1)

$$C : \begin{cases} x^2 + y^2 = 1 & \text{(superfície cilíndrica)} \\ x + y + z = 1 & \text{(plano)} \end{cases}$$

C é uma curva fechada; verifiquemos se a função de campo vectorial é gradiente.

$$\vec{F}(x,y,z) = (P, Q, R) = (-y, z, -1)$$

$$\frac{\partial Q}{\partial x} = 0 \; ; \; \frac{\partial P}{\partial y} = -1 \implies \vec{F} \text{ não é gradiente.}$$

<u>Formulação vectorial</u> : Consideremos a parametrização para a curva C

$$\vec{r}(\theta) = (\cos\theta, \operatorname{sen}\theta, 1 - \cos\theta - \operatorname{sen}\theta) \quad , \; \theta \in [0, 2\pi]$$

Como não é imposto o sentido de percurso de C, optemos por: $\theta \in \overrightarrow{[0, 2\pi]}$

$$\vec{F}[\vec{r}(\theta)] = (-\operatorname{sen}\theta, 1 - \cos\theta - \operatorname{sen}\theta, -1)$$

$$\vec{r}\,'(\theta) = (-\operatorname{sen}\theta, \cos\theta, +\operatorname{sen}\theta - \cos\theta)$$

$$\vec{F} \cdot \vec{r}\,'(\theta) = \operatorname{sen}^2\theta + \cos\theta - \cos^2\theta - \operatorname{sen}\theta\cos\theta - \operatorname{sen}\theta + \cos\theta =$$

$$= \cancel{\tfrac{1}{2}} - \tfrac{1}{2}\cos(2\theta) + 2\cos\theta - \cancel{\tfrac{1}{2}} - \tfrac{1}{2}\cos(2\theta) - \operatorname{sen}\theta\cos\theta - \operatorname{sen}\theta =$$

$$= -\cos(2\theta) + 2\cos\theta - \operatorname{sen}\theta\cos\theta - \operatorname{sen}\theta$$

Concluindo:

$$\int_C -y\,dx + z\,dy - dz = \int_0^{2\pi} \left(-\cos(2\theta) + 2\cos\theta - \operatorname{sen}\theta\cos\theta - \operatorname{sen}\theta\right) d\theta = 0$$

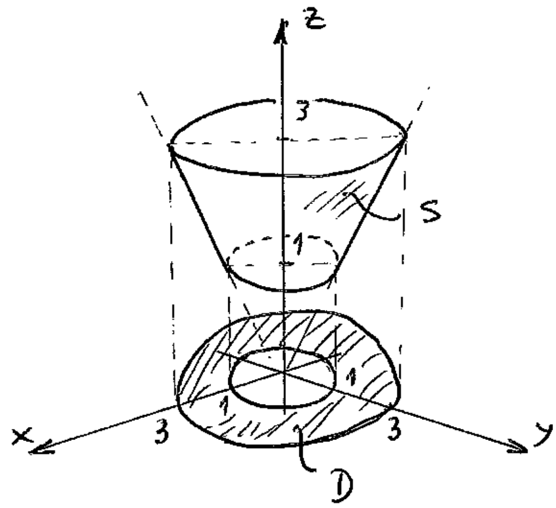
já que $\displaystyle\int_0^{2\pi} \cos(2\theta)\,d\theta = \int_0^{2\pi} \cos\theta\,d\theta = \int_0^{2\pi} \operatorname{sen}\theta\cos\theta\,d\theta = \int_0^{2\pi} \operatorname{sen}\theta\,d\theta = 0$

Nuis

2) $z = \sqrt{x^2+y^2}$, $z \in [1,3]$

Parametrização da superfície

S (coordenadas cartesianas)



$$\vec{r}(x,y) = \left(x, y, \sqrt{x^2+y^2}\right), \ (x,y) \in D$$

$$D = \{(x,y) \in \mathbb{R}^2 : 1 \le x^2+y^2 \le 3^2\}$$

$$\frac{\partial \vec{r}}{\partial x} = \left(1, 0, \frac{x}{\sqrt{x^2+y^2}}\right) \qquad \frac{\partial \vec{r}}{\partial y} = \left(0, 1, \frac{y}{\sqrt{x^2+y^2}}\right)$$

Vector Fundamental de S :

$$\vec{N}(x,y) = \frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & \frac{x}{\sqrt{x^2+y^2}} \\ 0 & 1 & \frac{y}{\sqrt{x^2+y^2}} \end{vmatrix} = \left(\frac{-x}{\sqrt{x^2+y^2}}, \frac{-y}{\sqrt{x^2+y^2}}, 1\right)$$

$$\|\vec{N}(x,y)\| = \left[\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} + 1\right]^{1/2} = \sqrt{2}$$

Área da superfície S :

$$A(S) = \iint_D \|\vec{N}(x,y)\| \, dx\,dy = \sqrt{2} \iint_D dx\,dy =$$

$$= \sqrt{2}\, A(D) = \sqrt{2}\, \pi \left[3^2 - 1^2\right] = 8\sqrt{2}\, \pi$$

3)

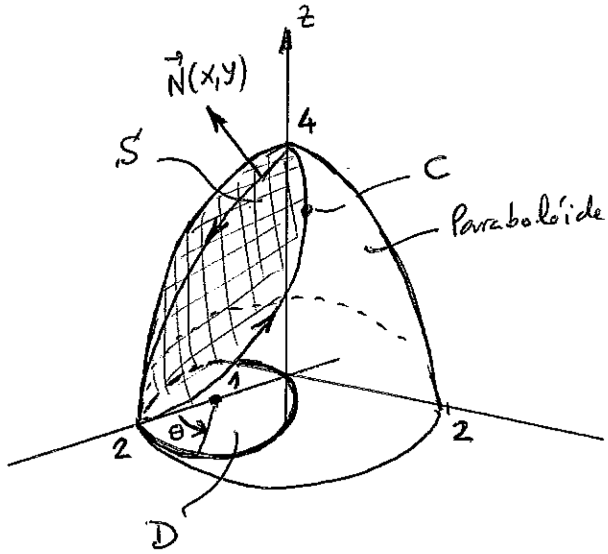
a) $C : \begin{cases} z = 4 - x^2 - y^2 & \text{(parabolóide)} \\ z = 4 - 2x & \text{(plano)} \end{cases}$

$z = 4 - x^2 - y^2 = 4 - (x^2 + y^2)$

Se $z = 0 \Rightarrow x^2 + y^2 = 4$

Considerando o ângulo θ referido na figura, tem-se

$$x = 1 + \cos\theta$$
$$y = \operatorname{sen}\theta$$
$$z = 4 - 2(1 + \cos\theta)$$



Então $\vec{r}(\theta) = (1 + \cos\theta, \operatorname{sen}\theta, 2 - 2\cos\theta)$, $\theta \in [\overrightarrow{0, 2\pi}]$

b)

Alternativa 1 :

Cálculo do fluxo :

$$\iint_S (\nabla \times \vec{F}) \cdot \vec{n} \, dS$$

$\vec{F}(x,y,z) = (-y, x, 0)$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & x & 0 \end{vmatrix} = (0, 0, 2)$$

Superfície S :

$\vec{r}_1(x,y) = (x, y, 4 - x^2 - y^2)$, $(x,y) \in D$

$D = \{ (x,y) \in \mathbb{R}^2 \quad 0 \leq (x-1)^2 + y^2 \leq 1 \}$

$\dfrac{\partial \vec{r}_1}{\partial x} = (1, 0, -2x)$; $\dfrac{\partial \vec{r}_1}{\partial y} = (0, 1, -2y)$

$$\vec{N}(x,y) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{vmatrix} = (2x, 2y, 1)$$

$(\nabla \times \vec{F})(\vec{r}_1(x,y)) = (0, 0, 2)$

Alternativa 2 :

Teorema de Stokes :

$$\iint_S (\nabla \times \vec{F}) \cdot \vec{n} \, dS = \oint_C \vec{F} \cdot d\vec{r}$$

$\vec{F}[\vec{r}(\theta)] = (-\operatorname{sen}\theta, 1 + \cos\theta, 0)$

$\vec{r}\,'(\theta) = (-\operatorname{sen}\theta, \cos\theta, 2\operatorname{sen}\theta)$

$\vec{F} \cdot \vec{r}\,'(\theta) = \operatorname{sen}^2\theta + \cos^2\theta + \cos\theta =$
$= 1 + \cos\theta$

$$\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} d\theta + \underbrace{\int_0^{2\pi} \cos\theta \, d\theta}_{=0} =$$

$= 2\pi$

Alternativa 1 (cont.):

$$(\nabla \times \vec{F})(\vec{r}_1(x,y)) \cdot \vec{N}(x,y) = 2$$

Concluindo

$$\iint_S (\nabla \times \vec{F}) \cdot \vec{n} \, dS = 2 \iint_D dx\,dy =$$

$$= 2\,A(D) = 2\pi$$

Visto

4) $y'' - 2y' + y = 0$: equação homogénea

$r^2 - 2r + 1 = 0 \Leftrightarrow (r-1)^2 = 0 \Leftrightarrow r = 1$ (raiz real dupla)

Solução da equação homogénea

$$y_h = c_1 e^x + c_2 x e^x$$

Método de variação das constantes

$$y = y_h + y_p = c_1(x)\, e^x + c_2(x)\, x e^x$$

$f_1(x) = e^x \qquad f_2(x) = x e^x \qquad g(x) = \dfrac{4e^x}{1+x}$

$f_1'(x) = e^x \qquad f_2'(x) = e^x + x e^x$

$$\begin{cases} f_1 c_1' + f_2 c_2' = 0 \\ f_1' c_1' + f_2' c_2' = g \end{cases} \Leftrightarrow \begin{bmatrix} e^x & x e^x \\ e^x & e^x + x e^x \end{bmatrix} \begin{bmatrix} c_1' \\ c_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \dfrac{4e^x}{1+x} \end{bmatrix}$$

$$\Delta = \begin{vmatrix} e^x & x e^x \\ e^x & e^x + x e^x \end{vmatrix} = e^{2x}$$

$$c_1' = \frac{1}{\Delta} \begin{vmatrix} 0 & x e^x \\ \dfrac{4e^x}{1+x} & e^x + x e^x \end{vmatrix} = \frac{1}{e^{2x}} \left[- \frac{4x e^{2x}}{1+x} \right] = \frac{-4x}{1+x}$$

$$c_1 = -4 \int \frac{x}{1+x}\, dx = -4 \int \frac{1+x-1}{1+x}\, dx = -4 \int dx + 4 \int \frac{1}{1+x}\, dx =$$

$$= -4x + 4 \ln(1+x) + k_1$$

$$c_2' = \frac{1}{\Delta} \begin{vmatrix} e^x & 0 \\ e^x & \dfrac{4e^x}{1+x} \end{vmatrix} = \frac{1}{e^{2x}} \left[\frac{4 e^{2x}}{1+x} \right] = \frac{4}{1+x}$$

$$c_2 = 4 \int \frac{1}{1+x}\, dx = 4 \ln(1+x) + k_2$$

Solução da equação não homogénea

$$y = c_1 e^x + c_2 x e^x + [-4x + 4 \ln(1+x)]\, e^x + 4 x e^x \ln(1+x) =$$

$$= c_1 e^x + c_2 x e^x + 4 e^x \ln(1+x)\, [1+x]$$

Wir

5 a)

$$f(t) = \mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left[\frac{2}{s^2+9}\right] + \mathcal{L}^{-1}\left[\frac{e^{-\pi s}}{s^2+2s+5}\right]$$

$$\mathcal{L}^{-1}\left[\frac{2}{s^2+9}\right] = \frac{2}{3}\mathcal{L}^{-1}\left[\frac{3}{s^2+9}\right] = \frac{2}{3}\operatorname{sen}(3t)$$

$$\mathcal{L}^{-1}\left[\frac{e^{-\pi s}}{s^2+2s+5}\right] = \mathcal{L}^{-1}\left[\frac{e^{-\pi s}}{(s+1)^2+4}\right] = \underset{t \to t-\pi}{\mathcal{L}^{-1}\left[\frac{1}{(s+1)^2+4}\right]} u(t-\pi) =$$

$$= \frac{1}{2}\underset{t \to t-\pi}{\mathcal{L}^{-1}\left[\frac{2}{(s+1)^2+4}\right]} u(t-\pi) = \frac{1}{2}\underset{t \to t-\pi}{\left[e^{-t}\mathcal{L}^{-1}\left[\frac{2}{s^2+4}\right]\right]} u(t-\pi) =$$

$$= \frac{1}{2}e^{-(t-\pi)}\operatorname{sen}(2(t-\pi))\, u(t-\pi) = \frac{1}{2}e^{\pi-t}\operatorname{sen}(2t)\, u(t-\pi)$$

Concluindo

$$f(t) = \frac{2}{3}\operatorname{sen}(3t) + \frac{1}{2}e^{\pi-t}\operatorname{sen}(2t)\, u(t-\pi) =$$

$$= \begin{cases} \frac{2}{3}\operatorname{sen}(3t), & 0 < t < \pi \\ \frac{2}{3}\operatorname{sen}(3t) + \frac{1}{2}e^{\pi-t}\operatorname{sen}(2t), & t > \pi \end{cases}$$

Wuj

5 b) $y'' + 4y = 4 - 4u(t-2)$ com $y(0) = y'(0) = 0$

$$\mathcal{L}[y] = Y(s)$$

$$\mathcal{L}[y''] = s^2 Y(s) - s\,y(0) - y'(0) = s^2 Y(s)$$

$$\mathcal{L}[4 - 4u(t-2)] = \mathcal{L}[4] - 4\mathcal{L}[u(t-2)] = \frac{4}{s} - \frac{4}{s}e^{-2s}$$

Então

$$(s^2+4)Y(s) = \frac{4}{s} - \frac{4}{s}e^{-2s} \Leftrightarrow Y(s) = \frac{4}{s(s^2+4)} - \frac{4}{s(s^2+4)}e^{-2s}$$

$$\frac{4}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4} = \frac{1}{s} - \frac{s}{s^2+4}$$

$4 = A(s^2+4) + Bs^2 + Cs$; $s = 0 \Rightarrow A = 1$

Termo em s^2 : $A + B = 0 \Rightarrow B = -1$

Termo em s : $C = 0$

$$Y(s) = \frac{1}{s} - \frac{s}{s^2+4} - \frac{e^{-2s}}{s} + \frac{s}{s^2+4}e^{-2s}$$

$\mathcal{L}^{-1}\left[\frac{1}{s}\right] = 1$; $\mathcal{L}^{-1}\left[\frac{s}{s^2+4}\right] = \cos(2t)$

$$\mathcal{L}^{-1}\left[\frac{e^{-2s}}{s}\right] = u(t-2)$$

$$\mathcal{L}^{-1}\left[\frac{s}{s^2+4}e^{-2s}\right] = \underset{t \to t-2}{\mathcal{L}^{-1}\left[\frac{s}{s^2+4}\right]}\, u(t-2) = \cos(2t-4)\,u(t-2)$$

Concluindo:

$$y(t) = \mathcal{L}^{-1}[Y(s)] = 1 - \cos(2t) - u(t-2) + \cos(2t-4)\,u(t-2) =$$

$$= \begin{cases} 1 - \cos(2t), & 0 < t < 2 \\ -\cos(2t) + \cos(2t-4), & t > 2 \end{cases}$$

6) Teorema de Green:

Sejam P e Q campos escalares em $\mathbb{R}^2$ contínuos e com derivadas contínuas num conjunto aberto $S \subseteq \mathbb{R}^2$. Seja C a curva de Jordan seccionalmente suave e seja $D \subset S$ a região constituída por C e o seu interior. Então:

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy = \oint_C P\, dx + Q\, dy \qquad (1)$$

Tendo em atenção a definição de integral duplo, sabe-se que a área de D, $A(D)$, é dada por

$$A(D) = \iint_D dx\, dy$$

Assim, se em (1) se considerar

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$$

verifica-se que

$$A(D) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy$$

Encontremos, então, uma função $\vec{F}(x,y) = (P, Q)$ que satisfaça esta condição.

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 \implies \frac{\partial Q}{\partial x} = \frac{1}{2} \quad e \quad \frac{\partial P}{\partial y} = -\frac{1}{2}$$

isto é, por exemplo,

$$Q(x,y) = \frac{1}{2}x \quad e \quad P(x,y) = -\frac{1}{2}y$$

Substituindo em (1), obtém-se

$$A(D) = \iint_D dx\, dy = \oint_C \left(-\frac{1}{2}y\right) dx + \left(\frac{1}{2}x\right) dy = \frac{1}{2}\oint_C -y\, dx + x\, dy$$

Wiz