Subtrees with small branching number

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- A tree T is *finitely branching* if it is $<\aleph_0$-branching. It is *infinitely branching* if $|I_T(x)| \geq \aleph_0$ for every $x \in T$.
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Proof.

If b is a cofinal branch, then b is an uncountable 1-branching subtree. $\square$

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If $X \subseteq T$, write $\uparrow X = \{y \in T : \exists x \in X (x \leq y)\}$.

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If $x \in T$*, the family* $\{\uparrow x \setminus \uparrow F : F \in [I(x)]^{<\omega}\}$ *is a local basis of open neighbourhoods of* x*. In particular, the topology is Hausdorff.*

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Let $x \in W = (\uparrow x_0) \cap \cdots \cap (\uparrow x_{n-1}) \cap (\uparrow y_0)^c \cap \cdots \cap (\uparrow y_{m-1})^c$.

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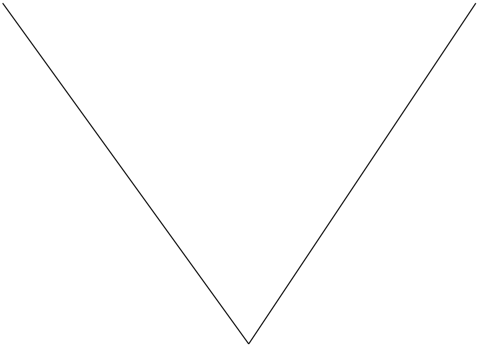

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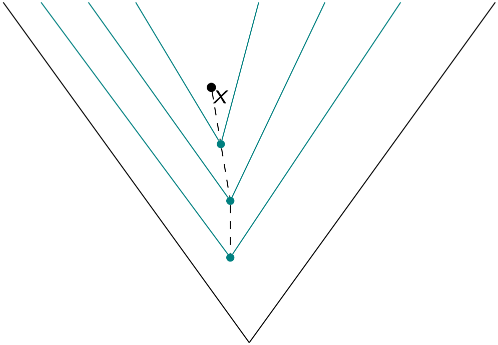

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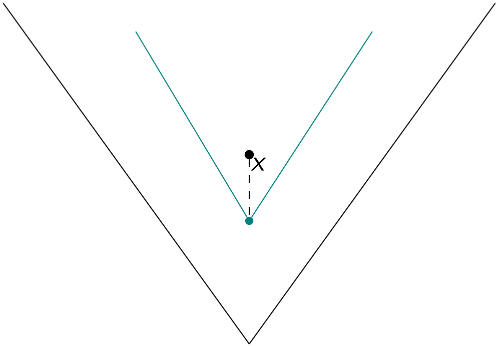

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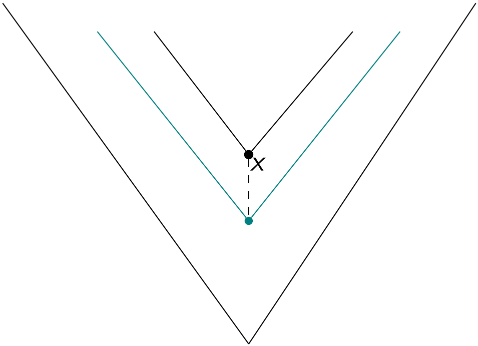

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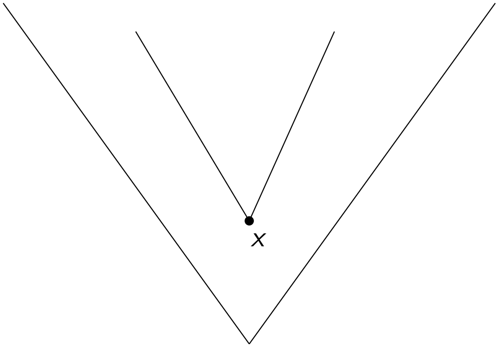

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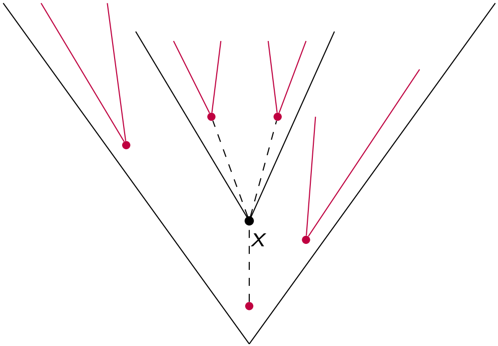

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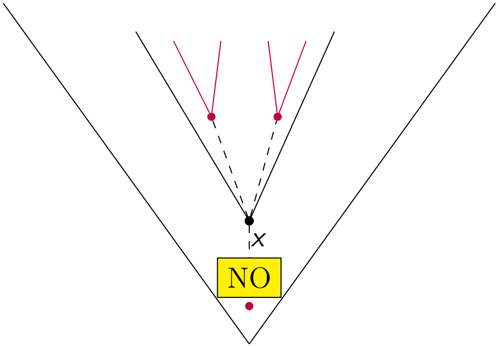

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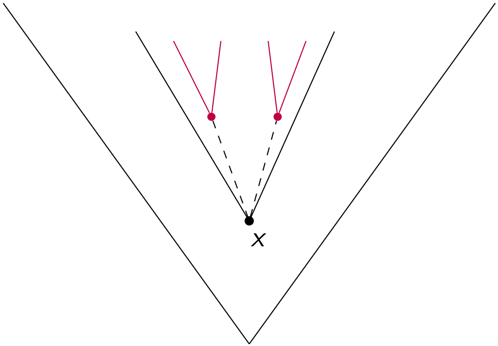

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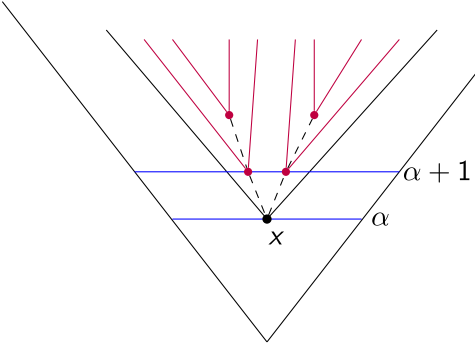

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It therefore suffices to study open covers of the form $\mathcal{U}_f = \{\uparrow x \setminus \uparrow f(x) : x \in T\}$, where $f \in \prod_{x \in T} [I(x)]^{<\omega}$.

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3. *The set of $\{ht(x) : x \text{ is safe}\}$ is uncountable.*

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Let T be an infinitely branching $\aleph_1$-tree. Then T is a Lindelöf tree if and only if it has the Lindelöf property with respect to the fine wedge topology.

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$\Leftarrow$) Let S be an uncountable finitely branching subtree of T. Define $f(x) = \emptyset$ for $x \in T \setminus S$ and $f(x) = I_S(x)$ for $x \in S$. If $x \in S$ and $y < x$, then $x \in \uparrow f(y)$, so x is safe for S. So $S \subseteq \{x \in T : x \text{ is safe}\}$, hence the latter is uncountable. $\square$

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Lemma (folklore [2])

Let T be a Suslin tree. The poset $\mathbb{P}_T$ has the ccc and is countably distributive. Moreover, if $D \subseteq \mathbb{P}_T$ is dense and open, then there exists $\alpha < \omega_1$ such that $T \restriction [\alpha, \omega_1) \subseteq D$.

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Let T be a Suslin tree in the universe V. If W is an outer model and $b \in W$ is a cofinal branch through T, then b is $\mathbb{P}_T$-generic over V.

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Suppose not. Let $S \subseteq T$ be a finitely branching uncountable subtree. Then S is also Suslin. Force with $\mathbb{P}_S$ to add a branch b. By the previous lemma, b is $\mathbb{P}_T$-generic over V. But S is finitely branching and T is infinitely branching, so b is disjoint from S above some node of T, by a density argument. $\square$

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$$\{\text{Suslin}\} \subseteq \{\text{Lindelöf}\} \subseteq \{\text{Aronszajn}\}$$

A non-Lindelöf Aronszajn tree

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Use a bijection $f : \omega_1 \times \omega \to \omega_1$ to transfer $\vec{e}$ to a coherent sequence $\langle f[e_\alpha] : \alpha < \omega_1 \rangle$, which is then used to build a binary Aronszajn tree T.

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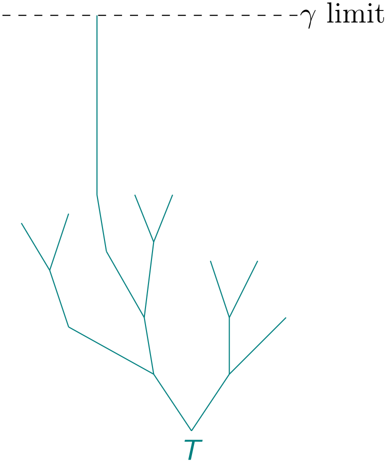
Now recursively build, level by level, an infinitely branching tree U which has T as a subtree.



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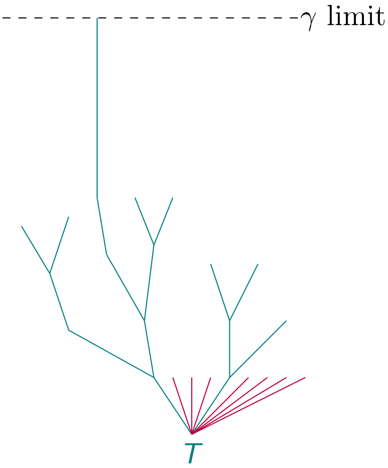

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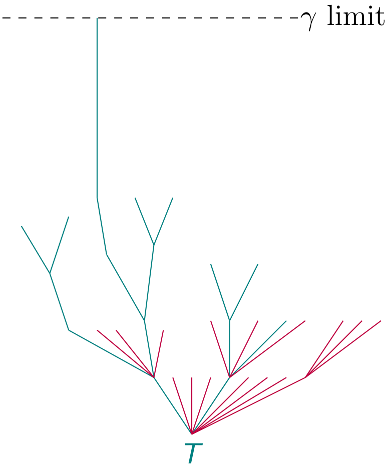

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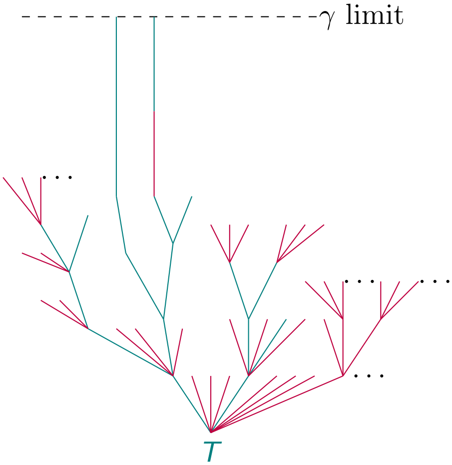

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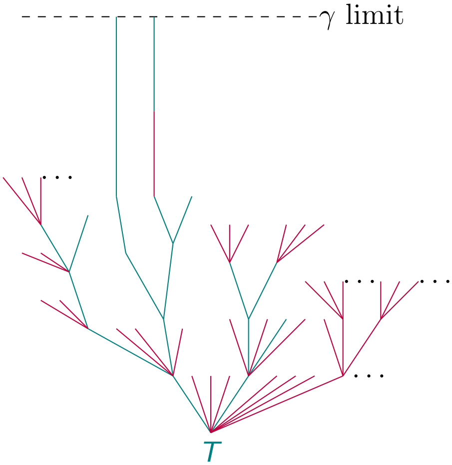

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($\Diamond$) A special Lindelöf tree

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Build an infinitely branching tree T with underlying set ω_1 together with a specializing function $\varphi : T \to \mathbb{Q}$ by recursion on levels, maintaining that if $\varphi(x) < q$ then there is some $y > x$ with $\varphi(y) = q$.

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Let $\vec{f} = \langle f_\alpha : \alpha \in \lim(\omega_1) \rangle$ be such $f_\alpha : \alpha \to [\alpha]^{<\omega}$ and $\vec{f}$ guesses any $f : \omega_1 \to [\omega_1]^{<\omega}$ stationarily often.

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Therefore, T is Lindelöf and so, under $\diamondsuit$,

$$\{\text{Suslin}\} \subsetneq \{\text{Lindelöf}\} \subsetneq \{\text{Aronszajn}\}$$

Definition

Let T be a normal infinitely branching $\aleph_1$-tree. Consider the following poset $\mathbb{P}$: conditions are functions $p \in \prod_{x \in F} [I(x)]^{<\omega}$, where $F \in [T]^{<\omega}$, such that:

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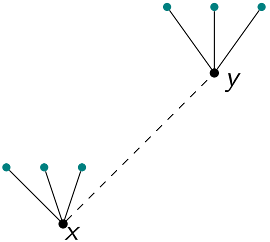

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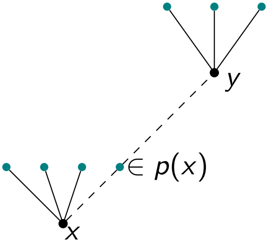

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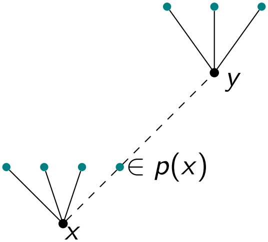

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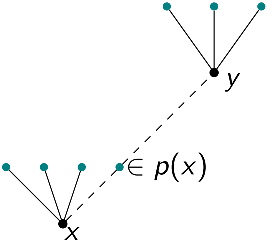

The generic subtree will be $\dot{S} = \bigcup\{\mathrm{dom}(p) : p \in \dot{G}\}$.
If $p \in \mathbb{P}$ and $x \in \mathrm{dom}(p)$, then p is a promise that $p(x) = I_{\dot{S}}(x)$.

Theorem (M.)

If T is a normal, infinitely branching Aronszajn tree, then $\mathbb{P}$ has the ccc and $\Vdash_{\mathbb{P}} \dot{S}$ is a finitely branching normal subtree of T.

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Is T still Aronszajn in $V^{\mathbb{P}}$?

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Corollary

Let $\mathbb{S}$ be Baumgartner's poset for specializing T with finite conditions. Then $\mathbb{S} \times \mathbb{P}$ is a ccc poset which forces that T is a special non-Lindelöf Aronszajn tree.

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Corollary

If $\mathrm{MA}_{\aleph_1}$ holds, then there are no Lindelöf trees.

Other ways of adding subtrees?

Theorem (M.)

Let T be an infinitely branching $\aleph_1$ tree. Suppose $\mathbb{P}$ is a poset, G is $\mathbb{P}$-generic over V and $S \in V[G]$ is a finitely branching subtree of T.

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Theorem (M.)

Let T be an infinitely branching $\aleph_1$ tree. Suppose $\mathbb{P}$ is a poset, G is $\mathbb{P}$-generic over V and $S \in V[G]$ is a finitely branching subtree of T. Suppose that $\mathbb{P}$ is either

- *countably closed*
- *strongly proper for a stationary set of countable elementary substructures of some (large) H_λ.*

Then $S \in V$.

Thank you :)

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