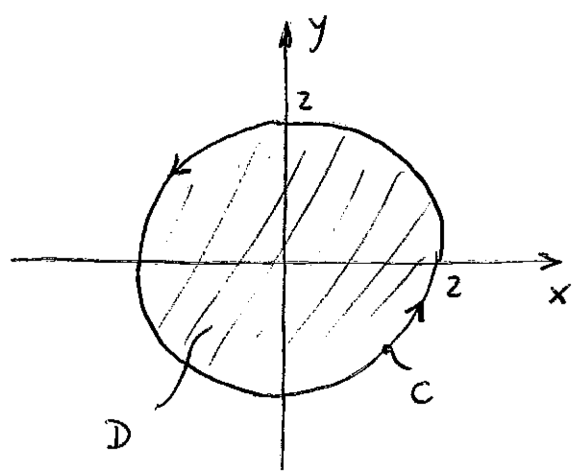
1)

$\vec{F}(x,y) = (P,Q) = \left(2y + \sqrt{1+x^5},\ 5x - e^{y^2}\right)$

A curva C é plana e fechada pelo que podemos recorrer ao Teorema de Green.



$$\frac{\partial P}{\partial y} = 2 \qquad \frac{\partial Q}{\partial x} = 5$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 3 \neq 0 \Rightarrow \vec{F} \text{ não é gradiente.}$$

$$\oint_C P\,dx + Q\,dy = \iint_D 3\,dx\,dy = 3\iint_D dx\,dy = 3\,A(D) =$$

$$= 3\pi\,(2^2) = 12\pi$$

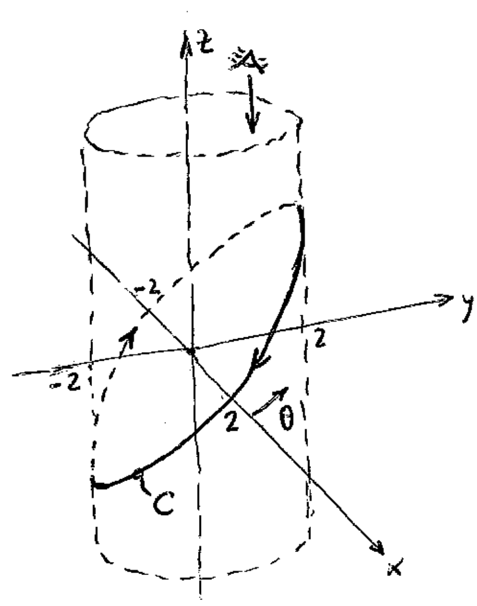
(Admite-se que a curva C é percorrida no sentido directo)

2) $\vec{F}(x,y,z) = (P,Q,R) = (2y, -2x, 1)$

$$\frac{\partial P}{\partial y} = 2 \neq \frac{\partial Q}{\partial x} = -2 \Rightarrow \vec{F} \text{ não é gradiente}$$

Curva C : $\begin{cases} x^2+y^2=4 \\ z = 2y \end{cases}$ $\rightarrow$ curva resultante da intersecção de uma superfície cilíndrica com um plano

Parametrizando a curva C em coordenadas polares:



$\vec{r}(\theta) = (2\cos\theta,\ 2\,\text{sen}\,\theta,\ 4\,\text{sen}\,\theta)$, $\theta \in [0, 2\pi]$

NOTA: O sentido de percurso da curva é contrário ao da marcação do ângulo θ.

[illegible]

$$\vec{r}'(\theta) = (-2\,\text{sen}\,\theta,\ 2\cos\theta,\ 4\cos\theta)$$

$$\vec{F}[\vec{r}(\theta)] = (4\,\text{sen}\,\theta,\ -4\cos\theta,\ 1)$$

$$\vec{F}[\vec{r}(\theta)]\cdot\vec{r}'(\theta) = -8\,\text{sen}^2\theta - 8\cos^2\theta + 4\cos\theta =$$

$$= 4\cos\theta - 8$$

$$\oint_C \vec{F}[\vec{r}(\theta)]\cdot\vec{r}'(\theta)\,d\theta = 4\underbrace{\int_{2\pi}^{0}\cos\theta\,d\theta}_{=0} - 8\int_{2\pi}^{0} d\theta =$$

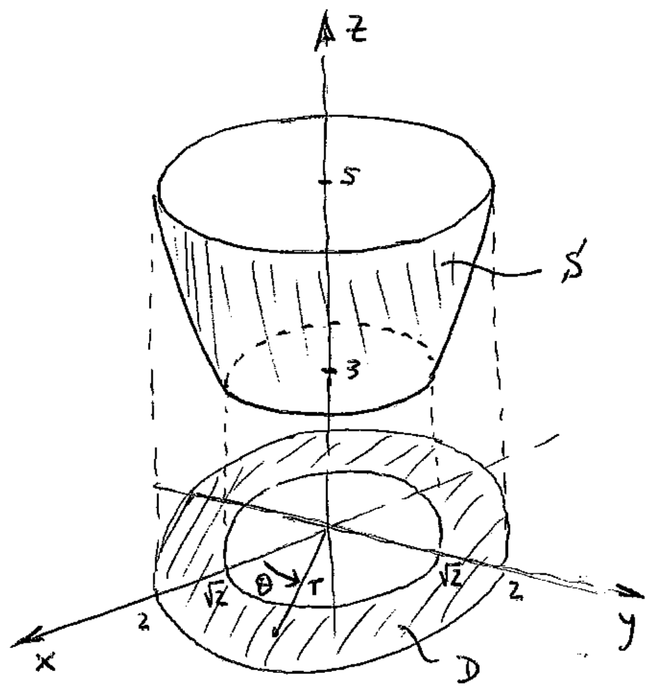
$$= 16\pi$$

3)

$$\begin{cases} z = x^2+y^2+1 \\ z = 5 \end{cases} \Leftrightarrow \begin{cases} x^2+y^2 = 4 \quad (\text{raio} = 2) \\ z = 5 \end{cases}$$

$$\begin{cases} z = x^2+y^2+1 \\ z = 3 \end{cases} \Leftrightarrow \begin{cases} x^2+y^2 = 2 \quad (\text{raio} = \sqrt{2}) \\ z = 3 \end{cases}$$

S : secção de um parabolóide



OPÇÃO I : Coordenadas cartesianas

$$\vec{r}(x,y) = (x, y, x^2+y^2+1) \,, \quad (x,y) \in D$$

$$D = \{ (x,y) \in \mathbb{R}^2 : \ 2 \leq x^2+y^2 \leq 4 \}$$

$$\frac{\partial \vec{r}}{\partial x} = (1, 0, 2x) \qquad \frac{\partial \vec{r}}{\partial y} = (0, 1, 2y)$$

$$\vec{N}(x,y) = \frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} = (-2x, -2y, 1)$$

$$\| \vec{N}(x,y) \| = \sqrt{1+4(x^2+y^2)} \quad \Rightarrow \quad A(S) = \iint_D \sqrt{1+4(x^2+y^2)}\, dx\, dy$$

Reescrevendo em coordenadas polares : $x = r\cos\theta$, $y = r \operatorname{sen}\theta$, $dx\,dy = r\,dr\,d\theta$
obtém-se

$$A(S) = \int_0^{2\pi} \int_{\sqrt{2}}^{2} r\sqrt{1+4r^2}\, dr\, d\theta = 2\pi \left(\frac{1}{8}\right)\left(\frac{2}{3}\right) \left[(1+4r^2)^{3/2} \right]_{\sqrt{2}}^{2} =$$

$$= \frac{\pi}{6} \left[17\sqrt{17} - 9\sqrt{9} \right] = \frac{\pi}{6} \left[17\sqrt{17} - 27 \right]$$

OPÇÃO II : Coordenadas polares

$$\vec{r}(r,\theta) = (r\cos\theta, r\operatorname{sen}\theta, r^2+1) \,, \quad r \in [\sqrt{2}, 2] \wedge \theta \in [0, 2\pi]$$

$$\frac{\partial \vec{r}}{\partial r} = (\cos\theta, \operatorname{sen}\theta, 2r) \qquad \frac{\partial \vec{r}}{\partial \theta} = (-r\operatorname{sen}\theta, r\cos\theta, 0)$$

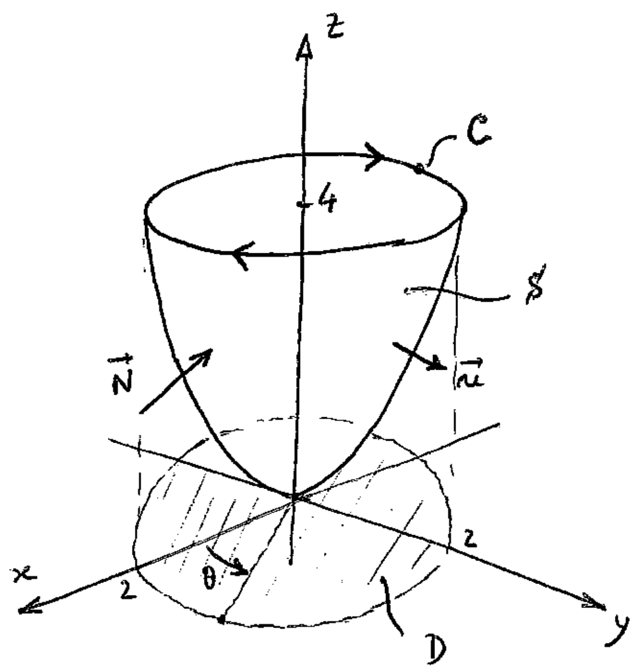
$$\vec{N}(r,\theta) = \frac{\partial \vec{r}}{\partial r} \times \frac{\partial \vec{r}}{\partial \theta} = (-2r^2\cos\theta, -2r^2\operatorname{sen}\theta, r)$$

$$\| \vec{N}(r,\theta) \| = \sqrt{4r^4 + r^2} = r\sqrt{1+4r^2}$$

$$A(S) = \int_0^{2\pi} \int_{\sqrt{2}}^{2} r\sqrt{1+4r^2}\, dr\, d\theta = \dots = \frac{\pi}{6} \left[17\sqrt{17} - 27 \right]$$

4) S : secção de um parabolóide

$$\begin{cases} z = x^2+y^2 \\ z = 4 \end{cases} \Leftrightarrow \begin{cases} x^2+y^2 = 4 \quad (\text{raio} = 2) \\ z = 4 \end{cases}$$



OPÇÃO I : Cálculo do fluxo através de definição

Parametrizando em coordenadas cartesianas

$$\vec{r}(x,y) = (x, y, x^2+y^2) \quad , (x,y) \in D$$

$$D = \{ (x,y) \in \mathbb{R}^2 : 0 \leq x^2+y^2 \leq 4 \}$$

$$\frac{\partial \vec{r}}{\partial x} = (1, 0, 2x) \qquad \frac{\partial \vec{r}}{\partial y} = (0, 1, 2y)$$

$$\vec{N}(x,y) = \frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} = (-2x, -2y, 1)$$: vector dirigido de fora para dentro de S !

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & z & 1 \end{vmatrix} = (-1, 0, -1) \quad (\text{constante})$$

$$(\nabla \times \vec{F})(\vec{r}(x,y)) = (-1, 0, -1) \qquad (\nabla \times \vec{F})(\vec{r}(x,y)) \cdot \vec{N}(x,y) = 2x - 1$$

Fluxo de dentro para fora de S :

$$-\iint_D (\nabla \times F)(\vec{r}(x,y)) \cdot \vec{N}(x,y)\, dx\, dy = -2 \iint_D x \, dx\, dy + \iint_D dx\, dy =$$

$$= -2 \underset{\underset{0}{\shortparallel}}{\bar{x}_D} A(D) + A(D) = A(D) = \pi (2^2) = 4\pi$$

OPÇÃO II : Teorema de Stokes

Parametrizando a linha C (bordo de S) em coordenadas polares :

$$\vec{r}(\theta) = (2\cos\theta, 2\,\text{sen}\,\theta, 4) \quad , \theta \in \overleftarrow{[0, 2\pi]} \quad (\text{sentido retrógrado})$$

$$\vec{r}'(\theta) = (-2\,\text{sen}\,\theta, 2\cos\theta, 0)$$

$$\vec{F}[\vec{r}(\theta)] = (2\,\text{sen}\,\theta, 4, 1) \qquad \vec{F}[\vec{r}(\theta)] \cdot \vec{r}'(\theta) = -4\,\text{sen}^2\theta + 8\cos\theta =$$

$$= -2 + 2\cos(2\theta) + 8\cos\theta$$

Fluxo de dentro para fora de S :

$$\int_{2\pi}^{0} (+2 \underbrace{\cancel{\cos(2\theta)}}_{\to 0} + 8 \underbrace{\cancel{\cos\theta}}_{\to 0} - 2)\, d\theta = -2 \int_{2\pi}^{0} d\theta = 4\pi$$

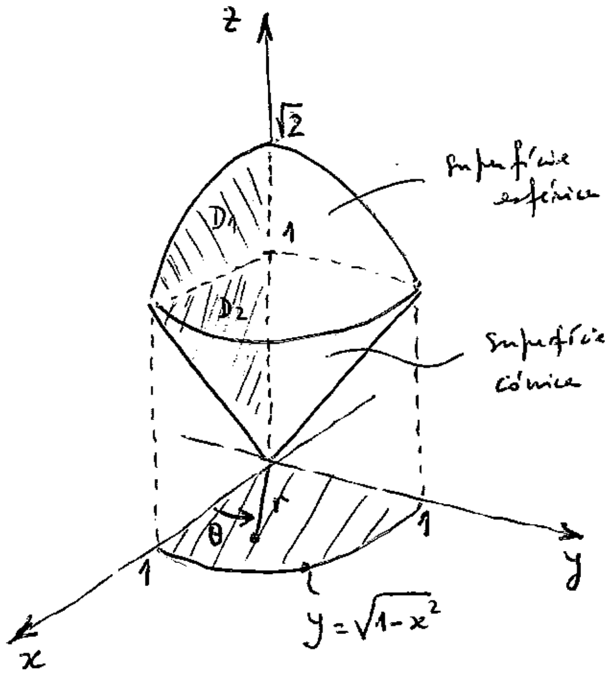
Nuno

5 a)

$x \in [0,1]$

$0 \leq y \leq \sqrt{1-x^2}$

$\sqrt{x^2+y^2} \leq z \leq \sqrt{2-x^2-y^2}$



$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} z \, dz \, dy \, dx =$$

$$= \frac{1}{2} \int_0^1 \int_0^{\sqrt{1-x^2}} \left[z^2 \right]_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} dy \, dx =$$

$$= \frac{1}{2} \int_0^1 \int_0^{\sqrt{1-x^2}} (2-x^2-y^2-x^2-y^2) \, dy \, dx = \int_0^1 \int_0^{\sqrt{1-x^2}} (1-x^2-y^2) \, dy \, dx = (*)$$

Passando para coordenadas polares :
$x = r\cos\theta$
$y = r \operatorname{sen}\theta$
$dx\,dy = r\,dr\,d\theta$
$1-x^2-y^2 = 1-r^2$
$r \in [0,1]$; $\theta \in [0, \pi/2]$

$$(*) = \int_0^{\pi/2} \int_0^1 (1-r^2)\, r \, dr \, d\theta = \frac{\pi}{2} \int_0^1 (r - r^3) \, dr =$$

$$= \frac{\pi}{2} \left[\frac{r^2}{2} - \frac{r^4}{4} \right]_0^1 = \frac{\pi}{2} \left(\frac{1}{2} - \frac{1}{4} \right) = \frac{\pi}{8}$$

b) Projectando o volume no plano xOz obtém-se o domínio de integração que integra as regiões D_1 e D_2 (ver figura na alínea a)).

Assim, ter-se-ia

$$\iint_{D_1} \int_0^{\sqrt{2-x^2-z^2}} z \, dy \, dz \, dx + \iint_{D_2} \int_0^{\sqrt{z^2-x^2}} z \, dy \, dz \, dx =$$

$$= \int_0^1 \int_1^{\sqrt{2-x^2}} \int_0^{\sqrt{2-x^2-y^2}} z \, dy \, dz \, dx + \int_0^1 \int_x^1 \int_0^{\sqrt{z^2-x^2}} z \, dy \, dz \, dx$$

NOTAS: <u>Superfície cónica</u>:

$$z = \sqrt{x^2+y^2} \quad , \quad z \geq 0 \wedge x \geq 0 \wedge y \geq 0$$

$$y = \sqrt{z^2-x^2}$$

<u>Superfície esférica</u>

$$z = \sqrt{2-x^2-y^2} \quad , \quad z \geq 0 \wedge x \geq 0 \wedge y \geq 0$$

$$y = \sqrt{2-x^2-z^2}$$