Methods of Stochastic Modelling
(Métodos de Modelação Estocástica)

Modelação e Desempenho de Redes e Serviços

Prof. Amaro de Sousa (asou@ua.pt)

DETI-UA, 2025/2026

Random experiment

- In a random experiment, the *sample space*, S, is the set of all possible results of the experiment
- Any subset E of the sample space S is named an event

- Given two events E and F :
  - The *union of the events*, $E \cup F$, is the set of possible results that belong to at least one of them
  - The *intersection of the events*, EF, is the set of possible results that belong to both events simultaneously

---

- When $EF = \varnothing$ ($\varnothing$ is the empty set), we say that the events are mutually exclusive
- The *complement of an event E*, E^c, is the set of all possible results (i.e., all results in the sample space S) that do not belong to E

Probabilities defined over events

- For each possible event E of a sample space S, the assignment of a real value $P(E)$ can represent the occurrence probability of event E, if it satisfies the following conditions:

  (1) $0 \leq P(E) \leq 1$

  (2) $P(S) = 1$

  (3) For any set of mutually exclusive events E_1, E_2, E_3, ...

$$P\left(\bigcup_i E_i\right) = \sum_i P(E_i)$$

- Corollaries:

  $P(E) + P(E^c) = 1$

  $P(E \cup F) = P(E) + P(F) - P(EF)$

Conditional probabilities

- Given two event E and F, the *conditional probability of E knowing that F has occured*, represented by $P(E|F)$, is given by

$$P(E|F) = P(EF) \,/\, P(F)$$

- Two events E and F are named *independent events* if

$$P(EF) = P(E)P(F)$$

- If E and F are independent events, then:

$$P(E|F) = P(EF) \,/\, P(F) = P(E)P(F) \,/\, P(F) = P(E)$$

$$P(F|E) = P(FE) \,/\, P(E) = P(F)P(E) \,/\, P(E) = P(F)$$

meaning that the occurrence probability of one event does not change if we know that the other event has occurred.

Bayes Rule

Consider a set of events F_1, F_2, …, F_n that are mutually exclusive, and their union is the sample space S of a random experiment. Consider any other event E.

The probability of event E is given by:

$$P(E) = \sum_{i=1}^{n} P(EF_i) = \sum_{i=1}^{n} P(E \mid F_i) P(F_i)$$

Moreover, knowing that event E has occurred, the occurrence probability of event F_j (with j = 1, 2, …, n) is given by:

$$P(F_j \mid E) = \frac{P(EF_j)}{P(E)} = \frac{P(E \mid F_j) P(F_j)}{P(E)} = \frac{P(E \mid F_j) P(F_j)}{\sum_{i=1}^{n} P(E \mid F_i) P(F_i)}$$

Example 1 – conditional probabilities

In a multiple-choice test, a student knows the answer with probability p (and, of course, guesses it with probability $1 - p$). When guessing the answer, the student answers correctly with probability $1/m$, where m is the number of multiple-choice answers.

Determine the probability of the student *(i)* to answer correctly each question and *(ii)* to know the answer when he answers correctly the question.

Events:
E – the student answers correctly
F_1 – the student knows the answer
F_2 – the student does not know the answer

(i) $P(E) = P(E|F_1)P(F_1) + P(E|F_2)P(F_2)$

$= 1 \times p + 1/m \times (1 - p) =$

$= p + (1 - p)/m$

(ii) $P(F_1|E) = P(E|F_1)P(F_1) / P(E)$

$= 1 \times p / [p + (1 - p)/m] =$

$= p\, m / [1 + (m - 1)\, p]$

If $p = 50\%$ and $m = 4$, then *(i)* $P(E) = 62.5\%$ and *(ii)* $P(F_1|E) = 80\%$

Example 2 – conditional probabilities

In a wireless link between two hosts, the probability of the transmitted data packets being received with errors is 0.1% in normal link conditions or 10% with external interferences. The probability of external interferences is 2%. In reception, the hosts are able detect if each data packet is or is not received with errors.

Determine: *(i)* the probability of a data packet being received with errors and *(ii)* the probability of the link being with interference when a data packet is received with errors.

Events:
- E – the packet is received with errors
- F_1 – the link is in the normal state
- F_2 – the link is with interference

(i) $P(E) = P(E|F_1)P(F_1) + P(E|F_2)P(F_2)$

$= 0.001\times(1 - 0.02) + 0.1\times0.02$

$= 0.00298 = 0.298\%$

(ii) $P(F_2|E) = P(E|F_2)P(F_2) / P(E)$

$= 0.1 \times 0.02 / 0.00298$

$= 0.671 = 67.1\%$

Random variables

- A *random variable X is a function* that assigns a real value to each possible result in S of a random experiment.

- The *distribution function* (or *cumulative distribution function*) of a random variable X is defined as:

$$F(x) = P(X \leq x) \quad , -\infty < x < +\infty$$

- *Properties* of the distribution function:

(1) $0 \leq F(x) \leq 1$ for all values of x

(2) if $x_1 \leq x_2$ than $F(x_1) \leq F(x_2)$ (non-decreasing function)

(3) $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to +\infty} F(x) = 1$

(4) $P(a < X \leq b) = F(b) - F(a)$, for $a < b$

Discrete random variables

- X is *a discrete random variable* if it can only take a finite number of values or a countable number of distinct separate values $x_1, x_2, \ldots, x_i, \ldots$

- The *probability function* (or *mass probability function*) of a discrete random variable X is defined as:

$$f(x_i) = P(X = x_i) \qquad \text{for all values of } i = 1,2,3,\ldots$$

- It is mandatory that: $$\sum_{i=1}^{\infty} f(x_i) = 1$$

- Consequently, the *distribution function* of X becomes:

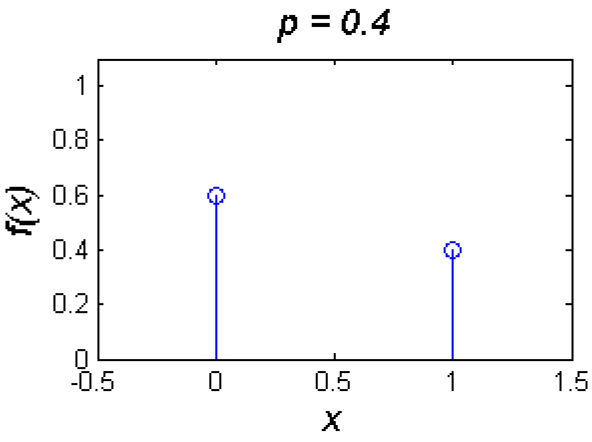
$$F(x) = \sum_{x_i \le x} f(x_i) \qquad , -\infty < x < +\infty$$

Examples of discrete random variables

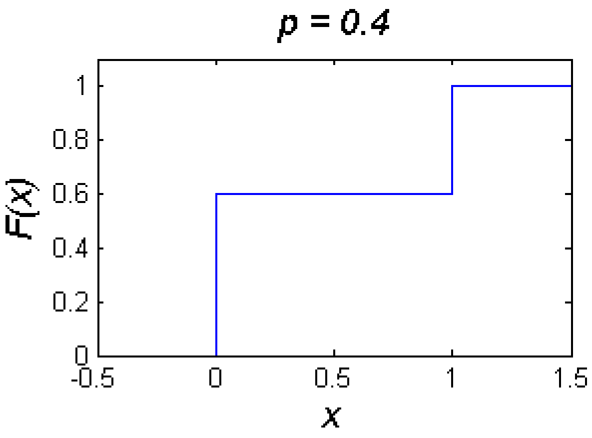
Bernoulli random variable represents an experiment that can be a success with probability *p* or a failure with probability 1 – *p*.

If *X* = 1 represents the success and *X* = 0 the failure, the probability function is:

$$f(i) = p^i (1-p)^{1-i}, i = 0,1$$



f(x) – probability function



F(x) – distribution function

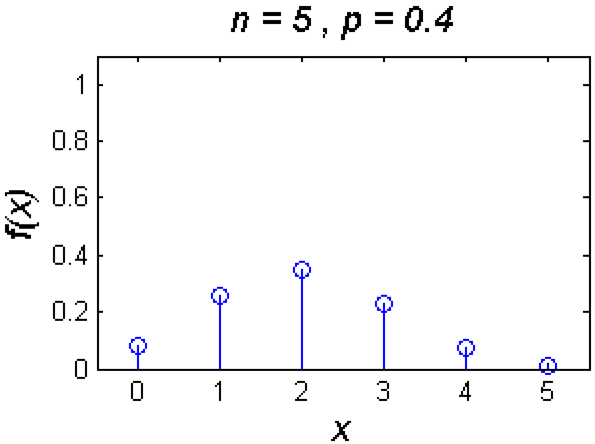
Examples of discrete random variables

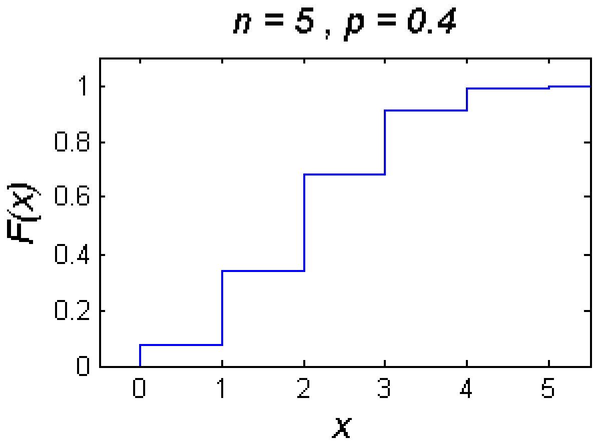
Binomial random variable represents a set of n independent Bernoulli experiments, each one that can be a success with probability p or a failure with probability $1 - p$.

If X represent the number of successes in n experiments, the probability function is:

$$f(i) = \binom{n}{i} p^i (1-p)^{n-i}, i = 0, 1, 2, \ldots, n$$

where $\binom{n}{i} = \frac{n!}{i!(n-i)!}$



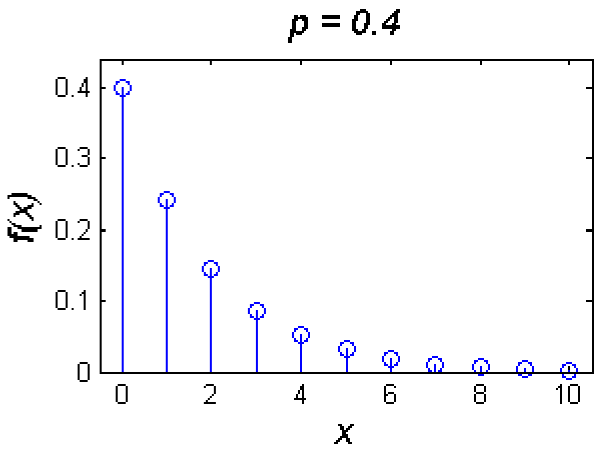


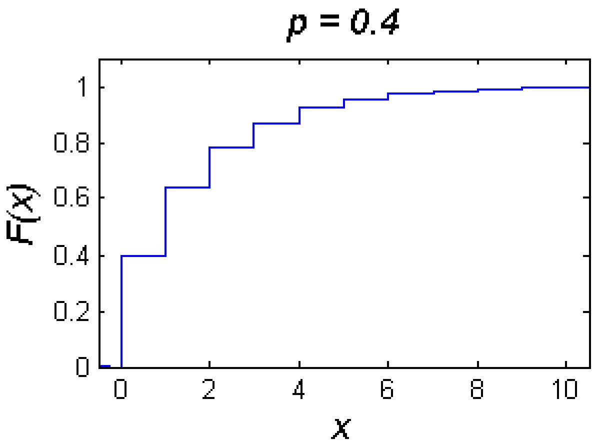
Examples of discrete random variables

<u>Geometric random variable</u>: represents a set independent Bernoulli experiments, all with the same success probability, until an experiment results in a success.

If X represents the number of failures before the success, the probability function is

$$f(i) = (1-p)^i p \ \ , \ i = 0, 1, 2, \ldots$$





If X represents the number of experiments until the success, the probability function is

$$f(i) = (1-p)^{i-1} p \ \ , \ i = 1, 2, \ldots$$

Example 3 – discrete random variables

On a given data link, the BER (*bit error rate*) is 10^{-5} and the errors in the different bits of a data packet are statistically independent.

Determine: *(i)* the probability of a data packet of size 100 Bytes to be received without errors and *(ii)* the probability of a data packet of size 1000 Bytes to be received with at least 2 errors.

The number of bits in error on a data packet is a binomial random variable with the probability of success given by the BER value and the number of Bernoulli experiments given by the number of bits of the packet

$$f(i) = \binom{n}{i} p^i (1-p)^{n-i}, i = 0, 1, 2, ..., n$$

(i) $$f(0) = \binom{n}{0} p^0 (1-p)^{n-0} = \binom{100 \times 8}{0} \times \left(1-10^{-5}\right)^{100\times 8} = 0.992 = 99.2\%$$

(ii) $$1 - f(0) - f(1) = 1 - \binom{n}{0} p^0 (1-p)^{n-0} - \binom{n}{1} p^1 (1-p)^{n-1}$$

$$= 1 - \left(1-10^{-5}\right)^{8000} - 8000 \times 10^{-5}\left(1-10^{-5}\right)^{7999} = 3.034\text{E-}3 = 0.3\%$$

Continuous random variables

- A *random variable X is said continuous* if it exists a non-negative function $f(x)$ such that for any interval of continuous values B:

$$P(X \in B) = \int_B f(x)dx \qquad \int_{-\infty}^{+\infty} f(x)dx = 1$$

$f(x)$ is the *probability density function* of the random variable X

- Therefore: $$P\{a \le X \le b\} = \int_a^b f(x)dx$$

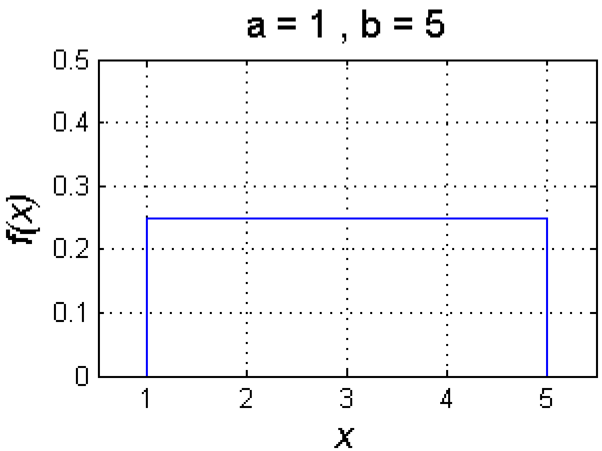
- The *distribution function* of the random variable X becomes:

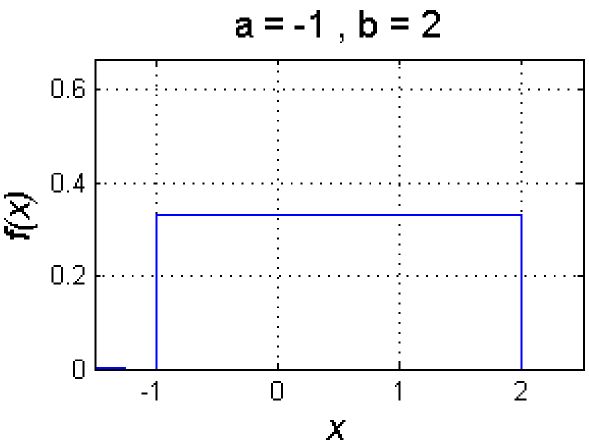
$$F(x) = P(X \in [-\infty, x]) = \int_{-\infty}^{x} f(y)dy$$

Examples of continuous random variables

Random variable with uniform distribution: a random variable is uniformly distributed in the interval [*a*,*b*] if its probability density function is given by

$$f(x) = \begin{cases} \dfrac{1}{b-a} & , a < x < b \\ 0 & , cc \end{cases}$$

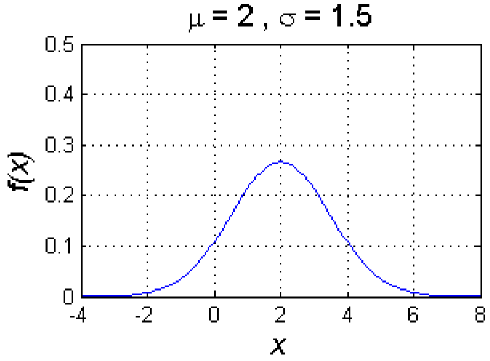




Examples of continuous random variables

<u>Random variable with Normal (ou Gaussian) distribution</u>: a random variable has a normal distribution with average μ and standard deviation σ if its probability density function is:

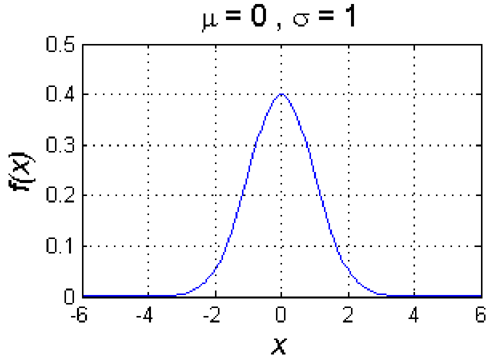
$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$$



The Standard Normal (or Gaussian) distribution is the normal distribution with average 0 and standard deviation 1.

In this case:

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$



Average (or expected value) of a random variable

- *Average* (or *expected value*), $E[X]$, of a random variable X:

$$E[X] = \begin{cases} \displaystyle\sum_{j=1}^{\infty} x_j \, f_X(x_j) & \text{if } X \text{ discrete} \\ \displaystyle\int_{-\infty}^{+\infty} x \, f_X(x_j) dx & \text{if } X \text{ continuous} \end{cases}$$

- Important property of the average: $E\left[\sum_{i=1}^{n} c_i \, X_i\right] = \sum_{i=1}^{n} c_i \, E[X_i]$

- Average of a random variable $Y = g(X)$:

$$E[g(X)] = \begin{cases} \displaystyle\sum_{j=1}^{\infty} g(x_j) \, f_X(x_j) & \text{if } X \text{ discrete} \\ \displaystyle\int_{-\infty}^{+\infty} g(x) \, f_X(x_j) dx & \text{if } X \text{ continuous} \end{cases}$$

Variance and standard deviation of a random variable

- *Variance* of a random variable X:

$$Var[X] = E\left[\left(X - E[X]\right)^2\right] = E\left[X^2\right] - E[X]^2$$

- Important properties of the variance:

2nd moment of X

$$Var[X] \geq 0$$

$$Var[cX] = c^2\, Var[X]$$

$$Var\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} Var[X_i] \quad \text{if } X_i \text{ are independent}$$

- *Standard deviation* of a random variable X:

$$\sigma[X] = \sqrt{Var[X]}$$

Example 4

A data link of 10 Mbps supports a flow of data packets whose size is 100 Bytes with probability 10%, 500 Bytes with probability 50% and 1500 Bytes with probability 40%. Consider the random variable *X* representing the packet transmission time.

Determine: *(i)* the average packet transmission time $E[X]$, *(ii)* the second moment of the packet transmission time $E[X^2]$ and *(iii)* the variance of the packet transmission time $Var[X]$.

(i) $$E[X]=\sum_{j=1}^{\infty} x_j f_X(x_j) = \frac{100\times 8}{10^7}\times 0.1+\frac{500\times 8}{10^7}\times 0.5+\frac{1500\times 8}{10^7}\times 0.4$$

$$= 0.688\times 10^{-3}\ \mathrm{sec}\ = 0.688\ \mathrm{msec}$$

(ii) $$E[X^2]=\sum_{j=1}^{\infty} (x_j)^2 f_X(x_j) = \left(\frac{100\times 8}{10^7}\right)^2\times 0.1+\left(\frac{500\times 8}{10^7}\right)^2\times 0.5+\left(\frac{1500\times 8}{10^7}\right)^2\times 0.4$$

$$= 6.5664\times 10^{-7}\ \mathrm{sec}^2$$

Example 4 - continuation

A data link of 10 Mbps supports a flow of data packets whose size is 100 Bytes with probability 10%, 500 Bytes with probability 50% and 1500 Bytes with probability 40%. Consider the random variable X representing the packet transmission time.

Determine: *(i)* the average packet transmission time $E[X]$, *(ii)* the second moment of the packet transmission time $E[X^2]$ and *(iii)* the variance of the packet transmission time $Var[X]$.

(iii) **1st alternative:** $Var[X] = E\left[\left(X - E[X]\right)^2\right]$

$$Var[X] = \left(\frac{100 \times 8}{10^7} - E[X]\right)^2 \times 0.1 + \left(\frac{500 \times 8}{10^7} - E[X]\right)^2 \times 0.5 + \left(\frac{1500 \times 8}{10^7} - E[X]\right)^2 \times 0.4$$

$$= 1.833 \times 10^{-7} \text{ sec}^2$$

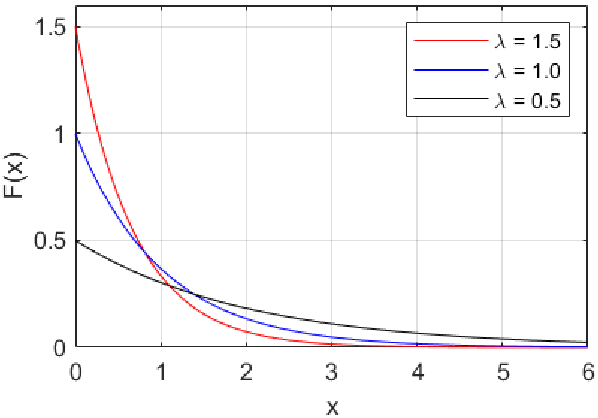
2nd alternative: $Var[X] = E\left[X^2\right] - E[X]^2$

$$Var[X] = 6.5664 \times 10^{-7} - (0.688 \times 10^{-3})^2 = 1.833 \times 10^{-7} \text{ sec}^2$$

Random variable with exponential distribution

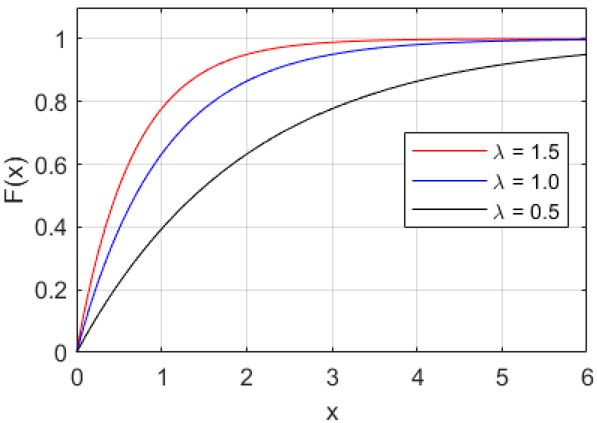
- A continuous random variable X following an exponential distribution with parameter λ, $\lambda > 0$, has the probability density function:

$$f(x)=\begin{cases}\lambda e^{-\lambda x}, & x \geq 0\\ 0, & x < 0\end{cases}$$



- The distribution function is:

$$F(x)=\begin{cases}1-e^{-\lambda x}, & x \geq 0\\ 0, & x < 0\end{cases}$$



Exponential distribution

- The average, variance and standard deviation of an exponentially distributed random variable X are only dependent on parameter λ :

$$E[X] = \frac{1}{\lambda} \qquad Var[X] = \left(\frac{1}{\lambda}\right)^2 \qquad \sigma[X] = \frac{1}{\lambda}$$

- The exponential distribution has no memory, meaning that:

$$P\{X > s + t \mid X > t\} = P\{X > s\}$$

- If the random variables X_1 and X_2 are independent and exponentially distributed with averages $1/\lambda_1$ and $1/\lambda_2$ respectively, than:

$$P\{X_1 < X_2\} = \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

Example 5 – exponential distribution

A data link of 10 Mbps supports a flow of data packets whose size is exponentially distributed with an average of 1000 Bytes. Consider the random variable *X* representing the packet transmission time.

Determine: *(i)* the average packet transmission time *E*[*X*], *(ii)* the variance of the packet transmission time *Var*[*X*] and *(iii)* the second moment of the packet transmission time *E*[X^2].

(i) $$E[X] = \frac{1000 \times 8}{10^7} = 8 \times 10^{-4} = 0.8 \text{ msec}$$

Capacity of the link in pps (packets per second)

$$E[X] = \frac{1}{\mu} \quad \Leftrightarrow \quad \mu = \frac{1}{E[X]} = \frac{1}{8 \times 10^{-4}} = 1250 \text{ pps}$$

(ii) $$Var[X] = \left(\frac{1}{\mu}\right)^2 = (8 \times 10^{-4})^2 = 6.4 \times 10^{-7} \text{ sec}^2$$

(iii) $$Var[X] = E[X^2] - E[X]^2 \quad \Leftrightarrow \quad E[X^2] = Var[X] + E[X]^2$$

$$E[X^2] = 6.4 \times 10^{-7} + (8 \times 10^{-4})^2 = 1.28 \times 10^{-6} \text{ sec}^2$$

Stochastic process

- A *stochastic process* $\{X(t), t \in T\}$ is a set of random variables: for each value $t \in T$, $X(t)$ is a random variable.

- Index t is frequently seen as a time instant. In this interpretation, the random variable $X(t)$ represents the *state* of the stochastic process on time instant t.

- Set T is the *set of all possible indices* of the stochastic process:
  (1) if T is a countable set, the stochastic process is designated as being in discrete time
  (2) if T is an interval of continuous values, the stochastic process is designated as being in continuous time

- The *state space* of the stochastic process is the set of all possible values that the random variables $X(t)$ can take.

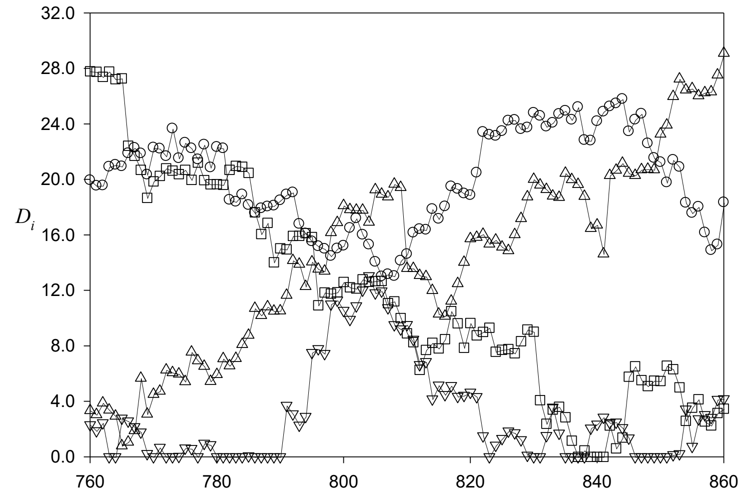
Examples of stochastic processes

Consider a system with a queue and a server. Clients arrive to the system and are either immediately served (if the server is empty) or go to the queue to wait to be served.

<u>*Queuing delay of clients*</u>

(1) is a stochastic process in discrete time (1st client, 2nd client, ...)

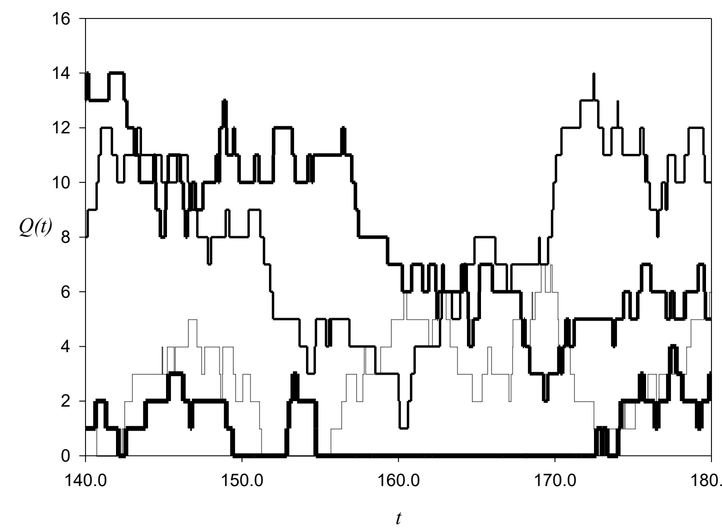
(2) the state is a continuous value (queuing delay value continuous)



<u>*Number of clients in the queue*</u>

(1) is a stochastic process in continuous time

(2) the state is a discrete value (0 clients, 1 client, 2 clients, ...)



Counting process

- A stochastic process $\{N(t),\ t \geq 0\}$ is a <u>*counting process*</u> if $N(t)$ represents the total number of events occurred until time instant t.
- A counting process satisfies the following conditions:

  (1) $N(t) \geq 0$.

  (2) $N(t)$ takes only non-negative integer values.

  (3) If $s < t$, then $N(s) \leq N(t)$.

  (4) If $s < t$, then $N(t) - N(s)$ is equal to the number of events occurred in the time interval $[s,t]$.
- A counting process has <u>*independent increments*</u> if the number of events in disjoint time intervals is independent.
- A counting process has <u>*stationary increments*</u> if the distribution function of the number of events occurred in any time interval is only dependent on the time interval length.

Poisson process

- A counting process is a *<u>Poisson process</u>* with rate λ, $\lambda > 0$, if the following conditions hold:

  (1) $N(0) = 0$;

  (2) the process has independent increments;

  (3) the number of events in a time interval of length t has a Poisson distribution with average λt, i.e., for all s, $t \geq 0$

$$P\{N(s+t)-N(s)=n\}=e^{-\lambda t}\frac{(\lambda t)^n}{n!}$$

- A Poisson process has stationary increments and the average number of events in a time interval of length t is:

$$E[N(t)] = \lambda t$$

the reason why the parameter λ is named the *<u>rate</u>* (i.e., the average number of events per time unit) of the Poisson process.

Properties of a Poisson process

- **Property 1:** Consider in a Poisson process with rate λ that:
  - T_1 is the time instant of the first event
  - T_n, $n > 1$, is the time interval length between the $(n-1)^{th}$ event and the n^{th} event
- Then, T_n, $n = 1, 2, \ldots$, are independent and identically distributed random variables with an *exponential distribution* with average $1/\lambda$.

- **Property 2:** Consider in a Poisson process $\{N(t),\ t \geq 0\}$ with rate λ that each event is independently classified as:
  - event of type 1 with probability p
  - event of type 2 with probability $1 - p$

  and that $\{N_1(t),\ t \geq 0\}$ and $\{N_2(t),\ t \geq 0\}$ represent the number of events of each type occurred in the time interval $[0,t]$.
- Then, $N_1(t)$ e $N_2(t)$ are *both independent Poisson processes* with rates λp and $\lambda(1-p)$, respectively.

Properties of a Poisson process

- **Property 3:** Consider $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$ as two independent Poisson processes with rates λ_1 and λ_2, respectively.
- Then, the process $N(t) = N_1(t) + N_2(t)$ is also *a Poisson process* with rate $\lambda = \lambda_1 + \lambda_2$.

- **Property 4:** Consider that we know that exactly n events have occurred from the beginning until time instant t in a Poisson process.
- Then, the occurrence time instants of the n events are independent random variables with a *uniform distribution* in the interval $[0, t]$.

Markov chain in continuous time

- Consider a stochastic process in continuous time $\{X(t),\ t \geq 0\}$ with a state space defined by the set of non-negative integer values, i.e., $\{0, 1, 2, \ldots\}$).
- $X(t)$ *is a Markov chain* if for all s, $t \geq 0$ and non-negative integer values i, j, $x(u)$, $0 \leq u < s$:

$$P\{X(s+t) = j \mid X(s) = i, X(u) = x(u), 0 \leq u < s\} = P\{X(s+t) = j \mid X(s) = i\}$$

- This property means that the probability of a future state $X(s+t)$ knowing the present state $X(s)$ and all past states $X(u)$, $0 \leq u < s$, depends only on the present state and is independent from the past.

- If $P\{X(s+t) = j \mid X(s) = i\}$ is independent of s, then it is said that the Markov chain has homogeneous transition probabilities:

$$P\{X(s+t) = j \mid X(s) = i\} = P\{X(t) = j \mid X(0) = i\}$$

Markov chain in continuous time

- A Markon chain in continuous time has the following properties:
  (1) The holding time of the process in each state *i* (i.e., the time interval from the moment the process enters state *i* until the moment the process leaves state *i*) is an exponentially distributed random variable with average $1/q_i$;

  NOTE: This property is equivalent to say that when the process is in state *i*, it jumps to another state with a transition rate q_i.

  (2) When the process leaves state *i*, it jumps to state *j* with a probability P_{ij} in accordance with the following conditions:

$$P_{ii} = 0 \qquad 0 \leq P_{ij} \leq 1 \quad , j \neq i \qquad \sum_j P_{ij} = 1$$

- In a Markov chain in continuous time, the holding time on each state and the next state to where the process jumps are independent random variables.

Markov chain in continuous time: transition rates

- For any pair of states *i* and *j*, consider:

$$q_{ij} = q_i P_{ij}$$

q_i - the transition rate from state *i* to another state (introduced in the previous slide)

P_{ij} - the probability of jumping to state *j* when the process leaves state *i* (introduced in the previous slide)

q_{ij} - the transition rate from state *i* to state *j*

- The state transition rates q_{ij} are the usual values represented in the state transition diagrams of Markov chains in continuous time.

- Since $$q_i = \sum_j q_i P_{ij} = \sum_j q_{ij} \qquad P_{ij} = \frac{q_{ij}}{q_i} = \frac{q_{ij}}{\sum_j q_{ij}}$$

we can always obtain all parameters of interest from the state transition rates q_{ij} in a Markov chain in continuous time.

State limit probabilities

- Consider $P_{ij}(t) = P\{X(s+t) = j \mid X(s) = i\}$

  the probability of state *j* after a time duration *t* when the Markov chain is in state *i* at the present time.

- The probability of a Markov chain in continuous time being on state *j* converges to a limit value which is independent of the initial state:

$$\pi_j \equiv \lim_{t \to \infty} P_{ij}(t)$$

- For a Markov chain to have state limit probabilities π_j , it must be:
  - irreducible (all states can reach each other),
  - aperiodic (there's no fixed cycle of states),
  - positive recurrent (starting on any state, the average time to return to it is finite to all states).
- In this course unit, all Markov chains of interest have these properties.

Computing state limit probabilities

- The state limit probabilities π_j can be computed by resolving the following set of equations:

$$q_j \pi_j = \sum_{k \neq j} q_{kj} \pi_k \qquad , \qquad \text{for all states } j$$

$$\sum_j \pi_j = 1$$

- These equations are known as the balance equations:

rate at which the system transits from state *j* to another state

=

rate at which the system transits from another state to state *j*

- The probability π_j gives also the percentage of time that the process is in state *j*.

- The state limit probabilities are also named stationary probabilities: if the initial state is characterized by the distribution $\{\pi_j\}$, then the probability of each state *j* is π_j, for all *t*.

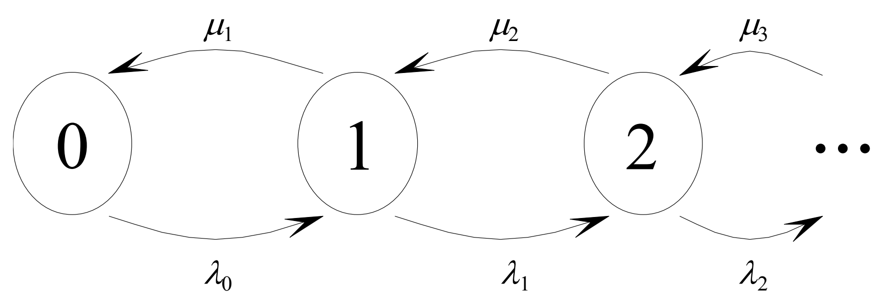
Birth-dead process

- Consider a system whose state represents the number of clients, and the system has an infinite capacity (i.e., the system can accommodate an infinite number of clients).
- When the system is in state n (i.e., it has n clients):

  (1) new clients arrive to the system at an exponential rate λ_n

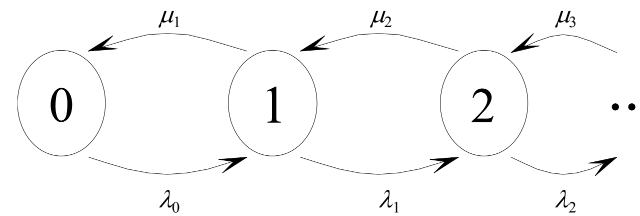
  (2) clients leave the system at an exponention rate μ_n

- This Markov chain is named a birth-dead process.
- Parameters λ_n (n = 0, 1, …) and μ_n (n = 1, 2, …) are referred to as the birth rates and the dead rates, respectively.



State transition diagram of a birth-dead process

Balance equations of a birth-dead process



In a birth-dead process, it is possible to determine the state limit probabilities π_n of each state n (= 0, 1, 2, …) as follows.

Balance equations: $\boxed{q_j \pi_j = \sum_{k \neq j} q_{kj} \pi_k \quad \text{, for each state } n}$

State	*rate from State = rate to State*
0	$\lambda_0 \pi_0 = \mu_1 \pi_1$
1	$(\lambda_1 + \mu_1)\pi_1 = \mu_2 \pi_2 + \lambda_0 \pi_0$
2	$(\lambda_2 + \mu_2)\pi_2 = \mu_3 \pi_3 + \lambda_1 \pi_1$
$n, n \geq 1$	$(\lambda_n + \mu_n)\pi_n = \mu_{n+1} \pi_{n+1} + \lambda_{n-1} \pi_{n-1}$

Or equivalently (by manipulation of the previous equations):

$$\lambda_n \pi_n = \mu_{n+1} \pi_{n+1}, \qquad n \geq 0$$

State limit probabilities of birth-dead processes

State limit probabilities:

$$\pi_0 = \frac{1}{1 + \sum_{i=1}^{\infty} \frac{\lambda_0 \lambda_1 \cdots \lambda_{i-1}}{\mu_1 \mu_2 \cdots \mu_i}}$$

$$\pi_n = \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n \left(1 + \sum_{i=1}^{\infty} \frac{\lambda_0 \lambda_1 \cdots \lambda_{i-1}}{\mu_1 \mu_2 \cdots \mu_i} \right)} = \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} \cdot \pi_0, \; n \geq 1$$

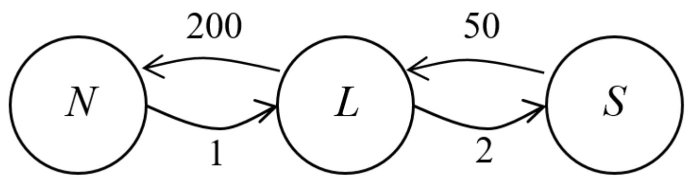
NOTE: If the process has a finite number N of states (i.e., n = 0, 1, …, N), the summations in the above expressions are from 0 to N.

In the case of an infinite number of states, the state limit probabilities exist only if:

$$\sum_{i=1}^{\infty} \frac{\lambda_0 \lambda_1 \cdots \lambda_{i-1}}{\mu_1 \mu_2 \cdots \mu_i} < \infty$$

Example 6

Consider a wireless link for packet transmission that can be in one of 3 possible states – Normal (*N*), Light Interference (*L*) or Severe Interferences (*S*) – according with the following Markov chain (rates in transitions per hour):



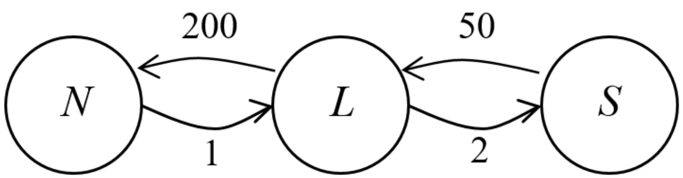
(a) Determine the probability of each state.

(b) Determine the average holding time of the link in each state (in minutes).

(c) Knowing that of each data packet being received with errors (i.e., with one or more errors) is 0.01% in state *N*, 0.1% in state *L* and 1% in state *S*, what is the probability of state *N* when a data packet is received with errors?

Example 6 – solution of (a)

Consider a wireless link for packet transmission that can be in one of 3 possible states – Normal (*N*), Light Interference (*L*) or Severe Interferences (*S*) – according with the following Markov chain (rates in transitions per hour):



(a) Determine the probability of each state.

$$P_N = \frac{1}{1 + \frac{1}{200} + \frac{1}{200} \times \frac{2}{50}} = 0.99483 = 99.483\%$$

$$\pi_0 = \frac{1}{1 + \sum_{i=1}^{\infty} \frac{\lambda_0 \lambda_1 \cdots \lambda_{i-1}}{\mu_1 \mu_2 \cdots \mu_i}}$$

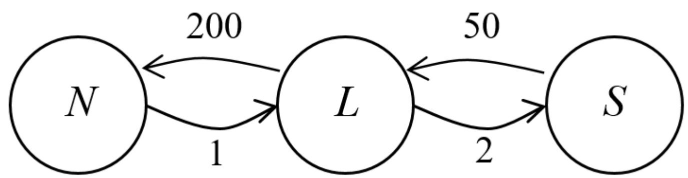
$$P_L = \frac{\frac{1}{200}}{1 + \frac{1}{200} + \frac{1}{200} \times \frac{2}{50}} = 0.00497 = 0.497\%$$

$$\pi_n = \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} \cdot \pi_0$$

$$P_S = \frac{\frac{1}{200} \times \frac{2}{50}}{1 + \frac{1}{200} + \frac{1}{200} \times \frac{2}{50}} = 0.0002 = 0.02\%$$

Example 6 – solution of (b)

Consider a wireless link for packet transmission that can be in one of 3 possible states – Normal (*N*), Light Interference (*L*) or Severe Interferences (*S*) – according with the following Markov chain (rates in transitions per hour):



(b) Determine the average holding time of the link in each state (in minutes).

$$T_N = \frac{1}{1} = 1 \text{ hour} = 60 \text{ minutes}$$

$$T_L = \frac{1}{2+200} = 0.00495 \text{ hours} = 0.3 \text{ minutes}$$

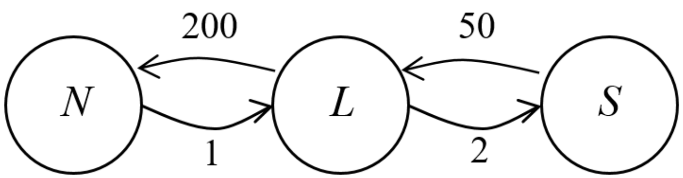
$$T_S = \frac{1}{50} = 0.02 \text{ hours} = 1.2 \text{ minutes}$$

Average holding time:
$T = 1/q_i$

$$q_i = \sum_j q_i P_{ij} = \sum_j q_{ij}$$

Example 6 – solution of (c)

Consider a wireless link for packet transmission that can be in one of 3 possible states – Normal (*N*), Light Interference (*L*) or Severe Interferences (*S*) – according with the following Markov chain (rates in transitions per hour):



(c) Knowing that of each data packet being received with errors (i.e., with one or more errors) is 0.01% in state *N*, 0.1% in state *L* and 1% in state *S*, what is the probability of state *N* when a data packet is received with errors?

by the Bayes rule

$$P(N|E) = \frac{P(E|N) \times P(N)}{P(E|N) \times P(N) + P(E|L) \times P(L) + P(E|S) \times P(S)}$$

$$= \frac{0.0001 \times 0.99483}{0.0001 \times 0.99483 + 0.001 \times 0.00497 + 0.01 \times 0.00020}$$

$$= 0.9346 = 93.46\%$$

Little's theorem definitions

- Consider the observation of a system from time instant t = 0. Consider:

  $L(t)$ – number of clients in the system in time instant t,

  $N(t)$ – number of clients that arrived at the system until time instant t,

  W_i – amount of time that the i^{th} client stays in the system.

- Average <u>number of clients</u> in the system until time instant t:

$$L_t = \frac{1}{t}\int_0^t L(\tau)d\tau \qquad L = \lim_{t\to\infty} L_t$$

- Average <u>arrival rate</u> of clients in the time interval [0,t]:

$$\lambda_t = N(t)/t \qquad \lambda = \lim_{t\to\infty} \lambda_t$$

- Average <u>amount of time</u> that clients stay in the system until time instant t:

$$W_t = \frac{\sum_{i=0}^{N(t)} W_i}{N(t)} \qquad W = \lim_{t\to\infty} W_t$$

Little's theorem

- The Little's theorem state that the long-term average number L of clients in a system is equal to the long-term average arrival rate λ multiplied by the average time W that a client spends in the system:

$$L = \lambda W$$

- This theorem translates the intuitive idea that, for the same client arrival rate λ, more congested systems (higher L) impose greater delays (higher W).

- Examples:
  – On a rainy day, the same car traffic rate λ is slower than normal (larger W) and consequently the streets are more congested (larger L).
  – A fast-food restaurant (smaller W) needs a smaller room (smaller L) than a regular restaurant, for the same customer arrival rate λ.

PASTA property
(*Poisson Arrivals always See Time Averages*)

- Consider a system such that each client arrives one at a time and is served one at a time.
- Consider $L(t)$ as the number of clients in the system in time instant t.
- Consider P_n, $n \geq 0$, defined as the stationary probability of exactly n clients being in the system (or the percentage of time the system has exactly n clients):

$$P_n = \lim_{t \to \infty} P\{L(t) = n\}$$

- Consider a_n as the percentage of clients that upon arrival to the system see the system with exactly n clients.

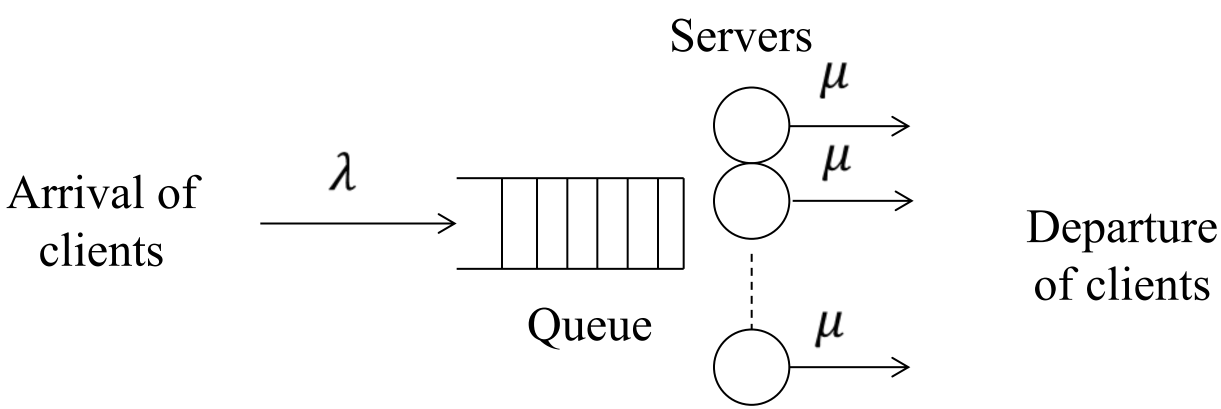
- **PASTA property:**

  Clients arriving according to a <u>Poisson process</u> such that <u>the serving time of each client is statistically independent of its arrival time instant</u> always see time averages:

$$a_n = P_n$$

Queuing system

- A queuing system is characterized by:
  - a set of c servers, each one serving clients with an average rate μ
  - a queue with a given capacity (in number of clients)
- Clients arrive at the system with an average rate λ
- When a client arrives:
  - if at least one of the servers is free, the client starts being served by an available server and departures from the system after being served
  - if all servers are busy, the client either goes to the queue (if the queue is not full) or is lost (if the queue is full)
- Clients in the queue are served with a FIFO (*First-In-First-Out*) queuing discipline



Queuing system

- A queuing system is generically represented by:

$$A/B/c/d$$

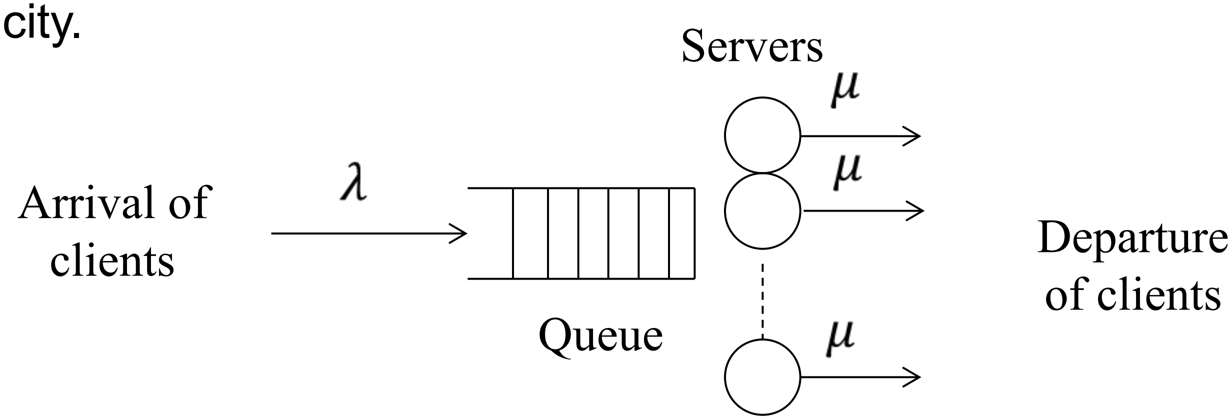
A – process of client arrivals:
M – Markovian, *D* – Deterministic, *G* – Generic

B – service time distribution:
M – Markovian, *D* – Deterministic, *G* – Generic

c – number of servers

d – capacity of the system in number of clients:
number of clients + capacity of the queue

- When *d* is not specified, it means that the queue has an infinite capacity.



M/*M*/1 queuing system

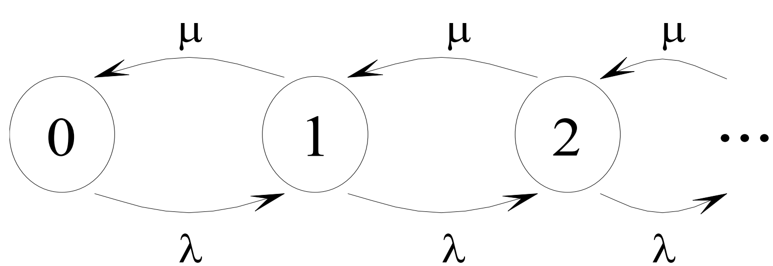
- Birth-dead process such that:

  (1) the client arrivals is a Poisson process with rate λ

  (2) the serving time of each server is exponentially distributed with average $1/\mu$ (i.e., the serving rate of each server is μ)

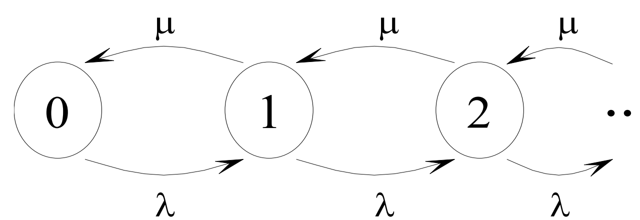
  (3) the system has 1 server

  (4) the system capacity is infinite (and, therefore, the birth-dead process has an infinite number of states)



- Example: a link with a capacity of μ packets/s and a very long queue such that data packets arrive with a Poisson rate of λ packets/s and data packets are exponentially distributed is modelled by an *M*/*M*/1 queuing system.

M/*M*/1 queuing system



$$P_0 = \frac{1}{1+\sum_{i=1}^{\infty}\frac{\lambda_0\lambda_1\cdots\lambda_{i-1}}{\mu_1\mu_2\cdots\mu_i}} = \frac{1}{1+\sum_{i=1}^{\infty}\left(\frac{\lambda}{\mu}\right)^i}$$

$$P_n = \frac{\lambda_0\lambda_1\cdots\lambda_{n-1}}{\mu_1\mu_2\cdots\mu_n}\cdot P_0 = \left(\frac{\lambda}{\mu}\right)^n \cdot P_0 = \frac{\left(\frac{\lambda}{\mu}\right)^n}{1+\sum_{i=1}^{\infty}\left(\frac{\lambda}{\mu}\right)^i}$$

- Average no. of clients in the system: $L = \sum_{n=0}^{\infty} nP_n = \frac{\lambda}{\mu-\lambda}$
- Average system delay of clients: $W = \frac{L}{\lambda} = \frac{1}{\mu-\lambda}$
- Average queuing delay of clients: $W_Q = W - \frac{1}{\mu} = \frac{\lambda}{\mu(\mu-\lambda)}$
- Average no. of clients in the queue: $L_Q = \lambda W_Q = \frac{\lambda^2}{\mu(\mu-\lambda)}$

by Little's theorem

M/*M*/1/*m* queuing system

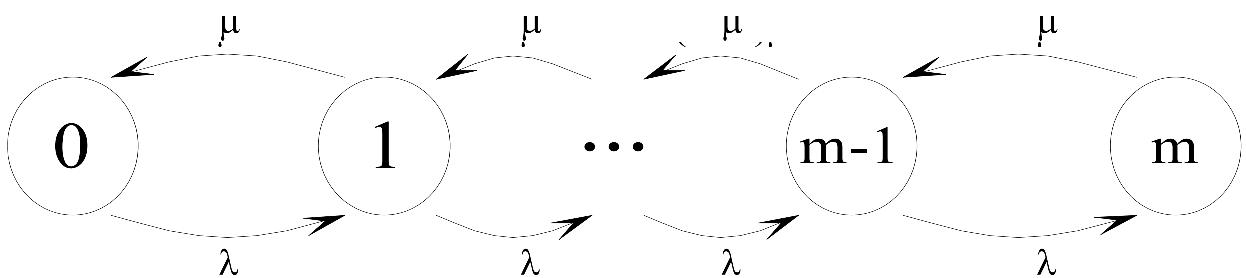
- Birth-dead process such that:

  (1) the client arrivals is a Poisson process with rate λ

  (2) the serving time of each server is exponentially distributed with average $1/\mu$ (i.e., the serving rate of each server is μ)

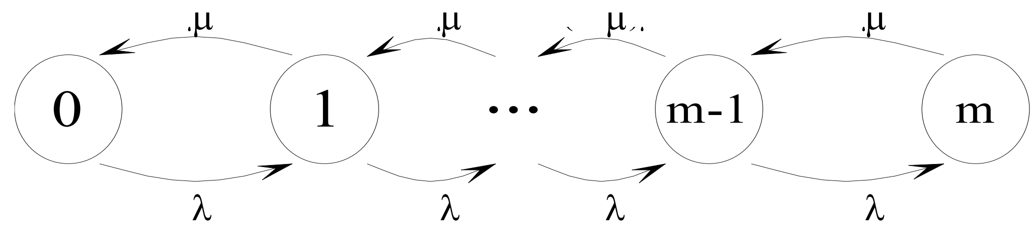
  (3) the system has 1 server

  (4) the system capacity is *m* clients (*i.e.*, the queue has a capacity of *m* – 1 clients)



The number of states of the birth-dead process is *m* + 1

M/M/*1/*m queuing system



- Balance equations:

$$\lambda P_{n-1} = \mu P_n, \qquad n = 1, 2, ..., m$$

- Stationary probability of *n* clients in the system:

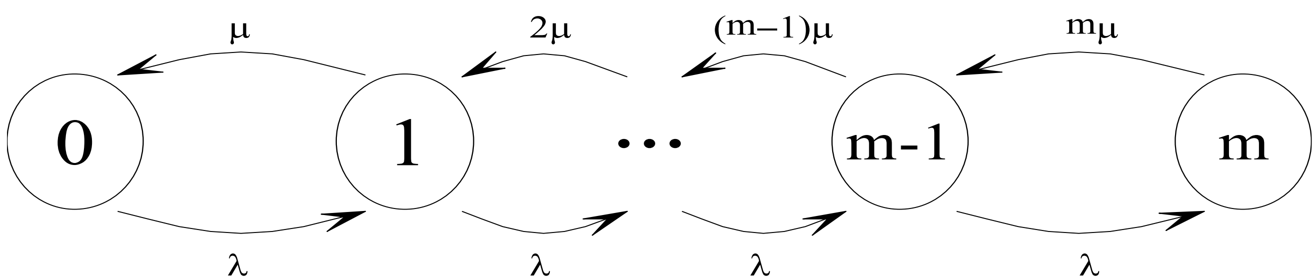
$$P_n = \frac{(\lambda/\mu)^n}{\sum_{i=0}^{m} (\lambda/\mu)^i} \qquad n = 0, 1, ..., m$$

- By PASTA probability, the probability of an arrival client seeing the system full (*i.e.*, the server not available and the queue fully occupied) is given by the probability of state *m*:

$$P_m = \frac{(\lambda/\mu)^m}{\sum_{i=0}^{m} (\lambda/\mu)^i}$$

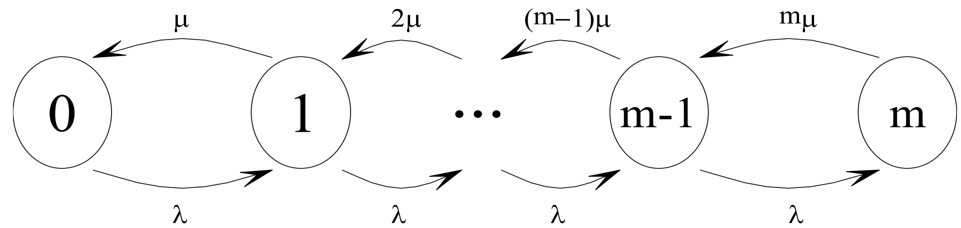
M/*M*/*m*/*m* queuing system

- Birth-dead process such that:
  - (1) the client arrivals is a Poisson process with rate λ
  - (2) the serving time of each server is exponentially distributed with average $1/\mu$ (i.e., the serving rate of each server is μ)
  - (3) the system has *m* servers
  - (4) the system capacity is *m* clients (*i.e.*, there is no queue)



- The number of states of the birth-dead process is $m + 1$.
- Since state n represents n servers attending one client each, the dead rate of state n is $n\mu$.

M/M/m/m queuing system



- Balance equations:

$$\lambda P_{n\text{-}1} = n\mu P_n, \qquad n = 1, 2, ..., m$$

- Stationary probability of *n* clients in the system:

$$P_n = \frac{(\lambda/\mu)^n / n!}{\sum_{i=0}^{m} (\lambda/\mu)^i / i!} \qquad n = 0, 1, ..., m$$

- By PASTA probability, the probability of an arrival client seeing the system full (*i.e.*, all servers unavailable) is given by the probability of state *m* (ErlangB formula):

$$P_m = \frac{(\lambda/\mu)^m / m!}{\sum_{i=0}^{m} (\lambda/\mu)^i / i!}$$

M/G/1 queuing system

- Birth-dead process such that:

  (1) the client arrivals is a Poisson process with rate λ

  (2) the serving time S of each client has a generic distribution and is independent of the client arriving time instant

  (3) the system has 1 server

  (4) the system capacity is infinite

- If the average, $E[S]$, and the 2nd moment, $E[S^2]$, of the serving time S is known, the average queuing delay of each client, W_Q, is given by the Pollaczek - Khintchine formula:

$$W_Q = \frac{\lambda E[S^2]}{2(1-\lambda E[S])}$$

- Therefore, the average client delay in the system, W, is given by the sum of the average queuing delay plus the average serving time:

$$W = \frac{\lambda E[S^2]}{2(1-\lambda E[S])} + E[S]$$

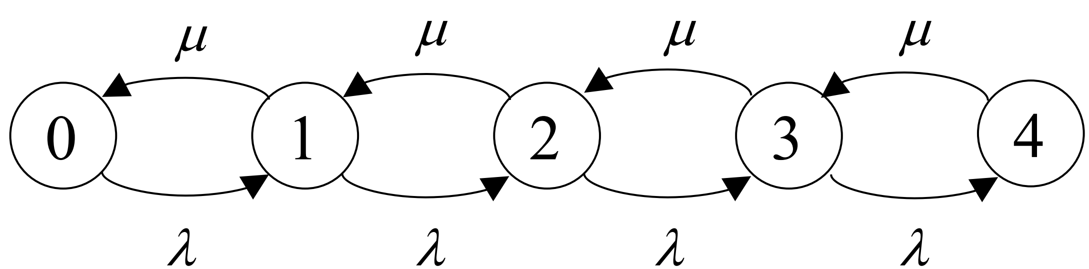
Example 7

Consider a data packet transmission system with a queue of capacity 3 packets and a link of 64 Kbps. The link is supporting a data flow of packets whose arrivals are a Poisson process with rate 15 pps (packets/sec). The packets length is exponentially distributed with average 400 bytes. Determine:

(a) the percentage of lost packets,

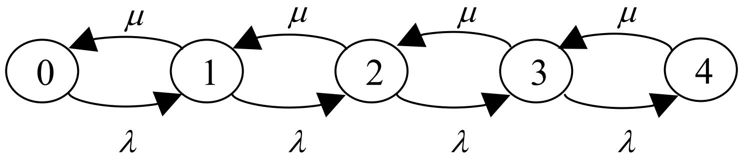
(b) the percentage of packet not suffering queuing delay,

(c) the percentual utilization of the link.



M/*M*/1/4 model

Example 7 – solution of (a)



Consider a data packet transmission system with a queue of capacity 3 packets and a link of 64 Kbps. The link is supporting a data flow of packets whose arrivals are a Poisson process with rate 15 pps (packets/sec). The packets length is exponentially distributed with average 400 bytes. Determine:

(a) the percentage of lost packets,

$$P_n = \frac{(\lambda/\mu)^n}{\sum_{i=0}^{m} (\lambda/\mu)^i} \qquad n = 0, 1, \ldots, m$$

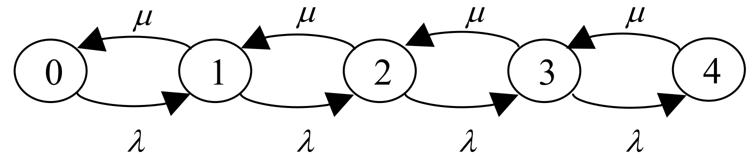
$$\mu = \frac{64000 \text{ bps}}{400 \times 8 \text{ bpp}} = 20 \text{ pps} \qquad \lambda = 15 \text{ pps}$$

$$P_4 = \frac{\left(\frac{\lambda}{\mu}\right)^4}{\left(\frac{\lambda}{\mu}\right)^0 + \left(\frac{\lambda}{\mu}\right)^1 + \left(\frac{\lambda}{\mu}\right)^2 + \left(\frac{\lambda}{\mu}\right)^3 + \left(\frac{\lambda}{\mu}\right)^4}$$

By PASTA property

$$P_4 = \frac{\left(\frac{15}{20}\right)^4}{1 + \frac{15}{20} + \left(\frac{15}{20}\right)^2 + \left(\frac{15}{20}\right)^3 + \left(\frac{15}{20}\right)^4} = 0.104 = 10.4\%$$

Example 7 – solution of (b)



Consider a data packet transmission system with a queue of capacity 3 packets and a link of 64 Kbps. The link is supporting a data flow of packets whose arrivals are a Poisson process with rate 15 pps (packets/sec). The packets length is exponentially distributed with average 400 bytes. Determine:

(b) the percentage of packet not suffering queuing delay,

$$P_n = \frac{(\lambda/\mu)^n}{\sum_{i=0}^{m} (\lambda/\mu)^i} \qquad n = 0, 1, \ldots, m$$

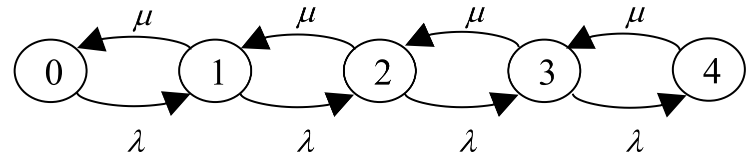
$$\mu = \frac{64000 \text{ bps}}{400 \times 8 \text{ bpp}} = 20 \text{ pps} \qquad \lambda = 15 \text{ pps}$$

$$P_0 = \frac{\left(\frac{\lambda}{\mu}\right)^0}{\left(\frac{\lambda}{\mu}\right)^0 + \left(\frac{\lambda}{\mu}\right)^1 + \left(\frac{\lambda}{\mu}\right)^2 + \left(\frac{\lambda}{\mu}\right)^3 + \left(\frac{\lambda}{\mu}\right)^4}$$

By PASTA property

$$P_0 = \frac{1}{1 + \frac{15}{20} + \left(\frac{15}{20}\right)^2 + \left(\frac{15}{20}\right)^3 + \left(\frac{15}{20}\right)^4} = 0.328 = 32.8\%$$

Example 7 – solution of (c)



Consider a data packet transmission system with a queue of capacity 3 packets and a link of 64 Kbps. The link is supporting a data flow of packets whose arrivals are a Poisson process with rate 15 pps (packets/sec). The packets length is exponentially distributed with average 400 bytes. Determine:

(c) the percentual utilization of the link.

$$U = 0 \times P_0 + 1 \times P_1 + 1 \times P_2 + 1 \times P_3 + 1 \times P_4$$

$$U = P_1 + P_2 + P_3 + P_4 = 1 - P_0$$

$$U = 1 - 0.328 = 0.672 = 67.2\%$$

Example 8

Consider a data packet transmission system with a very large queue and a link of 128 kbps. The system supports 2 data flows sharing the queue: in flow 1, packets are of constant size 128 Bytes and the packet arrivals is a Poisson process with rate 30 packets/sec; in flow 2, packets are of constant size 512 Bytes and the packet arrivals is a Poisson process with rate 10 packets/sec.

(a) Indicate and justify the type of queuing system that models this transmission system.

(b) Determine the average system delay of each data flow.

Example 8 – solution of (a)

Consider a data packet transmission system with a very large queue and a link of 128 kbps. The system supports 2 data flows sharing the queue: in flow 1, packets are of constant size 128 Bytes and the packet arrivals is a Poisson process with rate 30 packets/sec; in flow 2, packets are of constant size 512 Bytes and the packet arrivals is a Poisson process with rate 10 packets/sec.

(a) Indicate and justify the type of queuing system that models this transmission system.

This is a *M*/*G*/1 queuing system:

- The sum of 2 Poisson processes is a Poisson process ('*M*' in *M*/*G*/1) with rate 30 + 10 = 40 pps.
- The packet size distribution of the sum of the two flows is neither exponential nor constant and therefore is generic ('*G*' in *M*/*G*/1): packet size is 128 Bytes with probability 30/(30+10) = 0.75 or 512 Bytes with probability 10/(30+10) = 0.25.
- The number of servers is one ('1' em *M*/*G*/1) since the link is fully used to transmit one packet at a time.
- The queue is very large and, therefore, the system capacity is considered infinite.

Example 8 – solution of (b)

Consider a data packet transmission system with a very large queue and a link of 128 kbps. The system supports 2 data flows sharing the queue: in flow 1, packets are of constant size 128 Bytes and the packet arrivals is a Poisson process with rate 30 packets/sec; in flow 2, packets are of constant size 512 Bytes and the packet arrivals is a Poisson process with rate 10 packets/sec.

(b) Determine the average system delay of each data flow.

$$\boxed{W_Q = \frac{\lambda E[S^2]}{2(1-\lambda E[S])}}$$

$S_{128} = (128 \times 8)/128000 = 8 \times 10^{-3}$ sec $\qquad S_{512} = (512 \times 8)/128000 = 32 \times 10^{-3}$ sec

$E[S] = 0.75 \times S_{128} + 0.25 \times S_{512} =$

$= 0.75 \times 8 \times 10^{-3} + 0.25 \times 32 \times 10^{-3} = 14 \times 10^{-3}$ sec

$E[S^2] = 0.75 \times (S_{128})^2 + 0.25 \times (S_{512})^2 =$

$= 0.75 \times (8 \times 10^{-3})^2 + 0.25 \times (32 \times 10^{-3})^2 = 3.04 \times 10^{-4}$ sec^2

$$W_Q = \frac{\lambda E[S^2]}{2(1-\lambda E[S])} = \frac{40 \times 3.04 \times 10^{-4}}{2(1 - 40 \times 14 \times 10^{-3})} = 0.0143 = 14.3 \text{ msec}$$

$W_{128} = W_Q + S_{128} = 14.3 + 8 = 22.3$ msec

$W_{512} = W_Q + S_{512} = 14.3 + 32 = 46.3$ msec