SQUARE COMPACTNESS AND LINDELÖF TREES

PEDRO E. MARUN

Abstract. We prove that every weakly square compact cardinal is a strong limit cardinal. We also study Aronszajn trees with no uncountable finitely branching subtrees, characterizing them in terms of being Lindelöf with respect to a particular topology. We prove that the class of such trees is non-empty and lies strictly between the classes of Suslin and Aronszajn trees.

1. Introduction

Recall that a topological space is *Lindelöf* if and only if every open cover has a countable subcover. Unlike compactness, the Lindelöf property need not be presserved by finite products, as shown by the classical Sorgenfrey line example, which is the space X with underlying set $\mathbb{R}$ and topology generated by all left-closed right-open intervals. This space is Lindelöf, but the uncountable set $\{(x, -x) : x \in \mathbb{R}\}$ is closed and discrete in X^2, hence X^2 is not Lindelöf. For details, see [12, Countereexample 84].

Extending this to larger cardinals κ, we say that a topological space X is *κ-compact* if and only if every open cover of X has a subcover of size less than κ. So, compact is $\aleph_0$-compact and Lindelöf is $\aleph_1$-compact. In connection with this, Hajnal and Juhász [5] introduced the following large cardinal notion: an infinite cardinal κ is *square compact* if and only if for every κ-compact space X, X^2 is κ-compact. This is in fact equivalent to the product of any two κ-compact spaces being κ-compact. For the non-trivial direction, if X and Y are κ-compact, then so is their disjoint sum $X \oplus Y$. By assumption, $(X \oplus Y)^2$ is κ-compact. Since $X \times Y$ is a closed subset of $(X \oplus Y)^2$, it follows that $X \times Y$ is κ-compact too.

Recall that the *weight* $w(X)$ of a topological space X is the least size of a basis for the topology on X. A refined version of square compactness, graduated by weights, was introduced by Buhagiar and Džamonja in their 2021 paper [2]: given some cardinal λ, an infinite cardinal κ is *λ-square compact* if and only if for every space X of weight $\leq \lambda$, if X is κ-compact, then X^2 is κ-compact They (and we) say that κ is *weakly square compact* if and only if it is κ-square compact.

In this terminology, the results in [5] can be stated as:

Theorem (Hajnal-Juhász [5, Theorem 1])**.** *Every weakly square compact cardinal is regular.*

Theorem (Hajnal-Juhász [5, Theorem 2])**.** *Suppose that κ is uncountable. If κ is 2^κ-square compact, then it is weakly compact.*

2020 *Mathematics Subject Classification.* Primary 03E05; Secondary 54B10,03E04.

The results of this paper will form a part of the author's PhD thesis written under the supervision of James Cummings, to whom the author would like to express his gratitude.

This shows that the existence of κ which is 2^κ-square compact is already a large cardinal notion. In [2], Buhagiar and Džamonja undertake a closer study of weak square compactness, and give a variety of equivalent formulations. In particular, they proved the following:

Theorem (Buhagiar-Džamonja, [2, Theorem 5.1])**.** *Let κ be an uncountable cardinal. Suppose that $\kappa^{<\kappa} = \kappa$. Then κ is weakly compact if and only if it is weakly square compact.*

The first result we establish in this paper is that the assumption $\kappa^{<\kappa} = \kappa$ in the previous theorem can be removed. This is done by generalizing the Sorgenfrey line construction.

As far as we know, strong compactness continues to be the best upper bound for the consistency strength of square compactness. This is part of the folklore of the subject and attributed to Hajnal and Juhász in [2]. However, finding a proof in the published literature is non-trivial, so we provide one (see Theorem 2.13).

Having studied topologies on linear orders, we turn to looking at topologies on trees, with a view towards introducing new examples of Lindelöf spaces. In the survey [10], Nyikos considers a total of ten different topologies on trees. Of these, only two are always Hausdorff, and we adhere to the doctrine of only considering Hausdorff spaces. By [10, Theorem 3.6], the *coarse wedge topology* appears uninteresting for our purposes, since it is ω_1-compact[1] if and only if the underlying tree has countably many minimal elements. This leaves us with only the *fine wedge topology* to focus on. We give a tree-theoretic characterization of being Lindelöf with respect to this topology. First, some terminology: we say that a tree is *finitely branching* if and only if every point in the tree has finitely many immediate successors. A *subtree* of a tree T is a set $S \subseteq T$ such that for all $x \in S$ and $y \in T$, if $y < x$, then $y \in S$.

Theorem. *Let T be an $\aleph_1$-tree. Then T is Lindelöf with respect to the fine-wedge topology if and only if every finitely branching subtree of T is countable.*

We shall show that, in ZFC,

$$\{\text{Suslin}\} \subsetneq \{\text{Lindelöf}\} \subsetneq \{\text{Aronszajn}\}.$$

Here, by Lindelöf we mean Lindelöf with respect to the fine-wedge topology.

Given a partially ordered set X, we let X^* denote the *dual order* on X, that is $x <^* y$ if and only if $y < x$.

A *tree* is a pair $(T, <_T)$ such that $<_T$ is a strict partial order on T and $\{y \in T : y <_T x\}$ is well-ordered for every $x \in T$. We will usually suppress the subscript in $<_T$ and identify the tree with its underlying set when there is no danger of confusion.

Elements of a tree are referred to as *nodes* or *points*. We say $x, y \in T$ are *comparable*, denoted $x \parallel y$, if and only if $x \leq_T y$ or $y \leq_T x$. Otherwise, we say that x and y are *incomparable*, denoted $x \perp y$. The *height* of a node $x \in T$ is the order-type of the set $\{y \in T : y <_T x\}$, denoted $\mathrm{ht}_T(x)$ (or simply $\mathrm{ht}(x)$). Given an ordinal α, *level* α of the tree is the set $T_\alpha = \{x \in T : \mathrm{ht}_T(x) = \alpha\}$. The *height* $h_T(T)$ of T is defined by $\mathrm{ht}(T) = \min\{\alpha : T_\alpha = \emptyset\}$. Given an ordinal $\alpha < \mathrm{ht}(T)$, we let $T{\restriction}\alpha = \{x \in T : \mathrm{ht}(x) < \alpha\}$, which is of course a subtree of T of height α.

[1] We caution the reader that, in Nyikos' survey, X being ω_1-compact means that every closed and discrete subset of X is countable.

Given $x \in T$, we let $I_T(x)$ denote the set of *immediate successors* of x, and write $I(x)$ when there is no possibility of confusion.

2. Square compactness

We shall say a space X is *hereditarily κ-compact* if and only if every subspace of X is κ-compact. For example, any space of weight less than κ is hereditarily κ-compact.

A useful criterion for hereditary κ-compactness is

Lemma 2.1. *Let (X, τ) be a topological space. Then X is hereditarily κ-compact if and only if for every $\mathcal{U} \subseteq \tau$ there is some $\mathcal{U}_0 \in [\mathcal{U}]^{<\kappa}$ with $\bigcup \mathcal{U} = \bigcup \mathcal{U}_0$.*

Proof. $\Rightarrow$) Given $\mathcal{U}$, consider the subspace $\bigcup \mathcal{U}$.

$\Leftarrow$) Suppose $Y \subseteq X$ is not κ-compact. Fix some $\mathcal{U} \subseteq \tau$ such that $\bigcup \mathcal{U} \supseteq Y$ but there is no $\mathcal{U}_0 \in [\mathcal{U}]^{<\kappa}$ with $\bigcup \mathcal{U}_0 \supseteq Y$. Then $\bigcup \mathcal{U}_0 \neq \bigcup \mathcal{U}$ for every $\mathcal{U}_0 \in [\mathcal{U}]^{<\kappa}$. $\square$

The following is obvious:

Lemma 2.2. *Suppose that (X, τ) is κ-compact and $Y \subseteq X$ is closed. Then Y is κ-compact with the subspace topology.*

As mentioned in the introduction, Hajnal and Juhász already proved that weak square compactness entails regularity. To deal with the (strong) inaccessibility of κ, we will generalize the classical construction of the Sorgenfrey line to larger linear orders.

Definition 2.3. Let $(X, <)$ be a *dlo* (dense linear order without end-points). The *density* of X, denoted $d(X)$, is the cardinal

$$d(X) = \min\{|D| : D \text{ is dense in } X\}$$

This of course coincides with the density of X as a topological spacer under the order topology. $\lrcorner$

For example, $d(\mathbb{R}) = \aleph_0$. It is straightforward to show that $w(X)$, the weight of X with respect to the order topology, is exactly $d(X)$.

Definition 2.4. Given a dlo $(X, <)$, the family $\{[x, y) : x, y \in X \wedge x < y\}$ forms a basis for a topology on X, which we shall call the *Sorgenfrey* topology. $\lrcorner$

Lemma 2.5. *Let $(X, <)$ be a dlo with $d(X) < \kappa$. Then the Sorgenfrey topology on X is hereditarily κ-compact.*

Proof. Let $\mathcal{U} \subseteq \{[x, y) : x, y \in X\}$ and let $W = \bigcup\{(x, y) : [x, y) \in \mathcal{U}\}$. Obviously, W is open with respect to the order topology on X, which has weight less than κ. By Lemma 2.1, $W = \bigcup\{(x, y) : [x, y) \in \mathcal{U}_0\}$ for some $\mathcal{U}_0 \in [\mathcal{U}]^{<\kappa}$. Let $A := (\bigcup \mathcal{U}) \setminus W$.

Claim. $|A| < \kappa$.

Proof of claim. Fix $D \in [X]^{<\kappa}$ dense in the order topology. For each $x \in A$, find $[a_x, b_x) \in \mathcal{U}$ such that $x \in [a_x, b_x)$. Since $x \notin W$, we infer that $x = a_x < b_x$, so we can pick some $d_x \in D$ with $x < d_x < b_x$. Now suppose $x, y \in X$ with $x < y$. Since $y \notin W$, $b_x \leq y$. Then $d_x < b_x \leq y < d_y$, so $d_x < d_y$. Therefore, $x \mapsto d_x$ is an injective map from A into D. $\dashv$

For each $x \in A$, pick $U_x \in \mathcal{U}$ with $x \in U_x$. Let $\mathcal{U}_1 = \{U_x : x \in A\}$. Clearly, $|\mathcal{U}_1| < \kappa$. We now have that $\mathcal{U}_2 = \mathcal{U}_0 \cup \mathcal{U}_1 \in [\mathcal{U}]^{<\kappa}$ and $\bigcup \mathcal{U}_2 = \bigcup \mathcal{U}$. $\square$

Lemma 2.6. *Let $\kappa > \omega$ be a cardinal. Suppose that there is a dlo $(X, <)$ with $d(X) < \kappa = |X|$. Then κ is not κ-square compact.*

Proof. Replacing X by $X \oplus X^*$ if necessary, we may assume that $(X, <)$ admits an order reversing involution, which we shall suggestively denote by $x \mapsto -x$.

Let τ be the Sorgenfrey topology on X. Note that $w(X, \tau) \leq \kappa$, because $|X| \leq \kappa$, and that (X, τ) is κ-compact by lemma 2.5. It therefore suffices to show that X^2 is not κ-compact with respect to the product topology. Let

$$Y = \{(x, -x) : x \in X\}.$$

Since $x \mapsto -x$ is order-reversing, it is continuous with respect to the order topology $\tau_<$, hence Y is closed in $(X^2, \tau_< \otimes \tau_<)$. But $\tau_< \subseteq \tau$, so Y is closed in $(X^2, \tau \otimes \tau)$. For each $x \in X$, pick $u_x, v_x \in X$ with $u_x < x < v_x$. Now observe that

$$([x, v_x) \times [-x, -u_x)) \cap Y = \{(x, -x)\}.$$

We have shown that Y is discrete in $(X^2, \tau \otimes \tau)$. Since $|Y| = \kappa$, Y is not κ-compact, and so neither is $(X^2, \tau \otimes \tau)$ because Y is closed. □

The goal now is to build large dlo's with small density. This will be possible, under certain cardinal arithmetic constraints. Our original construction was rather convoluted, and we thank Will Brian for suggesting the following simpler approach.

Lemma 2.7. *Let $\kappa \geq \omega_1$. Suppose there exist infinite cardinals μ and θ such that $\mu^{<\theta} = \mu < \kappa \leq \mu^\theta$. Then there is a dlo X with $d(X) < \kappa = |X|$.*

Proof. Let $Y := {}^\theta\mu$, ordered lexicographically. Note that $|Y| = \mu^\theta \geq \kappa$. Let D be the set of sequences in Y which are eventually 0. Then D is dense in Y and $|D| = \mu^{<\theta} < \kappa$. By the Downward Lowenheim-Skölem theorem, find $X \prec Y$ with $D \subseteq X$ and $|X| = \kappa$. Since D is dense in X, $d(X) < \kappa$. □

Theorem 2.8. *Let $\kappa \geq \omega_1$. If there are cardinals μ and θ such that $\mu^{<\theta} = \mu < \kappa \leq \mu^\theta$, then κ is not κ-compact.*

Proof. Immediate from Lemmas 2.6 and 2.7. □

Corollary 2.9. *Suppose $\lambda \geq \omega$. Then λ^+ is not λ^+-square compact.*

Proof. Let $\theta := \min\{\nu : \lambda^\nu \geq \lambda\}$. By König's lemma, $\theta \leq \mathrm{cf}(\lambda)$, so $\lambda^{<\theta} = \lambda$ by the minimality of θ. Now apply Lemma 2.7 with $\kappa = \lambda^+$ and $\mu = \lambda$. □

In particular, if κ is κ-square compact, then κ is a limit cardinal, hence weakly inaccessible. In fact, this can be improved:

Corollary 2.10. *Suppose κ is κ-square compact. Then κ is strongly inaccessible.*

Proof. Suppose κ is not strong limit. Let

$$\theta = \min\{\nu : \exists\lambda\,(\nu \leq \lambda < \kappa \leq \lambda^\nu)\}.$$

To see that this is well defined, fix $\delta < \kappa$ so that $2^\delta \geq \kappa$, and take $\lambda = \nu = \delta$.

Having fixed θ, let $\lambda < \kappa$ be the least witness to the definition of θ, that is $\theta \leq \lambda < \kappa \leq \lambda^\theta$ and λ is least with these properties. Note that, if $\alpha < \theta$, then $\lambda^\alpha < \kappa$, since otherwise α contradicts the minimal choice of θ.

Claim 1. *θ is regular.*

Proof of claim 1. Suppose not, say $\theta^* = \mathrm{cf}(\theta) < \theta$. Fix $\langle \theta_\xi : \xi < \theta^* \rangle$ cofinal in θ. By the minimality of θ, $\lambda^{\theta_\xi} < \kappa$ for every $\xi < \theta^*$. Let $\lambda^* := \sup\{\lambda^{\theta_\xi} : \xi < \theta^*\}$. Since κ is regular and $\theta^* < \theta < \kappa$, it follows that $\lambda^* < \kappa$. We therefore have

$$\kappa \leq \lambda^\theta = \prod_{\xi < \theta^*} \lambda^{\theta_\xi} \leq (\lambda^*)^{\theta^*}$$

This contradicts the minimality of θ. ⊣

Put $\mu = \lambda^{<\theta}$. Again, $\mu < \kappa$, because $\lambda^\alpha < \kappa$ for $\alpha < \theta$ and $\theta < \kappa = \mathrm{cf}(\kappa)$. Also, $\mu^\theta \geq \lambda^\theta \geq \kappa$.

Claim 2. $\mu^{<\theta} = \mu$.

Proof of claim 2. We consider two separate cases.

Case 1: $\alpha \mapsto \lambda^\alpha$ is eventually constant for $\alpha < \theta$. Note that this includes the case when θ is a successor cardinal. By definition of μ, the eventual constant value must be μ, so $\lambda^\alpha = \mu$ for all large enough $\alpha < \theta$. But then $\mu^\alpha = \mu$ whenever $\alpha < \theta$ is sufficiently big, hence $\mu^{<\theta} = \mu$.

Case 2: λ^α is not eventually constant for $\alpha < \theta$. As θ is regular, $\mathrm{cf}(\mu) = \theta$. So, if $\alpha < \theta$, we have

$$\mu^\alpha = \sum_{\beta < \theta} (\lambda^\beta)^\alpha = \mu.$$

Therefore, $\mu^{<\theta} = \mu$. ⊣

Therefore, μ, θ and κ satisfy the conditions of Theorem 2.7, so κ is not κ-square compact. □

Theorem 2.11. *Let κ be an uncountable cardinal. Then κ is weakly compact if and only if it is κ-square compact.*

Proof. The forwards direction can be found in [5], Theorem 2. The backwards direction is in [2], Theorem 5.1, under the additional hypothesis that $\kappa^{<\kappa} = \kappa$. But this is redundant when κ is κ-square compact, because κ is strongly inaccessible by Corollary 2.10. □

To make the paper self-contained, we include a proof that strong compactness implies square compactness.

Recall that a *subbasis* for a topology τ (on a set X) is a family $\mathcal{S}$ such that τ is the smallest topology on X including $\mathcal{S}$. Equivalently, the set of finite intersections of members of $\mathcal{S}$ is a basis for τ.

Lemma 2.12. *Let κ be a strongly compact cardinal and X a topological space. Suppose that there exists a subbasis $\mathcal{S}$ such that for every cover of X using members of $\mathcal{S}$ there exists a subcover of size $< \kappa$. Then X is κ-compact.*

Proof. Let $\mathcal{B}$ be the collection of finite intersections of sets in $\mathcal{S}$, so $\mathcal{B}$ is a basis for X. If suffices to argue that every open cover of X consisting of members of $\mathcal{B}$ has a subcover of size $< \kappa$. Suppose towards a contradiction that this is not the case. Let $\mathcal{U} \subseteq \mathcal{B}$ be a cover of X such that no subset of $\mathcal{U}$ of size $< \kappa$ covers X. Let $\mathcal{I}$ be the κ-complete ideal generated by $\mathcal{U}$:

$$\mathcal{I} = \left\{ A \subseteq X : \exists \mathcal{U}_0 \in [\mathcal{U}]^{<\kappa} (A \subseteq \bigcup \mathcal{U}_0) \right\}$$

By our assumption on $\mathcal{U}$, $X \notin \mathcal{I}$, so $\mathcal{I}$ is proper. Since κ is strongly compact, there is a prime κ-complete ideal $\mathcal{J}$ on X such that $\mathcal{I} \subseteq \mathcal{J}$.

Claim. *If $x \in X$ then there is some $W_x \in \mathcal{J} \cap \mathcal{S}$ such that $x \in W_x$.*

Proof of claim. Fix $x \in X$. Since $\mathcal{U} \subseteq \mathcal{B}$ covers X, by definition of $\mathcal{B}$ there exists a finite sequence $\langle W_i^x : i < n_x \rangle \in \mathcal{S}^{n_x}$, where $n_x \in \omega$, such that $x \in \bigcap_{i<n_x} W_i^x$. By definition of $\mathcal{I}$, $\bigcap_{i<n_x} W_i^x \in \mathcal{J}$. Since $\mathcal{J}$ is prime, there exists $i_x < n_x$ such that $W_{i_x}^x \in \mathcal{J}$. Put $W_x = W_{i_x}^x$. This works. $\dashv$

Using the claim we can choose, for each $x \in X$, a set $W_x \in \mathcal{J} \cap \mathcal{S}$ such that $x \in W_x$. Obviously, $\{W_x : x \in X\}$ covers X. Since $\{W_x : x \in X\} \subseteq \mathcal{S}$, our hypothesis on $\mathcal{S}$ implies the existence of $Y \in [X]^{<\kappa}$ such that $\{W_x : x \in Y\}$ covers X. In symbols, $X = \bigcup_{x \in Y} W_x$. But $W_x \in \mathcal{J}$, $\mathcal{J}$ is κ-complete and $|Y| < \kappa$, so $X \in \mathcal{J}$. This contradicts that $\mathcal{J}$ is a proper ideal. $\square$

Corollary 2.13 (Hajnal-Juhász)**.** *Every strongly compact cardinal is square compact.*

Proof. Let κ be strongly compact. Suppose (X, τ) is a κ-compact space and let

$$\mathcal{S} := \{X \times U : U \in \tau\} \cup \{U \times X : U \in \tau\}.$$

It is clear that $\mathcal{S}$ is a subbasis for the product topology on X^2.

Let $\mathcal{U} \subseteq \mathcal{S}$ cover X, we argue that $\mathcal{U}$ has a subcover of size $< \kappa$. By Lemma 2.12, this is enough to complete the proof. Put $\mathcal{U}_0 := \mathcal{U} \cap (\tau \times \{X\})$ and $\mathcal{U}_1 = \mathcal{U} \cap (\{X\} \times \tau)$, so that $\mathcal{U} = \mathcal{U}_0 \cup \mathcal{U}_1$. Let $\mathcal{V}_0 = \{V : V \times X \in \mathcal{U}_0\}$ and $\mathcal{V}_1 = \{V : X \times V \in \mathcal{V}_1\}$.

Claim. *At least one of $\mathcal{V}_0$ or $\mathcal{V}_1$ covers X.*

Proof of claim. Suppose that $\bigcup \mathcal{V}_0 \neq X$ and $\bigcup \mathcal{V}_1 \neq X$. Pick $x_0 \in X \setminus \bigcup \mathcal{V}_0$ and $x_1 \in X \setminus \bigcup \mathcal{V}_1$. By assumption, $\mathcal{U}$ covers X^2, so $(x_0, x_1) \in U$ for some $U \in \mathcal{U}$. There are now two possibilities: either $U = V \times X$ for some $V \in \tau$, in which case $x_0 \in \bigcup \mathcal{V}_0$, or $U = X \times V$ for some $V \in \tau$, in which case $x_1 \in \bigcup \mathcal{V}_1$. In either case, we get a contradiction. $\dashv$

Suppose that $\mathcal{V}_0$ covers X, the other case is analogous. Let $\mathcal{V}$ be a subcover of $\mathcal{V}_0$ of size $< \kappa$. Then $\{V \times X : V \in \mathcal{V}\}$ is a subcover of $\mathcal{U}$ of size $< \kappa$, which completes the proof. $\square$

3. Preliminaries on trees

A *chain* in a tree T is a subset of T which is linearly ordered by $<_T$. A *branch* is a maximal chain. A *cofinal branch* is a branch which meets every level of T.

Given a regular cardinal κ, we say that T is a *κ-tree* if and only if $\mathrm{ht}(T) = \kappa$ and $|T_\alpha| < \kappa$ for every $\alpha < \kappa$. We say T is a *κ-Aronszajn tree* if and only if it is a κ-tree with no branches. An Aronszajn tree is just an $\aleph_1$-Aronszajn tree. Classically:

Theorem (König, [6])**.** *There are no $\aleph_0$-Aronszajn trees.*

Theorem (Aronszajn, see [13])**.** *There is an Aronszajn tree.*

Theorem (Specker, [13])**.** *If* CH *holds, then there is an $\aleph_2$-Aronszajn tree.*

Theorem (Mitchell-Silver, [9])**.** *The theories*

- ZFC + *"There is a weakly compact cardinal",*
- ZFC + *"There are no $\aleph_2$-Aronszajn trees"*

are equiconsistent.

An *antichain* in a tree is a set of pairwise incomparable elements of T. A κ-*Suslin tree* is a κ-tree which has no chains or antichains of size κ. A *Suslin tree* is an $\aleph_1$-Suslin tree.

A tree T is *normal* if and only if it satisfies the following conditions:

- It has a unique minimal element (called a *root*),
- for all $\alpha < \beta < \mathrm{ht}(T)$ and all $x \in T_\alpha$ there is some $y \in T_\beta$ such that $x < y$,
- for all $\alpha < \mathrm{ht}(T)$ and $x \in T_\alpha$ there exist $y, z \in T$ such that $x < y$, $x < z$, and $y \perp z$.

Lemma 3.1 (folklore)**.** *Let T be a normal κ-tree. If T is has no antichains of size κ, then T is κ-Suslin.*

Proof. If b is a branch through T of length κ, use the normality of T to pick, for each $x \in b$, some $y_x \in I(x)$ such that $y_x \notin b$. Then $A = \{y_x : x \in b\}$ is an antichain and $|A| = \kappa$. □

In view of Lemma 3.1, to check whether a given κ-tree is Suslin, one "only" has to argue that all of its antichains have size less than κ. We shall make use of this fact without any further mention.

We also recall that an $\aleph_1$-tree is *special* if and only if it can be written as a countable union of antichains. Equivalently, T is special if and only if there is an order preserving map $T \to \mathbb{Q}$, see [7, Lemma III.5.17].

Let $(T, <)$ be a tree. If $X \subseteq T$, we let $\uparrow X := \{y \in T : \exists x \in X(x \leq y)\}$. If $X = \{x\}$, we write $\uparrow x$ instead of $\uparrow\{x\}$. The symbols $\downarrow X$ and $\downarrow x$ are defined analogously.

4. The fine wedge topology

If T is a tree, the *fine wedge topology* on T is generated by the sets $\uparrow t$ and their complements, where $t \in T$.

Note that, if $x < y$, then $(\uparrow x) \setminus \uparrow y$ and $\uparrow y$ are disjoint open neighbourhoods of x and y, respectively. If $x \perp y$, then $\uparrow x$ and $\uparrow y$ are disjoint open neighbourhoods of x and y. We have thus shown that the topology is Hausdorff.

All topological notions below refer to the fine wedge topology.

If T is *finitely branching at x* (that is $|I(x)| < \aleph_0$), then the identity

$$\{x\} = (\uparrow x) \cap \bigcap_{y \in I(x)} (\uparrow y)^c$$

shows that x is isolated. Therefore, if T is finitely branching, the fine wedge topology is just the discrete topology on T. The interplay between finite and infinitely branching trees will play a key role in our work, see Theorem 4.9.

Recall that, if X is a topological space and $x \in X$, we say that a collection of open sets $\mathcal{B}$ is a *local basis at x* if and only if for every open set U with $x \in U$ there is some $B \in \mathcal{B}$ with $x \in B \subseteq U$. We define the *character of x* to be the cardinal $\chi(x, X) := \min\{|\mathcal{B}| : \mathcal{B} \text{ is a local basis at } x\}$.

Lemma 4.1. *Let T be a tree. Given $x \in T$, the sets*

$$(\uparrow x) \setminus \uparrow F,$$

where $F \in [I(x)]^{<\omega}$, form a local basis at x, and so $\chi(x, T) = |I(x)|$. In particular, if every node has $\aleph_0$ many immediate successors, then the fine-wedge topology is first countable.

Proof. Let U be a basic open neighbourhood of x, say

$$x \in U = \bigcap_{i<n} \uparrow x_i \setminus \bigcup_{j<m} \uparrow y_j$$

for some $x_i, y_j \in T$, $n, m \in \omega$. Let $J = \{j < m : x < y_j\}$. For each $j \in J$, let $z_j \in I(x)$ be the unique point with $z_j \leq y_j$. Then

$$x \in (\uparrow x) \setminus \bigcup_{j \in J} \uparrow z_j \subseteq U,$$

which completes the proof. □

Lemma 4.2. *Suppose κ is a regular cardinal. Let $(T, <)$ be a κ-tree which is κ-compact. Then T is κ-Aronszajn.*

Proof. Suppose that b is a cofinal branch through T. Then $\{(\uparrow x)^c : x \in b\}$ has no subcover of size $< \kappa$ by regularity. □

Lemma 4.3. *Let $(T, <)$ be a κ-Aronszajn tree. Then every cover of T by subbasic open sets has a subcover of size $< \kappa$.*

Proof. Let $\mathcal{U}$ be a cover of T by subbasic open sets. Observe that, if all nodes at some level of the tree belong to a cone from $\mathcal{U}$, then these cones give a subcover of size $< \kappa$ of the tree above that level. But there are less than κ many nodes below that level, so we're done. The idea is essentially to show that such a "good" level must exist.

Consider the sets

$$A = \{t \in T : \uparrow t \in \mathcal{U}\}$$

and

$$B = \{t \in T : (\uparrow t)^c \in \mathcal{U}\},$$

where ^{c} denotes complementation with respect to T. Suppose there are $s, t \in B$ such that $s \perp t$. Then $(\uparrow s) \cap (\uparrow t) = \emptyset$, so $(\uparrow s)^c \cup (\uparrow t)^c = T$ and we've found a finite subcover. Therefore, we may assume that B is linearly ordered, hence a branch. Put

$$X = \bigcap_{x \in B} \uparrow x$$

and observe that

$$T = X \cup \bigcup_{x \in B} (\uparrow x)^c.$$

Since the tree is κ-Aronszajn, $|B| < \kappa$, hence we only need to show that X is covered by some subset of $\mathcal{U}$ of size κ. Let α be the least height of a member of X (if X is empty, there's nothing to do), and pick $y \in X \cap T_\alpha$, where T_α denotes the α^{th} level of T. Since $\mathcal{U}$ covers T, there is some $U \in \mathcal{U}$ with $y \in U$. If $U = (\uparrow t)^c$ for some t, then $t \in B$, so $b < y$, contradicting that $y \in (\uparrow b)^c$. It follows that $y \in \uparrow t$ for some $t \in A$. This shows that $T_\alpha \cap X$ is on e of the good levels described in the first paragraph of the proof, and we're done. □

We isolate the following elementary result from point-set topology:

Lemma 4.4. *Let X be a topological space and κ an infinite cardinal. Suppose we have a sequence $\langle \mathcal{B}_x : x \in X \rangle$ such that $\mathcal{B}_x$ is a local basis at x for every $x \in X$. Then X is κ-compact if and only if for each $\Gamma \in \prod_{x \in X} \mathcal{B}_x$ there is some $Y \in [X]^{<\kappa}$ such that $X = \bigcup_{y \in Y} \Gamma(y)$.*

Proof. The forwards direction is easy. We prove the backwards implication. Let $\mathcal{U}$ be an open cover of X. Choose, for each $x \in X$, some $U_x \in \mathcal{U}$ such that $x \in U_x$. Given $x \in X$, we know that $\mathcal{B}_x$ is an open basis at x, hence we can find $\Gamma(x) \in \mathcal{B}_x$ such that $x \in \Gamma(x) \subseteq U_x$. This defines $\Gamma \in \prod_{x\in X} \mathcal{B}_x$. By assumption, there is some $Y \in [X]^{<\kappa}$ such that $X = \bigcup_{y\in Y} \Gamma(y)$. But then $X = \bigcup_{y\in Y} U_y$ and $|\{U_y : y \in Y\}| < \kappa$. □

In the tree context, we'll be looking at the system of local bases formed by the sets $\uparrow x \setminus \uparrow f(x)$, where $f \in \prod_{x\in T}[I(x)]^{<\omega}$. We shall say that such an f *codes* the cover $\mathcal{U}_f := \{\uparrow x \setminus \uparrow f(x) : x \in T\}$. Going forward, we shall blur the distinction between the function f and the open cover $\mathcal{U}_f$, and speak simply of the *cover* f. We shall also consider $\mathcal{U}_f$ for $f \in \prod_{x\in X}[I(x)]^{<\omega}$, where $X \subseteq T$ (of course, $\mathcal{U}_f$ might not cover T).

Note that covers of this kind have the following important property:

Lemma 4.5. *Let T be a tree and $f \in \prod_{x\in T}[I(x)]^{<\omega}$. The following are equivalent:*
(1) *f has a countable subcover.*
(2) *There is a limit ordinal $\alpha < \omega_1$ such that $f\restriction(T\restriction\alpha)$ covers T.*
(3) *There is an ordinal $\alpha < \omega_1$ such that for every $x \in T_\alpha$ there is some $y \in T\restriction\alpha$ such that $x \in \uparrow y \setminus \uparrow f(y)$.*

Proof. Trivial. □

Definition 4.6. Let T be an $\aleph_1$-tree and $f \in \prod_{x\in T}[I(x)]^{<\omega}$. We say that a point $x \in T$ is *safe* (for f) if and only if for all $y < x$, $x \in \uparrow f(y)$. ⌟

An immediate consequence of the definition is:

Lemma 4.7. *Let T be an $\aleph_1$-tree and $f \in \prod_{x\in T}[I(x)]^{<\omega}$. If $x \in T$ is safe, then so is every $y < x$. Also, if $z \in I(x)$ (with x safe), then z is safe if and only if $z \in f(x)$.*

Proof. Trivial. □

The key property of safe points is the following:

Lemma 4.8. *Let T be an $\aleph_1$-tree and $f \in \prod_{x\in T}[I(x)]^{<\omega}$. The following are equivalent:*
(1) *f has no countable subcover.*
(2) *For every $\alpha < \omega_1$ there is a safe point of height α.*
(3) *The set $\{\mathrm{ht}(x) : x \text{ is safe}\}$ is unbounded in ω_1.*

Proof. (1) ⇒ (2): Fix $\alpha < \omega_1$. By 4.5, there is some $x \in T_\alpha$ such that for all $y \in T\restriction\alpha$, $x \notin \uparrow y \setminus \uparrow f(y)$. In particular, if $y < x$, $x \in \uparrow f(y)$, so x is safe.

(2) ⇒ (3): Trivial.

(3) ⇒ (1): Suppose towards a contradiction that f has a countable subcover. By 4.5, there is some limit $\gamma < \omega_1$ such that $f\restriction(T\restriction\gamma)$ covers T. Choose a safe point x with $\gamma < \mathrm{ht}(x)$. Let $y \in T\restriction\gamma$. If $y \not< x$, then obviously $x \notin \uparrow y \setminus \uparrow f(y)$. If $y < x$, the safety of x implies that $x \in \uparrow f(y)$, so $x \notin \uparrow y \setminus \uparrow f(y)$. In either case, $x \notin \bigcup \mathcal{U}_{f\restriction(T\restriction\gamma)} = T$, contradiction. □

Recall that a subtree of a tree is a downwards closed subset. Note that, if S is a subtree of T and $\alpha < \mathrm{ht}(T)$, then $S_\alpha = S \cap T_\alpha$. Also, if $x \in S$, then $I_S(x) = I_T(x) \cap S$.

Theorem 4.9. *Let T be an $\aleph_1$-tree. Then T is Lindelöf if and only if every finitely branching subtree of T is countable.*

Proof. $\Rightarrow$) Suppose $S \subseteq T$ is a finitely branching subtree of T with $\mathrm{ht}(S) = \aleph_1$. Define $f(x) = I_T(x) \cap S$, where $x \in T$. We claim that the cover coded by f has no countable subcover. To see this, fix a limit ordinal $\alpha < \omega_1$. Choose $x \in S \cap T_\alpha$ and $y < x$ (in T). Since S is a subtree, $y \in S$. Let z be the unique element of $I_T(y) \cap {\downarrow} x$. As $z \leq x \in S$, we see that $z \in S$, hence $z \in f(y)$ and so $x \in {\uparrow} f(y)$. This shows that x is safe. There is therefore a safe point at every limit level, so we're done by Lemma 4.8.

$\Leftarrow$) Let f code a cover with no countable subcover. Let S be the set of points of S which are safe for f. It is clear that S is a subtree of T. Given $x \in S$, we see that $I_S(x) = f(x)$, so S is finitely branching. Since f has no countable subcover, $\mathrm{ht}(S) = \aleph_1$. $\square$

We end this section by showing that a Suslin tree is automatically Lindelöf. To do this, we need to recall some elementary facts on forcing with Suslin trees. Given a tree T, let $\mathbb{P}_T$ be T upside down. More precisely, the underlying set of $\mathbb{P}_T$ is (the underlying set of) T and the order on $\mathbb{P}_T$ is the dual order on T.

Lemma 4.10. *Let T be a Suslin tree. Then $\mathbb{P}_T$ is ccc and, for every dense open set $D \subseteq \mathbb{P}_T$ there is some $\alpha < \omega_1$ such that $\{x \in T : \mathrm{ht}(x) \geq \alpha\} \subseteq D$. In particular, $\mathbb{P}_T$ is* countably distributive *(every countable intersection of dense open sets is dense).*

Proof. The poset $\mathbb{P}_T$ being ccc is just a restatement of T having no uncountable antichains. To prove the remaining statements, let $D \subseteq \mathbb{P}_T$ be dense and open. Let $A \subseteq D$ be a maximal antichain, which might therefore be countable. Fix $\alpha < \omega_1$ such that $A \subseteq T{\upharpoonright}\alpha$. If $x \in T \setminus T{\upharpoonright}\alpha$, there is some $a \in A$ with $a \parallel x$, hence $a < x$. Since D is open, $x \in D$. $\square$

Lemma 4.11. *Let T be a Suslin tree in the universe V. Let W be an outer model of V. Suppose that $b \in W$ is a cofinal branch through T. Then b is $\mathbb{P}_T$-generic over V.*

Proof. Let $D \in V$ be dense and open in $\mathbb{P}_T$. Choose $\alpha < \omega_1^V$ with $T \setminus (T{\upharpoonright}\alpha) \subseteq D$. Then any point in b of height at least α must belong to D. $\square$

Theorem 4.12. *Let T be an infinitely branching Suslin tree. Then every finitely branching subtree of T is countable.*

Proof. Suppose that S is a finitely branching subtree of T with height $\aleph_1$. Since T is Suslin, so is S. Let G be $\mathbb{P}_S$-generic over V. By 4.11, G is $\mathbb{P}_T$-generic over V. But an easy density argument shows that every $\mathbb{P}_T$-generic branch is disjoint from S above some node of T, because S is finitely branching and T is not. This is a contradiction. $\square$

Corollary 4.13. *Every Suslin tree is Lindelöf.*

Proof. Immediate from theorems 4.9 and 4.12. $\square$

5. Examples of Lindelöf and non-Lindelöf trees

If s and t are two functions with domain α, we let $\Delta(s,t) := \{\xi < \alpha : s(\xi) \neq t(\xi)\}$. We write $s =^* t$ if and only if $\Delta(s,t)$ is finite.

We shall say that $\langle e_\alpha : \alpha < \omega_1 \rangle$ is a *coherent sequence of injections* if $e_\alpha : \alpha \to \omega$ is injective and $e_\alpha =^* e_\beta \restriction \alpha$ for all $\alpha, \beta < \omega_1$. Note that each e_α must have coinfinite range. Going forward, we shall speak simply of coherent sequences, which in the literature usually refers to finite to one functions. Let

$$T^{\vec{e}} = \bigcup_{\alpha < \omega_1} \{s \in {}^\alpha\omega : s \text{ is injective} \wedge s =^* e_\alpha\}.$$

It is clear that $T^{\vec{e}}$ is an infinitely branching Aronszajn tree.

In what follows $\mathbb{C}$ denotes Cohen forcing, instantiated as the set of finite partial functions from ω to ω.

Lemma 5.1 (Todorčević [14]). *Let $\vec{e}$ be a coherent sequence and $T = T^{\vec{e}}$. Let $\dot{r}$ be a $\mathbb{C}$-name for the generic real. Then*

$$\Vdash_{\mathbb{C}} \{\dot{r} \circ t : t \in \check{T}\} \text{ is Suslin.}$$

For a proof, see [1, p. 140, Lemma 3.3.7].

Lemma 5.2. *Let $X \subseteq \omega$ be an infinite set and $\dot{r}$ a $\mathbb{C}$-name for the generic real. Then $\Vdash_{\mathbb{C}} \dot{r}``X = \omega$.*

Proof. For each $n \in \omega$, the set $\{p \in \mathbb{C} : \exists n \in X \cap \mathrm{dom}(p)(p(i) = n)\}$ is dense in $\mathbb{C}$. $\square$

Lemma 5.3. *Let $\vec{e}$ be a coherent sequence. Then $T^{\vec{e}}$ is Lindelöf.*

Proof. Put $T = T^{\vec{e}}$ and suppose towards a contradiction that $S \subseteq T$ is an uncountable finitely branching subtree. Force to add a Cohen real r, so that $T^* := \{r \circ t : t \in T\}$ is Suslin in $V[r]$ by Lemma 5.1. Note that T^* is infinitely branching, by Lemma 5.2. But $S^* := \{r \circ s : s \in S\}$ is an finitely branching subtree of T^* and S^* has height $\aleph_1$, hence T^* is non-Lindelöf. This contradicts Lemma 4.13. $\square$

We remark that, while trees arising from coherent sequences are consistently special (under $\mathsf{MA}_{\aleph_1}$, for example), they are never Suslin, as the next lemma shows. In particular, $\{\text{Suslin}\} \subsetneq \{\text{Lindelöf}\}$.

Lemma 5.4. *Let T be an Aronszajn subtree of $\{s \in \omega^{<\omega_1} : s \text{ is injective}\}$. Then T is not Suslin.*

Proof. Let, for $n \in \omega$, $A_n := \{s \in T : \exists\, \alpha < \omega_1(\mathrm{dom}(s) = \alpha + 1 \wedge s(\alpha) = n)\}$. Note that each A_n is an antichain. Since

$$\bigcup_{\alpha < \omega_1} T_{\alpha+1} = \bigcup_{n \in \omega} A_n,$$

one of the A_n's must be uncountable, and we are done. $\square$

Corollary 5.5. *There is a special Lindelöf tree.*

Proof. Let $\vec{e}$ be a coherent sequence and put $T = T^{\vec{e}}$. Letting $S = \bigcup_{\alpha<\omega_1} T_{\alpha+1}$, the proof of 5.4 shows that S is special. To show that S is Lindelöf, force to add a Cohen real $r : \omega \to \omega$. By 5.1, $T^* := \{r \circ t : t \in T\}$ is Suslin in $V[r]$, hence so is $S^* = \{r \circ s : s \in S\}$. Going back to V, by the proof of 5.3, every finitely branching subtree of S is countable, and so S is Lindelöf. $\square$

Our next goal is to construct a non-Lindelöf Aronszajn tree. The idea is to build a finitely branching Aronszajn tree, and then make $\aleph_0$ many new nodes "sprout" at each node, producing a larger, infinitely branching tree. The constructions of Aronszajn trees that we're familiar with all produce infinitely branching trees, so our first task is to build a finitely branching one.

Recall that a tree if *splitting* if every node has at least two distinct immediate successors.

Lemma 5.6. *There is a splitting subtree of $2^{<\omega_1}$ which is Aronszajn.*

Proof. Fix a coherent sequence $\vec{e}$ and a bijection $f : \omega_1 \times \omega \to \omega_1$ such that for every limit ordinal $\gamma < \omega_1$, $f[\gamma \times \omega] = \gamma$. Put $\Gamma = \lim(\omega_1) \cup \{0\}$. Define $x_\alpha \in 2^\alpha$ for $\alpha \in \Gamma$ by $x_\alpha = \chi_{f[e_\alpha]}$, where χ_A denotes the characteristic function of A. This makes sense because $f[e_\alpha] \subseteq \alpha$.

Claim. *If $\alpha, \beta \in \Gamma$ and $\alpha < \beta$ then $x_\alpha =^* x_\beta \restriction \alpha$.*

Proof of claim. For readability, we extend our $\Delta(s,t)$ notation to allow functions with different domains. More precisely, if $\mathrm{dom}(s) \leq \mathrm{dom}(t)$, we write $\Delta(s,t)$ for $\Delta(s, t \restriction \mathrm{dom}(s))$.

If $\alpha = 0$ everything is trivial so we assume that $\alpha \geq \omega$. Fix $\eta \in \Delta(x_\alpha, x_\beta)$, so $x_\alpha(\eta) \neq x_\beta(\eta)$. Since $f[\alpha \times \omega] = \alpha$, there exist unique $\xi < \alpha$ and $n \in \omega$ with $f(\xi, n) = \eta$.

Since $x_\alpha(\eta) \neq x_\beta(\eta)$, there are two possibilities: $\eta \in f[e_\alpha] \setminus f[e_\beta]$ or $\eta \in f[e_\beta] \setminus f[e_\alpha]$. We consider the two cases separately:

1. Suppose that $\eta \in f[e_\alpha] \setminus f[e_\beta]$. Then $e_\alpha(\xi) = n$ but $e_\beta(\xi) \neq n$, so $\xi \in \Delta(e_\alpha, e_\beta)$ and $\eta = f(\xi, e_\alpha(\xi))$.
2. Suppose that $\eta \in f[e_\beta] \setminus f[e_\alpha]$. Then $e_\beta(\xi) = n$ but $e_\alpha(\xi) \neq n$, so $\xi \in \Delta(e_\alpha, e_\beta)$ and $\eta = f(\xi, e_\beta(\xi))$.

We have therefore established that

$$\Delta(x_\alpha, x_\beta) \subseteq \{f(\xi, e_\alpha(\xi)) : \xi \in \Delta(e_\alpha, e_\beta)\} \cup \{f(\xi, e_\beta(\xi)) : \xi \in \Delta(e_\alpha, e_\beta)\}.$$

Since $\vec{e}$ is coherent, the union on the right is finite, hence $\Delta(x_\alpha, x_\beta)$ is finite too. $\dashv$

Given $\alpha < \omega_1$, let γ_α be the unique ordinal in Γ such that $\gamma_\alpha \leq \alpha < \gamma_\alpha + \omega$. Note that $\alpha \in \Gamma$ if and only if $\alpha = \gamma_\alpha$. Define

$$T = \bigcup_{\alpha < \omega_1} \{x \in 2^\alpha : x \restriction \gamma_\alpha =^* x_{\gamma_\alpha}\}$$

By the claim, T is a subtree of $2^{<\omega_1}$. Indeed, if $x \in 2^\alpha$ and $y \in T \cap 2^\beta$ satisfy $x \subseteq y$, then $\gamma_\alpha \leq \gamma_\beta$, so $x \restriction \gamma_\alpha = (y \restriction \gamma_\beta) \restriction \gamma_\alpha =^* x_{\gamma_\beta} \restriction \gamma_\alpha =^* x_{\gamma_\alpha}$, where the last $=^*$ follows from the claim. The fact that T has countable levels is immediate from the claim. As $x_\alpha \in T_\alpha$ for every $\alpha \in \Gamma$, we see that T has height ω_1, and so T is an $\aleph_1$-tree.

We point out that that, if $x \in T$, then $x^\frown\langle 0 \rangle, x^\frown\langle 1 \rangle \in T$, because $\gamma_\alpha = \gamma_{\alpha+1}$ for every $\alpha < \omega_1$.

To see that T is Aronszajn, assume towards a contradiction that $\langle y_\alpha : \alpha < \omega_1 \rangle$ is a branch through T, so in particular $y_\alpha =^* x_\alpha$ for every $\alpha \in \lim(\omega_1)$. Find $\eta_\alpha < \alpha$ for each $\alpha \in \lim(\omega_1)$ so that $\Delta(y_\alpha, x_\alpha) \subseteq \eta_\alpha$. Since $\alpha \mapsto \eta_\alpha$ is regressive, by Fodor's Lemma there is some stationary set $E \subseteq \lim(\omega_1)$ and some $\eta < \omega_1$ such that $\eta_\alpha = \eta$ for every $\alpha \in E$. Since $|T_\eta| \leq \aleph_0$, we may find $E' \subseteq E$ stationary and $x, y \in 2^\eta$ such that $y_\alpha \restriction \eta = y$ and $x_\alpha \restriction \eta = x$ for every $\alpha \in E'$. If $\alpha, \beta \in E'$, $\alpha < \beta$,

$$x_\alpha \restriction [\eta, \alpha) = y_\alpha \restriction [\eta, \alpha) = y_\beta \restriction [\eta, \alpha) = x_\beta \restriction [\eta, \alpha)$$

by the choice of η. Since $x_\alpha \restriction \eta = x = x_\beta \restriction \eta$, it follows that $x_\alpha = x_\beta \restriction \alpha$. Finally, if $\xi < \alpha$ and $n := e_\alpha(\xi)$, then $x_\alpha(f(\xi, n)) = 1$, so $x_\beta(f(\xi, n)) = 1$, so $f(\xi, n) \in f[e_\beta]$, so $e_\beta(\xi) = n$. We have thus shown that $\langle e_\alpha : \alpha \in E' \rangle$ is a chain, which is absurd because E' is uncountable. □

Lemma 5.7. *There is an infinitely branching Aronszajn tree with a finitely branching subtree of height* $\aleph_1$*.*

Proof. Let $\vec{e}$ be a coherent sequence and let $T \subseteq 2^{<\omega_1}$ be the tree constructed from $\vec{e}$ in Lemma 5.6. We recursively define a tree $U \subseteq \omega^{<\omega_1}$ level by level, starting with $U_0 = \{\emptyset\}$. For successor stages, we let $U_{\alpha+1} = \{u^\frown\langle n\rangle : u \in U_\alpha \wedge n \in \omega\}$. If $\alpha < \omega_1$ is a limit ordinal, we let $U_\alpha = \{u \cup t\restriction[\mathrm{dom}(u), \alpha) : u \in U\restriction\alpha \wedge t \in T_\alpha\}$. An easy induction shows that U_α is countable and that $T_\alpha \subseteq U_\alpha$ for every α.

Claim 1. *If $\beta < \alpha$, $t \in T_\alpha$ and $u \in U_\beta$, then $u \cup t\restriction[\beta, \alpha) \in U_\alpha$.*

Proof of claim 1. By induction on α:

- If $\alpha = 0$, then it's obvious.
- Suppose that α is a successor ordinal, say $\alpha = \gamma + 1$. Then $u \cup (t\restriction[\beta, \alpha)) = (u \cup t\restriction[\beta, \gamma))^\frown\langle t(\gamma)\rangle$ which belongs to U_α by the inductive hypothesis and the definition of U_α.
- If α is a limit ordinal, this is immediate from the definition of U_α. ⊣

Claim 2. *If $\alpha < \omega_1$, $\beta < \alpha$ and $v \in U_\alpha$, then $v\restriction\beta \in U$.*

Proof of claim 2. By induction on α:

- If $\alpha = 0$ then it's vacuously true.
- Suppose that α is a successor ordinal, say $\alpha = \gamma + 1$. Fix $v \in U_\alpha$, so $v = u^\frown\langle n\rangle$ for some $u \in U_\gamma$ and $n \in \omega$. Let $\beta < \alpha$. If $\beta = \gamma$, then $v\restriction\beta = u \in U$ as desired. If $\beta < \gamma$, then $v\restriction\beta = u\restriction\beta \in U$ by the inductive hypothesis applied to γ.
- Suppose that α is a limit ordinal and let $v \in U_\alpha$, say $v = u \cup (t\restriction[\mathrm{dom}(u), \alpha))$, where $u \in U_\gamma$, $\gamma < \alpha$, and $t \in T$. Let $\beta < \alpha$. If $\beta = \gamma$, then $v\restriction\beta = u \in U_\beta$. If $\beta < \gamma$, then $v\restriction\beta = u\restriction\beta \in U$ by the inductive hypothesis applied to γ. If $\gamma < \beta$, then $v\restriction\beta = u \cup (t\restriction[\mathrm{dom}(u), \beta))$. We now induct on β. If β is a limit ordinal, then $v\restriction\beta \in U_\beta$ by definition of U_β. If β is a successor ordinal, say $\beta = \delta + 1$, then $v\restriction\beta = (u \cup t\restriction[\mathrm{dom}(u), \delta))^\frown\langle t(\delta)\rangle$. By Claim 1, $u \cup t\restriction[\mathrm{dom}(u), \delta) \in U_\delta$, and so $v\restriction\beta \in U_\beta$ by definition of U_β. ⊣

To see that U is Aronszajn, suppose towards a contradiction that b is a cofinal branch through U. Let x_α be the α^{th} point of b. For each limit ordinal $\alpha < \omega_1$, there exist $u_\alpha < x_\alpha$ and $t_\alpha \in T_\alpha$ such that $x_\alpha = u_\alpha \cup t_\alpha\restriction[\mathrm{dom}(u_\alpha), \alpha)$. Define $f : \lim(\omega_1) \to \omega_1$ by $f(\alpha) = \mathrm{dom}(u_\alpha)$. Since f is regressive, we may find a stationary set $\Gamma \subseteq \lim(\omega_1)$ and some $\eta < \omega_1$ such that $f``\Gamma = \{\eta\}$. As $|T_\eta| \leq \aleph_0$, we may assume, by shrinking Γ if necessary, that there is some $t \in T_\eta$ such that $t = t_\alpha\restriction\eta$ for all $\alpha \in \Gamma$. If $\alpha, \beta \in \Gamma$, $\alpha < \beta$, then $t_\alpha\restriction[\eta, \alpha) = x_\alpha\restriction[\eta, \alpha) = x_\beta\restriction[\eta, \alpha) = t_\beta\restriction[\eta, \alpha)$ by our choice of η. But then $t_\alpha = t_\beta\restriction\alpha$ because they both agree with t below η. This means that $\langle t_\alpha : \alpha \in \Gamma\rangle$ is an uncountable chain in T, contradicting that T is Aronszajn. □

Corollary 5.8. *There is a non-Lindelöf Aronszajn tree.*

Proof. Apply lemmas 5.6 and 5.7 to obtain an infinitely branching Aronszajn tree T with a finitely branching subtree of uncountable height. Then T is not Lindelöf by Theorem 4.9. $\square$

Remark 5.9. In the presence of $\diamondsuit$, one can imitate Jensen's classical construction of a Suslin tree to obtain a Lindelöf tree. By standard coding trickery, $\diamondsuit$ implies the existence of a sequence $\langle f_\alpha : \alpha < \omega_1 \rangle$, where each $f_\alpha : \alpha \to [\alpha]^{<\omega}$, wit the following guessing property: for each $f : \omega_1 \to [\omega_1]^{<\omega}$, the set $\{\alpha < \omega_1 : f \restriction \alpha = f_\alpha\}$ is stationary. So, the sequence $\langle f_\alpha : \alpha < \omega_1 \rangle$ lets us anticipate all possible covers of a tree with underlying set ω_1, and we can then argue as in the Jensen construction: at stage α, if $\alpha = \omega\alpha$, then put a safe point for f_α at level α. At the end of the construction, we will have destroyed every possible countable subcover.

If one instead wishes to construct a non-Lindelöf Aronszajn tree, a similar idea works, except we can do it with less. Recall that a *ladder system* is a sequence $\langle C_\delta : \delta \in \lim(\omega_1) \rangle$ such that $\sup(C_\delta) = \delta$ and $\mathrm{ot}(C_\delta) = \omega$ for every $\delta \in \lim(\omega_1)$. A $\clubsuit$-*sequence* is a ladder system $\langle C_\delta : \delta \in \lim(\omega_1) \rangle$ such that for every uncountable $X \subseteq \omega_1$ there is some $\delta \in \lim(\omega_1)$ such that $C_\delta \subseteq X$. *Ostaszweski's Principle* $\clubsuit$, introduced in [11], states the existence of a $\clubsuit$ sequence. It is known that $\diamondsuit$ is equivalent to $\mathsf{CH}+\clubsuit$, see [4, Theorem 4.2], so that $\clubsuit$ is weaker than $\diamondsuit$. The idea now is to build a tree with underlying set ω_1 and a cover $f : \omega_1 \to [\omega_1]^{<\omega}$ by recursion on levels, using a $\clubsuit$-sequence to anticipate branches and kill them from growing too tall. This ensures that the tree is Aronszajn. Throughout the construction, one ensures that the cover f has safe points on every level, by inductively requiring that the (sub)tree of safe points is normal. This ensures that the resulting cover has no countable subcover, hence the tree is not Lindelöf. $\lrcorner$

Open questions

1. Suppose $\lambda < \mu$ are infinite cardinals. Is it consistent, modulo large cardinals, that there exists a cardinal $\kappa \leq \lambda$ which is λ-square compact but not μ-square compact?
2. For which pairs of infinite cardinals κ, λ with $\kappa < \lambda$ can one find a Hausdorff κ-compact space of weight λ?
3. Does ZFC prove the existence of a non-Lindelöf special tree?
4. Is the topological square of a Lindelöf tree Lindelöf?

References

[1] Tomek Bartoszyński and Haim Judah. *Set Theory.* On the structure of the real line. A K Peters, Ltd., Wellesley, MA, 1995. ISBN: 978-1-56881-044-7.

[2] David Buhagiar and Mirna Džamonja. "Square Compactness and the Filter Extension Property". In: *Fundamenta Mathematicae* 252.3 (2021), pp. 325–342. ISSN: 0016-2736. DOI: 10.4064/fm787-4-2020.

[3] W. W. Comfort and S. Negrepontis. *The Theory of Ultrafilters.* Die Grundlehren Der Mathematischen Wissenschaften, Band 211. Springer-Verlag, New York-Heidelberg, 1974.

[4] Keith J. Devlin. "Variations on $\diamondsuit$". In: *The Journal of Symbolic Logic* 44.1 (1979), pp. 51–58. ISSN: 0022-4812. DOI: 10.2307/2273703.

[5] A. Hajnal and I. Juhász. “On Square-Compact Cardinals”. In: *Periodica Mathematica Hungarica. Journal of the János Bolyai Mathematical Society* 3.3-4 (1973), pp. 285–288. ISSN: 0031-5303. DOI: 10.1007/BF02018595.

[6] Dénes Kőnig. “Über Eine Schlussweise Aus Dem Endlichen Ins Unendliche”. In: *Acta Scientiarum Mathematicarum (Szeged)* 3 (1927), pp. 121–130.

[7] Kenneth Kunen. *Set Theory*. Vol. 34. Studies in Logic (London). College Publications, London, 2011, pp. viii+401. ISBN: 978-1-84890-050-9.

[8] Kenneth Kunen and Jerry E. Vaughan, eds. *Handbook of Set-Theoretic Topology*. North-Holland Publishing Co., Amsterdam, 1984. ISBN: 978-0-444-86580-9.

[9] William Mitchell. “Aronszajn Trees and the Independence of the Transfer Property”. In: *Annals of Mathematical Logic* 5 (1972), pp. 21–46. ISSN: 0003-4843. DOI: 10.1016/0003-4843(72)90017-4.

[10] Peter J. Nyikos. “Various Topologies on Trees”. In: *Proceedings of the Tennessee Topology Conference (Nashville, TN, 1996)*. World Sci. Publ., River Edge, NJ, 1997, pp. 167–198.

[11] A. J. Ostaszewski. “On Countably Compact, Perfectly Normal Spaces”. In: *Journal of the London Mathematical Society. Second Series* 14.3 (1976), pp. 505–516. ISSN: 0024-6107. DOI: 10.1112/jlms/s2-14.3.505.

[12] J Arthur Seebach and Lynn A Steen. *Counterexamples in Topology*. Vol. 18. Springer, 1978.

[13] Ernst Specker. “Sur Un Problème de Sikorski”. In: *Colloquium Mathematicae* 2 (1949), pp. 9–12.

[14] Stevo Todorčević. “Partitioning Pairs of Countable Ordinals”. In: *Acta Mathematica* 159.3-4 (1987), pp. 261–294. ISSN: 0001-5962. DOI: 10.1007/BF02392561.

DEPARTMENT OF MATHEMATICAL SCIENCES, CARNEGIE MELLON UNIVERSITY, PITTSBURGH, PA, 15213

Email address: pmarun@andrew.cmu.edu