ST104a Live Session Outline Solutions to Exercise 2

1. (a) Let B_1 and B_2 denote boxes 1 and 2, respectively, and let G denote a green ball and R a red ball. We have:

$$\begin{aligned} P(G) &= P(G \mid B_1)\, P(B_1) + P(G \mid B_2)\, P(B_2) \\ &= \frac{3}{4} \times \frac{1}{2} + \frac{2}{4} \times \frac{1}{2} = \frac{5}{8} = 0.625. \end{aligned}$$

   (b) This part can be found by using Bayes' theorem:

$$P(B_1 \mid G) = \frac{P(G \mid B_1)\, P(B_1)}{P(G)} = \frac{3/4 \times 1/2}{5/8} = \frac{3}{5} = 0.6.$$

2. (a) There are 24 females and 16 males in the class. Hence the answer is $24/(16+24) = 24/40 = 0.6$.

   (b) Using the total probability formula:

$$\frac{16}{40} \times \frac{24}{39} + \frac{24}{40} \times \frac{16}{39} = 0.4923.$$

   (c) This part can be answered in a similar way to (a) noting that there are now 16 males and 23 females in the class. Hence the probability is $16/39 = 0.4103$.

   (d) We have:

$$\begin{aligned} P(\text{female} \mid \text{pass}) &= \frac{P(\text{pass} \mid \text{female})\, P(\text{female})}{P(\text{pass})} \\ &= \frac{0.85 \times 24/40}{P(\text{pass} \cap \text{female}) + P(\text{pass} \cap \text{male})} \\ &= \frac{0.85 \times 24/40}{0.85 \times 24/40 + 0.80 \times 16/40} \\ &= 0.6145. \end{aligned}$$

3. (a) M can take the values 1, 2, 3, 4, 5 and 6. There are 36 possible outcomes. There is one way in which $M = 1$. Hence $P(M = 1) = 1/36$. The other values can be obtained in a similar manner. The distribution is given in the table below.

$M = m$	1	2	3	4	5	6
$P(M = m)$	1/36	3/36	5/36	7/36	9/36	11/36

   (b) There are six ways of having at most 4: $(3,1)$, $(1,3)$, $(2,1)$, $(1,2)$, $(2,2)$ and $(1,1)$. In three of them there is at least one 2, so the probability is $3/6 = 1/2$.

4. (a) We have:

$$P(X > 2) = P(X = 3) + P(X = 5) = 0.2 + 0.1 = 0.3.$$

(b) We have:

$$\mathrm{E}(X) = \sum_i x_i \, P(X = x_i) = 0 \times 0.5 + 1 \times 0.2 + 3 \times 0.2 + 5 \times 0.1 = 1.3.$$

(c) We have:

$$\mathrm{E}(X^2) = \sum_i x_i^2 \, P(X = x_i) = 0^2 \times 0.5 + 1^2 \times 0.2 + 3^2 \times 0.2 + 5^2 \times 0.1 = 4.5.$$

Hence:

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2 = 4.5 - (1.3)^2 = 2.81.$$

5. (a) Since $X \sim N(60,\, 25)$, we have:

$$\begin{aligned} P(X > 58) = P\left(\frac{X - 60}{\sqrt{25}} > \frac{58 - 60}{\sqrt{25}}\right) &= P(Z > -0.40) \\ &= \Phi(0.40) \\ &= 0.6554. \end{aligned}$$

(b) Since $\bar{X} \sim N(60,\, 5)$:

$$\begin{aligned} P(59 < \bar{X} < 61) &= P\left(\frac{59 - 60}{\sqrt{5}} < Z < \frac{61 - 60}{\sqrt{5}}\right) \\ &= P(-0.45 < Z < 0.45) \\ &= \Phi(0.45) - (1 - \Phi(0.45)) \\ &= 0.6736 - (1 - 0.6736) \\ &= 0.3472. \end{aligned}$$

6. We have:

$$P(X < 800) = P\left(\frac{X - 1{,}200}{400} < \frac{800 - 1{,}200}{400}\right) = P(Z < -1) = 0.1587$$

and:

$$P(Y < 800) = P\left(\frac{Y - 960}{200} < \frac{800 - 960}{200}\right) = P(Z < -0.80) = 0.2119.$$

So country B has a higher proportion of households spending less than \$800 per week.

7. (a) We have:

$$P(X + 4 < 4) = P\left(\frac{X + 4 - 4}{1} < \frac{4 - 4}{1}\right) = P(X < 0).$$

Since $X \sim N(0,\, 1)$, due to symmetry $P(X < 0) = 0.5$.

(b) We can write:

$$P(X - b < 0) = 0.975 \quad \Rightarrow \quad P(X - b + b < b) = 0.975 \quad \Rightarrow \quad P(X < b) = 0.975.$$

From tables we obtain $\Phi(1.96) = 0.975$, hence $b = 1.96$.