Mitigation vs. Suppression Social Distancing

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1 Single Compartment SIR Model

Euler-Lagrange Equation:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x_i}} - \frac{\partial \mathcal{L}}{\partial x_i} \tag{1}$$

which becomes:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x_i}} = \frac{\partial \mathcal{L}}{\partial x_i} \tag{2}$$

$$\dot{S} = b - \frac{\beta I S}{N} \tag{3}$$

$$\dot{I} = \frac{\beta I S}{N} - \gamma I \tag{4}$$

$$\dot{R} = \gamma I \tag{5}$$

Cost function:

$$-a\ln(\beta) + \mathrm{c}\beta + \frac{\beta \mathrm{IS}}{\mathrm{N}} \tag{6}$$

$$\begin{aligned}\mathcal{L} = -a\ln(\beta) + \mathrm{c}\beta + \frac{\beta \mathrm{IS}}{\mathrm{N}} + \\ \lambda_S(\dot{S} + \frac{\beta I S}{N} - b) + \\ \lambda_I(\dot{I} - \frac{\beta I S}{N} + \gamma I)\end{aligned} \tag{7}$$

We leave out the constraint for R because it doesn't factor into the cost equation.

Apply equation 1 $x_i = S$

$$\frac{d}{dt}\left(\lambda_s\right) - \left(\frac{\beta I}{N} + \lambda_S \frac{\beta I}{N}\right) \tag{8}$$

$$\dot{\lambda}_S = \frac{\beta I}{N}(1 + \lambda_S - b) \tag{9}$$

Apply equation 1 $x_i = I$:

$$\dot{\lambda}_I = \frac{\beta S}{N}(1 - \lambda_S - \lambda_I) + \gamma\lambda_I \tag{10}$$

Apply equation 1 $x_i = \beta$:

$$-\frac{a}{\beta} + c + \frac{IS}{N} - \lambda_S\frac{IS}{N} - \lambda_I\frac{IS}{N} \tag{11}$$

which becomes

$$\frac{a}{\beta} = c + \frac{IS}{N}(1 - \lambda_S - \lambda_I) \tag{12}$$

$$\beta = \frac{a}{c + \frac{IS}{N}(1 - \lambda_S - \lambda_I)} \tag{13}$$

When the infection rate is very low, we expect β to be:

$$\frac{a}{c} \tag{14}$$

Individual, myopic optimum:

$$\frac{d}{d\beta}\left[-\mathrm{aln}(\beta) + \mathrm{c}\beta + \beta\frac{\mathrm{IS}}{\mathrm{N}}\right] = 0 \tag{15}$$

which becomes

$$-\frac{a}{\beta} + c + \frac{IS}{N} = 0 \tag{16}$$

$$\frac{a}{\beta} = c + \frac{IS}{N} \tag{17}$$

$$\beta = \frac{a}{c + \frac{IS}{N}} \tag{18}$$

2 Exponential time weighting

Cost function:

$$\begin{aligned}\mathcal{L} = e^{-\alpha t}\left(-\mathrm{aln}(\beta) + \mathrm{c}\beta + \frac{\beta\mathrm{IS}}{\mathrm{N}}\right) + \\ \lambda_S(\dot{S} + \frac{\beta IS}{N} - b) + \\ \lambda_I(\dot{I} - \frac{\beta IS}{N} + \gamma I)\end{aligned} \tag{19}$$

3 Multi-Compartment SIR Model

$$\dot{S}_i = -\frac{1}{N}\sum_j \beta_{ij} I_j S_i \tag{20}$$

$$\dot{I}_i = \frac{1}{N}\sum_j \beta_{ij} I_j S_i - \gamma I_i \tag{21}$$

Cost function:

$$\mathcal{C} = \sum_i N_i \left(-a_{ij} \ln(\mathrm{\beta_{ij}}) + \mathrm{c\beta_{ij}} + \frac{\mathrm{\beta_{ij} I_j S_i}}{\mathrm{N_j}} \right) \tag{22}$$

Lagrangian:

$$\mathcal{L} = \sum_i N_i \left(-a_{ij} \ln(\mathrm{\beta_{ij}}) + \mathrm{c\beta_{ij}} + \frac{\mathrm{\beta_{ij} I_j S_i}}{\mathrm{N}} \right) + \sum_i \lambda_{S,i} \left(\dot{S}_i + \frac{1}{N}\sum_j \beta_{ij} I_j S_i \right) + \sum_i \lambda_{I,i} \left(\dot{I}_i - \frac{\beta_{ij} I_j S_i}{N} + \gamma I_i \right) \tag{23}$$