a) Primeiro, devemos notar que nunca teremos $C_t = 0$ ou $L_t = 0$, pois isso geraria utilidade $-\infty$. Além disso, a restrição orçamentária vale com igualdade. Se fosse estrita, aumentos de C_t seriam factíveis e gerariam mais utilidade.

Bom, considere primeiro o caso em que $L_0 > L_1$ e $L_0 > L_2$. O Lagrangeano é

$$\sum_{t=1}^{3} \beta^{t-1}\left[\alpha \log C_t + (1-\alpha)\log L_t\right] +$$

$$\lambda\left[-\sum_{t=1}^{3} \frac{1}{(1+r)^{t-1}} C_t + A_1 + \sum_{t=1}^{2} \frac{1}{(1+r)^{t-1}} w_t (L_0 - L_t)\right].$$

As CPOs são

$$\begin{cases} \dfrac{\alpha\beta^{t-1}}{C_t} = \dfrac{\lambda}{(1+r)^{t-1}}, & t = 1,2,3 \\[2ex] \dfrac{(1-\alpha)\beta^{t-1}}{L_t} = \dfrac{\lambda}{(1+r)^{t-1}} w_t, & t = 1,2 \end{cases}$$

A demanda frischiana é

$$C_t^F = \frac{\alpha}{\lambda} \cdot \left(\beta(1+r)\right)^{t-1}, \qquad L_t^F = w_t \frac{(1-\alpha)}{\lambda} \left(\beta(1+r)\right)^{t-1}$$

O valor ótimo de λ é

$$\sum_{t=1}^{3} \frac{1}{(1+r)^{t-1}} \frac{\alpha}{\lambda} \beta^{t-1} (1+r)^{t-1} = A_1 + \sum_{t=1}^{2} \frac{1}{(1+r)^{t-1}} w_t\left(L_0 - \frac{w_t (1-\alpha)\beta^{t-1}(1+r)^{t-1}}{\lambda}\right)$$

$$\Rightarrow \alpha \sum_{t=1}^{3} \beta^{t-1} = \lambda\left[A_1 + L_0 \sum_{t=1}^{2} \frac{w_t}{(1+r)^{t-1}}\right] - (1-\alpha)\sum_{t=1}^{2} w_t^2 \beta^{t-1}$$

$$\Rightarrow \lambda = \left[A_1 + L_0 \sum_{t=1}^{2} \frac{w_t}{(1+r)^{t-1}}\right]^{-1} \left[\alpha \sum_{t=1}^{3} \beta^{t-1} + (1-\alpha)\sum_{t=1}^{2} w_t^2 \beta^{t-1}\right]$$

As demandas marshallianas resultam da substituição de λ em C_t^F e L_t^F.

Agora, suponha $L_1 = L_2 = L_0$. Logo, a restrição vira $\sum_{t=1}^{3} \frac{1}{(1+r)^{t-1}} C_t \leq A_1$. O Lagrangeano é

$$\sum_{t=1}^{3} \beta^{t-1}\left[\alpha \log C_t + (1-\alpha)\log L_0\right] + \lambda\left[A_1 - \sum_{t=1}^{3} \frac{1}{(1+r)^{t-1}} C_t\right].$$

A demanda frischiana é a mesma, mas o λ será diferente. Neste caso,

$$A_1 = \frac{\alpha}{\lambda} \sum_{t=1}^{3} \beta^{t-1} \Rightarrow \lambda = \frac{\alpha}{A_1} \sum_{t=1}^{3} \beta^{t-1}$$

Agora, seja $L_i < L_0$ mas $L_j = L_0$.

Temos $\sum_{t=1}^{3} \beta^{t-1}\left[\alpha \log C_t + (1-\alpha)\log L_0\right] +$

$$\lambda\left[-\sum_{t=1}^{3} \frac{1}{(1+r)^{t-1}} C_t + A_1 + \frac{1}{(1+r)^{i-1}} w_i (L_0 - L_i)\right]$$ de Lagrangeano.

As demandas frischianas são as mesmas, mas o λ agora é

$$\frac{\alpha}{\lambda}\sum_{t=1}^{3} \beta^{t-1} = A_1 + \frac{1}{(1+r)^{i-1}} w_i L_0 - \frac{(1-\alpha)}{\lambda} w_i^2 \beta^{i-1}$$

$$\Rightarrow \left[\alpha \sum_{t=1}^{3} \beta^{t-1} + (1-\alpha) w_i^2 \beta^{i-1}\right] = \lambda\left[A_1 + \frac{1}{(1+r)^{i-1}} w_i L_0\right]$$

$$\Rightarrow \lambda = \left[\alpha \sum_{t=1}^{3} \beta^{t-1} + (1-\alpha) w_i^2 \beta^{i-1}\right]\left[A_1 + \frac{1}{(1+r)^{i-1}} w_i L_0\right]^{-1}$$

b) Bom, vamos ver caso a caso:

I) $L_1 < L_0$, $L_2 < L_0$.

$$\lambda = \left[A_1 + L_0 \sum_{t=1}^{2} \frac{w_t}{(1+r)^{t-1}}\right]^{-1} \left[\alpha \sum_{t=1}^{3} \beta^{t-1} + (1-\alpha)\sum_{t=1}^{2} w_t^2 \beta^{t-1}\right]$$

$$\frac{\partial \lambda}{\partial A_1} < 0$$

$$\frac{\partial \lambda}{\partial w_i} = \left[A_1 + L_0 \sum_{t=1}^{2} \frac{w_t}{(1+r)^{t-1}}\right]^{-1} (1-\alpha) 2 w_i \beta^{i-1}$$

$$- \left[\alpha \sum_{t=1}^{3} \beta^{t-1} + (1-\alpha)\sum_{t=1}^{2} w_t^2 \beta^{t-1}\right]\left[A_1 + L_0 \sum_{t=1}^{2} \frac{w_t}{(1+r)^{t-1}}\right]^{-2} \frac{L_0}{(1+r)^{i-1}}$$

II) $L_1 = L_2 = L_0$

$$\lambda = \frac{\alpha}{A_1} \sum_{t=1}^{3} \beta^{t-1}$$

$$\frac{\partial \lambda}{\partial A_1} < 0, \qquad \frac{\partial \lambda}{\partial w_i} = 0$$

III) $L_i < L_0$, $L_j = L_0$

$$\lambda = \left[\alpha \sum_{t=1}^{3} \beta^{t-1} + (1-\alpha) w_i^2 \beta^{i-1}\right]\left[A_1 + \frac{1}{(1+r)^{i-1}} w_i L_0\right]^{-1}$$

$$\frac{\partial \lambda}{\partial A_1} < 0$$

$$\frac{\partial \lambda}{\partial w_i} = \left[A_1 + \frac{1}{(1+r)^{i-1}} w_i L_0\right]^{-1} (1-\alpha) 2 w_i \beta^{i-1}$$

$$- \left[\alpha \sum_{t=1}^{3} \beta^{t-1} + (1-\alpha) w_i^2 \beta^{i-1}\right]\left[A_1 + \frac{1}{(1+r)^{i-1}} w_i L_0\right]^{-2} \frac{L_0}{(1+r)^{i-1}}$$

$$\frac{\partial \lambda}{\partial w_j} = 0.$$