Consistent solutions to the allocation of indivisible objects with general endowments*

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Abstract

We apply the consistency principle to examine various core concepts in a general allocation model that subsumes several familiar market design models as special cases. The conventional strong core is consistent but may be empty, whereas the exclusion core proposed by Balbuzanov and Kotowski (2019), although nonempty, is not consistent and may include unintuitive allocations. We therefore propose a refinement of the exclusion core, which is both nonempty and consistent. Our solution offers sharper predictions than existing alternatives and coincides with the strong core and/or the exclusion core in special cases that generalize familiar models.

Keywords: discrete exchange economy; complex endowments; core; consistency
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1 Introduction

This paper examines cooperative solution concepts in a general model of allocating indivisible objects without monetary transfers, where agents may hold complex endowments. Several special cases of the model have been extensively studied. These include the housing market model introduced by Shapley and Scarf (1974), the house allocation model introduced by Hylland and Zeckhauser (1979) and Abdulkadiroğlu and Sönmez (1998), and the house allocation with existing tenants model introduced by Abdulkadiroğlu and Sönmez (1999). However, as noted by Balbuzanov and Kotowski (2019) (BK hereafter), who first introduce the general model, these special cases involve overly simple endowment structures that fail to capture the complexity of property rights observed in practice. BK's model, which places complex endowments at the forefront, is therefore of both theoretical and practical interest. In the model, an agent may own multiple objects, none at all, or co-own objects with others in intricate ways. This enables the model to accommodate various environments. In this paper, we apply the *consistency* principle, which has been used in various fields to select desirable solutions, as a criterion to assess solution concepts in that general model. It leads us to propose a refinement of BK's exclusion core.

An important insight of BK is that, while standard core concepts work well in the special cases mentioned above, they fail to capture intuitive allocations in the general model. The weak core, based on strong domination that requires all coalition members must strictly benefit from blocking, is too inclusive and includes unintuitive allocations, whereas the strong core, based on weak domination that allows for indifferent members in the blocking coalition, is too stringent and can be empty. It is worth noting that between these two core concepts, the literature has generally favored the strong core since it guarantees Pareto efficiency, a key desideratum in many market design environments. To address these issues, BK propose the *exclusion core*, a novel concept based on the new interpretation of endowments as a distribution of exclusion rights. Specifically, a coalition directly controls its own endowments and has the right to exclude others from those endowments. By leveraging these direct exclusion rights, the coalition also indirectly controls the endowments of those who occupy the coalition's endowments, and so on. The exclusion core consists of allocations where no coalition can strictly benefit all its members by excluding others from the objects it directly or indirectly controls. The exclusion core is nonempty, Pareto efficient, and eliminates many of the unintuitive allocations admitted by the weak core.

Nevertheless, the exclusion core is not without shortcomings. As BK have already noted, it may fail to eliminate unintuitive allocations that the strong core can easily rule out.[1] For example, consider two agents $\{1,2\}$ who co-own an object a and prefers a, but a is assigned to another agent 3 who owns nothing, while nothing is assigned to 1 and 2. Since $\{1,2\}$ cannot simultaneously make themselves strictly better off by excluding 3 from a, this clearly unintuitive allocation is in the exclusion core according to BK's definition. The obvious weakness of the exclusion core in this simple example is due to its restriction that indifferent agents are not allowed to join blocking coalitions. We show that this restriction results in a fundamental shortcoming: the exclusion core fails to satisfy consistency, a property that the strong core satisfies. To demonstrate this, after introducing the model in Section 2, we prove the following result in Section 3: although the exclusion core coincides with the strong core in the special housing market model, both being singletons, if we add to the model an artificial agent who prefers nothing and co-owns every object with its initial owner, then the strong core remains a singleton, whereas the exclusion core expands to include all Pareto efficient allocations (Proposition 1). Interpreting the initial model as a reduced problem of the augmented model after removing the artificial agent who must receive nothing, this result shows that the exclusion core is not consistent.

In general, consistency is a property to test the behavior of solutions whose domains of definition contains problems involving varying populations. Consider a problem and an allocation chosen by a solution for it. Then, imagine some agents leave with their components of the allocation, and examine the reduced problem consisting of the remaining agents and remaining components of the allocation. The solution is consistent if, for this reduced problem, it chooses the restriction of the allocation to this reduced problem. In other words, once we start implementing the allocation recommended by a consistent solution, if a subgroup of agents leave with their assignments, there is no need to reevaluate the chosen allocation for the remaining agents. Consistency has been employed in various fields to select desirable solutions; see discussions in the related literature section.

In the usual definition, consistency compares any pair of economies where one is obtained from the other by removing a group of agents together with their assignments. In our model, however, we must ensure that the reduced economy has a consistent endowment structure with the original one, so that the former can be meaningfully viewed as a reduced version of the latter. For instance, in the housing market model, if an agent is removed with another agent's endowment, the reduced economy is no longer a housing market economy. Therefore, in Section 4, we define consistency by considering only the removal of a group of agents whose assignments come from their own endowments, in which sense we call the group *self-enforcing*. In the reduced economy, any remaining

[1] To rule out unintuitive allocations selected by the exclusion core, BK introduce a model of relational economies with more intricate exclusion rights.

object is owned by the set of its remaining owners, and if there exists a remaining object whose all owners have been removed, it is treated as publicly owned by all remaining agents. We also define a weaker version of consistency, where we remove only a minimal self-enforcing group. Proposition 2 proves that the strong core is consistent, whereas the exclusion core and the weak core are even not weakly consistent.[2]

We are then motivated to seek an improvement over the exclusion core concept. While BK argue that excluding indifferent agents from blocking coalitions is crucial for the exclusion core to be nonempty, we find that this is not strictly true. In Section 5, we propose the *strong exclusion core*, which allows indifferent agents to join an exclusion blocking coalition under the following condition: if any group of indifferent agents in the coalition is self-enforcing, their role must be limited to supporting the coalition's joint exclusion rights, rather than exercising their own exclusion rights. We formalize a definition that distinguishes a subcoalition's own exclusion rights from the coalition's exclusion rights. In the example above, we allow $\{1,2\}$ as a coalition to exclude 3 from a, since it reflects their joint exclusion right, but not any individual's exclusion right. In contrast, if a self-enforcing group of indifferent agents is allowed to join an exclusion blocking coalition, then the exclusion core would become empty, as warned by BK. We prove that the strong exclusion core is a nonempty and consistent subset of the exclusion core (Theorem 1). To prove its nonemptiness, in Section 6, we propose a generalization of the "you request my house - I get your turn" (YRMH-IGYT) mechanism, originally proposed by Abdulkadiroğlu and Sönmez (1999) for the house allocation with existing tenants model. We prove that all outcomes of this algorithm belong to the strong exclusion core (Lemma 1).

Following BK, Sun et al. (2025) propose two core concepts to address the shortcomings of the strong core. The strong core imposes no restrictions on indifferent agents in blocking coalitions, which results in its emptiness. Similar to this paper, Sun et al. (2025) introduce restrictions on the blocking behavior of indifferent agents. While Sun et al. (2025) show that their solutions are supersets of the strong core but neither include nor are included by the exclusion core, we prove in Section 7 that their solutions are supersets of the strong exclusion core (Proposition 3). Moreover, applying the consistency principle, we show that one of their solutions is consistent, while the other is only weakly consistent (Proposition 4).

To further explore the relationships between our solution and others, Section 8 analyzes several special cases of our model, which generalize familiar models in the literature and may be of independent interest. Although the strong exclusion core is a subset of the

[2] For solutions that might choose an empty set of allocations for an economy, consistency trivially has no restrictions, since no agents are supposed to leave with their assignments.

exclusion core and has no set inclusion relationship with the strong core in general cases, in some of these special cases, the strong exclusion core coincides with either the exclusion core or the strong core. In one case that generalizes the house allocation with existing tenants model, we show that the strong exclusion core satisfies a strong form of consistency, which enables us to predict the outcomes of a solution in economies with both public and private endowments based on its outcomes in economies with only private endowments. The strong core does not satisfy this property.

Section 9 concludes the paper by discussing a strong consistency property in general cases. This property, which strengthens consistency, further requires that every allocation selected by a solution for a reduced economy must equal the restriction of an allocation selected for the original economy. We show through examples that this property is too demanding. In particular, without assumptions on the preferences of removed agents, it may happen that a Pareto efficient allocation selected by a solution in the reduced economy is not the restriction of any Pareto efficient allocation in the original economy.

Related literature Numerous papers have investigated the consistency principle in various fields, including but not limited to bargaining theory (Lensberg, 1987), cooperative game (Peleg, 1985; Peleg and Tijs, 1996), bankruptcy problem (Aumann and Maschler, 1985; Young, 1987), cost allocation (Moulin, 1985), fair allocation (Thomson, 1988; Tadenuma and Thomson, 1991), and rationing problem (Moulin, 1999). See Driessen (1991) and Thomson (1990, 2011) for surveys on the use of consistency in various fields.

In the field of market design, consistency has been widely used to characterize mechanisms or solutions. To name a few, Sasaki and Toda (1992) use consistency to characterize the core (i.e., the set of stable matchings) in the marriage model of Gale and Shapley (1962). Ergin (2000) and Ehlers and Klaus (2007) both study consistent solutions in the house allocation model. Ergin (2002) characterizes the priority structures under which the deferred acceptance mechanism is consistent in the priority-based allocation model. Sönmez and Ünver (2010) use consistency with other axioms to characterize the YRMH-IGYT mechanism in the house allocation with existing tenants model; also see Karakaya et al. (2019). Ehlers (2014) uses consistency with other axioms to characterize the top trading cycle (TTC) mechanism with fixed tie-breaking in the housing market model with indifferent preferences. Doğan and Ehlers (2022) prove the minimal unstability of TTC among efficient and strategy-proof mechanisms for any stability comparison criterion that satisfies consistency and two other properties.

Notably, our notion of consistency is much weaker than the usual notion employed in models without individual endowments, where any set of agents can leave with their assignments. In our notion, only agents who receive assignments from their own endowments can leave. For instance, in the house allocation model, since all objects are publicly owned, our consistency imposes no restrictions, whereas the usual notion has nontrivial restrictions. Our definition is tailored to address endowments in our model. Similar definitions have been adopted in previous work on the housing market model and the house allocation with existing tenants model (e.g., Sönmez and Ünver, 2010 and Ehlers, 2014).

Finally, Balbuzanov and Kotowski (2024) generalize the exclusion core to a model that incorporates production. Whether our refinement of the exclusion core can apply to production networks is an open question. Ishida and Park (2025) study a special case of BK's model, where agents are partitioned into groups and each group collectively owns a corresponding partitioned group of objects. They show that the strong core and the exclusion core have a nonempty intersection. This special case belongs to one of the special types of economies discussed in Section 8.

2 The allocation model with general endowments

Let $\mathcal{I}$ be the grand set of agents and $\mathcal{O}$ the grand set of objects, both finite. An economy is represented by a tuple $\Gamma = (I, O, \succ_I, \{C_o\}_{o\in O})$, where $I \subseteq \mathcal{I}$ is the set of agents, $O \subseteq \mathcal{O}$ is the set of objects, $\succ_I = (\succ_i)_{i\in I}$ is the preference profile of agents, and $\{C_o\}_{o\in O}$ specifies the ownership structure. For each $o \in O$, $C_o \subseteq I$ is a nonempty set of agents who collectively own o. Each $i \in C_o$ is called an **owner** of o. If C_o is a singleton, o is privately owned by the agent in C_o. If $C_o = I$, o is publicly owned by all agents in I. Let o^* denote a null object with unlimited copies. Each $i \in I$ demands one object and has a strict preference relation $\succ_i$ over $O \cup \{o^*\}$. An object o is **acceptable** to i if $o \succ_i o^*$; otherwise, o is **unacceptable** to i. For any two objects o and o', we write $o \succsim_i o'$ if $o = o'$ or $o \succ_i o'$.

Given an economy $\Gamma = (I, O, \succ_I, \{C_o\}_{o\in O})$, an **allocation** is a mapping $\mu : I \to O \cup \{o^*\}$ such that $|\mu(i)| = 1$ for all $i \in I$ and $|\mu^{-1}(o)| \leq 1$ for all $o \in O$. In words, every agent receives exactly one object, and every object is assigned to at most one agent. If an agent receives o^*, it means that she receives nothing. An allocation μ is Pareto dominated by another allocation σ if $\sigma(i) \succsim_i \mu(i)$ for all $i \in I$, and $\sigma(j) \succ_j \mu(j)$ for some $j \in I$. An allocation is **Pareto efficient** if it is not Pareto dominated by any other allocation.

Given an economy $\Gamma = (I, O, \succ_I, \{C_o\}_{o\in O})$, every nonempty $C \subseteq I$ is called a **coalition**. A coalition C' is a **subcoalition** of C if $C' \subsetneq C$. We introduce an **endowment** function $\omega : 2^I \to 2^O$ such that, for each coalition C, $\omega(C) = \{o \in O : C_o \subseteq C\}$. That is, $\omega(C)$ is the set of objects owned by C. In an allocation μ, the set of objects assigned to a coalition C is denoted by $\mu(C)$, where $\mu(C) = \cup_{i\in C}\{\mu(i)\}$. We call a coalition C **self-enforcing** in an allocation μ if $\mu(C) \subseteq \omega(C) \cup \{o^*\}$. A self-enforcing coalition can obtain their assignments from their own endowments without relying on others'. For any two allocations μ and σ and a coalition C, we define $C_{\sigma>\mu} = \{i \in C : \sigma(i) \succ_i \mu(i)\}$ and $C_{\sigma=\mu} = \{i \in C : \sigma(i) = \mu(i)\}$. We call $C_{\sigma=\mu}$ the set of **unaffected agents** within C from μ to σ (or from σ to μ).

Let $\mathcal{E}$ denote the set of economies. For each $\Gamma \in \mathcal{E}$, let $\mathcal{A}(\Gamma)$ denote the set of allocations in Γ. A **solution** is a correspondence $f : \mathcal{E} \to \bigcup_{\Gamma \in \mathcal{E}} 2^{\mathcal{A}(\Gamma)}$ such that, for every $\Gamma \in \mathcal{E}$, $f(\Gamma) \in 2^{\mathcal{A}(\Gamma)}$ is a set of allocations in Γ. We allow $f(\Gamma)$ to be empty for some Γ. A solution f is *Pareto efficient* if, for every $\Gamma \in \mathcal{E}$, if $f(\Gamma)$ is nonempty, all elements of $f(\Gamma)$ are Pareto efficient allocations in Γ.

Several special cases of the model presented in this section have been extensively studied. Assume that all objects are acceptable to all agents. When $|I| = |O|$, $|C_o| = 1$ for all $o \in O$, and $C_o \cap C_{o'} = \emptyset$ for all distinct $o, o' \in O$, we obtain the *housing market model*. When $|I| = |O|$ and $C_o = I$ for all $o \in O$, we obtain the *house allocation model*. A hybrid of these two models is the *house allocation with existing tenants model*, where there exists a nonempty $I' \subseteq I$ and a nonempty $O' \subseteq O$ such that $|I'| = |O'|$ and each $i \in I'$ privately owns a distinct object in O', while objects in $O \backslash O'$ are publicly owned.

3 Standard core versus exclusion core

In the standard definition of the core, given an economy, a coalition blocks an allocation if its members can benefit from a reallocation of their endowments among themselves. Depending on whether all members must strictly benefit, the definition has two variants.

Definition 1. *In an economy, an allocation μ is* ***weakly blocked*** *by a coalition C via another allocation σ if*

1. *$\forall i \in C$, $\sigma(i) \succsim_i \mu(i)$ and $\exists j \in C$, $\sigma(j) \succ_j \mu(j)$;*
2. *$\sigma(C) \subseteq \omega(C) \cup \{o^*\}$.*

The ***strong core*** *consists of allocations that are not weakly blocked.*

In the above definition, if $\forall i \in C$, $\sigma(i) \succ_i \mu(i)$, then μ is ***strongly blocked*** *by C via σ. The* ***weak core*** *consists of allocations that are not strongly blocked.*

By definition, the strong core is a subset of the weak core. Both concepts have been widely used in the literature, and between them, the strong core is often favored since it provides more precise predictions and ensures Pareto efficiency. In the housing market model, the strong core is a singleton, consisting of the unique allocation found by the TTC mechanism. In the house allocation model, the strong core consists of all Pareto efficient allocations. The weak core often contains Pareto inefficient allocations.

However, for the model in Section 2, these two concepts may fail to predict intuitive allocations: the weak core may be too inclusive and contain unintuitive allocations, whereas the strong core may be too stringent and contain nothing. The following example due to BK illustrates these issues.

Example 1. *Consider an economy consisting of three agents* $\{1,2,3\}$ *and two objects* $\{a,b\}$*. Both objects are privately owned by agent* 1*. All agents accept both objects and prefer* a *over* b*. The following tables represent the ownership, preferences, and allocations under our consideration.*

	a	b
C_o:	1	1
μ:	1	
σ:	1	2
δ:	1	3

$\succ_1$	$\succ_2$	$\succ_3$
a	a	a
b	b	b

Since 1 *owns all objects, it is natural to assign her the best object* a*. Then, if we do not want* b *to be wasted,* b *must be assigned to one of the other agents. So,* σ *and* δ *are the two intuitive allocations in this example. The allocation* μ *is Pareto inefficient, since* b *is wasted.*

The weak core $= \{\mu, \sigma, \delta\}$*. Although* μ *is Pareto inefficient, it is not strongly blocked because any potential blocking coalition must include* 1*, who cannot be made strictly better off.*

The strong core $= \emptyset$*. It is not surprising that* μ *is weakly blocked by* $\{1,2,3\}$ *via* σ *or* δ*. However,* σ *and* δ *are also blocked:* $\{1,3\}$ *can weakly block* σ *via* δ*, by reallocating* b *from* 2 *to* 3*, while* $\{1,2\}$ *can weakly block* δ *via* σ*, by reallocating* b *from* 3 *to* 2*.*

To solve the issues with the standard core concepts, BK introduce the exclusion core as a new solution. This solution interprets endowments as a distribution of exclusion rights. Specifically, a coalition C directly controls their endowments $\omega(C)$. Given an allocation, C has the right to exclude any others who occupy their endowments. Then, by leveraging this exclusion right, C can extend their control to the endowments of those occupying $\omega(C)$, the endowments of those occupying these subsequent endowments, and so forth.

Formally, in an allocation μ, the set of objects controlled by a coalition C is defined as

$$\Omega(C|\omega,\mu) = \omega(\cup_{k=0}^{\infty} C^k),$$

where $C^0 = C$ and $C^k = C^{k-1} \cup \{i \in I \backslash C^{k-1} : \mu(i) \in \omega(C^{k-1})\}$ for every $k \geq 1$. The exclusion core consists of allocations where no coalition can make all members strictly better off by excluding others from their controlled objects.

Definition 2 (Balbuzanov and Kotowski, 2019)**.** *In an economy, an allocation μ is* ***exclusion blocked*** *by a coalition C via another allocation σ if*

1. $\forall i \in C$, $\sigma(i) \succ_i \mu(i)$;
2. $\forall j \in I \backslash C$, $\mu(j) \succ_j \sigma(j) \implies \mu(j) \in \Omega(C|\omega, \mu)$.

The ***exclusion core*** *consists of allocations that are not exclusion blocked.*

Every element of the exclusion core must be Pareto efficient, since otherwise it can be exclusion blocked by the set of agents who are strictly better off in a Pareto improvement.

Notably, an exclusion blocking coalition does not need to be self-enforcing. Another notable feature is that unaffected agents are banned from joining exclusion blocking coalitions. BK explain that, otherwise, the exclusion core would become empty. We apply the concept to Example 1 to illustrate it.

Example 1 revisited. *The exclusion core = $\{\sigma, \delta\}$. So, it consists of the two intuitive allocations. However, if we allow unaffected agents to join exclusion blocking coalitions, then $\{1,3\}$ would exclusion block σ via δ, and $\{1,2\}$ would exclusion block δ via σ, leading to an empty exclusion core.*

Nevertheless, precluding unaffected agents from joining exclusion blocking coalitions can cause the exclusion core to admit unintuitive allocations, which, however, might be easily eliminated by the strong core. See the following example for an illustration.

Example 2 (Balbuzanov and Kotowski, 2019)**.** *Consider the following economy where the only object a is owned by $\{1,2\}$, while 3 owns nothing. Every agent prefers a over o^*.*

	a
C_o:	1, 2
μ:	1
σ	2
δ	3
η	

$\succ_1$	$\succ_2$	$\succ_3$
a	a	a

There are four possible allocations: η is Pareto inefficient since the only object is unassigned, whereas the other three are Pareto efficient. However, in δ, object a is not assigned to any of its owners; instead, it is assigned to 3 who owns nothing. Therefore, δ is unintuitive.

The weak core = $\{\mu, \sigma, \delta, \eta\}$, the strong core = $\{\mu, \sigma\}$, and the exclusion core = $\{\mu, \sigma, \delta\}$. The coalition $\{1,2\}$ can weakly block δ via μ or σ, but it cannot exclusion block δ, because the two agents cannot be made strictly better off simultaneously.

In the rest of this section, we provide a result that demonstrates the weakness of the exclusion core, and the source of it is similar to the observation in Example 2.

In the housing market model, BK have shown that the exclusion core coincides with the strong core, both containing a unique allocation. Let $\Gamma = (I, O, \succ_I, \{i_o\}_{o \in O})$ denote a housing market economy where, for each $o \in O$, i_o is its unique owner. Now, we construct an economy Γ^* by adding an artificial agent i^* to Γ such that, for each $o \in O$, $C_o = \{i_o, i^*\}$, the agents in I maintain their preferences in Γ, while i^* views all objects as unacceptable. We call Γ^* an **augmented housing market**. Since i^* does not accept any object, to recommend allocations for Γ^*, it is intuitive to first assign i^* the null object o^* and then ignore i^* and consider allocations for the remaining agents. This would lead us to recommend the same set of allocations for both Γ^* and Γ. Indeed, the strong core satisfies this intuition: it recommends the same allocation for both Γ^* and Γ. In contrast, although the exclusion core coincides with the strong core in Γ, it expands to contain all Pareto efficient allocations in Γ^*. In all these allocations, i^* receives o^*. So, for the agents in I, the exclusion core in Γ^* coincides with the set of Pareto efficient allocations in Γ. On the other hand, if we ignore all private ownership in Γ and change to view Γ as a house allocation problem, the exclusion core in Γ would coincide with the set of Pareto efficient allocations. It means that, by adding an artificial agent to Γ, the exclusion core changes to recommend the same set of allocations as it would recommend by ignoring all agents' private ownership. The source of the problem is similar to Example 2: because i^* is an owner of every object in Γ^* but cannot be made strictly better off, every Pareto efficient allocation in Γ^* cannot be exclusion blocked.

Proposition 1. *In every augmented housing market, the strong core remains a singleton, whereas the exclusion core coincides with the set of Pareto efficient allocations.*

This result leads us to examine the consistency property of solutions in our model. We define consistency in the next section.

4 Consistency

This section defines the **consistency** property, which is widely used across various fields to evaluate the behavior of solutions defined over domains involving varying populations.

In the standard formulation, consistency compares any pair of economies in which one is derived from the other by removing a group of agents along with their assignments. In our model, however, we must ensure that the reduced economy has a consistent ownership structure with the original one, so that it can be meaningfully viewed as a reduced problem of the original economy. For example, in the housing market model, if an agent is removed with another agent's endowment, we would obtain an economy that is no longer a valid housing market problem. We therefore consider only removing a group of agents whose assignments come from their own endowments; in other words, the group is self-enforcing. In the reduced economy, for any remaining object, if it has remaining owners, it is now viewed as owned by those remaining owners; otherwise, it is viewed as publicly owned by all remaining agents. Our formulation is consistent with similar definitions adopted in previous work on the housing market model and the house allocation with existing tenants model (e.g., Sönmez and Ünver, 2010 and Ehlers, 2014).

Formally, in an economy $\Gamma = (I, O, \succ_I, \{C_o\}_{o\in O})$, for any allocation μ, if there exists a group of agents $I' \subsetneq I$ such that $\mu(I') \subseteq \omega(I') \cup \{o^*\}$, we then define an economy $\Gamma(\mu, I\backslash I') = (I\backslash I', O\backslash \mu(I'), \succ_{I\backslash I'}, \{C'_o\}_{o\in O\backslash\mu(I')})$ such that, for each $o \in O\backslash\mu(I')$, if $C_o \cap (I\backslash I') \neq \emptyset$, then $C'_o = C_o \cap (I\backslash I')$; otherwise, $C'_o = I\backslash I'$. We use $\mu_{I\backslash I'}$ to denote the restriction of μ to $I\backslash I'$; that is, for each $i \in I\backslash I'$, $\mu_{I\backslash I'}(i) = \mu(i)$.[3] So, $\mu_{I\backslash I'}$ is a well-defined allocation in $\Gamma(\mu, I\backslash I')$.

Definition 3. *A solution f is **consistent** if, for every $\Gamma = (I, O, \succ_I, \{C_o\}_{o\in O})$, if $f(\Gamma) \neq \emptyset$, then for every $\mu \in f(\Gamma)$ and every $I' \subsetneq I$ such that $\mu(I') \subseteq \omega(I') \cup \{o^*\}$, $\mu_{I\backslash I'} \in f(\Gamma(\mu, I\backslash I'))$.*

We also define a weaker version of consistency, where the reduced economy is obtained by removing a minimal self-enforcing coalition. A self-enforcing coalition C is **minimal** if no subcoalition of C is self-enforcing.

Definition 4. *A solution f is **weakly consistent** if, for every $\Gamma = (I, O, \succ_I, \{C_o\}_{o\in O})$, if $f(\Gamma) \neq \emptyset$, then for every $\mu \in f(\Gamma)$ and every $I' \subsetneq I$ such that $\mu(I') \subseteq \omega(I') \cup \{o^*\}$ but no subcoalition of I' satisfies the same condition, $\mu_{I\backslash I'} \in f(\Gamma(\mu, I\backslash I'))$.*

Proposition 2 proves that the strong core is consistent. In contrast, Proposition 1 implies that the exclusion core is even not weakly consistent. In an augmented housing market Γ^*, the exclusion core equals the set of Pareto efficient allocations, in all of which i^* receives o^*. However, if removing i^* with o^*, we obtain the original economy Γ in which the exclusion core becomes a singleton. The weak core is neither weakly consistent. In Example 1, suppose that agent 1 owns an additional object c, which is the least preferred acceptable object for all agents. Then, μ, in which 1 receives a and the other objects are unassigned, is in the weak core. However, after removing 1 with a, the restriction of μ is not in the weak core in the reduced economy.

Proposition 2. *The strong core is consistent, whereas the exclusion core and the weak core are even not weakly consistent.*

[3] More generally, we use μ_C to denote the restriction of an allocation μ to a coalition C.

5 The strong exclusion core

The inadequacy of the exclusion core in eliminating unintuitive allocations (Example 2) and its violation of consistency (Proposition 2) motivate us to propose a refinement of it. Our key insight is that unaffected agents should be allowed to join exclusion blocking coalitions under appropriate conditions.

To gain the insight, let us compare Example 1 and Example 2. In Example 1, agent 1 owns all objects. If 1 is allowed to join exclusion blocking coalitions, 1 would alternate between the other two agents in forming blocking coalitions, rendering the exclusion core empty. In these blocking, 1 would remain unaffected by receiving one of his endowments and the blocking would only reflect 1's own exclusion right. In contrast, in Example 2, it would be convenient to eliminate unintuitive allocations if unaffected agents are allowed to join exclusion blocking coalitions. In the example, if $\{1,2\}$ are allowed to block δ via μ or σ, the unaffected agent in the coalition would receive o^* and the blocking would reflect the coalition's joint exclusion right.

This comparison suggests a scenario where unaffected agents might be allowed to join exclusion blocking coalitions: when these agents rely only on their own endowments (and o^*) to remain unaffected, and their participation serves to help the coalition execute the joint exclusion rights, rather than their own exclusion rights.

To differentiate between the joint exclusion rights of a coalition and the own exclusion rights of a subcoalition, we introduce the following definition. In any economy, given any allocation μ, for any coalition C and any subcoalition $C' \subsetneq C$, the set of objects that are controlled by C' but are independent of the ownership of $C \backslash C'$ is defined to be

$$\Omega^*(C'|C,\omega,\mu) = \omega(\cup_{k=0}^{\infty} C^k),$$

where $C^0 = C'$, $C^1 = C^0 \cup \{i \in I \backslash C : \mu(i) \in \omega(C^0)\}$, and $C^k = C^{k-1} \cup \{i \in I \backslash (C \cup C^{k-1}) : \mu(i) \in \omega(C^{k-1})\}$ for every $k \geq 2$. In words, when we extend the control rights of C' to the agents outside C, we exclude the ownership of the agents in $C \backslash C'$.[4]

The following condition formalizes the case that the set of unaffected agents in an exclusion blocking coalition C relies only on their own endowments to remain unaffected, and their participation only helps the coalition execute the joint exclusion rights, rather than their own exclusion rights: if $C_{\sigma=\mu} \neq \emptyset$, then $\sigma(C_{\sigma=\mu}) \subseteq \omega(C_{\sigma=\mu}) \cup \{o^*\}$, and $\forall j \in I \backslash C$, $\mu(j) \succ_j \sigma(j) \implies \mu(j) \in \Omega(C|\omega,\mu)$ but $\mu(j) \notin \Omega^*(C_{\sigma=\mu}|C,\omega,\mu)$.

However, through the following example, we show that there exists another scenario

[4] Example 8 in Appendix D illustrates the difference between $\Omega^*(C'|C,\omega,\mu)$ and $\Omega(C'|\omega,\mu)$.

where it is reasonable to allow unaffected agents to join exclusion blocking coalitions: when $C_{\sigma=\mu}$ relies on the ownership of $C_{\sigma>\mu}$ to remain unaffected. This justifies the participation of $C_{\sigma=\mu}$ in the coalition. In this case, they may help the coalition execute the joint exclusion rights, but they may also execute their own exclusion rights.

Example 3. *Consider an economy combining the features of Example 1 and Example 2.*

	a	b
C_o:	1, 2	1
μ:	1	2
σ:	2	1
δ:	1	3

$\succ_1$	$\succ_2$	$\succ_3$
a	a	a
b	b	b

Any Pareto efficient allocation must assign both objects to agents.

Allocations μ and σ are intuitive. Since 1 and 2 co-own object a and both prefer a over b, it is intuitive to assign a to one of them. If 2 receives a, then 1 should receive his private endowment, b. If 1 receives a, it is intuitive to assign b to 2, which can be viewed as the outcome of an exchange between 1 and 2 for their ownership. In contrast, δ is unintuitive, because 3 owns nothing but receives b, whereas 2 co-owns a but receives nothing.

The strong core = $\{\mu, \sigma\}$, whereas the exclusion core = $\{\mu, \sigma, \delta\}$. To rule out δ from the exclusion core, we need to allow $\{1, 2\}$ to form a coalition to exclusion block δ via μ. This blocking would only execute the exclusion right of 1, who is unaffected by the blocking. However, to remain unaffected, 1 must receive a, which is jointly owned by $\{1, 2\}$. This justifies 1's participation in the coalition and the execution of her own exclusion right.

To unify the above scenarios and address more general situations, our final proposal adds the following condition to the definition of exclusion blocking: if $C_{\sigma=\mu} \neq \emptyset$, then $\sigma(C_{\sigma=\mu}) \subseteq \omega(C) \cup \{o^*\}$, and $\forall j \in I \setminus C$, $\mu(j) \succ_j \sigma(j) \implies \mu(j) \in \Omega(C|\omega, \mu)$ but $\mu(j) \notin \Omega^*(C'|C, \omega, \mu)$ for every $C' \subseteq C_{\sigma=\mu}$ such that $\sigma(C') \subseteq \omega(C') \cup \{o^*\}$.

This condition encompasses two cases. If $\sigma(C_{\sigma=\mu}) \subseteq \omega(C_{\sigma=\mu}) \cup \{o^*\}$, then the condition reduces to the first scenario discussed above. That is, the participation of $C_{\sigma=\mu}$ is only to help the coalition execute their joint exclusion rights, rather than their own exclusion rights. If $C_{\sigma=\mu}$ is not self-enforcing, then $\sigma(C_{\sigma=\mu}) \subseteq \omega(C) \cup \{o^*\}$ implies that $C_{\sigma=\mu}$ relies on the ownership of $C_{\sigma>\mu}$ to remain unaffected. Then, the condition requires that, similar to the first scenario, if any subcoalition of $C_{\sigma=\mu}$ is self-enforcing, then the subcoalition does not execute their own exclusion rights in the blocking. Thus, their participation is only to help the coalition execute their joint exclusion rights.

Definition 5. *In an economy, an allocation μ is* ***weakly exclusion blocked*** *by a coalition C via another allocation σ if*

1. $\forall i \in C$, $\sigma(i) \succsim_i \mu(i)$ *and* $\exists j \in C$, $\sigma(j) \succ_j \mu(j)$;

2. $\sigma(C_{\sigma=\mu}) \subseteq \omega(C) \cup \{o^*\}$;

3. $\forall j \in I \backslash C$, $\mu(j) \succ_j \sigma(j) \implies \mu(j) \in \Omega(C|\omega,\mu)$, *but* $\mu(j) \notin \Omega^*(C'|C,\omega,\mu)$ *for every* $C' \subseteq C_{\sigma=\mu}$ *such that* $\sigma(C') \subseteq \omega(C') \cup \{o^*\}$.

The ***strong exclusion core*** *consists of allocations that are not weakly exclusion blocked.*

Applying this definition to the above examples, in Example 1, agent 1 is a self-enforcing unaffected agent from σ to δ (and from δ to σ). So, she cannot form a weak exclusion blocking coalition with others in these two allocations. In Example 2, $\{1,2\}$ can weakly exclusion block δ via μ or σ, because the agent receiving o^* is a self-enforcing unaffected agent and does not execute her own exclusion right in the blocking. In Example 3, $\{1,2\}$ can weakly exclusion block δ via μ, because 1 is unaffected but not self-enforcing and thus is allowed to execute her own exclusion right in the blocking.

By definition, the strong exclusion core is a subset of the exclusion core: if C exclusion blocks μ via σ, since $C_{\sigma=\mu} = \emptyset$, C also weakly exclusion blocks μ via σ. We prove that the strong exclusion core is nonempty and consistent.

Theorem 1. *The strong exclusion core is a nonempty subset of the exclusion core, and it is consistent.*

A mechanism defined in Section 6 can find elements of the strong exclusion core.

6 The "you request my house – I get your turn" mechanism

Our model encompasses the house allocation with existing tenants model as a special case. For that special model, Abdulkadiroğlu and Sönmez (1999) introduce the YRMH-IGYT mechanism. This section proposes a generalization of this mechanism to find the elements of the strong exclusion core.

YRMH-IGYT proceeds as follows in the house allocation with existing tenants model. Fix a linear order of agents, and let private endowments point to their owners. Then, let the first agent in the order point to her favorite object. If the object is not privately owned, or its private owner has been removed, let the agent obtain that object and then remove them. Otherwise, move the owner of that object to the top of the order and then repeat the above step. If, in some step, a cycle forms, then the cycle must involve existing tenants who point to each other's private endowments. Let them exchange their private endowments as indicated by the cycle, and then remove them.

Our generalization of YRMH-IGYT differs from the above definition in two respects. **First**, we allow agents to share ownership under certain conditions.[5] Specifically, after a cycle is removed in a step, if any agent in the cycle co-owns an object that remains unassigned and any object in the cycle has owners that are not removed, then in subsequent steps, we let the remaining owners of the object in the cycle acquire shared ownership of the remaining objects co-owned by the agent in the cycle. **Second**, in each step, for each remaining object, if it has remaining owners, we let it point to one of its remaining owners; otherwise, we let it point to one of the remaining agents who have acquired shared ownership of the object, provided such agents exist. In both cases, ties are broken according to a fixed order of agents. If both types of agents do not exist, we let the object not point to any agent. For simplicity, we continue to refer to our mechanism as YRMH-IGYT.

YRMH-IGYT

Notation: After step t, let $I(t)$ denote the set of remaining agents and $O(t)$ the set of remaining objects. For each $o \in O(t)$, let $C_o(t)$ denote the set of remaining owners and $S_o(t)$ the set of remaining agents who have acquired shared ownership of o.

Initialization: Let $I(0) = I$, $O(0) = O$, and for each $o \in O$, $C_o(0) = C_o$ and $S_o(0) = \emptyset$. Fix a linear order of agents $\rhd$. Let each $o \in O$ initially point to the $\rhd$-highest agent in C_o.

Step $t \geq 1$: If there exist arcs between remaining agents and objects from the previous step, maintain all of them. If the agent pointed by any $o \in O(t-1)$ is removed in the previous step, then:

- If $C_o(t-1) \neq \emptyset$, let o point to the $\rhd$-highest agent in $C_o(t-1)$;[6]
- If $C_o(t-1) = \emptyset$ and $S_o(t-1) \neq \emptyset$, let o point to the $\rhd$-highest agent in $S_o(t-1)$;
- If $C_o(t-1) = \emptyset$ and $S_o(t-1) = \emptyset$, let o not point to any agent.

Let the highest-ranked agent i in the current order (in step one, this order is $\rhd$) point to her favorite remaining object, denoted by o. There are three cases.

- If o does not point to any agent, or $o = o^*$, let i obtain o and remove them.

[5] This feature follows the mechanism in Sun et al. (2025). However, the outcomes of their mechanism may not belong to our solution.

[6] Although we update the order of agents in the procedure, $\rhd$ denotes a fixed order.

- If o points to an agent and a cycle forms, let every agent in the cycle obtain the object she points to and then remove them.

  For every agent j and every object a involved in the cycle, if there exists $b \in O(t)$ and $j' \in I(t)$ such that $j \in C_b(t-1) \cup S_b(t-1)$, $j' \in C_a(t-1) \cup S_a(t-1)$, and $j' \notin C_b(t-1) \cup S_b(t-1)$, then let $j' \in S_b(t)$. That is, j' acquires the shared ownership of b in following steps.

- If o points to an agent but a cycle does not form, move the agent pointed by o to the top of the current order of agents.

In each step, either at least one agent is removed, or no agent is removed but the order of agents is updated. If no agent is removed in a step, then after finite steps, either an agent will be removed or a cycle will form. So, the algorithm must stop in finite steps.

By choosing different initial order $\rhd$, YRMH-IGYT finds different allocations. We call them the *outcomes* of YRMH-IGYT and prove that they are in the strong exclusion core.[7]

Lemma 1. *Every outcome of YRMH-IGYT belongs to the strong exclusion core.*

Example 4 illustrates the procedure of YRMH-IGYT.

Example 4. *Consider four agents $\{1,2,3,4\}$ and four objects $\{a,b,c,d\}$.*

	a	b	c	d
C_o:	$1,2,3$	2	3	4
σ:	2	3	1	4

$\succ_1$	$\succ_2$	$\succ_3$	$\succ_4$
a	a	b	a
b	c	d	b
c	b	c	d
d	d	a	c

Let the order be $4 \rhd 2 \rhd 3 \rhd 1$. We show that YRMH-IGYT finds the allocation σ. Before step one, b, c, and d point to their unique owners, and a points to its $\rhd$-highest owner, 2.

*In **step one**, 4 points to a. Because a points to 2, move 2 to the top of the order. In **step two**, 2 points to a. So, 2 forms a cycle with a, and they are removed. Then, because 1 and 3 are remaining owners of a, and b is 2's remaining endowment, 1 and 3 acquire the shared ownership of b. In **step three**, 4 points to b. Because b points to 3, move 3 to the top of the order. In **step four**, 3 points to b and forms a cycle with b. So, they are removed. In **step five**, 4 points to d and forms a cycle with d. So, they are removed. In **step six**, 1 points to c. Because the owner of c has been removed, 1 obtains c.*

[7] Example 9 in Appendix D shows that YRMH-IGYT may not find all elements of the strong exclusion core.

7 Relationship with the solutions of Sun et al. (2025)

Following BK, Sun et al. (2025) propose two variants of the standard core concepts based on the conventional interpretation of endowments as items that their owners must consume or exchange with others. Both solutions add restrictions on the blocking behavior of a coalition that involves self-enforcing unaffected agents. We introduce their definitions and then clarify their relationship with our solution.

Definition 6. *In an economy, an allocation μ is **effectively blocked** by a coalition C via another allocation σ if*

1. *$\forall i \in C$, $\sigma(i) \succsim_i \mu(i)$ and $\exists j \in C$, $\sigma(j) \succ_j \mu(j)$;*
2. *$\sigma(C) \subseteq \omega(C) \cup \{o^*\}$;*
3. *For every $C' \subseteq C_{\sigma=\mu}$ such that $\sigma(C') \subseteq \omega(C') \cup \{o^*\}$, if $o \in \omega(C') \backslash \sigma(C')$ and $C \neq I$, then $o \notin \sigma(C)$.*

*The **effective core** consists of allocations that are not effectively blocked.*

*If condition 3 is replaced by the following 3', then μ is **rectification blocked** by C via σ:*

3'. *For every $C' \subseteq C_{\sigma=\mu}$ such that $\sigma(C') \subseteq \omega(C') \cup \{o^*\}$, if $o \in \omega(C') \backslash \sigma(C')$ and $\mu^{-1}(o) \in I \backslash C$, then $o \notin \sigma(C)$.*

*The **rectified core** consists of allocations that are not rectification blocked.*

Effective blocking requires that, in a weak blocking coalition C, if a subcoalition $C' \subseteq C_{\sigma=\mu}$ is self-enforcing, then C' cannot assign any of their redundant endowments, $o \in \omega(C') \backslash \sigma(C')$, to others in the coalition, unless $C = I$ (i.e., when all agents weakly benefit from the blocking). Rectification blocking relaxes effective blocking by allowing C' to assign their redundant endowments to others in the coalition, as long as these objects are not initially assigned to agents outside the coalition.

The rectified core is a superset of the strong core and a subset of the effective core, and the effective core is a Pareto efficient subset of the weak core. BK have shown that the strong core and the exclusion core do not include each other. Sun et al. (2025) have shown that their two core concepts do not have set inclusion relationships with the exclusion core. However, we prove that the strong exclusion core is a subset of the rectified core.[8] Thus, the strong exclusion core is in the intersection of the exclusion core and the two

[8] In Example 5, the strong exclusion core, $\{\sigma_1, \sigma_2\}$, is a strict subset of the rectified core, $\{\mu, \sigma_1, \sigma_2\}$.

core concepts of Sun et al. (2025). However, the strong exclusion core and the strong core still do not include each other.[9]

Proposition 3. *The strong exclusion core is a subset of the rectified core (and of the effective core).*

We provide a result showing the difference between the two core concepts of Sun et al. (2025): the rectified core is consistent, while the effective core is only weakly consistent.

Proposition 4. *The rectified core is consistent. The effective core is weakly consistent, but not consistent.*

8 Special economies

This section compares the various core concepts in special cases of our model. It deepens our understanding of these concepts and also provides new results for these special cases. In this section, we treat the set of the YRMH-IGYT outcomes as a solution, referring to it as YRMH-IGYT, and include it in our results. This clarifies the relationship between the mechanism and the various core concepts.

We first analyze a type of economies where agents may co-own objects, but every coalition never owns more objects than their demands. Formally, an economy is called **no-redundant-ownership** if every agent regards all objects as acceptable, and for every coalition C, $|\omega(C)| \leq |C|$. For this type of economies, Sun et al. (2025) have proved that the strong core coincides with the effective core.[10] Since the strong exclusion core is a subset of the effective core, it implies that the strong exclusion core is a subset of the strong core. However, the exclusion core and the strong core still do not include each other.[11]

Proposition 5. *In no-redundant-ownership economies, YRMH-IGYT $\subseteq$ strong exclusion core $\subseteq$ [strong core $\cap$ exclusion core].*

We then analyze a special type of no-redundant-ownership economies that we call **no-overlapping-ownership**: agents regard all objects as acceptable and for every two objects a and b, $C_a \cap C_b = \emptyset$. So, every agent is the owner of at most one object. This type can be viewed as a generalization of the housing market model where every agent privately

[9] In Example 1, the strong core is empty, while the strong exclusion core is nonempty. In Example 5 in Section 8, the strong exclusion core is a strict subset of the strong core.

[10] To be precise, Sun et al. (2025) only require $|\omega(C)| \leq |C|$ for all C with $|C| < |I| - 1$.

[11] Example 2 belongs to this type of economies, where δ is in the exclusion core but not in the strong core. Example 5 also belongs to this type, where μ is in the strong core but not in the exclusion core.

owns one object. For this type, the outcomes of YRMH-IGYT can be equivalently found by first endowing every object to one of its owners and then running TTC. We prove that YRMH-IGYT can find all allocations in the strong core. The intuition is essentially the same as the well-known result in the housing market model that TTC finds the unique element of the strong core. Therefore, for this type, the strong core coincides with the strong exclusion core and is a subset of the exclusion core. Example 2 belongs to this type and shows that the strong core can be a strict subset of the exclusion core.

Proposition 6. *In no-overlapping-ownership economies, YRMH-IGYT =strong exclusion core =strong core $\subseteq$ exclusion core.*

We next consider a type of economies called **private-ownership**: for each $o \in O$, $|C_o| = 1$. That is, every object is owned by one agent. We allow an agent to own multiple objects and some agents may regard some objects as unacceptable. This type overlaps with the no-redundant-ownership type. For instance, the housing market model belongs to both types. But because agents may have redundant endowments in this type, the strong core may be empty. For this type, BK have proved that the exclusion core coincides with the strong core whenever the latter is nonempty. We prove that the strong exclusion core coincides with the exclusion core, and all of their elements can be found by YRMH-IGYT.

Proposition 7. *In private-ownership economies, YRMH-IGYT=strong exclusion core=exclusion core, and they coincide with the strong core when the strong core is nonempty.*

After considering private ownership, we turn to consider another type called **public-ownership**: for each $o \in O$, $C_o = I$. That is, all objects are publicly owned. For this type, BK have proved that the exclusion core coincides with the strong core, both consisting of all Pareto efficient allocations. It is obvious that YRMH-IGYT can find all Pareto efficient allocations for this type. So, we have the following result.

Proposition 8. *In public-ownership economies, YRMH-IGYT=strong exclusion core=exclusion core=strong core.*

Finally, we analyze a hybrid of the former two types called **private-public-ownership**: for each $o \in O$, $|C_o| = 1$ or $C_o = I$. That is, each object is either privately owned or publicly owned. This type can be viewed as an generalization of the house allocation with existing tenants model. Although the strong core coincides with the (strong) exclusion core and YRMH-IGYT in public-ownership economies, and when it is nonempty, they also coincide in private-ownership economies, we show that the strong core differs from those solutions in private-public-ownership economies, even when it is nonempty. See the following example.

Example 5. *We first consider a house allocation with existing tenants economy with four agents $\{1,2,3,4\}$ and four objects $\{a,b,c,d\}$. Object d is publicly owned.*

	a	b	c	d
C_o:	1	2	3	1,2,3,4
μ:	1	3	2	4
σ_1	4	1	3	2
σ_2	1	3	4	2

$\succ_1$	$\succ_2$	$\succ_3$	$\succ_4$
b	d	b	a
a	c	c	d
c	b	a	c
d	a	d	b

The strong core = $\{\mu,\sigma_1,\sigma_2\}$, whereas the (strong) exclusion core = YRMH-IGYT = $\{\sigma_1,\sigma_2\}$. The allocation μ is exclusion blocked by $\{1,2,4\}$ via σ_1, and it cannot be found by YRMH-IGYT.

We then consider another economy that is obtained by adding an agent 5 to the above economy and letting 5 privately own d but accept only o^*. Therefore, d is no longer publicly owned in the second economy. The allocations for the first economy can be extended to the second by letting 5 obtain o^*. In the second economy, ***the strong core is empty****: μ is weakly blocked by $\{1,2,5\}$ via σ_1, σ_1 is weakly blocked by $\{2,3,5\}$ via σ_2, and σ_2 is weakly blocked by $\{1,2,5\}$ via σ_1. However, the (strong) exclusion core = YRMH-IGYT = $\{\sigma_1,\sigma_2\}$, as in the first economy.*

The first economy is private-public-ownership, while the second is private-ownership. The first is a reduced problem of the second by removing 5 with o^*. The (strong) exclusion core and YRMH-IGYT are invariant across the two economies, whereas the strong core differs.

Every private-public-ownership economy can be viewed as the reduced problem of a private-ownership economy in which public endowments in the first economy are privately owned by an additional agent in the second economy who accepts only o^*. Every Pareto efficient solution must assign o^* to the additional agent. After removing the agent with o^* from the second economy, we obtain the first economy. We prove that the (strong) exclusion core and YRMH-IGYT satisfy a stronger version of consistency for such economies, which requires that the set of allocations recommended by a solution for the first economy exactly coincides with the set of allocations recommended by the solution for the second economy after removing i^* with her assignment. However, the strong core violates this property.

Formally, let $\mathcal{E}^0$ denote the set of private-public-ownership economies. In every such economy $\Gamma = (I, O, \succ_I, \{C_o\}_{o\in O})$, let O^P denote the set of public endowments. Define an augmented economy $\Gamma^* = (I\cup\{i^*\}, O, \succ_{I\cup\{i^*\}}, \{C^*_o\}_{o\in O})$ such that, for every $o \in O\backslash O^p$, $C^*_o = C_o$, for every $o \in O^P$, $C^*_o = \{i^*\}$, and i^* accepts only o^*. For every allocation μ in Γ^*, let μ^R denotes its restriction to Γ by removing i^* with her assignment.

Definition 7. *A solution f is* ***strongly consistent*** *on $\mathcal{E}^0$ if, for every $\Gamma \in \mathcal{E}^0$, $f(\Gamma) = \{\mu^R : \mu \in f(\Gamma^*)\}$.*

Lemma 2 proves that the exclusion core and YRMH-IGYT are strongly consistent on $\mathcal{E}^0$, whereas Example 5 has shown that the strong core is not. The intuition behind the result is that, in any $\Gamma \in \mathcal{E}^0$, if an agent receives a public endowment, she is impossible to be excluded by any coalition. So, it does not change the exclusion core if we regard public endowments as privately owned by i^*, who never joins any exclusion blocking coalition since she cannot be made strictly better off. It neither changes the outcome of YRMH-IGYT associated with any $\rhd$, because i^* must be removed with o^* before any public endowments are assigned, and after i^* is removed, the remaining procedures of YRMH-IGYT in the two economies coincide.

Given Lemma 2, by Proposition 7, the exclusion core and YRMH-IGYT still coincide in private-public-ownership economies. Since the strong exclusion core lies between them, the three solutions coincide in such economies, which we formalize as Proposition 9. This also implies that the strong exclusion core is strongly consistent on $\mathcal{E}^0$.

Lemma 2. *The (strong) exclusion core and YRMH-IGTY are strongly consistent on $\mathcal{E}^0$, whereas the strong core is not.*

Proposition 9. *In private-public-ownership economies, YRMH-IGYT=strong exclusion core =exclusion core.*

BK have proved that in the house allocation with existing tenants model, YRMH-IGYT coincides with the exclusion core. Proposition 9 generalizes their result by showing that the result holds in all private-public-ownership economies.

Since the strong core is consistent, for every $\Gamma \in \mathcal{E}^0$, if the strong core in Γ^* is nonempty, its restriction to Γ is a subset of the strong core in Γ. In that case, Lemma 2 and Proposition 7 imply that, in Γ, the exclusion core is a subset of the strong core.

We omit the rectified core and the effective core from the results in this section, as their relationships with other solutions follow directly from known results. In Proposition 5 and Proposition 6, they coincide with the strong core. In Proposition 7 and Proposition 9, they are a superset of the exclusion core. In Proposition 8, they coincide with other solutions, all consisting of all Pareto efficient allocations.

9 Remark on a stronger form of consistency

The *consistency* property requires that a solution must not shrink as a self-enforcing coalition in any allocation selected by the solution leaves with their assignments. However, it does not preclude the possibility that the solution expands in a reduced economy. Section 8 defines strong consistency on $\mathcal{E}^0$, which precludes a solution's expansion in the reduced economy of every Γ^*, where $\Gamma \in \mathcal{E}^0$, when i^* leaves with o^*. We prove that the strong exclusion core and some other solutions satisfy this property. This section defines a general form of strong consistency and examines whether those solutions still satisfy it. Through two examples, we show that strong consistency in the general form is too demanding in our model. Solutions discussed in this paper do not satisfy it.

Definition 8. *A solution f is **strongly consistent** if, for every $\Gamma = (I, O, \succ_I, \{C_o\}_{o\in O})$, if $f(\Gamma) \neq \emptyset$, then for every $\mu \in f(\Gamma)$ and every $I' \subsetneq I$ such that $\mu(I') \subseteq \omega(I') \cup \{o^*\}$, $f(\Gamma(\mu, I\backslash I')) = \{\delta_{I\backslash I'} : \delta \in f(\Gamma)$ and $\delta_{I'} = \mu_{I'}\}$.*

In words, strong consistency requires that, if a self-enforcing coalition I' leaves with their assignments in an allocation μ recommended by a solution f for an economy Γ, then the set of allocations recommend by f for the reduced economy $\Gamma(\mu, I\backslash I')$ must coincide with the set of restrictions of allocations recommend by f for Γ in which I' receive the same assignments as they do under μ.

For any allocation μ' in $\Gamma(\mu, I\backslash I')$, we use $\mu' \oplus \mu_{I'}$ to denote an allocation in Γ such that the agents in $I\backslash I'$ receive their assignments in μ' and the agents in I' receive their assignments in μ. Then, in addition to consistency, strong consistency requires that, for every $\mu' \in f(\Gamma(\mu, I\backslash I'))$, $\mu' \oplus \mu_{I'} \in f(\Gamma)$.

The following example shows that, if we do not impose any restriction on the preferences of the leaving agents in I', even though an allocation μ' is in the strong exclusion core in $\Gamma(\mu, I\backslash I')$, $\mu' \oplus \mu_{I'}$ may not be Pareto efficient in Γ. Since the economies in the example are no-overlapping-ownership, the strong core and the two solutions of Sun et al. (2025) coincide with the strong exclusion core in the example. So, the analysis also holds for these solutions.

Example 6. *Consider the following economy where a is owned by $\{1, 2\}$ and b is owned by 3.*

	a	b
C_o:	$1,2$	3
μ:	1	3
σ:	3	2

$\succ_1$	$\succ_2$	$\succ_3$
a	b	a
b	a	b

Among the three agents, 1 has identical preferences with 3, while 2 has opposite preferences. So, in any Pareto efficient allocation, if 3 obtains b, 1 must obtain a, while if 3 obtains a, either 1 or 2 can obtain b. However, if 3 obtains a and 1 obtains b, then $\{1, 2\}$ can weakly exclusion block the allocation via μ, while if 3 obtains nothing, then 3 can block the allocation by claiming b. Thus, the strong exclusion core $= \{\mu, \sigma\}$.

In μ, 3 is self-enforcing. Removing 3 with b, we obtain a reduced economy where $\{1,2\}$ own a. The restriction of μ, where 1 receives a and 2 receives o^*, is in the strong exclusion core. However, in the reduced economy, another allocation δ, where 1 receives o^* and 2 receives a, is also in the strong exclusion core. But $\delta \oplus \mu_{\{3\}}$ is not Pareto efficient in the original economy.

In Example 6, agent 3 is self-enforcing in μ, but she prefers a remaining object in the reduced economy to the object she leaves with.

However, even though we require that the leaving agents in I' must prefer their assignments over the remaining objects, and I' is restricted to be a minimal self-enforcing coalition, the strong exclusion core is still not strongly consistent. See Example 7.

Example 7. *Consider the following economy with four agents and three objects.*

	a	b	c
C_o:	1	1	1, 2, 4
μ:	1	2	3
σ:	1	2	4

$\succ_1$	$\succ_2$	$\succ_3$	$\succ_4$
a	b	c	c
b	a	o^*	o^*
c	c		

The allocation μ is not in the strong exclusion core. It can be weakly exclusion blocked by $\{1,2,4\}$ via σ, since $\{1,2\}$ are unaffected and self-enforcing, yet the blocking executes the joint exclusion right of $\{1,2,4\}$.

The allocation σ is in the strong exclusion core. In σ, 1 is self-enforcing and prefers her assignment to remaining objects. Removing 1 with object a, we obtain a reduced economy where b is publicly owned and c is owned by $\{2,4\}$. However, in the reduced economy, the restriction of μ is in the strong exclusion core: $\{2,4\}$ cannot form a coalition to weakly exclusion block the restriction of μ, because 2 must receive b to remain unaffected and b is publicly owned. Since $\mu_{\{2,3,4\}} \oplus \sigma_{\{1\}} = \mu$, it means that the strong exclusion core is not strongly consistent.

To see that the other solutions discussed in the paper are neither strongly consistent, recall that the exclusion core and the weak core are not weakly consistent (Proposition 2), the effective core is not consistent (Proposition 4), and the strong core and the rectified core are not strongly consistent on $\mathcal{E}^0$ (Example 5).[12]

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Appendix

A Proofs of Lemma 1 and Theorem 1

Proof of Lemma 1. For any economy $\Gamma = (I, O, \succ_I, \{C_o\}_{o \in O})$ and any order of agents $\rhd$, let μ denote the outcome of YRMH-IGYT with $\rhd$. We prove that μ belongs to the strong exclusion core.

Suppose that μ is weakly exclusion blocked by a coalition C via another σ. Without loss of generality, let C contain all agents who are strictly better off in σ. Since YRMH-IGYT is a special case of the Generalized TTC algorithm defined by BK, μ is in the exclusion core. So, it must be that $C_{\sigma=\mu} \neq \emptyset$, and there are agents who are worse off in σ.

Among the agents who are worse off in σ, let $i^\diamond$ be an agent who is removed earliest in YRMH-IGYT. Suppose that $i^\diamond$ is removed in step $t^\diamond$. Among the agents who are strictly better off in σ, let $i^\star$ be an agent who is removed earliest in YRMH-IGYT. Suppose that $i^\star$ is removed in step $t^\star$. Because $\sigma(i^\star)$ is better than $\mu(i^\star)$ for $i^\star$, $\sigma(i^\star)$ must be removed before step $t^\star$. Let j be the agent such that $\mu(j) = \sigma(i^\star)$. Thus, $\sigma(j) \neq \mu(j)$, and j is removed earlier than $i^\star$. So, j must not belong to C and must be worse off in σ. Then, j is removed no earlier than $i^\diamond$. It implies that $t^\star > t^\diamond$. Thus, all agents who are strictly better off in σ must be removed no earlier than step $t^\star$, all agents who are worse off in σ must be removed no earlier than step $t^\diamond$, and all agents who are removed before step $t^\diamond$ must be unaffected.

Let $o^\diamond = \mu(i^\diamond)$. Because $i^\diamond$ is worse off in σ, $o^\diamond \in \Omega(C|\omega, \mu)$. However, we prove that there must exist a self-enforcing $C' \subseteq C_{\sigma=\mu}$ such that $o^\diamond \in \Omega^*(C'|C, \omega, \mu)$, which leads to a contradiction. We prove this result through four claims.

Claim 1. *In any step $t \leq t^\diamond$ of YRMH-IGYT, if a cycle is removed and the cycle involves an agent from C or an object from $\omega(C)$, then all agents involved in the cycle belong to $C_{\sigma=\mu}$ and all objects involved in the cycle belong to $\omega(C)$.*

Proof. Let $\{t_1, t_2, \ldots, t_N\}$ be the sequence of steps $t \leq t^\diamond$ in which a cycle is removed and the cycle involves an agent from C or an object from $\omega(C)$.

In step t_1, without loss of generality, represent the cycle removed in that step by

$$i \to o \to i_1 \to o_2 \to i_2 \to \cdots \to o_\ell \to i.$$

Without loss of generality, we assume that either $i \in C$ or $o \in \omega(C)$. If $i \in C$, because all agents in $C_{\sigma>\mu}$ are removed after step $t^\diamond$, $i \in C_{\sigma=\mu}$. Because $\mu(i) = o$ and $\mu(C_{\sigma=\mu}) \subseteq \omega(C) \cup \{o^*\}$, $o \in \omega(C)$. Then, i_1 must be an owner of o, since otherwise i_1 must acquire the shared ownership of o and thus o must be involved in a cycle earlier than step t_1, which contradicts the definition of step t_1. So, $i_1 \in C$, and, similar to i, $i_1 \in C_{\sigma=\mu}$. Applying the arguments inductively to the remaining agents and objects in the cycle, we conclude that all agents in the cycle belong to $C_{\sigma=\mu}$ and all objects in the cycle belong to $\omega(C)$. If $o \in \omega(C)$, the arguments similarly apply.

In step t_2, without loss of generality, represent the cycle removed in that step still by

$$i \to o \to i_1 \to o_2 \to i_2 \to \cdots \to o_\ell \to i.$$

Without loss of generality, we assume that either $i \in C$ or $o \in \omega(C)$. If $i \in C$, similarly as above, $i \in C_{\sigma=\mu}$, and thus $o \in \omega(C)$. If i_1 is an owner of o, then $i_1 \in C$, and thus $i_1 \in C_{\sigma=\mu}$. If i_1 acquires the shared ownership of o, then all owners of o must be removed before step t_2 and at least one owner of o is involved in a cycle such that the ownership of o is shared with others. Then, that cycle must be removed in step t_1, since step t_1 is the first step in which a cycle involving an agent from C is removed. Then, by the result in step t_1, all agents in that cycle belong to $C_{\sigma=\mu}$. Since i_1 acquires the shared ownership of o, i_1 must be involved in that cycle, implying that $i_1 \in C_{\sigma=\mu}$. Thus, in any case, we must have $i_1 \in C_{\sigma=\mu}$. Applying these arguments inductively to the remaining agents and objects in the cycle, we conclude that all agents in the cycle belong to $C_{\sigma=\mu}$ and all objects in the cycle belong to $\omega(C)$. If $o \in \omega(C)$, the arguments similarly apply.

The proof inductively holds for the remaining steps. ■

Claim 2. *In any step $t \leq t^\diamond$ of YRMH-IGYT, if any $o \in \omega(C)$ is removed without being involved in a cycle, then there exists a self-enforcing $C' \subseteq C_{\sigma=\mu}$ such that $o \in \omega(C')$.*

Proof. Since o is removed without being involved in a cycle, o does not point to any agent in step t. So, all owners of o and those who acquire the shared ownership of o are removed before step t. Since $o \in \omega(C)$, it implies that $C_o \subseteq C_{\sigma=\mu}$.

Let C' consist of the agents in C_o, the owners of $\mu(C_o)$ (i.e., $\cup_{o' \in \mu(C_o)} C_{o'}$), the owners of $\mu(\cup_{o' \in \mu(C_o)} C_{o'})$, and so on. Formally, let $C' = \cup_{k=0}^{\infty} C^k$ where $C^0 = C_o$ and $C^k = \cup_{o' \in \mu(C^{k-1})} C_{o'}$ for all $k \geq 1$. Then, $\mu(C') \subseteq \omega(C') \cup \{o^*\}$ and $o \in \omega(C')$. If any agent in C' remains in step t, then the agent must acquire the shared ownership of o. Thus, o must point to some agent in step t, which is a contradiction. So, all agents in C' must be removed before step t. This implies that these agents are unaffected from μ to σ. Since $C^0 = C_o \subseteq C_{\sigma=\mu}$ and $\mu(C_{\sigma=\mu}) \subseteq \omega(C) \cup \{o^*\}$, $C^1 = \cup_{o' \in \mu(C^0)} C_{o'} \subseteq C$. Thus, $C^1 \subseteq C_{\sigma=\mu}$. Applying this argument inductively to C^k, we conclude that $C^k \subseteq C_{\sigma=\mu}$ for all $k \geq 1$. So, $C' \subseteq C_{\sigma=\mu}$. Thus, C' is a self-enforcing subset of $C_{\sigma=\mu}$ and $o \in \omega(C')$. ■

Claim 3. *In any step $t \le t^{\diamond}$ of YRMH-IGYT, if any $o \notin \omega(C)$ is removed without being involved in a cycle, and $C_o \cap C \neq \emptyset$, then for any $j \in C_o \cap C$, there exists a self-enforcing $C' \subseteq C_{\sigma=\mu}$ such that $\mu(j) \in \omega(C')$.*

Proof. Since o is removed without being involved in a cycle, o does not point to any agent in step t. This means that all owners of o and those who acquire the shared ownership of o are removed before step t.

Consider any $j \in C_o \cap C$. Since j is removed before step t, $j \in C_{\sigma=\mu}$, and thus $\mu(j) \in \omega(C)$. If j and $\mu(j)$ are removed without being involved in a cycle, by Claim 2, there exists a self-enforcing $C' \subseteq C_{\sigma=\mu}$ such that $\mu(j) \in \omega(C')$. If j and $\mu(j)$ are removed by being involved in a cycle, let C' consist of the agents in $C_{\mu(j)}$, the owners of $\mu(C_{\mu(j)})$, and so on. Formally, let $C' = \cup_{k=0}^{\infty} C^k$ where $C^0 = C_{\mu(j)}$ and $C^k = \cup_{o' \in \mu(C^{k-1})} C_{o'}$ for all $k \ge 1$. Then, $\mu(C') \subseteq \omega(C') \cup \{o^*\}$ and $\mu(j) \in \omega(C')$. If any agent in C' remains in step t, then the agent must acquire the shared ownership of o. Thus, o must point to some agent in step t, which is a contradiction. So, all agents in C' must be removed before step t. This implies that these agents are unaffected from μ to σ. Thus, $C_{\mu(j)} \subseteq C_{\sigma=\mu}$. Since $\mu(C_{\sigma=\mu}) \subseteq \omega(C) \cup \{o^*\}$, $C^1 = \cup_{o' \in \mu(C^0)} C_{o'} \subseteq C$. Thus, $C^1 \subseteq C_{\sigma=\mu}$. Applying this argument inductively to C^k, we conclude that $C^k \subseteq C_{\sigma=\mu}$ for all $k \ge 1$. So, $C' \subseteq C_{\sigma=\mu}$. Thus, C' is a self-enforcing subset of $C_{\sigma=\mu}$ and $\mu(j) \in \omega(C')$. ■

Recall that $\Omega\big(C|\omega,\mu\big) = \omega(\cup_{k=0}^{\infty} C^k)$ where $C^0 = C$ and $C^k = C^{k-1} \cup \{i \in I \backslash C^{k-1} : \mu(i) \in \omega(C^{k-1})\}$ for every $k \ge 1$.

Claim 4. *In any step $t \le t^{\diamond}$ of YRMH-IGYT, if any $i \in C^k \backslash C^{k-1}$ for any $k \ge 1$ is removed with an object o, then i and o are not involved in a cycle, and there exists a self-enforcing $C' \subseteq C_{\sigma=\mu}$ such that $o \in \Omega^*(C'|C,\omega,\mu)$.*

Proof. We prove the lemma by induction.

Base step: Consider any $i \in C^1 \backslash C^0$. Then, $o \in \omega(C)$.

If i and o are involved in a cycle in step t, since $o \in \omega(C)$, by Claim 1, all agents in the cycle belong to $C_{\sigma=\mu}$, which is a contradiction. So, i and o are not involved in a cycle. Then, by Claim 2, there exists a self-enforcing $C' \subseteq C_{\sigma=\mu}$ such that $o \in \omega(C')$.

Induction step: Suppose that the lemma holds for all $i \in C^{k-1} \backslash C^0$ with some $k \ge 2$. We then prove that it also holds for all $i \in C^k \backslash C^{k-1}$.

Consider any $i \in C^k \backslash C^{k-1}$. Then, $o \in \omega(C^{k-1})$ but $o \notin \omega(C^{k-2})$.

We first prove that i and o are not involved in a cycle in step t. Suppose that they are involved in a cycle. Let i_1 be the agent pointed by o in the cycle. If i_1 is an owner of o, there are two cases. If $i_1 \in C$, by Claim 1, all agents in the cycle belong to $C_{\sigma=\mu}$, which is a contradiction. If $i_1 \in C^{k-1} \backslash C$, by the induction assumption, i_1 and $\mu(i_1)$ are not involved in a cycle in step t, which is a contradiction.

So, i_1 must not be an owner of o. Then, all owners of o must be removed before step t and i_1 must acquire the shared ownership of o. Note that $C_o \subseteq C^{k-1}$.

For every $j \in C_o$ such that $j \in C^{k-1} \backslash C$, by the induction assumption, j and $\mu(j)$ are not involved in a cycle when they are removed. Therefore, j does not share the ownership of o with other agents.

For every $j \in C_o$ such that $j \in C$, since j is removed before step t, $j \in C_{\sigma=\mu}$ and thus $\mu(j) \in \omega(C)$. By Claim 1, if j shares the ownership of o with other agents before step t, those agents must belong to $C_{\sigma=\mu}$. By the same argument, those agents also only share the ownership of o with other agents who belong to $C_{\sigma=\mu}$. Thus, since i_1 acquires the shared ownership of o, $i_1 \in C_{\sigma=\mu}$. However, by Claim 1, it means that all agents in the cycle in step t belong to $C_{\sigma=\mu}$, which is a contradiction.

Therefore, i and o are not involved in a cycle in step t. So, o does not point to any agent when it is removed. This means that all owners of o and those who acquire the shared ownership of o are removed before step t.

For every $j \in C_o$ such that $j \in C$, since j is removed before step t, $j \in C_{\sigma=\mu}$ and thus $\mu(j) \in \omega(C)$. There are two cases. If j and $\mu(j)$ are not involved in a cycle when they are removed, by Claim 2, there exists a self-enforcing $C' \subseteq C_{\sigma=\mu}$ such that $\mu(j) \in \omega(C')$. Let $C'_j = C' \cup \{j\}$. Then, C'_j is a self-enforcing subset of $C_{\sigma=\mu}$ such that $j \in C'_j$ and $\mu(j) \in \omega(C'_j)$. If j and $\mu(j)$ are involved in a cycle when they are removed, since $j \in C_o$ and o is removed without being involved in a cycle, by Claim 3, there exists a self-enforcing $C' \subseteq C_{\sigma=\mu}$ such that $\mu(j) \in \omega(C')$. Let $C'_j = C' \cup \{j\}$. Then, C'_j is a self-enforcing subset of $C_{\sigma=\mu}$ such that $j \in C'_j$ and $\mu(j) \in \omega(C'_j)$.

For every $j \in C_o$ such that $j \in C^{k-1} \backslash C$, by the induction assumption, there exists a self-enforcing $C'_j \subseteq C_{\sigma=\mu}$ such that $\mu(j) \in \Omega^*(C'_j | C, \omega, \mu)$.

Thus, $o \in \Omega^*(\cup_{j \in C_o} C'_j | C, \omega, \mu)$, where $\cup_{j \in C_o} C'_j$ is a self-enforcing subset of $C_{\sigma=\mu}$. ■

Since $o^\diamond \in \Omega(C|\omega, \mu)$, there exists $k \geq 1$ such that $o^\diamond \in \omega(C^{k-1})$ and $i^\diamond \in C^k \backslash C^{k-1}$. Because $i^\diamond$ is removed with $o^\diamond$ in step $t^\diamond$, by Claim 4, there exists a self-enforcing $C' \subseteq C_{\sigma=\mu}$ such that $o^\diamond \in \Omega^*(C'|C, \omega, \mu)$. This means that μ is not weakly exclusion blocked. □

Proof of Theorem 1. Lemma 1 has implied that the strong exclusion core is nonempty. It is Pareto efficient because it is a subset of the exclusion core, which is Pareto efficient. In the following, we prove that the strong exclusion core is consistent.

Consider any economy $\Gamma = (I, O, \succ_I, \{C_o\}_{o \in O})$. Let μ be any allocation in the strong exclusion core. For every self-enforcing $I' \subsetneq I$ in μ, we want to prove that $\mu_{I \backslash I'}$ is in the strong exclusion core in $\Gamma(\mu, I\backslash I')$. Let ω' denote the endowment function in $\Gamma(\mu, I\backslash I')$.

Suppose that in $\Gamma(\mu, I\backslash I')$, $\mu_{I\backslash I'}$ is weakly exclusion blocked by a coalition C via another σ'. Since μ is Pareto efficient in Γ, $\mu_{I\backslash I'}$ is Pareto efficient in $\Gamma(\mu, I\backslash I')$. So, $C \subsetneq I\backslash I'$. We then prove that in Γ, μ must be weakly exclusion blocked by $C^* = C \cup I'$ via σ, where $\sigma_{I\backslash I'} = \sigma'$ and $\sigma_{I'} = \mu_{I'}$. This is a contradiction.

We verify the following conditions.

(1) It obviously holds that, $\forall i \in C^*$, $\sigma(i) \succsim_i \mu(i)$, and $\exists i \in C$ such that $\sigma(i) \succ_i \mu(i)$.

(2) Since $C^*_{\sigma=\mu} = C_{\sigma=\mu} \cup I'$, $C^*_{\sigma=\mu}$ is nonempty. Then, $\sigma(C^*_{\sigma=\mu}) = \sigma(C_{\sigma=\mu}) \cup \sigma(I')$. Since $\sigma(C_{\sigma=\mu}) = \sigma'(C_{\sigma=\mu}) \subseteq \omega'(C) \cup \{o^*\} \subseteq \omega(C^*) \cup \{o^*\}$ and $\sigma(I') = \mu(I') \subseteq \omega(I') \cup \{o^*\} \subseteq \omega(C^*) \cup \{o^*\}$, we have $\sigma(C^*_{\sigma=\mu}) \subseteq \omega(C^*) \cup \{o^*\}$.

(3) $\forall j \in I\backslash C^*$, if $\mu(j) \succ_j \sigma(j)$, then $j \in I\backslash I'$ and $\sigma(j) = \sigma'(j)$. Since C weakly exclusion blocks $\mu_{I\backslash I'}$ via σ' in $\Gamma(\mu, I\backslash I')$, $\mu(j) \in \Omega(C|\omega', \mu_{I\backslash I'})$. It is easy to prove that $\Omega(C|\omega', \mu_{I\backslash I'}) \subseteq \Omega(C^*|\omega, \mu)$.[13] Thus, $\mu(j) \in \Omega(C^*|\omega, \mu)$. It remains to prove that $\mu(j) \notin \Omega^*(C'|C^*, \omega, \mu)$ for every $C' \subseteq C^*_{\sigma=\mu}$ such that $\sigma(C') \subseteq \omega(C') \cup \{o^*\}$. There are three cases.

Case 1: $C' \subseteq C$. Then, C' is self-enforcing in $\Gamma(\mu, I\backslash I')$. So, $\mu(j) \notin \Omega^*(C'|C, \omega', \mu_{I\backslash I'})$. It is easy to prove that $\Omega^*(C'|C^*, \omega, \mu) \subseteq \Omega^*(C'|C, \omega', \mu_{I\backslash I'})$.[14] So, $\mu(j) \notin \Omega^*(C'|C^*, \omega, \mu)$.

Case 2: $C' \subseteq I'$. Let $\Omega^*(C'|C^*, \omega, \mu) = \omega(\cup_{k=0}^{\infty} C^k)$, where $C^0 = C'$, $C^1 = C^0 \cup \{i \in I\backslash C^* : \mu(i) \in \omega(C^0)\}$, and $C^k = C^{k-1} \cup \{i \in I\backslash(C^* \cup C^{k-1}) : \mu(i) \in \omega(C^{k-1})\}$ for every $k \geq 1$. For every $i \in C^1\backslash C^0$, $\mu(i) \in \omega(C')$. So, $\mu(i)$ is publicly owned in $\Gamma(\mu, I\backslash I')$.

Suppose that $\mu(j) \in \Omega^*(C'|C^*, \omega, \mu)$. Then, $\mu(j) \in \omega(C')$, or $\mu(j) \in \omega(C^k)$ for some $k \geq 1$. If $\mu(j) \in \omega(C')$, then $\mu(j)$ is publicly owned in $\Gamma(\mu, I\backslash I')$. So, $\mu(j) \notin \Omega(C|\omega', \mu_{I\backslash I'})$, since $C \subsetneq I\backslash I'$. If $\mu(j) \in \omega(C^k)$ for some $k \geq 1$, then because $C^1\backslash C^0 \subseteq C^k$ and $\mu(C^1\backslash C^0) \subseteq \omega(C')$, $\omega(C^k)$ is not controlled by C in $\Gamma(\mu, I\backslash I')$. So, $\mu(j) \notin \Omega(C|\omega', \mu_{I\backslash I'})$. In any case, there is a contradiction.

Case 3: $C' \cap I' \neq \emptyset$ and $C' \cap C \neq \emptyset$. Since $\sigma(C' \cap I') \subseteq \omega(C') \cup \{o^*\}$ and $\sigma(C' \cap I') \subseteq \omega(I') \cup \{o^*\}$, it must be that $\sigma(C' \cap I') \subseteq \omega(C' \cap I') \cup \{o^*\}$. So, $C' \cap I'$ is self-enforcing in Γ. Since $\sigma(C' \cap C) \subseteq \omega(C') \cup \{o^*\}$ and $\sigma(C' \cap C) \subseteq \sigma(C_{\sigma=\mu}) \subseteq \omega'(C) \cup \{o^*\}$, it must be that $\sigma(C' \cap C) \subseteq \omega'(C' \cap C) \cup \{o^*\}$. That is, $C' \cap C$ is self-enforcing in $\Gamma(\mu, I\backslash I')$.

Suppose that $\mu(j) \in \Omega^*(C'|C^*, \omega, \mu)$. Given that $\mu(j) \in \Omega(C|\omega', \mu_{I\backslash I'})$, in the following,

[13] Let $\Omega(C^*|\omega, \mu) = \omega(\cup_{k=0}^{\infty} C^k)$, where $C^0 = C^*$ and $C^k = C^{k-1} \cup \{i \in I\backslash C^{k-1} : \mu(i) \in \omega(C^{k-1})\}$ for every $k \geq 1$. Let $\Omega(C|\omega', \mu_{I\backslash I'}) = \omega'(\cup_{k=0}^{\infty} \tilde{C}^k)$, where $\tilde{C}^0 = C$ and $\tilde{C}^k = \tilde{C}^{k-1} \cup \{i \in (I\backslash I')\backslash \tilde{C}^{k-1} : \mu(i) \in \omega'(\tilde{C}^{k-1})\}$ for every $k \geq 1$. It is obvious that $\omega'(C) \subseteq \omega(C^*)$. Then, we obtain $\tilde{C}^1 \subseteq C^1$. Since $I' \subseteq C^1$, we similarly obtain $\omega'(\tilde{C}^1) \subseteq \omega(C^1)$. It inductively holds that $\tilde{C}^k \subseteq C^k$ and $\omega'(\tilde{C}^k) \subseteq \omega(C^k)$ for all $k \geq 1$.

[14] Let $\Omega^*(C'|C^*, \omega, \mu) = \omega(\cup_{k=0}^{\infty} C^k)$, where $C^0 = C'$, $C^1 = C^0 \cup \{i \in I\backslash C^* : \mu(i) \in \omega(C^0)\}$, and $C^k = C^{k-1} \cup \{i \in I\backslash(C^* \cup C^{k-1}) : \mu(i) \in \omega(C^{k-1})\}$ for every $k \geq 1$. Let $\Omega^*(C'|C, \omega', \mu_{I\backslash I'}) = \omega'(\cup_{k=0}^{\infty} \tilde{C}^k)$, where $\tilde{C}^0 = C'$, $\tilde{C}^1 = \tilde{C}^0 \cup \{i \in (I\backslash I')\backslash C : \mu(i) \in \omega'(C^0)\}$, and $\tilde{C}^k = \tilde{C}^{k-1} \cup \{i \in (I\backslash I')\backslash(C \cup C^{k-1}) : \mu(i) \in \omega'(\tilde{C}^{k-1})\}$ for every $k \geq 1$. It is obvious that $\omega(C') \subseteq \omega'(C')$. Then, we obtain $C^1 \subseteq \tilde{C}^1$, and similarly, $\omega(C^1) \subseteq \omega'(\tilde{C}^1)$. It inductively holds that $C^k \subseteq \tilde{C}^k$ and $\omega(C^k) \subseteq \omega'(\tilde{C}^k)$ for all $k \geq 1$.

we prove that $\mu(j) \in \Omega^*(C' \cap C|C, \omega', \mu)$, which leads to a contradiction.

Let $\Omega(C|\omega', \mu_{I\setminus I'}) = \omega'(\cup_{k=0}^{\infty} \tilde{C}^k)$, where $\tilde{C}^0 = C$ and $\tilde{C}^k = \tilde{C}^{k-1} \cup \{i \in (I\setminus I')\setminus \tilde{C}^{k-1} : \mu(i) \in \omega'(\tilde{C}^{k-1})\}$ for every $k \geq 1$. Let $\Omega^*(C'|C^*, \omega, \mu) = \omega(\cup_{k=0}^{\infty} C^k)$, where $C^0 = C'$, $C^1 = C^0 \cup \{i \in I\setminus C^* : \mu(i) \in \omega(C^0)\}$, and $C^k = C^{k-1} \cup \{i \in I\setminus(C^* \cup C^{k-1}) : \mu(i) \in \omega(C^{k-1})\}$ for every $k \geq 1$.

Since $\mu(j) \in \Omega(C|\omega', \mu_{I\setminus I'})$, there exists $k \geq 0$ such that $\mu(j) \in \omega'(\tilde{C}^k)$.

If $\mu(j) \in \omega'(\tilde{C}^0)$, since $\tilde{C}^0 = C$, $C_{\mu(j)} \subseteq C \cup I'$ and $C_{\mu(j)} \cap C \neq \emptyset$. Then, since $\mu(j) \in \Omega^*(C'|C^*, \omega, \mu)$, we must have $C_{\mu(j)} \subseteq C'$. Thus, $C_{\mu(j)} \cap C \subseteq C' \cap C$, and $\mu(j) \in \omega'(C' \cap C)$.

If $\mu(j) \in \omega'(\tilde{C}^1)\setminus\omega'(\tilde{C}^0)$, then $C_{\mu(j)} \subseteq \tilde{C}^1 \cup I'$ and $C_{\mu(j)} \cap [\tilde{C}^1\setminus\tilde{C}^0] \neq \emptyset$. Then, since $\mu(j) \in \Omega^*(C'|C^*, \omega, \mu)$, for every $i \in C_{\mu(j)}$ such that $i \notin C'$, we must have $i \in I\setminus C^*$ and $\mu(i) \in \omega(C')$. Then, since $C_{\mu(j)} \subseteq \tilde{C}^1 \cup I'$, $\mu(i) \in \omega'(C)$. Then, similar to the above case, $\mu(i) \in \omega'(C' \cap C)$. So, $\mu(j) \in \Omega^*(C' \cap C|C, \omega', \mu)$.

Inductively, suppose that for some $k \geq 1$ and all $i \in I\setminus C^*$ such that $\mu(i) \in \omega'(\tilde{C}^{k-1})$ and $\mu(i) \in \Omega^*(C'|C^*, \omega, \mu)$, we have proved that $\mu(i) \in \Omega^*(C' \cap C|C, \omega', \mu)$. We then consider the case that $\mu(j) \in \omega'(\tilde{C}^k)\setminus\omega'(\tilde{C}^{k-1})$. Then, $C_{\mu(j)} \subseteq \tilde{C}^k \cup I'$ and $C_{\mu(j)} \cap [\tilde{C}^k\setminus\tilde{C}^{k-1}] \neq \emptyset$. Since $\mu(j) \in \Omega^*(C'|C^*, \omega, \mu)$, for every $i \in C_{\mu(j)}$ such that $i \notin C'$, $i \in I\setminus C^*$ and $\mu(i) \in \Omega^*(C'|C^*, \omega, \mu)$. Then, since $C_{\mu(j)} \subseteq \tilde{C}^k \cup I'$, $\mu(i) \in \omega'(\tilde{C}^{k-1})$. By the induction assumption, $\mu(i) \in \Omega^*(C' \cap C|C, \omega', \mu)$. So, $\mu(j) \in \Omega^*(C' \cap C|C, \omega', \mu)$. □

B Proofs of Propositions 1 to 4

Proof of Proposition 1. We first prove that the strong core is a singleton in any augmented housing market. In every Pareto efficient allocation, i^* must receive o^*. Let μ denote the allocation in which i^* receives o^* and μ_I coincides with the unique strong core allocation in the original housing market. To prove that μ is in the strong core, suppose that μ is weakly blocked by some $C \subseteq I \cup \{i^*\}$ via some σ'. Then, it must be that $i^* \in C$, and thus $\sigma'(i^*) = \mu(i^*) = o^*$. So, in the original housing market, μ_I is weakly blocked by $C\setminus\{i^*\}$ via σ'_I, which is a contradiction. Let μ' be any other Pareto efficient allocation. In the original housing market, μ'_I is weakly blocked by a coalition $C \subseteq I$ via some σ. Then, in the augmented housing market, μ' must be weakly blocked by $C \cup \{i^*\}$ via σ' where $\sigma'(i^*) = o^*$ and $\sigma'_I = \sigma$. So, μ' is not in the strong core.

We then prove that the exclusion core equals the set of Pareto efficient allocations in any augmented housing market. Let μ be any Pareto efficient allocation. If μ is exclusion blocked by a coalition C via another σ, there must exist an agent who is worse off in σ. Therefore, C must contain i^*, because otherwise $\omega(C) = \emptyset$. However, since i^* receives o^* in μ, i^* cannot be made strictly better off. Thus, i^* cannot join C, which is a contradiction. □

Proof of Proposition 2. Proposition 1 has implied that the exclusion core is not weakly consistent. We prove that the strong core is consistent. Consider any economy $\Gamma = (I, O, \succ_I, \{C_o\}_{o\in O})$. Suppose that the strong core is nonempty in Γ. Let μ be any allocation in the strong core and $I' \subsetneq I$ be a self-enforcing coalition in μ. We want to prove that $\mu_{I\backslash I'}$ is in the strong core in the reduced economy $\Gamma(\mu, I\backslash I')$. Since μ is Pareto efficient in Γ, $\mu_{I\backslash I'}$ must be Pareto efficient in the reduced economy.

Suppose that in the reduced economy, $\mu_{I\backslash I'}$ is weakly blocked by a coalition $C \subseteq I\backslash I'$ via another σ'. Because $\mu_{I\backslash I'}$ is Pareto efficient in the reduced economy, $C \neq I\backslash I'$. In the following, we prove that in Γ, μ is weakly blocked by $C \cup I'$ via an allocation σ, where $\sigma_{I\backslash I'} = \sigma'$ and $\sigma_{I'} = \mu_{I'}$. This is a contradiction.

We verify the following conditions. First, it is obvious that all members of $C \cup I'$ are weakly better off in σ compared to μ and at least one member from C is strictly better off.

Second, since $\mu_{I\backslash I'}$ is weakly blocked by C via σ' in the reduced economy, $\sigma'(C) \subseteq \omega'(C) \cup \{o^*\}$, where ω' denotes the endowment function in the reduced economy. Since $\omega'(C) \subseteq \omega(C\cup I')$ and $\omega(I') \subseteq \omega(C\cup I')$, $\omega'(C)\cup\omega(I') \subseteq \omega(C\cup I')$. So, $\sigma(C\cup I') = \sigma'(C)\cup\mu(I') \subseteq \omega'(C) \cup \omega(I') \cup \{o^*\} \subseteq \omega(C \cup I') \cup \{o^*\}$. □

Proof of Proposition 3. In any economy, suppose that an allocation μ is rectification blocked by a coalition C via another σ. We then prove that μ is also weakly exclusion blocked.

Define $I_1 = \{i \in I\backslash C : \mu(i) \in \sigma(C)\}$. Let σ' be an allocation in which, $\forall i \in I\backslash(I_1 \cup C)$, $\sigma'(i) = \mu(i)$, $\forall i \in C$, $\sigma'(i) = \sigma(i)$, and $\forall i \in I_1$, $\sigma'(i) = o^*$. We then prove that μ is weakly exclusion blocked by C via σ'. We verify the following conditions:

(1) It holds that, $\forall i \in C$, $\sigma'(i) \succsim_i \mu(i)$, and $\exists i \in C$ such that $\sigma'(i) \succ_i \mu(i)$.

(2) If $C_{\sigma'=\mu} \neq \emptyset$, then $\sigma'(C_{\sigma'=\mu}) = \sigma(C_{\sigma=\mu}) \subseteq \omega(C) \cup \{o^*\}$, since $\sigma(C) \subseteq \omega(C) \cup \{o^*\}$.

(3) If $\mu(j) \succ_j \sigma'(j)$, then $j \in I_1$, implying that $\mu(j) \in \sigma(C) \subseteq \omega(C)$. Suppose that there exists $C' \subseteq C_{\sigma'=\mu}$ such that $\sigma'(C') \subseteq \omega(C') \cup \{o^*\}$ and $\mu(j) \in \Omega^*(C'|C, \omega, \mu)$. Then, it means that $\mu(j) \in \omega(C')\backslash\sigma'(C')$. However, rectification blocking requires that $\mu(j) \notin \sigma(C)$, which is a contradiction. □

Proof of Proposition 4. We first prove that the rectified core is consistent. Consider any economy $\Gamma = (I, O, \succ_I, \{C_o\}_{o\in O})$. Let μ be any allocation in the rectified core. For any self-enforcing $I' \subsetneq I$ in μ, we want to prove that $\mu_{I\backslash I'}$ is in the rectified core in $\Gamma(\mu, I\backslash I')$. Let ω' denote the endowment function in $\Gamma(\mu, I\backslash I')$.

Suppose that $\mu_{I\backslash I'}$ is rectification blocked by a coalition C via another σ' in $\Gamma(\mu, I\backslash I')$. Since μ is Pareto efficient in Γ, $\mu_{I\backslash I'}$ is Pareto efficient in $\Gamma(\mu, I\backslash I')$. Therefore, it must be that $C \subsetneq I\backslash I'$. Moreover, since $\sigma'(C) \subseteq \omega'(C) \cup \{o^*\}$, for every $i \in C$, $\sigma'(i)$ is not publicly owned in the reduced economy. Therefore, $\sigma'(C) \cap [\omega(I')\backslash\mu(I')] = \emptyset$.

We then prove that in Γ, μ is rectification blocked by $C^* = C \cup I'$ via σ, where $\sigma_{I\backslash I'} = \sigma'$ and $\sigma_{I'} = \mu_{I'}$. We verify the following conditions.

(1) It obviously holds that, $\forall i \in C^*$, $\sigma(i) \succsim_i \mu(i)$, and $\exists i \in C$ such that $\sigma(i) \succ_i \mu(i)$.

(2) Since $\omega'(C) \subseteq \omega(C \cup I')$ and $\omega(I') \subseteq \omega(C \cup I')$, $\omega'(C) \cup \omega(I') \subseteq \omega(C \cup I')$. So, $\sigma(C \cup I') = \sigma'(C) \cup \mu(I') \subseteq \omega'(C) \cup \omega(I') \cup \{o^*\} \subseteq \omega(C \cup I') \cup \{o^*\}$.

(3) For every $C' \subseteq C^*_{\sigma=\mu}$ such that $\sigma(C') \subseteq \omega(C') \cup \{o^*\}$, if $o \in \omega(C') \backslash \sigma(C')$ and $\mu^{-1}(o) \in I \backslash C^*$, we want to prove that $o \notin \sigma(C^*)$. We consider three cases.

Case 1: $C' \subsetneq C_{\sigma=\mu}$. Since $o \in \omega(C') \backslash \sigma(C')$ and $\mu^{-1}(o) \in I \backslash C^*$, in the reduced economy, $o \in \omega'(C') \backslash \sigma'(C')$ and $\mu^{-1}_{I\backslash I'}(o) \in I \backslash (I' \cup C)$. Then, since C rectification blocks $\mu_{I\backslash I'}$ via σ' in $\Gamma(\mu, I \backslash I')$, $o \notin \sigma'(C)$. Since I' is self-enforcing, $o \notin \sigma(I')$. So, $o \notin \sigma(C^*)$.

Case 2: $C' \subseteq I'$. Since $o \in \omega(C') \backslash \sigma(C')$ and $\mu^{-1}(o) \in I \backslash C^*$, in the reduced economy, o is publicly owned. Since $C \subsetneq I \backslash I'$, $o \notin \sigma'(C)$. Since $\mu^{-1}(o) \in I \backslash C^*$, $o \notin \sigma(I')$. So, $o \notin \sigma(C^*)$.

Case 3: $C' \cap I' \neq \emptyset$ and $C' \cap C \neq \emptyset$. Since $\sigma(C' \cap I') \subseteq \omega(C') \cup \{o^*\}$ and $\sigma(C' \cap I') \subseteq \omega(I') \cup \{o^*\}$, it must be that $\sigma(C' \cap I') \subseteq \omega(C' \cap I') \cup \{o^*\}$. So, $C' \cap I'$ is self-enforcing in the original economy. Since $\sigma(C' \cap C) \subseteq \omega(C') \cup \{o^*\}$ and $\sigma(C' \cap C) \cap [\omega(I') \backslash \mu(I')] = \emptyset$, it must be that $\sigma(C' \cap C) \subseteq \omega'(C' \cap C) \cup \{o^*\}$. That is, $C' \cap C$ is self-enforcing in the reduced economy.

Given $o \in \omega(C') \backslash \sigma(C')$ and $\mu^{-1}(o) \in I \backslash C^*$, there are two cases. If $o \in \omega(C' \cap I') \backslash \sigma(C' \cap I')$, then as in Case 2, $o \notin \sigma(C^*)$. If $o \in \omega'(C' \cap C) \backslash \sigma(C' \cap C)$, because C rectification blocks $\mu_{I\backslash I'}$ via σ' in the reduced economy, $o \notin \sigma'(C)$. So, $o \notin \sigma(C^*)$.

We then prove that the effective core is weakly consistent. Consider the same situation in the above proof but assume that I' is a minimal self-enforcing coalition. We then prove that in the original economy, μ is effectively blocked by $C^* = C \cup I'$ via σ. We only need to verify the following condition, since the other two conditions hold as before.

(3) For every $C' \subseteq C^*_{\sigma=\mu}$ such that $\sigma(C') \subseteq \omega(C') \cup \{o^*\}$, if $o \in \omega(C') \backslash \sigma(C')$, we want to prove that $o \notin \sigma(C^*)$. Since I' is a minimal self-enforcing coalition, if $C' \cap I' \neq \emptyset$, it must be that $I' \subseteq C'$. So, we consider only two cases.

Case 1: $C' \subsetneq C_{\sigma=\mu}$. Since $o \in \omega(C') \backslash \sigma(C')$, in the reduced economy, it must be that $o \in \omega'(C') \backslash \sigma'(C')$. Since $C \neq I \backslash I'$ and C effectively blocks $\mu_{I'}$ via σ' in the reduced economy, $o \notin \sigma'(C)$. So, $o \notin \sigma(C^*)$.

Case 2: $I' \subseteq C'$. Then, $C' \backslash I'$ must be self-enforcing in the reduced economy. Given $o \in \omega(C') \backslash \sigma(C')$, there are two cases. If $o \in \omega(I') \backslash \sigma(I')$, then o is publicly owned in the reduced economy. Since $C \neq I \backslash I'$, $o \notin \sigma(C^*)$. If $o \in \omega'(C' \backslash I') \backslash \sigma(C' \backslash I')$, because C effectively blocks $\mu_{I'}$ via σ' in the reduced economy, $o \notin \sigma'(C)$. So, $o \notin \sigma(C^*)$.

Example 10 in Appendix D shows that the effective core is not consistent. $\square$

C Proofs of Propositions in Section 8

Proof of Proposition 5. This proposition is implied by Proposition 3 and Sun et al.'s (2025) result that the effective core coincides with the strong core in no-redundant-ownership economies. □

Proof of Proposition 6. It suffices to prove that all allocations in the strong core can be found by YRMH-IGYT. Let μ be any allocation in the strong core. For every $o \in O$, there must exist one and only one agent within C_o, denoted by i_o, who obtains an object in μ. Otherwise, there would exist some $o' \in O$ such that all agents in $C_{o'}$ obtain nothing in μ. But then, $C_{o'}$ would weakly block μ by allocating o' to one member of $C_{o'}$, which is a contradiction. Then, μ can be found by YRMH-IGYT with an order $\rhd$ in which all $\{i_o\}_{o \in O}$ are ranked above the other agents. By Proposition 5, we obtain the coincidence between strong core, strong exclusion core, and the outcomes of YRMH-IGYT. □

Proof of Proposition 7. It suffices to prove that YRMH-IGYT can find all allocations in the exclusion core. Let μ be any allocation in the exclusion core. We present an algorithm to find an order $\rhd$ of agents such that μ is found by YRMH-IGYT with $\rhd$.

Initialization: Start with the set of all agents and the set of all objects.

Step $t \geq 1$: Given the set of remaining agents and the set of remaining objects from the previous step, we conduct two operations.

– **Operation A**: Let remaining agents point to their favorite remaining objects. Let remaining objects point to their owners if their owners remain and otherwise point to nothing. Let o^* point to all agents. If there exist cycles, these cycles must be disjoint. Remove these cycles. Repeat this operation until no cycles are generated. All of these cycles must appear in YRMH-IGYT with any order of agents. The first part of Claim 5 below proves that the object pointed by every agent in every cycle must be her assignment in μ. Denote the set of these agents by D_t. If $t = 1$, place the agents in D_1 at the bottom of $\rhd$ and rank them arbitrarily. If $t > 1$, place the agents in D_t right above those in D_{t-1} in $\rhd$ and rank them arbitrarily. Go to Operation B if there exist remaining agents.

– **Operation B**: After Operation A, we obtain a graph in which all remaining agents point to their favorite remaining objects, all remaining objects point to their owners if their owners remain and otherwise point to nothing, but there are no cycles. Because every agent points to one object and every object points to at most one agent, the agents and objects in the graph must form disjoint trees such that the root of each tree is an object that points to nothing and every remaining agent is linked to one root through a unique directed path in the tree.[15] Denote these roots by $o_1, o_2, \ldots, o_m$. . For every root o_ℓ, let I_ℓ denote the set of agents who directly point to o_ℓ. The second part of Claim 5 proves that there exists at least one root o_ℓ and an agent $i \in I_\ell$ such that $\mu(i) = o_\ell$. Denote the set of such i by U_t. If $t = 1$, place the agents in U_1 at the top of $\rhd$ and rank them arbitrarily. If $t > 1$, place the agents in U_t right below those in U_{t-1} in $\rhd$ and rank them arbitrarily. Remove U_t with their assignments in μ. Move to step $t+1$ if there exist remaining agents. Otherwise, stop.

Since at least one agent is removed in each step, the above algorithm must stop in finite steps. Claim 5 implies that the outcome of YRMH-IGYT with the order $\rhd$ is μ.

Claim 5. *In each step of the above algorithm:*

> *(1) In Operation A, the object pointed by every agent in every generated cycle must be her assignment in μ.*
>
> *(2) In Operation B, there exists at least one root o_ℓ and an agent $i \in I_\ell$ such that $\mu(i) = o_\ell$.*

Proof. We prove the claim by induction. Suppose that for all students removed before step t, they are removed with their assignments in μ. We then consider step t.

(**Operation A**) Let $(C_1, C_2, \ldots, C_K)$ be the order of cycles removed in Operation A. That is, each C_k represents the set of agents involved in the corresponding cycle, and C_k is removed no later than C_{k+1}. If multiple cycles are removed simultaneously, their relative ranking in the order is arbitrary.

Suppose that there exists an agent in C_1 who points to an object different from her assignment in μ. Denote by C_1' the set of such agents in the cycle. We prove that C_1' can exclusion block μ via another allocation σ in which, for all $i \in C_1$, $\sigma(i)$ is the object pointed by i in the cycle, for all $j \in I \backslash C_1$ with $\mu(j) \in \sigma(C_1)$, $\sigma(j) = o^*$, and for all other j, $\sigma(j) = \mu(j)$. First, since the objects pointed by all agents in C_1 are their most preferred objects among the remaining ones in step t, all agents in C_1' must be strictly better off in σ. Second, for any $j \in I \backslash C_1'$ with $\mu(j) \succ_j \sigma(j)$, it must be that $\mu(j) \in \sigma(C_1)$. Because C_1 forms a cycle, we have $\sigma(C_1) \subseteq \Omega(C_1'|\omega, \mu)$. Therefore, $\mu(j) \succ_j \sigma(j)$ implies $\mu(j) \in \Omega(C_1'|\omega, \mu)$. However, this contradicts the assumption that μ is in the exclusion core.

Inductively, suppose that for each C_ℓ with $\ell < k$, all agents in the cycle point to their assignments in μ. Now, consider C_k. Suppose that there exists an agent in C_k who points to an object different from her assignment in μ. Denote by C_k' the set of such agents in the cycle. We prove that C_k' can exclusion block μ via another allocation σ in which, for all

[15] Formally, an agent i is linked to an object o_ℓ via a directed path if there exist distinct agents $\{i_1, \ldots, i_k\}$ and distinct objects $\{o_1, \ldots, o_k\}$ such that $i \to o_1 \to i_1 \to o_2 \to i_2 \to \cdots \to i_{k-1} \to o_k \to i_k \to o_\ell$.

$i \in C_k$, $\sigma(i)$ is the object pointed by i, for all $j \in I \backslash C_k$ with $\mu(j) \in \sigma(C_k)$, $\sigma(j) = o^*$, and for all other j, $\sigma(j) = \mu(j)$. First, since the objects pointed by all agents in C_k are their most preferred objects among the remaining ones, all agents in C'_k must be strictly better off in σ. Second, for any $j \in I \backslash C'_k$ with $\mu(j) \succ_j \sigma(j)$, it must be that $\mu(j) \in \sigma(C_k)$. Since the agents in C_k form a cycle, $\sigma(C_k) \subseteq \Omega(C'_k|\omega, \mu)$. Therefore, $\mu(j) \succ_j \sigma(j)$ implies $\mu(j) \in \Omega(C'_k|\omega, \mu)$. However, this contradicts the assumption that μ is in the exclusion core.

(**Operation B**) Since μ is Pareto efficient, every root o_ℓ must be assigned to some agent in μ. Suppose that the claim is not true. So, every root o_ℓ is assigned to some $i_\ell \notin I_\ell$ in μ. We then prove that μ is exclusion blocked, which is a contradiction. There are two cases.

Case 1: There exists a root o_ℓ such that i_ℓ is linked to o_ℓ, but i_ℓ does not directly point to o_ℓ. Suppose that i_ℓ is linked to o_ℓ through the following directed path:

$$i_\ell \to o_1 \to i_1 \to o_2 \to i_2 \to \cdots \to i_{k-1} \to o_k \to i_k \to o_\ell.$$

Define $C = \{i_1, i_2, \ldots, i_k, i_\ell\}$. Let C' consist of the agents in C who point to objects different from their assignments in μ. Since $i_\ell, i_k \in C'$, C' is nonempty. Then, we prove that C' can exclusion block μ via another allocation σ in which, for all $i \in C$, $\sigma(i)$ is the object pointed by i, for all $j \notin C$ with $\mu(j) \in \sigma(C)$, $\sigma(j) = o^*$, and for all other j, $\sigma(j) = \mu(j)$. First, since all agents point to their most preferred objects among the remaining ones, all agents in C' are strictly better off in σ. Second, for every j with $\mu(j) \succ_j \sigma(j)$, it must be that $\mu(j) \in \sigma(C)$. Since the agents in C form a chain and $i_k \in C'$, $\sigma(C) \subseteq \Omega(C'|\omega, \mu)$. Therefore, $\mu(j) \succ_j \sigma(j)$ implies $\mu(j) \in \Omega(C'|\omega, \mu)$.

Case 2: For every root o_ℓ, i_ℓ is not linked to o_ℓ. Then, every i_ℓ must be linked to some $o_k \neq o_\ell$. Let $\{o_1, o_2, \ldots, o_x\}$ be a smallest subset of root objects such that, for every o_y in the subset, i_y is linked to some o_z in the subset. Similar to Case 1, every i_y is linked to o_z through a directed path. Denote by C the set of all agents in those directed paths (see Figure 1 for an illustration). Among these agents, denote by C' the set of agents who point to objects different from their assignments in μ. C' is nonempty because, for every o_y in the subset, every i_y belongs to C', and every agent who directly points to o_y also belongs to C'. We then prove that C' can exclusion block μ via another allocation σ in which, for every $i \in C$, $\sigma(i)$ is the object pointed by i, for every $j \notin C$ with $\mu(j) \in \sigma(C)$, $\sigma(j) = o^*$, and for every other j, $\sigma(j) = \mu(j)$. First, since all agents point to their most preferred objects among the remaining ones, all agents in C' are strictly better off in σ. Second, for every j with $\mu(j) \succ_j \sigma(j)$, it must be that $\mu(j) \in \sigma(C)$. Similar to Case 1, since the agents in C are involved in disjoint paths and the agent in each path who directly points to o_y belongs to C', $\sigma(C) \subseteq \Omega(C'|\omega, \mu)$. Therefore, $\mu(j) \succ_j \sigma(j)$ implies $\mu(j) \in \Omega(C'|\omega, \mu)$. ■

$$i_y \to o_1 \to i_1 \to o_2 \to i_2 \to \cdots \to i_{k-1} \to o_k \to i_k \to o_z.$$
$$i_z \to o'_1 \to i'_1 \to o'_2 \to i'_2 \to \cdots \to i'_{k-1} \to o'_k \to i'_k \to o_y.$$

Figure 1: Suppose that $\{o_y, o_z\}$ is a subset of root objects such that o_y is assigned to i_y who is linked to o_z and o_z is assigned to i_z who is linked to o_y. Then, C is the set of the agents in the two directed paths and C' is the subset of those agents who point to objects different from their assignments in μ. Therefore, $\{i_y, i_z, i_k, i'_k\} \subseteq C'$. Then, all objects in the two paths are controlled by C'.

This finishes the proof of Proposition 7. □

Proof of Proposition 8. For this type of economies, YRMH-IGYT is equivalent to the serial dictatorship algorithm, which can find all Pareto efficient allocations by varying the order of agents. This implies the proposition. □

Proof of Lemma 2. For every allocation μ in any $\Gamma \in \mathcal{E}^0$, let μ^A be the allocation for Γ^* in which i^* receives o^* and the other agents receive their assignments in μ.

We first prove that the exclusion core is strongly consistent on $\mathcal{E}^0$. For every $\Gamma \in \mathcal{E}^0$ and every allocation μ in the exclusion core in Γ^*, we prove that μ^R is in the exclusion core in Γ. Because μ is Pareto efficient in Γ^*, μ must assign o^* to i^* and μ^R must be Pareto efficient in Γ. Suppose that μ^R is exclusion blocked by a coalition C via another allocation σ in Γ. Because μ^R is Pareto efficient, $C \subsetneq I$. For every $j \in I$ with $\mu(j) \succ_j \sigma(j)$, because $\mu(j) \in \Omega(C|\omega, \mu^R)$, $\mu(j)$ cannot be a public endowment in Γ. Let ω^* be the endowment function in Γ^*. Then, we have $\Omega(C|\omega^*, \mu) = \Omega(C|\omega, \mu^R)$, since $i^* \notin C$. But this means that in Γ^*, μ is exclusion blocked by C via σ^A, which is a contradiction.

Symmetrically, consider any μ in the exclusion core in Γ. We prove that μ^A is in the exclusion core in Γ^*. Since μ is Pareto efficient in Γ, μ^A is Pareto efficient in Γ^*. Suppose that μ^A is exclusion blocked by a coalition C via another allocation σ in Γ^*. Then, we have $C \subsetneq I$, and therefore $\Omega(C|\omega, \mu) = \Omega(C|\omega^*, \mu^A)$. But this means that in Γ, μ is exclusion blocked by C via σ^R, which is a contradiction.

We then prove that YRMH-IGYT is strongly consistent on $\mathcal{E}^0$. To prove it, we prove that, in any $\Gamma \in \mathcal{E}^0$, the outcome of YRMH-IGYT associated with any $\rhd$ equals the restriction of the outcome of YRMH-IGYT with any $\rhd^*$ in Γ^*, if $\rhd$ and $\rhd^*$ rank the agents in I identically.

Specifically, in Γ, given $\rhd$, let t be the first step of YRMH-IGYT in which an agent points to a public endowment and is then removed. Denote such an agent by i and the public endowment by o. This means that all agents who are removed before step t are involved in cycles and are removed with private endowments. In Γ^*, let $\rhd^*$ be any order that ranks the agents in I identically to $\rhd$. Then, the agents who are involved in cycles before step t in Γ must be involved in the same cycles in Γ^* before i is removed. So, in Γ^*, after these agents are removed, o is the best object for i. When i points to o, i^* either has been removed or not. If i^* has been removed, then the remaining steps of YRMH-IGYT coincide in the two economies, with the only difference that public endowments in Γ point to the $\triangleright$-highest remaining agent, while they do not point to any agents in Γ^*. If i^* has not been removed, then after i points to o, i^* will be moved to the top of the current order. In the next step, i^* will point to o^* and be removed. After that, the remaining steps of YRMH-IGYT still coincide in the two economies.

As proved in Proposition 9, the strong exclusion core coincides with the exclusion core and YRMH-IGYT in private-public-ownership economies. So, the strong exclusion core is also strongly consistent on $\mathcal{E}^0$. Example 5 has shown that the strong core is not strongly consistent on $\mathcal{E}^0$. □

Proof of Proposition 9. For every $\Gamma \in \mathcal{E}^0$, since Γ^* is private-ownership, by Proposition 7, YRMH-IGYT and the exclusion core coincide in Γ^*. By Lemma 2, YRMH-IGYT and the exclusion core are strongly consistent on $\mathcal{E}^0$. So, the two solutions coincide in Γ. Because the strong exclusion core lies between them, the three solutions coincide. □

D Additional Examples

Example 8 (Illustration of $\Omega^*(C'|C,\omega,\mu)$)**.** *Consider the following economy.*

	a	b	c
C_o:	1, 2	1	1
μ:	3	1	2
σ:	2	1	3

$\succ_1$	$\succ_2$	$\succ_3$
b	a	a
c	b	b
a	c	c

Allocation μ is unintuitive: 1 receives her private endowment b, which is her favorite object. However, although 2 co-owns and most prefers a, 2 receives c, while a is assigned to 3, who owns nothing. We show that μ does not belong to the strong exclusion core. It is weakly exclusion blocked by $C = \{1,2\}$ via σ, through excluding 3 from a. To see this, note that $C_{\sigma=\mu} = \{1\}$, $\sigma(C_{\sigma=\mu}) \subsetneq \omega(C_{\sigma=\mu})$, and $\Omega^*(C_{\sigma=\mu}|C,\omega,\mu) = \{b,c\}$. So, the blocking reflects $\{1,2\}$'s joint exclusion right.

However, because $\mu(2) = c$, which is privately owned by 1, $\Omega(C_{\sigma=\mu}|\omega,\mu) = \Omega(C|\omega,\mu) = \{a,b,c\}$. If we replace $\Omega^*(C'|C,\omega,\mu)$ with $\Omega(C'|\omega,\mu)$ in the definition of the strong exclusion core, then although the above blocking reflects $\{1,2\}$'s joint exclusion right, it would seem that the blocking also reflects 1's own exclusion right, which would disqualify the blocking. That is,

μ would belong to the strong exclusion core. This shows the difference between $\Omega^*(C'|C,\omega,\mu)$ and $\Omega(C'|\omega,\mu)$, and the necessity of introducing $\Omega^*(C'|C,\omega,\mu)$.

Example 9 (YRMH-IGYT $\subsetneq$ strong exclusion core)**.** *Consider the following economy.*

	a	b	c
C_o:	$1,4$	$1,2,3$	$1,2,3$
μ:	1	2	4
σ:	1	2	3

$\succ_1$	$\succ_2$	$\succ_3$	$\succ_4$
a	b	c	c
b	c	b	b
c	a	a	a

We explain that μ cannot be found by YRMH-IGYT, but it is in the strong exclusion core.

Suppose that μ is an outcome of YRMH-IGYT. Because 3 most prefers c, 3 cannot be removed when c is still available. Thus, when 4 points to c in some step, because 3 remains and is an owner of c, c must point to one of its owners. If c does not point to 3 in that step, because 1 obtains a and 2 obtains b, after they are removed, c will point to 3. So, in any case, c must point to 3 in some step, and 3 will then obtain c, which is a contradiction.

Suppose that μ is weakly exclusion blocked by some coalition C. The coalition must contain 3 because she is the only agent who can be made strictly better off. To make 3 strictly better off, 4 must be excluded from c. So, the coalition cannot contain 4. Then, the only possible coalition is $\{1,2,3\}$, because any other coalition does not control any object. However, because 1 is an unaffected agent in the coalition yet her assignment, a, is not owned by the coalition, the blocking does not hold.

Example 10 (Effective core is not consistent)**.** *Consider the following economy.*

	a	b	c
C_o:	$1,3$	4	4
μ:	2	3	4
σ:	1	3	4

$\succ_1$	$\succ_2$	$\succ_3$	$\succ_4$
a	a	b	c
b	b	c	b
c	c	a	a

Allocation μ is in the effective core. In particular, μ cannot be effectively blocked by $\{1,3,4\}$ via σ, because 4 is unaffected and self-enforcing, but 3 receives the redundant endowment of 4.

The coalition $\{3,4\}$ is self-enforcing in μ. If we remove $\{3,4\}$ with their assignments, we obtain a reduced economy in which 1 owns a and 2 owns nothing. In the restriction of μ to this reduced economy, 2 receives a while 1 receives nothing, which is not in the effective core in the reduced economy. So, the effective core is not consistent.