & p^{a'}_{17} & p^b_{18} & p^b_{19} & p^b_{1\,10} & p^{a'}_{1\,11} & p^{a'}_{1\,12} \\
p^a_{21} & p^b_{22} & p^b_{23} & p^b_{24} & p^a_{25} & p^a_{26} & p^{a'}_{7} & p^b_{28} & p^b_{29} & p^b_{2\,10} & p^{a'}_{2\,11} & p^{a'}_{2\,12} \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
p^a_{41} & 0 & 1 & 0 & p^a_{45} & p^a_{46} & p^{a'}_{47} & p^b_{48} & p^b_{49} & p^b_{4\,10} & p^{a'}_{4\,11} & p^{a'}_{4\,12} \\
p^a_{51} & p^b_{52} & p^b_{53} & p^b_{54} & p^a_{55} & p^a_{56} & p^{a'}_{57} & p^b_{58} & p^b_{59} & p^b_{5\,10} & p^{a'}_{5\,11} & p^{a'}_{5\,12} \\
p^a_{61} & p^b_{62} & p^b_{63} & p^b_{64} & p^a_{65} & p^a_{66} & p^{a'}_{67} & p^b_{68} & p^b_{69} & p^b_{6\,10} & p^{a'}_{6\,11} & p^{a'}_{6\,12} \\
p^a_{7,1} & 0 & 0 & 1 & p^a_{7,5} & p^a_{76} & p^{a'}_{77} & p^b_{78} & p^b_{79} & p^b_{7\,10} & p^{a'}_{7\,11} & p^a_{7\,12}
\end{array}\right].$$

Take the elementary matrix $Y^{(1)}_{II,3,4}$ with $D^{(1)}_{34} = -\begin{bmatrix} p^a_{15} & p^a_{16} \\ p^a_{45} & p^a_{46} \\ p^a_{75} & p^a_{76} \end{bmatrix}$. Then

$$P^a_1 Y^b_I Y^{(1)}_{II,3,4} = \left[\begin{array}{cccccc|cccc|c|c} p^a_{11} & 1 & 0 & 0 & 0 & 0 & p^{a'}_{17} & p^b_{18} & p^b_{19} & p^b_{1\,10} & p^{a'}_{1\,11} & p^{a'}_{1\,12} \\ p^a_{21} & p^b_{22} & p^b_{23} & p^b_{24} & p^{a'}_{25} & p^{a'}_{26} & p^{a'}_{27} & p^b_{,8} & p^b_{29} & p^b_{2\,10} & p^{a'}_{2\,11} & p^{a'}_{2\,12} \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ p^a_{41} & 0 & 1 & 0 & 0 & 0 & p^{a'}_{47} & p^b_{,8} & p^b_{49} & p^b_{4\,10} & p^{a'}_{4\,11} & p^{a'}_{4\,12} \\ p^a_{51} & p^b_{52} & p^b_{53} & p^b_{54} & p^{a'}_{55} & p^{a'}_{56} & p^{a'}_{57} & p^b_{58} & p^b_{59} & p^b_{5\,10} & p^{a'}_{5\,11} & p^{a'}_{5\,12} \\ p^a_{61} & p^b_{62} & p^b_{63} & p^b_{64} & p^{a'}_{65} & p^{a'}_{66} & p^{a'}_{67} & p^b_{68} & p^b_{69} & p^b_{6\,10} & p^{a'}_{6\,11} & p^{a'}_{6\,12} \\ p^a_{71} & 0 & 0 & 1 & 0 & 0 & p^{a'}_{77} & p^b_{78} & p^b_{79} & p^b_{7\,10} & p^{a'}_{7\,11} & p^a_{7\,12} \end{array}\right].$$

Repeating the process with appropriate elementary matices $Y^{(2)}_{II,3,3}$, $Y^{(3)}_{II,3,1}$ and $Y^{(4)}_{II,3,1}$ of type II we obtain

$$P^b_1 = \left[\begin{array}{cccccc|cccc|c|c} p^a_{11} & 1 & 0 & 0 & 0 & 0 & p^{a'}_{17} & 0 & 0 & 0 & 0 & 0 \\ p^a_{21} & p^b_{22} & p^b_{23} & p^b_{24} & p^{a'}_{25} & p^{a'}_{26} & p^{a'}_{27} & p^{b'}_{28} & p^{b'}_{29} & p^{b'}_{2\,10} & p^{a'}_{2\,11} & p^b_{2\,12} \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ p^a_{41} & 0 & 1 & 0 & 0 & 0 & p^{a'}_{47} & 0 & 0 & 0 & 0 & 0 \\ p^a_{51} & p^b_{52} & p^b_{53} & p^b_{54} & p^{a'}_{55} & p^{a'}_{56} & p^{a'}_{57} & p^{b'}_{58} & p^{b'}_{59} & p^{b'}_{5\,10} & p^{a'}_{5\,11} & p^b_{5\,12} \\ p^a_{61} & p^b_{62} & p^b_{63} & p^b_{64} & p^{a'}_{65} & p^{a'}_{66} & p^{a'}_{67} & p^{b'}_{68} & p^{b'}_{69} & p^{b'}_{6\,10} & p^{a'}_{6\,11} & p^b_{6\,12} \\ p^a_{71} & 0 & 0 & 1 & 0 & 0 & p^{a'}_{77} & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Finally, let $P^b = \begin{bmatrix} P^b_1 \\ P^b_2 \\ P^b_3 \\ P^b_4 \end{bmatrix} \in \mathcal{P}_{(W;\underline{r})}$. Then $P^b \sim P$. As before, by Proposition 5.5, $\underline{\mathcal{I}}$ is an admissible sequence of indices for $P^b$ and $T_4 = \begin{bmatrix} p^{a'}_{55} & p^{a'}_{56} \\ p^{a'}_{65} & p^{a'}_{66} \end{bmatrix} \in \mathrm{Gl}(2)$. Putting $Y^c_{11} = \mathrm{diag}(1, I_3, T_4^{-1})$, $Y^c_{22} = \mathrm{diag}(1, I_3)$, $Y^c_{33} = Y^c_{44} = 1$ and $Y^c_I = \mathrm{diag}(Y^c_{11}, Y^c_{22}, Y^c_{33}, Y^c_{44})$,

$$P^b_1 Y^c_I = \left[\begin{array}{cccccc|cccc|c|c} p^a_{11} & 1 & 0 & 0 & 0 & 0 & p^{a'}_{17} & 0 & 0 & 0 & 0 & 0 \\ p^a_{21} & p^b_{22} & p^b_{23} & p^b_{24} & p^c_{2,5} & p^c_{26} & p^{a'}_{2,7} & p^{b'}_{2,8} & p^{b'}_{29} & p^{b'}_{2\,10} & p^b_{2\,11} & p^b_{2\,12} \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ p^a_{41} & 0 & 1 & 0 & 0 & 0 & p^{a'}_{47} & 0 & 0 & 0 & 0 & 0 \\ p^a_{51} & p^b_{52} & p^b_{53} & p^b_{54} & 1 & 0 & p^{a'}_{57} & p^{b'}_{58} & p^{b'}_{59} & p^{b'}_{5,10} & p^b_{5\,11} & p^b_{5\,12} \\ p^a_{61} & p^b_{62} & p^b_{63} & p^b_{64} & 0 & 1 & p^{a'}_{67} & p^{b'}_{68} & p^{b'}_{69} & p^{b'}_{6\,10} & p^b_{6\,11} & p^b_{6\,12} \\ p^a_{71} & 0 & 0 & 0 & 1 & 0 & p^{a'}_{77} & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Using appropriate elementary matrices $Y^{(2)}_{II,4,3}$ and $Y^{(4)}_{II,4,1}$, we get

$$P^{(\mathrm{re})}_1 = \left[\begin{array}{cccccc|cccc|c|c} p^r_{11} & 1 & 0 & 0 & 0 & 0 & p^r_{17} & 0 & 0 & 0 & 0 & 0 \\ p^r_{21} & p^r_{22} & p^r_{23} & p^r_{24} & p^r_{25} & p^r_{26} & p^r_{27} & p^r_{28} & p^r_{29} & p^r_{2\,10} & p^r_{2\,11} & p^r_{2\,12} \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ p^r_{41} & 0 & 1 & 0 & 0 & 0 & p^r_{47} & 0 & 0 & 0 & 0 & 0 \\ p^r_{51} & p^r_{52} & p^r_{53} & p^r_{54} & 1 & 0 & p^r_{57} & 0 & 0 & 0 & p^r_{5\,11} & 0 \\ p^r_{61} & p^r_{62} & p^r_{63} & p^r_{64} & 0 & 1 & p^r_{67} & 0 & 0 & 0 & p^r_{6\,11} & 0 \\ p^r_{71} & 0 & 0 & 1 & 0 & 0 & p^r_{77} & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Let

$$P^{(\mathrm{re})} = \begin{bmatrix} P^{(r)}_1 \\ P^{(r)}_2 \\ P^{(r)}_3 \\ P^{(r)}_4 \end{bmatrix} = \begin{bmatrix} P^{(r)}_1 \\ I^T_{r_2,r_1} P^{(r)}_1 \\ I^T_{r_3,r_1} P^{(r)\,2}_1 \\ I^T_{r_4,r_1} P^{(r)\,3}_1 \end{bmatrix}.$$

We will say that $P^{(\mathrm{re})}$ is a matrix in *reduced form*. Observe that

$$P^{(\mathrm{re})}_1((3,1,4,7,5,6),:) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ p^r_{11} & 1 & 0 & 0 & 0 & 0 & p^r_{17} & 0 & 0 & 0 & 0 & 0 \\ p^r_{41} & 0 & 1 & 0 & 0 & 0 & p^r_{47} & 0 & 0 & 0 & 0 & 0 \\ p^r_{71} & 0 & 0 & 1 & 0 & 0 & p^r_{77} & 0 & 0 & 0 & 0 & 0 \\ p^r_{51} & p^r_{52} & p^r_{53} & p^r_{54} & 1 & 0 & p^r_{5,7} & 0 & 0 & 0 & p^r_{5\,11} & 0 \\ p^r_{61} & p^r_{62} & p^r_{63} & p^r_{64} & 0 & 1 & p^r_{67} & 0 & 0 & 0 & p_{6\,11} & 0 \end{bmatrix}$$

$$
= \left[\begin{array}{c|ccc|cc||c|ccc||c||c}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
p^r_{11} & 1 & 0 & 0 & 0 & 0 & p^r_{17} & 0 & 0 & 0 & 0 & 0 \\
p^r_{41} & 0 & 1 & 0 & 0 & 0 & p^r_{47} & 0 & 0 & 0 & 0 & 0 \\
p^r_{71} & 0 & 0 & 1 & 0 & 0 & p^r_{77} & 0 & 0 & 0 & 0 & 0 \\
\hline
p^r_{51} & p^r_{52} & p^r_{53} & p^r_{54} & 1 & 0 & p^r_{57} & 0 & 0 & 0 & p^r_{5\,11} & 0 \\
p^r_{61} & p^r_{62} & p^r_{63} & p^r_{64} & 0 & 1 & p^r_{67} & 0 & 0 & 0 & p_{6\,11} & 0
\end{array}\right]
= \left[\begin{array}{ccc|cc|c|c}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
P^{(r,1)}_{3,1} & I_3 & 0 & P^{(r,2)}_{3,1} & 0 & 0 & 0 \\
P^{(r,1)}_{4,1} & P^{(r,1)}_{4,3} & I_2 & P^{(r,2)}_{4,1} & 0 & P^{(r,3)}_{4,1} & 0
\end{array}\right],
$$

with $P^{(r,j)}_{i,k} \in \mathbb{R}^{(\tau_i - \tau_{i-1}) \times (\tau_k - \tau_{k-1})}$, $1 \le i \le m = 4$, $1 \le k \le m-1 = 3$, $1 \le k \le m - j$, and the number of parameters in $P^{(\mathrm{re})}$ is $30 = sr - N = 84 - 54$. $\square$

With this example in mind we define the notion of *reduced form* of a matrix in $\mathcal{P}_{(W;\underline{r})}$ and show that any matrix in this open set is $\widetilde{C}_W$-equivalent to a matrix in reduced form. Recall that $r = r_1$.

Let $P = \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_k \end{bmatrix} \in \mathcal{P}_{(W;\underline{r})}$ with $P_1 = \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1m} \end{bmatrix}$ and $P_{1j} \in \mathbb{R}^{r \times w_j}$, $1 \le j \le m$. Let $\underline{\mathcal{I}} = (\mathcal{I}_1, \ldots, \mathcal{I}_m)$ be an admissible sequence of indices for $P$, with $\mathcal{I}_j = (i_1, \ldots, i_{\tau_j})$, $1 \le j \le m$; in particular, $\mathcal{I}_m = (i_1, \ldots, i_{w_1})$. A matrix $R = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_k \end{bmatrix} \in \mathcal{P}_{(W;\underline{r})}$ with $R_1 = \begin{bmatrix} R_{11} & R_{12} & \cdots & R_{1m} \end{bmatrix}$ and $R_{1j} \in \mathbb{R}^{r \times w_j}$, $1 \le j \le m$, is said to be a *$\widetilde{C}_W$-reduced form of $P$ with respect to $\underline{\mathcal{I}}$* if

$$
R_{11}(\mathcal{I}_m, :) = \begin{bmatrix}
I_{\tau_1} & 0 & 0 & \cdots & 0 \\
R^{(1)}_{21} & I_{(\tau_2 - \tau_1)} & 0 & \cdots & 0 \\
R^{(1)}_{31} & R^{(1)}_{32} & I_{(\tau_3 - \tau_2)} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
R^{(1)}_{m1} & R^{(1)}_{m2} & R^{(1)}_{m3} & \cdots & I(\tau_m - \tau_{m-1})
\end{bmatrix} \tag{39}
$$

and for $j = 2, 3, \ldots, m$,

$$
R_{1j}(\mathcal{I}_m, :) =
$$

$$
\begin{array}{cc}
\begin{array}{cccccc}
\tau_1 & \tau_2 - \tau_1 & \tau_3 - \tau_2 & \cdots & \tau_{m-j} - \tau_{m-j-1} & \tau_{m-j+1} - \tau_{m-j}
\end{array} & \\
\begin{bmatrix}
0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 \\
R^{(j)}_{j+1\,1} & 0 & 0 & \cdots & 0 & 0 \\
R^{(j)}_{j+2\,1} & R^{(j)}_{j+2\,2} & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
R^{(j)}_{m1} & R^{(j)}_{m2} & R^{(j)}_{m3} & \cdots & R^{(j)}_{m\,m-j} & 0
\end{bmatrix} &
\begin{array}{c}
\tau_1 \\ \vdots \\ \tau_j - \tau_{j-1} \\ \tau_{j+1} - \tau_j \\ \tau_{j+2} - \tau_{j+1} \\ \vdots \\ \tau_m - \tau_{m-1}
\end{array}
\end{array} \tag{40}
$$

**Remark 5.7** (i) Note that $\mathcal{I}_m = (i_1, \ldots, i_{w_1})$ and so $R_{1j}(\mathcal{I}_m, :) \in \mathbb{R}^{w_1 \times w_j}$, $1 \le j \le m$; in particular, $R_{1m}(\mathcal{I}_m, :) = 0 \in \mathbb{R}^{w_1 \times w_m}$.

(ii) Since $R$ is assumed to be in $\mathcal{P}_{(W,\underline{r})}$, $R_j = I^T_{r_1, r_j} R_1 W^{j-1}$ for $1 \le j \le k$. Therefore the $\widetilde{C}_W$-reduced forms of $P$ with respect to $\underline{\mathcal{I}}$ are completely determined by $R_1$.

(iii) A detailed analysis of the zero-nonzero block pattern of $R_1$ yields the following characterization of the $\widetilde{C}_W$-reduced forms of $P$ with respect to $\underline{\mathcal{I}}$:

$$\begin{array}{ll} R^{(1)}_{ii} = I_{(\tau_i-\tau_{i-1})}, & 1 \leq i \leq m, \\ R^{(1)}_{ik} = 0, & 1 \leq i \leq m-1,\ i < k \leq m, \\ R^{(j)}_{ik} = 0, & 1 \leq i \leq m,\ 2 \leq j \leq m, \\ & \max\{i-j+1, 1\} \leq k \leq m-j+1. \end{array} \tag{41}$$

The two first conditions mean that $R_{11}(\underline{\mathcal{I}}_m, :)$ has the form of (39) and the third condition means that $R_{1j}(\underline{\mathcal{I}}_m, :)$ has the form of (40).

□

**Theorem 5.8** *Let $P \in \mathcal{P}_{(W;\underline{\tau})}$ and let $\underline{\mathcal{I}} = (\mathcal{I}_1, \ldots, \mathcal{I}_m)$ be an admissible sequence of indices for $P$. Then $P$ is $\widetilde{C}_W$-equivalent to a unique $\widetilde{C}_W$-reduced form with respect to $\underline{\mathcal{I}}$.*

**Proof.** Assume that $\mathcal{I}_j = (i_1, \ldots, i_{\tau_j})$, $1 \leq j \leq m$, let $s_j = \sum_{i=1}^{j} w_i$, $1 \leq j \leq m$ and put $s_m = s$. Write

$$P(\mathcal{I}_m, :) = \left[\begin{array}{ccc|ccc|c|c} P^{(1)}_{11} & \cdots & P^{(1)}_{1m} & P^{(2)}_{11} & \cdots & P^{(2)}_{1m-1} & \cdots & P^{(m)}_{11} \\ P^{(1)}_{21} & \cdots & P^{(1)}_{2m} & P^{(2)}_{21} & \cdots & P^{(2)}_{2m-1} & \cdots & P^{(m)}_{21} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ P^{(1)}_{m1} & \cdots & P^{(1)}_{mm} & P^{(2)}_{m1} & \cdots & P^{(2)}_{mm-1} & \cdots & P^{(m)}_{m1} \end{array}\right],$$

with $P^{(j)}_{ik} \in \mathbb{R}^{(\tau_i-\tau_{i-1})\times(\tau_k-\tau_{k-1})}$, $1 \leq i, j \leq m$, $1 \leq k \leq m-j+1$.

If $\widehat{P} = PY$ with $Y \in \widetilde{C}_W$, then $\widehat{P}(\mathcal{I}_m, :) = P(\mathcal{I}_m, :)Y$ and, by Proposition 5.5, $\underline{\mathcal{I}}$ is an admissible sequence of indices for $\widehat{P}$.

Since $\underline{\mathcal{I}}$ is an admissible sequence of indices for $P$, by (37), $\begin{bmatrix} P^{(1)}_{11} & \cdots & P^{(1)}_{1j} \\ \vdots & \vdots & \vdots \\ P^{(1)}_{j1} & \cdots & P^{(1)}_{jj} \end{bmatrix} = P(\mathcal{I}_j, 1:\tau_j) \in \mathrm{Gl}(\tau_j)$, $1 \leq j \leq m$. We will prove by induction on $\ell$ that, for $1 \leq \ell \leq m$, $P \sim P^{(\ell)}$ with

$$\begin{array}{ll} P^{(\ell,1)}_{ii} = I_{(\tau_i-\tau_{i-1})}, & 1 \leq i \leq \ell, \\ P^{(\ell,1)}_{ik} = 0, & 1 \leq i \leq \ell, \quad i < k \leq m, \\ P^{(\ell,j)}_{ik} = 0, & 1 \leq i \leq \ell,\ 2 \leq j \leq m, \\ & \max\{i-j+1, 1\} \leq k \leq m-j+1. \end{array} \tag{42}$$

Taking into account (41), this will prove that $P^{(m)}$ is a $\widetilde{C}_W$-reduced form of $P$ with respect to $\underline{\mathcal{I}}$.

- For $\ell = 1$, we have $P^{(1)}_{11} \in \mathrm{Gl}(\tau_1)$. Let $T_1 = {P^{(1)}_{11}}^{-1}$, $Y_{ii} = \mathrm{diag}(T_1, I_{(\tau_2-\tau_1)}, \ldots, I_{(\tau_{m-i+1}-\tau_{m-i})})$, $1 \leq i \leq m$ and $Y^{(a)}_I = \mathrm{diag}(Y_{11}, Y_{22}, \ldots, Y_{mm})$. Then $Y^{(a)}_I$ is an elementary matrix of type I and

$$P(\mathcal{I}_m, :)Y^{(a)}_I = \left[\begin{array}{cccc|ccc|c|c} I_{\tau_1} & P^{(a,1)}_{12} & \cdots & P^{(a,1)}_{1m} & P^{(a,2)}_{11} & \cdots & P^{(a,2)}_{1m-1} & \cdots & P^{(a,m)}_{11} \\ P^{(a,1)}_{21} & P^{(a,1)}_{22} & \cdots & P^{(a,1)}_{2m} & P^{(a,2)}_{21} & \cdots & P^{(a,2)}_{2m-1} & \cdots & P^{(a,m)}_{21} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ P^{(a,1)}_{m1} & P^{(a,1)}_{m2} & \cdots & P^{(a,1)}_{mm} & P^{(a,2)}_{m1} & \cdots & P^{(a,2)}_{mm-1} & \cdots & P^{(a,m)}_{m1} \end{array}\right],$$

where $P_{i1}^{(a,j)} = P_{i1}^{(j)}T_1$, $1 \le i,j \le m$ and $P_{ik}^{(a,j)}{=}P_{ik}^{(j)}$, $1 \le i,j \le m$, $2 \le k \le m-j+1$.

Now, take the elementary matrix $Y_{II,1,2}^{(1)}$ with $D_{12}^{(1)} = -P_{12}^{(a,1)}$. Then

$$P(\mathcal{I}_m,:)Y_I^{(a)}Y_{II,1,2}^{(1)} = \left[\begin{array}{cccc|ccc|c|c} I_{\tau_1} & 0 & \dots & P_{1m}^{(b,1)} & P_{1,1}^{(b,2)} & \cdots & P_{1m-1}^{(b,2)} & \cdots & P_{11}^{(b,m)} \\ P_{21}^{(b,1)} & P_{22}^{(b,1)} & \dots & P_{2m}^{(b,1)} & P_{21}^{(b,2)} & \cdots & P_{2m-1}^{(b,2)} & \cdots & P_{21}^{(b,m)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ P_{m1}^{(b,1)} & P_{m2}^{(b,1)} & \dots & P_{mm}^{(b,1)} & P_{m1}^{(b,2)} & \dots & P_{mm-1}^{(b,2)} & \cdots & P_{m1}^{(b,m)} \end{array}\right],$$

where $P_{i,2}^{(b,j)} = P_{i,2}^{(a,j)} - P_{i,1}^{(a,j)}P_{1,2}^{(a,1)}$, $1 \le i,j \le m$ and $P_{i,k}^{(b,j)}{=}P_{i,k}^{(a,j)}$, $1 \le i,j \le m$, $1 \le k \le m-j+1$, $k \ne 2$.

Repeating the same transformation with the appropriate elementary matrices $Y_{II,1,k}^{(1)}$, $3 \le k \le m$ and $Y_{II,1,k}^{(j)}$, $2 \le j \le m$, $1 \le k \le m-j+1$, we get

$$P^{(1)}(\mathcal{I}_m,:) = \left[\begin{array}{cccc|ccc|c|c} I_{\tau_1} & 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 \\ P_{21}^{(1,1)} & P_{22}^{(1,1)} & \dots & P_{2m}^{(1,1)} & P_{21}^{(1,2)} & \cdots & P_{2m-1}^{(1,2)} & \cdots & P_{21}^{(1,m)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ P_{m1}^{(1,1)} & P_{m2}^{(1,1)} & \dots & P_{mm}^{(1,1)} & P_{m1}^{(1,2)} & \dots & P_{mm-1}^{(1,2)} & \cdots & P_{m,1}^{(1,m)} \end{array}\right].$$

Then, $P^{(1)}$ satisfies (42) for $\ell = 1$.

- Assume now that $\ell \in \{2,\dots,m-1\}$ and $P \sim P^{(\ell)}$ with $P^{(\ell)}$ satisfying (42), i.e.

$$P^{(\ell)}(\underline{\mathcal{I}}_m,:) = \begin{bmatrix} P_{11}^{(\ell)} & P_{12}^{(\ell)} & \cdots & P_{1m}^{(\ell)} \end{bmatrix}$$

where

$$P_{11}^{(\ell)} = \begin{bmatrix} I_{\tau_1} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ P_{21}^{(\ell,1)} & I_{(\tau_2-\tau_1)} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ P_{\ell 1}^{(\ell,1)} & P_{\ell 2}^{(\ell,1)} & \cdots & I_{(\tau_\ell-\tau_{\ell-1})} & 0 & \cdots & 0 \\ P_{\ell+1\,1}^{(\ell,1)} & P_{\ell+1\,2}^{(\ell,1)} & \cdots & P_{\ell+1\,\ell}^{(\ell,1)} & P_{\ell+1\,\ell+1}^{(\ell,1)} & \cdots & P_{\ell+1\,m}^{(\ell,1)} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ P_{m1}^{(\ell,1)} & P_{m2}^{(\ell,1)} & \cdots & P_{m\ell}^{(\ell,1)} & P_{m\,\ell+1}^{(\ell,1)} & \cdots & P_{mm}^{(\ell)} \end{bmatrix},$$

$$P_{1j}^{(\ell)} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ P_{j+1\,1}^{(\ell,j)} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ P_{\ell\,1}^{(\ell,j)} & P_{\ell\,2}^{(\ell,j)} & \cdots & P_{\ell\,\ell-j}^{(\ell,j)} & 0 & \cdots & 0 \\ P_{\ell+1\,1}^{(\ell,j)} & P_{\ell+1\,2}^{(\ell,j)} & \cdots & P_{\ell+1\,\ell-j}^{(\ell,j)} & P_{\ell+1\,\ell-j+1}^{(\ell,j)} & \cdots & P_{\ell+1\,m-j+1}^{(\ell,j)} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ P_{m1}^{(\ell,j)} & P_{m2}^{(\ell,j)} & \cdots & P_{m\,\ell-j}^{(\ell,j)} & P_{m\,\ell-j+1}^{(\ell,j)} & \cdots & P_{m\,m-j+1}^{(\ell,j)} \end{bmatrix}, \quad 2 \le j \le \ell-1,$$

and

$$P_{1j}^{(\ell)} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \\ P_{\ell+1\,1}^{(\ell,j)} & P_{\ell+1\,2}^{(\ell,j)} & \cdots & P_{\ell+1\,m-j+1}^{(\ell,j)} \\ \vdots & \vdots & & \vdots \\ P_{m1}^{(\ell,j)} & P_{m2}^{(\ell,j)} & \cdots & P_{m\,m-j+1}^{(\ell,j)} \end{bmatrix}, \quad \ell \le j \le m.$$

By Proposition 5.5, $\underline{\mathcal{I}}$ is an admissible sequence of indices for $P^{(\ell)}$. Thus

$$P_{11}^{(\ell)}((1,\ldots,\tau_{\ell+1}),(1,\ldots,\tau_{\ell+1})) = \begin{bmatrix} I_{\tau_1} & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ P_{\ell 1}^{(\ell,1)} & \cdots & I_{(\tau_\ell-\tau_{\ell-1})} & 0 \\ P_{\ell+1\,1}^{(\ell,1)} & \cdots & P_{\ell+1\,\ell}^{(\ell,1)} & P_{\ell+1\,\ell+1}^{(\ell,1)} \end{bmatrix} \in \mathrm{Gl}(\tau_{\ell+1}),$$

and, consequently, $P_{\ell+1\,\ell+1}^{(\ell,1)} \in \mathrm{Gl}(\tau_{\ell+1}-\tau_\ell)$. Let $T_{\ell+1} = {P_{\ell+1\,\ell+1}^{(\ell,1)}}^{-1}$,

$$\begin{array}{rll} Y'_{jj} = & \mathrm{diag}(I_{\tau_1},\ldots,T_{\ell+1},\ldots,I_{(\tau_{m-j+1}-\tau_{m-j})}), & 1\le j\le m-\ell, \\ Y'_{jj} = & \mathrm{diag}(I_{\tau_1},\ldots,I_{(\tau_{m-j+1}-\tau_{m-j})}), & m-\ell+1\le j\le m, \end{array}$$

and $Y_I^{(a')} = \mathrm{diag}(Y'_{11},Y'_{22},\ldots,Y'_{mm})$. For $1\le j\le m-\ell$, the matrix $P_{1j}^{(\ell)}Y'_{jj}$ is obtained form $P_{1j}^{(\ell)}$ by replacing the block $P_{i\,\ell+1}^{(\ell,j)}$ by the block $P_{i\,\ell+1}^{(\ell,j)}T_{\ell+1}$ $(\ell+1\le i\le m)$, and for $m-\ell+1\le j\le m$, $P_{1j}^{(\ell)}Y'_{jj} = P_{1j}^{(\ell)}$.

Now, using successively appropriate matrices $Y_{II,\ell+1,k}^{(1)}$, $\ell+2\le k\le m$, we can annihilate the blocks $P_{\ell+1\,\ell+2}^{(\ell,1)},\ldots,P_{\ell+1\,m}^{(\ell,1)}$. Similarly, with appropriate matrices $Y_{II,\ell+1,k}^{(j)}$, $2\le j\le m$, $\max\{\ell-j+2,1\}\le k\le m-j+1$, we can annihilate the blocks $P_{\ell+1\,\ell-j+2}^{(\ell,j)}$, $\ldots$, $P_{\ell+1\,m-j+1}^{(\ell,1)}$ for $j=2,\ldots\ell$ and $P_{\ell+1\,1}^{(\ell,j)},\ldots,$ $P_{\ell+1\,m-j+1}^{(\ell,j)}$ for $j=\ell+1,\ldots m$. Therefore (42) holds for $1\le\ell\le m$.

Setting $P^{(\mathrm{re})} = P^{(m)}$, $P^{(\mathrm{re})}$ satisfies (41) and so it is a $\widetilde{C}_W$-reduced form of $P$ with respect to $\underline{\mathcal{I}}$.

Let us see now that the matrix $P^{(\mathrm{re})}$ is unique, that is to say, that if $P\sim\widehat{P}^{(\mathrm{re})}$ with

$$\begin{array}{lll} \widehat{P}_{ii}^{(\mathrm{re},1)} = I_{(\tau_i-\tau_{i-1})}, & 1\le i\le m, & \\ \widehat{P}_{ik}^{(\mathrm{re},1)} = 0, & 1\le i\le m-1, & i<k\le m, \\ \widehat{P}_{ik}^{(\mathrm{re},j)} = 0, & 1\le i\le m, & 2\le j\le m, \\ & \max\{i-j+1,1\}\le k\le m-j+1. & \end{array}$$

then $P^{(\mathrm{re})} = \widehat{P}^{(\mathrm{re})}$.

As $P\sim\widehat{P}^{(\mathrm{re})}$, there exists $Y\in\widetilde{C}_W$ such that $P^{(\mathrm{re})}Y = \widehat{P}^{(\mathrm{re})}$; in particular,

$$P^{(\mathrm{re})}(\mathcal{I}_m,:)Y = \widehat{P}^{(\mathrm{re}}(\mathcal{I}_m,:),$$

where $Y$ is the matrix of (17) and so, it satisfies the properties (18), (19) and (20). It is then enough to prove that $Y_{11} = I_{w_1}$ and $Y_{1j} = 0$ for $j=2,\ldots,m$ because, by (20), this would imply that $Y = I_s$ (recall that $s=\sum_{i=1}^m w_i$).

We can split the columns of $P^{(\mathrm{re})}(\mathcal{I}_m,:)$ and $\widehat{P}^{(\mathrm{re}}(\mathcal{I}_m,:)$ as follows

$$P^{(\mathrm{re})}(\mathcal{I}_m,:) = \begin{bmatrix} R_1 & R_2 & \cdots & R_m \end{bmatrix}, \quad \widehat{P}^{(\mathrm{re})}(\mathcal{I}_m,:) = \begin{bmatrix} \widehat{R}_1 & \widehat{R}_2 & \cdots & \widehat{R}_m \end{bmatrix},$$

with $R_j,\widehat{R}_j\in\mathbb{R}^{w_1\times w_j}$, $1\le j\le m$. Then $R_1$ and $\widehat{R}_1$ are lower block-triangular matrices with identity matrices as diagonal blocks (cf. (39)) and $Y_{11} = R_1^{-1}\widehat{R}_1$ is also a lower block-triangular matrices with identity matrices as diagonal blocks. However, by definition (see (18)), $Y_{11}$ is an upper block-triangular matrix whose blocks are of the same size as the blocks of $R_1$ and $\widehat{R}_1$. Hence $Y_{11} = I_{w_1}$ and by (20), $Y_{jj} = I_{w_j}$ for $1 \le j \le m$.

Let us prove now by induction that, for $j \in \{2, 3, \dots, m\}$, $Y_{1j} = 0$. In fact, $\widehat{R}_2 = R_1Y_{12} + R_2Y_{22} = R_1Y_{12} + R_2$ because $Y_{22} = I_{w_2}$. Thus, $Y_{12} = R_1^{-1}(\widehat{R}_2 - R_2)$. Now, by (18) and (40) $Y_{12}$ and $\widehat{R}_2 - R_2$ are matrices with the form of (18) and (40) with $j = 2$, respectively. Since $R_1^{-1}$ is a lower block-triangular matrix, $R_1^{-1}(\widehat{R}_2 - R_2)$ has the same zero-nonzero block pattern as $\widehat{R}_2 - R_2$. Therefore $Y_{12} = \widehat{R}_2 - R_2 = 0$.

Assume that $Y_{1j} = 0$ for $j = 1, \dots, \ell - 1$ with $3 \le \ell \le m$. By (20), $Y_{i\ell} = 0$ for $i = 2, \dots, \ell - 1$ and so $\widehat{R}_\ell = R_1Y_{1\ell} + R_\ell Y_{\ell\ell} = R_1Y_{1\ell} + R_\ell$. As above, $Y_{1\ell} = R_1^{-1}(\widehat{R}_\ell - R_\ell)$ and since $Y_{1\ell}$ and $R_1^{-1}(\widehat{R}_\ell - R_\ell)$ have complementary zero-nonzero block structures, $Y_{1\ell} = \widehat{R}_\ell - R_\ell = 0$. Therefore $Y = I_s$ and $P^{(\mathrm{re})} = \widehat{P}^{(\mathrm{re})}$. □

**Remark 5.9** The number of parameters of $P^{(\mathrm{re})}$ is the number of parameters of $P$ minus the number of parameters of $Y$. That is to say $\dim \mathcal{P}_{(W;\underline{r})} - \dim C_W = rs - N_W$, where $s = \sum_{i=1}^m w_i$ and $N_W = \sum_{i=1}^m w_i^2$ (see Remark 2.5). Note that it follows from Proposition 4.2 that $r = r_1 \ge w_i$, $1 \le i \le m$ and so $rs \ge N_W$. □

## 5.2 Reduced form when $M$ has two conjugated complex eigenvalues

Let $\widehat{W} = \widehat{W}(\lambda, \overline{\lambda}))$, $\lambda = a+bi \in \mathbb{C}\backslash\mathbb{R}$, with Weyr characteristic $(w_1, \dots, w_m)$ and $B = \left[\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right]$. As before, $\tau_i = w_{m-i+1}$, $0 \le i \le m$ $(w_{m+1} = 0)$, $s_j = \sum_{i=1}^j w_i$, $1 \le j \le m$ $(s_m = s)$ and assume that $\mathcal{P}_{(\widehat{W};\underline{r})} \ne \emptyset$ (see Proposition 4.2); in particular, $r_1 \ge w_1$.

We will use additional notation. Let $B_0 = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$. Given a row vector $z = \begin{bmatrix} z_1 & z_2 \end{bmatrix} \in \mathbb{R}^{1\times 2}$, $Z^\diamond$ denotes the matrix $Z^\diamond = \left[\begin{smallmatrix} z \\ zB_0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} z_1 & z_2 \\ -z_2 & z_1 \end{smallmatrix}\right] \in \mathbb{R}^{2\times 2}$. As noted in Section 2.3 (see the note after Lemma 2.7), $C_B = C_{B_0} = \{Z^\diamond : z \in \mathbb{R}^{1\times 2}\}$. Since $\det Z^\diamond = z_1^2 + z_2^2$, $Z^\diamond \in \widetilde{C}_B$ if and only if $z \ne 0$.

If $Z = \begin{bmatrix} z_{11} & \cdots & z_{1n} \\ \vdots & & \vdots \\ z_{m1} & \cdots & z_{mn} \end{bmatrix} \in \mathbb{R}^{m\times 2n}$, with $z_{ij} \in \mathbb{R}^{1\times 2}$, $1 \le i \le m$, $1 \le j \le n$, $Z^\diamond$ is the matrix $Z^\diamond = \begin{bmatrix} Z_{11}^\diamond & \cdots & Z_{1n}^\diamond \\ \vdots & & \vdots \\ Z_{m1}^\diamond & \cdots & Z_{mn}^\diamond \end{bmatrix} \in \mathbb{R}^{2m\times 2n}$. Recall (see (12)) that $I_n \otimes B = \operatorname{diag}(\overbrace{B, \dots, B}^{n})$. Since $C_B = C_{B_0} = \{Z^\diamond : z \in \mathbb{R}^{1\times 2}\}$ it is easy to see that

$$C_{B^{(n)}} = C_{B_0^{(n)}} = \{Z^\diamond : Z \in \mathbb{R}^{n\times 2n}\}$$

Recall (Lemma 2.7) that $Y \in C_{\widehat{W}}$ if and only if $Y$ has the structure of (17) satisfying the properties (18), (20) and for $1 \le i, j \le m$ and $\max\{i-j+1, 1\} \le k \le m-j+1$, (see (22)) $D_{i,k}^{(j)} = \begin{bmatrix} T_{\alpha\,\beta}^{(j)} \end{bmatrix}_{\substack{\tau_{i-1}+1 \le \alpha \le \tau_i \\ \tau_{k-1}+1 \le \beta \le \tau_k}} \in \mathbb{R}^{2(\tau_i - \tau_{i-1})\times 2(\tau_k - \tau_{k-1})}$ and $T_{\alpha,\beta}^{(j)} = \begin{bmatrix} x_{\alpha\,\beta}^{(j)} & y_{\alpha\,\beta}^{(j)} \\ -y_{\alpha\,\beta}^{(j)} & x_{\alpha\,\beta}^{(j)} \end{bmatrix} \in \mathbb{R}^{2\times 2}$. Therefore, we can write $D_{i,k}^{(j)} = Z_{i,k}^{(j)\diamond} \in \mathbb{R}^{2(\tau_i - \tau_{i-1})\times 2(\tau_k - \tau_{k-1})}$ with $Z_{i,k}^{(j)} \in \mathbb{R}^{(\tau_i - \tau_{i-1})\times 2(\tau_k - \tau_{k-1})}$, $1 \le i, j \le m$, $\max\{i-$

$j+1,1\} \leq k \leq m-j+1$. In addition $Y \in \widetilde{C}_{\widehat{W}}$ if and only if

$$D_{i,i}^{(1)} \in \widetilde{C}_{B^{(\tau_i-\tau_{i-1})}} \quad 1 \leq i \leq m.$$

**Definition 5.10**

1. *Let* $T_i \in \widetilde{C}_{B^{(\tau_i-\tau_{i-1})}}$, $1 \leq i \leq m$ *and* $\widehat{Y_I} = \operatorname{diag}(\widehat{Y_{11}}, \ldots, \widehat{Y_{mm}})$ *with* $\widehat{Y_{ii}} = \operatorname{diag}(T_1, \ldots, T_{m-i+1})$, $1 \leq i \leq m$. *The matrices of this type will be called* elementary matrices of type I *and they form a subgroup of* $\widetilde{C}_{\widehat{W}}$.
2. *For* $j=1$, $1 \leq i < k \leq m$, *and for* $2 \leq j \leq m$, $1 \leq k \leq m-j+1$ $1 \leq i \leq k+j-1$, *let* $\widehat{Y_{II,i,k}^{(j)}}$ *be a matrix of* (18) *with, perhaps,* $D_{ik}^{(j)} \neq 0$,

$$D_{ii}^{(1)} = I_{2(\tau_i-\tau_{i-1})}, \quad 1 \leq i \leq m,$$

*and all the other blocks zero. This type of matrices will be called* elementary matrices of type II *and they form a subgroup of* $\widetilde{C}_{\widehat{W}}$.

**Lemma 5.11** *Given an integer* $i \geq 2$ *and* $Z \in \mathbb{R}^{m \times 2n}$,

$$\operatorname{rank}\begin{bmatrix} Z \\ ZB^{(n)} \\ \vdots \\ Z{B^{(n)}}^{i-1} \end{bmatrix} = \operatorname{rank}\begin{bmatrix} Z \\ ZB^{(n)} \end{bmatrix}.$$

**Proof.** If $Z = \begin{bmatrix} Z_1 & \ldots & Z_n \end{bmatrix} \in \mathbb{R}^{m \times 2n}$, with $Z_j \in \mathbb{R}^{m \times 2}$, $1 \leq j \leq n$, then

$$\begin{bmatrix} Z \\ ZB^{(n)} \\ \vdots \\ Z{B^{(n)}}^{i-1} \end{bmatrix} = \begin{bmatrix} Z_1 & \cdots & Z_n \\ Z_1 B & \cdots & Z_n B \\ \vdots & & \vdots \\ Z_1 B^{i-1} & \cdots & Z_n B^{i-1} \end{bmatrix}.$$

It follows from $B^2 = 2aB-(a^2+b^2)I_2$ that $Z_jB^k = 2aZ_jB^{k-1}-(a^2+b^2)Z_jB^{k-2}$, for $1 \leq j \leq n$ and $k \geq 2$. Therefore there exists $S \in \mathrm{Gl}(mi)$ such that

$$S\begin{bmatrix} Z \\ ZB^{(n)} \\ \vdots \\ Z{B^{(n)}}^{i-1} \end{bmatrix} = \begin{bmatrix} Z \\ ZB^{(n)} \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

and the lemma follows. $\square$

**Proposition 5.12** *Let*

$$P = \begin{bmatrix} P_1 \\ \vdots \\ P_k \end{bmatrix} \in \mathcal{P}_{(\widehat{W},\underline{r})}, \; P_i = \begin{bmatrix} P_{i1} & P_{i2} & \cdots & P_{im} \end{bmatrix} \tag{43}$$

*with* $P_{ij} \in \mathbb{R}^{r_i \times 2w_j}$, $1 \leq i \leq k$ *and* $1 \leq j \leq m$. *Then* $\operatorname{rank} P_{11}^{\diamond} = 2w_1$.

**Proof.** Since $P \in \mathcal{P}_{(\widehat{W},\underline{r})}$, $\operatorname{rank} P = \sum_{j=1}^{m} 2w_j$ and so $\operatorname{rank}\begin{bmatrix} P_{1j} \\ \vdots \\ P_{kj} \end{bmatrix} = 2w_j$, $1 \leq j \leq m$. On the other hand, it follows from $P_{i+1} = I_{r_1,r_{i+1}}^T P_1 \widehat{W}^i$, $1 \leq i \leq k-1$, that $P_{i+1\,1} = I^T_{r_1,r_{i+1}} P_{11} B^{(w_1)^i}$, $1 \le i \le k-1$. Thus,

$$\begin{bmatrix} P_{11} \\ \vdots \\ P_{k1} \end{bmatrix} = \begin{bmatrix} P_{11} \\ I^T_{r_1,r_2} P_{11} B^{(w_1)} \\ \vdots \\ I^T_{r_1,r_k} P_{11} B^{(w_1)^{k-1}} \end{bmatrix} = \operatorname{diag}(I_{r_1}, I^T_{r_1,r_2} \dots, I^T_{r_1,r_k}) \begin{bmatrix} P_{11} \\ P_{11} B^{(w_1)} \\ \vdots \\ P_{11} B^{(w_1)^{k-1}} \end{bmatrix}.$$

Hence

$$2w_1 = \operatorname{rank} \begin{bmatrix} P_{11} \\ \vdots \\ P_{k1} \end{bmatrix} = \operatorname{rank} \begin{bmatrix} P_{11} \\ P_{11} B^{(w_1)} \\ \vdots \\ P_{11} B^{(w_1)^{k-1}} \end{bmatrix}.$$

By Lemma 5.11, $\operatorname{rank} \begin{bmatrix} P_{11} \\ P_{11} B^{(w_1)} \end{bmatrix} = 2w_1$.

Put

$$P_{11} = \begin{bmatrix} z_{11} & \cdots & z_{1w_1} \\ \vdots & & \vdots \\ z_{r1} & \cdots & z_{rw_1} \end{bmatrix}, \quad z_{ij} \in \mathbb{R}^{1\times 2}, \quad 1 \le i \le r, \, 1 \le j \le w_1. \tag{44}$$

Then

$$2w_1 = \operatorname{rank} \begin{bmatrix} P_{11} \\ P_{11} B^{(w_1)} \end{bmatrix} = \operatorname{rank} \left[\begin{array}{ccc} z_{11} & \cdots & z_{1w_1} \\ \vdots & & \vdots \\ z_{r1} & \cdots & z_{rw_1} \\ \hline z_{11}B & \cdots & z_{1w_1}B \\ \vdots & & \vdots \\ z_{r1}B & \cdots & z_{rw_1}B \end{array}\right] = \operatorname{rank} \left[\begin{array}{ccc} z_{11} & \cdots & z_{1w_1} \\ z_{11}B & \cdots & z_{1w_1}B \\ \hline \vdots & & \vdots \\ \hline z_{r1} & \cdots & z_{rw_1} \\ z_{r1}B & \cdots & z_{rw_1}B \end{array}\right].$$

Let $T = \begin{bmatrix} 1 & 0 \\ -\frac{a}{b} & \frac{1}{b} \end{bmatrix}$. Then

$$T \begin{bmatrix} z_{ij} \\ z_{ij}B \end{bmatrix} = \begin{bmatrix} z_{ij} \\ z_{ij}B_0 \end{bmatrix}, \quad 1 \le i \le r, \quad 1 \le j \le w_1.$$

Therefore

$$\operatorname{diag}(\overbrace{T, \dots, T}^{r}) \left[\begin{array}{ccc} z_{11} & \cdots & z_{1w_1} \\ z_{11}B & \cdots & z_{1w_1}B \\ \hline \vdots & & \vdots \\ \hline z_{r1} & \cdots & z_{rw_1} \\ z_{r1}B & \cdots & z_{rw_1}B \end{array}\right] = \left[\begin{array}{ccc} z_{11} & \cdots & z_{1w_1} \\ z_{11}B_0 & \cdots & z_{1,w_1}B_0 \\ \hline \vdots & & \vdots \\ \hline z_{r1} & \cdots & z_{rw_1} \\ z_{r1}B_0 & \cdots & z_{rw_1}B_0 \end{array}\right] = \begin{bmatrix} Z^\diamond_{11} & \cdots & Z^\diamond_{1w_1} \\ \vdots & & \vdots \\ Z^\diamond_{r1} & \cdots & Z^\diamond_{rw_1} \end{bmatrix} = P^\diamond_{11}.$$

As $\operatorname{diag}(\overbrace{T, \dots, T}^{r}) \in \mathrm{Gl}(2r)$, $\operatorname{rank} P^\diamond_{11} = 2w_1$. □

**Proposition 5.13** *Let $P \in \mathcal{P}_{(W;\underline{r})}$. Then, for each $j = 1, \dots, m$, there is a sequence of $\tau_j$ indices $\mathcal{I}_j \subseteq \{1, \dots, r\}$ satisfying* (35), (36) *and*

$$P(\mathcal{I}_j, 1 : 2\tau_j)^\diamond \in \mathrm{Gl}(2\tau_j), \quad 1 \le j \le m \tag{45}$$

**Proof.** Let $P \in \mathcal{P}_{(\widehat{W};\underline{r})}$ be the matrix of (43) and let $P_{11} \in \mathbb{R}^{r\times 2w_1}$ be that of (44). Write $P_{11} = \begin{bmatrix} P^{(1)}_{11} & P^{(2)}_{11} & \cdots & P^{(m)}_{11} \end{bmatrix}$ with $P^{(j)}_{11} \in \mathbb{R}^{r_1 \times 2(\tau_j - \tau_{j-1})}$, $1 \le j \le m$ and $X_j = \begin{bmatrix} P^{(1)}_{11} & P^{(2)}_{11} & \cdots & P^{(j)}_{11} \end{bmatrix} \in \mathbb{R}^{r\times 2\tau_j}$, $1 \le j \le m$. We claim that $\operatorname{rank} X_j = \tau_j$, $1 \le j \le m$. In fact, as in the proof of Proposition 5.12,

$\operatorname{rank} P_{11}^{\diamond}(:, 1:2\tau_j) = \operatorname{rank}\begin{bmatrix} X_j \\ X_j B_0^{(\tau_j)} \end{bmatrix}$. Since $\operatorname{rank} P_{11}^{\diamond} = 2w_1$, bearing in mind that $B_0$ is invertible,

$$\left.\begin{array}{rcl} 2\tau_j & = & \operatorname{rank} P_{11}^{\diamond}(:, 1:2\tau_j) = \operatorname{rank}\begin{bmatrix} X_j \\ X_j B_0^{(\tau_j)} \end{bmatrix} \\ & = & \operatorname{rank}\left(\begin{bmatrix} X_j & 0 \\ 0 & X_j \end{bmatrix}\begin{bmatrix} I_{2\tau_j} & 0 \\ 0 & B_0^{(\tau_j)} \end{bmatrix}\right) = 2\operatorname{rank} X_j. \end{array}\right\}, \quad 1 \le j \le m.$$

Since $\operatorname{rank} P_{11}^{(1)} = \tau_1$, in $P_{11}^{(1)}$ there must be $\tau_1$ linearly independent rows $i_1 < \cdots < i_{\tau_1}$. Then $\mathcal{I}_1 = (i_1, \ldots, i_{\tau_1}) \in Q_{\tau_1, r} = Q_{\tau_1 - \tau_0, r}$ and $\operatorname{rank} P(\mathcal{I}_1, 2\tau_1)^{\diamond} = \operatorname{rank} P_{11}^{(1)}(\mathcal{I}_1, :)^{\diamond} = \operatorname{rank}\begin{bmatrix} P_{11}^{(1)}(\mathcal{I}_1, :) \\ P_{11}^{(1)}(\mathcal{I}_1 :) B_0^{(\tau_1)} \end{bmatrix} = 2\tau_1$.

Now, $\operatorname{rank}\begin{bmatrix} P_{11}^{(1)} & P_{11}^{(2)} \end{bmatrix} = \operatorname{rank} X_2 = \tau_2$. Thus, in $P_{11}^{(2)}$ there must be $\tau_2 - \tau_1$ rows $i_{\tau_1+1} < i_{\tau_1+2} < \cdots < i_{\tau_2}$ such that the rows $i_1 < \cdots < i_{\tau_1}$, $i_{\tau_1+1} < \cdots < i_{\tau_2}$ of $X_2$ are linearly independent. Put $\mathcal{I}_2 = (i_1, \ldots, i_{\tau_2})$. Then $I_1 \subseteq I_2$, $I_2 \setminus I_1 = (i_{\tau_1+1}, \ldots, i_{\tau_2}) \in Q_{\tau_2 - \tau_1, r}$, and $\operatorname{rank} P(\mathcal{I}_2, 2\tau_2)^{\diamond} = \operatorname{rank} X_2(\mathcal{I}_2, :)^{\diamond} = \operatorname{rank}\begin{bmatrix} X_2(\mathcal{I}_2, :) \\ X_2(\mathcal{I}_2 :) B_0^{(\tau_2)} \end{bmatrix} = 2\tau_2$. Continuing the process, we can obtain $m$ sequences $\mathcal{I}_1, \ldots, \mathcal{I}_m$ satisfying (35), (36) and (45). □

**Definition 5.14** *Given $P \in \mathcal{P}_{(\widehat{W}; \underline{r})}$, let $\mathcal{I}_i$, $1 \le i \le m$, be sequences of indices satisfying* (35), (36) *and* (45). *Then we say that $\underline{\mathcal{I}} = (\mathcal{I}_1, \ldots, \mathcal{I}_m)$ is an* admissible sequence of indices *for $P$.*

**Proposition 5.15** *Let $P, \widehat{P} \in \mathcal{P}_{(\widehat{W}; \underline{r}}$ be matrices such that $\widehat{P} \sim P$ and let $\underline{\mathcal{I}} = (\mathcal{I}_1, \ldots, \mathcal{I}_m)$ be an admissible sequence of indices for $P$. Then $\underline{\mathcal{I}}$ is also an admissible sequence of indices for $\widehat{P}$.*

**Proof.** The proof is analogous to that of Proposition 5.5. □

Let

$$\mathcal{A}_{\widehat{W}} = \{\underline{\mathcal{I}} = (\mathcal{I}_1, \ldots, \mathcal{I}_m) \,:\, \mathcal{I}_j \text{ satisfies (35) and (36)}, 1 \le j \le m\}. \tag{46}$$

For $\underline{\mathcal{I}} \in \mathcal{A}_{\widehat{W}}$, $\mathcal{U}_{\underline{\mathcal{I}}}$ denotes the open subset of $\mathcal{P}_{(\widehat{W}; \underline{r})}$ formed by the matrices of $\mathcal{P}_{(\widehat{W}; \underline{r})}$ with $\underline{\mathcal{I}}$ as an admissible sequence of indices.

As in the case of only one real eigenvalue (Section 5.1) we introduce now the notion of reduced form of a matrix in $\mathcal{P}_{(\widehat{W}; \underline{r})}$.

Let $P = \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_k \end{bmatrix} \in \mathcal{P}_{(\widehat{W}; \underline{r})}$ with $P_1 = \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1m} \end{bmatrix}$ and $P_{1j} \in \mathbb{R}^{r \times 2w_j}$, $1 \le j \le m$. Let $\underline{\mathcal{I}} = (\mathcal{I}_1, \ldots, \mathcal{I}_m)$ be an admissible sequence of indices for $P$, with $\mathcal{I}_j = (i_1, \ldots, i_{\tau_j})$, $1 \le j \le m$. A matrix $R = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_k \end{bmatrix} \in \mathcal{P}_{(\widehat{W}; \underline{r})}$ with $R_1 = \begin{bmatrix} R_{11} & R_{12} & \cdots & R_{1m} \end{bmatrix}$ and $R_{1j} \in \mathbb{R}^{r \times 2w_j}$, $1 \le j \le m$, is said to be a *$\widetilde{C}_{\widehat{W}}$-reduced form of $P$ with respect to $\underline{\mathcal{I}}$* if

$$R_{11}(\mathcal{I}_m,:)^\diamond = \begin{bmatrix} I_{2\tau_1} & 0 & 0 & \cdots & 0 \\ R_{21}^{(1)\diamond} & I_{2(\tau_2-\tau_1)} & 0 & \cdots & 0 \\ R_{31}^{(1)\diamond} & R_{32}^{(1)\diamond} & I_{2(\tau_3-\tau_2)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ R_{m1}^{(1)\diamond} & R_{m2}^{(1)\diamond} & R_{m3}^{(1)\diamond} & \cdots & I_{2(\tau_m-\tau_{m-1})} \end{bmatrix}$$

and for $j = 2, 3, \ldots, m$,

$$R_{1j}(\mathcal{I}_m,:)^\diamond =$$

$$\begin{array}{cc} \begin{array}{cccccc} 2\tau_1 & 2(\tau_2-\tau_1) & 2(\tau_3-\tau_2) & \cdots & 2(\tau_{m-j}-\tau_{m-j-1}) & 2(\tau_{m-j+1}-\tau_{m-j}) \end{array} & \\ \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ R_{j+1\,1}^{(j)\diamond} & 0 & 0 & \cdots & 0 & 0 \\ R_{j+2\,1}^{(j)\diamond} & R_{j+2\,2}^{(j)\diamond} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ R_{m1}^{(j)\diamond} & R_{m2}^{(j)\diamond} & R_{m3}^{(j)\diamond} & \cdots & R_{m\,m-j}^{(j)\diamond} & 0 \end{bmatrix} & \begin{array}{c} 2\tau_1 \\ \vdots \\ 2(\tau_j-\tau_{j-1}) \\ 2(\tau_{j+1}-\tau_j) \\ 2(\tau_{j+2}-\tau_{j+1}) \\ \vdots \\ 2(\tau_m-\tau_{m-j}) \end{array} \end{array}$$

**Theorem 5.16** *Let $P \in \mathcal{P}_{(\widehat{W};\underline{r})}$ and let $\underline{\mathcal{I}} = (\mathcal{I}_1, \ldots, \mathcal{I}_m)$ be an admissible sequence of indices for $P$. Then $P$ is $\widetilde{C}_{\widehat{W}}$-equivalent to a unique $\widetilde{C}_{\widehat{W}}$-reduced form $P^{(\mathrm{re})} \in \mathcal{P}_{(\widehat{W};\underline{r})}$ with respect to $\underline{\mathcal{I}}$.*

**Proof.** The proof is analogous to that of Theorem 5.8, replacing $P_{ik}^{(j)}$ by $P_{ik}^{(j)\diamond}$ in (42) and using the elementary matrices $\widehat{Y_I}$ and $\widehat{Y_{II,i,k}^{(j)}}$ instead of $Y_I$ and $Y_{II,i,k}^{(j)}$. □

**Remark 5.17** The number of parameters of $P^{(\mathrm{re})}$ is $2sr - N_{\widehat{W}} = \dim \mathcal{P}_{(\widehat{W},\underline{r})} - \dim \widetilde{C}_{\widehat{W}}$.

## 5.3 Local parameterization and local system of coordinates of $\mathcal{P}_{(A;\underline{r})}/\widetilde{C}_A$

We consider now the general case: let $\underline{\alpha} : \alpha_1(s) \mid \cdots \mid \alpha_n(s)$ be monic polynomials such that $\sum_{i=1}^n \deg(\alpha_i(s)) = n$ and assume that (9) is the prime factorization of $\alpha_{n-i+1}(s)$. Let $A$ be the associated real Weyr canonical form of (33). Assume also that condition (24) holds true. It follows from Remark 4.3 that $\mathcal{P}_{(A;\underline{r})} \neq \emptyset$.

Given $P, \widehat{P} \in \mathcal{P}_{(A;\underline{r})}$, we can partition $P$ and $\widehat{P}$ in the form

$$P = \begin{bmatrix} P_1 & \ldots & P_p & P_{p+1} & \ldots & P_{p+q} \end{bmatrix}, \ \widehat{P} = \begin{bmatrix} \widehat{P}_1 & \ldots & \widehat{P}_p & \widehat{P}_{p+1} & \ldots & \widehat{P}_{p+q} \end{bmatrix},$$

with $P_i, \widehat{P}_i \in \mathcal{P}_{(W_i;\underline{r})}$, $1 \le i \le p$ and $P_i, \widehat{P}_i \in \mathcal{P}_{(\widehat{W}_i;\underline{r})}$, $p+1 \le i \le p+q$.

Then $P \overset{\widetilde{C}_A}{\sim} \widehat{P}$ if and only if $P_i \overset{\widetilde{C}_{W_i}}{\sim} \widehat{P}_i$, $1 \le i \le p$, and $P_i \overset{\widetilde{C}_{\widehat{W}_i}}{\sim} \widehat{P}_i$, $p+1 \le i \le p+q$.

**Definition 5.18** *With the above notation, let* $P = \begin{bmatrix} P_1 & \dots & P_p & P_{p+1} & \dots & P_{p+q} \end{bmatrix} \in \mathcal{P}_{(A;\underline{r})}$ *and let* $\underline{\mathcal{I}}^{(i)}$ *be an admissible sequence of indices for* $P_i$, $1 \le i \le p+q$. *Then we say that* $\underline{\mathcal{I}} = (\underline{\mathcal{I}}^{(1)}, \dots, \underline{\mathcal{I}}^{(p+q)})$ *is a* multi-index *for* $P$.

**Definition 5.19** *Let* $P = \begin{bmatrix} P_1 & \dots & P_p & P_{p+1} & \dots & P_{p+q} \end{bmatrix} \in \mathcal{P}_{(A;\underline{r})}$ *and let* $\underline{\mathcal{I}} = (\underline{\mathcal{I}}^{(1)}, \dots, \underline{\mathcal{I}}^{(p+q)})$ *be a* multi-index *for* $P$. *Let* $P_i^{(\mathrm{re})}$ *be the* $\widetilde{C}_{W_i}$*-reduced form of* $P_i$ *with respect to* $\underline{\mathcal{I}}^{(i)}$, $1 \le i \le p$, *and let* $P_i^{(\mathrm{re})}$ *be the* $\widetilde{C}_{\widehat{W}_i}$ *reduced form of* $P_i$ *with respect to* $\underline{\mathcal{I}}^{(i)}$ , $p+1 \le i \le p+q$. *Then*

$$P^{(\mathrm{re})} = \begin{bmatrix} P_1^{(\mathrm{re})} & \dots & P_p^{(\mathrm{re})} & P_{p+1}^{(\mathrm{re})} & \dots & P_{p+q}^{(\mathrm{re})} \end{bmatrix}$$

*is said to be the* $\widetilde{C}_A$-reduced form *of* $P$.

Let $s^{(i)} = \sum_{j=1}^{m_{i,1}} w_{i,j}$, $1 \le i \le p+q$, $N_i = \dim \widetilde{C}_{W_i}$, $1 \le i \le p$, and $N_i = \dim \widetilde{C}_{\widehat{W}_i}$, $p+1 \le i \le p+q$, then $n = \sum_{i=1}^{p} s^{(i)} + 2\sum_{i=p+1}^{p+q} s^{(i)}$ and $N = \dim \widetilde{C}_A = \sum_{i=1}^{p+q} N_i$. The number of parameters in $P^{(\mathrm{re})}$ is $\sum_{i=1}^{p}(s^{(i)}r - N_i) + \sum_{i=p+1}^{p+q}(2s^{(i)}r - N_i) = nr - N$.

Recalling the definitions of $\mathcal{A}_W$ and $\mathcal{A}_{\widehat{W}}$ in (38) and (46), respectively, let

$$\mathcal{A}_A = \mathcal{A}_{W_1} \times \cdots \times \mathcal{A}_{W_p} \times \mathcal{A}_{\widehat{W}_{p+1}} \times \cdots \times \mathcal{A}_{\widehat{W}_{p+q}}.$$

Given $\underline{\mathcal{I}} = (\underline{\mathcal{I}}^{(1)}, \dots, \underline{\mathcal{I}}^{(p+q)}) \in \mathcal{A}_A$, let $\mathcal{U}_{\underline{\mathcal{I}}} = \mathcal{U}_{\underline{\mathcal{I}}^{(1)}} \times \cdots \times \mathcal{U}_{\underline{\mathcal{I}}^{(p+q)}}$. Note that the matrices in $\mathcal{U}_{\underline{\mathcal{I}}^{(j)}}$ are, for $1 \le j \le p+q$, full column rank matrices; however, there may be matrices in $\mathcal{U}_{\underline{\mathcal{I}}}$ which do not have full column rank. Thus, we must define $\mathcal{V}_{\underline{\mathcal{I}}} = \mathcal{P}_{(A;\underline{r})} \cap \mathcal{U}_{\underline{\mathcal{I}}}$. This is an open subset of $\mathcal{P}_{(A;\underline{r})}$.

Let $\mathcal{R}_{\underline{\mathcal{I}}^{(i)}}$ denote the subset of $\mathcal{U}_{\underline{\mathcal{I}}^{(i)}}$ formed by the matrices $P_i^{(\mathrm{re})} \in \mathcal{U}_{\underline{\mathcal{I}}^{(i)}}$ in $\widetilde{C}_{W_i}$-reduced form or $\widetilde{C}_{\widehat{W}_i}$-reduced form with respect to $\underline{\mathcal{I}}^{(i)}$ according as $1 \le i \le p$ or $p+1 \le i \le p+q$. Now, for $1 \le i \le p$, let $\nu_i : \mathbb{R}^{s^{(i)}r - N_i} \longrightarrow \mathcal{R}_{\underline{\mathcal{I}}^{(i)}}$ be defined as follows: Let $x \in \mathbb{R}^{s^{(i)}r - N_i}$. Use the $(\tau_2 - \tau_1)\tau_1$ first elements of $x$ to successively construct, row by row, a $(\tau_2 - \tau_1) \times \tau_1$ matrix and call it $R_{2,1}^{(1)}$ as in (39). Then use the following $(\tau_3 - \tau_2)\tau_1$ elements of $x$ to successively construct, row by row, a $(\tau_3 - \tau_2) \times \tau_1$ matrix and call it $R_{31}^{(1)}$ as in (39). Use the following $(\tau_3 - \tau_2)(\tau_2 - \tau_1)$ to construct the matrix $R_{32}^{(1)}$ of (39) . Use the same rules to successively construct the remaining blocks of the lower block-triangular matrix $R_{11}(\mathcal{I}_{m_i}, :)$ of (39). Then use the remaining elements of $x$ to construct the matrices $R_{1j}(\mathcal{I}_{m_i}, :)$ of (40), $2 \le j \le m_i - 1$ (note that $R_{1m_i}(\mathcal{I}_{m_i}, :) = 0$). Next, for $p+1 \le i \le p+q$ define $\nu_i : \mathbb{R}^{2s^{(i)}r - N_i} \longrightarrow \mathcal{R}_{\underline{\mathcal{I}}^{(i)}}$ to map $x \in \mathbb{R}^{2s^{(i)}r - N_i}$ into $(R_{11}(\mathcal{I}_{m_i}, :)^{\diamond}, R_{12}(\mathcal{I}_{m_i}, :)^{\diamond}, \dots, R_{1m_i - 1}(\mathcal{I}_{m_i}, :)^{\diamond})$ where these matrices are constructed as in the previous case.

Define now $\mathcal{R}_{\underline{\mathcal{I}}} = \mathcal{R}_{\underline{\mathcal{I}}^{(1)}} \times \cdots \times \mathcal{R}_{\underline{\mathcal{I}}^{(p+q)}}$ and $\nu_{\underline{\mathcal{I}}} : \mathbb{R}^{nr-N} \longrightarrow \mathcal{R}_{\underline{\mathcal{I}}}$ as $\nu(x) = (\nu_1(x_1), \dots, \nu_p(x_p), \nu_{p+1}(x_{p+1}), \dots, \nu_{p+q}(x_{p+q}))$ where $x = (x_1, \dots, x_p, x_{p+1}, \dots, x_{p+q})$ and $x_i \in \mathbb{R}^{s^{(i)}r - N_i}$ if $1 \le i \le p$ and $x_i \in \mathbb{R}^{2s^{(i)}r - N_i}$ if $p+1 \le i \le q$ . It is plain that $\nu_{\underline{\mathcal{I}}}$ is a diffeomorphism and we can identify $\mathcal{R}_{\underline{\mathcal{I}}}$ with $\mathbb{R}^{nr-N}$.

Let $\mathcal{W}_{\underline{\mathcal{I}}} = \nu_{\underline{\mathcal{I}}}^{-1}(\mathcal{R}_{\underline{\mathcal{I}}} \cap \mathcal{P}_{(A;\underline{r})})$. Then $\mathcal{W}_{\underline{\mathcal{I}}}$ is an open subset of $\mathbb{R}^{nr-N}$. If $x \in \mathcal{W}_{\underline{\mathcal{I}}}$, we will will denote $\nu_{\underline{\mathcal{I}}}(x)$ by $P_x^{(\mathrm{re})}$

Let $\pi$ be the submersion $\pi : \mathcal{P}_{(A;\underline{r})} : \longrightarrow \mathcal{P}_{(A;\underline{r})}/\widetilde{C}_A$. Then $\pi$ is an open map (see, for example, [15, Theorem 7.16]) and so $\widetilde{\mathcal{V}_{\underline{\mathcal{I}}}} = \pi(\mathcal{V}_{\underline{\mathcal{I}}})$ is an open subset of $\mathcal{P}_{(A;\underline{r})}/\widetilde{C}_A$.

**Theorem 5.20** *The map $\psi_{\underline{\mathcal{I}}} : \mathcal{W}_{\underline{\mathcal{I}}} \longrightarrow \widetilde{\mathcal{V}}_{\underline{\mathcal{I}}}$, defined by $\psi_{\underline{\mathcal{I}}}(x) = \widetilde{P_x^{(\mathrm{re})}}$, where $\widetilde{P_x^{(\mathrm{re})}}$ is the orbit of $P_x^{(\mathrm{re})}$ under the action of $\widetilde{C}_A$, is a diffeomorphism.*

**Proof.** It is clear that $\psi_{\underline{\mathcal{I}}}$ is well defined and bijective.

On one hand, the map $\varphi_{\underline{\mathcal{I}}} : \mathcal{W}_{\underline{\mathcal{I}}} \longrightarrow \mathcal{V}_{\underline{\mathcal{I}}}$ defined by $\varphi_{\underline{\mathcal{I}}}(x) = P_x^{(\mathrm{re})}$ is differentiable. Hence, the map $\psi_{\underline{\mathcal{I}}} = \pi \mid_{\mathcal{V}_{\underline{\mathcal{I}}}} \circ \varphi_{\underline{\mathcal{I}}}$ is also differentiable.

On the other hand, the map $\alpha_{\underline{\mathcal{I}}} : \mathcal{V}_{\underline{\mathcal{I}}} \longrightarrow \mathcal{R}_{\underline{\mathcal{I}}} \cap \mathcal{P}_{(A;\underline{r})}$ defined by $\alpha_{\underline{\mathcal{I}}}(P) = P^{(\mathrm{re})}$, where $P^{(\mathrm{re})}$ is the $\widetilde{C}_A$-reduced form of $P$ with respect to $\underline{\mathcal{I}}$, is differentiable. Hence, the map $\eta_{\underline{\mathcal{I}}} = \nu_{\underline{\mathcal{I}}}^{-1} \circ \alpha_{\underline{\mathcal{I}}} : \mathcal{V}_{\underline{\mathcal{I}}} \longrightarrow \mathcal{W}_{\underline{\mathcal{I}}}$ is also differenciable. Since $\eta_{\underline{\mathcal{I}}} = \psi_{\underline{\mathcal{I}}}^{-1} \circ \pi \mid_{\mathcal{V}_{\underline{\mathcal{I}}}}$, by Proposition 7.17 of [15], we conclude that $\psi_{\underline{\mathcal{I}}}^{-1}$ is differentiable. $\square$

**Remark 5.21** The map $\psi_{\underline{\mathcal{I}}}$ defined in Theorem 5.20 is a local parameterization and $\psi_{\underline{\mathcal{I}}}^{-1}$ is a local system of coordinates for $\mathcal{P}_{(A;\underline{r})}/\widetilde{C}_A$.

# 6 Local parameterization and local system of coordinates of $\mathcal{H}_{(F,G)}$

Assume that we are given a sequence of monic polynomials $\underline{\alpha} : \alpha_1(s) \mid \cdots \mid \alpha_n(s)$ such that $\sum_{i=1}^n \deg(\alpha_i) = n$ and a controllable pair $(F,G) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$ in $p$-Brunovsky canonical form (cf. (23)) with $\underline{r} : r_1 \cdots \geq r_k > 0 = r_{k+1} = \cdots = r_n$ as Brunovsky indices. Let $\underline{k} : k_1 \geq \cdots \geq k_r > 0 = k_{r+1} = \cdots = k_m$ be its controllability indices and assume that condition (24) holds. Then, by Proposition 2.9, $\mathcal{H}_{(F,G)} \neq \emptyset$. Assume also (Remark 4.6) that $G = \begin{bmatrix} G_1 & 0 \end{bmatrix}$, $G_1 \in \mathbb{R}^{n\times r}$ and $\operatorname{rank} G_1 = r$. Also, let $\underline{r} : r_1 \cdots \geq r_k > 0 = r_{k+1} = \cdots = r_n$ be the Brunovsky indices of $(F,G)$ and let $A \in \mathcal{O}(\underline{\alpha})$. Then the map $\phi : \mathcal{P}_{(A;\underline{r})}/\widetilde{C}_A \longrightarrow \mathcal{H}_{(F,G_1)}$ defined in Theorem 4.8, is a diffeomorphism.

Assume that $A$ is in real Weyr canonical form (cf. (33)) and with the notation of Section 5.3, let $\underline{\mathcal{I}} \in \mathcal{A}_A$ and $\widehat{\mathcal{V}}_I = \phi(\widetilde{\mathcal{V}}_{\underline{\mathcal{I}}})$. Then $\widehat{\mathcal{V}}_I$ is an open subset of $\mathcal{H}_{(F,G_1)}$. Hence, if $\psi_{\underline{\mathcal{I}}} : \mathcal{W}_{\underline{\mathcal{I}}} \longrightarrow \widetilde{\mathcal{V}}_{\underline{\mathcal{I}}}$ is the diffeomorphism defined in Theorem 5.20, then $\alpha_{\underline{\mathcal{I}}} = \phi \circ \psi_{\underline{\mathcal{I}}} : \mathcal{W}_{\underline{\mathcal{I}}} \longrightarrow \widehat{\mathcal{V}}_{\underline{\mathcal{I}}}$ is also a diffeomorphism.

Recall that $\mathcal{H}_{(F,G)} = \mathcal{H}_{(F,G_1)} \times \mathbb{R}^{(m-r)\times n}$. Then $\widehat{\mathcal{V}'}_I = \widehat{\mathcal{V}}_I \times \mathbb{R}^{(m-r)\times n}$ is an open subset of $\mathcal{H}_{(F,G)}$ and

$$\begin{array}{rccc} \alpha'_{\underline{\mathcal{I}}} : & \mathcal{W}_{\underline{\mathcal{I}}} \times \mathbb{R}^{(m-r)\times n} & \longrightarrow & \widehat{\mathcal{V}}'_{\underline{\mathcal{I}}} \\ & (x, K_2) & \mapsto & \begin{bmatrix} \alpha_{\underline{\mathcal{I}}}(x) \\ K_2 \end{bmatrix} \end{array}$$

is a diffeomorphism. Therefore, $\alpha'_{\underline{\mathcal{I}}}$ is a local parameterization and $\psi_{\underline{\mathcal{I}}}^{-1}$ is a local system of coordinates of $\mathcal{H}_{(F,G)}$.

**Remark 6.1** If $m = n$ and $r_1 = n$, then $(F,G)$ is feedback equivalent to $(0, I_n)$ and $\mathcal{H}_{(F,G)} = \mathcal{O}(\underline{\alpha})$. In this case we obtain a parameterization of $\mathcal{O}(\underline{\alpha})$.

We finish with an example illustrating the whole procedure to obtain a parametrization of $\mathcal{H}_{(F,G)}$ when $(F,G)$ is in $p$-Brunovsky canonical form.

**Example 6.2** Let $n = 5$, $\alpha_1(s) = \alpha_2(s) = \alpha_3(s) = 1$, $\alpha_4(s) = s$, $\alpha_5(s) = s^2(s^2+1)$, and let $\underline{r} = (r_1, r_2, r_3) = (2,2,1)$. Then $\underline{\alpha} = 1 \mid 1 \mid 1 \mid s \mid s^2(s^2+1)$ and the Segre characteristic of any $A \in \mathcal{O}(\underline{\alpha})$ for the eigenvalue 0 is $(2,1)$ and that for the eigenvalues $i$ and $-i$ is $(1)$. Thus their Weyr characteristics are the conjugate partitions of $(2,1)$ and $(1)$, respectivley:

$$w(0) = (2,1), \quad w(i) = w(-i) = (1).$$

Also $\tau_1 = 2$, $\tau_2 = 1$ for the eigenvalue 0 and $\tau_1 = 1$ for the eigenvalues $i$ and $-i$.

On the other hand, the controllability indices of $(F,G)$ are $\underline{k} = (3,2)$. Therefore, $(3,2) \prec (4,1)$ and, by Proposition 2.9, $\mathcal{H}_{(F,G)} \neq \emptyset$.

Let

$$A = \left[\begin{array}{cc|c||cc} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline\hline 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{array}\right] = \begin{bmatrix} W(0) & 0 \\ 0 & \widehat{W}(i,-i) \end{bmatrix},$$

and

$$\begin{bmatrix} F & G \end{bmatrix} = \left[\begin{array}{cc|cc|c||cc} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right].$$

Note that

$$w(0) \cup w(i) \cup w(-i) = (2,1,1,1) \prec \underline{r} = (2,2,1).$$

It follows from (26) and Proposition 2.9 that $\mathcal{H}_{(F,G)} \neq \emptyset$ and from Remark 4.3 that $\mathcal{P}_{(A;\underline{r})} \neq \emptyset$.

$$\mathcal{P}_{(A;\underline{r})} = \left\{ \left[\begin{array}{c} p_1 \\ p_2 \\ \hline p_1 A \\ p_2 A \\ \hline p_1 A^2 \end{array}\right] = \left[\begin{array}{cc|c||cc} p_{11}^{(1)} & p_{12}^{(1)} & p_{11}^{(2)} & p_{14} & p_{15} \\ p_{21}^{(1)} & p_{22}^{(1)} & p_{21}^{(2)} & p_{24} & p_{25} \\ \hline 0 & 0 & p_{11}^{(1)} & -p_{15} & p_{14} \\ 0 & 0 & p_{21}^{(1)} & -p_{25} & p_{24} \\ \hline 0 & 0 & 0 & -p_{14} & -p_{15} \end{array}\right] \in \mathrm{Gl}(5) \right\}.$$

The map

$$\begin{array}{rccc} \phi: & \mathcal{P}_{(A;\underline{r})}/\widetilde{C}_A & \longrightarrow & \mathcal{H}_{(F,G)} \\ & \widetilde{P} & \mapsto & \begin{bmatrix} p_1 A^3 \\ p_2 A^2 \end{bmatrix} P^{-1}, \end{array}$$

is a diffeomorphism. The possible multi-indices for the matrices in $\mathcal{P}_{(A;\underline{r})}$ are $\underline{\mathcal{I}} = (\underline{\mathcal{I}}^{(1)}, \underline{\mathcal{I}}^{(2)})$ with $\underline{\mathcal{I}}^{(1)} = ((1),(1,2))$ or $\underline{\mathcal{I}}^{(1)} = ((2),(2,1))$, and $\underline{\mathcal{I}}^{(2)} = (1)$ or $\underline{\mathcal{I}}^{(2)} = (2)$.

Assume that $\underline{\mathcal{I}} = (\underline{\mathcal{I}}^{(1)}, \underline{\mathcal{I}}^{(2)})$ with $\underline{\mathcal{I}}^{(1)} = ((2),(2,1))$ and $\underline{\mathcal{I}}^{(2)} = (1)$; i.e., $p_{21}^{(1)} \neq 0$, $\begin{bmatrix} p_{11}^{(1)} & p_{12}^{(1)} \\ p_{21}^{(1)} & p_{22}^{(1)} \end{bmatrix} \in \mathrm{Gl}(2)$, and $\begin{bmatrix} p_{14} & p_{15} \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \end{bmatrix}$. Let $\mathcal{V}_{\underline{\mathcal{I}}}$ be the open subset of matrices in $\mathcal{P}_{(A;\underline{r})}$ which admit $\underline{\mathcal{I}}$ as a multi-index, and $\widehat{V}_{\underline{\mathcal{I}}} = \phi \circ \pi(\mathcal{V}_{\underline{\mathcal{I}}}) \subseteq \mathcal{H}_{(F,G)}$.

In order to obtain the $\widetilde{C}_A$-reduced form with respect to $\underline{\mathcal{I}}$ of the matrices in $\mathcal{V}_{\underline{\mathcal{I}}}$ we proceed as follows. Let

$$P_1 = \left[\begin{array}{cc|c} p_{11}^{(1)} & p_{12}^{(1)} & p_{11}^{(2)} \\ p_{21}^{(1)} & p_{22}^{(1)} & p_{21}^{(2)} \\ \hline 0 & 0 & p_{11}^{(1)} \\ 0 & 0 & p_{21}^{(1)} \\ \hline 0 & 0 & 0 \end{array}\right], \quad P_2 = \left[\begin{array}{cc} p_{14} & p_{15} \\ p_{24} & p_{25} \\ \hline -p_{15} & p_{14} \\ -p_{25} & p_{24} \\ \hline -p_{14} & -p_{15} \end{array}\right].$$

According to Theorem 5.8, $P_1$ is equivalent to a unique matrix $P_1^{(\mathrm{re})}$ such that

$$P_1^{(\mathrm{re})}((2,1),:) = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ p_{2,1}^{(\mathrm{re})} & 1 & 0 \end{array}\right]; \text{ i.e., } P_1^{(\mathrm{re})} = \left[\begin{array}{cc|c} p_{21}^{(\mathrm{re})} & 1 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 0 & p_{21}^{(\mathrm{re})} \\ 0 & 0 & 1 \\ \hline 0 & 0 & 0 \end{array}\right].$$

Therefore there exists a matrix $Y_1 = \begin{bmatrix} d_{11}^{(1)} & d_{12}^{(1)} & d_{11}^{(2)} \\ 0 & d_{22}^{(1)} & d_{21}^{(2)} \\ 0 & 0 & d_{11}^{(1)} \end{bmatrix} \in \widetilde{C}_{W(0)}$ such that $P_1 Y_1 = P_1^{(\mathrm{re})}$. It follows from this that $p_{21}^{(\mathrm{re})} = \frac{p_{11}}{p_{21}}$.

Analogously, there exists a matrix $Y_2 = \begin{bmatrix} z_{11} & z_{12} \\ -z_{12} & z_{11} \end{bmatrix} \in \widetilde{C}_{\widehat{W}(i,-i)}$ such that $P_2 Y_2 = \left[\begin{array}{cc} 1 & 0 \\ p_{24}^{(\mathrm{re})} & p_{25}^{(\mathrm{re})} \\ \hline 0 & 1 \\ -p_{25}^{(\mathrm{re})} & p_{24}^{(\mathrm{re})} \\ \hline -1 & 0 \end{array}\right] = P_2^{(\mathrm{re})}$. In fact, $Y_2 = \left[\begin{smallmatrix} p_{14} & p_{15} \\ -p_{15} & p_{14} \end{smallmatrix}\right]^{-1}$. So, $z_{11} = \frac{p_{14}}{p_{14}^2+p_{15}^2}$, $z_{12} = -\frac{p_{15}}{p_{14}^2+p_{15}^2}$, $p_{24}^{(\mathrm{re})} = \frac{1}{p_{14}^2+p_{15}^2}(p_{14}p_{24} + p_{15}p_{25})$ and $p_{25}^{(\mathrm{re})} = \frac{1}{p_{14}^2+p_{15}^2}(p_{14}p_{25} - p_{15}p_{24})$.

Summarizing,

$$P^{(\mathrm{re})} = \left[\begin{array}{c} p_1^{(\mathrm{re})} \\ p_2^{(\mathrm{re})} \\ \hline p_1^{(\mathrm{re})} A \\ p_2^{(\mathrm{re})} A \\ \hline p_1^{(\mathrm{re})} A^2 \end{array}\right] = \left[\begin{array}{cc|c||cc} p_{21}^{(\mathrm{re})} & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & p_{24}^{(\mathrm{re})} & p_{25}^{(\mathrm{re})} \\ \hline 0 & 0 & p_{21}^{(\mathrm{re})} & 0 & 1 \\ 0 & 0 & 1 & -p_{25}^{(\mathrm{re})} & p_{24}^{(\mathrm{re})} \\ \hline 0 & 0 & 0 & -1 & 0 \end{array}\right].$$

The free parameters of $P^{(\mathrm{re})}$ are $p_{21}^{(\mathrm{re})}$, $p_{24}^{(\mathrm{re})}$ and $p_{25}^{(\mathrm{re})}$ (recall that, by Theorem 3.2, $\dim \mathcal{H}(F,G) = nr - N$ where $n = \sum_{i=1}^{n} \deg(\alpha_i(s))$, $r = r_1$ and $N$ is given by (16); in this case $\dim \mathcal{H}(F,G) = 5 \cdot 2 - 7 = 3$. Since $P^{(\mathrm{re})}$ must be invertible, the free parameters must satisfy $p_{21}^{(\mathrm{re})} p_{24}^{(\mathrm{re})} \neq 1$. Then, recalling the definition of $\phi$ in (32),

$$\begin{bmatrix} p_1^{(\mathrm{re})} A^3 \\ p_2^{(\mathrm{re})} A^2 \end{bmatrix} (P^{(\mathrm{re})})^{-1} = \begin{bmatrix} 0 & 0 & \frac{1}{p_{21}^{(\mathrm{re})} p_{24}^{(\mathrm{re})} - 1} & -\frac{p_{21}^{(\mathrm{re})}}{p_{21}^{(\mathrm{re})} p_{24}^{(\mathrm{re})} - 1} & \frac{p_{21}^{(\mathrm{re})} p_{25}^{(\mathrm{re})}}{p_{21}^{(\mathrm{re})} p_{24}^{(\mathrm{re})} - 1} \\ 0 & 0 & \frac{p_{25}^{(\mathrm{re})}}{p_{21}^{(\mathrm{re})} p_{24}^{(\mathrm{re})} - 1} & -\frac{p_{21}^{(\mathrm{re})} p_{25}^{(\mathrm{re})}}{p_{21}^{(\mathrm{re})} p_{24}^{(\mathrm{re})} - 1} & p_{24}^{(\mathrm{re})} + \frac{p_{21}^{(\mathrm{re}} p_{25}^{(\mathrm{re})2}}{p_{21}^{(\mathrm{re})} p_{24}^{(\mathrm{re})} - 1} \end{bmatrix}.$$

Taking $\mathcal{W}_{\underline{\mathcal{I}}} = \left\{ \left[\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right] \in \mathbb{R}^3 \ : \ xy \neq 1 \right\}$, then $\mathcal{W}_{\underline{\mathcal{I}}}$ is an open set of $\mathbb{R}^3$ and

$$\begin{array}{cccc} \alpha_{\underline{\mathcal{I}}} : & \mathcal{W}_{\underline{\mathcal{I}}} & \longrightarrow & \widehat{V}_{\underline{\mathcal{I}}} \\ & \begin{bmatrix} x \\ y \\ z \end{bmatrix} & \mapsto & \begin{bmatrix} 0 & 0 & \frac{1}{xy-1} & -\frac{x}{xy-1} & \frac{xz}{xy-1} \\ 0 & 0 & \frac{z}{xy-1} & -\frac{xz}{xy-1} & y+\frac{xz^2}{xy-1} \end{bmatrix}, \end{array}$$

is a parameterization of $\widehat{V}_{\underline{\mathcal{I}}}$. □

# 7 Conclusions

Given a sequence $\alpha_1(s) \mid \cdots \mid \alpha_n(s)$ of monic polynomials with $\sum_{i=1}^n \deg(\alpha_i(s)) = n$ and a controllable linear control system $(F, G)$, the geometry of the set $\mathcal{H}_{(F,G)}$ of feedback matrices $K$ such that the state matrix of the closed loop system $F + GK$ has $\alpha_1(s) \mid \cdots \mid \alpha_n(s)$ as invariant polynomials, has been studied. It is proved that $\mathcal{H}_{(F,G)}$ is a differentiable manifold diffeomorphic to an orbit space by the action of a Lee group. Namely, the orbit space is an orbit space of truncated observability matrices whose state matrix is fixed and has the given sequence of polynomials as invariant polynomials; and the Lee group is the centralizer of that matrix. Then the dimension, a local parametrization and a local system of coordinates of $\mathcal{H}_{(F,G)}$ are provided.

# A Proof of Proposition 4.2

The first step of the proof is to prove that $\mathcal{P}_{(A;\underline{r})} \neq \emptyset$ if and only if there exist nonnegative integers $k'_1 \leq k_1, \ldots, k'_r \leq k_r$ (recall that $(k_1, \ldots k_r)$ is the conjugate partition of $(r_1, \ldots, r_k)$) such that

$$(k'_{\sigma(1)}, \ldots, k'_{\sigma(r)}) \prec (\deg(\alpha_n(s)), \ldots, \deg(\alpha_d(s))) \tag{47}$$

where $(k'_{\sigma(1)}, \ldots, k'_{\sigma(r)})$ is a permutation of $(k'_1, \ldots, k'_r)$ rearranged in nonincreasing order; i. e., $k'_{\sigma(1)} \geq \cdots \geq k'_{\sigma(r)}$, and $\alpha_1(s) \mid \cdots \mid \alpha_d(s)$ are the invariant polynomials of $A$. In fact, bearing in mind the relationship of the Antoulas' truncated and permuted observability matrices and the matrices of $\mathcal{P}_{(A;\underline{r})}$, when $n = \sum_{i=1}^d r_i = d$, $P \in \mathcal{P}_{(A;\underline{r})}$ if and only if (using Antoulas' notation of [1, Section 2.2]) $\left\{p_1, p_1A, \ldots, p_1A^{k_1-1}, \ \ldots, p_r, p_rA, \ldots, p_rA^{k_r-1}\right\}$ form a *nice basis* of $\mathbb{R}^d$. In that case, $k_1 \geq \cdots \geq k_r$ are said to be *nice indices* of $(P_1, A)$. Hence $\mathcal{P}_{(A;\underline{r})} \neq \emptyset$ if and only if there exists a matrix $P_1 \in \mathbb{R}^{r\times d}$ such that $k_1 \geq \cdots \geq k_r$ are nice indices of $(P_1, A)$. Thus, when $n = d$, it follows from [24, Corollary 2.7] that $\mathcal{P}_{(A;\underline{r})} \neq \emptyset$ if and only if

$$(k_1, \ldots, k_r) \prec (\deg(\alpha_n(s)), \ldots, \deg(\alpha_d(s)))$$

or, equivalenty, by Proposition 2.1,

$$(\deg(\alpha_n(s)), \ldots, \deg(\alpha_d(s)))^* \prec (r_1, \ldots, r_k).$$

If $n = \sum_{i=1}^d r_i > d$ then some rows of $P \in \mathcal{P}_{(A;\underline{r})}$ must be linear dependent on other rows. If we take the first $d$ linearly independent rows of $P$ then they form a nice basis of $\mathbb{R}^d$ because if, for some indices $1 \leq i \leq r$ and $0 \leq q \leq$

$k_i - 1$, $p_i A^q$ linearly depends on the rows preceding it in $P$ then $p_i A^{q+1}$ also depend on the rows preceding it in $P$. Hence there are nonnegative integers $k'_1 \leq k_1, \ldots, k'_r \leq k_r$ such that they are nice indices of $(P_1, A)$ and so they satisfy (47). And conversely, if there are indices $k'_1 \leq k_1, \ldots, k'_r \leq k_r$ satisfying (47) then there is $P' \in \mathcal{P}_{(A;\underline{r}')}$ where $\underline{r}' = (r'_1, \ldots, r'_k)$ is the conjugate partition of $\underline{k}' = (k'_1, \ldots, k'_r)$. Then we can add the rows $p_i A^{k'_i}, \ldots p_i A^{k_i - 1}$, $1 \leq i \leq r$, in the appropriate positions to obtain a matrix $P \in \mathcal{P}_{(A;\underline{r})}$.

The second part of the proof is to show that (29) and (30) are equivalent. Put $x = n - d = \sum_{j=1}^{r} k_j - \sum_{j=1}^{d} \deg(\alpha_j(s))$. Then, for $i \geq 1$,

$$\begin{aligned}
&\textstyle\sum_{j=i+1}^{r} k_j \geq \sum_{j=1}^{d-i} \deg(\alpha_j(s)) \Leftrightarrow n - \sum_{j=1}^{i} k_j \geq d - \sum_{j=d-i+1}^{d} \deg(\alpha_j(s)) \\
&\Leftrightarrow x + \textstyle\sum_{j=d-i+1}^{d} \deg(\alpha_j(s)) \geq \sum_{j=1}^{i} k_j \\
&\Leftrightarrow x + \deg(\alpha_d(s)) + \cdots + \deg(\alpha_{d-i+1}(s)) \geq k_1 + \cdots + k_i \\
&\Leftrightarrow (k_1, \ldots, k_r) \prec (x + \deg(\alpha_d(s)), \deg(\alpha_{d-1}(s), \ldots, \deg(\alpha_1(s)))
\end{aligned}$$

where we have used that $x + \sum_{i=1}^{d} \deg(\alpha_i(s)) = \sum_{i=1}^{r} k_i$. Taking into account that $(r_1, \ldots, r_k) = (k_1, \ldots, k_r)^*$, $(w_1, \ldots, w_d) = (\deg(\alpha_d(s)), \ldots, \deg(\alpha_1(s)))^*$ and using item (ii) of Proposition 2.1, we get

$$(w_1, \ldots, w_d) \cup (x)^* \prec (r_1, \ldots, r_d).$$

In conclusion, (29) and (30) are equivalent conditions.

Finally, we are to prove that $\mathcal{P}_{(A;\underline{r})} \neq \emptyset$ if and only if (30) holds.

We have seen in the first step of the proof that $\mathcal{P}_{(A;\underline{r})} \neq \emptyset$ if and only if there exist indices $k'_1 \leq k_1, \ldots, k'_r \leq k_r$ satisfying (47) where $k'_{\sigma(1)} \geq \cdots \geq k'_{\sigma(r)}$. Let us see that $k'_{\sigma(j)} \leq k_j$, $1 \leq j \leq r$. Let $j \in \{1, \ldots, r\}$ and assume that $k'_{\sigma(j)} > k_j$. Then $k'_{\sigma(1)} \geq \cdots \geq k'_{\sigma(j)} > k_j \geq k'_j$. This means that $j \notin \{\sigma(1), \ldots, \sigma(j)\}$ and so, there exists $\ell > j$ such that $\ell \in \{\sigma(1), \ldots, \sigma(j)\}$. Thus, $k_j \geq k_\ell \geq k'_\ell \geq k'_{\sigma(j)} > k_j$, which is a contradiction.

Taking $\widetilde{k}_j = k'_{\sigma(j)}$, $1 \leq j \leq r$ we can conclude that $\mathcal{P}_{(A;\underline{r})} \neq \emptyset$ if and only if there exist indices $\widetilde{k}_1 \geq \cdots \geq \widetilde{k}_r$ such that $\widetilde{k}_j \leq k_j$, $1 \leq j \leq r$, and

$$(\widetilde{k}_1, \ldots, \widetilde{k}_r) \prec (\deg(\alpha_d(s)), \ldots, \deg(\alpha_1(s))). \tag{48}$$

Equivalently, there exist nonnegative integers $\widetilde{r}_1 \geq \cdots \geq \widetilde{r}_d$ (the conjugate partition of $(\widetilde{k}_1, \ldots, \widetilde{k}_r)$) such that

$$\widetilde{r}_j \leq r_j, \quad 1 \leq j \leq d, \tag{49}$$

$$(w_1, \ldots, w_d) \prec (\widetilde{r}_1, \ldots, \widetilde{r}_d). \tag{50}$$

Note that (49) is equivalent to $\widetilde{k}_j \leq k_j$ because $\widetilde{r}_i = \#\{j : \widetilde{k}_j \geq i\} \leq \#\{j : k_j \geq i\} = r_i$ and by item (ii) of Proposition 2.1, (50) is equivalent to (48).

The last step of the proof is to show that condition (30) holds if and only if the exist indices $\widetilde{r}_1 \geq \cdots \geq \widetilde{r}_d$ satisfying (49) and (50).

The "if" part es immediate: it follows from (49) and (50) that $\sum_{j=1}^{i} w_j \leq \sum_{j=1}^{i} \widetilde{r}_j \leq \sum_{j=1}^{i} r_j, \quad 1 \leq i \leq d$.

Conversely, assume that (30) holds and let $h = \min\{i : d \leq \sum_{j=1}^{i} r_j\}$. Then $\sum_{j=1}^{h-1} r_j < d \leq \sum_{j=1}^{h} r_j$. Define

$$\begin{array}{ll} \widetilde{r}_j = r_j, & 1 \leq j \leq h-1, \\ \widetilde{r}_h = d - \sum_{i=1}^{h-1} r_j, & \\ \widetilde{r}_j = 0, & h+1 \leq j \leq d. \end{array}$$

Then $\widetilde{r}_h = d - \sum_{i=1}^{h-1} r_j \leq r_h \leq r_{h-1} = \widetilde{r}_{h-1}$. Therefore $\widetilde{r}_1 \geq \cdots \geq \widetilde{r}_d$ and (49) holds. Since $\sum_{i=1}^{d} \widetilde{r}_j = d = \sum_{i=1}^{d} \deg(\alpha_j)$, (50) follows from (30). □

# Disclosure statement

The authors report there are no competing interests to declare.

# References


[1] Antoulas, A. C. New results on the algebraic theory of linear systems: The solution of the cover problems. *Linear Algebra Appl. 50* (1983), 1–43.

[2] Baragaña, I., and Puerta, F. Versal deformation of realisable Markov parameters. *International Journal of Control 92*, 8 (2019), 1846–1857.

[3] Baragaña, I., Puerta, F., Puerta, X., and Zaballa, I. On the geometry of the generalized partial realization problem. *Math. Control Signals Systems 22* (2010), 39–89.

[4] Baragaña, I., and Zaballa, I. Column completion of a pair of matrices. *Linear and Multilinear Algebra 27*, 4 (1990), 243–273.

[5] Brunovsky, P. Classification of linear controllable systems. *Kybernetica 3*, 6 (1970), 173–188.

[6] Demmel, J., and Edelman, A. The dimension of matrices (matrix pencils) with given Jordan (Kronecker) canonical forms. *Linear Algebra and its Applications 230* (1995), 61–87.

[7] Estrada, M. B., Loiseau, J.-J., and Baquero, R. A geometric proof of Rosenbrock's theorem on pole assignment. *Kybernetika 33*, 4 (1997), 357–370.

[8] Flamm, D. A new proof of Rosenbrock's theorem on pole assignment. *IEEE Transactions on Automatic Control 25*, 6 (1980), 1128–1133.

[9] Gantmacher, F. R. *The Theory of Matrices, Vol. I.* Chelsea Publishing Company, New York, 1959.

[10] Gohberg, I., Lancaster, P., and Rodman, L. *Invariant Subspaces of Matrices with Applications.* John Wiley & Sons, New York, 1986.

[11] Guillemin, V., and Pollack, A. *Differential topology.* Prentice Hall, New Jersey, 1974.

[12] Hardy, G. H., Littlewood, J. E., and Pólya, G. *Inequalities*. Cambridge Univ. Press, Cambridge, 1967.

[13] Lancaster, P., and Tismenetsky, M. *The Theory of Matrices*. Academic Press, New York, 1985.

[14] Lancaster, P., and Zaballa, I. Parametrizing structure preserving transformations of matrix polynomials. *Operator Theory: Advances and Applications 218* (2012), 403–424.

[15] Lee, J. M. *Introduction to Smooth Manifolds*. Springer Verlag, New York, 2003.

[16] Macdonald, I. *Symmetric Functions and Hall Polynomials*. Oxford University Press, Oxford, 1995.

[17] Mammadov, K. Pole placement parameterisation for full–state feedback with minimal dimensionality and range. *International Journal of Control 94*, 2 (2019), 382–389.

[18] Marshall, A. W., Olkin, I., and Arnold, B. C. *Inequalities: Theory of Majorization and its Applications*, second ed., vol. 143. Springer, 2011.

[19] Miguenza, D., Montoro, E., and Roca, A. The centralizer of an endomorphism over an arbitrary field. *Linear Algebra and its Applications 591* (2020), 322–351.

[20] O'Meara, K. C., Clark, J., and Vinsonhaler, C. I. *Advanced Topics in Linear Algebra*. Oxford University Press, New York, 2011.

[21] Rosenbrock, H. H. *State-Space and Multiariable Theory*. Thomas Nelson, London, 1970.

[22] Shapiro, H. The Weyr characteristic. *American Mathematical Monthly 106*, 10 (1999), 919–929.

[23] Zaballa, I. Interlacing and majorization in invariant factor assignment problems. *Linear Algebra and its Applications 121* (1989), 409–421.

[24] Zaballa, I. Controllability and Hermite indices of matrix pairs. *International Journal of Control 68*, 1 (1997), 61–86.