Subtrees with small branching number

Pedro Marun
Carnegie Mellon University

March 23, 2023

Definition

- A *tree* is a partially ordered set $(T, \leq)$ such that for every $x \in T$, the set $\{y \in T : y < x\}$ is well-ordered.

Definition

- A *tree* is a partially ordered set $(T, \leq)$ such that for every $x \in T$, the set $\{y \in T : y < x\}$ is well-ordered.
- The *height* of a point $x \in T$, denoted $\mathrm{ht}_T(x)$, is the order type of $\{y \in T : y < x\}$.

Definition

- A *tree* is a partially ordered set $(T, \leq)$ such that for every $x \in T$, the set $\{y \in T : y < x\}$ is well-ordered.
- The *height* of a point $x \in T$, denoted $\mathrm{ht}_T(x)$, is the order type of $\{y \in T : y < x\}$.
- For an ordinal α, the α*th level* of T is the set $T_\alpha = \{x \in T : \mathrm{ht}(x) = \alpha\}$.

Definition

- A *tree* is a partially ordered set $(T, \leq)$ such that for every $x \in T$, the set $\{y \in T : y < x\}$ is well-ordered.
- The *height* of a point $x \in T$, denoted $\mathrm{ht}_T(x)$, is the order type of $\{y \in T : y < x\}$.
- For an ordinal α, the α*th level* of T is the set $T_\alpha = \{x \in T : \mathrm{ht}(x) = \alpha\}$.
- The *height of the tree* T is the ordinal $\mathrm{ht}(T) = \min\{\alpha : T_\alpha = \emptyset\}$.

Definition

- A *tree* is a partially ordered set $(T, \leq)$ such that for every $x \in T$, the set $\{y \in T : y < x\}$ is well-ordered.
- The *height* of a point $x \in T$, denoted $\mathrm{ht}_T(x)$, is the order type of $\{y \in T : y < x\}$.
- For an ordinal α, the *αth level* of T is the set $T_\alpha = \{x \in T : \mathrm{ht}(x) = \alpha\}$.
- The *height of the tree* T is the ordinal $\mathrm{ht}(T) = \min\{\alpha : T_\alpha = \emptyset\}$.
- For $X \subseteq \mathrm{ht}(T)$, define $T{\restriction}X = \{x \in T : \mathrm{ht}_T(x) \in X\}$.

Definition

Let T be a tree.

- Given a regular cardinal κ, T is a κ-*tree* iff $\mathrm{ht}(T) = \kappa$ and $|T_\alpha| < \kappa$ for all $\alpha < \kappa$.

Definition

Let T be a tree.

- Given a regular cardinal κ, T is a *κ-tree* iff $\mathrm{ht}(T) = \kappa$ and $|T_\alpha| < \kappa$ for all $\alpha < \kappa$.
- A *chain* in T is a linearly ordered subset of T.

Definition

Let T be a tree.

- Given a regular cardinal κ, T is a *κ-tree* iff $\mathrm{ht}(T) = \kappa$ and $|T_\alpha| < \kappa$ for all $\alpha < \kappa$.
- A *chain* in T is a linearly ordered subset of T.
- A *cofinal branch* in T is a chain that meets every level.

Definition

Let T be a tree.

- Given a regular cardinal κ, T is a *κ-tree* iff $\mathrm{ht}(T) = \kappa$ and $|T_\alpha| < \kappa$ for all $\alpha < \kappa$.
- A *chain* in T is a linearly ordered subset of T.
- A *cofinal branch* in T is a chain that meets every level.

Lemma (König [3])

Let T be an $\aleph_0$-tree. Then T has a cofinal branch.

Definition

Let T be a tree.

- Given a regular cardinal κ, T is a *κ-tree* iff $\mathrm{ht}(T) = \kappa$ and $|T_\alpha| < \kappa$ for all $\alpha < \kappa$.
- A *chain* in T is a linearly ordered subset of T.
- A *cofinal branch* in T is a chain that meets every level.

Lemma (König [3])

Let T be an $\aleph_0$-tree. Then T has a cofinal branch.

Theorem (Aronszajn [6])

There exists an $\aleph_1$-tree with no cofinal branches.

Definition

- Let T be a tree and $x \in T$. The set of *immediate successors* of T is denoted by $I_T(x)$.

Definition

- Let T be a tree and $x \in T$. The set of *immediate successors* of T is denoted by $I_T(x)$.
- Let λ be a cardinal. A tree T is λ-*branching* [respectively $<\lambda$-*branching*] iff for every $x \in T$, $|I_T(x)| = \lambda$ [respectively $|I_T(x)| < \lambda$].

Definition

- Let T be a tree and $x \in T$. The set of *immediate successors* of T is denoted by $I_T(x)$.
- Let λ be a cardinal. A tree T is λ-*branching* [respectively $<\lambda$-*branching*] iff for every $x \in T$, $|I_T(x)| = \lambda$ [respectively $|I_T(x)| < \lambda$].
- A *subtree* of a tree T is a subset S of T such that for every $s \in S$ and $t \in T$, if $t < s$, then $t \in S$.

Definition

- Let T be a tree and $x \in T$. The set of *immediate successors* of T is denoted by $I_T(x)$.
- Let λ be a cardinal. A tree T is λ-*branching* [respectively $<\lambda$-*branching*] iff for every $x \in T$, $|I_T(x)| = \lambda$ [respectively $|I_T(x)| < \lambda$].
- A *subtree* of a tree T is a subset S of T such that for every $s \in S$ and $t \in T$, if $t < s$, then $t \in S$.

Lemma

Let T be an $\aleph_1$-tree. If T has no uncountable 1*-branching subtrees, then T is Aronszajn.*

Definition

- Let T be a tree and $x \in T$. The set of *immediate successors* of T is denoted by $I_T(x)$.
- Let λ be a cardinal. A tree T is λ-*branching* [respectively $<\lambda$-*branching*] iff for every $x \in T$, $|I_T(x)| = \lambda$ [respectively $|I_T(x)| < \lambda$].
- A *subtree* of a tree T is a subset S of T such that for every $s \in S$ and $t \in T$, if $t < s$, then $t \in S$.

Lemma

Let T be an $\aleph_1$-tree. If T has no uncountable 1*-branching subtrees, then T is Aronszajn.*

Proof.

If b is a cofinal branch, then b is an uncountable 1-branching subtree. $\square$

Definition

An infinitely branching $\aleph_1$-tree T is *Lindelöf* iff every finitely branching subtree of T is countable.

Definition

An infinitely branching $\aleph_1$-tree T is *Lindelöf* iff every finitely branching subtree of T is countable.

The previous lemma shows that

$$\{\text{Lindelöf}\} \subseteq \{\text{Aronszajn}\}$$

Definition

An infinitely branching $\aleph_1$-tree T is *Lindelöf* iff every finitely branching subtree of T is countable.

The previous lemma shows that

$$\{\text{Lindelöf}\} \subseteq \{\text{Aronszajn}\}$$

We will show that the inclusion is proper.

Definition

An infinitely branching $\aleph_1$-tree T is *Lindelöf* iff every finitely branching subtree of T is countable.

The previous lemma shows that

$$\{\text{Lindelöf}\} \subseteq \{\text{Aronszajn}\}$$

We will show that the inclusion is proper.
First, we explain the terminology:

Definition

An infinitely branching $\aleph_1$-tree T is *Lindelöf* iff every finitely branching subtree of T is countable.

The previous lemma shows that

$$\{\text{Lindelöf}\} \subseteq \{\text{Aronszajn}\}$$

We will show that the inclusion is proper.
First, we explain the terminology: Trees are Lindelöf if and only if they have they are Lindelöf spaces with respect to a natural topology.

Definition

An infinitely branching $\aleph_1$-tree T is *Lindelöf* iff every finitely branching subtree of T is countable.

The previous lemma shows that

$$\{\text{Lindelöf}\} \subseteq \{\text{Aronszajn}\}$$

We will show that the inclusion is proper.
First, we explain the terminology: Trees are Lindelöf if and only if they have they are Lindelöf spaces with respect to a natural topology.

Definition (Nyikos)

Let T be a tree. The *fine wedge topology* on T is generated by all sets of the form $\uparrow x$ and their complements, where $\uparrow x = \{y \in T : x \leq y\}$ and $x \in T$.

Definition

An infinitely branching $\aleph_1$-tree T is *Lindelöf* iff every finitely branching subtree of T is countable.

The previous lemma shows that

$$\{\text{Lindelöf}\} \subseteq \{\text{Aronszajn}\}$$

We will show that the inclusion is proper.
First, we explain the terminology: Trees are Lindelöf if and only if they have they are Lindelöf spaces with respect to a natural topology.

Definition (Nyikos)

Let T be a tree. The *fine wedge topology* on T is generated by all sets of the form $\uparrow x$ and their complements, where $\uparrow x = \{y \in T : x \leq y\}$ and $x \in T$.

If $X \subseteq T$, write $\uparrow X = \{y \in T : \exists x \in X (x \leq y)\}$.

Lemma

If $x \in T$, the family $\{\uparrow x \setminus \uparrow F : F \in [I(x)]^{<\omega}\}$ is a local basis of open neighbourhoods of x. In particular, the topology is Hausdorff.

Lemma

If $x \in T$, the family $\{\uparrow x \setminus \uparrow F : F \in [I(x)]^{<\omega}\}$ is a local basis of open neighbourhoods of x. In particular, the topology is Hausdorff.

Proof.

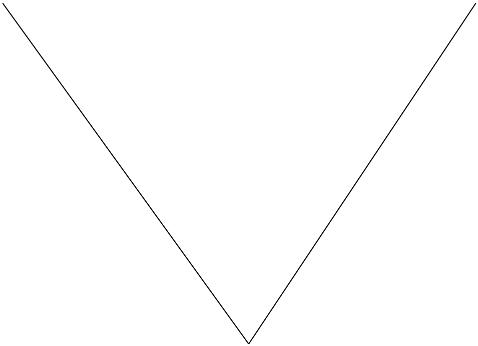
Let $x \in W = (\uparrow x_0) \cap \cdots \cap (\uparrow x_{n-1}) \cap (\uparrow y_0)^c \cap \cdots \cap (\uparrow y_{m-1})^c$.

Lemma

If $x \in T$, the family $\{\uparrow x \setminus \uparrow F : F \in [I(x)]^{<\omega}\}$ is a local basis of open neighbourhoods of x. In particular, the topology is Hausdorff.

Proof.

Let $x \in W = (\uparrow x_0) \cap \cdots \cap (\uparrow x_{n-1}) \cap (\uparrow y_0)^c \cap \cdots \cap (\uparrow y_{m-1})^c$.



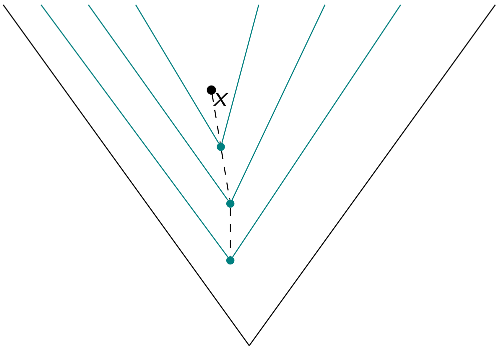
□

Lemma

If $x \in T$, the family $\{\uparrow x \setminus \uparrow F : F \in [I(x)]^{<\omega}\}$ is a local basis of open neighbourhoods of x. In particular, the topology is Hausdorff.

Proof.

Let $x \in W = (\uparrow x_0) \cap \cdots \cap (\uparrow x_{n-1}) \cap (\uparrow y_0)^c \cap \cdots \cap (\uparrow y_{m-1})^c$.
Since T is a tree, $\{x_i : i < n\}$ is a chain, say with maximum x_0.



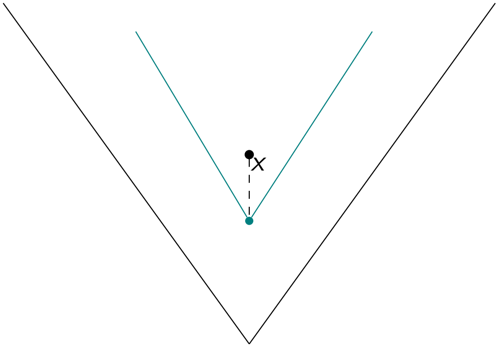
$\square$

Lemma

If $x \in T$, the family $\{\uparrow x \setminus \uparrow F : F \in [I(x)]^{<\omega}\}$ is a local basis of open neighbourhoods of x. In particular, the topology is Hausdorff.

Proof.

Let $x \in W = (\uparrow x_0) \cap (\uparrow y_0)^c \cap \cdots \cap (\uparrow y_{m-1})^c$.



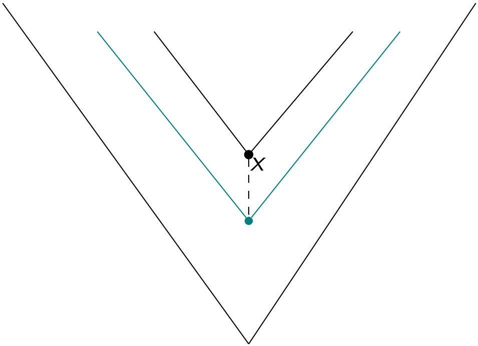
$\square$

Lemma

If $x \in T$, the family $\{\uparrow x \setminus \uparrow F : F \in [I(x)]^{<\omega}\}$ is a local basis of open neighbourhoods of x. In particular, the topology is Hausdorff.

Proof.

Let $x \in W = (\uparrow x_0) \cap (\uparrow y_0)^c \cap \cdots \cap (\uparrow y_{m-1})^c$.



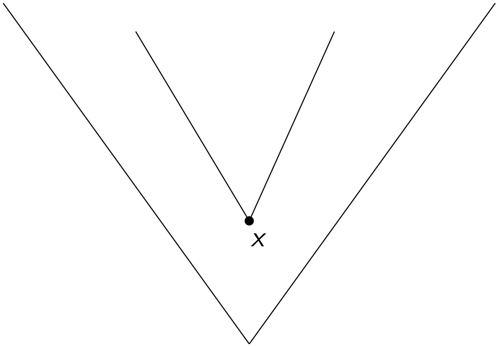
$\square$

Lemma

If $x \in T$, the family $\{\uparrow x \setminus \uparrow F : F \in [I(x)]^{<\omega}\}$ is a local basis of open neighbourhoods of x. In particular, the topology is Hausdorff.

Proof.

Let $x \in W = (\uparrow x) \cap (\uparrow y_0)^c \cap \cdots \cap (\uparrow y_{m-1})^c$.



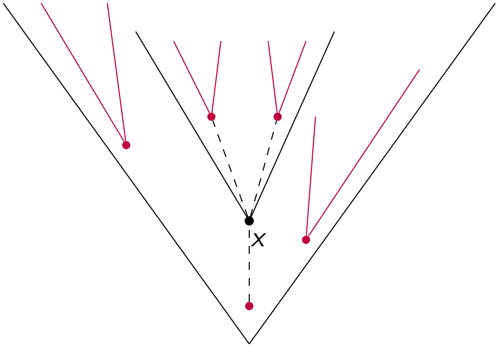
$\square$

Lemma

If $x \in T$, the family $\{\uparrow x \setminus \uparrow F : F \in [I(x)]^{<\omega}\}$ is a local basis of open neighbourhoods of x. In particular, the topology is Hausdorff.

Proof.

Let $x \in W = (\uparrow x) \cap (\uparrow y_0)^c \cap \cdots \cap (\uparrow y_{m-1})^c$.



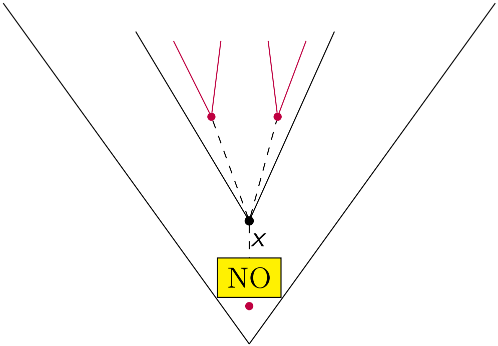
$\square$

Lemma

If $x \in T$, the family $\{\uparrow x \setminus \uparrow F : F \in [I(x)]^{<\omega}\}$ is a local basis of open neighbourhoods of x. In particular, the topology is Hausdorff.

Proof.

Let $x \in W = (\uparrow x) \cap (\uparrow y_0)^c \cap \cdots \cap (\uparrow y_{m-1})^c$.



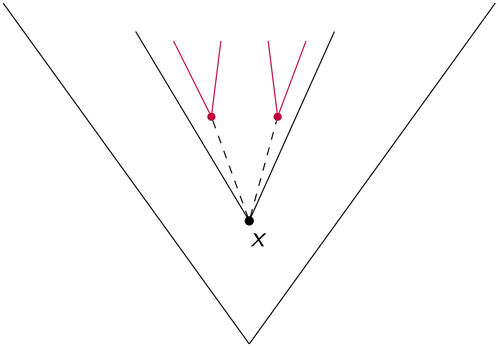
$\square$

Lemma

If $x \in T$, the family $\{\uparrow x \setminus \uparrow F : F \in [I(x)]^{<\omega}\}$ is a local basis of open neighbourhoods of x. In particular, the topology is Hausdorff.

Proof.

Let $x \in W = (\uparrow x) \cap (\uparrow y_0)^c \cap \cdots \cap (\uparrow y_{m-1})^c$.



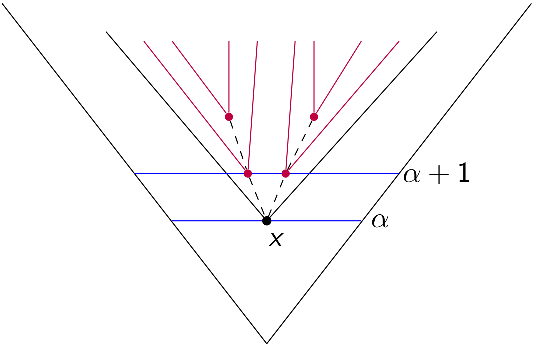
□

Lemma

If $x \in T$, the family $\{\uparrow x \setminus \uparrow F : F \in [I(x)]^{<\omega}\}$ is a local basis of open neighbourhoods of x. In particular, the topology is Hausdorff.

Proof.

Let $x \in W = (\uparrow x) \cap (\uparrow y_0)^c \cap \cdots \cap (\uparrow y_{m-1})^c$.



$\square$

It therefore suffices to study open covers of the form $\mathcal{U}_f = \{\uparrow x \setminus \uparrow f(x) : x \in T\}$, where $f \in \prod_{x \in T} [I(x)]^{<\omega}$.

It therefore suffices to study open covers of the form $\mathcal{U}_f = \{\uparrow x \setminus \uparrow f(x) : x \in T\}$, where $f \in \prod_{x \in T}[I(x)]^{<\omega}$.

Definition

Given $f \in \prod_{x \in T}[I(x)]^{<\omega}$, we will say that $x \in T$ is *safe* iff for every $y < x$, $x \in \uparrow f(y)$.

It therefore suffices to study open covers of the form $\mathcal{U}_f = \{\uparrow x \setminus \uparrow f(x) : x \in T\}$, where $f \in \prod_{x \in T}[I(x)]^{<\omega}$.

Definition

Given $f \in \prod_{x \in T}[I(x)]^{<\omega}$, we will say that $x \in T$ is *safe* iff for every $y < x$, $x \in \uparrow f(y)$.

Note that the set of safe points is a finitely branching subtree of T.

It therefore suffices to study open covers of the form $\mathcal{U}_f = \{\uparrow x \setminus \uparrow f(x) : x \in T\}$, where $f \in \prod_{x \in T} [I(x)]^{<\omega}$.

Definition

Given $f \in \prod_{x \in T} [I(x)]^{<\omega}$, we will say that $x \in T$ is *safe* iff for every $y < x$, $x \in \uparrow f(y)$.

Note that the set of safe points is a finitely branching subtree of T.

Lemma

Let T be an infinitely branching $\aleph_1$-tree and $f \in \prod_{x \in T} [I(x)]^{<\omega}$. The following are equivalent:

It therefore suffices to study open covers of the form $\mathcal{U}_f = \{\uparrow x \setminus \uparrow f(x) : x \in T\}$, where $f \in \prod_{x \in T} [I(x)]^{<\omega}$.

Definition

Given $f \in \prod_{x \in T} [I(x)]^{<\omega}$, we will say that $x \in T$ is *safe* iff for every $y < x$, $x \in \uparrow f(y)$.

Note that the set of safe points is a finitely branching subtree of T.

Lemma

Let T be an infinitely branching $\aleph_1$-tree and $f \in \prod_{x \in T} [I(x)]^{<\omega}$. The following are equivalent:

1. *f has no countable subcover.*

It therefore suffices to study open covers of the form $\mathcal{U}_f = \{\uparrow x \setminus \uparrow f(x) : x \in T\}$, where $f \in \prod_{x\in T}[I(x)]^{<\omega}$.

Definition

Given $f \in \prod_{x\in T}[I(x)]^{<\omega}$, we will say that $x \in T$ is *safe* iff for every $y < x$, $x \in \uparrow f(y)$.

Note that the set of safe points is a finitely branching subtree of T.

Lemma

Let T be an infinitely branching $\aleph_1$-tree and $f \in \prod_{x\in T}[I(x)]^{<\omega}$. The following are equivalent:

1. *f has no countable subcover.*
2. *Every level has a safe point.*

It therefore suffices to study open covers of the form
$\mathcal{U}_f = \{\uparrow x \setminus \uparrow f(x) : x \in T\}$, where $f \in \prod_{x \in T}[I(x)]^{<\omega}$.

Definition

Given $f \in \prod_{x \in T}[I(x)]^{<\omega}$, we will say that $x \in T$ is *safe* iff for every $y < x$, $x \in \uparrow f(y)$.

Note that the set of safe points is a finitely branching subtree of T.

Lemma

Let T be an infinitely branching $\aleph_1$-tree and $f \in \prod_{x \in T}[I(x)]^{<\omega}$. The following are equivalent:

1. *f has no countable subcover.*
2. *Every level has a safe point.*
3. *The set of $\{ht(x) : x \text{ is safe}\}$ is uncountable.*

Theorem (M.)

Let T be an infinitely branching $\aleph_1$-tree. Then T is a Lindelöf tree if and only if it has the Lindelöf property with respect to the fine wedge topology.

Theorem (M.)

Let T be an infinitely branching $\aleph_1$-tree. Then T is a Lindelöf tree if and only if it has the Lindelöf property with respect to the fine wedge topology.

Proof.

$\Rightarrow$) Let f code a cover with no countable subcover. Let S be the set of safe points. By the previous lemma, S is uncountable.

Theorem (M.)

Let T be an infinitely branching $\aleph_1$-tree. Then T is a Lindelöf tree if and only if it has the Lindelöf property with respect to the fine wedge topology.

Proof.

$\Rightarrow$) Let f code a cover with no countable subcover. Let S be the set of safe points. By the previous lemma, S is uncountable.
$\Leftarrow$) Let S be an uncountable finitely branching subtree of T.

Theorem (M.)

Let T be an infinitely branching $\aleph_1$-tree. Then T is a Lindelöf tree if and only if it has the Lindelöf property with respect to the fine wedge topology.

Proof.

$\Rightarrow$) Let f code a cover with no countable subcover. Let S be the set of safe points. By the previous lemma, S is uncountable.
$\Leftarrow$) Let S be an uncountable finitely branching subtree of T. Define $f(x) = \emptyset$ for $x \in T \setminus S$ and $f(x) = I_S(x)$ for $x \in S$.

Theorem (M.)

Let T be an infinitely branching $\aleph_1$-tree. Then T is a Lindelöf tree if and only if it has the Lindelöf property with respect to the fine wedge topology.

Proof.

$\Rightarrow$) Let f code a cover with no countable subcover. Let S be the set of safe points. By the previous lemma, S is uncountable.
$\Leftarrow$) Let S be an uncountable finitely branching subtree of T. Define $f(x) = \emptyset$ for $x \in T \setminus S$ and $f(x) = I_S(x)$ for $x \in S$. If $x \in S$ and $y < x$, then $x \in \uparrow f(y)$, so x is safe for S. So $S \subseteq \{x \in T : x \text{ is safe}\}$, hence the latter is uncountable. $\square$

Recall that if T is a tree, an *antichain* is a set of pairwise incomparable elements of T. A *Suslin tree* is a tree with no uncountable chains or antichains.

Recall that if T is a tree, an *antichain* is a set of pairwise incomparable elements of T. A *Suslin tree* is a tree with no uncountable chains or antichains.
If T is a tree, $\mathbb{P}_T$ is the dual order.

Recall that if T is a tree, an *antichain* is a set of pairwise incomparable elements of T. A *Suslin tree* is a tree with no uncountable chains or antichains.
If T is a tree, $\mathbb{P}_T$ is the dual order.

Lemma (folklore [2])

Let T be a Suslin tree. The poset $\mathbb{P}_T$ has the ccc and is countably distributive. Moreover, if $D \subseteq \mathbb{P}_T$ is dense and open, then there exists $\alpha < \omega_1$ such that $T \restriction [\alpha, \omega_1) \subseteq D$.

Recall that if T is a tree, an *antichain* is a set of pairwise incomparable elements of T. A *Suslin tree* is a tree with no uncountable chains or antichains.
If T is a tree, $\mathbb{P}_T$ is the dual order.

Lemma (folklore [2])

Let T be a Suslin tree. The poset $\mathbb{P}_T$ has the ccc and is countably distributive. Moreover, if $D \subseteq \mathbb{P}_T$ is dense and open, then there exists $\alpha < \omega_1$ such that $T \restriction [\alpha, \omega_1) \subseteq D$.

Lemma (folklore [2])

Let T be a Suslin tree in the universe V. If W is an outer model and $b \in W$ is a cofinal branch through T, then b is $\mathbb{P}_T$-generic over V.

Theorem (M.)

Let T be an infinitely branching Suslin tree. Then T is Lindelöf.

Theorem (M.)

Let T be an infinitely branching Suslin tree. Then T is Lindelöf.

Proof.

Suppose not. Let $S \subseteq T$ be a finitely branching uncountable subtree. Then S is also Suslin. Force with $\mathbb{P}_S$ to add a branch b. By the previous lemma, b is $\mathbb{P}_T$-generic over V. But S is finitely branching and T is infinitely branching, so b is disjoint from S above some node of T, by a density argument. □

Theorem (M.)

Let T be an infinitely branching Suslin tree. Then T is Lindelöf.

Proof.

Suppose not. Let $S \subseteq T$ be a finitely branching uncountable subtree. Then S is also Suslin. Force with $\mathbb{P}_S$ to add a branch b. By the previous lemma, b is $\mathbb{P}_T$-generic over V. But S is finitely branching and T is infinitely branching, so b is disjoint from S above some node of T, by a density argument. $\square$

$$\{\text{Suslin}\} \subseteq \{\text{Lindelöf}\} \subseteq \{\text{Aronszajn}\}$$

An example of a Lindelöf tree

Let $\vec{e} = \langle e_\alpha : \alpha < \omega_1 \rangle$ be a *coherent sequence of injections*, so $e_\alpha : \alpha \to \omega$ is injective and $\alpha < \beta$ implies $e_\alpha =^* e_\beta$, that is $\{\xi < \alpha : e_\alpha(\xi) = e_\beta(\xi)\}$ is finite.

An example of a Lindelöf tree

Let $\vec{e} = \langle e_\alpha : \alpha < \omega_1 \rangle$ be a *coherent sequence of injections*, so $e_\alpha : \alpha \to \omega$ is injective and $\alpha < \beta$ implies $e_\alpha =^* e_\beta$, that is $\{\xi < \alpha : e_\alpha(\xi) = e_\beta(\xi)\}$ is finite.
It is well-known that the set

$$T^{\vec{e}} = \bigcup_{\alpha < \omega_1} \{t \in {}^\alpha\omega : t \text{ is injective } \wedge t =^* e_\alpha\}$$

is an infinitely branching Aronszajn tree.

An example of a Lindelöf tree

Let $\vec{e} = \langle e_\alpha : \alpha < \omega_1 \rangle$ be a *coherent sequence of injections*, so $e_\alpha : \alpha \to \omega$ is injective and $\alpha < \beta$ implies $e_\alpha =^* e_\beta$, that is $\{\xi < \alpha : e_\alpha(\xi) = e_\beta(\xi)\}$ is finite.
It is well-known that the set

$$T^{\vec{e}} = \bigcup_{\alpha < \omega_1} \{t \in {}^\alpha\omega : t \text{ is injective } \wedge t =^* e_\alpha\}$$

is an infinitely branching Aronszajn tree.
Let $\mathbb{C}$ be Cohen forcing, instantiated as the set of finite partial functions from ω to ω.

An example of a Lindelöf tree

Let $\vec{e} = \langle e_\alpha : \alpha < \omega_1 \rangle$ be a *coherent sequence of injections*, so $e_\alpha : \alpha \to \omega$ is injective and $\alpha < \beta$ implies $e_\alpha =^* e_\beta$, that is $\{\xi < \alpha : e_\alpha(\xi) = e_\beta(\xi)\}$ is finite.
It is well-known that the set

$$T^{\vec{e}} = \bigcup_{\alpha < \omega_1} \{t \in {}^\alpha\omega : t \text{ is injective } \wedge t =^* e_\alpha\}$$

is an infinitely branching Aronszajn tree.
Let $\mathbb{C}$ be Cohen forcing, instantiated as the set of finite partial functions from ω to ω.

Theorem (Todorčević)

If $\vec{e}$ is a coherent sequence of injections and $r : \omega \to \omega$ is Cohen generic over V, then

$$\{r \circ t : t \in T^{\vec{e}}\}$$

is an infinitely branching Suslin tree in $V[r]$.

An example of a Lindelöf tree

Theorem (M.)

Let $\vec{e}$ be a coherent sequence of injections. Then $T^{\vec{e}}$ is a Lindelöf tree.

An example of a Lindelöf tree

Theorem (M.)

Let $\vec{e}$ be a coherent sequence of injections. Then $T^{\vec{e}}$ is a Lindelöf tree.

Proof.

It is clear that $T^{\vec{e}}$ is infinitely branching. Suppose now that $S \subseteq T$ is an uncountable finitely branching subtree. Force to add a Cohen real $r : \omega \to \omega$. In $V[r]$, $T^* = \{r \circ t : t \in T\}$ is infinitely branching by genericity and $S^* = \{r \circ s : s \in S\}$ is a finitely branching uncountable subtree of T^*. But T^* is a Suslin tree in $V[r]$, which is a contradiction. $\square$

An example of a Lindelöf tree

Theorem (M.)

Let $\vec{e}$ be a coherent sequence of injections. Then $T^{\vec{e}}$ is a Lindelöf tree.

Proof.

It is clear that $T^{\vec{e}}$ is infinitely branching. Suppose now that $S \subseteq T$ is an uncountable finitely branching subtree. Force to add a Cohen real $r : \omega \to \omega$. In $V[r]$, $T^* = \{r \circ t : t \in T\}$ is infinitely branching by genericity and $S^* = \{r \circ s : s \in S\}$ is a finitely branching uncountable subtree of T^*. But T^* is a Suslin tree in $V[r]$, which is a contradiction. $\square$

Since $T^{\vec{e}}$ is never Suslin,

$$\{\text{Suslin}\} \subsetneq \{\text{Lindelöf}\} \subseteq \{\text{Aronszajn}\}$$

Theorem

There exists an Aronszajn non-Lindelöf tree.

Theorem

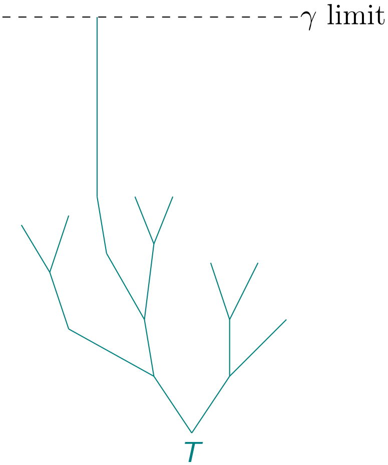
There exists an Aronszajn non-Lindelöf tree.

Proof sketch.

Fix a coherent sequence of injection $\vec{e}$ and use a bijection $f : \omega_1 \times \omega \to \omega_1$ to transfer $\vec{e}$ to a coherent sequence $\langle f[e_\alpha] : \alpha < \omega_1 \rangle$, which is then used to build a binary Aronszajn tree T.

Theorem

There exists an Aronszajn non-Lindelöf tree.



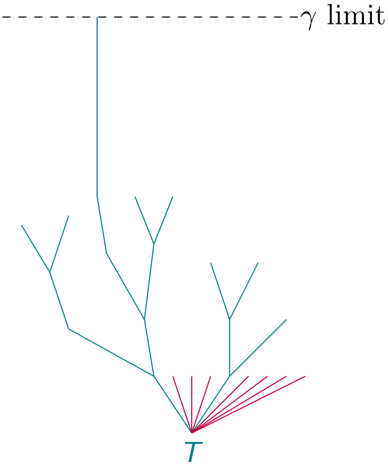
Proof sketch.

Fix a coherent sequence of injection $\vec{e}$ and use a bijection $f : \omega_1 \times \omega \to \omega_1$ to transfer $\vec{e}$ to a coherent sequence $\langle f[e_\alpha] : \alpha < \omega_1 \rangle$, which is then used to build a binary Aronszajn tree T.

Now recursively build, level by level, an infinitely branching tree U which has T as a subtree. $\square$

Theorem

There exists an Aronszajn non-Lindelöf tree.

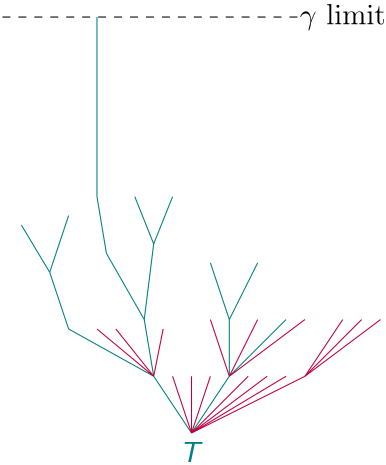


Proof sketch.

Fix a coherent sequence of injection $\vec{e}$ and use a bijection $f : \omega_1 \times \omega \to \omega_1$ to transfer $\vec{e}$ to a coherent sequence $\langle f[e_\alpha] : \alpha < \omega_1 \rangle$, which is then used to build a binary Aronszajn tree T.
Now recursively build, level by level, an infinitely branching tree U which has T as a subtree. $\square$

Theorem

There exists an Aronszajn non-Lindelöf tree.



Proof sketch.

Fix a coherent sequence of injection $\vec{e}$ and use a bijection $f : \omega_1 \times \omega \to \omega_1$ to transfer $\vec{e}$ to a coherent sequence $\langle f[e_\alpha] : \alpha < \omega_1 \rangle$, which is then used to build a binary Aronszajn tree T.

Now recursively build, level by level, an infinitely branching tree U which has T as a subtree. $\square$

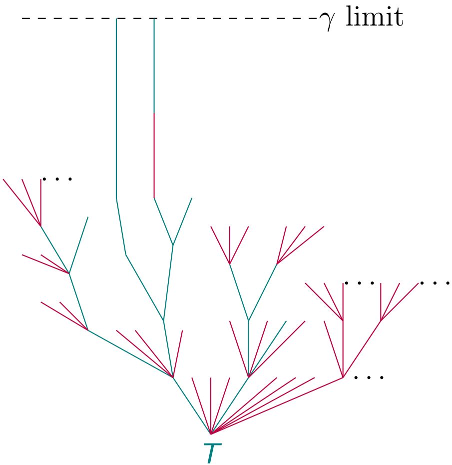
Theorem

There exists an Aronszajn non-Lindelöf tree.

Proof sketch.

Fix a coherent sequence of injections $\vec{e}$ and use a bijection $f : \omega_1 \times \omega \to \omega_1$ to transfer $\vec{e}$ to a coherent sequence $\langle f[e_\alpha] : \alpha < \omega_1 \rangle$, which is then used to build a binary Aronszajn tree T.

Now recursively build, level by level, an infinitely branching tree U which has T as a subtree. $\square$



$$\{\text{Suslin}\} \subsetneq \{\text{Lindelöf}\} \subsetneq \{\text{Aronszajn}\}$$

Thank you :)

References

[1] Tomek Bartoszyński and Haim Judah. *Set Theory*. A K Peters, Ltd., Wellesley, MA, 1995. xii+546.

[2] Keith J. Devlin and Håvard Johnsbråten. *The Souslin Problem*. Lecture Notes in Mathematics 405. Springer, 1974. 132 pp.

[3] Dénes Kőnig. "Über Eine Schlussweise Aus Dem Endlichen Ins Unendliche". In: *Acta Scientiarum Mathematicarum (Szeged)* 3 (1927).

[4] Pedro E. Marun. "Square Compactness and Lindelöf Trees". In: *Submitted for publication* (2023).

[5] Peter J. Nyikos. "Various Topologies on Trees". In: *Proceedings of the Tennessee Topology Conference (Nashville, TN, 1996)*. 1997.

[6] Ernst Specker. "Sur Un Problème de Sikorski". In: *Colloquium Mathematicae* 2 (1949).