Nonlinear Control
Nonlinear systems study
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Simulation Project

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1. Nonlinear Actuators in Feedback Loop

Considering the following linear time-variant system, $G(s)$:

$$G(s) = \frac{20}{s(s+2)(s+5)} \tag{1}$$

1.1 Known Actuators

The Describing Function method is initiated by obtaining the Describing Function of the nonlinearity, which gave the following expressions:

- **Actuator 1 - Dead-zone**

$$N(x) = k - \frac{2k}{\pi}\left[\arcsin\frac{\Delta}{x} + \frac{\Delta}{x}\sqrt{1-\left(\frac{\Delta}{x}\right)^2}\right] \tag{2}$$

Substituting $\Delta = 0.5$ and $k = 2$ in equation 2:

$$N(x) = 2 - \frac{4}{\pi}\left[\arcsin\frac{1}{2x} + \frac{1}{2x}\sqrt{1-\frac{1}{4x^2}}\right] \tag{3}$$

Equation 3 results in the following graphic representation:

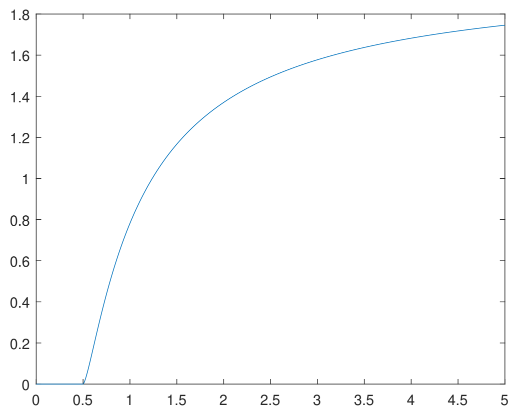

Figure 1: $N(x)$ of Actuator 1.

- **Actuator 2 - Saturation with dead-zone**

$$N(x) = \frac{4M}{\pi x}\sqrt{1-\left(\frac{\Delta}{x}\right)^2} \tag{4}$$

Substituting $\Delta = 1$ and $M = 1$ in 4:

$$N(x) \;=\; \frac{4}{\pi x}\sqrt{1 \;-\; \frac{1}{x^2}} \tag{5}$$

Equation 5 results in the following graphic representation:

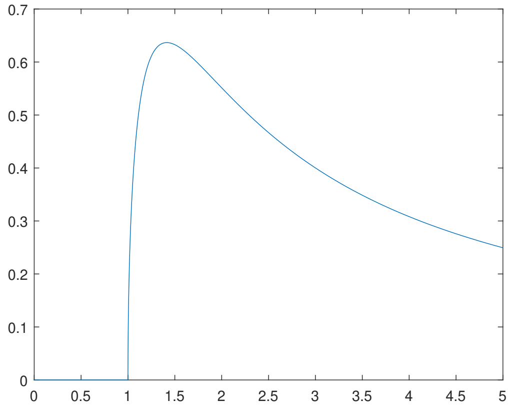

Figure 2: $N(x)$ of Actuator 2.

- **Actuator 3 - Hysteresis**

$$N(x) \;=\; \frac{4M\sin\alpha}{\pi x}e^{-j\Phi} \tag{6}$$

Where,

$$\begin{cases} \alpha &= 0.5\left(\pi - \arcsin\frac{c}{x} - \arcsin\frac{b}{x}\right) \\ \Phi &= \alpha + \arcsin\frac{c}{x} - \frac{\pi}{2} \end{cases} \quad \text{and} \quad \begin{cases} b \;=\; \Delta \;-\; h \\ c \;=\; \Delta \;+\; h \end{cases} \tag{7}$$

Since this Hysteresis has no dead-zone, $\Delta \;=\; 0$. Substituting in system 7:

$$\begin{cases} b \;=\; -h \\ c \;=\; h \end{cases} \quad \text{and so} \quad \begin{cases} \alpha \;=\; 0.5(\pi \;-\; \arcsin\frac{h}{x} \;-\; \arcsin -\frac{h}{x}) \;=\; \frac{\pi}{2} \\ \Phi \;=\; \frac{\pi}{2} \;+\; \arcsin\frac{h}{x} \;-\; \frac{\pi}{2} \;=\; \arcsin\frac{h}{x} \end{cases} \tag{8}$$

Finally, using the expressions from system 8 in equation 6:

$$N(x) \;=\; \frac{2\sin\frac{\pi}{2}}{\pi x}e^{-j\arcsin\frac{2}{x}} \;=\; \frac{2}{\pi x}e^{-j\arcsin\frac{2}{x}} \tag{9}$$

Equation 9 results in the following graphic representation:

The graphic below illustrates these 3 non-linearity being applied to $G(s)$:

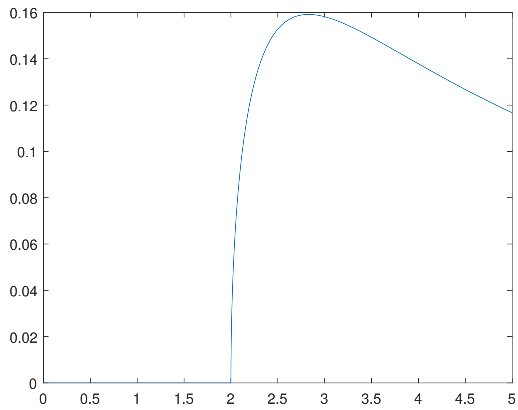

Figure 3: $N(x)$ of Actuator 3.

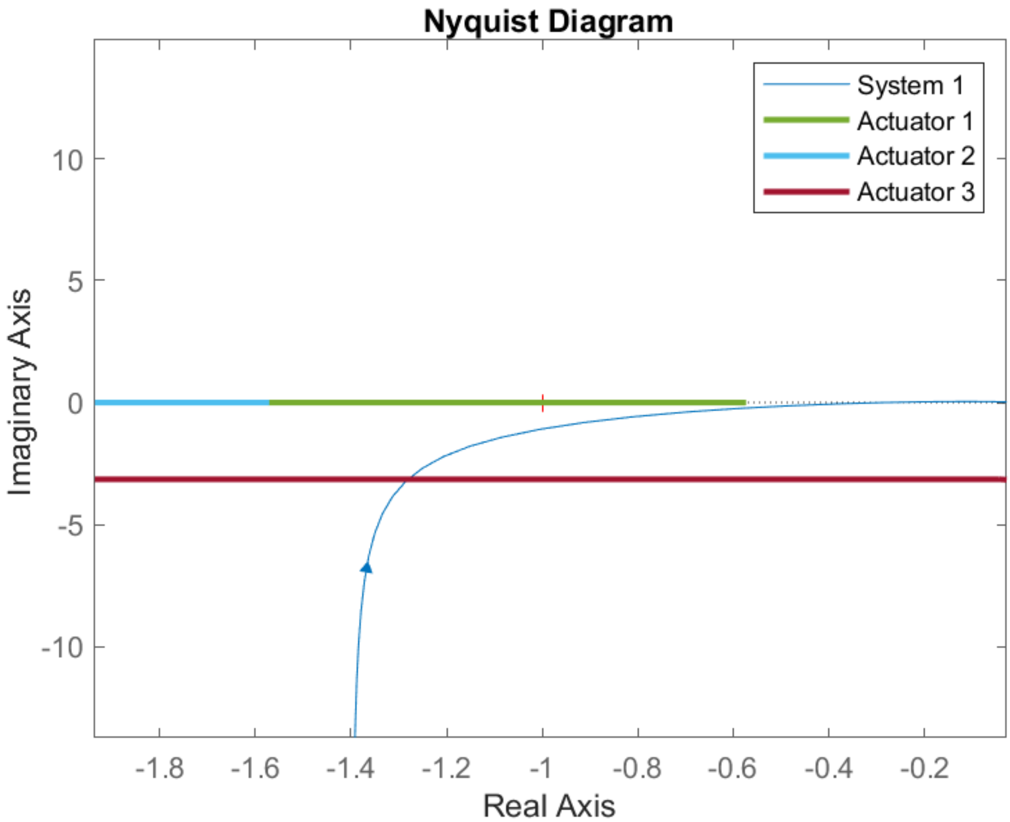

Figure 4: $\frac{1}{N(x)}$ of all Actuators and $G(s)$.

Analyzing figure 4, it is possible to conclude that there is one limit cycle related to the third actuator. For further inspection of this specific limit cycle, the figure below was captured.

The limit cycle detected in figure 5 has, approximately, a real part of -1.28 and an imaginary part of -3.14. To characterize it's amplitude and frequency a MATLAB script was developed to assist with the calculations, from which the results were, approximately, an amplitude of 2.16

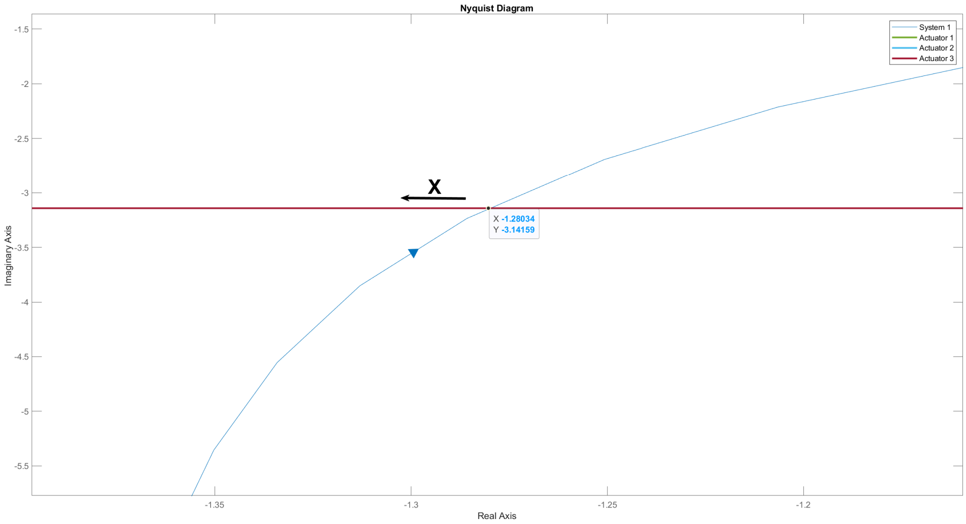

Figure 5: Limit Cycle.

and a frequency of 0.56 rad/s. Looking at it's stability, the part encircled by the Nyquist plot, the closed-loop is unstable, thus, X increases, however the part that is not encircled by the Nyquist plot, the closed-loop is stable, thus, X decreases. Therefore, taking this in mind it's safe to conclude that the limit cycle in the intersection point is stable.

Therefore, in way to demonstrate the occurrence of limit cycles, the system was tested with Actuator 3, using an input signal consisting of a sine wave with a frequency of 0.56 rad/s and an amplitude of 2.16. As shown in Graph 6, the system exhibited a limit cycle in response to this input.

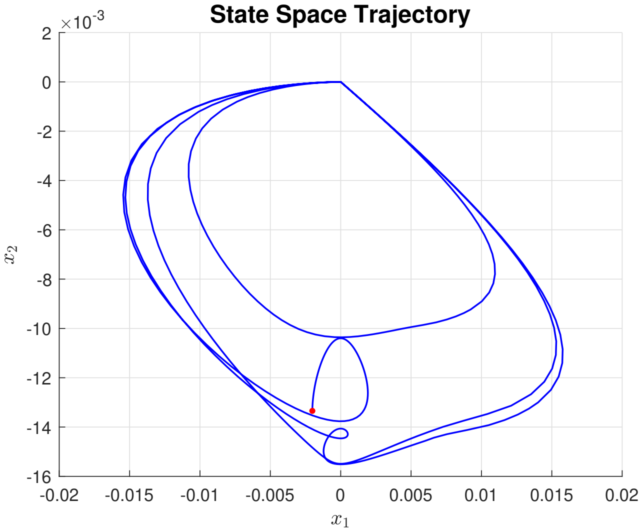

Figure 6: Limit Cycle behavior represented in the state space plan.

1.2 Unknown Actuators

Consider that the nonlinearity of the actuator is confined to a sector $K \in [K_1, K_2]$, where $G(s)$ represents the linear, time-invariant system. It is strictly proper and strictly stable, as demonstrated by the poles of the transfer function. Under these conditions, the Popov Criterion can be applied to determine the sector defined by K.

The key to applying the Popov Criterion is analyzing the following graph 7.

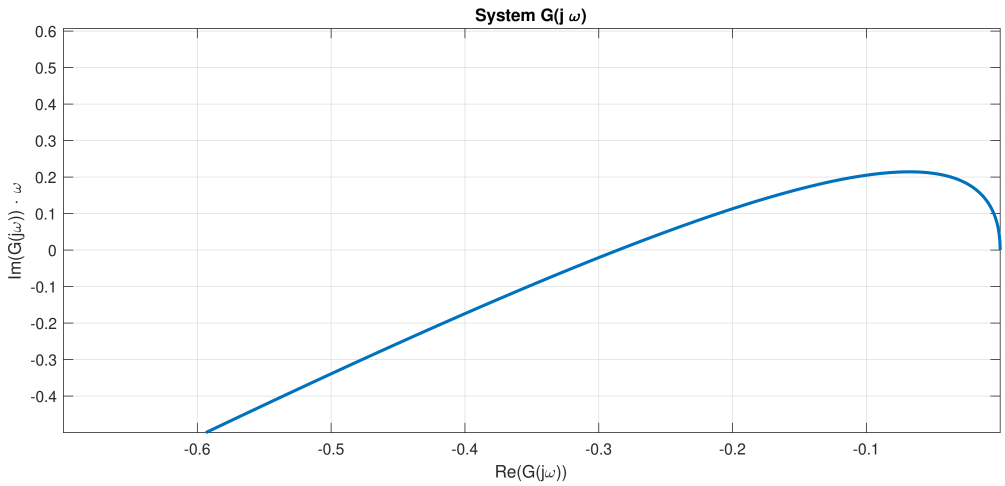

Figure 7: System G represented in the Popov domain using $X(G(j\omega))$ (real part of $G(j\omega)$) and $Y(G(j\omega))$ (imaginary part of $G(j\omega)$ scaled by ω).

According to the Popov Criterion, the sector in which K lies is determined by the values that satisfy the following inequality:

$$X(j\omega) \; - \; qY(j\omega) + \frac{1}{K} > 0 \quad \forall\, \omega \tag{10}$$

Where, $\frac{1}{q}$ represents the slope of the Popov line, $X(j\omega) = \text{Re}(G(j\omega))$ and $Y(j\omega) = \text{Im}(G(j\omega)) \cdot \omega$. The value where $Y = 0$ corresponds to the position on the X-axis at $-\frac{1}{K}$. As shown in figure 7, the minimum value of K, denoted K_1, is 0. This is because, no matter how small K_1 is, there will always be a line that satisfies the Popov inequality.

The main challenge now is to determine the maximum value of K, denoted as K_2. Graphically, this involves identifying the largest value of K where the slope of the line best approximates the leftmost point of the curve defined in figure 7. Since the equation 10 involves two variables, q and K, and substituting values does not yield a trivial solution, a MATLAB script was used to solve the equation and find the maximum value of K.

The MATLAB script essentially defines a range over which q and K can vary. For each value of q, the script finds the values of K that satisfy the equation 10 and stores the maximum value of K for each line defined by the slope $\frac{1}{q}$. After finding all the maximum values for each q, the overall maximum value of K is the largest of these individual maximum values. Using this approach, the figure 8 is obtained and the Popov lines that represents the maximum value of K for each q is showed and the influence of q in this value.

So, it can be concluded from figure 8 that the value of K_2 is equal to 3.475, and the slope of the corresponding line is determined by q, which is equal to 0.684. The following line, along

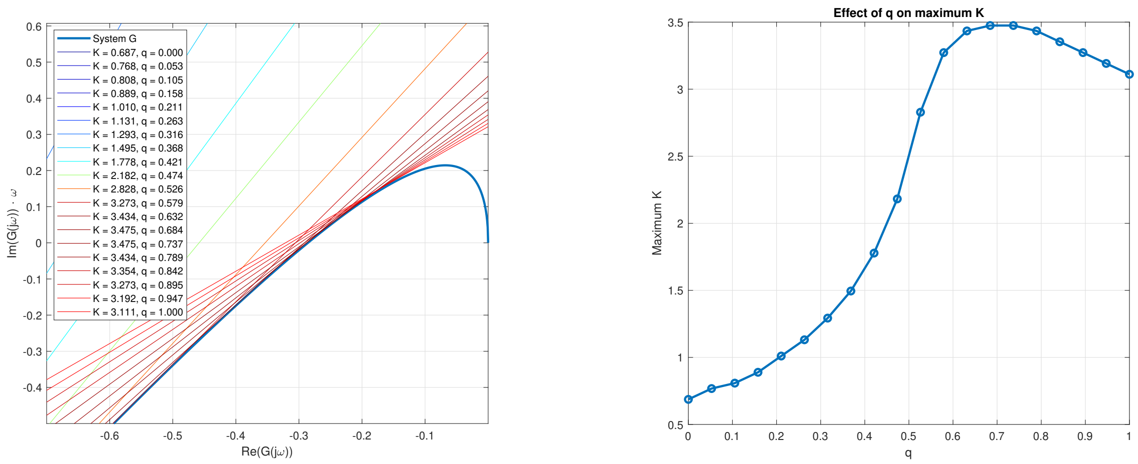

Figure 8: Popov lines illustrating the influence of the line's slope on the maximum value of K.

with its point of intersection with the X-axis, is shown in figure 9.

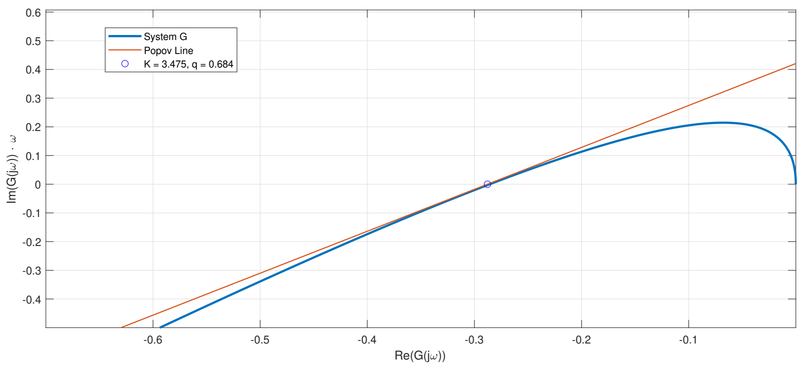

Figure 9: Maximum value of K and its popov line representation.

Therefore, according to the Popov Criterion, the sector K is defined by:

$$K \in [0 \quad 3.475] \tag{11}$$

The figure 10 shows a simulation of the system when different gain values are implemented, and as expected, when the gain is greater than 3.475, the system starts to become unstable.

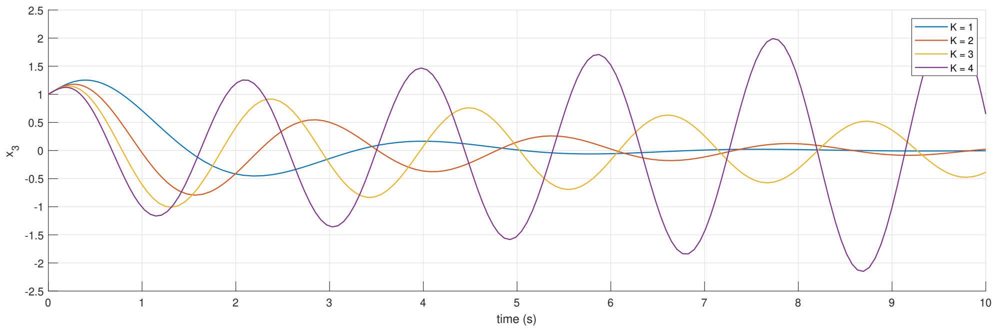

Figure 10: System simulation for diferent K gain values.

2. Controller Design

Considering the following nonlinear time-variant system:

$$\begin{cases} \dot{x}_1 = -x_1 + u \\ \dot{x}_2 = -x_2 - x_1 x_3 + u \\ \dot{x}_3 = x_1 x_2 \\ y = x_2 \end{cases} \tag{12}$$

2.1 Input-State Feedback Linearization

The process of designing a suitable controller for the system using input-state feedback linearization is detailed in the following points:

1. **Define the state equation**

$$\dot{x} = f(x) + g(x)u \text{ , where } f(x) = \begin{bmatrix} -x_1 \\ -x_2 - x_1 x_3 \\ x_1 x_2 \end{bmatrix} \text{ and } g(x) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \tag{13}$$

2. **Check controllability and involutivity**

   For both conditions it is important to note that there are 3 variables in this system, so $n = 3$.

   - **Controllability**

     The condition to be satisfied is the following:

$$\text{rank}\left(\left[\ g(x) \quad ad_f^1 g(x) \quad ad_f^2 g(x)\ \right]\right) = n \tag{14}$$

     The first step is to determine $ad_f^1 g(x)$:

$$ad_f^1 g(x) = [f(x),\ g(x)] = \nabla g(x) \cdot f(x) - \nabla f(x) \cdot g(x) \Leftrightarrow$$

$$\Leftrightarrow ad_f^1 g(x) = 0 - \begin{bmatrix} \frac{\partial f(x)_1}{\partial x_1} & \frac{\partial f(x)_1}{\partial x_2} & \frac{\partial f(x)_1}{\partial x_3} \\ \frac{\partial f(x)_2}{\partial x_1} & \frac{\partial f(x)_2}{\partial x_2} & \frac{\partial f(x)_2}{\partial x_3} \\ \frac{\partial f(x)_3}{\partial x_1} & \frac{\partial f(x)_3}{\partial x_2} & \frac{\partial f(x)_3}{\partial x_3} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \Leftrightarrow \tag{15}$$

$$\Leftrightarrow ad_f^1 g(x) = - \begin{bmatrix} -1 & 0 & 0 \\ -x_3 & -1 & -x_1 \\ x_2 & x_1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ x_3 + 1 \\ -x_2 - x_1 \end{bmatrix}$$

Next is to calculate $ad_f^2 g(x)$:

$$ad_f^2 g(x) = \left[f(x),\ ad_f^2 g(x)\right] = 0 - \nabla f(x) \cdot \begin{bmatrix} 1 \\ x_3 + 1 \\ -x_2 - x_1 \end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow\ ad_f^2 g(x) = -\begin{bmatrix} -1 & 0 & 0 \\ -x_3 & -1 & -x_1 \\ x_2 & x_1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ x_3 + 1 \\ -x_2 - x_1 \end{bmatrix} \Leftrightarrow \tag{16}$$

$$\Leftrightarrow\ ad_f^2 g(x) = -\begin{bmatrix} -1 \\ -2x_3 - 1 \\ x_1 + x_2 + x_1 x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2x_3 + 1 \\ -x_1 - x_2 - x_1 x_3 \end{bmatrix}$$

And so this results in a matrix that satisfies the controllability condition:

$$\begin{gathered}\operatorname{rank}\left(\begin{bmatrix} 1 & 1 & 1 \\ 1 & x_3 + 1 & 2x_3 + 1 \\ 0 & -x_1 - x_2 & -x_1 - x_2 - x_1 x_3 \end{bmatrix}\right) = 3\ (= n)\ , \\ \textit{Iff}:\ (x_3 \neq 0 \wedge x_1 \neq 0)\ \vee\ (x_3 \neq 0 \wedge x_2 \neq 0)\end{gathered} \tag{17}$$

- **Involutivity**

  The condition to be satisfied is the following:

$$\operatorname{rank}\left(\begin{bmatrix} g(x) & ad_f^1 g(x) \end{bmatrix}\right) = n - 1 \tag{18}$$

  Remembering equations 13 and 15, it results in a matrix that satisfies the involutivity condition:

$$\operatorname{rank}\left(\begin{bmatrix} 1 & 1 \\ 1 & x_3 + 1 \\ 0 & -x_1 - x_2 \end{bmatrix}\right) = 2\ (= n - 1)\,, \textit{Iff}:\ x_3 \neq 0 \wedge x_1 \neq -x_2 \tag{19}$$

- **Obtain the coordinate transformation**

  This coordinate transformation transforms the system into its controllable canonical form:

$$z = \Phi(x) = \begin{bmatrix} q(x) \\ L_f q(x) \\ L_f^2 q(x) \end{bmatrix} \tag{20}$$

Where $q(x)$ should verify 3 conditions:

$$L_f q(x) \ = \ 0 \ \Leftrightarrow \ \frac{\partial q(x)}{\partial x_1} \ + \ \frac{\partial q(x)}{\partial x_2} \ = \ 0 \tag{21}$$

$$L_{ad_f} g(x) q(x) \ = \ 0 \ \Leftrightarrow \ \frac{\partial q(x)}{\partial x_1} + (x_3 + 1)\frac{\partial q(x)}{\partial x_2} + (-x_2 - x_1)\frac{\partial q(x)}{\partial x_3} \ = \ 0 \tag{22}$$

$$\begin{gathered} L^2_{ad_f} g(x) q(x) \ \neq \ 0 \ \Leftrightarrow \\ \Leftrightarrow \frac{\partial q(x)}{\partial x_1} \ + \ (2x_3 \ + \ 1)\frac{\partial q(x)}{\partial x_2} \ + \ (-x_1 \ - x_2 \ - x_2 x_3)\frac{\partial q(x)}{\partial x_3} \ \neq \ 0 \end{gathered} \tag{23}$$

From which it can be conclude the following:

$$\begin{cases} \frac{\partial q(x)}{\partial x_1} \ = \ - \ \frac{\partial q(x)}{\partial x_2} \\ \\ (- \ x_3)\frac{\partial q(x)}{\partial x_1} \ + \ (- \ x_2 \ - \ x1)\frac{\partial q(x)}{\partial x_3} \ = \ 0 \\ \\ \frac{\partial q(x)}{\partial x_1} \ + \ x_1 \frac{\partial q(x)}{\partial x_3} \ \neq \ 0 \end{cases} \tag{24}$$

Assuming $q(x) = x_1 x_3 \ - \ x_2 x_3$ *iff* $x_1 \neq 1 \ \wedge \ x_2 \ \neq 2 \ \wedge \ x_3 \ \neq \ 1$, the next step is to determine $L_f q(x)$ and $L^2_f q(x)$:

$$L_f q(x) \ = \ \nabla q(x) \cdot f(x) \ = \ \begin{bmatrix} x_3 & -x_3 & x_1 - x_2 \end{bmatrix} \cdot \begin{bmatrix} - \ x_1 \\ - \ x_2 \ - \ x_1 x_3 \\ x_1 x_2 \end{bmatrix} \ \Leftrightarrow$$

$$\Leftrightarrow \ L_f q(x) \ = \ -x_1 x_3 \ + \ x_3(x_2 \ + \ x_1 x_3) \ + \ (x_1 \ - \ x_2) x_1 x_2 \tag{25}$$

$$L^2_f q(x) \ = \ \nabla(L_f q(x)) \cdot f(x) \ =$$

$$= \ \begin{bmatrix} -x_3 \ + x_3^2 \ + 2x_1 x_2 \ - x_2^2 & -x_3 \ + x_1^2 \ - x_1 x_2 & -x_1 \ + x_2 \ + 2x_1 x_3 \end{bmatrix} \cdot f(x) \ =$$

$$= -4x_1^2 x_2 - x_1 x_3^2 - 2x_1^2 x_3 + 2x_1 x_2^2 + x_1 x_3(1 - x_2) + 2x_2^2 - x_2 x_3(1 + x_1) + 2x_1^2 x_2 x_3 \tag{26}$$

This results in the following transformation:

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} x_1 x_3 \ - \ x_2 x_3 \\ -x_1 x_3 \ + \ x_3(x_2 \ + \ x_1 x_3) \ + \ (x_1 \ - \ x_2) x_1 x_2 \\ -4x_1^2 x_2 - x_1 x_3^2 - 2x_1^2 x_3 + 2x_1 x_2^2 + x_1 x_3(1 - x_2) + 2x_2^2 - x_2 x_3(1 + x_1) + 2x_1^2 x_2 x_3 \end{bmatrix} \tag{27}$$

Unfortunately, this matrix is **not invertible**. This was conclude by both analytically and a MATLAB script. Which means that it is not possible to design a suitable controller, for this dynamic system, using the control technique input-state feedback linearization.

2.2 Input-Output Feedback Linearization

For the purpose of using the control technique input-output feedback linearization the system is decomposed in the vector fields $f(x)$ and $g(x)$, from equation 13, and the scalar field $h(x)$:

$$h(x) \;=\; x_2 \tag{28}$$

The design process is detailed in the steps below:

1. **Obtain the system relative degree,** r

   The relative degree is minimum number of differentiations required for the input u to explicitly appear in the output y:

$$\begin{aligned} y \;&=\; x_2 \\ \dot{y} \;&=\dot{x}_2 \;=\; -x_2 \;-\; x_1 x_3 \;+\; u \end{aligned} \tag{29}$$

   Which for this system is $r = 1$.

2. **Obtain the state space transformation**

   This coordinate transformation transforms the system to its **normal form**

$$\xi = \Phi(x) = \begin{bmatrix} h(x) \\ \phi_2(x) \\ \phi_3(x) \end{bmatrix}, \text{where} \begin{cases} \quad h(x) \;=\; x_2 \\ \\ \phi_2(x): \;\; L_g\phi_2(x) = 0 \Leftrightarrow \frac{\partial\phi_2(x)}{\partial x_1} + \frac{\partial\phi_2(x)}{\partial x_2} = 0 \\ \\ \phi_3(x): \;\; L_g\phi_3(x) = 0 \Leftrightarrow \frac{\partial\phi_3(x)}{\partial x_1} + \frac{\partial\phi_3(x)}{\partial x_2} = 0 \end{cases} \tag{30}$$

   Assuming $\phi_2(x) = x_3$ and $\phi_3(x) = x_1 - x_2$, results an **invertible** state transformation:

$$\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ x_1 \;-\; x_2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \xi_3 \;+\; \xi_1 \\ \xi_1 \\ \xi_2 \end{bmatrix} \tag{31}$$

3. **Obtain the representation in Byrnes-Isidori form (normal form):**

$$\dot{\xi} \;=\; \frac{d\Phi(x)}{dx}\dot{x} \;=\; \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \dot{x} \Leftrightarrow \begin{cases} \dot{\xi}_1 \;=\; -\,\xi_1 \;-\; (\xi_3 \;+\; \xi_1)\xi_2 \;+\; u \\ \\ \dot{\xi}_2 \;=\; (\xi_3 \;+\; \xi_1)\xi_1 \\ \\ \dot{\xi}_3 \;=\; -\,\xi_3 \;+\; (\xi_3 \;+\; \xi_1)\xi_2 \\ \\ \;y \;=\; \xi_1 \end{cases} \tag{32}$$

4. **Check the internal stability zero dynamics**

   For a **stable** feedback law:

$$\begin{gathered} u = v + \xi_1 + (\xi_3 + \xi_1)\xi_2 + K\xi_1 \\ \Leftrightarrow \\ u = v + x_2 + x_1 x_3 + K x_2 \end{gathered} \tag{33}$$

   Zeroing the output gives these direct implications:

$$y = 0 \Leftrightarrow \xi_1 = 0 \Rightarrow \dot{\xi}_2 = 0 \tag{34}$$

   Looking at equation 32 with this in mind is possible deduce that $u = (\xi_3 + \xi_1)\xi_2$, which also means that $\dot{\xi}_3 = -\xi_3 + u$. So assuming a stable input $u = 0$, it results in the following zero dynamics:

$$\dot{\xi}_3 = F_3(\xi_1, \xi_2, \xi_3) = F_3(0, 0, \xi_3) = -\xi_3 \tag{35}$$

   Since $F_3(\xi_1, \xi_2, \xi_3) < 0$, the internal dynamics is asymptotically stable, which implies the system is input-output feedback linearizable.

For the purpose of having a more graphical comprehension, of this assumed stability regarding the feedback law showed in equation 33, a variety of pole placement were performed, as you can see in the figure 11.

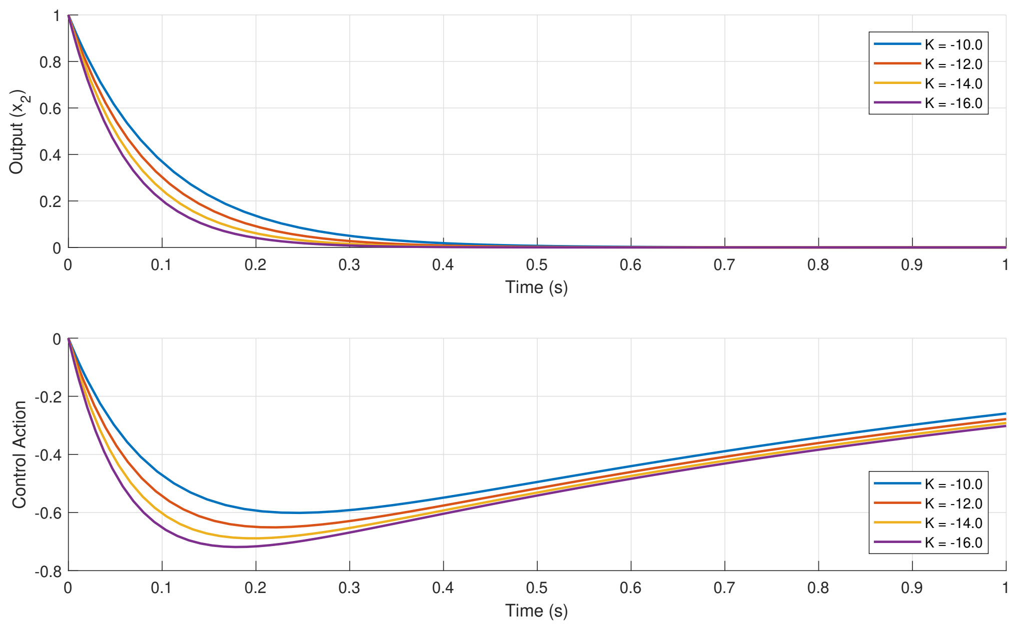

Figure 11: System response for different values of gain K.

2.3 Sliding Mode Control

The design process for a sliding mode control can be divided into three parts, each of which is detailed below:

1. **Define the sliding surface**

   Since the system described in equation 12 has three states, the general sliding surface can be defined as:
   $$s(x) = s_1 x_1 + s_2 x_2 + s_3 x_3 \tag{36}$$
   Considering $s_1 , s_2 , s_3 > 0$.

2. **Guarantee that the system states are attracted to the sliding surface**

   To prove that the system is attracted to the sliding surface, two conditions must be verified: $s(x)$ must form an invariant set, and the state convergence must occur in finite time.

   - **Invariant set**
     For the first condition, considering an energy function defined as $V(x) = \frac{1}{2}s^2(x)$, $s(x)$ will form an invariant set if the following conditions are satisfied:
     $$\begin{cases} V(x) = \frac{1}{2}s^2(x) > 0 & \text{(a)} \\ \dot{V}(x) = s(x)\dot{s}(x) < 0 & \text{(b)} \end{cases} \tag{37}$$
     Changing the value of s(x) as defined in 36 in the function given by 37.a, it is possible to have the following:
     $$V(x) = \frac{1}{2}s^2(x) = \frac{1}{2}\mathbf{x}^T\mathbf{P}\mathbf{x} \Longrightarrow \mathbf{P} = \begin{bmatrix} s_1^2 & s_1 s_2 & s_1 s_3 \\ s_1 s_2 & s_2^2 & s_2 s_3 \\ s_1 s_3 & s_2 s_3 & s_3^2 \end{bmatrix} \tag{38}$$
     As $s_1 , s_2 , s_3 > 0$, the matrix $\mathbf{P} > 0$, so $V(x) > 0 \, \forall \, \mathbf{x}$.
     Developing now the inequality given by equation 37.b and considering a feedback law $u = K_1 x_1 + K_2 x_2 + K_3 x_3$, the following expression is obtained:
     $$\begin{aligned} \dot{V}(x) &= s(x)\dot{s}(x) \\ \dot{V}(x) &= s(x)\left(s_1\dot{x}_1 + s_2\dot{x}_2 + s_3\dot{x}_3\right) \\ \dot{V}(x) &= s(x)x_1\left(-s_1 + s_1K_1 + s_3K_1 - s_2x_3 + s_3x_2\right) + \\ &\quad s(x)x_2\left(-s_2 + s_1K_2 + s_3K_2\right) + \\ &\quad s(x)x_3\left(s_1K_3 + s_3K_3\right) \end{aligned} \tag{39}$$
     To clarify the analysis of the sign of $\dot{V}(x)$, the following terms are defined:
     $$\begin{aligned} A &= s(x)x_1\left(-s_1 + s_1K_1 + s_3K_1 - s_2x_3 + s_3x_2\right) \\ B &= s(x)x_2\left(-s_2 + s_1K_2 + s_3K_2\right) \\ C &= s(x)x_3\left(s_1K_3 + s_3K_3\right) \end{aligned} \tag{40}$$

So, for each of these terms:

$$
\begin{array}{l}
A<0 \Longleftrightarrow \begin{cases} s(x)x_1>0 \wedge & -s_1+K_1(s_1+s_3)-s_2x_3-s_3x_2<0 \\ & \Longrightarrow K_1<\frac{s_1+s_2x_3+s_3x_2}{s_1+s_3} \qquad \text{(a)} \\ s(x)x_1<0 \wedge & -s_1+K_1(s_1+s_3)-s_2x_3-s_3x_2>0 \\ & \Longrightarrow K_1>\frac{s_1+s_2x_3+s_3x_2}{s_1+s_3} \qquad \text{(b)} \end{cases} \\
B<0 \Longleftrightarrow \begin{cases} s(x)x_2>0 \wedge & -s_2+K_2(s_1+s_3)<0 \\ & \Longrightarrow K_2<\frac{s_2}{s_1+s_3} \qquad \text{(c)} \\ s(x)x_2<0 \wedge & -s_2+K_2(s_1+s_3)>0 \\ & \Longrightarrow K_2>\frac{s_2}{s_1+s_3} \qquad \text{(d)} \end{cases} \\
C<0 \Longleftrightarrow \begin{cases} s(x)x_3>0 \wedge & K_3(s_1+s_3)<0 \\ & \Longrightarrow K_3<0 \qquad \text{(e)} \\ s(x)x_3<0 \wedge & K_3(s_1+s_3)>0 \\ & \Longrightarrow K_3>0 \qquad \text{(f)} \end{cases}
\end{array}
\tag{41}
$$

Considering a sliding surface defined by $s_1 = 1$, $s_2 = 1$, $s_3 = 1$, so the sliding equation 36 is equal to:

$$s(x) = x_1 + x_2 + x_3 \tag{42}$$

Beyond this, given a state space $\mathbf{x} = [x_1\ x_2\ x_3] = [2\ 2\ 2]$, it is possible to conclude the following:

$$
\begin{array}{lllll}
s(x)x_1 = & (2+2+2)\cdot 2 = & 12 > 0 & \Longrightarrow & \text{equation 41.a} \\
s(x)x_2 = & (2+2+2)\cdot 2 = & 12 > 0 & \Longrightarrow & \text{equation 41.c} \\
s(x)x_3 = & (2+2+2)\cdot 2 = & 12 > 0 & \Longrightarrow & \text{equation 41.e}
\end{array}
\tag{43}
$$

So, the possible values of K_1, K_2 and K_3 are:

$$
\begin{array}{llll}
K_1 & < \frac{s_1+s_2x_3+s_3x_2}{s_1+s_3} & \Longrightarrow K_1<2.5 & \Longrightarrow K_1=0.25 \\
K_2 & < \frac{s_2}{s_1+s_3} & \Longrightarrow K_2<0.5 & \Longrightarrow K_2=0.25 \\
K_3 & < 0 & \Longrightarrow K_3<0 & \Longrightarrow K_3=-1
\end{array}
\tag{44}
$$

It is important to note that these gain values are valid for the entire trajectory. Therefore, the control action at this point $\mathbf{x} = [2\ 2\ 2]$ could be:

$$u = 0.25x_1 + 0.25x_2 - x_3 \tag{45}$$

- **State convergence in finite time**

  To guarantee that the control law makes the system converges in a finite time, the following term is added to control action given by equation 45:

$$u = K_1x_1 + K_2x_2 + K_3x_3 + u_N \tag{46}$$

where K_1, K_2 and K_3, are the terms already calculated before, and the new control need to be designed such that $\dot{V}(x) \leq -\eta|s(x)|$, $\eta > 0$.
So, developing the equation of $\dot{V}(x)$, the new term that need to be calculated is given by:

$$D = u_N s(x)(s_1 + s_3) = -\eta|s(x)| \implies u_N = -\frac{\eta}{s_1 + s_3}\text{sign}(s(x)) \quad (47)$$

As $s_1 = 1$ and $s_3 = 1$, $u_N = -\frac{\eta}{2}\text{sign}(s(x))$

Therefore, a possible control action that guarantee that the system states are attracted to the sliding surface is given by:

$$u = 0.25x_1 + 0.25x_2 - x_3 - \frac{\eta}{2}\text{sign}(s(x)) \quad (48)$$

3. **Guarantee that the system states remain on the sliding surface**

   Once the system is attracted to the sliding surface, the control law must ensure that it remains on the sliding surface defined by Equation 42. The control input u is expressed as:

$$u = u_{eq} + u_N \quad (49)$$

   where u_{eq} is defined as:

$$\dot{s}(x) = 0 \implies u_{eq} = \frac{-L_f s}{L_g s} = \frac{-x_1 - x_2 - x_1 x_3}{2} \quad (50)$$

   Considering an uncertainty parameter constant ρ equal to 0.1, the u_N expression is given by:

$$u_N = -(\rho + \eta)\ sat\left(\frac{s(x)}{\Phi}\right) \quad (51)$$

In order to testing robustness to parameters, the following control law was applied for different values of η and Φ, in a initial state space equal to $\mathbf{x} = [2\ 2\ 2]$:

- Before reaching the surface

$$u = 0.25x_1 + 0.25x_2 - x_3 - \eta\ sat\left(\frac{s(x)}{\Phi}\right) \quad (52)$$

- After reaching the surface ($|s(x)| \leq \Phi$)

$$u = \frac{-x_1 - x_2 - x_1 x_3}{2} - (\rho + \eta)\ sat\left(\frac{s(x)}{\Phi}\right) \quad (53)$$

Each parameter was varied within a specified range, while the others were maintained constant. The resulting state-space trajectory for x_2, control actions, and sliding surfaces were then analyzed as follows:

1. $\mathbf{\Phi} = [\mathbf{0.1}\ \ \mathbf{1}\ \ \mathbf{2}\ \ \mathbf{4}]$ and $\eta = \mathbf{0.1}$

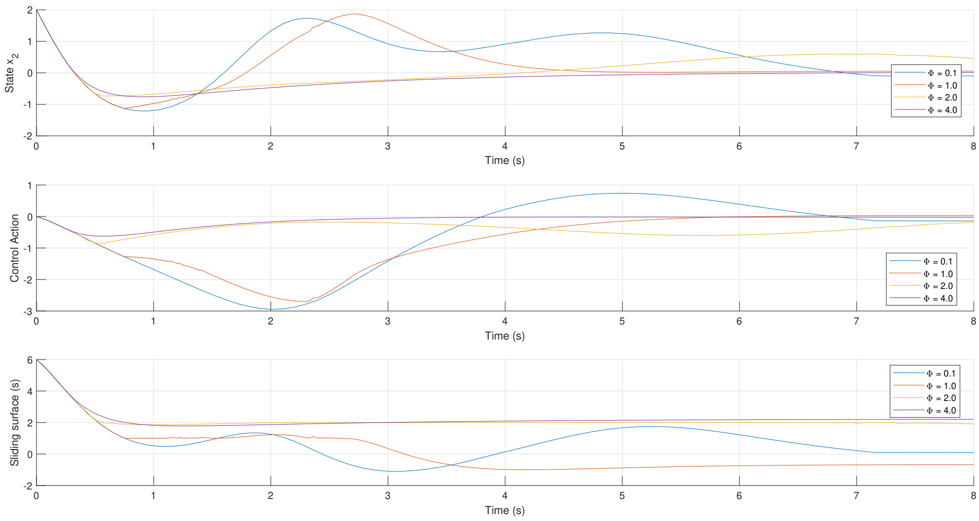

Figure 12: Influence of Φ in the feedback law control system.

As seen in the figure 12, as Φ increases, the chattering tends to decrease, resulting in a smoother control action. However, this also causes the states to deviate further from the defined sliding surface. Therefore, the choice of Φ represents a trade-off between the performance of the control action and the system's adherence to the sliding surface. Considering this, a reasonable choice for $\Phi = 1$ could be made, as it strikes a balance by minimizing chattering while ensuring the system remains close to the sliding surface.

2. $\eta = [\mathbf{0.1}\ \ \mathbf{0.5}\ \ \mathbf{1}\ \ \mathbf{1.5}]$ and $\mathbf{\Phi} = \mathbf{1}$

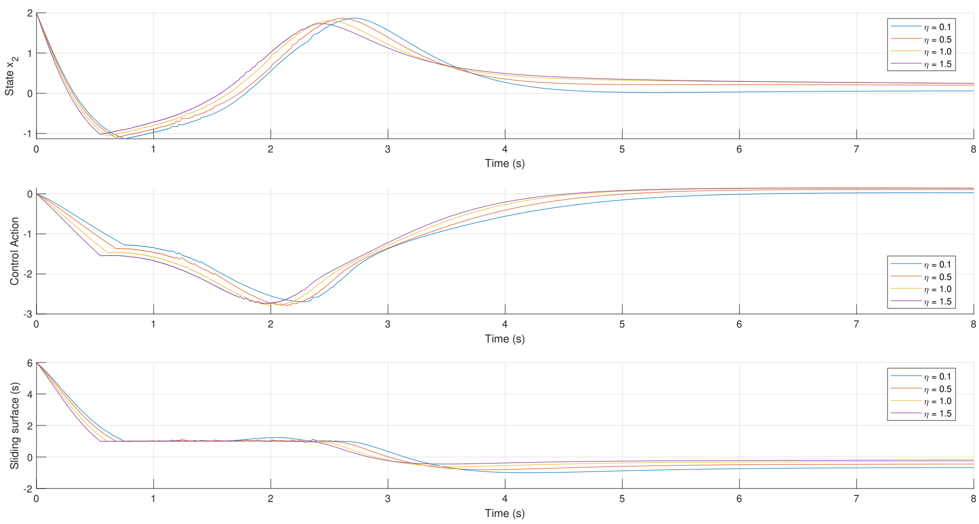

Figure 13: Influence of η in the feedback law control system.

The influence of η is related to the time of convergence to the sliding surface. As can be seen in Figure 13, as the value of η increases, the states converge to the surface more quickly. However, this needs to be considered alongside the potential increase in chattering. Therefore, based on the results shown above, a reasonable choice for $\eta = 1.5$ could be made, as it provides a good balance between fast convergence and manageable chattering.

Therefore, after analyzing each parameter, the best pair chosen was $\Phi = 1$ and $\eta = 1.5$. The behavior of the system under these conditions is shown in Figure 14.

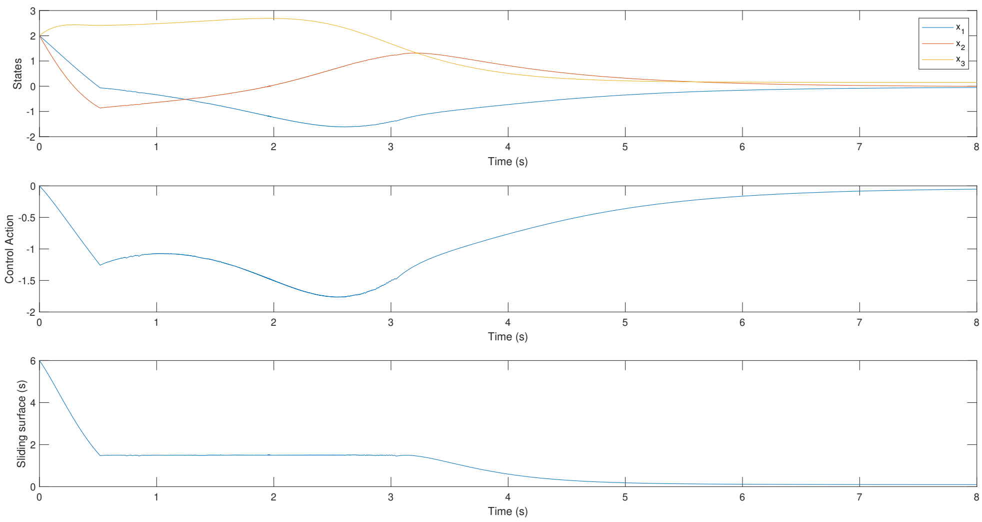

Figure 14: System behavior for $\Phi = 1$ and $\eta = 1.5$.