Recursive State Estimation for Multiple Switching Models with Unknown Transition Probabilities

This work considers hybrid systems with continuous-valued target states and discrete-valued regime variable. The changes (switches) of the regime variable are modeled by a finite state Markov chain with unknown and random transition probabilities following Dirichlet distributions. Our work analytically derives the marginal posterior distribution of the states and regime variables, the transition probabilities being integrated out. This leads to a variety of recursive hybrid state estimation schemes which are an appealing intuitive and straightforward extension of standard algorithms. Their performance is illustrated by a maneuvering target tracking example.

I. INTRODUCTION

Target tracking is often formulated as a hybrid state estimation problem, characterized by a continuous-valued target state and a discrete-valued regime (mode) variable [1]. The target state vector $x_t \in \mathbb{R}^n$, which consists of kinematic and possibly attribute components, evolves according to the dynamic model derived from a stochastic difference (or differential) equation. The discrete regime variable r_t is governed by a discrete stochastic process. It takes one of a finite number of possible models each corresponding to a behavior mode. The model switching is usually modeled by a finite state homogeneous Markov chain, with a priori known transition probabilities. Various practical solutions to the hybrid estimation problem have been proposed in the literature [1, ch. 11, 2, ch. 10], with the interactive multiple model (IMM) algorithm [3, 4] being arguably the most popular one.

In practice, transition probabilities of the underlying Markov chain are rarely known exactly and therefore are treated by engineers as "design parameters." For example, diagonal elements of the transition probability matrix (TPM) are computed assuming prior knowledge of the mean sojourn time [5]. Clearly in situations where this prior knowledge is poor or even lacking, it would be desirable to estimate transition probabilities recursively, from the measurement data. The problem has been studied by several authors [6, 7]. Tugnait [6] developed a truncate maximum likelihood technique whereas Jilkov and Li [7] recently derived Bayesian estimates of transition probabilities assuming prior knowledge of all moments of the TPM prior probability density function (pdf). These algorithms rely on the knowledge of the "current" time-step model likelihoods which are unknown and need to be approximated. Various approximations are discussed in [7].

We propose an alternative Bayesian approach. As in [7], we assume that the TPM is unknown and random, but we treat here the TPM as a nuisance parameter and do not assume the knowledge of "current" time-step model likelihoods. By using Dirichlet priors on the transition probabilities, the TPM can be integrated out analytically so as to obtain the expression for the marginal posterior distribution of the state and regime variables. We then show how to incorporate this expression into standard recursive state estimation schemes.

The paper is organized as follows. Section II describes the mathematical models for Bayesian estimation. Section III derives the marginal posterior distribution of the states and regime variables, the TPM being integrated out analytically. This section also shows how one can use this expression to generalize standard hybrid state estimation schemes to the case where the transition probabilities are unknown. Section IV demonstrates the proposed algorithms on a numerical example, where we track a target with three levels of acceleration, switching according to a Markov chain. Finally Section V is devoted to conclusions.

II. BACKGROUND

Let r_t denote a discrete-time time-homogeneous s-state first-order Markov chain with transition probabilities $\pi_{ij} \triangleq \Pr\{r_{t+1} = j \mid r_t = i\}$ for any $i,j \in S$ where $S \triangleq \{1,2,\ldots,s\}$. The TPM $\mathbf{\Pi} = [\pi_{ij}]$, is thus an $s \times s$ matrix, with elements satisfying $\pi_{ij} \geq 0$ and $\sum_{j=1}^{s} \pi_{ij} = 1$, for each $i \in S$. Denote the initial probability distribution as $\mu_i \triangleq \Pr\{r_1 = i\}$, for $i \in S$, such that $\mu_i \geq 0$, $\forall i \in S$ and $\sum_{i=1}^{s} \mu_i = 1$. We denote $\boldsymbol{\mu} = [\mu_1,\ldots,\mu_s]^{\mathrm{T}}$, where $^{\mathrm{T}}$ stands for vector or matrix transpose.

The joint density of all target states $X_t \triangleq \{x_0,\ldots,x_t\}$ and regime variables $R_t \triangleq \{r_1,\ldots,r_t\}$ up to time t, expressed conditioned upon the values of $\boldsymbol{\mu}$ and $\mathbf{\Pi}$ is as follows:

$$p(X_t,R_t \mid \boldsymbol{\mu},\mathbf{\Pi}) = \left[\prod_{k=2}^{t} p(x_k \mid x_{k-1},r_k)p(r_k \mid r_{k-1},\mathbf{\Pi})\right] \times p(r_1 \mid \boldsymbol{\mu})p(x_1 \mid x_0,r_1)p(x_0) \qquad (1)$$

$$= \left[\prod_{k=2}^{t} p(x_k \mid x_{k-1},r_k)\pi_{r_{k-1}r_k}\right] \times \mu_{r_1} p(x_1 \mid x_0,r_1)p(x_0). \qquad (2)$$

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Both x_t and r_t are unobserved. Let us denote the observation at time t as y_t, and let $Y_t \triangleq \{y_1,\ldots,y_t\}$. It is assumed that the observations are conditionally independent upon (x_t,r_t), i.e.,

$$p(Y_t \mid X_t,R_t) = \prod_{k=1}^{t} p(y_k \mid x_k,r_k). \tag{3}$$

This class of hybrid models includes for example jump Markov linear systems [1]:

$$x_t = A(r_t)x_{t-1} + B(r_t)w_t \tag{4}$$

$$y_t = C(r_t)x_t + D(r_t)v_t. \tag{5}$$

where v_t and w_t are mutually independent sequences of independent Gaussian random variables. We assumed here that $\boldsymbol{\mu}$ and $\boldsymbol{\Pi}$ are *unknown* and *random*. We assign a prior distribution on $\boldsymbol{\mu}$ and on the rows of the matrix $\boldsymbol{\Pi} \triangleq [\boldsymbol{\pi}_1,\boldsymbol{\pi}_2,\ldots,\boldsymbol{\pi}_s]^{\mathrm{T}}$ with $\boldsymbol{\pi}_i = [\pi_{i1},\ldots,\pi_{is}]^{\mathrm{T}}$ for $i = 1,\ldots,s$. Furthermore we assume that

$$p(\boldsymbol{\mu},\boldsymbol{\pi}_1,\ldots,\boldsymbol{\pi}_s) = p(\boldsymbol{\mu})\prod_{i=1}^{s} p(\boldsymbol{\pi}_i) \tag{6}$$

and that $\boldsymbol{\mu}$ as well as each $\boldsymbol{\pi}_i$ admit an s-dimensional Dirichlet distribution:

$$\boldsymbol{\mu} \sim \mathcal{D}(\boldsymbol{\alpha}_0); \qquad \boldsymbol{\pi}_i \sim \mathcal{D}(\boldsymbol{\alpha}_i) \tag{7}$$

where $\boldsymbol{\alpha}_i \triangleq (\alpha_{i1},\ldots,\alpha_{is})$. If $\alpha_{ij} = 1$ for $i \in \{0,\ldots,s\}$ and $j \in \{1,\ldots,s\}$ we obtain uninformative uniform distributions.

A note on the Dirichlet distribution follows next [8]. Assume $\mathbf{z} = (z_1,\ldots,z_s) \sim \mathcal{D}(\mathbf{a}) = \mathcal{D}(a_1,\ldots,a_s)$ then the pdf of $\mathbf{z}$ is

$$p(\mathbf{z}) = \frac{1}{C(\mathbf{a})}\prod_{j=1}^{s} z_j^{a_j-1}\mathbb{I}_{\left\{\sum_{j=1}^{s} z_j=1\right\}}(\mathbf{z}) \tag{8}$$

where

$$\mathbb{I}_A(\mathbf{z}) = \begin{cases} 1 & \text{if} \quad \mathbf{z} \in A \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

is the indicator function of the set A, and $C(\mathbf{a})$ is a normalizing constant chosen so that $\int p(\mathbf{z})d\mathbf{z} = 1$. It can be shown that

$$C(\mathbf{a}) = \frac{\prod_{j=1}^{s}\Gamma(a_j)}{\Gamma\left(\sum_{j=1}^{s} a_j\right)} \tag{10}$$

where $\Gamma(\cdot)$ is the standard Gamma function. The parameters of Dirichlet distribution a_i can be interpreted as *prior observation counts*. Denoting $a_0 \triangleq \sum_{j=1}^{s} a_j$, one has

$$E[z_i] = \frac{a_i}{a_0}$$

$$\mathrm{var}[z_i] = \frac{(a_0-a_i)a_i}{a_0^2(a_0+1)} \tag{11}$$

$$\mathrm{cov}[z_i,z_j] = \frac{-a_i a_j}{a_0^2(a_0+1)}.$$

It means one can fix the parameter $(a_1,\ldots,a_s)$ so as to get a specific mean and variance for each component if some prior knowledge is available.

III. BAYESIAN ESTIMATION

A. Marginal Posterior Distribution

When $(\boldsymbol{\mu},\boldsymbol{\Pi})$ are known, the tracking algorithms (such as IMM [1]) aim at estimating $p(x_t,r_t \mid Y_t,\boldsymbol{\mu},\boldsymbol{\Pi})$ which is the marginal distribution of

$$p(X_t,R_t \mid Y_t,\boldsymbol{\mu},\boldsymbol{\Pi}) \propto p(Y_t \mid X_t,R_t)p(X_t,R_t \mid \boldsymbol{\mu},\boldsymbol{\Pi}). \tag{12}$$

A term which appears in tracking algorithms is then

$$p(r_k \mid r_{k-1}) = \pi_{r_{k-1}r_k}$$

as well as $p(r_1) = \mu_{r_1}$ at $t = 1$.

When $\boldsymbol{\Pi},\boldsymbol{\mu}$ are unknown, an intuitive solution would be to extend the state (x_t,r_t) with $\boldsymbol{\Pi}$ and $\boldsymbol{\mu}$. Then one would estimate $p(x_t,r_t,\boldsymbol{\Pi},\boldsymbol{\mu} \mid Y_t)$ which is a marginal of $p(X_t,R_t,\boldsymbol{\Pi},\boldsymbol{\mu} \mid Y_t)$. However, a simpler alternative consists of integrating out the unknown parameters $(\boldsymbol{\Pi},\boldsymbol{\mu})$ analytically. Consider the conditional density

$$p(X_t,R_t,\boldsymbol{\Pi},\boldsymbol{\mu} \mid Y_t) \propto p(Y_t \mid X_t,R_t)p(X_t,R_t \mid \boldsymbol{\Pi},\boldsymbol{\mu})p(\boldsymbol{\Pi},\boldsymbol{\mu}) \tag{13}$$

$$\propto p(Y_t \mid X_t,R_t)p(X_t \mid R_t)p(R_t \mid \boldsymbol{\Pi},\boldsymbol{\mu})p(\boldsymbol{\Pi},\boldsymbol{\mu}). \tag{14}$$

Note that only the term $p(R_t \mid \boldsymbol{\Pi},\boldsymbol{\mu})$ depends on $(\boldsymbol{\Pi},\boldsymbol{\mu})$ and so we can write

$$p(X_t,R_t \mid Y_t) \propto p(Y_t \mid X_t,R_t)p(X_t \mid R_t)p(R_t) \tag{15}$$

where

$$p(R_t) = \int p(R_t \mid \boldsymbol{\Pi},\boldsymbol{\mu})p(\boldsymbol{\Pi},\boldsymbol{\mu})\,d\boldsymbol{\Pi}\,d\boldsymbol{\mu}. \tag{16}$$

Next we show that $p(R_t)$ of (16) can be derived analytically. Observe first that $p(R_t \mid \boldsymbol{\Pi},\boldsymbol{\mu})$ can be expressed as follows:

$$p(R_t \mid \boldsymbol{\Pi},\boldsymbol{\mu}) = \left[\prod_{j=1}^{s}\mu_j^{\delta(r_1,j)}\right]\prod_{i=1}^{s}\left[\prod_{j=1}^{s}\pi_{ij}^{n_{ij}(R_t)}\right] \tag{17}$$

where $\delta(r_k,j) = 1$ if $r_k = j$ and zero otherwise and

$$n_{ij}(R_t) = \sum_{k=2}^{t}\delta(r_{k-1},i)\delta(r_k,j) \tag{18}$$

is the number of transitions from regime i to regime j in the sequence R_t. Using (6), (8), (10), and (17), (16) can be developed as follows:

$$
\begin{aligned}
p(R_t) &= \int p(R_t \mid \boldsymbol{\Pi}, \boldsymbol{\mu}) p(\boldsymbol{\mu}, \boldsymbol{\pi}_1, \ldots, \boldsymbol{\pi}_s) d\boldsymbol{\mu} d\boldsymbol{\Pi} \\
&= \int \prod_{j=1}^{s} \mu_j^{\alpha_{0j}+\delta(r_1,j)-1} \mathbb{I}_{\left\{\sum_{j=1}^{s}\mu_j=1\right\}}(\boldsymbol{\mu}) \frac{\Gamma\left(\sum_{j=1}^{s}\alpha_{0j}\right)}{\prod_{j=1}^{s}\Gamma(\alpha_{0j})} \\
&\quad \times \prod_{i=1}^{s}\left(\left[\prod_{j=1}^{s}\pi_{ij}^{\alpha_{ij}+n_{ij}(R_t)-1}\right] \frac{\Gamma\left(\sum_{j=1}^{s}\alpha_{ij}\right)}{\prod_{j=1}^{s}\Gamma(\alpha_{ij})} \right. \\
&\quad \left. \mathbb{I}_{\left\{\sum_{j=1}^{s}\pi_{ij}=1\right\}}(\boldsymbol{\pi}_i)\right) d\boldsymbol{\mu} d\boldsymbol{\pi}_1 \ldots d\boldsymbol{\pi}_s \\
&= \frac{\Gamma\left(\sum_{j=1}^{s}\alpha_{0j}\right)}{\prod_{j=1}^{s}\Gamma(\alpha_{0j})} \times \frac{\prod_{j=1}^{s}\Gamma(\alpha_{0j}+\delta(r_1,j))}{\Gamma\left(\sum_{j=1}^{s}(\alpha_{0j}+\delta(r_1,j))\right)} \\
&\quad \times \prod_{i=1}^{s}\left(\frac{\Gamma\left(\sum_{j=1}^{s}\alpha_{ij}\right)}{\prod_{j=1}^{s}\Gamma(\alpha_{ij})} \times \frac{\prod_{j=1}^{s}\Gamma(\alpha_{ij}+n_{ij}(R_t))}{\Gamma\left(\sum_{j=1}^{s}(\alpha_{ij}+n_{ij}(R_t))\right)}\right)
\end{aligned}
$$

that is

$$
\begin{aligned}
p(R_t) &= \frac{\Gamma\left(\sum_{j=1}^{s}\alpha_{0j}\right)}{\prod_{j=1}^{s}\Gamma(\alpha_{0j})} \times \frac{\prod_{j=1}^{s}\Gamma(\alpha_{0j}+\delta(r_1,j))}{\Gamma\left(1+\sum_{j=1}^{s}\alpha_{0j}\right)} \\
&\quad \times \prod_{i=1}^{s}\left(\frac{\Gamma\left(\sum_{j=1}^{s}\alpha_{ij}\right)}{\prod_{j=1}^{s}\Gamma(\alpha_{ij})} \times \frac{\prod_{j=1}^{s}\Gamma(\alpha_{ij}+n_{ij}(R_t))}{\Gamma\left(n_i(R_t)+\sum_{j=1}^{s}\alpha_{ij}\right)}\right) \\
&\propto \prod_{j=1}^{s}\Gamma(\alpha_{0j}+\delta(r_1,j)) \times \prod_{i=1}^{s}\left(\frac{\prod_{j=1}^{s}\Gamma(\alpha_{ij}+n_{ij}(R_t))}{\Gamma\left(n_i(R_t)+\sum_{j=1}^{s}\alpha_{ij}\right)}\right)
\end{aligned} \tag{19}
$$

where

$$
n_i(R_t) = \sum_{j=1}^{s} n_{ij}(R_t) = \sum_{k=2}^{t}\delta(r_{k-1}, i) \tag{20}
$$

is the number of times r_k is equal to i from time 1 to $t-1$.

After integration of the TPM, the term which appears in tracking algorithms is no more $p(r_t \mid r_{t-1}, \boldsymbol{\Pi})$ but $p(r_t \mid R_{t-1})$ as the Markov property is lost. Using Bayes rule we have

$$
p(r_t \mid R_{t-1}) = \frac{p(R_t)}{p(R_{t-1})} = \frac{\prod_{i=1}^{s}\left(\prod_{j=1}^{s}\Gamma(\alpha_{ij}+n_{ij}(R_t))\right) \times \prod_{i=1}^{s}\Gamma\left(n_i(R_{t-1})+\sum_{j=1}^{s}\alpha_{ij}\right)}{\prod_{i=1}^{s}\left(\prod_{j=1}^{s}\Gamma(\alpha_{ij}+n_{ij}(R_{t-1}))\right) \times \prod_{i=1}^{s}\Gamma\left(n_i(R_t)+\sum_{j=1}^{s}\alpha_{ij}\right)}. \tag{21}
$$

Assuming that $r_{t-1} = k$ and $r_t = l$, one has the following identities:

$$
\begin{aligned}
n_i(R_t) &= n_i(R_{t-1}), \qquad \text{for} \quad i \neq k \\
n_k(R_t) &= n_k(R_{t-1}) + 1 \\
n_{ij}(R_t) &= n_{ij}(R_{t-1}), \qquad \text{for} \quad i \neq k \\
n_{kj}(R_t) &= n_{kj}(R_{t-1}), \qquad \text{for} \quad j \neq l \\
n_{kl}(R_t) &= n_{kl}(R_{t-1}) + 1.
\end{aligned}
$$

The transition probability can now be expressed as

$$
\begin{aligned}
&p(r_t = l \mid R_{t-2}, r_{t-1} = k) \\
&\quad = \frac{\Gamma(\alpha_{kl}+n_{kl}(R_t))}{\Gamma(\alpha_{kl}+n_{kl}(R_{t-1}))} \times \frac{\Gamma\left(n_k(R_{t-1})+\sum_{j=1}^{s}\alpha_{kj}\right)}{\Gamma\left(n_k(R_t)+\sum_{j=1}^{s}\alpha_{kj}\right)} \\
&\quad = \frac{\Gamma(\alpha_{kl}+n_{kl}(R_{t-1})+1)}{\Gamma(\alpha_{kl}+n_{kl}(R_{t-1}))} \times \frac{\Gamma\left(n_k(R_{t-1})+\sum_{j=1}^{s}\alpha_{kj}\right)}{\Gamma\left(n_k(R_{t-1})+\sum_{j=1}^{s}\alpha_{kj}+1\right)}.
\end{aligned}
$$

Using a property of the Gamma function, that is $\Gamma(x+1) = x\Gamma(x)$, one gets

$$
p(r_t = l \mid R_{t-2}, r_{t-1} = k) = \frac{n_{kl}(R_{t-1}) + \alpha_{kl}}{n_k(R_{t-1}) + \sum_{j=1}^{s}\alpha_{kj}}. \tag{22}
$$

Formula (22) is the "key" result of this work. This formula represents a very intuitive result as it tells that the transition probability is estimated asymptotically from the "empirical" average of the previous transitions, with the influence of "prior observation counts" (parameters of the Dirichlet distribution). Numbers $n_{kl}(R_{t-1})$ and $n_k(R_{t-1})$ can be computed recursively as follows:

$$
n_{kl}(R_{t-1}) = n_{kl}(R_{t-2}) + \delta(r_{t-2}, k)\delta(r_{t-1}, l) \tag{23}
$$

$$
n_k(R_{t-1}) = n_k(R_{t-2}) + \delta(r_{t-2}, k). \tag{24}
$$

This means that (22) can be implemented recursively with the computational and memory requirements independent of time index t *if* R_t *was observed.*

Given R_t, one can perform inference on $(\boldsymbol{\Pi}, \boldsymbol{\mu})$ based on

$$
p(\boldsymbol{\Pi}, \boldsymbol{\mu} \mid R_t) \propto p(R_t \mid \boldsymbol{\Pi}, \boldsymbol{\mu}) p(\boldsymbol{\Pi}, \boldsymbol{\mu}).
$$

Straightforward calculations give

$$
\boldsymbol{\mu} \mid R_t \sim D(\alpha_{01} + \delta(r_1, 1), \ldots, \alpha_{0s} + \delta(r_1, s)) \tag{25}
$$

$$
\boldsymbol{\pi}_i \mid R_t \sim D(\alpha_{i1} + n_{i1}(R_t), \ldots, \alpha_{is} + n_{is}(R_t)). \tag{26}
$$

Conditional means and variances of the parameters can be computed using (11).

B. Applications to Recursive State Estimation Schemes

The marginal posterior distribution is given by (15) where $p(R_t)$ is given by (16). Based on these expressions, we can derive a variety of recursive state estimation schemes which are a direct extension of standard hybrid state estimation schemes. We present some of them here.

For a system with multiple switching models, when the transition probabilities (and the initial distribution) of the Markov chain are known, a standard pruning approximation can be described as follows. Propagate over time t a fixed number (say N) of hypothesis R_t characterized by the highest posterior values

$$p(R_t \mid Y_t, \boldsymbol{\mu}, \boldsymbol{\Pi}) \propto p(Y_t \mid R_t) p(R_t \mid \boldsymbol{\mu}, \boldsymbol{\Pi}). \qquad (27)$$

The likelihood term $p(Y_t \mid R_t)$ in (27) is computed through the Kalman filter if the system is a Jump Markov linear system or by an approximate nonlinear filtering technique otherwise [2, pp. 385–387]. In the case of unknown transition probabilities (and initial distribution), the extension is straightforward: one simply has to substitute $p(R_t \mid Y_t, \boldsymbol{\mu}, \boldsymbol{\Pi})$ by $p(R_t \mid Y_t) \propto p(Y_t \mid R_t) p(R_t)$.

Note that given N distinct sequences (hypotheses) at time t: $R_t^{(i)}$ $i = 1, \ldots, N$, we make the following approximation

$$p(R_t \mid Y_t) \approx \frac{\sum_{i=1}^N p(R_t^{(i)} \mid Y_t) \delta(R_t - R_t^{(i)})}{\sum_{i=1}^N p(R_t^{(i)} \mid Y_t)}$$

so that

$$p(\boldsymbol{\Pi} \mid Y_t) = \sum_{R_t} p(\boldsymbol{\Pi}, R_t \mid Y_t) = \sum_{R_t} p(\boldsymbol{\Pi} \mid R_t) p(R_t \mid Y_t)$$

$$\approx \frac{\sum_{i=1}^N p(R_t^{(i)} \mid Y_t) p(\boldsymbol{\Pi} \mid R_t^{(i)})}{\sum_{i=1}^N p(R_t^{(i)} \mid Y_t)}.$$

where $p(\boldsymbol{\Pi} \mid R_t^{(i)})$ is given by (26).

In the case of IMM and other merging type algorithms, the extension to unknown transition probabilities is more complex. We propose here a simple suboptimal approach which at each time t performs a hard-decision $\hat{r}_t = \arg\max \hat{p}(r_t \mid Y_t)$ where $\hat{p}(r_t \mid Y_t)$ is given by the IMM algorithm using the prior $p(r_t \mid \hat{R}_{t-1})$ given in (22) where $\hat{R}_{t-1} \triangleq (\hat{r}_1, \ldots, \hat{r}_{t-1})$.

IV. NUMERICAL RESULTS

We illustrate the proposed estimation algorithms with an example of a target moving in one dimension with a Markov switching acceleration (similar to example in [7]). The target state vector is defined as $x = [p \;\; v]^{\mathrm{T}}$ where p and v stand for target position and velocity. Target dynamics are given by

$$\begin{bmatrix} p \\ v \end{bmatrix}_{t+1} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p \\ v \end{bmatrix}_t + \begin{bmatrix} T^2/2 \\ T \end{bmatrix} (a_{t+1} + w_{t+1}) \qquad (28)$$

which represents a slightly modified form of (4) with a being the acceleration (maneuver input) and w modelling small perturbations in acceleration. The initial target position and velocity are distributed according to $p_0 \sim \mathcal{N}(80000, 100^2)$ and $v_0 \sim \mathcal{N}(400, 100^2)$, respectively. The process noise $w_t \sim \mathcal{N}(0, 2^2)$ is a white noise sequence independent of p_0 and v_0. The acceleration process a_t is a Markov chain with $s = 3$ states: $a_1 = 0$, $a_2 = 25$, and $a_3 = -25$ and with initial probabilities $\mu_1 = 0.8$, $\mu_2 = \mu_3 = 0.1$. Its TPM is as follows:

$$\boldsymbol{\Pi} = \begin{bmatrix} 0.95 & 0.025 & 0.025 \\ 0.1 & 0.8 & 0.1 \\ 0.05 & 0.05 & 0.9 \end{bmatrix}. \qquad (29)$$

The sampling interval is $T = 10$ and $t = 0, 1, \ldots, 150$.

The measurement equation is given by (5) with $C(r_t) = [1 \;\; 0]$ and $D(r_t) = 1$. The measurement noise $v_t \sim \mathcal{N}(0, 100^2)$ is white and independent of the process noise w_t.

A. Multiple Model Pruning

The implemented multiple model pruning (MMP) algorithm is described first. Assume at time $t-1$ one has say N distinct sequences $R_{t-1}^{(i)}$ $i = 1, \ldots, N$, with their posterior values $p(R_{t-1}^{(i)} \mid Y_{t-1})$ known (up to a normalizing constant). Then the MMP algorithm proceeds as follows at time t.

For $i = 1, \ldots, N$
For $r_t = 1, \ldots, s$
Compute $p(R_{t-1}^{(i)}, r_t \mid Y_t) \propto p(y_t \mid Y_{t-1}, R_{t-1}^{(i)}, r_t)$
$\cdot\, p(r_t \mid R_{t-1}^{(i)}) p(R_{t-1}^{(i)} \mid Y_{t-1})$, where
$p(y_t \mid Y_{t-1}, R_{t-1}^{(i)}, r_t)$ is the pdf of innovation computed by the Kalman filter associated to the linear system (28) and the sequence of regime variables $(R_{t-1}^{(i)}, r_t)$,
$p(r_t \mid R_{t-1}^{(i)})$ is computed using (22).
Select and preserve only the N "best" assumptions amongst the $N \times s$ hypotheses, i.e., those with the highest values of posterior $p(R_{t-1}^{(i)}, r_t \mid Y_t)$.

In simulations we compared the results obtained using three different versions of the MMP algorithm. The first two (referred to as MMP-A and MMP-B) both assume complete ignorance of the transition probability matrix, by using $\pi_{ij} = 1/s$ ($s = 3$ in our example). However, while the MMP-A has a built-in on-line reestimation of transition probabilities based on (22), the MMP-B algorithm has no such capability. The parameters of Dirichlet distribution set initially in the MMP-A are $\alpha_1 = \alpha_2 = \alpha_3 = (1, 1, 1)$. The third MMP version (referred to as the MMP-C) has a complete knowledge of the TPM in (29), hence its performance represents the best achievable performance. All three MMPs preserved at each time step only $N = 5$ best hypotheses; performance in theory improves as N increases. Figs. 1 and 2 show the simulation results obtained using all three MMP algorithms, averaged over 200 Monte Carlo runs. Fig. 1 shows the estimation error (mean absolute error) in (a) position and (b) velocity, while Fig. 2 displays the mean transition probability estimates versus time. The results are discussed in Subsection IVC.

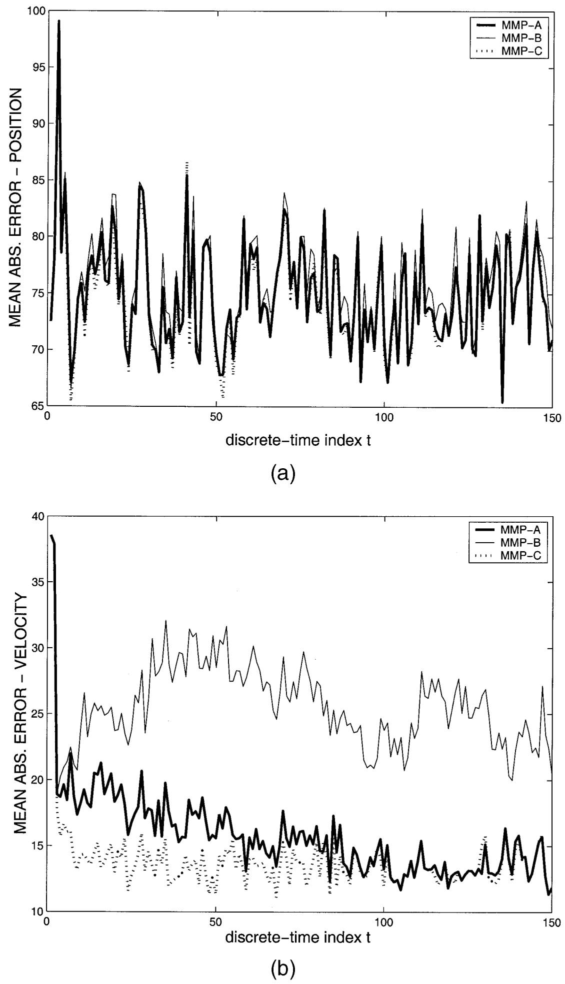

Fig. 1. MMP filter errors in (a) position and (b) velocity: thick line is MMP-A, thin solid line is MMP-B, dotted line is MMP-C. State estimate is approximate minimum mean square estimate.

B. Interacting Multiple Models

The recursive estimation of target state and acceleration (regime or mode variable) is illustrated in this section with the standard IMM algorithm [3]. Note that the IMM provides only a "soft" decision on the mode, given by mode probabilities $\Pr\{r_t = i\}$, $i \in S$. For estimation of transition probabilities using (22), we base inference on $p(\mathbf{\Pi} \mid \hat{R}_t)$ where $\hat{R}_t$ is the sequence of regime variables estimated using the IMM algorithm. This (hard decision) estimate in simulations is made by selecting at each time t the mode of the IMM with the highest probability.

Following the same methodology as in Section IVA, in simulations we compared the results obtained using three different versions of the IMM. IMM-A and IMM-B assume complete ignorance of the TPM, by using $\pi_{ij} = 1/3$ for all $i, j \in S$. The IMM-A has a built-in on-line estimation of transition probabilities according to (22), while the IMM-B is using a TPM with fixed values $\pi_{ij} = 1/3$ all the time. IMM-C has a complete knowledge of the TPM given by (29). Figs. 3 and 4 present the simulation results obtained using 200 Monte Carlo runs. The results are discussed in the next subsection.

C. Discussion

Since the target position is measured fairly accurately, the performance curves showing the mean absolute error in position (Figs. 1(a) and 3(a)) are all very similar. These figures are not informative and we concentrate instead on estimation errors in velocity, displayed in Figs. 1(b) and 3(b). For both MMP

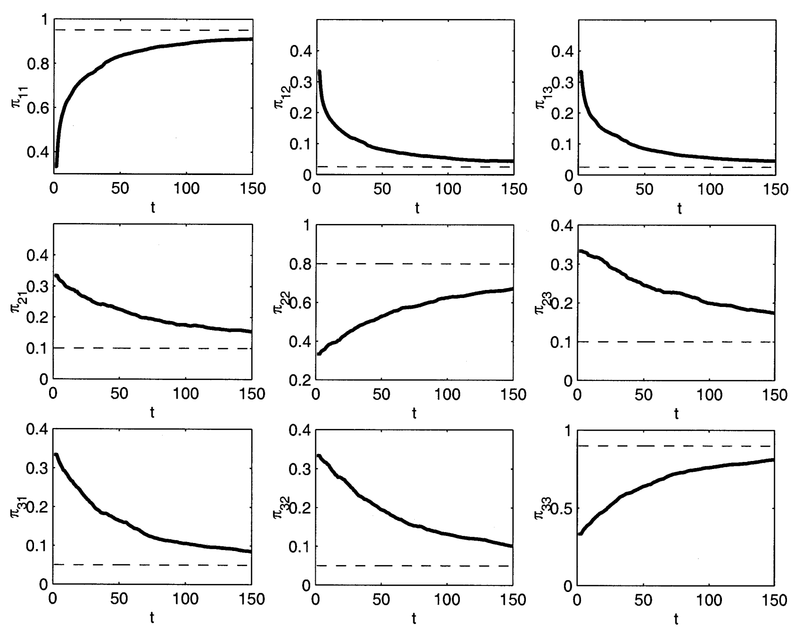

Fig. 2. Approximate conditional mean of transition probabilities compared with MMP-A algorithm: dashed lines are true values, solid thick lines are estimates.

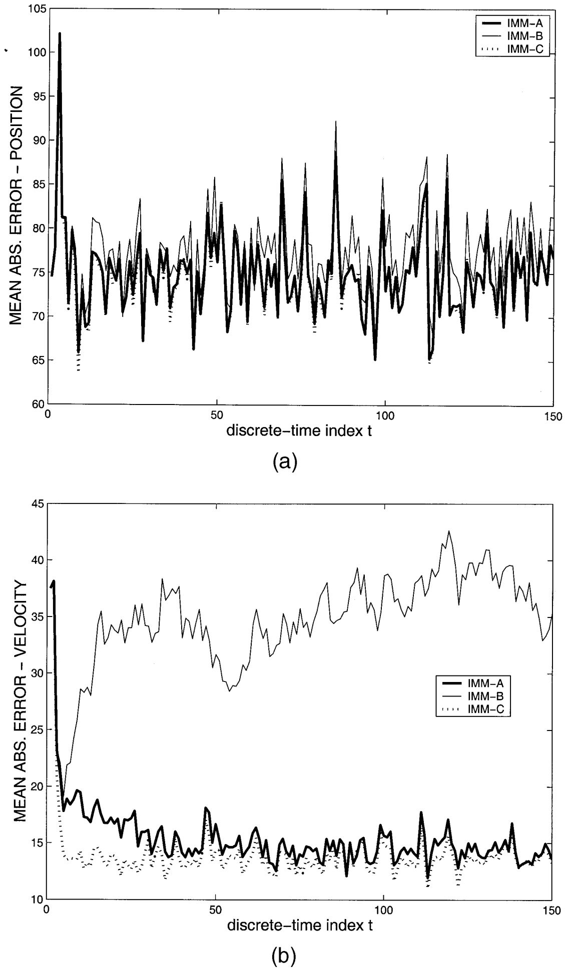

Fig. 3. IMM filter errors in (a) position and (b) velocity: thick solid line is MMP-A, thin solid line is IMM-B, dotted line is IMM-C. State estimate is approximate minimum mean square estimate.

and IMM schemes, the estimation of velocity is by far the worst with version B, based on fixed TPM with values $\pi_{ij} = 1/3$. As one would expect, the error of both MMP-B and IMM-B is at an approximately constant level during the observation period. Note however that the error of IMM-B is somewhat higher than the error of the MMP-B, suggesting that the MMP algorithm is more robust to the incorrect choice of the TPM than the IMM algorithm.

Next let us consider the performance of MMP-A and IMM-A, both which incorporate on-line estimation of transition probabilities based on the algorithm proposed in this work. Their errors in velocity are decreasing steadily as their transition probability estimates converge towards the true values (as indicated in Figs. 2 and 4, respectively). Thus, towards the end of the observation period, the MMP-A (Fig.1.b) approaches the performance of the MMP-C (which represents the best achievable performance). Similarly the IMM-A (Fig. 3(b)) approaches the performance of the IMM-C (the best achievable performance). Both pruning (MMP) and merging (IMM) algorithms are characterized by similar performance in version B (estimation of TPM) and version C (known TPM).

V. SUMMARY

The paper presented a simple framework to perform state estimation in hybrid systems when the transition probabilities are unknown. Using the Dirichlet distribution, it is possible to derive analytically the marginal posterior distribution

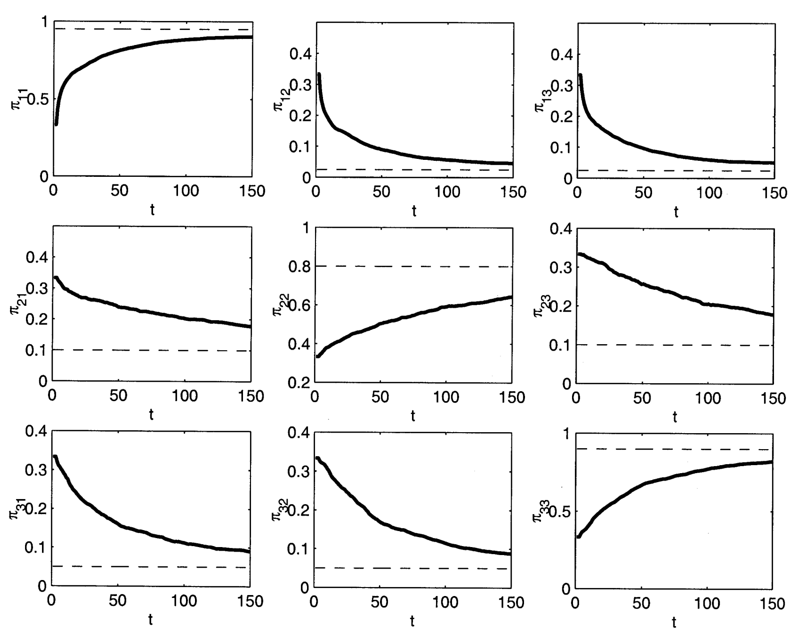

Fig. 4. Mean of transition probability estimates $E[\pi_{ij} \mid \hat{R}_t]$ obtained with IMM-A algorithm: dashed lines are true values, solid thick lines are estimates.

of the states and regime variables. Based on this expression, one can derive a variety of recursive state estimation schemes which are an appealing intuitive and straightforward extension of standard hybrid state estimation schemes. The proposed algorithms are particularly useful in situations where prior knowledge of transition probabilities is poor or completely lacking. The performance of the algorithms has been demonstrated with a numerical example which considered maneuvering target with three levels of acceleration. Further work will consider the case where a dynamic model is incorporated into the transition probabilities.

ARNAUD DOUCET
Dept. of Electrical and Electronic Engineering
University of Melbourne
VIC 3010
Australia

BRANKO RISTIC
Defence Science and Technology Organisation
200 Labs
P.O. Box 1500
Edinburgh SA 5111
Australia
E-mail: (branko.ristic@dsto.defence.gov.au)

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Small-Signal Modeling of Variable-Frequency Pulsewidth Modulators

Small-signal linear models are presented for variable-frequency constant on-time and variable-frequency constant off-time pulsewidth modulators. Both modulation methods are shown to have leading phase characteristics and nonlinear amplitude responses. The analytical results are validated by numerical simulation results and compared with previously published empirical models. The models are intended for use in analysis and control design of pulsewidth modulated switching power converters.

I. INTRODUCTION

Modeling of the pulsewidth modulation (PWM) process is the key to correct and accurate close-loop behavior prediction for PWM switching-mode power converters. Assume that a PWM converter is described by state-space models $\mathbf{x}' = \mathbf{A}_1\mathbf{x} + \mathbf{b}_1 v_{\text{in}}$ and $\mathbf{x}' = \mathbf{A}_2\mathbf{x} + \mathbf{b}_2 v_{\text{in}}$ when the swith is on and off, respectively. Averaging theory states that the dynamics of such a converter can be described by an averaged model

$$\frac{d\mathbf{x}}{dt} = [d\mathbf{A}_1 + (1-d)\mathbf{A}_2]\mathbf{x} + [d\mathbf{b}_1 + (1-d)\mathbf{b}_2]v_{\text{in}} \tag{1}$$

where d is the duty ratio of the switch. To complete the model for close-loop operation, a modulator model is needed to relate the duty ratio d to the controller output (which serves as the input to the pulsewidth modulator).

Various PWM schemes are being used in power electronics, especially for dc-dc conversion applications. Earlier work on pulse modulator modeling has focused on output spectrum analysis of different constant-frequency modulation processes, such as pulse-amplitude modulation, pulse-position modulation, and pulsewidth modulation using natural and uniform sampling [1–2]. Reference [3] analyzed a special case of constant-frequency PWM for control applications where the input is a sinusoidal signal (without a dc component). Similar results were reported in [4] for PWM inverters for motor drive applications. Responses of constant-frequency pulsewidth modulators to a small sinusoidal perturbation around the steady-state input were analyzed in [5] to generate small-signal modulator

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