

AN EXACT METHOD FOR A PROBLEM OF TIME SLOT PRICING

OLIVIER BILENNE AND FRÉDÉRIC MEUNIER

ABSTRACT. A company provides a service at different time slots, each slot being endowed with a capacity. A non-atomic population of users is willing to purchase this service. The population is modeled as a continuous measure over the preferred times. Every user looks at the time slot that minimizes the sum of the price assigned by the company to this time slot and the distance to their preferred time. If this sum is non-negative, then the user chooses this time slot for getting the service. If this sum is positive, then the user rejects the service.

We address the problem of finding prices that ensure that the volume of users choosing each time slot is below capacity, while maximizing the revenue of the company. For the case where the distance function is convex, we propose an exact algorithm for solving this problem in time $O(n^3|P|^3)$, where P is the set of possible prices and n is the number of time slots. For the case where the prices can be any real numbers, this algorithm can also be used to find asymptotically optimal solutions in polynomial time under mild extra assumptions on the distance function and the measure modeling the population.

1. INTRODUCTION

1.1. Motivation. Consider a company offering a service to a large population of users at different time slots, each with a limited capacity. The company can assign a distinct price to each slot, aiming at simultaneously keeping the number of users served on each time slot below some threshold, and maximizing its profit. This is exactly the problem addressed in the present paper.

This work results from a collaboration with Électricité de France (EDF), the French electricity producer. In anticipation of the growing popularity of electric vehicles, EDF has explored various aspects of this topic. One of them is whether carefully designed pricing strategies could help smooth out demand at charging stations. More precisely, assuming a perfect forecast of the demand, is it possible to adjust the time slot prices so as to incentivize users to change the time of their recharge and to keep the number of users always below some threshold, while maximizing the revenue? More generally, the question of smoothing demand at charging stations belongs to the broader task of efficiently managing electricity consumption through dynamic pricing schemes.

This question actually looks relevant across diverse domains where users express preferences for receiving services at specific times, and their decisions are influenced by price and inconvenience incurred due to deviations from their preferred timing. Among these we find for instance ridesharing services, public transportation systems, streaming platforms, etc. along with all domains where pricing strategies can be used to match supply and demand while maximizing revenue.

2020 *Mathematics Subject Classification.* 90B50.

Key words and phrases. pricing, optimal transport, dynamic programming, smoothing demand.

This research benefited from the support of the FMJH Program Gaspard Monge for optimization and operations research and their interactions with data science.

In this work, we introduce a modeling approach for problems of this nature and an efficient strategy to address them. This kind of optimization problems more generally relies on revenue management for which a rich literature exists (see, e.g., [11] for an introduction to revenue management). Several studies have also examined the question of pricing time slots [1, 3, 5, 10], some of which exploring the specific context of charging stations [2, 12]. However, we are unaware of any research that addresses the problem examined in this work.

1.2. Problem formulation. A company is providing a service at different time slots and a population of users is willing to benefit from this service. Each user is characterized by a preferred time s . The preferred times are modeled by a finite absolutely continuous measure μ over \mathbb{R} (which throughout the paper is endowed with the Lebesgue measure): for a measurable subset A of \mathbb{R} , the quantity $\mu(A)$ is the volume of users whose preferred time for being served is in A . The cost incurred by a user for being served at time t while preferring time s is modeled as $d(s - t)$, where d is a strictly convex function (and thus continuous) with its unique minimum at 0.

Service availability is restricted to n distinct time slots $1, 2, \dots, n$. Each time slot j is characterized by a time of service t_j and a limited capacity ν_j : at time t_j , there is a maximum volume $\nu_j \in \mathbb{R}_+$ of users that can be served.

The company assigns each time slot j a price p_j , taken from a given set P of possible prices. The users are free to choose the time slot j over which they are served, or to choose not to be served at all. Yet, the assumption for the decision process is the following. Each user with preferred time s considers a time slot j that minimizes the total cost $d(s - t_j) + p_j$: if the minimal value of the total cost is non-positive, then the user chooses to be served at time t_j ; otherwise, the user chooses not to be served.

The objective for the company is to maximize its revenue defined as the integral of the price paid by the users getting the service. We formulate the problem as the following mathematical program:

$$\begin{aligned}
 \text{(U)} \quad & \underset{p_1, \dots, p_n}{\text{maximize}} && \sum_{j=1}^n p_j \mu(I_j(p)) \\
 & \text{subject to} && \mu(I_j(p)) \leq \nu_j \quad \forall j \in [n] \\
 & && p_j \in P \quad \forall j \in [n],
 \end{aligned}$$

where $I_j(p) = \{s \in \mathbb{R} : d(s - t_j) + p_j \leq \min(0, d(s - t_k) + p_k) \ \forall k \in [n]\}$ is formed by the users choosing time slot j for getting the service, and $\mu(I_j(p))$ forms the service load at j . We denote by $\text{Opt}(P)$ the optimal value of the revenue, with an emphasis on its dependency on the price set P .

Formally, this problem is *bilevel*: there is the optimization problem of the company (upper level), but the users, by choosing a time slot or deciding not to be served, are also solving their own elementary optimization problem (lower level). Regarding (U) as a bilevel program, it is not difficult to see that optimistic and pessimistic interpretations of the problem — which would respectively assume cooperation and non-cooperation of the users with the company — coincide, due to the strict convexity of the function d and the non-atomic nature of the measure μ . This ensures that the interiors of the sets I_1, \dots, I_n do not overlap and that problem (U) is well formulated. See, e.g., [4] for further discussions about bilevel programming.

We consider two special cases of the problem:

- the case where P is finite.
- the case where P is \mathbb{R} , μ has bounded support, and d is strongly convex.

Remark 1. The users actually solve what is called in optimal transport theory a *semi-discrete transport problem* (see, e.g., [8]). It is well-known that the dual variables of such problems provide prices that lead to an “automatic” satisfaction of the capacity constraints. However, the presence of the revenue as an objective function makes unlikely that straightforward adaptations of results from optimal transport theory could lead to optimal solutions of (U).

1.3. Contributions. Our contributions are mainly algorithmic. We assume that the following operations take constant time: computing the value of d at any point; inverting and minimizing functions of the form $s \mapsto d(s)$ or $s \mapsto d(s) - d(s - t)$; computing the value of μ on any interval. It is worth noting that the strict convexity of function d lends plausibility to the computational assumption; in particular, $s \mapsto d(s) - d(s - t)$ is increasing (see Fact 2.1 below). Under this computation assumption, we prove the following two theorems.

Theorem 1.1. *When P is finite, the problem can be solved in time $O(n^3|P|^3)$.*

Our second main result addresses the case when P is \mathbb{R} . The main message is that, under mild assumptions, we can efficiently find solutions within arbitrary optimality gaps. Suppose that μ has a bounded support $\text{supp}(\mu)$, and define

$$p^{\max} := -d(0) \quad \text{and} \quad p^{\min} := \min \left(p^{\max}, \inf_{s \in \text{supp}(\mu), j, k \in [n]} \{d(s - t_j) - d(s - t_k)\} \right).$$

Proposition 1.2. *We have $\text{Opt}(\mathbb{R}) = \text{Opt}([p^{\min}, p^{\max}])$. Moreover, for every $\delta > 0$, we have $\text{Opt}(\delta\mathbb{Z}) = \text{Opt}([p^{\min}, p^{\max} + \delta) \cap \delta\mathbb{Z})$.*

Note that by continuity of the objective and the constraints and by compactness, an immediate consequence of this proposition is that there always exists an optimal solution to problem (U) on $[p^{\min}, p^{\max}]$ when $P = \mathbb{R}$, and one on $[p^{\min}, p^{\max} + \delta) \cap \delta\mathbb{Z}$ when $P = \delta\mathbb{Z}$.

Proof of Proposition 1.2. We first deal with the case $P = \mathbb{R}$. Setting p_k to p^{\max} for every k provides a feasible solution giving 0 to the objective function, which implies $\text{Opt}(\mathbb{R}) \geq 0$. If $\text{Opt}(\mathbb{R}) = 0$, then the solution with all p_k equal to p^{\max} is optimal and we have $\text{Opt}(\mathbb{R}) = \text{Opt}([p^{\min}, p^{\max}])$. We can thus suppose that $\text{Opt}(\mathbb{R}) > 0$ and consider a feasible solution $p \in \mathbb{R}^n$ giving a positive value to the objective function. Pick a $j \in [n]$ such that $\mu(I_j(p)) > 0$ and $p_j > 0$. By the definition of j , there is a point $s \in \text{supp}(\mu)$ such that $d(s - t_j) + p_j \leq d(s - t_k) + p_k$ for every $k \in [n]$, yielding $p_k \geq d(s - t_j) - d(s - t_k) + p_j \geq p^{\min} + p_j > p^{\min}$. Thus, $p_k > p^{\min}$ for every $k \in [n]$. Since setting $p'_k := \min(p_k, p^{\max})$ provides a feasible solution p' that does not decrease the objective function, we get the equality $\text{Opt}(\mathbb{R}) = \text{Opt}([p^{\min}, p^{\max}])$.

We then deal with the case $P = \delta\mathbb{Z}$. Similarly as for the previous case, setting p_k to $q := \lceil p^{\max}/\delta \rceil \delta$ provides a feasible solution giving 0 to the objective function, which implies $\text{Opt}(\delta\mathbb{Z}) \geq 0$. If $\text{Opt}(\delta\mathbb{Z}) = 0$, then $\text{Opt}(\delta\mathbb{Z}) = \text{Opt}([p^{\min}, p^{\max} + \delta) \cap \delta\mathbb{Z})$ for the same reasons as above. We can thus suppose $\text{Opt}(\delta\mathbb{Z}) > 0$ and consider a feasible solution $p \in (\delta\mathbb{Z})^n$ giving a positive value to the objective function. Again, as above, we necessarily have $p_k > p^{\min}$ for every $k \in [n]$. Since setting $p'_k := \min(p_k, q)$ provides a feasible solution p' that does not decrease the objective function, we get the equality $\text{Opt}(\delta\mathbb{Z}) = \text{Opt}([p^{\min}, p^{\max} + \delta) \cap \delta\mathbb{Z})$. \square

Since $\delta\mathbb{Z}$ is a subset of \mathbb{R} , we have $\text{Opt}(\delta\mathbb{Z}) \leq \text{Opt}(\mathbb{R})$. Combining the above proposition with Theorem 1.1 implies that the computation of the lower bound $\text{LB}(\delta) = \text{Opt}(\delta\mathbb{Z})$ can be

done in time $O(\frac{1}{\delta^3}n^3)$, and the next theorem shows that a close upper bound can be computed with the same time complexity at the price of a small extra assumption. This upper bound only depends on the support of μ , the lower and upper bounds on μ , the function d , and the times t_1, \dots, t_n , and is independent of the values given to the measures μ and ν .

Theorem 1.3. *Assume that d is strongly convex and the density of μ is interval supported and lower- and upper-bounded above zero. Then an upper bound $\text{UB}(\delta)$ on $\text{Opt}(\mathbb{R})$ can be computed in time $O(\frac{1}{\delta^3}n^3)$, such that $\text{UB}(\delta) - \text{LB}(\delta) = O(\delta^{1/8})$.*

Acknowledgments. We are grateful to our colleagues at EDF R&D, Guilhem Dupuis and Cheng Wan, for many stimulating discussions that played a key role in identifying the problem and clarifying its industrial relevance. This research benefited from the support of the FMJH Program Gaspard Monge for optimization and operations research and their interactions with data science.

2. PRELIMINARY RESULTS

We start with a direct consequence of the strict convexity of d , which we state as an observation for future references.

Fact 2.1. *The map $s \mapsto d(s - t_j) - d(s - t_k)$ is increasing for $j < k$.*

It is convenient in the problem (U) to rewrite the sets $I_j(p)$ as

$$(1) \quad I_j(p) = J_j(p) \cap \mathcal{L}_j(p_j) \quad \forall p \in P^n, j \in [n],$$

where $J_j(p) = \{s \in \mathbb{R} : d(s - t_j) + p_j \leq d(s - t_k) + p_k \ \forall k \in [n]\}$, and $\mathcal{L}_j(q) = \{s \in \mathbb{R} : d(s - t_j) + q \leq 0\}$ denotes the sublevel set of $s \mapsto d(s - t_j)$. The upcoming result is well known in the context of optimal transport; see, for instance, [8, Theorem 2.9] which, together with [8, Theorem 2.5], characterizes the solutions of a convex optimal transport problem.

Lemma 2.2 (Monotonicity). *Let $p \in P^n$ and $s, s' \in \mathbb{R}$. If $s < s'$, then $k \leq k'$ for all $k, k' \in [n]$ such that $s \in J_k(p)$ and $s' \in J_{k'}(p)$.*

Proof. Let $k, k' \in [n]$, $s \in J_k(p)$, $s' \in J_{k'}(p)$, and assume that $s < s'$. By definition of $J_k(p)$ and $J_{k'}(p)$, we have $d(s - t_k) + p_k \leq d(s - t_{k'}) + p_{k'}$ and $d(s' - t_{k'}) + p_{k'} \leq d(s' - t_k) + p_k$. Adding up the previous two inequalities yields $d(s - t_k) + d(s' - t_{k'}) \leq d(s - t_{k'}) + d(s' - t_k)$, which rewrites as $d(s - t_k) - d(s - t_{k'}) \leq d(s' - t_k) - d(s' - t_{k'})$. It follows from Fact 2.1 that $k \leq k'$. \square

Proposition 2.3 shows that the sets $I_j(p)$, $J_j(p)$, $\mathcal{L}_j(q)$ follow a particular arrangement. Its proof relies on Lemma 2.2. Notice that any of these sets can be empty. See Fig. 1 for an illustration of Proposition 2.3 and its relation with the service loads at the time slots.

Proposition 2.3. *The sets $I_j(p)$, $J_j(p)$, $\mathcal{L}_j(q)$ are closed intervals for every $j \in [n]$ and every $p \in P^n$ and $q \in P$. Moreover, for every $p \in P^n$, the sets $J_1(p), \dots, J_n(p)$ cover \mathbb{R} , have pairwise disjoint interiors, and are ordered by increasing index j .*

Proof. The sets $I_j(p)$, $J_j(p)$, $\mathcal{L}_j(q)$ are closed as they contain all their limit points. For each $k \in [n]$, the set $\{s \in \mathbb{R} : d(s - t_j) + p_j \leq d(s - t_k) + p_k\}$ is an interval by the monotonicity of $s \mapsto d(s - t_j) - d(s - t_k)$ (Fact 2.1). Hence, $J_j(p)$ is an interval as the intersection of all

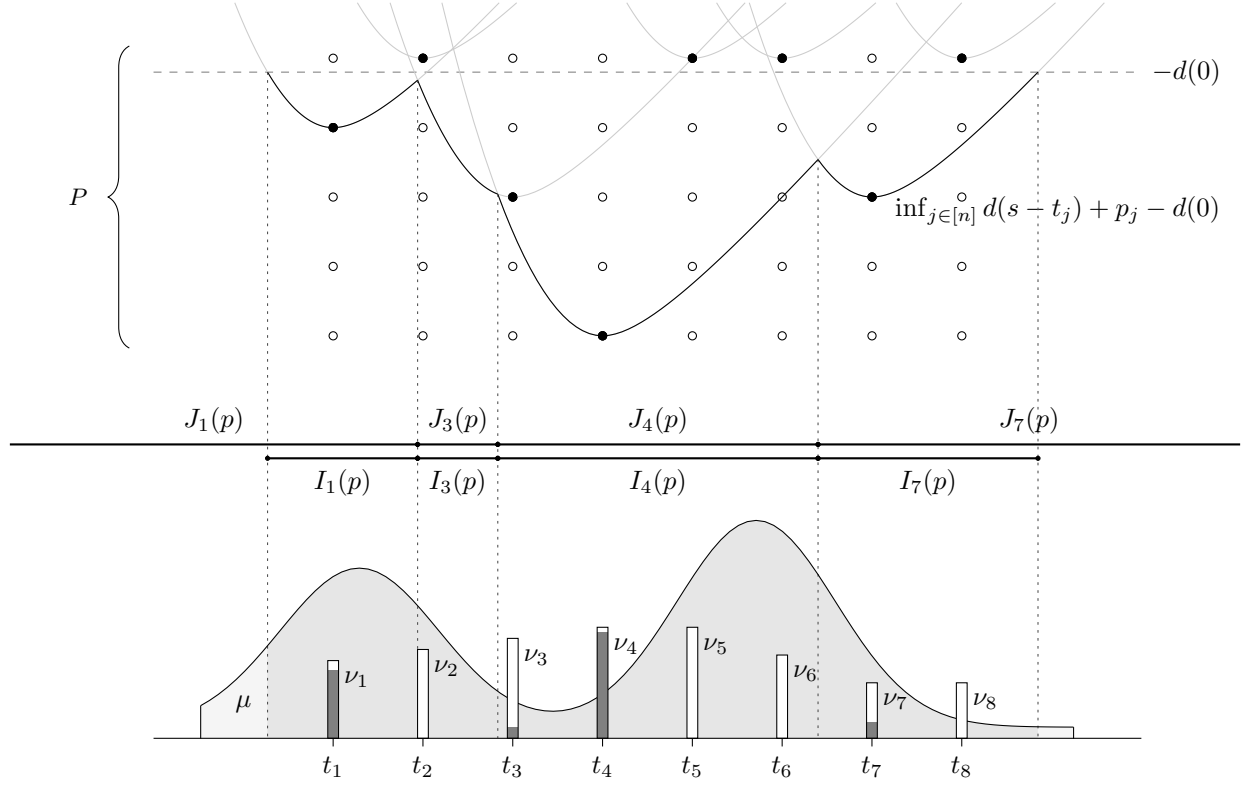


FIGURE 1. The black dots form an example of a feasible price profile when $n = 8$ and P has five elements. The lower envelope of the functions $d(s - t_j) + p_j$ determines the intervals $J_j(p)$, which are ordered as in Proposition 2.3. Its nonpositivity further delimits the intervals $I_j(p)$. All time slots with prices above the threshold $-d(0)$ are therefore avoided. The solution is feasible as the service loads $\mu(I_j(p))$ are less than the capacities ν_j .

these sets for k ranging over $[n]$. The set $\mathcal{L}_j(p)$ is an interval as the sublevel set of a convex function. It follows from (1) that $I_j(p)$ is an interval, as intersection of two intervals.

Further, $s \in \mathbb{R}$ belongs to $J_j(p)$ whenever j minimizes $d(s - t_j) + p_j$ over $[n]$. Since every finite set has minimal elements, at least one such j exists, and therefore the sets $J_1(p), \dots, J_n(p)$ cover \mathbb{R} . The last two claims are consequences of Lemma 2.2. \square

3. ALGORITHM WHEN P IS FINITE

The purpose of this section is to prove Theorem 1.1 and to explicitly describe the algorithm guaranteed by this theorem. This is done in a first subsection, assuming two lemmas, which are eventually proved in a second subsection.

3.1. Description of the algorithm and proof of Theorem 1.1. The algorithm for solving the problem when P is finite uses a directed graph. To ease the description of this graph, we will say that a time slot ℓ is *covered* by a pair $(i, q) \in [n] \times P$ at $s \in \mathbb{R}$ if

$$d(s - t_i) + q \leq d(s - t_\ell) + \max P.$$

By extension, we consider that $\ell \in [n]$ is covered by (i, q) at $-\infty$ if $\lim_{s' \rightarrow -\infty} d(s' - t_i) - d(s' - t_\ell) \leq \max P - q$, and at $+\infty$ if $\lim_{s' \rightarrow +\infty} d(s' - t_i) - d(s' - t_\ell) \leq \max P - q$. These limits are well defined by Fact 2.1. From Fact 2.1 we also know that, for $i, j \in [n]$ with $i < j$ and $q \in P$, the equation $d(s - t_i) - d(s - t_j) = q$ has at most one solution. We write this solution $s_{ij}(q)$, where s_{ij} is the inverse of $s \mapsto d(s - t_i) - d(s - t_j)$, with the convention $s_{ij}(q) = -\infty$ when $d(s - t_i) - d(s - t_j) > q$ for all $s \in \mathbb{R}$, and $s_{ij}(q) = +\infty$ when $d(s - t_i) - d(s - t_j) < q$ for all $s \in \mathbb{R}$. Let $D = (V, A)$ be the directed digraph with $V := ([n] \times P) \cup \{(0, \cdot), (n+1, \cdot)\}$ and with $A := A_1 \cup A_2 \cup A_3$, where

$$A_1 := \{((0, \cdot), (j, q)) : \text{every } \ell \text{ in } \{1, \dots, j\} \text{ is covered by } (j, q) \text{ at } -\infty\}$$

$$A_2 := \{((i, r), (j, q)) : i < j, s_{ij}(q - r) \neq \pm\infty \text{ and every } \ell \text{ in } \{i, \dots, j\} \text{ is covered by } (i, r) \text{ and by } (j, q) \text{ at } s_{ij}(q - r)\}$$

$$A_3 := \{((j, q), (n+1, \cdot)) : \text{every } \ell \text{ in } \{j, \dots, n\} \text{ is covered by } (j, q) \text{ at } +\infty\}.$$

In D , the vertices $(0, \cdot)$ and $(n+1, \cdot)$ are respectively seen as the source and the sink. To each pair of consecutive arcs a, a' , we assign a *reward* $w(a, a')$ as follows.

- When $a = ((0, \cdot), (j, q)) \in A_1$ and $a' = ((j, q), (k, r')) \in A_2$. Then set

$$w(a, a') := \begin{cases} qv(a, a') & \text{if } v(a, a') \leq \nu_j, \\ -\infty & \text{otherwise,} \end{cases}$$

where $v(a, a') := \mu(\{s \in (-\infty, s_{jk}(r' - q)] : d(s - t_j) + q \leq 0\})$.

- When $a = ((i, r), (j, q)) \in A_2$ and $a' = ((j, q), (k, r')) \in A_2$. Then set

$$w(a, a') := \begin{cases} qv(a, a') & \text{if } s_{ij}(q - r) \leq s_{jk}(r' - q) \text{ and } v(a, a') \leq \nu_j, \\ -\infty & \text{otherwise,} \end{cases}$$

where $v(a, a') = \mu(\{s \in [s_{ij}(q - r), s_{jk}(r' - q)] : d(s - t_j) + q \leq 0\})$.

- When $a = ((i, r), (j, q)) \in A_2$ and $a' = ((j, q), (n+1, \cdot)) \in A_3$. Then set

$$w(a, a') := \begin{cases} qv(a, a') & \text{if } v(a, a') \leq \nu_j, \\ -\infty & \text{otherwise,} \end{cases}$$

where $v(a, a') := \mu(\{s \in [s_{ij}(q - r), +\infty) : d(s - t_j) + q \leq 0\})$.

- When $a = ((0, \cdot), (j, q)) \in A_1$ and $a' = ((j, q), (n+1, \cdot)) \in A_3$. Then set

$$w(a, a') := \begin{cases} qv(a, a') & \text{if } v(a, a') \leq \nu_j, \\ -\infty & \text{otherwise,} \end{cases}$$

where $v(a, a') := \mu(\{s \in \mathbb{R} : d(s - t_j) + q \leq 0\})$.

We consider this reward to be attached to the corresponding arc in the line digraph of D , denoted by $L(D)$. (We remind the reader that the *line digraph* has A as its vertex set and an arc (a, a') for each pair of consecutive arcs a, a' in D .) As usual, the reward $w(C)$ of a path C in $L(D)$ is defined as the sum of the rewards on its arcs. We are thus interested in paths in $L(D)$ whose induced paths in D go from the source to the sink. To lighten the text, we call source-to-sink path in $L(D)$ the line path of any source-to-sink path in D .

The idea of the construction is to model the price profiles as source-to-sink paths in D . Roughly speaking, the vertices visited by such a path indicate which time slots should be

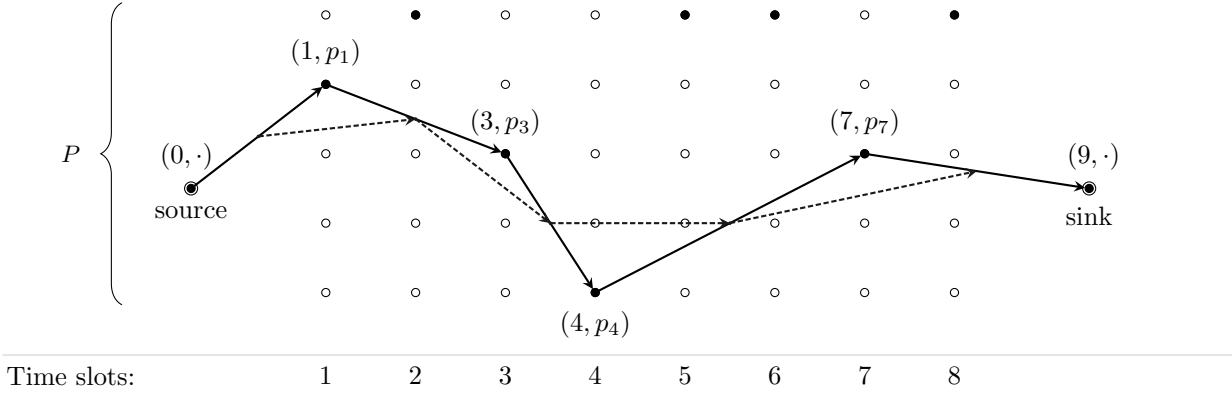


FIGURE 2. Example of a source-to-sink path in D with its associated path in $L(D)$ (dashed lines) when $n = 8$ and P has five elements. The corresponding price profile derived in Lemma 3.2 is given by the sequence of black dots.

considered for selection by the users, and which prices may be attached to this time slots. The conditions in the definition of A_1 , A_2 , and A_3 ensure that it is always possible to choose prices for unused time slots that keep them unattractive to all users. The revenue generated by a vertex of such a path depends on the vertices just before and just after on the path, and this is the reason why it is more convenient to consider $L(D)$. See Fig. 2 for an illustration of a source-to-sink path in D and its corresponding line path in $L(D)$.

For the next statements, we denote by $R(p)$ the revenue $\sum_{j=1}^n \mu(I_j(p))$ when p is a feasible solution of (U). We set $R(p) := -\infty$ when p is not a feasible solution of (U).

Lemma 3.1. *Given a price profile $p \in P^n$, there exists a source-to-sink path C in $L(D)$ with $R(p) \leq w(C)$.*

Lemma 3.2. *Given a source-to-sink path C in $L(D)$, there exists a price profile $p \in P^n$ with $R(p) \geq w(C)$. Such a profile can be built in linear time.*

Lemmas 3.1 and 3.2 show that computing an optimal price profile boils down to computing a source-to-sink path in $L(D)$ of maximal reward. Since D is acyclic, $L(D)$ is acyclic as well, and such a computation can be done in linear time in the number of arcs of $L(D)$ (see, e.g., [6],[9, Theorem 8.16]), i.e., in $O(|A||V|) = O(n^3|P|^3)$. The construction of D itself can be done in the same time complexity. This suffices to prove Theorem 1.1. Observe that the price profile of Fig. 1 and the source-to-sink path of Fig. 2 are related through Lemmas 3.1 and 3.2.

3.2. Proofs of the lemmas.

Proof of Lemma 3.1. Let $p = (p_1, \dots, p_n) \in P^n$. If $\mu(I_j(p)) > \nu_j$ for at least one j , then we have $R(p) = -\infty$, and in particular $R(p) \leq w(C)$ for any source-to-sink path C . So we assume that $\mu(I_j(p)) \leq \nu_j$ for all j . Consider the m time slots j_1, \dots, j_m such that $j_1 < j_2 < \dots < j_m$ and $J_{j_i}(p)$ is an interval of non-zero length for every i , where $1 \leq m \leq n$. The sequence of vertices $(0, \cdot), (j_1, p_{j_1}), \dots, (j_m, p_{j_m}), (n+1, \cdot)$ induces a source-to-sink path in D by Proposition 2.3: the endpoints of the intervals $J_{j_1}(p), \dots, J_{j_m}(p)$ are precisely the points $-\infty, s_{j_1 j_2}(p_{j_2} - p_{j_1}), \dots, s_{j_{m-1} j_m}(p_{j_m} - p_{j_{m-1}}), +\infty$. Set $(j_0, p_0) = (0, \cdot)$ and $(j_{m+1}, p_{n+1}) = (n+1, \cdot)$, and consider two consecutive arcs $a_i = ((j_{i-1}, p_{j_{i-1}}), (j_i, p_{j_i}))$

and $a_{i+1} = ((j_i, p_{j_i}), (j_{i+1}, p_{j_{i+1}}))$ of this path. By construction, $v(a_i, a_{i+1}) = \mu(I_{j_i}(p))$ is the service load at j_i and $w(a_i, a_{i+1}) = p_{j_i} \mu(I_{j_i}(p))$. The path induces in turn a source-to-sink path C in $L(D)$ such that $R(p) = w(C)$. \square

Proof of Lemma 3.2. Let C be a source-to-sink path in $L(D)$ with m arcs denoted by $(a_0, a_1), \dots, (a_{m-1}, a_m)$, where $1 \leq m \leq n$. Let $(j_1, p_{j_1}), \dots, (j_m, p_{j_m})$ be the internal vertices of the path induced by C in D , and set $U = \{j_1, \dots, j_m\}$. For each $j \notin U$, set $p_j := \max P$. This defines a price profile $p = (p_1, \dots, p_n)$. If $w(a, a') = -\infty$ for an arc (a, a') of C , then we have $w(C) = -\infty$, and in particular $R(p) \geq w(C)$. So we assume that $w(a, a') > -\infty$ for every arc (a, a') of C .

First assume that $m > 1$. Set $s_1 := s_{j_1 j_2}(p_{j_2} - p_{j_1}), \dots, s_{m-1} := s_{j_{m-1} j_m}(p_{j_m} - p_{j_{m-1}})$. Since for $i \in [m-1]$ it holds that $((j_i, p_{j_i}), ((j_{i+1}, p_{j_{i+1}}))) \in A_2$ and $w((j_i, p_{j_i}), ((j_{i+1}, p_{j_{i+1}}))) > -\infty$, we have $-\infty < s_1 \leq \dots \leq s_{m-1} < +\infty$. We claim that these $m-1$ points are the endpoints of $J_{j_1}(p), \dots, J_{j_m}(p)$. To prove this, it is enough to show that the inequality

$$(2) \quad d(s - t_{j_i}) + p_{j_i} \leq d(s - t_\ell) + p_\ell$$

holds for $j_i \leq \ell \leq n$ when $i \in [m-1]$ and $s \leq s_i$, and when $i = m$ and $s \in \mathbb{R}$; and the inequality $d(s - t_{j_i}) + p_{j_i} \leq d(s - t_\ell) + p_\ell$ holds for $1 \leq \ell \leq j_i$ when $i = 1$ and $s \in \mathbb{R}$, and when $i-1 \in [m-1]$ and $s \geq s_{i-1}$. Indeed, combining these inequalities yields $J_{j_1}(p) \supseteq (-\infty, s_1], J_{j_2}(p) \supseteq [s_1, s_2], \dots, J_{j_{m-1}}(p) \supseteq [s_{m-2}, s_{m-1}], J_{j_m}(p) \supseteq [s_{m-1}, +\infty)$. The claim then follows from Proposition 2.3. Suppose now that $m = 1$. We claim that $J_{j_1}(p) = \mathbb{R}$. To prove this, it is enough to show that the same two inequalities hold for $i = 1 = m$ and $s \in \mathbb{R}$. To establish the two claims, we only verify the inequality (2); the second one can be checked similarly. We do this by considering $i \in [m]$, $j_i \leq \ell \leq n$, and $s \in \mathbb{R}$ in the following three cases:

- *When $i = m$ and $s \in \mathbb{R}$.* Then, we have $\ell \geq j_m$, which implies $\ell \notin U$ and $p_\ell = \max P$. Since C is a source-to-sink path in $L(D)$, we infer from the definition of the arcs of D that ℓ is covered by (j_m, p_{j_m}) at $+\infty$. It follows from Fact 2.1 and $j_m \leq \ell$ that $d(s - t_{j_m}) - d(s - t_\ell) \leq d(s' - t_{j_m}) - d(s' - t_\ell)$ holds for all $s' \geq s$. Equivalently, we have $d(s - t_{j_m}) + p_{j_m} \leq d(s - t_\ell) + p_{j_m} + d(s' - t_{j_m}) - d(s' - t_\ell)$ for all $s' \geq s$. Using the fact that ℓ is covered and letting $s' \rightarrow +\infty$ yields $d(s - t_{j_m}) + p_{j_m} \leq d(s - t_\ell) + p_{j_m} + \max P - p_{j_m} = d(s - t_\ell) + p_\ell$. Hence, the inequality (2) holds.
- *When $1 \leq i < m$, $\ell \in U$, and $s \leq s_i$.* Then, we show (2) by induction on k . For $k = i$, the inequality (2) is immediate. Now, suppose it holds for $k \in \{i, \dots, m-1\}$. Using successively Fact 2.1 with $s \leq s_i \leq s_k$ and the definition of $s_{j_k j_{k+1}}(p_{j_{k+1}} - p_{j_k})$, we find $d(s - t_{j_k}) - d(s - t_{j_{k+1}}) \leq d(s_k - t_{j_k}) - d(s_k - t_{j_{k+1}}) = p_{j_{k+1}} - p_{j_k}$. Then, combining the induction hypothesis with the last inequality, we find $d(s - t_{j_i}) + p_{j_i} \leq d(s - t_{j_k}) + p_{j_k} \leq d(s - t_{j_{k+1}}) + p_{j_{k+1}}$. Hence, the inequality (2) holds.
- *When $1 \leq i < m$, $\ell \notin U$, and $s \leq s_i$.* Then, we have $p_\ell = \max P$. Since C is a source-to-sink path in $L(D)$, we infer from the definition of the arcs of D that there is $k \in \{i, \dots, m\}$ such that $j_k \leq \ell$ and ℓ is covered either by (j_k, p_{j_k}) at s_k when $i \leq k < m$, or by (j_m, p_{j_m}) at $+\infty$ when $k = m$. In the first case, it follows from Fact 2.1, $j_k \leq \ell$, and $s \leq s_k$ that $d(s - t_{j_k}) - d(s - t_\ell) \leq d(s_k - t_{j_k}) - d(s_k - t_\ell)$. Applying successively the inequality (2) at j_k , the fact that ℓ is covered, and the last inequality, we find $d(s - t_{j_i}) + p_{j_i} \leq d(s - t_{j_k}) + p_{j_k} \leq d(s_k - t_\ell) + \max P + d(s - t_{j_k}) - d(s_k - t_{j_k}) \leq d(s - t_\ell) + p_\ell$. In the second case, applying

the inequality (2) successively at $s \leq s_i$ for $j_i \leq j_m$ and then at $s \in \mathbb{R}$ for $j_m \leq \ell$, we find $d(s - t_{j_i}) + p_{j_i} \leq d(s - t_{j_m}) + p_{j_m} \leq d(s - t_\ell) + p_\ell$. Hence, the inequality (2) holds.

Thus, we have shown

$$J_{j_1}(p) = (-\infty, s_1], \quad J_{j_2}(p) = [s_1, s_2], \quad \dots, \quad J_{j_{m-1}}(p) = [s_{m-2}, s_{m-1}], \quad J_{j_m}(p) = [s_{m-1}, +\infty).$$

This implies that, for $i = 1, \dots, m$, we have $v(a_{i-1}, a_i) = \mu(\{s \in J_{j_i}(p) : d(s - t_{j_i}) + p_{j_i} \leq 0\}) = \mu(I_{j_i}(p))$, and therefore $\mu(I_{j_i}(p)) \leq \nu_{j_i}$ and $p_{j_i} \mu(I_{j_i}(p)) = w(a_{i-1}, a_i)$. Since $J_{j_1}(p), \dots, J_{j_m}(p)$ cover \mathbb{R} , for every ℓ that is not realized as j_i we have $\mu(I_\ell(p)) = 0$. Hence $R(p) = \sum_{\ell=1}^n p_\ell \mu(I_\ell(p)) = w(C)$. \square

4. ALGORITHM WHEN P IS \mathbb{R}

This section is devoted to the proof of Theorem 1.3, which deals with the case $P = \mathbb{R}$. Throughout Section 4 we assume that d is strongly convex with parameter M and that the density of μ is interval supported and lower- and upper-bounded above 0. We respectively denote the corresponding bounds on the density by $\mu > 0$ and $\bar{\mu}$. We also define the constant $L = 2 \max \{ [M(t_{i+1} - t_i)]^{-1} : i \in [n-1] \}$, which plays a role in our analysis. Lastly, we adopt the following notation: for any $q \in \mathbb{R}$, we write $q^\delta = \lfloor q/\delta \rfloor \delta$; for any $p \in \mathbb{R}^n$, we write $p^\delta = (p_1^\delta, \dots, p_n^\delta)$.

4.1. Proof of Theorem 1.3. Given $\delta > 0$, consider the following ‘‘approximate’’ version of (U):

$$(U^\delta) \quad \begin{array}{ll} \underset{p_1, \dots, p_n}{\text{maximize}} & \sum_{j=1}^n (p_j^\delta + \delta) \mu(I_j(p^\delta)) \\ \text{subject to} & \mu(I_j(p^\delta)) \leq \nu_j + \bar{\mu} L \delta + 2\bar{\mu} \sqrt{2\delta/M} \quad \forall j \in [n] \\ & p_j \in \mathbb{R} \quad \forall j \in [n]. \end{array}$$

Denote by $\text{Opt}^\delta(\mathbb{R})$ its optimal value.

We first state a counterpart to Proposition 1.2 for the modified problem (U^δ) .

Proposition 4.1. *For every $\delta > 0$, we have $\text{Opt}^\delta(\mathbb{R}) + n\bar{\mu}L\delta p^{\max} \geq \text{Opt}(\mathbb{R})$. Moreover, (U^δ) admits an optimal solution in $([p^{\min} - \delta, p^{\max} + \delta] \cap \delta\mathbb{Z})^n$.*

Our proof for Theorem 1.3 then rests on the following result.

Proposition 4.2. *We have $\text{Opt}^\delta(\mathbb{R}) + n\bar{\mu}L\delta p^{\max} - \text{Opt}(\delta\mathbb{Z}) = O(\delta^{1/8})$.*

With this proposition, and the algorithm of Section 3, the proof of Theorem 1.3 is somehow routine.

Proof of Theorem 1.3. Proposition 4.1 ensures that $\text{UB}(\delta) = \text{Opt}^\delta(\mathbb{R}) + n\bar{\mu}L\delta p^{\max}$ is an upper bound on $\text{Opt}(\mathbb{R})$. On the other hand, Proposition 4.2 ensures that $\text{Opt}^\delta(\mathbb{R}) + n\bar{\mu}L\delta p^{\max} - \text{Opt}(\delta\mathbb{Z}) = O(\delta^{1/8})$. Finally, the algorithm of Section 3.1 can be adapted in a straightforward way to solve (U^δ) in $O(n^3|P|^3)$. \square

4.2. Proofs of Propositions 4.1 and 4.2.

4.2.1. *Preliminaries.* We start by establishing two basic lemmas about strongly convex functions. Recall that a function f is strongly convex with parameter α if

$$(3) \quad f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y) - \frac{\alpha}{2}\theta(1 - \theta)(x - y)^2$$

holds for all $x, y \in \mathbb{R}$ and $\theta \in [0, 1]$; see [7].

Lemma 4.3. *Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a strongly convex function with parameter α and reaching its global minimum at 0. Then $f(y) - f(x) \geq \frac{1}{2}\alpha(y - x)^2$ for all $0 \leq x \leq y$.*

Proof. For $\theta \in [0, 1)$, we have by (3)

$$f(y) - f(x) \geq \frac{1}{1 - \theta} (f(\theta x + (1 - \theta)y) - f(x)) + \frac{\alpha}{2}\theta(x - y)^2 \geq \frac{\alpha}{2}\theta(y - x)^2,$$

where we used $\theta x + (1 - \theta)y \geq x$ and the fact that f does not decrease on nonnegative numbers. We get the desired inequality by making θ go to 1. \square

Lemma 4.4. *Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a strongly convex function with parameter α . Then for all $x \leq y$ and $h > 0$, we have $f(y + h) - f(y) \geq f(x + h) - f(x) + \alpha h(y - x)$.*

Proof. Set $\theta = h/(y + h - x)$, and observe that $x + h = \theta(y + h) + (1 - \theta)x$. Applying (3) at $x + h$, we find $f(x + h) \leq \theta f(y + h) + (1 - \theta)f(x) - \frac{\alpha}{2}h(y - x)$, where we have used $\theta(1 - \theta)(y + h - x)^2 = h(y - x)$. Similarly, $y = \theta x + (1 - \theta)(y + h)$ yields $f(y) \leq \theta f(x) + (1 - \theta)f(y + h) - \frac{\alpha}{2}h(y - x)$. The desired inequality follows by adding the last two results and rearranging the terms. \square

Our proofs for Propositions 4.1 and 4.2 rest on a series of ancillary results. We emphasize that the sets considered in the sequel will be measurable thanks to Proposition 2.3. In the upcoming sections, we use $|X|$ to denote the Lebesgue measure of a measurable set X , and $A \triangle B = (A \setminus B) \cup (B \setminus A)$ to denote the symmetric difference between two sets A and B .

4.2.2. *Proof of Proposition 4.1.* The next result relies on Lemma 4.4. We recall that $I_j(p^\delta) = J_j(p^\delta) \cap \mathcal{L}_j(p_j^\delta)$, where J_j and \mathcal{L}_j are defined in Section 2.

Lemma 4.5. *For all $\eta, \eta' \geq 0$, all $p, p' \in \mathbb{R}^n$ such that $p_k - \eta' \leq p'_k \leq p_k + \eta$ for $k \in [n]$, and all $j \in [n]$, we have $|J_j(p') \setminus J_j(p)| \leq (\eta + \eta')L$ and $|J_j(p) \setminus J_j(p')| \leq (\eta + \eta')L$.*

Proof. Since $p_k - \eta' \leq p'_k \leq p_k + \eta$ rewrites as $p'_k - \eta \leq p_k \leq p'_k + \eta'$, we only show the first inequality. The second inequality can be shown similarly by exchanging the roles of p and p' .

We focus on $J_j(p') \setminus J_j(p)$, which we rewrite as $E \cup F$, where

$$\begin{aligned} E &= \{s \in J_j(p') : \exists i < j, d(s - t_i) + p_i < d(s - t_j) + p_j\}, \\ F &= \{s \in J_j(p') : \exists k > j, d(s - t_k) + p_k < d(s - t_j) + p_j\}. \end{aligned}$$

Let $s \in E$ and $\xi \geq \eta + \eta'$. By successively using the assumption on p and p' , the definition of E , and Lemma 4.4 (with $x = s - \frac{1}{2}L\xi - t_j$, $y = s - t_j$, and $h = t_j - t_i$), we find $i < j$ such that $p'_j - p'_i \geq p_j - p_i - \eta - \eta' > d(s - t_i) - d(s - t_j) - \eta - \eta' \geq d(s - \frac{1}{2}L\xi - t_i) - d(s - \frac{1}{2}L\xi - t_j) + \frac{1}{2}LM(t_j - t_i)\xi - \eta - \eta' \geq d(s - \frac{1}{2}L\xi - t_i) - d(s - \frac{1}{2}L\xi - t_j)$, which implies $s - \frac{1}{2}L\xi \notin J_j(p')$ and therefore $s - \frac{1}{2}L\xi \notin E$. Since this holds for all $\xi \geq \eta + \eta'$, we have $|E| \leq \frac{1}{2}L(\eta + \eta')$. Similarly, let $s \in F$ and $\xi \geq \eta + \eta'$. By successively using Lemma 4.4 (with $x = s - t_k$, $y = s + \frac{1}{2}L\xi - t_k$, and $h = t_k - t_j$), the definition of F , and the assumption on p and p' , we find $k > j$ such that $d(s + \frac{1}{2}L\xi - t_j) - d(s + \frac{1}{2}L\xi - t_k) \geq d(s - t_j) - d(s - t_k) + \frac{1}{2}LM(t_k - t_j)\xi > p_k - p_j + \eta + \eta' \geq p'_k - p'_j$,

which implies $s + \frac{1}{2}L\xi \notin J_j(p')$ and therefore $s + \frac{1}{2}L\xi \notin F$. Since this holds for all $\xi \geq \eta + \eta'$, we have $|F| \leq \frac{1}{2}L(\eta + \eta')$. Thus, $|J_j(p') \setminus J_j(p)| \leq |E| + |F| \leq L(\eta + \eta')$. \square

Next, we show that the length of the sublevel set \mathcal{L}_j is Hölder continuous with exponent $1/2$.

Lemma 4.6. *For all $q, q' \in \mathbb{R}$ and all $j \in [n]$, we have $|\mathcal{L}_j(q) \triangle \mathcal{L}_j(q')| \leq 2\sqrt{2|q' - q|/M}$.*

Proof. Without loss of generality, we assume $q \leq q'$. We have $\mathcal{L}_j(q) \supseteq \mathcal{L}_j(q')$ and it follows from Proposition 2.3 that $|\mathcal{L}_j(q) \triangle \mathcal{L}_j(q')| = |\mathcal{L}_j(q) \setminus \mathcal{L}_j(q')| = |\mathcal{L}_j(q)| - |\mathcal{L}_j(q')|$. It remains to show that $|\mathcal{L}_j(q)| - |\mathcal{L}_j(q')| \leq 2\sqrt{2(q' - q)/M}$.

We first assume that $q \leq q' \leq -\min d$, so that $-q$ and $-q'$ belong to the codomain of d . Since $s \mapsto d(s - t_j) + q$ is strongly convex, its sublevel sets are compact sets and therefore finite intervals of \mathbb{R} . Hence, we can find $z \leq z' \leq t_j \leq s' \leq s$ such that $d(z - t_j) = d(s - t_j) = -q$ and $d(z' - t_j) = d(s' - t_j) = -q'$. Using Lemma 4.3 with $f = d$, $x = s' - t_j$, and $y = s - t_j$, we find $q' - q = d(s - t_j) - d(s' - t_j) \geq \frac{1}{2}M(s - s')^2$, which implies $s - s' \leq \sqrt{2(q' - q)/M}$. Using Lemma 4.3 with $f: s \mapsto d(-s)$, $x = t_j - z'$, and $y = t_j - z$, we find $q' - q = d(z - t_j) - d(z' - t_j) \geq \frac{1}{2}M(z' - z)^2$, which implies $z' - z \leq \sqrt{2(q' - q)/M}$. Combining the last two results gives $|\mathcal{L}_j(q)| - |\mathcal{L}_j(q')| = (s - z) - (s' - z') \leq 2\sqrt{2(q' - q)/M}$.

In the general case, consider $\tilde{q} = \min(q, -\min d)$ and $\tilde{q}' = \min(q', -\min d)$. Note that $\tilde{q} \leq \tilde{q}'$. By construction, we have $|\mathcal{L}_j(q)| = |\mathcal{L}_j(\tilde{q})|$ and $|\mathcal{L}_j(q')| = |\mathcal{L}_j(\tilde{q}')|$. Since the projection $x \mapsto \min(x, -\min d)$ is non-expansive, we find $\tilde{q}' - \tilde{q} \leq q' - q$. Applying the previous result to $\tilde{q} \leq \tilde{q}' \leq -\min d$ yields $|\mathcal{L}_j(q)| - |\mathcal{L}_j(q')| = |\mathcal{L}_j(\tilde{q})| - |\mathcal{L}_j(\tilde{q}')| \leq 2\sqrt{2(\tilde{q}' - \tilde{q})/M} \leq 2\sqrt{2(q' - q)/M}$. \square

The next lemma is a consequence of Lemmas 4.5 and 4.6. It tells us to what extent a feasible price profile of the modified problem (U^δ) may violate the capacity constraints in the original problem (U) .

Lemma 4.7. *Let $\delta > 0$. For all $p \in \mathbb{R}^n$ and $j \in [n]$, we have*

$$\mu(I_j(p)) - \bar{\mu}L\delta \leq \mu(I_j(p^\delta)) \leq \mu(I_j(p)) + \bar{\mu}L\delta + 2\bar{\mu}\sqrt{2\delta/M}.$$

Proof. Applying Lemma 4.5 to p and p^δ with parameters $\eta = 0$ and $\eta' = \delta$, we find $|J_j(p) \setminus J_j(p^\delta)| \leq L\delta$ and $|J_j(p^\delta) \setminus J_j(p)| \leq L\delta$, which imply $\mu(J_j(p) \setminus J_j(p^\delta)) \leq \bar{\mu}L\delta$ and $\mu(J_j(p^\delta) \setminus J_j(p)) \leq \bar{\mu}L\delta$.

Now, it follows from $p_j^\delta \leq p_j$ and the definition of \mathcal{L}_j that $\mathcal{L}_j(p_j) \subseteq \mathcal{L}_j(p_j^\delta)$. Using this inclusion and the identity $A \cap B \subseteq (C \cap B) \cup (A \setminus C)$, we find

$$\begin{aligned} I_j(p) &\stackrel{(1)}{=} J_j(p) \cap \mathcal{L}_j(p_j) \subseteq J_j(p) \cap \mathcal{L}_j(p_j^\delta) \subseteq (J_j(p^\delta) \cap \mathcal{L}_j(p_j^\delta)) \cup (J_j(p) \setminus J_j(p^\delta)) \\ &= I_j(p^\delta) \cup (J_j(p) \setminus J_j(p^\delta)). \end{aligned}$$

It follows from $\mu(J_j(p) \setminus J_j(p^\delta)) \leq \bar{\mu}L\delta$ that $\mu(I_j(p)) \leq \mu(I_j(p^\delta)) + \mu(J_j(p) \setminus J_j(p^\delta)) \leq \mu(I_j(p^\delta)) + \bar{\mu}L\delta$, which is the first inequality. Similarly, using the same identity twice, we find

$$\begin{aligned} I_j(p^\delta) &\stackrel{(1)}{=} J_j(p^\delta) \cap \mathcal{L}_j(p_j^\delta) \subseteq (J_j(p) \cap \mathcal{L}_j(p_j^\delta)) \cup (J_j(p^\delta) \setminus J_j(p)) \\ &= (\mathcal{L}_j(p_j^\delta) \cap J_j(p)) \cup (J_j(p^\delta) \setminus J_j(p)) \\ &\subseteq (\mathcal{L}_j(p_j) \cap J_j(p)) \cup (\mathcal{L}_j(p_j^\delta) \setminus \mathcal{L}_j(p_j)) \cup (J_j(p^\delta) \setminus J_j(p)) \\ &= I_j(p) \cup (\mathcal{L}_j(p_j^\delta) \setminus \mathcal{L}_j(p_j)) \cup (J_j(p^\delta) \setminus J_j(p)). \end{aligned}$$

It follows from $\mu(J_j(p^\delta) \setminus J_j(p)) \leq \bar{\mu}L\delta$ and Lemma 4.6 that $\mu(I_j(p^\delta)) \leq \mu(I_j(p)) + \mu(J_j(p^\delta) \setminus J_j(p)) + \mu(\mathcal{L}_j(p_j^\delta) \setminus \mathcal{L}_j(p_j)) \leq \mu(I_j(p)) + \bar{\mu}L\delta + 2\bar{\mu}\sqrt{2\delta/M}$, which is the second inequality. \square

The $\bar{\mu}L\delta$ terms in Lemma 4.7 are due to deviations of the preference intervals in the modified problem (U^δ) after discretization of the prices. An additional term $2\bar{\mu}\sqrt{2\delta/M}$ appears in the second inequality because rounded down prices in (U^δ) may attract users that refuse service in (U) . Note that in the particular case $p \in (\delta\mathbb{Z})^n$, we have $I_j(p^\delta) = I_j(p)$ and thus $\mu(I_j(p^\delta)) = \mu(I_j(p))$.

Our proof for Proposition 4.1 follows from Lemma 4.7.

Proof of Proposition 4.1. Following the exact lines of the proof given for Proposition 1.2, we can show that (U^δ) admits an optimal solution p in $[p^{\min} - \delta, p^{\max} + \delta]^n$. Since by construction $\tilde{p} = (p_1^\delta, \dots, p_n^\delta)$ satisfies $I_j(p^\delta) = I_j(\tilde{p}^\delta)$ for $j \in [n]$, we find that $\tilde{p} \in ([p^{\min} - \delta, p^{\max} + \delta] \cap \delta\mathbb{Z})^n$ is an optimal solution of (U^δ) too. Hence $\text{Opt}^\delta(\mathbb{R}) = \text{Opt}^\delta(\delta\mathbb{Z})$.

It remains to show the inequality $\text{Opt}^\delta(\delta\mathbb{Z}) + n\bar{\mu}L\delta p^{\max} \geq \text{Opt}(\mathbb{R})$. Let $p \in [p^{\min}, p^{\max}]^n$ be an optimal solution of (U) , which exists according to Proposition 1.2, and consider its discrete approximation p^δ . Using the second inequality in Lemma 4.7, we find $\mu(I_j(p^\delta)) \leq \mu(I_j(p)) + \bar{\mu}L\delta + 2\bar{\mu}\sqrt{2\delta/M} \leq \nu_j + \bar{\mu}L\delta + 2\bar{\mu}\sqrt{2\delta/M}$ for all $j \in [n]$, which implies that p is a feasible solution of (U^δ) . Using $p_j^\delta + \delta \geq p_j$ and the first inequality in Lemma 4.7, we find $\text{Opt}^\delta(\delta\mathbb{Z}) \geq \sum_{j=1}^n (p_j^\delta + \delta)\mu(I_j(p^\delta)) \geq \sum_{j=1}^n p_j(\mu(I_j(p)) - \bar{\mu}L\delta) \geq \text{Opt}(\mathbb{R}) - n\bar{\mu}L\delta p^{\max}$. \square

4.2.3. *Proof of Proposition 4.2.* The proof of Proposition 4.2 requires a technical result.

Lemma 4.8. *There exists $\gamma > 0$ such that, for all $p \in [p^{\min}, +\infty)^n$, $j \in [n]$, and $\Delta \geq 0$, the following holds:*

- $[s, s + \gamma\Delta] \subseteq I_j(p)$ for all $s \in I_j(p')$, where $p' = (p_1 + \Delta, \dots, p_{j-1} + \Delta, p_j + \Delta, p_{j+1}, \dots, p_n)$.
- $[s - \gamma\Delta, s] \subseteq I_j(p)$ for all $s \in I_j(p')$, where $p' = (p_1, \dots, p_{j-1}, p_j + \Delta, p_{j+1} + \Delta, \dots, p_n + \Delta)$.

Proof. Let $p \in \mathbb{R}^n$, $j \in [n]$, and $\Delta \geq 0$. We only derive the result for $p' = (p_1 + \Delta, \dots, p_{j-1} + \Delta, p_j + \Delta, p_{j+1}, \dots, p_n)$. A proof for $p' = (p_1, \dots, p_{j-1}, p_j + \Delta, p_{j+1} + \Delta, \dots, p_n + \Delta)$ is obtained similarly.

Consider $S = \{s \in \mathbb{R} : d(s) + p^{\min} \leq 0\}$. If $p^{\min} = p^{\max}$, then $S = \{0\}$ and we have either $\Delta = 0$, or $p'_j > p^{\max}$ and $I_j(p') = \emptyset$. Since the result is immediate in both cases, we can assume that S is a nondegenerate interval. Let L_d be the Lipschitz constant of d over S , and let λ be the maximum Lipschitz constant among the functions $s \mapsto d(s - t_i) - d(s - t_j)$ over the intervals $S_{i,j} = \{s + t : s \in S, t \in [t_i, t_j]\}$ for $i, j \in [n]$ with $i < j$. We show the result for $\gamma = 1/\max(L_d, \lambda)$.

Let $s \in I_j(p')$. It follows from (1) that $s \in J_j(p')$ and $s \in \mathcal{L}_j(p'_j)$. Since $s \in \mathcal{L}_j(p'_j)$, we have $d(s - t_j) + p^{\min} \leq d(s - t_j) + p_j = d(s - t_j) + p'_j - \Delta \leq -\Delta$, and $\Delta \geq 0$ yields $s - t_j \in S$. Since the result is immediate when $I_j(p') = \emptyset$, we assume both $J_j(p') \neq \emptyset$ and $\mathcal{L}_j(p'_j) \neq \emptyset$, and we show that, for all $s \in I_j(p')$ and $s' \in [s, s + \gamma\Delta]$, we have $s' \in I_j(p)$.

We first establish the preliminary property that every $s' \in [s - \Delta/L_d, s + \Delta/L_d]$ satisfies $s' - t_j \in S$ and $s' \in S_{j,k}$ for all $k \in [n]$ with $j < k$. Consider any $s' \in [s - \Delta/L_d, s + \Delta/L_d]$ and suppose $d(s' - t_j) + p^{\min} > 0$. By continuity of d , we can find $s'' \in (s - \Delta/L_d, s + \Delta/L_d)$ such that $d(s'' - t_j) + p^{\min} = 0$, which implies $s'' - t_j \in S$. It follows from $s - t_j \in S$, the convexity of S , and the definition of L_d that $d(s'' - t_j) + p^{\min} \leq d(s - t_j) + p^{\min} + L_d|s'' - s| \leq -\Delta + L_d|s'' - s| < 0$, which is a contradiction. Hence, $d(s' - t_j) + p^{\min} \leq 0$, which implies the desired property.

For $s' \in [s, s + \gamma\Delta]$, we have $s' \in [s - \Delta/L_d, s + \Delta/L_d]$, and the preliminary property yields $s' - t_j \in S$. It follows from $s - t_j \in S$, the convexity of S , and the definition of L_d that $d(s' - t_j) + p_j \leq d(s - t_j) + L_d|s' - s| + p_j \leq d(s - t_j) + L_d\gamma\Delta + p_j \leq d(s - t_j) + \Delta + p_j = d(s - t_j) + p'_j \leq 0$. Thus, for $s' \in [s, s + \gamma\Delta]$, we have $s' \in \mathcal{L}_j(p_j)$.

Now, consider $k \in [n]$. First suppose $k \leq j$. For $s' \geq s$, Fact 2.1 yields $d(s' - t_k) - d(s' - t_j) \geq d(s - t_k) - d(s - t_j)$. It follows that

$$d(s' - t_j) + p_j \leq d(s' - t_k) + p_k + [d(s - t_j) + p'_j] - [d(s - t_k) + p'_k] \leq d(s' - t_k) + p_k,$$

where we use $p_j = p_k + p'_j - p'_k$ and $s \in J_j(p')$. Suppose now $k > j$. For $s' \in [s, s + \gamma\Delta]$, we have $s' \in [s - \Delta/L_d, s + \Delta/L_d]$, and the preliminary property yields $s' \in S_{j,k}$. It follows from $s \in S_{j,k}$, the convexity of $S_{j,k}$, and the definition of λ that $d(s' - t_j) - d(s' - t_k) \leq d(s - t_j) - d(s - t_k) + \lambda(s' - s) \leq d(s - t_j) - d(s - t_k) + \lambda\gamma\Delta \leq d(s - t_j) - d(s - t_k) + \Delta$. Consequently,

$$d(s' - t_j) + p_j \leq d(s' - t_k) + p_k + [d(s - t_j) + p'_j] - [d(s - t_k) + p'_k] \leq d(s' - t_k) + p_k,$$

where we use $p_j = p_k + p'_j - p'_k - \Delta$ and $s \in J_j(p')$. Thus, for $s' \in [s, s + \Delta/\lambda]$ we have $d(s' - t_j) + p_j \leq d(s' - t_k) + p_k$ for all $k \in [n]$, and $s' \in J_j(p)$.

Combining our last two conclusions, we find from (1) that $s' \in J_j(p) \cap \mathcal{L}_j(p_j) = I_j(p)$ for all $s \in I_j(p')$ and all $s' \in [s, s + \gamma\Delta]$. \square

In order to show Proposition 4.2, we proceed as follows. We consider a discrete optimal solution p of (U^δ) in accordance with Proposition 4.1, and then we alter the individual prices one time slot at a time until a new discrete price profile is obtained that satisfies the constraints of (U) , thus providing us with a lower bound to $\text{Opt}(\delta\mathbb{Z})$. To do so, we use two main tools: Lemma 4.9 identifies for p a particular time slot j^* where a substantial decrease in service load I_{j^*} can be obtained through simultaneous increase of all prices, and Lemma 4.10 provides bounds for the variations in service load at the time slots resulting from the successive price alterations. Starting from p , our strategy is then to progressively reduce service load through successive applications of Lemma 4.10 from the extreme time slots towards the specified slot j^* , where initially the service load had been sufficiently reduced so as to compensate for the cumulated side effects of the future price jumps at the other time slots.

We set $\delta^{\max} = 8/(L^2M)$, as this constant plays a role in the upcoming results.

Lemma 4.9. *Let $h > 0$. There exist $f(h) > 0$ and $0 < \bar{\delta}(h) \leq \delta^{\max}$ such that, for every $\delta \in (0, \bar{\delta}(h)]$ and every optimal solution $p \in ([p^{\min} - \delta, p^{\max} + \delta] \cap \delta\mathbb{Z})^n$ of (U^δ) , setting $N = \lceil f(h)\delta^{-3/4} \rceil$ yields one of the following: $\sum_{j=1}^n \mu(I_j(p + N\delta)) = 0$, or one can find a time slot $j^* \in [n]$ such that $\mu(I_{j^*}(p + N\delta)) \leq \mu(I_{j^*}(p)) - h\sqrt{\delta}$.*

In Lemma 4.9 and its proof, we use the notation $p + N\delta = (p_1 + N\delta, \dots, p_n + N\delta)$. We note that the existence of an optimal solution $p \in ([p^{\min} - \delta, p^{\max} + \delta] \cap \delta\mathbb{Z})^n$ of the modified problem (U^δ) follows from Proposition 4.1. The message of Lemma 4.9 is that the solution p of (U^δ) is either $O(\delta^{1/4})$ close to a feasible profile $\tilde{p} \in (\delta\mathbb{Z})^n$ of (U) under which no user decides to be served, or that there is one particular time slot j^* where any overhead $h\sqrt{\delta}$ in service load can be cut by simultaneously increasing all prices by a quantity $N\delta = O(\delta^{1/4})$.

Proof of Lemma 4.9. We show the lemma for $\bar{\delta}(h) = \min(\delta^{\max}, [L'_d \mu(\mathbb{R})/(\underline{\mu} f(h))]^4)$ and $f(h) = [n\pi(L'_d/\underline{\mu})h]^{1/2}$, where $\pi = \max(0, p^{\max} + 2\delta^{\max}) + p^{\max} - p^{\min} + \delta^{\max} > 0$ and L'_d denotes the Lipschitz constant of d over the interval $S' = \{s \in \mathbb{R} : d(s) + p^{\min} - \delta^{\max} \leq 0\}$. Consider δ , p , and N as introduced in the statement of the lemma, and set $\tau = (\underline{\mu}/L'_d)f(h)\sqrt[4]{\delta}$.

First consider the case $N\delta > p^{\max} - p^{\min} + \delta$. By definition of p and δ , we have $p_j + N\delta > p^{\max}$ for all $j \in [n]$, and therefore no user is served in (U) under $p + N\delta$. Hence, $\sum_{j=1}^n \mu(I_j(p + N\delta)) = 0$, which is the first alternative.

Now, consider the case $\sum_{j=1}^n \mu(I_j(p)) < \tau$. Again, we prove that the first alternative $\sum_{j=1}^n \mu(I_j(p + N\delta)) = 0$ holds by showing that no user is served under $p + N\delta$. To do this, we take $s \in \text{supp}(\mu)$ and $k \in [n]$, for which we show that $d(s - t_k) + p_k + N\delta > 0$. Since $d(s - t_k) + p_k + N\delta > 0$ is immediate when $d(s - t_k) + p_k > 0$, we can suppose that $d(s - t_k) + p_k \leq 0$, which implies $s - t_k \in S'$. Since $\delta \leq \bar{\delta}(h) \leq [L'_d \mu(\mathbb{R})/(\underline{\mu} f(h))]^4$, we have $\tau \leq \mu(\mathbb{R})$, and therefore $\sum_{j=1}^n \mu(I_j(p)) < \mu(\mathbb{R})$. Hence, one can find $s' \in \text{supp}(\mu)$ such that $d(s' - t_j) + p_j > 0$ for all $j \in [n]$. In particular, $d(s' - t_k) + p_k > 0$, and it follows from the continuity of d and the convexity of $\text{supp}(\mu)$ that there exists $s'' \in \text{supp}(\mu)$ such that $d(s'' - t_k) + p_k = 0$, $s'' - t_k \in S'$, and $d(\hat{s} - t_k) + p_k \leq 0$ for all \hat{s} located between s and s'' . This last property yields $\hat{s} \in \bigcup_{j=1}^n I_j(p)$ for all such \hat{s} , and consequently $|s - s''| \leq \sum_{j=1}^n |I_j(p)|$. Since $N\delta \geq L'_d \tau / \underline{\mu}$, we have $N\delta > (L'_d / \underline{\mu}) \sum_{j=1}^n \mu(I_j(p)) \geq L'_d \sum_{j=1}^n |I_j(p)| \geq L'_d |s - s''|$. Consequently, $d(s - t_k) + p_k + N\delta \geq d(s'' - t_k) + p_k - L'_d |s - s''| + N\delta > d(s'' - t_k) + p_k = 0$, where the first inequality results from the definition of L'_d .

Lastly, consider the case $N\delta \leq p^{\max} - p^{\min} + \delta$ and $\sum_{j=1}^n \mu(I_j(p)) \geq \tau$. Let the price profile \tilde{p} be defined for $j \in [n]$ by $\tilde{p}_j = p_j + N\delta$, and let $D_j = \mu(I_j(p)) - \mu(I_j(\tilde{p}))$. By definition of p and \tilde{p} , we have $J_j(p) = J_j(\tilde{p})$ and $\mathcal{L}_j(p_j) \supseteq \mathcal{L}_j(\tilde{p}_j)$. It follows from (1) that $I_j(p) \supseteq I_j(\tilde{p})$, and therefore $D_j \geq 0$. Optimality of p in (U^δ) and $p, \tilde{p} \in (\delta\mathbb{Z})^n$ then imply

$$0 \geq \sum_{j=1}^n (\tilde{p}_j + \delta) \mu(I_j(\tilde{p})) - \sum_{j=1}^n (p_j + \delta) \mu(I_j(p)) = N\delta \sum_{j=1}^n \mu(I_j(p)) - \sum_{j=1}^n (\tilde{p}_j + \delta) D_j.$$

Using $\delta \leq \delta^{\max}$, the definition of p and \tilde{p} , and the assumption on $N\delta$, we find $\tilde{p}_j + \delta = p_j + \delta + N\delta \leq (p^{\max} + \delta) + \delta + (p^{\max} - p^{\min} + \delta) \leq \pi$ for all $j \in [n]$. It follows under our assumption that

$$\tau \leq \sum_{j=1}^n \mu(I_j(p)) \leq \frac{1}{N\delta} \sum_{j=1}^n (\tilde{p}_j + \delta) D_j \leq \frac{\pi}{N\delta} \sum_{j=1}^n D_j \leq \frac{n\pi}{N\delta} \max_{j \in [n]} D_j.$$

Consequently, we can find $j^* \in [n]$ such that $D_{j^*} \geq \tau N\delta / (n\pi)$. It follows from $p, \tilde{p} \in (\delta\mathbb{Z})^n$ that $\mu(I_{j^*}(\tilde{p})) = \mu(I_{j^*}(p)) - D_{j^*} \leq \mu(I_{j^*}(p)) - \tau N\delta / (n\pi)$. Introducing the expressions for τ and $f(h)$ into the last inequality and using $N \geq f(h)\delta^{-3/4}$, we find $\mu(I_{j^*}(\tilde{p})) \leq \mu(I_{j^*}(p)) - h\sqrt{\delta}$, and we get the second alternative. \square

Given a price profile, suppose we simultaneously raise the prices at the first j (or last j) time slots in order to reduce attendance at those time slots. Lemma 4.10 provides bounds for the decrease in their individual service loads, and bounds for the variations in service load at the remaining time slots.

Lemma 4.10. *There exists $K > 0$ such that for every $\delta \in (0, \delta^{\max}]$, $p \in ([p^{\min}, +\infty) \cap \delta\mathbb{Z})^n$, and $j^* \in [n]$ with $\mu(J_{j^*}(p)) > 0$, the following holds:*

(i) For $j < j^*$, $\Delta > 0$, and $p' = (p_1 + K\Delta, \dots, p_{j-1} + K\Delta, p_j + K\Delta, p_{j+1}, \dots, p_n)$, we have $\mu(J_{j^*}(p')) > 0$, and

$$\begin{aligned} \mu(I_k(p')) &\leq \mu(I_k(p)) && \text{if } 1 \leq k < j, \\ \mu(I_j(p')) &\leq \max(0, \mu(I_j(p)) - \Delta), \\ \mu(I_k(p')) &\leq \mu(I_k(p)) + \bar{\mu}KL\Delta && \text{if } j < k \leq j^*, \\ \mu(I_k(p')) &= \mu(I_k(p)) && \text{if } j^* < k \leq n. \end{aligned}$$

(ii) For $j > j^*$, $\Delta > 0$, and $p' = (p_1, \dots, p_{j-1}, p_j + K\Delta, p_{j+1} + K\Delta, \dots, p_n + K\Delta)$, we have $\mu(J_{j^*}(p')) > 0$, and

$$\begin{aligned} \mu(I_k(p')) &= \mu(I_k(p)) && \text{if } 1 \leq k < j^*, \\ \mu(I_k(p')) &\leq \mu(I_k(p)) + \bar{\mu}KL\Delta && \text{if } j^* \leq k < j, \\ \mu(I_j(p')) &\leq \max(0, \mu(I_j(p)) - \Delta), \\ \mu(I_k(p')) &\leq \mu(I_k(p)) && \text{if } j < k \leq n. \end{aligned}$$

Proof. We show (i) for the constants $K = \lceil 1/(\underline{\mu}\gamma) \rceil$, where γ is specified by Lemma 4.8. The proof of (ii) is omitted as it can be done similarly and with the same constants. Let $j < j^*$, $\Delta > 0$, and $p' = (p_1 + K\Delta, \dots, p_{j-1} + K\Delta, p_j + K\Delta, p_{j+1}, \dots, p_n)$. First consider $s \in J_{j^*}(p)$. For $\ell \in [n]$, we have $p_\ell \leq p'_\ell$ and $p_{j^*} = p'_{j^*}$, which yield $d(s - t_{j^*}) + p'_{j^*} = d(s - t_{j^*}) + p_{j^*} \leq d(s - t_\ell) + p_\ell \leq d(s - t_\ell) + p'_\ell$, and therefore $s \in J_{j^*}(p')$. Hence, $J_{j^*}(p) \subseteq J_{j^*}(p')$. Since $\mu(J_{j^*}(p)) > 0$, we also have $\mu(J_{j^*}(p')) > 0$. We now verify each remaining claim of (i) individually. Consider $k \in [n]$.

Suppose $1 \leq k < j$. Then, for all $s \in I_k(p')$ and $\ell \in [n]$, it holds that $d(s - t_k) + p_k = d(s - t_k) + p'_k - K\Delta \leq \min(0, d(s - t_\ell) + p'_\ell) - K\Delta \leq \min(0, d(s - t_\ell) + p_\ell)$, so that $s \in I_k(p)$. Hence, $I_k(p') \subseteq I_k(p)$, and $\mu(I_k(p')) \leq \mu(I_k(p))$.

Suppose $k = j$. In the case $I_j(p') = \emptyset$, we have $\mu(I_j(p')) = 0$. Now, consider the case $I_j(p') \neq \emptyset$. Lemma 4.8 yields $[s', s' + \gamma K\Delta] \subseteq I_j(p)$ for all $s' \in I_j(p')$. Since $\mu(J_{j^*}(p)) > 0$, $j < j^*$, and μ is interval supported, it follows from Proposition 2.3 that $\mu(I_j(p)) \geq \mu(I_j(p')) + \underline{\mu}\gamma K\Delta \geq \mu(I_j(p')) + \Delta$. In any case, we have $\mu(I_j(p')) \leq \max(0, \mu(I_j(p)) - \Delta)$.

Suppose $j < k \leq j^*$. Using Lemma 4.5 with parameters $\eta = K\Delta$ and $\eta' = 0$, we find $|J_k(p') \setminus J_k(p)| \leq LK\Delta$. Since $p'_k = p_k$, we also have $\mathcal{L}_k(p'_k) = \mathcal{L}_k(p_k)$. It follows from (1) that $|I_k(p') \setminus I_k(p)| = |(J_k(p') \setminus J_k(p)) \cap \mathcal{L}_k(p_k)| \leq |J_k(p') \setminus J_k(p)| \leq KL$, and $\mu(I_k(p')) \leq \mu(I_k(p)) + \mu(I_k(p') \setminus I_k(p)) \leq \mu(I_k(p)) + \bar{\mu}|I_k(p') \setminus I_k(p)| \leq \mu(I_k(p)) + \bar{\mu}KL\Delta$.

Suppose $j^* < k \leq n$. We successively show $I_k(p) \subseteq I_k(p')$ and $I_k(p') \subseteq I_k(p)$, which imply $I_k(p) = I_k(p')$ and $\mu(I_k(p)) = \mu(I_k(p'))$. First, consider $s \in I_k(p)$. For $\ell \in [n]$, we have $d(s - t_k) + p'_k = d(s - t_k) + p_k \leq \min(0, d(s - t_\ell) + p_\ell) \leq \min(0, d(s - t_\ell) + p'_\ell)$. Hence, $s \in I_k(p')$, and therefore $I_k(p) \subseteq I_k(p')$. Now, consider $s \in I_k(p')$, and pick $s' \in J_{j^*}(p)$. For $\ell \in [n]$, we have $d(s' - t_{j^*}) + p'_{j^*} = d(s' - t_{j^*}) + p_{j^*} \leq d(s' - t_\ell) + p_\ell \leq d(s' - t_\ell) + p'_\ell$ and therefore $s' \in J_{j^*}(p')$. It follows from Lemma 2.2 that $s' \leq s$. For $1 \leq \ell < j^*$, using $s' \leq s$ and Fact 2.1 yields $d(s' - t_\ell) - d(s' - t_{j^*}) \leq d(s - t_\ell) - d(s - t_{j^*})$, and it follows that $d(s' - t_{j^*}) + p_{j^*} \leq d(s' - t_\ell) + p_\ell \leq d(s - t_\ell) + p_\ell + d(s' - t_{j^*}) - d(s - t_{j^*})$. Consequently, $d(s - t_k) + p_k = d(s - t_k) + p'_k \leq \min(0, d(s - t_{j^*}) + p'_{j^*}) = \min(0, d(s' - t_{j^*}) + p_{j^*} + d(s - t_{j^*}) - d(s' - t_{j^*})) \leq \min(0, d(s - t_\ell) + p_\ell)$. For $j^* \leq \ell \leq n$, we have $d(s - t_k) + p_k = d(s - t_k) + p'_k \leq \min(0, d(s - t_\ell) + p'_\ell) = \min(0, d(s - t_\ell) + p_\ell)$. Hence, $s \in I_k(p)$, and therefore $I_k(p') \subseteq I_k(p)$. \square

The next lemma claims the existence of a feasible solution of (U) at distance $O(\delta^{1/4})$ from an optimal solution of (U^δ) . Lemma 4.11 relies on Lemmas 4.9 and 4.10, and it is the basis of our proof to Proposition 4.2.

Lemma 4.11. *There exist $C, \zeta > 0$, and maps $p, \tilde{p}: (0, \zeta] \rightarrow \mathbb{R}^n$ such that $p(\delta) \in ([p^{\min} - \delta, p^{\max} + \delta] \cap \delta\mathbb{Z})^n$ is an optimal solution of (U^δ) , $\tilde{p}(\delta)$ is a feasible solution of (U), and we have $\|p(\delta) - \tilde{p}(\delta)\|_\infty \leq C\sqrt[4]{\delta}$.*

We emphasize that the proof of Lemma 4.11 will make clear that the constants C and ζ involved in the lemma only depend on the function d , the support of μ and its lower and upper bounds, and the slot times t_1, \dots, t_n , and are independent of the values of μ and ν .

Proof of Lemma 4.11. Applying Lemma 4.9 with parameter value $h = 4\bar{\mu}(1+T)^{n-1}\sqrt{2/M}$, where $T = \bar{\mu}KL$ and K is specified by Lemma 4.10, yields specific constants $f(h), \bar{\delta}(h) > 0$. We show the lemma for $C = 2(3 + (1+T)^{n-1})(2L^2M)^{-3/4} + f(h)$, $\zeta = \bar{\delta}(h)$, and for maps p, \tilde{p} that we specify as follows. In view of Proposition 4.1, we set $p(\delta)$ to any discrete optimal solution of (U^δ) in $([p^{\min} - \delta, p^{\max} + \delta] \cap \delta\mathbb{Z})^n$. We are going to construct $\tilde{p}(\delta)$ based on $p(\delta)$, for the range $\delta \in (0, \bar{\delta}(h)]$ in which Lemma 4.9 applies. To lighten the text, we drop the argument δ and write $p \equiv p(\delta)$. Set $N = \lceil f(h)\delta^{-3/4} \rceil$, $p' = p + N\delta$, and $\Delta = 4\bar{\mu}\sqrt{2\delta/M}$. By definition of p and p' , we have $J_j(p) = J_j(p')$ and $\mathcal{L}_j(p_j) \supseteq \mathcal{L}_j(p'_j)$ for $j \in [n]$. It follows from (1) that $I_j(p) \supseteq I_j(p')$ and $\mu(I_j(p)) \geq \mu(I_j(p'))$ for $j \in [n]$. Since $p \in (\delta\mathbb{Z})^n$ and p is an optimal solution of (U^δ) , we get, for $j \in [n]$,

$$(4) \quad \mu(I_j(p')) \leq \mu(I_j(p)) = \mu(I_j(p^\delta)) \leq \nu_j + \bar{\mu}L\delta + 2\bar{\mu}\sqrt{2\delta/M} \leq \nu_j + \Delta,$$

where we use $\delta \leq \sqrt{\delta^{\max}\delta} = (2/L)\sqrt{2\delta/M}$.

First suppose that $\sum_{j=1}^n \mu(I_j(p')) = 0$. Then, we have $\mu(I_j(p')) = 0$ for all $j \in [n]$. Hence, $\tilde{p} = p'$ is a feasible price profile of (U).

Otherwise, according to Lemma 4.9, there is a time slot $j^* \in [n]$ such that

$$(5) \quad \mu(I_{j^*}(p')) + h\sqrt{\delta} \leq \mu(I_{j^*}(p)) = \mu(I_{j^*}(p^\delta)) \stackrel{(4)}{\leq} \nu_{j^*} + \Delta.$$

From (1) and the first inequality in (5), we find $\mu(J_{j^*}(p)) \geq \mu(I_{j^*}(p)) \geq \mu(I_{j^*}(p')) + h\sqrt{\delta} > 0$. Hence, the service load $\mu(J_{j^*}(p))$ is a positive quantity. Now, consider the vectors

$$v_j = (\underbrace{1, \dots, 1}_j, 0, \dots, 0) \quad \text{and} \quad w_j = (\underbrace{0, \dots, 0}_j, 1, 1, \dots, 1) \quad \text{for } j \in [n].$$

We derive the price profile $\tilde{p} = p' + K \sum_{j=1}^{j^*-1} \Delta_j v_j + K \sum_{j=j^*+1}^n \Delta_j w_j$, where $\Delta_j = (1+T)^{j-1} \Delta$ for $j = 1, \dots, j^* - 1$ and $\Delta_j = (1+T)^{n-j} \Delta$ for $j = j^* + 1, \dots, n$. We apply $j^* - 1$ times Lemma 4.10(i) with successive parameters $\Delta_1, \dots, \Delta_{j^*-1}$, and then $n - j^*$ times Lemma 4.10(ii) with successive parameters $\Delta_n, \dots, \Delta_{j^*+1}$. This is possible because $\mu(J_{j^*}(p)) > 0$, $J_{j^*}(p') = J_{j^*}(p)$ (all the prices are translated upward by the same quantity), and by induction the $n - 1$ price profiles successively considered for Lemma 4.10 meet this condition as well, with prices not lower than p^{\min} . Using $\sum_{k=1}^{j-1} T\Delta_k - \Delta_j = -\Delta$ when $1 \leq j < j^*$, $\sum_{k=j+1}^n T\Delta_k - \Delta_j = -\Delta$ when $j^* < j \leq n$, and $\sum_{k=1}^{j^*-1} T\Delta_k + \sum_{k=j^*+1}^n T\Delta_k = (1+T)^{j^*-1} \Delta + (1+T)^{n-j^*} \Delta - 2\Delta \leq$

$(1+T)^{n-1}\Delta - \Delta = h\sqrt{\delta} - \Delta$, we find

$$\begin{aligned}\mu(I_j(\tilde{p})) &\leq \max(0, \mu(I_j(p')) + \sum_{k=1}^{j-1} T\Delta_k - \Delta_j) \stackrel{(4)}{\leq} \nu_j & \text{if } 1 \leq j < j^*, \\ \mu(I_j(\tilde{p})) &\leq \max(0, \mu(I_j(p')) + \sum_{k=j+1}^n T\Delta_k - \Delta_j) \stackrel{(4)}{\leq} \nu_j & \text{if } j^* < j \leq n, \\ \mu(I_{j^*}(\tilde{p})) &\leq \max(0, \mu(I_{j^*}(p')) + \sum_{k=1}^{j^*-1} T\Delta_k + \sum_{k=j^*+1}^n T\Delta_k) \stackrel{(5)}{\leq} \nu_{j^*}.\end{aligned}$$

Hence, we have $\mu(I_j(\tilde{p})) \leq \nu_j$ for all $j \in [n]$, and \tilde{p} is a feasible price profile of (U).

In any case, $\tilde{p}(\delta) \equiv \tilde{p}$ is a feasible price profile of (U). Given $\delta \in (0, \bar{\delta}(h)]$, we have $\delta \leq \sqrt[4]{(\delta^{\max})^3 \delta} = 8\sqrt[4]{\delta/(2L^2M)^3}$, $\sqrt{\delta} \leq \sqrt[4]{\delta^{\max} \delta} = 2\sqrt[4]{\delta/(2L^2M)}$ and, after computations, $\|p(\delta) - \tilde{p}(\delta)\|_\infty \leq [8(2(1+T)^{n-1} - 1)(2L^2M)^{-3/4} + f(h)]\delta^{1/4}$, which completes the proof. \square

We are now in a position to show Proposition 4.2.

Proof of Proposition 4.2. Using Lemma 4.11, we derive C , ζ , and maps $p, \tilde{p}: (0, \zeta] \rightarrow \mathbb{R}^n$ such that $p(\delta) \in ([p^{\min} - \delta, p^{\max} + \delta] \cap \delta\mathbb{Z})^n$ is an optimal solution of (U^δ) , $\tilde{p}(\delta)$ is a feasible solution of (U), and $z := \|p(\delta) - \tilde{p}(\delta)\|_\infty \leq C\sqrt[4]{\delta}$ holds for all $\delta \in (0, \zeta]$.

Let $\delta \in (0, \zeta]$. To lighten the text, we drop the argument δ and write $p \equiv p(\delta)$ and $\tilde{p} \equiv \tilde{p}(\delta)$. We compute the distance between $R(\tilde{p}) := \sum_{k=1}^n \tilde{p}_k \mu(I_k(\tilde{p}))$, which defines the revenue induced by \tilde{p} in (U), and the upper bound $R^\delta(p) + n\bar{\mu}L\delta p^{\max}$, where $R^\delta(p) := \sum_{k=1}^n (p_k + \delta)\mu(I_k(p)) = \text{Opt}^\delta(\mathbb{R})$ defines the revenue induced by p in (U^δ) . We know from (1) that $I_k(p) \triangle I_k(\tilde{p}) \subseteq (J_k(p) \triangle J_k(\tilde{p})) \cup (\mathcal{L}_k(p_k) \triangle \mathcal{L}_k(\tilde{p}_k))$. Using Lemma 4.5 with parameters $\eta = \eta' = z$, we find $|J_j(\tilde{p}) \setminus J_j(p)| \leq 2zL$ and $|J_j(p) \setminus J_j(\tilde{p})| \leq 2zL$, which imply $|J_j(p) \triangle J_j(\tilde{p})| = |J_j(\tilde{p}) \setminus J_j(p)| + |J_j(p) \setminus J_j(\tilde{p})| \leq 4zL$. From Lemma 4.6, we also have $|\mathcal{L}_k(p_k) \triangle \mathcal{L}_k(\tilde{p}_k)| \leq 2\sqrt{2/M}|p_k - \tilde{p}_k|^{1/2} \leq 2\sqrt{2z/M}$. It follows that $|I_k(p) \triangle I_k(\tilde{p})| \leq |J_k(p) \triangle J_k(\tilde{p})| + |\mathcal{L}_k(p_k) \triangle \mathcal{L}_k(\tilde{p}_k)| \leq 4zL + 2\sqrt{2z/M}$ and, consequently, $\mu(I_k(p)) \leq \mu(I_k(\tilde{p})) + \mu(I_k(p) \triangle I_k(\tilde{p})) \leq \mu(I_k(\tilde{p})) + 4\bar{\mu}zL + 2\bar{\mu}\sqrt{2z/M}$ holds for $k \in [n]$. Using $p \in (\delta\mathbb{Z})^n$, we find

$$\begin{aligned}R^\delta(p) &= \sum_{k=1}^n (p_k + \delta)\mu(I_k(p)) \\ &\leq \sum_{k=1}^n (p_k + \delta)(\mu(I_k(\tilde{p})) + 4\bar{\mu}zL + 2\bar{\mu}\sqrt{2z/M}) \\ &= \sum_{k=1}^n (\tilde{p}_k + (p_k - \tilde{p}_k) + \delta)\mu(I_k(\tilde{p})) + 2\bar{\mu}(2zL + \sqrt{2z/M}) \sum_{k=1}^n (p_k + \delta) \\ &= R(\tilde{p}) + \sum_{k=1}^n ((p_k - \tilde{p}_k) + \delta)\mu(I_k(\tilde{p})) + 2\bar{\mu}(2zL + \sqrt{2z/M})(n\delta + \sum_{k=1}^n p_k),\end{aligned}$$

where $p_k \leq p^{\max} + \delta$ for all $k \in [n]$. For all $\delta \in (0, \zeta]$, we thus find $R^\delta(p(\delta)) + n\bar{\mu}L\delta p^{\max} \leq R(\tilde{p}(\delta)) + h(\delta)$, where $h(\delta) = n\mu(\mathbb{R})(z + \delta) + 2n\bar{\mu}(2zL + \sqrt{2z/M})(p^{\max} + 2\delta) + n\bar{\mu}L\delta p^{\max}$. It follows from $z \leq C\sqrt[4]{\delta}$ that $h(\delta) = O(\delta^{1/8})$. Finally, $\text{Opt}^\delta(\mathbb{R}) + n\bar{\mu}L\delta p^{\max} = R^\delta(p(\delta)) + n\bar{\mu}L\delta p^{\max} \leq R(\tilde{p}(\delta)) + h(\delta) \leq \text{Opt}(\delta\mathbb{Z}) + h(\delta)$ completes the proof. \square

5. CONCLUDING REMARKS

The solutions developed in this paper rest on a series of assumptions. When the set of prices P is finite, the users' ability to choose a unique (μ -almost everywhere) time slot follows from the strict convexity of the cost function d —on which Lemma 2.2 and Proposition 2.3 directly rely—and from the non-atomic nature of the population μ . Without these two assumptions, the bilevel problem becomes ambiguous as to which time slots indecisive users may choose. An optimistic approach then is to introduce additional incentives to assist the indecisive users in their final decisions. A more conservative approach is to stick to

the assumption that each user may choose any time slot that minimizes their total cost, in which case the sets $J_1(p), \dots, J_n(p)$ remain ordered as in Proposition 2.3 but cease to have disjoint interiors, and the construction of the graph for solving (U) must be amended accordingly. Another basic assumption made in this work is that users share a unique cost function. User diversity can be modeled, for instance, by considering ℓ user types with distinct cost functions d_1, \dots, d_ℓ , populations μ_1, \dots, μ_ℓ , interpretations of Proposition 2.3 for $J_{1,1}(p), \dots, J_{1,n}(p); \dots; J_{\ell,1}(p), \dots, J_{\ell,n}(p)$, and graphs $D_1 = (V, A_1), \dots, D_\ell = (V, A_\ell)$; a naive extension of our graph-based solution would then require time $O(n^{3\ell}|P|^{3\ell})$ over a subset of the Cartesian product of $L(D_1), \dots, L(D_\ell)$. Another possible refinement of the pricing problem would assume that service duration may differ for the users. This feature could be integrated into our model by considering m user types requiring 1 to m consecutive time slots for service, thus enabling graph-based solutions in time $O(n^{m+1}|P|^{m+1})$ over the m th line digraph $L(\dots L(L(D)) \dots)$. We leave open the question of time complexity in more elaborate settings of the time slot pricing problem.

REFERENCES

1. Fabian Akkerman, Martijn Mes, and Eduardo Lalla-Ruiz, *Dynamic time slot pricing using delivery costs approximations*, Computational Logistics: 13th International Conference, ICCL 2022, Barcelona, Spain, September 21–23, 2022, Proceedings (Berlin, Heidelberg), Springer-Verlag, 2022, p. 214–230.
2. Miguel F Anjos, Ikram Bouras, Luce Brotcorne, Alemseged G Weldeyesus, Clémence Alasseur, and Riadh Zorgati, *Integrated location, sizing, and pricing for EV charging stations*, Combinatorial Optimization and Applications: A Tribute to Bernard Gendron, Springer, 2024, pp. 431–448.
3. Kursad Asdemir, Varghese S Jacob, and Ramayya Krishnan, *Dynamic pricing of multiple home delivery options*, European Journal of Operational Research **196** (2009), 246–257.
4. Benoît Colson, Patrice Marcotte, and Gilles Savard, *An overview of bilevel optimization*, Annals of Operations Research **153** (2007), 235–256.
5. Denis Lebedev, Paul Goulart, and Kostas Margellos, *A dynamic programming framework for optimal delivery time slot pricing*, European Journal of Operational Research **292** (2021), 456–468.
6. Jaroslav Morávek, *A note upon minimal path problem*, Journal of Mathematical Analysis and Applications **30** (1970), 702–717.
7. Yurii Nesterov, *Introductory lectures on convex optimization: A basic course*, 1 ed., Springer Publishing Company, Incorporated, 2014.
8. Filippo Santambrogio, *Optimal transport for applied mathematicians*, Birkhäuser, NY **55** (2015), no. 58-63, 94.
9. Alexander Schrijver, *Combinatorial optimization: Polyhedra and efficiency*, Springer, 2003.
10. Arne Strauss, Nalan Gülpınar, and Yijun Zheng, *Dynamic pricing of flexible time slots for attended home delivery*, European Journal of Operational Research **294** (2021), 1022–1041.
11. Garrett J Van Ryzin and Kalyan T Talluri, *An introduction to revenue management*, Emerging theory, methods, and applications, Informs, 2005, pp. 142–194.
12. Wei Yuan, Jianwei Huang, and Ying Jun Angela Zhang, *Competitive charging station pricing for plug-in electric vehicles*, IEEE Transactions on Smart Grid **8** (2015), 627–639.

(Olivier Bilenne) CONSERVATOIRE NATIONAL DES ARTS ET MÉTIERS, CÉDRIC, FRANCE.
Email address: `olivier-stephane.bilenne@lecnam.net`

(Frédéric Meunier) CERMICS, ÉCOLE NATIONALE DES PONTS ET CHAUSSÉES, FRANCE.
Email address: `frederic.meunier@enpc.fr`