Recap: Model Predictive Control (MPC)

- MPC solves a N-step lookahead problem (called **Optimal Control** problem) in a receding horizon fashion, in order to approximate the infinite-horizon DP:

Terminal Cost Approximation

$$
\begin{aligned}
J_0(\bar{x}_0) = \min_{X,U} \quad & \hat{J}_N(x_N) + \sum_{i=0}^{N-1} x_i^\top Q x_i + u_i^\top R u_i \\
\text{s.t.} \quad & x_{i+1} = Ax_i + Bu_i, \quad \forall i \in \{0, 1, 2, ...\} \\
& x_i \in \mathcal{X}, \quad \forall i \in \{0, 1, 2, ...\} \\
& u_i \in \mathcal{U}, \quad \forall i \in \{0, 1, 2, ...\} \\
& x_0 = \bar{x}_0 \\
& x_N \in \mathcal{X}_f
\end{aligned}
$$

Terminal set constraint

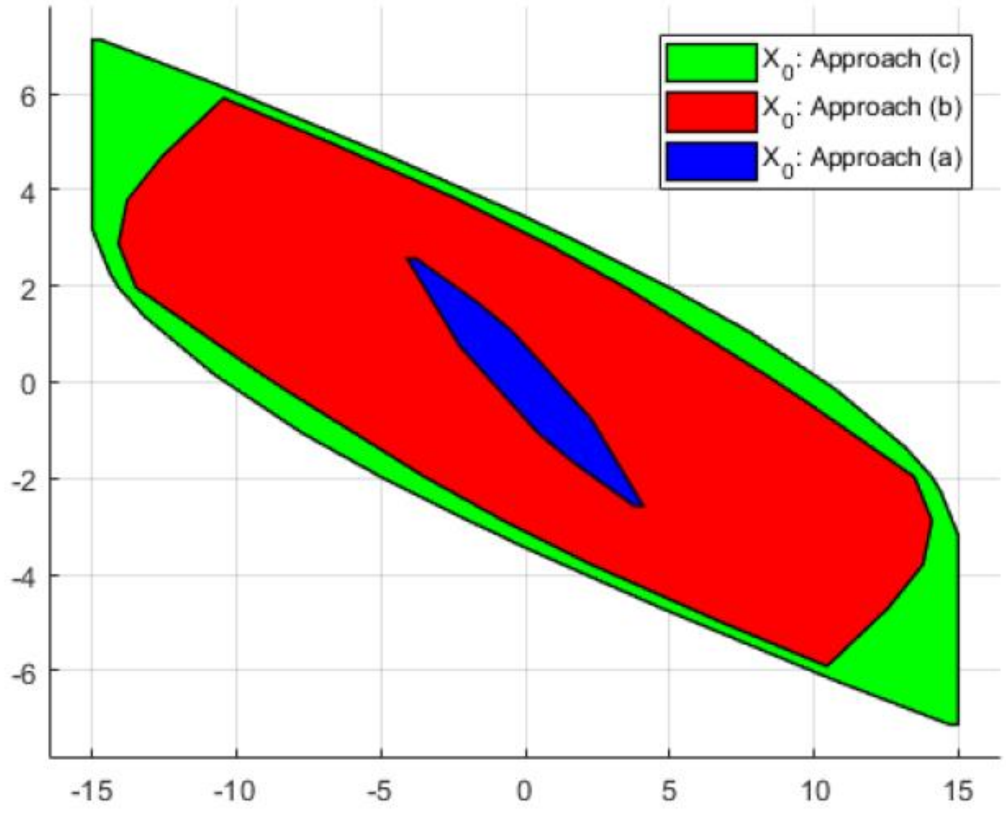
Recap: Model Predictive Control (MPC)

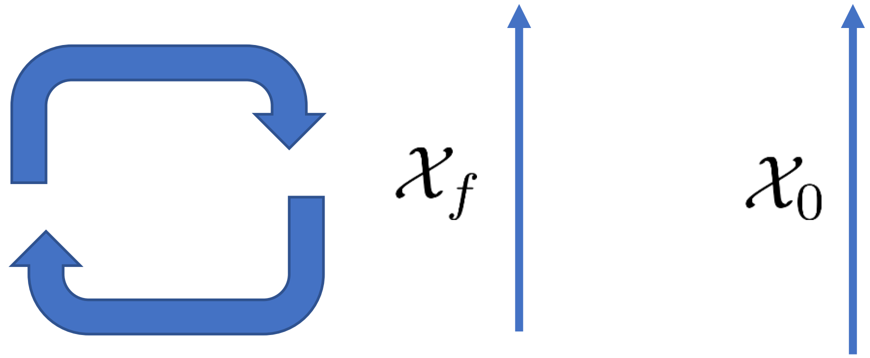
- In order to use MPC as our Approximation Algorithm, we need to show that the following two properties are satisfied:
  - **(1) Recursive Feasibility:** As we apply the MPC Algorithm, we want the Optimal Control problem to be feasible at every stage.
  - **(2) Asymptotic Stability:** As we apply the MPC Algorithm, we want to get closer and closer to the origin at the system evolves.
- Last time we showed that this is true if we let the terminal set be: $\mathcal{X}_f = 0$
- And if we start from x_0 picked from the following set:

$$\mathcal{X}_0 = \left\{ x \in \mathcal{X} : \exists (u_0, u_1, ..., u_{N-1}) \text{ s.t.: } \begin{cases} x_{k+1} = Ax_k + Bu_k, \forall k \in \{0, ..., N-1\} \\ x_k \in \mathcal{X}, \, u_k \in \mathcal{U}, \forall k \in \{0, ..., N-1\} \\ x_0 = x, x_N = 0 \end{cases} \right\}$$

Trade-off between terminal and initial sets

- There is a trade-off between both sets





MPC Algorithm for larger terminal sets

- In order to extend our results to larger terminal sets, we need a few definitions.

- We call a set $\mathcal{O} \subseteq \mathcal{X}$ **positively invariant** for any closed-loop systems $x_{k+1} = f(x_k)$ we have:

$$x_0 \in \mathcal{O} \Rightarrow x_k \in \mathcal{O},\ k \geq 1$$

- In words: "a set is invariant if once our systems starts from it, it stays in it forever."

- The invariant set that contains every positively invariant is called the maximal positively invariant set, and we called it $\mathcal{O}_\infty$.

Example: 2nd-order autonomous system

- Consider the following system:

$$x_{k+1} = Ax_k = \begin{bmatrix} 0.5 & 0 \\ 1.0 & -0.5 \end{bmatrix} x_k$$

- Subject to the following constraints:

$$x_k \in \mathcal{X} = \left\{ x \in \mathbb{R}^2 : \begin{bmatrix} -10 \\ -10 \end{bmatrix} \leq \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 10 \\ 10 \end{bmatrix} \right\}, \forall k \geq 0$$

- Let's find a set $P(\mathcal{X})$ that is the set of all points that if we start at $P(\mathcal{X})$ we do not leave $\mathcal{X}$:

$$P(\mathcal{X}) = \{ x \in \mathbb{R}^2 : Ax \in \mathcal{X} \}$$

Example: 2nd-order autonomous system

- To find P, let's re-write the constraints as follows:

$$x_k \in \mathcal{X} = \left\{ x \in \mathbb{R}^2 : \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} x \leq \begin{bmatrix} 10 \\ 10 \\ 10 \\ 10 \end{bmatrix} \right\}, \forall k \geq 0$$

- Or compactly:

$$\mathcal{X} = \left\{ x \in \mathbb{R}^2 : Hx \leq h \right\}$$

- So, we use linearity to find :

$$P(\mathcal{X}) = \{x \in \mathbb{R}^2 : Ax \in \mathcal{X}\} = \{x \in \mathbb{R}^2 : HAx \leq h\}$$

Example: 2^{nd}-order autonomous system

- Explicitly:

$$P(\mathcal{X}) = \left\{ x \in \mathbb{R}^2 : \begin{bmatrix} 0.5 & 0 \\ 1 & -0.5 \\ -0.5 & 0 \\ -1 & -0.5 \end{bmatrix} x \leq \begin{bmatrix} 10 \\ 10 \\ 10 \\ 10 \end{bmatrix} \right\}$$

- Now, let’s take the intersection:

$$P(\mathcal{X}) \cap \mathcal{X}$$

- Compactly:

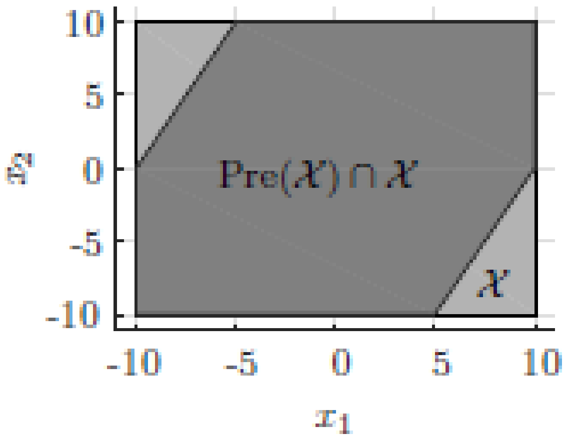
$$P(\mathcal{X}) \cap \mathcal{X} = \left\{ x \in \mathbb{R}^2 : HAx \leq h, Hx \leq h \right\}$$

Example: 2nd-order autonomous system

- Explicitly:

$$P(\mathcal{X}) \cap \mathcal{X} = \left\{ x \in \mathbb{R}^2 : \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & -0.5 \\ -1 & 0.5 \end{bmatrix} x \leq \begin{bmatrix} 10 \\ 10 \\ 10 \\ 10 \\ 10 \\ 10 \end{bmatrix} \right\}$$

- In a figure:



(figure taken from Borelli and Morari)

Example: 2nd-order autonomous system

- Now let's suppose we take $P(P(\mathcal{X}))$
- And we take the intersection again:

$$P(P(\mathcal{X})) \cap P(\mathcal{X})$$

- It turns out that we have the same set as before:

$$P(P(\mathcal{X})) \cap P(\mathcal{X}) = \left\{ x \in \mathbb{R}^2 : \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & -0.5 \\ -1 & 0.5 \end{bmatrix} x \leq \begin{bmatrix} 10 \\ 10 \\ 10 \\ 10 \\ 10 \\ 10 \end{bmatrix} \right\}$$

Example: 2^{nd}-order autonomous system

- So we reached a "convergence":

- If we start at:

$$\left\{x \in \mathbb{R}^2 : \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & -0.5 \\ -1 & 0.5 \end{bmatrix} x \leq \begin{bmatrix} 10 \\ 10 \\ 10 \\ 10 \\ 10 \\ 10 \end{bmatrix} \right\}$$

- We stay in it forever. Hence the above set is the invariance set:

$$\mathcal{O}_\infty = \left\{x \in \mathbb{R}^2 : \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & -0.5 \\ -1 & 0.5 \end{bmatrix} x \leq \begin{bmatrix} 10 \\ 10 \\ 10 \\ 10 \\ 10 \\ 10 \end{bmatrix} \right\}$$

MPC Algorithm for larger terminal sets

- Next we define a set $\mathcal{C} \subseteq \mathcal{X}$ to be the **control invariant** for a system $x_{k+1} = f(x_k, u_k)$ if:

$$x_0 \in \mathcal{C} \Rightarrow \exists u_k \in \mathcal{U} \text{ s.t.: } x_k \in \mathcal{C},\ k \geq 1$$

- In words: "a set is control invariant if once our systems starts from it, there is always a feasible control such that once applied the system inside the set."

- The control invariant set that contains every control invariant is called the maximal control invariant set, and we called it $\mathcal{C}_\infty$.

Example: 2^{nd}-order unstable system

- Consider the following unstable system:

$$x_{k+1} = Ax_k + Bu_k = \begin{bmatrix} 1.5 & 0 \\ 1.0 & -1.5 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_k$$

- Subject to the following constraints:

$$x_k \in \mathcal{X} = \left\{ x \in \mathbb{R}^2 : \begin{bmatrix} -10 \\ -10 \end{bmatrix} \leq \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 10 \\ 10 \end{bmatrix} \right\}, \forall k \geq 0$$

$$u_k \in \mathcal{U} = \left\{ u \in \mathbb{R} : -5 \leq u \leq 5 \right\}, \forall k \geq 0$$

Example: 2nd-order unstable system

- Again consider the Precursor set $P(\mathcal{X})$

$$P(\mathcal{X}) = \{x \in \mathbb{R}^2 : \ \exists u \in \mathcal{U} \text{ s.t.: } Ax + Bu \in \mathcal{X}\}$$

- Again let's rewrite the constraint sets:

$$x_k \in \mathcal{X} = \left\{ x \in \mathbb{R}^2 : \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} x \leq \begin{bmatrix} 10 \\ 10 \\ 10 \\ 10 \end{bmatrix} \right\} = \left\{ x \in \mathbb{R}^2 : Hx \leq h \right\}, \forall k \geq 0$$

$$u_k \in \mathcal{U} = \left\{ u \in \mathbb{R} : \begin{bmatrix} 1 \\ -1 \end{bmatrix} u \leq \begin{bmatrix} 5 \\ 5 \end{bmatrix} \right\} = \left\{ u \in \mathbb{R} : H_u u \leq h_u \right\}, \forall k \geq 0$$

Example: 2nd-order unstable system

- So we can write:

$$P(\mathcal{X}) = \left\{x \in \mathbb{R}^2 : \; \exists u \in \mathbb{R} \text{ s.t.: } \begin{bmatrix} HA & HB \\ 0 & H_u \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \leq \begin{bmatrix} h \\ h_u \end{bmatrix} \right\}$$

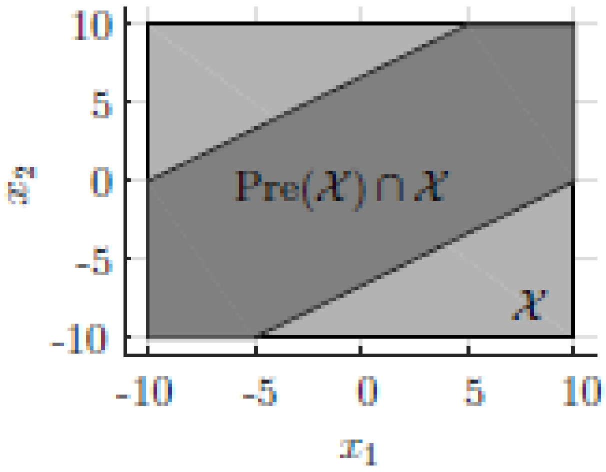
- This set is in fact a Projection. Now if we compute the intersection $P(\mathcal{X}) \cap \mathcal{X}$:

- We get:

$$P(\mathcal{X}) \cap \mathcal{X} = \left\{ x \in \mathbb{R}^2 : \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & -1.5 \\ -1 & 1.5 \end{bmatrix} x \leq \begin{bmatrix} 10 \\ 10 \\ 10 \\ 10 \\ 10 \\ 10 \end{bmatrix} \right\}$$

Example: 2nd-order unstable system

- In a figure we get:



- As before let's apply the Precursor to the Precursor $P(P(\mathcal{X}))$

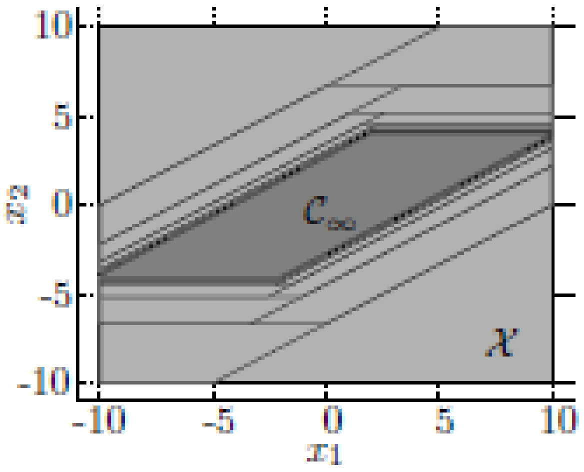
- It turns out that now, we do not converge in 1 step. In fact we converge after 45 iterations...

Example: 2nd-order unstable system

- Still, the end result is the maximal control invariant set $\mathcal{C}_\infty$:

$$C_\infty = \left\{ x \in \mathbb{R}^2 : \begin{bmatrix} 0 & 1 \\ 0 & -1 \\ 0.55 & -0.83 \\ -0.55 & 0.83 \\ 1 & 0 \\ -1 & 0 \end{bmatrix} x \leq \begin{bmatrix} 4 \\ 4 \\ 2.22 \\ 2.22 \\ 10 \\ 10 \end{bmatrix} \right\}$$

- In a figure:



MPC Algorithm for larger terminal sets

- Lastly recall our initial set $\mathcal{X}_0$

$$\mathcal{X}_0 = \left\{ x \in \mathcal{X} : \exists (u_0, u_1, ..., u_{N-1}) \text{ s.t.: } \begin{cases} x_{k+1} = Ax_k + Bu_k, \ \forall k \in \{0, ..., N-1\} \\ x_k \in \mathcal{X}, \ u_k \in \mathcal{U}, \ \forall k \in \{0, ..., N-1\} \\ x_0 = x, x_N \in \mathcal{X}_f \end{cases} \right\}$$

- Note that this set does not depend on the objective function of the Optimal Control Problem. But it depends on N and on $\mathcal{X}_f$.

- Now, consider the positively invariant set $\mathcal{O}_\infty$ associated with the closed-loop system that comes from following the policy:

$$\mu_{\text{MPC}}(\bar{x}_0) = u_0^*$$

MPC Algorithm for larger terminal sets

- By using our definitions of the three sets we can establish the following relations:

$$\mathcal{O}_\infty \subseteq \mathcal{X}_0$$

Since we start if we start in $\mathcal{O}_\infty$ we stay in it, so we are feasible forever, so we are also in $\mathcal{X}_0$

$$\mathcal{O}_\infty \subseteq \mathcal{C}_\infty$$

Since the MPC policy is only one out of many to control the system

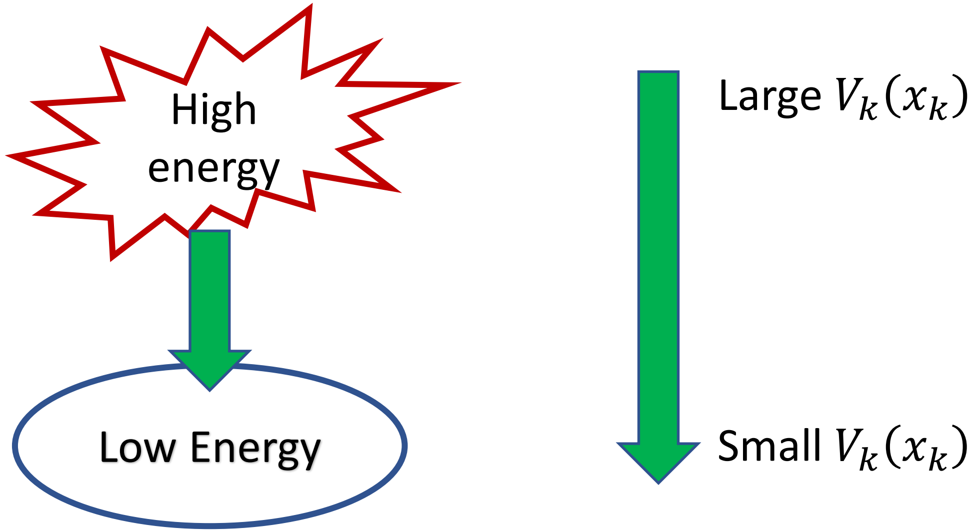
- What about: $\cancel{\mathcal{C}_\infty \subseteq \mathcal{X}_0}$ **?**
- In general there are no inclusion-type relations between $\mathcal{X}_0$ and $\mathcal{C}_\infty$.

MPC Algorithm for larger terminal sets

- Now let's introduce our last ingredient: **The Lyapunov Function**

- A time-varying function $V_k(x_k)$ is called a Lyapunov Function if the following hold:
  - $V_k(0) = 0$ and $V_k(x) > 0$ for all $x \neq 0$.
  - $\alpha_1(||x||) \leq V_k(x) \leq \alpha_2(||x||)$, where α_1, α_2 are strictly increasing functions
  - $V_k(x_{k+1}) - V_k(x_k) \leq 0$

- The intuition is that the Lyapunov Function captures the "energy" of the system:



Example: Linear Systems

- Consider the following linear system:

$$x_{k+1} = Ax_k$$

- Consider the following function:

$$V_k(x_k) = x_k^\top P x_k$$

- If $P \succ 0$ (positive definite) and solves:

$$A^\top PA - P \prec 0 \Rightarrow x^\top A^\top PAx - x^\top Px < 0, \forall x \neq 0$$

- Then $V_k(x_k)$ is a Lyapunov function.

Example: Linear Systems

- This is easy to verify: It is clear that

$$V_k(0) = 0$$

$$V_k(x_k) = x_k^\top P x_k > 0,\ \forall x_k \neq 0 \qquad \lambda_{\min}(P)||x_k||_2^2 \leq V_k(x_k) \leq \lambda_{\max}(P)||x_k||_2^2$$

- And that:

$$V_k(x_{k+1}) - V_k(x_k) = x_{k+1}^\top P x_{k+1} - x_k^\top P x_k = x_k^\top A^\top P A x_k - x_k^\top P x_k =$$
$$x_k(A^\top P A - P)x_k < 0$$

- Then the “energy” of the system decreases as the system evolves.

- Hence it follows that $||x_k|| \to 0$, as $k \to \infty$

Example: Linear Systems

- This example shows a nice property of linear systems. If we can find a positive matrix P such that:

$$A^\top P A - P \prec 0$$

- Then the system is asymptotically stable.
  - Note that if spectral radius of A is strictly less than unity, we can use P as the identity matrix and the result holds.

- This result can be generalized for any dynamical system $\mathrm{x}_{k+1} = f(x_k)$:

- If there exists a Lyapunov Function $V_k(x_k)$ where $V_k(x_{k+1}) - V_k(x_k) < 0$ then the system is asymptotically stable: $||x_k|| \to 0$, as $k \to \infty$.

Example: LQR problem

- This example that we have seen is the LQR problem:

$$J^*(\bar{x}_0) = \min_{X,U} \quad \sum_{i=0}^{\infty} x_i^\top Q x_i + u_i^\top R u_i$$
$$\text{s.t. } x_{i+1} = Ax_i + Bu_i, \quad \forall i \in \{0, 1, 2, ...\}$$
$$x_0 = \bar{x}_0$$

- For this problem, given the controllability and observability conditions, the solution to the above problem is given by the Algebraic Riccati Equation

$$\mu^*(x) = Kx, \qquad \begin{cases} K = -(B^\top K B + R)^{-1} B^\top P A \\ P = A^\top (P - PB(B^\top P B + R)^{-1} B^\top P) A + Q \end{cases}$$

- And the closed-loop system is:

$$x_{k+1} = Ax_k + Bu_k = (A + BK)x_k, \quad \forall k \in \{0, 1, 2, ...\}$$

Example: LQR problem

- And we saw that the optimal Value Function ("cost-to-go") is a quadratic function:

$$J_0^*(x_0) = x_0^\top P x_0$$

- The Value function $J^*(x_0)$ is a Lyapunov Function!

- Let's check the properties:

$$J_0^*(0) = 0 \qquad J_0^*(x) = x^\top P x > 0,\, \forall x \neq 0$$

$$\lambda_{\min}(P)||x||_2^2 \leq J_0^*(x) \leq \lambda_{\max}(P)||x||_2^2$$

Example: LQR problem

- And finally:

$$J_0^*(x_{k+1}) - J_0^*(x_k) = x_k^\top (A+BK)^\top P(A+BK)x_k - x_k^\top P x_k$$

- Omitting x_k

$$J_0^*(x_{k+1}) - J_0^*(x_k) = x_k^\top (A+BK)^\top P(A+BK)x_k - x_k^\top P x_k =$$

$$A^\top PA + K^\top B^\top PA + A^\top PBK + K^\top B^\top PBK - P =$$

$$A^\top PA + A^\top PBK + K^\top B^\top PA + K^\top (R + B^\top PB)K - K^\top RK - P =$$

$$A^\top PA + A^\top PBK + K^\top B^\top PA - K^\top B^\top PA - K^\top RK =$$

$$A^\top PA^\top + A^\top PBK - P - K^\top RK$$

- Now using the Riccati Equation: $P = Q + A^\top PA + A^\top PBK$:

$$A^\top PA^\top + A^\top PBK - P - K^\top RK = -Q - K^\top RK \prec 0$$

Establishing MPC Properties

- Now let's return to MPC, we will prove that two properties:
  - **<u>(1) Recursive Feasibility</u>**
  - **<u>(2) Asymptotic Stability</u>**

- Hold under the following set of assumptions:

- Stage costs are positive definite: strictly positive and only zero at the origin

- The terminal set $\mathcal{X}_f$ is a **<u>invariant set</u>** under some local control policy $v(x_k)$:

$$x_{k+1} = f(x_k, v(x_k)) \in \mathcal{X}_f, \, \forall x_k \in \mathcal{X}_f$$

$$\mathcal{X}_f \subseteq \mathcal{X}, \, v(x_k) \in \mathcal{U}, \, \forall x_k \in \mathcal{X}_f$$

Establishing MPC Properties

- The terminal cost approximation is **a Lyapunov Function** in the terminal set $\mathcal{X}_f$

$$\hat{J}_k(x_{k+1}) - \hat{J}_k(x_k) \leq 0,\ \forall x_k \in \mathcal{X}_f$$

- Without loss of generality, this is equivalent to:

$$\hat{J}_k(x_{k+1}) - \hat{J}_k(x_k) \leq -g(x_k, v(x_k)),\ \forall x_k \in \mathcal{X}_f$$

- Where $g_k(x_k, v(x_k))$ is the stage cost.

- Under those three assumptions, the MPC policy:

$$\mu_{\text{MPC}}(\bar{x}_0) = u_0^*$$

- Is **Recursive Feasible** and **Asymptotically Stable** with initial feasible set $\mathcal{X}_0$

Establishing MPC Properties

- As we did before, let's start by proving feasibility. We start from a point $x_0 \in \mathcal{X}_0$

- So for the very first time step, the Optimal Control Problem is feasible with solution:

$$(x_0, u_0^*, x_1, u_1^*, ..., x_{N-1}, u_{N-1}^*, x_N)$$

- We apply the first-stage control u_0^* and discards the rest. The system evolves to:

$$x_1 = Ax_0 + Bu_0^*$$

- Now at x_1 consider the following control sequence:

$$(u_1^*, ..., u_{N-1}^*, v(x_N)) \Longrightarrow \begin{array}{l} x_N \in \mathcal{X}_f \to v(x_N) \text{ is feasible} \\ x_{N+1} = f(x_N, v(x_N)) \in \mathcal{X}_f \end{array}$$

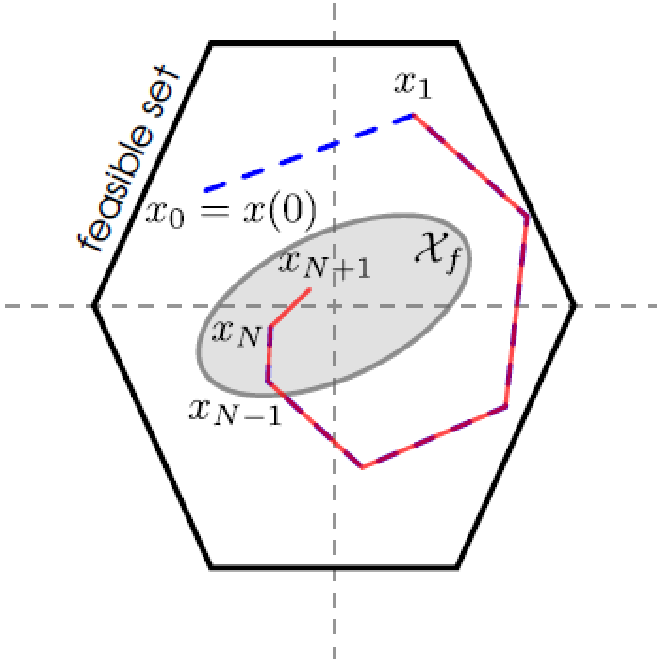
Establishing MPC Properties

- So the control sequence:

$$(u_1^*, ..., u_{N-1}^*, v(x_N))$$

- Is feasible (maybe not optimal) for the Optimal Control problem that starts at x_1.

- This can be illustrated with a picture:



(figure taken from Borelli and Morari)

Establishing MPC Properties

- We can conclude the proof by induction to establish recursive feasibility for all time steps $k \geq 0$.

- Now let's focus on stability. Our goal is to show that the entire cost-to-go function:

$$J_0^*(x_0) = \sum_{i=0}^{N-1} g(x_i, u_i^*) + \hat{J}_N(x_N)$$

- is a Lyapunov Function. By our set of assumptions, it is clear that:

$$J_0^*(0) = 0 \qquad J_0^*(x) = x^\top P x > 0,\ \forall x \neq 0$$

Establishing MPC Properties

- All it remains to show is that:

$$J_0^*(x_1) - J_0^*(x_0) < 0$$

- We verify this explicitly by writing:

$$J_0^*(x_1) \leq \sum_{i=1}^{N} g(x_i, u_i^*) + \hat{J}_N(f(x_N, v(x_N)) =$$

$$\sum_{i=1}^{N-1} g(x_i, u_i^*) + \hat{J}_N(x_N) - g(x_0, u_0^*) + \hat{J}_n(f(x_N, v(x_N))) - \hat{J}_N(x_N) + g(x_N, v(x_N)) =$$

$$J_0^*(x_0) - g(x_0, u_0^*) + \underbrace{\hat{J}_N(f(x_N, v(x_N))) - \hat{J}_N(x_N) + g(x_N, v(x_N))}_{\leq 0}$$

Establishing MPC Properties

- So it follows that:

$$J_0^*(x_1) - J_0^*(x_0) \leq -g(x_0, u_0^*) < 0$$

- So $J_0^*(x_0)$ is a Lyapunov Function w.r.t to the closed-loop system that follows the MPC policy:

$$\mu_{\text{MPC}}(\bar{x}_0) = u_0^*$$

- Hence the MPC Algorithm is asymptotically stable.

- Hence, when using the MPC Algorithm it is imperative that we choose:
  - **(1)** the set $\mathcal{X}_f$ needs to be an invariant set.
  - **(2)** the terminal cost-to-go approximation $\hat{J}_N(x_N)$ needs to be a Lyapunov Function over $\mathcal{X}_f$.

Establishing MPC Properties

- Notice how interesting this is:

- As long as we pick an invariant set $\mathcal{X}_f$ and a Lyapunov function $\hat{J}_N(x_N)$ we are **guaranteed** to succeed:
  - We will stay feasible always
  - Eventually we will reach the origin (i.e.: reach our goal)

- So, MPC is an Approximate Dynamic Programming that can converge to the optimal policy, in Infinite-Horizon problems.

- Different MPC Algorithms (using different sets $\mathcal{X}_f$ and different $\hat{J}_N(x_N)$ will display:
  - Different **transient periods** (that is until we reach the "steady-state" around the origin
  - Different initial feasible set $\mathcal{X}_0$ (also called **region of attraction**)

Example: Linear MPC

- Let's return to our Linear MPC case:

$$
\begin{aligned}
J_0(\bar{x}_0) = \min_{X,\,U} \quad & \hat{J}_N(x_N) + \sum_{i=0}^{N-1} x_i^\top Q x_i + u_i^\top R u_i \\
\text{s.t.} \quad & x_{i+1} = A x_i + B u_i, \quad \forall i \in \{0, 1, 2, ...\} \\
& x_i \in \mathcal{X}, \quad \forall i \in \{0, 1, 2, ...\} \\
& u_i \in \mathcal{U}, \quad \forall i \in \{0, 1, 2, ...\} \\
& x_0 = \bar{x}_0 \\
& x_N \in \mathcal{X}_f
\end{aligned}
$$

- Let's use a quadratic function to approximate the terminal cost:

$$\hat{J}_N(x_N) = x_N^\top P x_N$$

Example: Linear MPC

- Suppose we solve an unconstrained version of the problem, via LQR, and we obtain the matrices:

$$K = -(B^\top K B + R)^{-1} B^\top P A$$

$$P = A^\top (P - PB(B^\top P B + R)^{-1} B^\top P) A + Q$$

- Take $\mathcal{X}_f$ to be the maximum invariant set for the closed-loop system:

$$x_{k+1} = (A + BK) x_k \in \mathcal{X}_f, \quad \forall k \in \{0, 1, 2, ...\}$$

$$\mathcal{X}_f \subseteq \mathcal{X},\, K x_k \in \mathcal{U},\, \forall x_k \in \mathcal{X}_f$$

Example: Linear MPC

- It is a direct verification to see that

$$\hat{J}_N(x_N) = x_N^\top P x_N$$

- Is a Lyapunov Function over $\mathcal{X}_f$

$$x_{k+1}^\top P x_{k+1}^\top - x_k^\top P x_k =$$
$$x_k^\top(-P + A^\top PA - A^\top PB(B^\top PB + R)^{-1}B^\top PAx_k = -x_k^\top Q x_k$$

- So all conditions are met, and the linear MPC algorithm is both recursively feasible and asymptotically stable under those choices for $\mathcal{X}_f$ and $\hat{J}_N(x_N)$.

Trade-off between terminal and initial sets

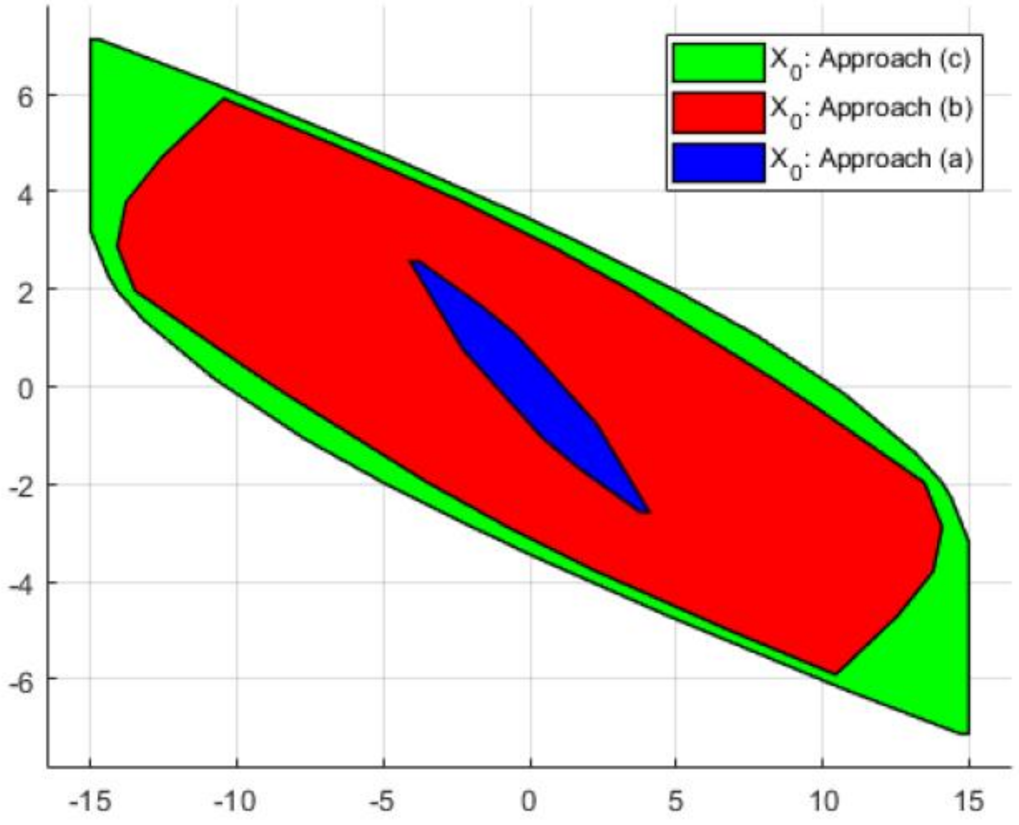
- Consider the following problem:

$$x_{k+1} = Ax_k + Bu_k = \begin{bmatrix} 1.2 & 1 \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k$$

$$x_k \in \mathcal{X} = \left\{ x \in \mathbb{R}^2 : \begin{bmatrix} -15 \\ -15 \end{bmatrix} \leq \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 15 \\ 15 \end{bmatrix} \right\}, \forall k \geq 0$$

$$u_k \in \mathcal{U} = \left\{ u \in \mathbb{R} : -1 \leq u \leq 1 \right\}, \forall k \geq 0$$

Trade-off between terminal and initial sets



- Approach (a):

$$\mathcal{X}_f = 0$$

- Approach (b):

$$\mathcal{X}_f = \mathcal{O}_\infty, \text{ for: } x_{k+1} = (A + BK)x_k$$

- Where K is some arbitrary matrix that stabilizes the system

- Approach (c):

$$\mathcal{X}_f = \mathcal{O}_\infty, \text{ for: } x_{k+1} = (A + BK)x_k$$

- Where K is the solution of the Riccati Equation

Remarks about MPC and Set Computation

- Computing invariants sets is not an easy task in general
  - In fact it can be VERY hard, for non-linear systems

- On the next lecture we will generalize the concepts we saw today to design algorithms based on the backwards recursion to compute these sets ($\mathcal{O}_\infty$,$\mathcal{C}_\infty$,$\mathcal{X}_0$) efficiently when the constraints are given by **polyhedrons**.

- In practice these sets are not often used:
  - Often, people “tune” their MPC by trial and error; increasing horizon length; changing costs; etc.
  - People “check” stability by simulation/sampling.
  - Often, other types of policies (e.g.: model-free) are used from where we are in regions, where MPC feasibility cannot be maintained.