

# A Generalized Control Function Approach to Production Function Estimation

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## Abstract

We generalize the control function approach to production function estimation. Our generalization accommodates scenarios in which productivity evolves jointly with other unobservable factors such as latent demand shocks and the invertibility assumption underpinning the traditional proxy variable approach fails. We provide conditions under which the output elasticity of the variable input—and hence the markup—is nonparametrically point-identified. A Neyman orthogonal moment condition ensures oracle efficiency of our GMM estimator. A Monte Carlo exercise shows a large bias for the traditional proxy variable approach that decreases rapidly and nearly vanishes for our generalized control function approach.

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The proxy variable approach to production function estimation following G. Steven Olley & Ariel Pakes (1996) (henceforth, OP) rests on two key assumptions. First, the invertibility assumption requires that there exists a function  $\omega_{it} = \kappa(x_{it})$  that maps observables  $x_{it}$  into the productivity  $\omega_{it}$  of firm  $i$  in period  $t$ . It is well understood that invertibility can fail because of unobserved demand heterogeneity or in imperfectly competitive environments with partially or fully unobserved rivals or changes in firm conduct (see Ulrich Doraszelski & Lixiong Li (2025) and the references therein). In this paper, we denote these unobservable factors collectively as  $\delta_{it}$ . Besides latent demand shocks,  $\delta_{it}$  may capture unobserved variation in investment opportunities or financial constraints across firms or time.

Second, the proxy variable approach rests on the assumption that productivity is governed by a Markov process with law of motion  $\omega_{it} = g(\omega_{it-1}) + \xi_{it}$ , where  $g(\omega_{it-1}) = E[\omega_{it} | \omega_{it-1}]$  and the productivity innovation  $\xi_{it}$  is outside the firm's information set in period  $t - 1$ . Replacing the autonomous Markov process with a controlled Markov process is straightforward as long as the control is observed (see Jan De Loecker (2013) for learning-by-exporting and Ulrich Doraszelski & Jordi Jaumandreu (2013) for R&D).

In this paper, we allow productivity  $\omega_{it}$  and any other unobservables  $\delta_{it}$  to evolve jointly and generalize the law of motion for productivity from  $\omega_{it} = g(\omega_{it-1}) + \xi_{it}$  to  $\omega_{it} = g(\omega_{it-1}, \delta_{it-1}) + \xi_{it}$ . This accommodates scenarios in which productivity and demand are linked through product innovations, foreign market access, or the firm's R&D and advertising decisions. For example, a firm may move downmarket with products that are cheaper to produce but appeal to a broader set of consumers or it may move upmarket with products that are more costly to produce but appeal to more quality-conscious consumers.

One approach to dealing with additional unobservables is to extend the univariate inversion in OP to a multivariate inversion. Ensuring the existence of a function  $(\omega_{it}, \delta_{it}) = \kappa(x_{it})$  that recovers all unobservables, however, requires making specific assumptions on the firm's profit maximization problem (see Ulrich Doraszelski & Jordi Jaumandreu (2018) for labor-augmenting productivity and Paul L. E. Grieco, Shengyu Li & Hongsong Zhang (2016) for unobserved intermediate input prices).

In this paper, we pursue another approach that does not require a multivariate inversion. We focus on estimating the production function  $f(k_{it}, v_{it})$ , where  $k_{it}$  and  $v_{it}$  are inputs, and make no attempt to identify or estimate the law of motion for productivity. We provide conditions under which the output elasticity of the variable input  $v_{it}$ —and hence the markup following Jan De Loecker & Frederic Warzynski (2012)—is nonparametrically point-identified and can be estimated by solving a straightforward GMM problem.

We start by showing that the proxy variable approach to production function estimation following OP can be seen as controlling for “enough” of the variation in productivity to facilitate finding instruments relative to a direct instrumental variables approach. We then generalize this control function approach by adding variables to the control function. We show that this allows handling empirically relevant settings that are outside the scope of the proxy variable approach. By adding variables to the control function, our approach entails a tradeoff between increasing robustness on the one hand and decreasing efficiency and identification power on the other hand.

## 1 Setup

Our setup follows Doraszelski & Li (2025) (henceforth, DL), except that we generalize the law of motion for productivity to  $\omega_{it} = E[\omega_{it} | \omega_{it-1}, \delta_{it-1}] + \xi_{it} = g(\omega_{it-1}, \delta_{it-1}) + \xi_{it}$ . For concreteness we think of  $\delta_{it}$  as latent demand shocks. Firm  $i$  in period  $t$  uses inputs  $k_{it}$  and  $v_{it}$  to produce output  $q_{it}$  according to the production function  $q_{it} = f(k_{it}, v_{it}) + \omega_{it} + \varepsilon_{it}$ , where lower case letters denote logs. Capital  $k_{it}$  is a predetermined input that is chosen in period  $t - 1$  whereas  $v_{it}$  is freely variable and decided on in period  $t$  after the firm observes  $\omega_{it}$  and  $\delta_{it}$ . The disturbance  $\varepsilon_{it}$  sits between the firm’s output  $q_{it}$  as recorded in the data and the output  $q_{it}^* = q_{it} - \varepsilon_{it} = f(k_{it}, v_{it}) + \omega_{it}$  that the firm planned on when it decided on the variable input  $v_{it}$ . It can be interpreted alternatively as measurement error or as the untransmitted component of productivity.

We assume  $E[\varepsilon_{it} | x_{it}] = 0$  for observables  $x_{it} = (k_{it}, v_{it}, \dots)$  and  $E[\xi_{it} + \varepsilon_{it} | z_{it}] = 0$  for instruments  $z_{it} = (k_{it}, k_{it-1}, v_{it-1}, \dots)$ . The invertibility assumption in OP ensures  $E[\omega_{it-1} | x_{it-1}] =$

$\omega_{it-1}$  or, equivalently,  $E[q_{it-1}|x_{it-1}] = f(k_{it-1}, v_{it-1}) + \omega_{it-1} = q_{it-1}^*$ . The timing and Markov process assumptions in OP further ensure  $E[\xi_{it} + \varepsilon_{it}|z_{it}, \omega_{it-1}] = 0$  and thus  $E[\xi_{it} + \varepsilon_{it}|z_{it}, E[q_{it-1}|x_{it-1}]] = 0$ . This, in turn, ensures that  $E[q_{it-1}|x_{it-1}]$  can be used as an additional instrument. We refer the reader to DL, OP, James Levinsohn & Amil Petrin (2003) (henceforth, LP), and Daniel A. Ackerberg, Kevin Caves & Garth Frazer (2015) (henceforth, ACF) for a justification of these assumptions and further details on the setup.

**Notation.** We write  $z_{it} \setminus z_{it}^\dagger$  for the elements of the vector  $z_{it}$  that are not contained in the vector  $z_{it}^\dagger$ . To avoid clutter, all equalities involving random variables and conditional expectations are understood to hold almost surely. Proofs are deferred to the Supplemental Appendix.

## 2 Results

The proxy variable approach to production function estimation following OP can be seen as a control function approach. In particular, the control function  $h(k_{it-1}, v_{it-1}, E[q_{it-1}|x_{it-1}]) = g(E[q_{it-1}|x_{it-1}] - f(k_{it-1}, v_{it-1}))$  ensures that the moment condition

$$E[\omega_{it} + \varepsilon_{it} - h(k_{it-1}, v_{it-1}, E[q_{it-1}|x_{it-1}])|z_{it}, E[q_{it-1}|x_{it-1}]] = 0, \quad (1)$$

where  $\omega_{it} + \varepsilon_{it} = q_{it} - f(k_{it}, v_{it})$ , holds at the true production function.

Moment condition (1) follows because the law of motion for productivity is  $\omega_{it} = g(\omega_{it-1}) + \xi_{it}$  in the OP/LP/ACF framework and the invertibility assumption ensures  $E[q_{it-1}|x_{it-1}] = f(k_{it-1}, v_{it-1}) + \omega_{it-1}$ . Hence, we have

$$\begin{aligned} & E[\omega_{it} + \varepsilon_{it} - g(E[q_{it-1}|x_{it-1}] - f(k_{it-1}, v_{it-1}))|z_{it}, E[q_{it-1}|x_{it-1}]] \\ &= E[\omega_{it} + \varepsilon_{it} - g(\omega_{it-1})|z_{it}, E[q_{it-1}|x_{it-1}]] = E[\xi_{it} + \varepsilon_{it}|z_{it}, E[q_{it-1}|x_{it-1}]] = 0. \end{aligned}$$

Following DL, moment condition (1) can alternatively be derived without invertibility if we choose  $x_{it-1} = z_{it}$ . Without invertibility, however, the control function  $h(k_{it-1}, v_{it-1}, E[q_{it-1}|x_{it-1}])$  generally no longer equals the law of motion  $g(E[q_{it-1}|x_{it-1}] - f(k_{it-1}, v_{it-1}))$ .

As our notation for the control function  $h(k_{it-1}, v_{it-1}, E[q_{it-1}|x_{it-1}])$  emphasizes, moment condition (1) requires that given  $(k_{it-1}, v_{it-1}, E[q_{it-1}|x_{it-1}])$  all other variables in  $z_{it}$  are uncorrelated with the sum of the transmitted and untransmitted components of productivity  $\omega_{it} + \varepsilon_{it}$ . Moment condition (1) is less demanding than requiring that  $E[\omega_{it} + \varepsilon_{it}|z_{it}] = 0$  for instruments  $z_{it}$  in a direct instrumental variables approach to production function estimation. It is widely understood that finding instruments in the latter approach is difficult, or perhaps even impossible, in practice (Zvi Griliches & Jacques Mairesse 1998). A key insight of OP is that all that remains of current productivity  $\omega_{it}$  after controlling for lagged productivity  $\omega_{it-1}$  via the control function  $h(k_{it-1}, v_{it-1}, E[q_{it-1}|x_{it-1}])$  is the productivity innovation  $\xi_{it}$ . Because  $\xi_{it}$  is an independent shock, this facilitates finding instruments.

Moment condition (1) rules out some empirically relevant cases, including the joint evolution of productivity  $\omega_{it}$  and latent demand shocks  $\delta_{it}$ .<sup>1</sup> To accommodate these cases and  $\omega_{it} = g(\omega_{it-1}, \delta_{it-1}) + \xi_{it}$  as the law of motion for productivity, we generalize the control function approach in OP. To this end, we choose the *special instrument*  $z_{it}^\dagger \subseteq z_{it}$  and replace moment condition (1) with the following assumption:

**Assumption 1.**  $E\left[\omega_{it} + \varepsilon_{it} - h\left(z_{it} \setminus z_{it}^\dagger, E[q_{it-1}|x_{it-1}]\right) \middle| z_{it}, E[q_{it-1}|x_{it-1}]\right] = 0$  for the control function  $h\left(z_{it} \setminus z_{it}^\dagger, E[q_{it-1}|x_{it-1}]\right) = E\left[\omega_{it} + \varepsilon_{it} | z_{it} \setminus z_{it}^\dagger, E[q_{it-1}|x_{it-1}]\right]$ .

The special instrument  $z_{it}^\dagger$  can encompass one or several components of the instruments  $z_{it}$ . Assumption 1 requires that given  $(z_{it} \setminus z_{it}^\dagger, E[q_{it-1}|x_{it-1}])$ , all other variables in  $z_{it}$  are uncorrelated with the sum of the transmitted and untransmitted components of productivity  $\omega_{it} + \varepsilon_{it}$ . The restrictiveness of Assumption 1 depends on the choice of  $z_{it}^\dagger$ . If we choose  $z_{it}^\dagger = z_{it} \setminus (k_{it-1}, v_{it-1})$ , then Assumption 1 coincides with moment condition (1). If we instead choose  $z_{it}^\dagger \subsetneq z_{it} \setminus (k_{it-1}, v_{it-1})$ , then Assumption 1 is less demanding: as  $z_{it}^\dagger$  contains fewer variables, the conditioning set  $z_{it} \setminus z_{it}^\dagger$  contains more variables and thus controls for more of the variation in productivity.

As discussed above, Assumption 1 is satisfied in the OP/LP/ACF framework. More importantly,

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<sup>1</sup>DL show that invertibility can fail in the presence of latent demand shocks. This, in turn, invalidates current capital  $k_{it}$  as an instrument even if productivity  $\omega_{it}$  evolves as an AR(1) process separately from latent demand shocks  $\delta_{it}$ . DL show how to proceed without invertibility under the maintained assumption that  $\omega_{it}$  evolves separately from  $\delta_{it}$ .

it enables going beyond this framework by not relying on invertibility and by shifting the focus from the law of motion for productivity to specifying a model for the special instrument  $z_{it}^\dagger$ . As the researcher chooses the special instrument, one can leverage institutional features or auxiliary data to justify Assumption 1. Example 1 illustrates this point.

**Example 1** (Independent input price shocks). Assume that the price of the variable input  $p_{it}^V$  evolves according to  $p_{it}^V = \kappa(p_{it-1}^V, \eta_{it}, \tau_i)$  for some function  $\kappa$ , where  $\eta_{it}$  is a firm- and time-specific input price shock and  $\tau_i$  is a firm-specific shifter such as the type of inputs the firm uses. The shifter  $\tau_i$  may be correlated with  $\delta_{it-1}$  in the law of motion for productivity. This accommodates scenarios in which the type of inputs links productivity and demand.

Assume that  $\eta_{it}$  is an independent shock and therefore in particular independent of  $(\xi_{it}, \varepsilon_{it}, \varepsilon_{it-1})$  and of any variables known or chosen by firm in period  $t - 1$ . Because  $\omega_{it} = g(\omega_{it-1}, \delta_{it-1}) + \xi_{it}$ , it follows that

$$(\omega_{it} + \varepsilon_{it}, \omega_{it-1} + \varepsilon_{it-1}) \perp\!\!\!\perp \eta_{it} \mid k_{it}, k_{it-1}, v_{it-1}, (p_{it'}^V)_{t' < t}.$$

Assume further that the shifter  $\tau_i$  can be identified from  $(p_{it'}^V)_{t' < t}$ .<sup>2</sup> It follows that

$$(\omega_{it} + \varepsilon_{it}, \omega_{it-1} + \varepsilon_{it-1}) \perp\!\!\!\perp p_{it}^V \mid k_{it}, k_{it-1}, v_{it-1}, (p_{it'}^V)_{t' < t}.$$

Choosing  $z_{it} = (k_{it}, k_{it-1}, v_{it-1}, p_{it}^V, (p_{it'}^V)_{t' < t})$ ,  $x_{it-1} = z_{it}$ , and  $z_{it}^\dagger = p_{it}^V$ , we have

$$(\omega_{it} + \varepsilon_{it}, \omega_{it-1} + \varepsilon_{it-1}) \perp\!\!\!\perp z_{it}^\dagger \mid z_{it} \setminus z_{it}^\dagger. \quad (2)$$

Because  $\omega_{it-1} + \varepsilon_{it-1} \perp\!\!\!\perp z_{it}^\dagger \mid z_{it} \setminus z_{it}^\dagger$ , we know that  $q_{it-1} \perp\!\!\!\perp z_{it}^\dagger \mid z_{it} \setminus z_{it}^\dagger$  and thus that  $E[q_{it-1} \mid x_{it-1}]$  depends only on  $z_{it} \setminus z_{it}^\dagger$ . Equation (2) therefore implies

$$\omega_{it} + \varepsilon_{it} \perp\!\!\!\perp z_{it}^\dagger \mid z_{it} \setminus z_{it}^\dagger, E[q_{it-1} \mid x_{it-1}]$$

and Assumption 1 holds. Furthermore, because  $E[q_{it-1} \mid x_{it-1}]$  depends only on  $z_{it} \setminus z_{it}^\dagger$ , the control function simplifies from  $h(z_{it} \setminus z_{it}^\dagger, E[q_{it-1} \mid x_{it-1}])$  to  $h(z_{it} \setminus z_{it}^\dagger)$ . ■

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<sup>2</sup>Alternatively, assume that we have other information that pins down  $\tau_i$ . For example,  $\tau_i$  may be the location of the firm if there are regional differences in input markets or it may be the countries from which the firm imports inputs.

Assumption 1 ensures that the moment condition

$$E \left[ q_{it} - f(k_{it}, v_{it}) - h \left( z_{it} \setminus z_{it}^\dagger, E[q_{it-1}|x_{it-1}] \right) \middle| z_{it}, E[q_{it-1}|x_{it-1}] \right] = 0 \quad (3)$$

holds at the true production function.

We are interested in the output elasticity  $\frac{\partial f(k_{it}, v_{it})}{\partial v_{it}}$  of the variable input  $v_{it}$  as the key to estimating the markup. The choice of the special instrument  $z_{it}^\dagger$  entails a tradeoff. As noted above, as  $z_{it}^\dagger$  contains fewer variables, the conditioning set  $z_{it} \setminus z_{it}^\dagger$  contains more variables and thus controls for more of the variation in productivity. This leaves less exogenous variation in  $z_{it}^\dagger$  for identifying  $\frac{\partial f(k_{it}, v_{it})}{\partial v_{it}}$ . To ensure that  $z_{it}^\dagger$  has sufficient identification power, we make the following assumption:

**Assumption 2.** Let  $k_{it} \in z_{it} \setminus z_{it}^\dagger$ . Conditional on  $(z_{it} \setminus z_{it}^\dagger, E[q_{it-1}|x_{it-1}])$ ,  $z_{it}^\dagger$  is a complete instrument for  $v_{it}$ .

Assumption 2 places restrictions on the underlying economic model. In particular, it rules out the case where the law of motion for productivity is  $\omega_{it} = g(\omega_{it-1}) + \xi_{it}$  and the variable input demand  $v_{it} = \kappa(k_{it}, \omega_{it})$  is some function  $\kappa$  that depends only on capital and productivity and can be inverted for productivity.<sup>3</sup> In this case, for any choice of the special instrument  $z_{it}^\dagger \subseteq z_{it} \setminus (k_{it}, k_{it-1}, v_{it-1})$ ,  $z_{it}^\dagger$  is independent of  $\omega_{it}$  conditional on  $(z_{it} \setminus z_{it}^\dagger, E[q_{it-1}|x_{it-1}])$  because  $\omega_{it-1}$  is pinned down by  $(z_{it} \setminus z_{it}^\dagger, E[q_{it-1}|x_{it-1}])$  and the productivity innovation  $\xi_{it}$  is an independent shock. Consequently,  $z_{it}^\dagger$  cannot generate any variation in  $v_{it} = \kappa(k_{it}, \omega_{it})$  after controlling for  $(z_{it} \setminus z_{it}^\dagger, E[q_{it-1}|x_{it-1}])$ . Note that this lack-of-variation argument breaks down if latent demand shocks  $\delta_{it}$  enter the law of motion for productivity or the variable input demand. Hence, latent demand shocks  $\delta_{it}$  make room for Assumption 2 to hold.

Our main identification result is the following:

**Theorem 1.** Under Assumptions 1 and 2, moment condition (3) nonparametrically point-identifies  $\frac{\partial f(k_{it}, v_{it})}{\partial v_{it}}$ .

Turning from identification to estimation, for any weighting function  $\varphi(z_{it})$  of the instruments

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<sup>3</sup> Assumption 2 therefore also rules out the nonidentification result in Amit Gandhi, Salvador Navarro & David A. Rivers (2020).

$z_{it}$ , moment condition (3) implies that the moment condition

$$E \left[ \varphi(z_{it}) \left( q_{it} - f(k_{it}, v_{it}) - h \left( z_{it} \setminus z_{it}^\dagger, E[q_{it-1}|x_{it-1}] \right) \right) \right] = 0 \quad (4)$$

holds at the true production function. GMM estimation based on moment condition (4) can be conducted by viewing the control function  $h \left( z_{it} \setminus z_{it}^\dagger, E[q_{it-1}|x_{it-1}] \right)$  as a nuisance parameter and estimating it alongside the production function, as in the OP/LP/ACF framework.<sup>4</sup> However, the dimension of the control function may be high, especially if  $z_{it} \setminus z_{it}^\dagger$  comprises many variables. This increases the asymptotic variance of the estimates and can create numerical challenges for minimizing the GMM objective function.

Our main estimation result shows that these drawbacks can be avoided:

**Theorem 2.** *For any weighting function  $\varphi(z_{it})$  of the instruments  $z_{it}$ , moment condition (3) implies that the moment condition*

$$E \left[ (\varphi(z_{it}) - \tilde{\varphi}_{it}) \left( q_{it} - f(k_{it}, v_{it}) - (\tilde{q}_{it} - \tilde{f}_{it}) \right) \right] = 0, \quad (5)$$

where

$$\tilde{\varphi}_{it} = E \left[ \varphi(z_{it}) \middle| z_{it} \setminus z_{it}^\dagger, E[q_{it-1}|x_{it-1}] \right], \quad \tilde{q}_{it} - \tilde{f}_{it} = E \left[ q_{it} - f(k_{it}, v_{it}) \middle| z_{it} \setminus z_{it}^\dagger, E[q_{it-1}|x_{it-1}] \right],$$

holds at the true production function. Moreover, if a production function satisfies moment condition (5), then it satisfies moment condition (4) for the control function  $h \left( z_{it} \setminus z_{it}^\dagger, E[q_{it-1}|x_{it-1}] \right) = \tilde{q}_{it} - \tilde{f}_{it}$ . Finally, if  $\varphi(z_{it})$  and  $q_{it} - f(k_{it}, v_{it})$  have finite  $L^2$  norms, then moment condition (5) is Neyman orthogonal with respect to  $L^2$ -integrable perturbations of  $(\tilde{\varphi}_{it}, \tilde{q}_{it} - \tilde{f}_{it})$ .

Applying results from the double-debiased machine learning literature (Victor Chernozhukov, Denis Chetverikov, Mert Demirer, Esther Duflo, Christian Hansen, Whitney Newey & James Robins 2018), Neyman orthogonality ensures that the GMM estimator based on moment condition (5) is oracle efficient as long as the estimators for  $\tilde{\varphi}_{it}$ ,  $\tilde{q}_{it}$ , and  $\tilde{f}_{it}$  converge sufficiently fast. This

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<sup>4</sup>The law of motion for productivity cannot generally be recovered from estimates of the production and control functions.

means that the asymptotic distribution of the GMM estimator is *as if* the true values of  $\tilde{\varphi}_{it}$  and  $\tilde{q}_{it} - \tilde{f}_{it}$  are known.

A complication arises because estimating  $\tilde{\varphi}_{it}$  and  $\tilde{q}_{it} - \tilde{f}_{it}$  requires itself a plugin estimator for  $E[q_{it-1}|x_{it-1}]$ . To the best of our knowledge, the literature has not yet developed a treatment for such “double plugin” estimators. In what follows, we therefore focus on the case where the control function simplifies from  $h(z_{it} \setminus z_{it}^\dagger, E[q_{it-1}|x_{it-1}])$  to  $h(z_{it} \setminus z_{it}^\dagger)$  as in Example 1. Thus, the plugin estimator for  $E[q_{it-1}|x_{it-1}]$  is no longer required, and  $\tilde{\varphi}_{it} = E[\varphi(z_{it})|z_{it} \setminus z_{it}^\dagger]$  and  $\tilde{q}_{it} - \tilde{f}_{it} = E[q_{it} - f(k_{it}, v_{it})|z_{it} \setminus z_{it}^\dagger]$ .

While our generalized control function approach accommodates the joint evolution of productivity  $\omega_{it}$  and latent demand shocks  $\delta_{it}$ , it is not costless. If moment condition (1) holds, then for any choice of the special instrument  $z_{it}^\dagger \subsetneq z_{it} \setminus (k_{it-1}, v_{it-1})$  it is less efficient than the OP/LP/ACF procedure. This reflects a general tradeoff between robustness and efficiency in nonparametric estimation. Our generalized control function approach makes this tradeoff explicit by allowing a researcher to target greater robustness (smaller  $z_{it}^\dagger$ ) or greater efficiency (larger  $z_{it}^\dagger$ ).

### 3 Monte Carlo Exercise

**Data generating process.** Similar to DL, we specify the CES production function

$$f(k_{it}, v_{it}) = \frac{\nu}{\rho} \ln (\alpha \exp(\rho k_{it}) + (1 - \alpha) \exp(\rho v_{it}))$$

with  $\alpha = 0.3$ ,  $\rho = -1$ , and  $\nu = 0.95$ , the disturbance  $\varepsilon_{it} \sim N(0, 0.5^2)$ , and the CES demand  $q_{it}^* = \delta_{1it} - (1 + \exp(-\delta_{2it}))p_{it}$ , where  $p_{it}$  is the output price and  $\delta_{it} = (\delta_{1it}, \delta_{2it})$  captures shocks to the demand the firm faces and unobserved rivals.

Different from DL, the price of capital  $p_{it}^K$ , the price of the variable input  $p_{it}^V$ , and the latent demand shocks  $\delta_{1it}$  and  $\delta_{2it}$  follow Gaussian AR(1) processes. We parameterize these processes so that  $E[p_{it}^K] = E[p_{it}^V] = 0$ ,  $E[\delta_{1it}] = 10$ ,  $E[\delta_{2it}] = -1.3543$ ,  $\text{Var}(p_{it}^K) = \text{Var}(p_{it}^V) = \text{Var}(\delta_{2it}) = 0.5^2$ ,  $\text{Var}(\delta_{1it}) = 5^2$ , and the autocorrelation is 0.7. Short-run profit maximization implies the

markup  $\mu_{it} = \frac{P_{it}}{MC_{it}} = 1 + \exp(\delta_{2i})$ , where  $MC_{it}$  is marginal cost, and thus  $E[\ln \mu_{it}] = 0.25$  and  $\text{Var}(\ln \mu_{it}) = 0.0126$ .

We specify the law of motion for productivity

$$g(\omega_{it-1}, \delta_{it-1}) = \mu_\omega + \rho_\omega \omega_{it-1} + \rho_{\delta_1} \delta_{1it-1} + \rho_{\delta_2} \delta_{2it-1}$$

and the productivity innovation  $\xi_{it} \sim N(0, \sigma_\omega^2)$ . We parameterize  $\mu_\omega$ ,  $\rho_\omega$ ,  $\rho_{\delta_1}$ ,  $\rho_{\delta_2}$ , and  $\sigma_\omega^2$  so that  $E[\omega_{it}] = 0$ ,  $\text{Var}(\omega_{it}) = 0.5^2$ ,  $\text{corr}(\omega_{it}, \omega_{it-1}) = 0.7$ ,  $\text{corr}(\omega_{it}, \delta_{1it}) = 0.3$ , and  $\text{corr}(\omega_{it}, \delta_{2it}) = -0.3$ . This aligns with the notion that more productive firms participate in larger and more competitive markets.

We simulate  $S = 1,000$  datasets with  $N = 5,000$  firms and  $T = 20$  periods. We refer the reader to DL for further details on the data generating process.

**Estimation.** We use GMM estimation based on moment condition (5) to estimate the production function parameters  $\theta = (\alpha, \rho, \nu)$ . Our instruments are  $z_{it} = (k_{it}, k_{it-1}, v_{it-1}, p_{it-1}, p_{it}^V, p_{it-1}^V)$  and our special instrument is  $z_{it}^\dagger = p_{it}^V$  as in Example 1. Our weighting function  $\varphi(z_{it})$  is the complete set of Hermite polynomials of total degree 4 in the variables in  $z_{it}$ . We estimate  $\tilde{\varphi}_{it} = E[\varphi(z_{it})|z_{it} \setminus z_{it}^\dagger]$  by OLS using the complete set of Hermite polynomials of total degree  $d$  in the variables in  $z_{it} \setminus z_{it}^\dagger$ . We proceed similarly to estimate  $\tilde{q}_{it} - \tilde{f}_{it} = E[q_{it} - f(k_{it}, v_{it})|z_{it} \setminus z_{it}^\dagger]$ . The latter must be re-estimated at each iteration of the GMM problem. Even though the control function  $h(z_{it} \setminus z_{it}^\dagger) = E[\omega_{it} + \varepsilon_{it}|z_{it} \setminus z_{it}^\dagger]$  is absent from moment condition (5), the total degree  $d$  implicitly determines how well we can approximate it. We accordingly explore  $d \in \{2, 3, 4, 5\}$ . We provide further details on the GMM estimator in the Supplemental Appendix.

With an estimate of  $\theta$  in hand, we estimate the markup  $\mu_{it}$  of firm  $i$  in period  $t$  as

$$\ln \mu_{it} + \varepsilon_{it} = p_{it} + q_{it} - p_{it}^V - v_{it} + \ln \frac{\partial f(k_{it}, v_{it})}{\partial v_{it}}.$$

The right-hand side is the log of the output elasticity minus the log of the expenditure share of the variable input. Noting that the disturbance  $\varepsilon_{it}$  averages out as  $E[\varepsilon_{it}] = 0$ , we refer to the average of

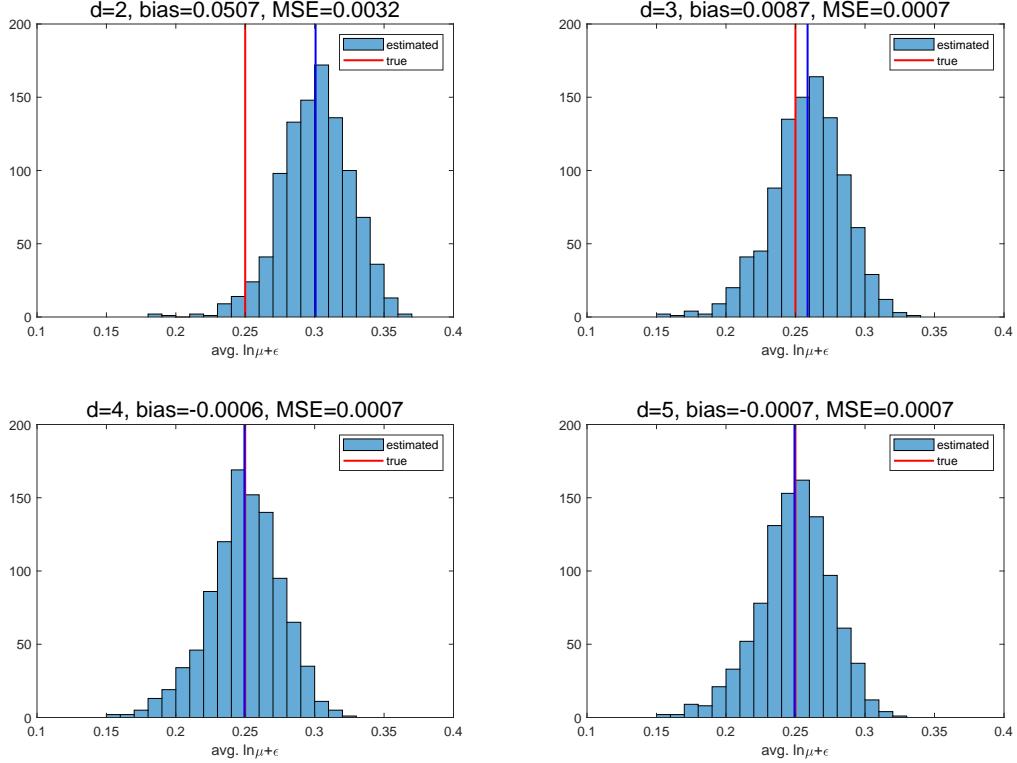


Figure 1: Distribution of average log markup for Hermite polynomials of total degree  $d \in \{2, 3, 4, 5\}$ .

$\ln \mu_{it} + \varepsilon_{it}$  across firms and time simply as the average log markup.

**Results.** As a baseline, we implement the modified OP/LP/ACF procedure described in DL by choosing  $x_{it-1} = z_{it}$  and explicitly including the first-order bias correction. We use a univariate Hermite polynomial of order 4 to approximate the law of motion for productivity. The bias in the average log markup is 0.1277 (compared to its true value of 0.25) and the mean squared error is 0.0164. The large bias is not surprising given that the joint evolution of productivity  $\omega_{it}$  and latent demand shocks  $\delta_{it}$  is outside the scope of the OP/LP/ACF procedure.

Turning to the generalized control function approach in this paper, Figure 1 shows the distribution of the average log markup. As can be seen, the results rapidly improve with the total degree  $d$  of the complete set of Hermite polynomials in the variables in  $z_{it} \setminus z_{it}^\dagger$ . The bias decreases from 0.0507 for  $d = 2$  to  $-0.0006$  for  $d = 4$  and the mean squared error from 0.0032 to 0.0007. There

are no further improvements going from  $d = 4$  to  $d = 5$ .

## 4 Concluding Remarks

We provide conditions for consistently estimating the production function in empirically relevant settings that are outside the scope of the OP/LP/ACF procedure. As such, our approach complements the OP/LP/ACF framework. Our approach generalizes the control function that is already present in OP and requires solving a straightforward GMM problem. We hope that it proves valuable for applied researchers seeking to estimate the production function and the markup from it.

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# Supplemental Appendix: A Generalized Control Function Approach to Production Function Estimation

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**Proof of Theorem 1.** Let  $f^0(k_{it}, v_{it})$  denote the true production function. Because  $q_{it} - f^0(k_{it}, v_{it}) = \omega_{it} + \varepsilon_{it}$ , Assumption 1 implies that  $f^0(k_{it}, v_{it})$  satisfies moment condition (3) for the control function  $h^0(z_{it} \setminus z_{it}^\dagger, E[q_{it-1}|x_{it-1}]) = E[q_{it} - f^0(k_{it}, v_{it}) | z_{it} \setminus z_{it}^\dagger, E[q_{it-1}|x_{it-1}]]$ . Let  $\tilde{f}(k_{it}, v_{it})$  be some production function that also satisfies moment condition (3) for some control function  $\tilde{h}(z_{it} \setminus z_{it}^\dagger, E[q_{it-1}|x_{it-1}])$ . To show that  $\frac{\partial f^0(k_{it}, v_{it})}{\partial v_{it}}$  is nonparametrically point-identified, we show that, for any  $k_{it}$ , the difference  $f^0(k_{it}, v_{it}) - \tilde{f}(k_{it}, v_{it})$  does not change with  $v_{it}$  almost surely.

Because both  $(f^0, h^0)$  and  $(\tilde{f}, \tilde{h})$  satisfy moment condition (3), taking the difference yields

$$\begin{aligned} & E \left[ f^0(k_{it}, v_{it}) - \tilde{f}(k_{it}, v_{it}) \middle| z_{it}, E[q_{it-1}|x_{it-1}] \right] \\ &= E \left[ \tilde{h}(z_{it} \setminus z_{it}^\dagger, E[q_{it-1}|x_{it-1}]) - h^0(z_{it} \setminus z_{it}^\dagger, E[q_{it-1}|x_{it-1}]) \middle| z_{it}, E[q_{it-1}|x_{it-1}] \right] \\ &= \tilde{h}(z_{it} \setminus z_{it}^\dagger, E[q_{it-1}|x_{it-1}]) - h^0(z_{it} \setminus z_{it}^\dagger, E[q_{it-1}|x_{it-1}]). \end{aligned}$$

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This implies the exclusion restriction

$$\begin{aligned} & \mathbb{E} \left[ f^0(k_{it}, v_{it}) - \tilde{f}(k_{it}, v_{it}) \middle| z_{it}, \mathbb{E}[q_{it-1}|x_{it-1}] \right] \\ = & \mathbb{E} \left[ f^0(k_{it}, v_{it}) - \tilde{f}(k_{it}, v_{it}) \middle| z_{it} \setminus z_{it}^\dagger, \mathbb{E}[q_{it-1}|x_{it-1}] \right]. \end{aligned}$$

Assumption 2 ensures that, for any function  $\kappa$ , the exclusion restriction

$$\begin{aligned} & \mathbb{E} \left[ \kappa \left( v_{it}, z_{it} \setminus z_{it}^\dagger, \mathbb{E}[q_{it-1}|x_{it-1}] \right) \middle| z_{it}, \mathbb{E}[q_{it-1}|x_{it-1}] \right] \\ = & \mathbb{E} \left[ \kappa \left( v_{it}, z_{it} \setminus z_{it}^\dagger, \mathbb{E}[q_{it-1}|x_{it-1}] \right) \middle| z_{it} \setminus z_{it}^\dagger, \mathbb{E}[q_{it-1}|x_{it-1}] \right] \end{aligned}$$

holds only if  $\kappa$  does not change with  $v_{it}$  almost surely. Recalling that  $k_{it} \in z_{it} \setminus z_{it}^\dagger$  and setting  $\kappa(v_{it}, z_{it} \setminus z_{it}^\dagger, \mathbb{E}[q_{it-1}|x_{it-1}]) = f^0(k_{it}, v_{it}) - \tilde{f}(k_{it}, v_{it})$  therefore establishes the result.

**Proof of Theorem 2.** We divide the proof into three parts.

We first prove that moment condition (3) implies moment condition (5). Recalling the definitions of  $\tilde{\varphi}_{it}$  and  $\tilde{q}_{it} - \tilde{f}_{it}$  in Theorem 2, moment condition (3) implies

$$\mathbb{E} \left[ q_{it} - f(k_{it}, v_{it}) - (\tilde{q}_{it} - \tilde{f}_{it}) \middle| z_{it}, \mathbb{E}[q_{it-1}|x_{it-1}] \right] = 0. \quad (6)$$

Because  $\varphi(z_{it}) - \tilde{\varphi}_{it}$  is pinned down by  $(z_{it}, \mathbb{E}[q_{it-1}|x_{it-1}])$ , we have

$$\begin{aligned} & \mathbb{E} \left[ (\varphi(z_{it}) - \tilde{\varphi}_{it}) \left( q_{it} - f(k_{it}, v_{it}) - (\tilde{q}_{it} - \tilde{f}_{it}) \right) \right] \\ = & \mathbb{E} \left[ (\varphi(z_{it}) - \tilde{\varphi}_{it}) \mathbb{E} \left[ q_{it} - f(k_{it}, v_{it}) - (\tilde{q}_{it} - \tilde{f}_{it}) \middle| z_{it}, \mathbb{E}[q_{it-1}|x_{it-1}] \right] \right] \\ = & 0, \end{aligned}$$

where the first equality is due to the law of iterated expectations and the second equality to equation (6).

Next, we prove that moment condition (5) implies moment condition (4) with  $h(z_{it} \setminus z_{it}^\dagger, \mathbb{E}[q_{it-1}|x_{it-1}]) =$

$\tilde{q}_{it} - \tilde{f}_{it}$ . Note that

$$\begin{aligned} & \mathbb{E} \left[ \tilde{\varphi}_{it} \left( q_{it} - f(k_{it}, v_{it}) - (\tilde{q}_{it} - \tilde{f}_{it}) \right) \right] \\ = & \mathbb{E} \left[ \tilde{\varphi}_{it} \mathbb{E} \left[ q_{it} - f(k_{it}, v_{it}) - (\tilde{q}_{it} - \tilde{f}_{it}) \mid z_{it} \setminus z_{it}^\dagger, \mathbb{E}[q_{it-1}|x_{it-1}] \right] \right] \\ = & 0, \end{aligned} \tag{7}$$

where the first equality is due to the law of iterated expectations and the second equality to  $\tilde{q}_{it} - \tilde{f}_{it} = \mathbb{E}[q_{it} - f(k_{it}, v_{it}) | z_{it} \setminus z_{it}^\dagger, \mathbb{E}[q_{it-1}|x_{it-1}]] = 0$  by the definition of  $\tilde{q}_{it} - \tilde{f}_{it}$ . Adding moment condition (5) and moment condition (7) implies moment condition (4) with  $h(z_{it} \setminus z_{it}^\dagger, \mathbb{E}[q_{it-1}|x_{it-1}]) = \tilde{q}_{it} - \tilde{f}_{it}$ .

Finally, we prove that moment condition (5) is Neyman orthogonal. Define the shorthand  $\tilde{z}_{it} = (z_{it} \setminus z_{it}^\dagger, \mathbb{E}[q_{it-1}|x_{it-1}])$ . Define  $\mathcal{F}$  to be the set of function tuples  $(\eta, \zeta)$  such that  $\mathbb{E}[\eta^2(\tilde{z}_{it})] < \infty$  and  $\mathbb{E}[\zeta^2(\tilde{z}_{it})] < \infty$ . Because  $\varphi(z_{it})$  and  $q_{it} - f(k_{it}, v_{it})$  have finite  $L^2$  norms, for any  $(\eta, \zeta) \in \mathcal{F}$  and for any  $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2$ , we have

$$\mathbb{E} \left| (\varphi(z_{it}) - \tilde{\varphi}_{it} - \lambda_1 \eta(\tilde{z}_{it})) \left( q_{it} - f(k_{it}, v_{it}) - (\tilde{q}_{it} - \tilde{f}_{it} + \lambda_2 \zeta(\tilde{z}_{it})) \right) \right| < \infty.$$

Moment condition (5) is therefore well-defined for any  $L^2$ -integrable perturbation of  $(\tilde{\varphi}_{it}, \tilde{q}_{it} - \tilde{f}_{it})$ .

Next, note that

$$\begin{aligned} & \frac{\partial \mathbb{E}(\varphi(z_{it}) - \tilde{\varphi}_{it} - \lambda_1 \eta(\tilde{z}_{it}))(q_{it} - f(k_{it}, v_{it}) - (\tilde{q}_{it} - \tilde{f}_{it} + \lambda_2 \zeta(\tilde{z}_{it})))}{\partial \lambda} \Big|_{\lambda=0} \\ = & \begin{pmatrix} \mathbb{E} \left[ \eta(\tilde{z}_{it}) \left( q_{it} - f(k_{it}, v_{it}) - (\tilde{q}_{it} - \tilde{f}_{it}) \right) \right] \\ - \mathbb{E}[(\varphi(z_{it}) - \tilde{\varphi}_{it}) \zeta(\tilde{z}_{it})] \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \end{aligned}$$

where the second equality is due to the law of iterated expectations and the definitions of  $\tilde{\varphi}_{it}$  and  $\tilde{q}_{it} - \tilde{f}_{it}$ .

**GMM estimator.** Corresponding to moment condition (5), define the moment function

$$m_{it}(\theta) = q_{it} - f(k_{it}, v_{it}; \theta) - (\tilde{q}_{it} - \tilde{f}_{it}(\theta)), \tag{8}$$

where we make the parameterization of the production function explicit.

We solve the GMM problem

$$\min_{\theta} \left( \frac{1}{NT} \sum_{i,t} (\varphi(z_{it}) - \tilde{\varphi}_{it}) m_{it}(\theta) \right)^{\top} W \left( \frac{1}{NT} \sum_{i,t} (\varphi(z_{it}) - \tilde{\varphi}_{it}) m_{it}(\theta) \right),$$

where the superscript  $\top$  denotes the transpose. We use the weighting matrix

$$W = \left( \frac{1}{NT-1} \sum_{i,t} ((\varphi(z_{it}) - \tilde{\varphi}_{it}) m_{it}(\theta^0) - \hat{\mu})^{\top} ((\varphi(z_{it}) - \tilde{\varphi}_{it}) m_{it}(\theta^0) - \hat{\mu}) \right)^{-1},$$

where

$$\hat{\mu} = \frac{1}{NT} \sum_{i,t} (\varphi(z_{it}) - \tilde{\varphi}_{it}) m_{it}(\theta^0)$$

and  $\theta^0$  denotes the true parameters.

Recall that  $\tilde{\varphi}_{it} = E[\varphi(z_{it})|z_{it} \setminus z_{it}^{\dagger}]$ . To the extent that terms in  $\varphi(z_{it})$  can be perfectly predicted by the complete set of Hermite polynomials of total degree  $d$  in the variables in  $z_{it} \setminus z_{it}^{\dagger}$ , the matrix  $\Phi$  with rows  $\varphi(z_{it}) - \tilde{\varphi}_{it}$  is rank deficient. In solving the GMM problem, we therefore use a selection of columns from the matrix  $\Phi$  that has full rank. To construct this selection, we start with an empty matrix and keep adding columns from the matrix  $\Phi$  as long as this increases the rank.