ON THE CONSISTENCY STRENGTH OF MM(ω_1)

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Abstract. We prove that the consistency of Martin's Maximum restricted to partial orders of cardinality ω_1 follows from the consistency of ZFC.

1. Introduction

Given the profusion of independence results which followed Cohen's discovery of the method of forcing, it has become a major objective of set theory to find natural axiomatic extensions of ZFC which decide Cantor's Continuum Problem as well as other important questions undecidable in ZFC. For example, in the last five decades forcing axioms have been widely studied and shown to have very interesting consequences regarding the continuum. Intuitively, the idea behind them is that the universe of set theory must be somehow saturated under forcing. More precisely, given a class Γ of partial orders and a cardinal κ, the *forcing axiom for* Γ *and* κ, $\mathsf{FA}(\Gamma, \kappa)$, is the assertion that for every $P \in \Gamma$ and every collection $\mathcal{D}$ of size at most κ consisting of dense subsets of P, there is a filter $G \subset P$ such that $G \cap D \neq \emptyset$ for every $D \in \mathcal{D}$.

Some forcing axioms which are especially significant for their wide range of applications not only in set theory, but also in algebra, analysis, topology, and other fields, are Martin's Axiom for ω_1-many dense sets (MA_{ω_1}) introduced by Martin, Solovay and Tennenbaum in the mid 1960's, the Proper Forcing Axiom (PFA) introduced by Baumgartner and Shelah in the early 1980's, and the Semiproper Forcing Axiom (SPFA) and Martin's Maximum (MM) introduced by Foreman, Magidor and Shelah in the mid 1980's. They are defined as $\mathsf{FA}(\Gamma, \omega_1)$ for Γ being, respectively, the class of all posets with the countable chain condition, the class of all proper posets, the class of all semiproper posets, and the class of all posets preserving stationary subsets of ω_1, where these four classes are being presented in increasing order.

The forcing axioms MA_{ω_1}, PFA and SPFA are known to be relatively consistent (from ZFC in the first case and modulo large cardinals in the other two) by means of forcing iterations which fall in the same class Γ being considered. So, this kind of construction heavily depends on certain preservation criteria. One of them is the central theorem of Shelah stating that if P_α is a countable support forcing iteration of $\{\dot{Q}_\beta : \beta < \alpha\}$ such that every that every $\dot{Q}_\beta$ is a proper forcing notion in $V^{P_\alpha \restriction \beta}$,

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then P_α is proper (in particular, P_α does not collapse ω_1). Another one, also due to Shelah, holds in the context of revised countable support forcing iterations and semiproper forcings. On the other hand, there is no such preservation result for stationary set preserving posets and the classical argument for the consistency of MM (see [3] and [6]) goes in a slightly different way: it passes by showing that SPFA implies that every stationary set preserving notion of forcing is semiproper, which in turn implies the equivalence between SPFA and MM.

Let us denote by $\mathsf{PFA}(\omega_1)$ and $\mathsf{MM}(\omega_1)$ the respective restrictions of MM and PFA to posets of cardinality ω_1. So, $\mathsf{PFA}(\omega_1)$ and $\mathsf{MM}(\omega_1)$ are defined as $\mathsf{FA}(\Gamma, \omega_1)$, for Γ being the class of all posets of cardinality ω_1 which are proper in the first case, and the class of all posets of cardinality ω_1 which preserve stationary subsets of ω_1 in the second case. It is well-known that ZFC and $\mathsf{ZFC}+\mathsf{PFA}(\omega_1)$ are equiconsistent, which follows from the fact that under CH, forcings of size ω_1 can be iterated with countable support up to length ω_2 with an ω_2-c.c. forcing iteration (see Lemmas 2.4 and 2.5 of Chapter VIII of [5]). Shelah proved that $\mathsf{ZFC}+$ "there exists a strongly inaccessible cardinal" implies the consistency of $\mathsf{ZFC}+\mathsf{MM}(\omega_1)$ (see Theorem 4.3 of Chapter III of [5]). The main theorem of this article states that Shelah's inaccessible can be removed from this consistency result.

Theorem 1.1. *Assume CH and $2^{\omega_1} = \omega_2$. Then there is a countable support forcing iteration P_{ω_2} of $\{\dot{Q}_\beta \colon \beta < \omega_2\}$ with the following properties:*

1. *Every $\dot{Q}_\beta$ is, in $V^{P_{\omega_2} \restriction \beta}$, a proper poset;*
2. *P_{ω_2} is proper and has the ω_2-chain condition;*
3. *P_{ω_2} forces that every forcing of cardinality ω_1 which preserves stationary subsets of ω_1 is proper;*
4. *P_{ω_2} forces $\mathsf{MM}(\omega_1)$.*

Theorem 1.1 immediately implies that the theories ZFC and $\mathsf{ZFC}+\mathsf{MM}(\omega_1)$ are equiconsistent.

2. Stationary Set Preserving But Not Proper

Before proving the main theorem, we give a brief sketch of the consistency that there exists a forcing poset of size ω_1 which preserves stationary subsets of ω_1 but is not proper. This fact means that the work done in the next section for destroying stationary set preserving posets which are not proper is necessary. Another proof was given previously by Sakai [4], who introduced a new combinatorial principle called $\diamondsuit^{++}$ and showed that it implies the existence of a non-proper poset of size ω_1 preserving stationary subsets of ω_1. Since $\diamondsuit^{++}$ is forceable and holds in L, the existence of such a poset is independent of ZFC. Similar to the construction in the present section, Sakai's forcing is actually a Kurepa tree, although it is not presented as such.

It is worth pointing out in this context that a poset of size ω_1 is proper iff it is semiproper. Namely, if Q is a poset of size ω_1 and N is a countable elementary submodel, then by a straightforward argument any condition in Q is a master condition for N iff it is a semi-master condition for N. Thus, we are actually proving the consistency of the existence of a poset of size ω_1 which preserves stationary sets and is not semiproper.

The forcing for introducing such a poset is the standard forcing for adding a Kurepa tree, and the poset which is stationary set preserving but not proper in the generic extension is the generic Kurepa tree. For any ordinal α, let $P(\alpha)$ be the poset consisting of conditions which are pairs (s, f), where s is a countable tree of height a countable successor ordinal satisfying the usual normality properties, and f is a function whose domain is a countable subset of α so that for all $\gamma \in \mathrm{dom}(f)$, $f(\gamma)$ is an element of the top level of s. Let $(t, g) \leq (s, f)$ if t end-extends s, $\mathrm{dom}(f) \subset \mathrm{dom}(g)$, and for all $\gamma \in \mathrm{dom}(f)$, $f(\gamma) \leq_t g(\gamma)$.

The basic properties of $P(\alpha)$ are as follows.

(1) $P(\alpha)$ is countably closed and, assuming CH, ω_2-c.c.
(2) Assume that G is a generic filter on $P(\alpha)$. The union of the trees s, where $(s, f) \in G$ for some f, is a normal ω_1-tree which we will denote by T, and the canonical name by $\dot{T}$.
(3) For each $i < \alpha$, the downwards closure of $\{f(i) : \exists s\ (s, f) \in G,\ i \in \mathrm{dom}(f)\}$ is a cofinal branch of T, which we will denote by b_i, and the canonical name by $\dot{b}_i$.
(4) For all distinct $i, j < \alpha$, the set of $(s, f) \in P(\alpha)$ such that $i, j \in \mathrm{dom}(f)$ and $f(i) \neq f(j)$ is dense open. Hence, $b_i \neq b_j$ in $V[G]$. In particular, if $\alpha \geq \omega_2$ then T is a Kurepa tree in $V[G]$.

We also note that if $\alpha < \beta$ then $P(\alpha)$ is a regular suborder of $P(\beta)$, and $P(\alpha+1)$ is forcing equivalent to $P(\alpha) * \dot{T}$.

The next two lemmas complete the proof.

Lemma 2.1. *The poset $P(\omega_2)$ forces that $\dot{T}$ is not proper.*

Proof. Fix a large enough regular cardinal λ. Let G be a generic filter on $P(\omega_2)$. In $V[G]$, let $\mathcal{X}$ denote the collection of all countable $M \prec H(\lambda)$ satisfying: (a) $M = N[G]$ for some countable $N \prec H(\lambda)^V$, (b) there exists a master condition $q_N = (s_N, f_N) \in G$ for N, and (c) for all $a \in T$ of height $N \cap \omega_1$, there exists some $\gamma \in \mathrm{dom}(f_N) \cap N$ such that $f_N(\gamma) = a$.

Working in V, for any countable elementary submodel $N \prec H(\lambda)^V$ and $p \in N \cap P(\omega_2)$, it is straightforward to build a master condition $q \leq p$ for N satisfying property (c) above. It follows that in $V[G]$, $\mathcal{X}$ is stationary in $[H(\lambda)]^\omega$. Let $M = N[G] \in \mathcal{X}$ and we claim that there does not exist a master condition in T for M. Note that an element a of T is a master condition for $N[G]$ iff for any dense open set $D \subset T$ in N, there is some $x <_T a$ such that $x \in D \cap N$. Suppose for a contradiction that a is such an element. By dropping a down to the height $N \cap \omega_1$ if necessary, assume without loss of generality that a has height $N \cap \omega_1$. By the choice of q_N, fix $\gamma \in \mathrm{dom}(f_N) \cap N$ such that $f_N(\gamma) = a$. Since $q_N \in G$, it follows that $a \in b_\gamma$.

Let $\dot{a}$ be a name which is forced by some $r \leq q_N$ in G to be master condition in $\dot{T}$ for $N[G]$, has height $N \cap \omega_1$, and is in $\dot{b}_\gamma$. Then $r * \dot{a}$ is a master condition in $P(\omega_2) * \dot{T}$ for N. But by property (4) of $P(\omega_2 + 1)$ the generic branches at coordinates γ and ω_2 must diverge. Since the dense set described in property (4) is in N and $P(\omega_2 + 1)$ and $P(\omega_2) * \dot{T}$ are forcing equivalent, the fact that $r * \dot{a}$ is a master condition for N implies that the branches with coordinates γ and ω_2 must diverge below height $N \cap \omega_1$. Hence, $r * \dot{a}$ forces that a is not in $\dot{b}_\gamma$, which contradicts that $a \in b_\gamma$. $\square$

Lemma 2.2. *The poset $P(\omega_2)$ forces that $\dot{T}$ preserves stationary subsets of ω_1.*

Proof. Suppose for a contradiction that for some generic filter G on $P(\omega_2)$, in $V[G]$ there exists a stationary set $S \subset \omega_1$, a T-name $\dot{C}$ for a club subset of ω_1, and some $x \in T$ such that $x \Vdash_T^{V[G]} \check{S} \cap \dot{C} = \emptyset$. Since $P(\omega_2)$ is ω_2-c.c., using nice names and a density argument we can find some $\gamma < \omega_2$ such that $S = \dot{S}^{G \restriction \gamma}$ for some $P(\gamma)$-name $\dot{S}$, the T-name $\dot{C}$ is in $V[G \restriction \gamma]$, and there is a condition $(t, g) \in G$ such that $x \leq_t g(\gamma)$.

Now the forcing $P(\omega_2)$ can be factored as $P(\gamma) * \dot{T} * P(\omega_2)/\dot{G}_{\gamma+1}$, and we can correspondingly write $V[G] = V[G \restriction \gamma][b_\gamma][H]$ for some H, where $x \in b_\gamma$ since $(t, g) \in G$. By an absoluteness argument, $x \Vdash_T^{V[G_\gamma]} \check{S} \cap \dot{C} = \emptyset$. Hence, S is disjoint from the club $\dot{C}^{b_\gamma}$ in $V[G_{\gamma+1}]$, which contradicts that S is stationary in $V[G]$. $\square$

3. Proving the Main Theorem

The forcing iteration described in Theorem 1.1 will involve forcing two types of posets: (1) proper posets of size ω_1, bookkeeping so that all such posets in the final model will have been forced with ω_2-many times in the iteration, and (2) forcing notions which destroy posets which are stationary set preserving but not proper. Hence, in the final model every stationary set preserving forcing is proper.

In order to prove that the forcing iteration is ω_2-c.c., we will use a property introduced by Shelah (see Chapter VIII Section 2 of [5]).

Definition 3.1. A poset R satisfies the *ω_2-properness isomorphism condition* (*ω_2-p.i.c.* for short) if and only if for every large enough regular cardinal θ, for every well-ordering $<$ of H_θ and for all ordinals $\alpha < \beta < \omega_2$ the following holds: if N_α and N_β are countable elementary submodels of $(H_\theta, \in, <, R)$ such that $\alpha \in N_\alpha$, $\beta \in N_\beta$, $N_\alpha \cap \omega_2 \subset \beta$, $N_\alpha \cap \alpha = N_\beta \cap \beta$, $p \in N_\alpha \cap R$ and $\pi : N_\alpha \to N_\beta$ is an isomorphism satisfying $\pi(\alpha) = \beta$ and $\pi \restriction (N_\alpha \cap N_\beta) = id$, then there exists a master condition q for N_α, extending p and $\pi(p)$, such that

$$q \Vdash_R \pi``(\dot{G} \cap \check{N}_\alpha) = \dot{G} \cap \check{N}_\beta.$$

Every proper poset of size ω_1 has the ω_2-p.i.c., and if CH holds, then every ω_2-p.i.c. poset satisfies the ω_2-chain condition. Moreover, by Lemma 2.4 of Chapter VIII of [5], under the assumption of CH, if P_{ω_2} is a countable support forcing iteration of $\{\dot{Q}_\beta : \beta < \omega_2\}$ such that every that every $\dot{Q}_\beta$ has the ω_2-p.i.c. in $V^{P_{\omega_2} \restriction \beta}$, then P_{ω_2} has the ω_2-chain condition. Therefore, CH implies that P_{ω_2} does not collapse cardinals. In the context of our specific iteration, we will apply this result by taking each $\dot{Q}_\beta$ to be either a name for a proper poset of size ω_1 or a name for a poset Q as described in the next theorem.

Theorem 3.2. *There exists a proper countably distributive poset Q of cardinality 2^{ω_1} with the ω_2-p.i.c. satisfying that for every poset P of cardinality ω_1, if P is not proper, then*

$$\Vdash_Q \check{P} \textit{ does not preserve stationary subsets of } \omega_1.$$

With this new ingredient, and assuming CH together with $2^{\omega_1} = \omega_2$, the construction of a forcing iteration P_{ω_2} witnessing Theorem 1.1 is very natural. Since $2^{\omega_1} = \omega_2$, we can fix a function $\Phi : \omega_2 \to H_{\omega_2}$ with the property that $\{\beta \in \omega_2 : \Phi(\beta) = x\}$ is unbounded in ω_2 for each $x \in H_{\omega_2}$. At stage $\beta < \omega_2$, if $\Phi(\beta)$ is a $P_{\omega_2} \restriction \beta$-name for a proper poset of cardinality ω_1, then let $\dot{Q}_\beta = \Phi(\beta)$. Otherwise, let $\dot{Q}_\beta$ be a $P_{\omega_2} \restriction \beta$-name for a poset Q as in Theorem 3.2.

The poset P_{ω_2} clearly forces that every poset of size ω_1 which preserves stationary sets is proper. Finally, note that P_{ω_2} forces $\mathsf{MM}(\omega_1)$ since for every P_{ω_2}-name $\dot{\mathcal{P}}$ for a poset of cardinality ω_1 and for every sequence $(\dot{D}_i)_{i<\omega_1}$ of $\mathcal{P}_{\omega_2}$-names for dense subsets of $\dot{\mathcal{P}}$, there is a high enough $\beta < \omega_2$ such that $\dot{\mathcal{P}}$ and all members of $(\dot{D}_i)_{i<\omega_1}$ are $P_{\omega_2}\restriction\beta$-names and $\Phi(\beta) = \dot{\mathcal{P}}$. So, $\dot{Q}_\beta$ is either $\dot{\mathcal{P}}$ or a $P_{\omega_2}\restriction\beta$-name for a poset Q as in Theorem 3.2 depending on whether or not $\dot{\mathcal{P}}$ is a $P_{\omega_2}\restriction\beta$-name for a proper poset. This is possible thanks to the ω_2-chain condition of P_{ω_2}, the fact that P_{ω_2} has cardinality $2^{\omega_1} = \omega_2$, and the unboundedness assumption on the bookkeeping function Φ.

The rest of this section is devoted to proving Theorem 3.2.

Definition 3.3. In an ω_1-preserving forcing extension $V[G]$, a *continuous V-reflection sequence* is a sequence $\langle \bar{M}_\alpha \colon \alpha \in C\rangle$ such that:

1. $C \subset \omega_1$ is a closed unbounded set;
2. for each $\alpha \in C$, $\bar{M}_\alpha$ is the transitive collapse of some elementary submodel of $(H^V_{\omega_2}, \in)$ such that $\alpha = \omega_1^{\bar{M}_\alpha}$;
3. (*continuity*) for every $\alpha \in C$ and every function $x \colon \alpha^{<\omega} \to \alpha$ in the model $\bar{M}_\alpha$ there is $\gamma \in \alpha$ such that for every ordinal $\delta \in C$ between γ and α, $x\restriction\delta^{<\omega} \in \bar{M}_\delta$ (which implies by (2) above that δ is closed under x);
4. (*reflection*) for every stationary set $S \subset [H^V_{\omega_2}]^\omega$ in V, the set $\{\alpha \in C \colon \bar{M}_\alpha$ is the transitive collapse of some element of $S\} \subset \omega_1$ is stationary.

Proposition 3.4. *Let $V[G]$ be an ω_1-preserving forcing extension in which there exists a continuous V-reflection sequence. In V, let P be a forcing of cardinality ω_1 which is not proper. Then $V[G] \models P$ does not preserve stationary subsets of ω_1.*

Proof. First, move to $V[G]$ and fix a continuous V-reflection sequence $\langle \bar{M}_\alpha \colon \alpha \in C\rangle$.

Claim 3.5. *For every function $x \colon \omega_1^{<\omega} \to \omega_1$ in V, for all but countably many $\alpha \in C$, $x\restriction\alpha^{<\omega} \in \bar{M}_\alpha$ holds.*

Proof. By the reflection property of the sequence, the set $S = \{\alpha \in C \colon x\restriction\alpha^{<\omega} \in \bar{M}_\alpha\} \subset \omega_1$ is stationary. Use the continuity property of the sequence to find a regressive function $f \colon S \to \omega_1$ such that for every ordinal $\alpha \in S$ and every ordinal $\delta \in C$ between $f(\alpha)$ and α, $x\restriction\delta^{<\omega} \in \bar{M}_\delta$. Use Fodor's lemma to find an ordinal $\gamma \in \omega_1$ such that the set $\{\alpha \in S \colon f(\alpha) = \gamma\}$ is stationary. It is immediate from the definitions that for every ordinal $\delta \in C$ greater than γ, $x\restriction\delta^{<\omega} \in \bar{M}_\delta$. $\square$

Now, let P be a poset of cardinality ω_1 which is not proper, which we may assume without loss of generality has underlying set ω_1. If P collapses ω_1, then it also collapses ω_1 in $V[G]$, and hence is not stationary set preserving. So assume that P preserves ω_1. Note that P can be coded in $(H^V_{\omega_2}, \in)$ by a function $x \colon \omega_1^{<\omega} \to \omega_1$ in V (for example, by the characteristic function of its partial ordering). By the claim, thinning out C to a closed unbounded subset if necessary, we may assume that for every $\alpha \in C$, $P\restriction\alpha \in \bar{M}_\alpha$.

Now, return to V and observe that since P is not proper, by a pigeonhole argument there must be a condition $p \in P$ and a stationary set $S \subset [H_{\omega_2}]^\omega$ such that no model in S has a master condition below p. Move to $V[G]$ and use the reflection property of the sequence to conclude that the set $T = \{\alpha \in C \colon \bar{M}_\alpha$ is the transitive collapse of some model in $S\}$ is stationary. It will be enough to show that in $V[G]$, $p \Vdash_P \check{T}$ is nonstationary.

To this end, let $\dot{E}$ be the P-name for the set $\{\alpha \in C\colon$ the P-generic filter has nonempty intersection with every maximal antichain of $P\restriction\alpha$ in the model $\bar{M}_\alpha\}$.

Claim 3.6. *$\Vdash_P \dot{E} \subset \omega_1^V$ is a closed unbounded set.*

Proof. First, argue for the unboundedness. Let $q \in P$ be a condition and $\gamma \in \omega_1$ be an ordinal. Back in V, consider the set $U \subset [H_{\omega_2}]^\omega$ of all models which contain γ and have a master condition below q. Let us prove that the set U is stationary.

To this end, let $f\colon H_{\omega_2}^{<\omega} \to H_{\omega_2}$ be a function. To find a model $M \in U$ closed under the function f, let θ be a large enough regular cardinal and let $X = \langle N_\alpha\colon \alpha \in \omega_1\rangle$ be a continuous increasing tower of countable elementary submodels of H_θ containing f as an element, and let $N = \bigcup_\alpha N_\alpha$. Let $H \subset P$ be a generic filter containing q, and consider the models $N_\alpha[H]$ for $\alpha \in \omega_1$ and $N[H]$. Since the poset P is a subset of N, it is clear that $N[H] \cap V = N$. The models $\langle N_\alpha[H]\colon \alpha \in \omega_1\rangle$ form a continuous increasing sequence of countable subsets of $N[H]$, so $Y = \langle N_\alpha[H] \cap V\colon \alpha \in \omega_1\rangle$ is a continuous increasing sequence of countable subsets of $N[H] \cap V = N$. Since ω_1 is preserved passing to $V[H]$, the sequences X and Y must intersect at some point, i.e. there must be an ordinal $\alpha \in \omega_1$ such that $N_\alpha[H] \cap V = N_\alpha$. Fix a condition $r \leq q$ in H such that $r \Vdash_P^V N_\alpha[\dot{H}] \cap V = N_\alpha$. Then r is a master condition for $N_\alpha \cap H_{\omega_2}$, and $N_\alpha \cap H_{\omega_2}$ is a model in the set U closed under the function f.

In $V[G]$ again, use the reflection property to find an ordinal $\alpha \in C$ which is greater than γ and such that $\bar{M}_\alpha$ is the transitive collapse of some model M in U. By the definition of U, fix a master condition $r \leq q$ for M. It easily follows that r forces that $\check{\alpha} \in \dot{E}$. This completes the proof of the unboundedness of $\dot{E}$.

For the closure, suppose that some condition $q \in P$ forces an ordinal $\alpha \in C$ to be a limit point of $\dot{E}$. To show that $q \Vdash \check{\alpha} \in \dot{E}$, let $A \in \bar{M}_\alpha$ be a maximal antichain of $P\restriction\alpha$ in the model $\bar{M}_\alpha$ and let $r \leq q$; we must find a condition in $A \cap \alpha$ compatible with r. To do this, apply the continuity property to a suitable function to find an ordinal $\gamma \in \alpha$ such that for every ordinal $\delta \in C$ between γ and α, $A \cap \delta$ is a maximal antichain of $P\restriction\delta$ in the model $\bar{M}_\delta$. Since r forces that α is a limit point of $\dot{E}$, find a condition $s \leq r$ and an ordinal $\delta \in C$ between γ and α such that $s \Vdash \check{\delta} \in \dot{E}$. By the definition of the name $\dot{E}$, there must be an element of $A \cap \delta$ compatible with the condition s, and hence with r. This completes the proof. $\square$

It is now clear from the definitions that in $V[G]$, $p \Vdash_P \dot{E} \cap \check{T} = \emptyset$. The proof of the proposition is complete. $\square$

We now define the poset for adding a continuous V-reflection sequence.

Definition 3.7. Q is the set of all pairs $q = \langle a_q, b_q\rangle$ where

(1) a_q is a function whose domain is a closed countable subset of ω_1 called the *support of* q, $\mathrm{supp}(q)$;
(2) for every ordinal $\alpha \in \mathrm{supp}(q)$, writing $M = a_q(\alpha)$, we have that M is the transitive collapse of an elementary submodel of $(H_{\omega_2}^V, \in)$ such that $\omega_1^M = \alpha$;
(3) (continuity) for every $\alpha \in \mathrm{supp}(q)$ and every function $x\colon \alpha^{<\omega} \to \alpha$ in the model $a_q(\alpha)$, there is $\gamma \in \alpha$ such that for every ordinal $\delta \in \mathrm{supp}(q)$ between γ and α, $x\restriction\delta^{<\omega} \in a_q(\delta)$;
(4) b_q is a countable set of functions from $\omega_1^{<\omega}$ to ω_1.

The ordering is given by $r \le q$ if $\mathrm{supp}(r)$ is an end-extension of $\mathrm{supp}(q)$, $a_q \subset a_r$, $b_q \subset b_r$, and for every ordinal $\alpha \in \mathrm{supp}(r) \setminus \mathrm{supp}(q)$ and every function $x \in b_q$, $x \restriction \alpha^{<\omega} \in a_r(\alpha)$.

It is not difficult to see that the forcing Q has cardinality 2^{ω_1}, and the relation $\le$ is transitive and reflexive. The forcing properties of Q are all derived from the following consideration:

Definition 3.8. Let M be a countable elementary submodel of $H(\kappa)$, for a large enough regular cardinal κ. Let $g \subset M$ be a filter meeting all open dense subsets of Q in M.

(1) Let $a = \bigcup_{s \in g} a_s \cup \{\langle M \cap \omega_1, \bar{M} \rangle\}$, where $\bar{M}$ is the transitive collapse of the model $M \cap H_{\omega_2}$;
(2) let b be the set of all functions from $\omega_1^{<\omega}$ to ω_1 belonging to the model M;
(3) let $r(M, g) = \langle a, b \rangle$.

Proposition 3.9. *$r(M, g)$ is a condition in Q which is a common lower bound of all conditions in g.*

Proof. Write $\alpha = M \cap \omega_1$. First of all, the fact that g is a filter shows that $\bigcup_{s \in g} a_s$ is a function, and its domain c is a subset of $M \cap \omega_1$ which is closed except perhaps at its supremum. A simple density argument shows that in fact $\sup(c) = \alpha$. Thus, to verify that $r(M, g)$ is a condition, it is only necessary to check the continuity of a at α. Let $y \colon \alpha^{<\omega} \to \alpha$ be any function in the model $\bar{M}$, and let $x \in M$ be the function whose collapse is y. By a density argument, there must be a condition $s \in g$ such that $x \in b_s$. The definition of the ordering on Q then shows that the ordinal $\gamma = \max \mathrm{supp}(s)$ witnesses the continuity condition for α and x.

To verify that for every condition $s \in g$, $r(M, g) \le s$ holds, it is enough to verify that for every ordinal $\delta \in \mathrm{dom}(a) \setminus \mathrm{supp}(s)$ and every $x \in b_s$, it is the case that $x \restriction \delta^{<\omega} \in a(\delta)$. For $\delta \in \alpha$ this is immediately clear from the assumption that g is a filter. If $\delta = \alpha$, then $x \in M$ since $x \in b_s$ and $s \in M$; by the elementarity of M we conclude again that $x \restriction \delta^{<\omega}$ belongs to $a(\alpha)$, since it is the transitive collapse image of the function x. $\square$

Corollary 3.10. *The poset Q satisfies the following properties:*

(1) *proper;*
(2) *countably distributive;*
(3) *ω_2-p.i.c.*

Proof. For (1), let $q \in Q$ be a condition and let M be a countable elementary submodel of $H(\kappa)$, for a large enough regular cardinal κ, such that q and Q are in M. Construct a filter $g \subset M \cap Q$ containing the condition q and meeting all dense open subsets of Q which belong to the model M. It is immediate that $r(M, g)$ is a master condition for the model M below q. For (2), if in addition $\{D_n \colon n \in \omega\}$ is a countable collection of open dense subsets of Q and M is selected in such a way that each D_n is in M, then $r(M, g)$ is a condition below q in the intersection $\bigcap_n D_n$.

For (3), suppose that M, N are two isomorphic countable elementary submodels. By the Mostowski collapse lemma, the isomorphism is unique, and we denote it by $\pi \colon M \to N$. Let $q \in M \cap Q$ be an arbitrary condition and let $g \subset M \cap Q$ be a filter having q as an element and meeting all open dense subsets of Q which belong to the model M. It will be enough to show that there is a condition r extending all the elements of the set $g \cup \pi''g$. To find r, write $r(M,g) = \langle a_M, b_M \rangle$ and $r(N, \pi''g) = \langle a_N, b_N \rangle$, and observe that $a_M = a_N$ since the isomorphism π fixes $M \cap \omega_1 = N \cap \omega_1$ and because the two models M and N have the same transitive collapse. So, $r(M,g)$ and $r(N,\pi''g)$ are compatible as witnessed by the common extension $r = \langle a_M, b_M \cup b_N \rangle$ and r works as desired. $\square$

Corollary 3.11. *Let $G \subset Q$ be a generic filter. In the model $V[G]$, let $F = \bigcup\{a \colon \exists r \in G\ a = a_r\}$. Then F is a continuous V-reflection sequence.*

Proof. It is immediately clear that $\mathrm{dom}(F)$ is a closed unbounded subset of ω_1 and that F satisfies the continuity property. Thus, it will be enough to verify the reflection property. For this, return to the ground model, let $q \in Q$ and let $S \subset [H_{\omega_2}]^\omega$ be a stationary set. Let also $\dot{E}$ be a Q-name for a closed unbounded subset of ω_1. It will be enough to find a condition $r \leq q$ and an ordinal $\alpha \in \mathrm{supp}(r)$ such that $a_r(\alpha)$ is the transitive collapse of a model in S and $r \Vdash \check{\alpha} \in \dot{E}$.

To this end, use the stationarity of the set S to find a countable elementary submodel M of $H(\kappa)$ for some large enough regular cardinal κ, containing both q and $\dot{E}$ such that $M \cap H_{\omega_2} \in S$. Find a filter $g \subset Q \cap M$ generic over M containing the condition q, and let $r = r(M,g)$ and $\alpha = M \cap \omega_1$. It is clear that $r \leq q$, $r \Vdash \check{\alpha} \in \dot{E}$ since r is a master condition for M, and $a_r(\alpha)$ is a model isomorphic to $M \cap H_{\omega_2} \in S$. $\square$

The proofs of Theorems 3.2 and 1.1 are complete.

4. Final Remarks

We finish this article with a couple of open questions. The first of them is motivated by the size of the poset Q used in Theorem 3.2 and the second one comes from the search for fragments of MM implying (as PFA does) that the continuum is equal to ω_2.

Question 4.1. *Does $\mathsf{PFA}(\omega_1)$ imply $\mathsf{MM}(\omega_1)$?.*

Question 4.2. *Does $\mathsf{MM}(\omega_1)$ imply $2^\omega = \omega_2$?.*

In [2], Asperó and Mota proved that the forcing axiom $\mathsf{FA}(\Gamma, \omega_1)$, for Γ being the class of all finitely proper posets of cardinality ω_1, is consistent with the continuum being arbitrarily large. Very recently, Asperó and Golshani have improved that result by showing that $\mathsf{PFA}(\omega_1)$ is also compatible with $2^\omega > \omega_2$ (see [1]). Therefore, a positive answer for our first question would imply a negative answer for the second one.

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