Econometrics - Problem Sets

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Problem Set 1

Question 1

$$\mathrm{E}(z_i u_i) = \mathrm{E}_z(\mathrm{E}(z_i u_i|z_i)) = \mathrm{E}_z(z_i\,\mathrm{E}(u_i|z_i)) = 0$$

Question 2

Given that we know that $\mathrm{E}(u_i|x_i) = 0$, we can use the implication of this fact, which is:

$$\mathrm{E}(u_i|x_i) = 0 \Longrightarrow \mathrm{E}(x_i u_i) = 0$$

Hence, we can simply input the regression errors and retrieve the moment equations:

$$\mathrm{E}(x_i(y_i - \beta log(x_i)) = 0$$

The sample analog is given by:

$$\overline{g}_n(\beta) = n^{-1}\sum_{i=1}^{n} g_i(\beta) = n^{-1}\sum_{i=1}^{n} x_i(y_i - \beta log(x_i)) = 0$$

We can solve the sample analog equation for β:

$$\hat{\beta}_{GMM} = \frac{\sum_{i=1}^{n} x_i w_i y_i}{\sum_{i=1}^{n}(x_i w_i log(x_i))}$$

for some weighting matrix $\mathbf{W}$

Question 3

Let $f(x_i) = \frac{dg(x_i,\beta)}{d\beta}$, where f is continuous at x_i

Since we are assuming that $\mathrm{E}[u|x] = 0$, we know that for any continuous function f, $\mathrm{E}[f(x)u] =$ is valid.

Then, we have the following moment condition:

$$\mathrm{E}[f(x_i)(y_i - g(x_i, \beta)] = 0$$

With the sample analog:

$$n^{-1} \sum_{i=1}^{n} f(x_i)(y_i - g(x_i, \beta)) = 0$$

For the problem:

$$\min_{\beta} \quad RSS(\beta) = n^{-1} \sum_{i=1}^{n} (y_i - g(x_i, \beta))^2$$

The first order condition of this minimization problem is:

$$\frac{dRSS(\beta)}{\beta} = \frac{-2}{n} \sum_{i=1}^{n} (y_i - g(x_i, \beta)) \frac{dg(.)}{d\beta} = 0$$

Rearranging the terms of the FOC results in the same sample analog of our assumed moment equation.

Question 4

Assume $y_i = x_i\beta + u_i$ and $\mathrm{E}(u_i|x_i) = 0$. From earlier statements, we know that $\mathrm{E}(x_i u_i) = 0$. Hence, the sample analog of the moments equation is given by:

$$\overline{g}_n(\beta) = n^{-1} \sum_{i=1}^{n} g_i(\beta) = n^{-1} \sum_{i=1}^{n} x_i(y_i - \beta x_i) = 0$$

The GMM estimator minimizes:

$$\hat{\beta}_{GMM} \in \min_{\beta} \quad n\overline{g}_n(\beta)'\overline{g}_n(\beta)$$

We can rewrite it as:

$$\min_{\beta} \quad n(X'y - X'X\beta)'W(X'y - X'X\beta)$$

$$\begin{aligned} Q_N(\beta) &= (X'y - X'X\beta)'W(X'y - X'X\beta) \\ &= (y'X - \beta'X'X)[WX'y - WX'X\beta] \\ &= y'XWX'y - y'XWX'X\beta - \beta'X'XWX'y + \beta'X'XWX'X\beta \end{aligned}$$

The first condition order yields:

$$= -2X'XWX'y + 2X'XWX'X\beta = 0$$

Hence, the GMM estimator is:

$$\hat{\beta}_{GMM} = (X'XWX'X)^{-1}(X'XWX'y)$$

Notice that if $W = (X'X)^{-1}$, then:

$$\hat{\beta}_{GMM} = \hat{\beta}_{OLS}$$

And given that $\mathrm{E}[X'u] = 0$, we know that the OLS estimator is consistent, then the GMM estimator is also consistent.

But, we have to proof the consistency for any positive definite matrix. Thus

$$\begin{aligned}
\hat{\beta}_{GMM} &= (X'XWX'X)^{-1}(X'XWX'y) \\
&= (X'XWX'X)^{-1}(X'XWX'(X\beta + u)) \\
&= (X'XWX'X)^{-1}(X'XWX'X\beta) + (X'XWXX')^{-1}(X'XWX'u) \\
&= \beta + (X'XWX'X)^{-1}(X'XWX'u) \\
&= \beta + (n^{-1}X'XWX'X)^{-1}n^{-1}(X'XWX'u)
\end{aligned}$$

By WLLN:

$$n^{-1}X'u \xrightarrow{p} \mathrm{E}[X'u]$$

Hence:

$$\hat{\beta}_{GMM} \xrightarrow{p} \beta$$

Question 5

Can an estimator be unbiased and inconsistent? Find an example or argue verbally? Short answer: **Yes.**

Example: Take a i.i.d sample $\{X_1, X_2, ..., X_n\}$. Now use $T(X) = X_1$ as an estimator for the mean $\mathrm{E}[X] = \mu$. It is unbiased, but not consistent.

Problem Set 2

Question 1

We know that the function **a** is of the order $\mathcal{O}(h^k)$ if and only if:

$$\frac{a(h)}{h^k} \longrightarrow t, t \in \mathbb{R}, h \longrightarrow 0$$

Thus, take $h^* = (\frac{1}{n})^{0.2}$
Now we have:

$$\sqrt{nh^*} = \sqrt{n(\frac{1}{n})^{0.2}} = \sqrt{\frac{n}{n^{0.2}}} = \sqrt{n^{0.8}} = (n^{0.8})^{0.5} = n^{0.4}$$

Hence, $\sqrt{nh^*}$ is of order $\mathcal{O}(n^{0.4})$

Question 2

We know that the function a is said to be $o(h^k)$ if and only if:

$$\frac{a(h)}{h^k} \longrightarrow t, t \in \mathbb{R}, h \longrightarrow 0$$

i.e, $a(h)$ goes to zero "way faster" than h^k.
Then given the functions f and g of order $o(n^2)$, $o(n)$, respectively, we have:

$$\frac{f(n)}{n^2} \longrightarrow t, t \in \mathbb{R}$$

$$\frac{g(n)}{n} \longrightarrow t, t \in \mathbb{R}$$

Thus, we have:

$$\frac{f(n)}{n^2}\frac{g(n)}{n} = \frac{f(n)g(n)}{n^3} \longrightarrow t, t \in \mathbb{R}$$

Hence, fg is of order $o(n^3)$.

Question 3

First, we define the Integrated Squared Error (ISE):

$$ISE(h) = \int (\hat{f}(x_0) - f(x_0))^2 dx_0$$

The Mean Integrated Squared Error (MISE) is:

$$MISE(h) = E\left(\int (\hat{f}(x_0) - f(x_0))^2 dx_0\right) = \int MSE(\hat{f}(x_0))dx_0$$

Rewriting the MISE, we have:

$$MISE(h) = \int MSE(\hat{f}(x_0))dx_0 = \int b(x_0)^2 + Var(\hat{f}(x_0))dx_0$$

The bias term is:

$$b(x_0) = \frac{1}{2}h^2 f''(x_0) \int z^2 K(z)dz + \mathcal{O}^3$$

And the variance is:

$$Var(\hat{f}(x_0)) = \frac{1}{nh} f(x_0) \int K(z)^2 dz + o(\frac{1}{nh})$$

By disregarding the asymptotic terms, we can rewrite the MISE as:

$$MISE(h) = \int (\frac{1}{2} f''(x_0) \int z^2 K(z)dz)^2 + \frac{1}{nh} f(x_0) \int K(z)^2 dz$$

$$MISE(h) = \int \frac{1}{4}h^4 f''(x_0)^2 \left(\int z^2 K(z)dz\right)^2 dx_0 + \int \frac{1}{nh} f(x_0) \int K(z)^2 dz dx_0$$

$$MISE(h) = \int \frac{1}{4}h^4 f''(x_0)^2 \left(\int z^2 K(z)dz\right)^2 dx_0 + \frac{1}{nh} \int f(x_0)dx_0 \int K(z)^2 dz$$

Since $\int f(x_0)dx_0 = 1$, then:

$$MISE(h) = \int \frac{1}{4}h^4 f''(x_0)^2 \left(\int z^2 K(z)dz\right)^2 dx_0 + \frac{1}{nh} \int K(z)^2 dz$$

$$MISE(h) = \frac{1}{4}h^4 \left(\int f''(x_0)^2 dx_0\right) \left(\int z^2 K(z)dz\right)^2 + \frac{1}{nh} \int K(z)^2 dz$$

The globally optimal bandwidth h^* minimizes MISE(h). Then the FOC is:

$$h^3 \left(\int f''(x_0)^2 dx_0\right) \left(\int z^2 K(z)dz\right)^2 - \frac{1}{nh^2} \int K(z)^2 dz = 0$$

$$h^5 \left(\int f''(x_0)^2 dx_0\right) \left(\int z^2 K(z)dz\right)^2 = \left(\frac{1}{n} \int K(z)^2 dz\right)$$

$$h^5 = n^{-1} \left(\int f''(x_0)^2 dx_0\right)^{-1} \left(\int z^2 K(z)dz\right)^{-2} \left(\int K(z)^2 dz\right)$$

$$h^* = n^{-0.2} \delta \left(\int f''(x_0)^2 dx_0\right)^{-0.2}$$

where δ is as it is defined in Cameron et Trived (2005).

Question 4

First, we can rewrite the cross-validation function:

$$CV = \frac{1}{n}\sum(y_i - \bar{m}_{-i}(x_i, h))^2$$

$$= \frac{1}{n}\sum(y_i - \bar{m}_{-i}(x_i, h) + m(x_i) - m(x_i))^2$$

$$= \frac{1}{n}\sum(\varepsilon_i - \bar{m}_{-i}(x_i, h) + m(x_i))^2$$

$$= \frac{1}{n}\sum(\varepsilon_i^2 + 2(m(x_i) - \bar{m}_{-i}(x_i, h))\varepsilon_i + (m(x_i) - \bar{m}_{-i}(x_i, h))^2)$$

Taking the expectation is:

$$\mathbb{E}_{\mathbf{x},\varepsilon}(CV) = \mathbb{E}_{\mathbf{x},\varepsilon}[\frac{1}{n}\sum(\varepsilon_i^2 + 2(m(x_i) - \bar{m}_{-i}(x_i, h))\varepsilon_i + (m(x_i) - \bar{m}_{-i}(x_i, h))^2)]$$

Since we are assuming the data is i.i.d, we have:

$$= \frac{1}{n}\sum \mathbb{E}_{\mathbf{x},\varepsilon}[(\varepsilon_i^2 + 2(m(x_i) - \bar{m}_{-i}(x_i, h))\varepsilon_i + (m(x_i) - \bar{m}_{-i}(x_i, h))^2)]$$

Redistributing:

$$= \mathbb{E}_{\mathbf{x},\varepsilon}[\varepsilon_i^2] + 2\mathbb{E}_{\mathbf{x},\varepsilon}[(m(x_i) - \bar{m}_{-i}(x_i, h))\varepsilon_i] + \mathbb{E}_{\mathbf{x},\varepsilon}[(m(x_i) - \bar{m}_{-i}(x_i, h))^2].$$

From previous discussions (statistics course), given the data is i.i.d, we have the following property:

$$2\mathbb{E}_{\mathbf{x},\varepsilon}[(m(x_i) - \bar{m}_{-i}(x_i, h))\varepsilon_i] = 0$$

Hence:

$$\mathbb{E}_{\mathbf{x},\varepsilon}[CV] = \mathbb{E}_{\mathbf{x},\varepsilon}[\varepsilon_i^2] + \mathbb{E}_{\mathbf{x},\varepsilon}[(m(x_i) - \bar{m}_{-i}(x_i, h))^2]$$

$$= \varepsilon^2 + \mathbb{E}_{\mathbf{x},\varepsilon}[((\bar{m}_{-i}(x_i, h) - m(x_i))^2]$$

This resembles a function of type $f(x) = a + bg(x)$. Thus we can minimize only in terms of the function $g(x)$.

Now we have to prove that minimizing our $g(.)$ is equivalent to minizing the Integrated Mean Squared Error (IMSE).