Theory of Computation

L.EIC, 2nd Year

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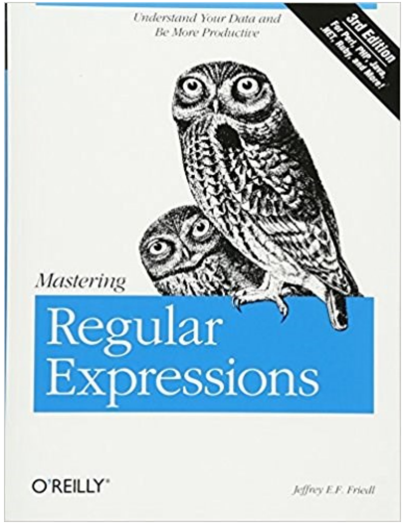
Outline

- Regular Expressions
- Conversion of regular expressions into ε-NFAs
- Conversions of FAs into regular expressions

Regular Expressions

- Useful for searching in text (e.g., grep and sed of Linux/Unix), in compiler generators (Lex, Flex, lexical analyzers)
- Useful in applications that need to search for patterns (e.g., intrusion detection systems, anti-virus)
- Built-in in some programming languages (e.g., Perl)
  - PCRE (Perl-compatible regular expressions) format
  - POSIX regex format
- Supported by using APIs, such as in Java (java.util.regex)
- And…



Regular Expressions

- Alternative to NFAs (including ε-NFAs) and DFAs
- Equivalent to NFAs (including ε-NFAs) and DFAs
- Algebraic characteristics allow the use of expressions to specify the strings of the language
- Regular expressions define languages
  - **Example: 01*+10***
  - L(**01*+10***): Language of the binary strings starting with a 0 followed by zero or more 1s, or starting with a 1 followed by zero or more 0s

Operators over Languages

- **Union** of two languages L and M (**L ∪ M**), is the set of the strings that belong to L, to M, or to both
  - L = {001, 10, 111} M = {ε, 001} L ∪ M = {ε, 001, 10, 111}
- **Concatenation** of two languages L and M (**LM** or **L.M**), is the set of strings obtained by concatenating any string in L with any string in M
  - LM = {001, 10, 111, 001001, 10001, 111001}
- **Closure** of a language L (**L***) is the set of strings obtained concatenating na arbitrary number of strings of L, including repetitions, i.e., $L^* = \cup_{i \geq 0} L^i$, in which $L^0=\{\varepsilon\}$
  - L= {0,1}, L* is the language of the binary strings

Closure Examples

- $L = \{0, 11\}$
  - $L^0 = \{\varepsilon\}$
  - $L^1 = L = \{0, 11\}$
  - $L^2 = LL = \{00, 011, 110, 1111\}$
  - …
  - $L^* = \{\varepsilon, 0, 11, 00, 011, 110, 1111, \ldots\}$
  - Although L is a finite language, as well as each L^i, L^* is infinite
- L = {all strings with only 0s}
  - $L^* = L$
  - L is infinite, such that L^*
- $L = \varnothing$
  - $L^* = L^0 = \{\varepsilon\}$

Construction of Regular Expressions

- Basis
  - The special symbols ε e $\emptyset$ are regular expressions
    - $L(\varepsilon) = \{\varepsilon\}$ and $L(\emptyset) = \emptyset$
  - If a is a symbol, **a** is a regular expression
    - $L(\mathbf{a}) = \{a\}$
  - A variable (e.g., L) is a regular expression
    - Represents any language specified by regular expressions
- Induction
  - IF E and F are regular expressions, E + F is a regular expression
    - $L(E + F) = L(E) \cup L(F)$
  - If E and F are regular expressions, EF is a regular expression
    - $L(EF) = L(E)L(F)$
  - If E is a regular expression, E* is a regular expression
    - $L(E^*) = (L(E))^*$
  - If E is a regular expression, (E) is a regular expression
    - $L((E)) = L(E)$

Regular Expressions Operators

- * (zero or more occurrences)
- . (concatenation: symbol can be omitted)
- + (or |, or $\cup$)
- Operator precedence (from highest to lowest)
  - *
  - .
  - +
- Parenthesis can be used to “force” a certain order
- $^+$ used to represent 1 or more occurrences

Example

- Write a regular expression for the set of strings consisting of alternating 0s and 1s
  - Example: **01**   L(**01**) = {01}
  - First tentative: (**01**)*
    - $\neq$ **01***
    - L((**01**)*) = {ε, 01, 0101, 0101, …}
    - We miss many strings!
  - Second tentative:
    - (**01**)*+(**10**)*+**0**(**10**)*+**1**(**01**)*
      - Right?
    - (ε+**1**)(**01**)*(ε+**0**)
      - Right?

Exercise 1

1) Write regular expressions for the following languages:

   a) the set of strings over {a,b,c} with at least one ‘a’ and at least one ‘b’

   b) binary strings where all the pairs of adjacent 0s appear before all the pairs of adjacent 1’s

2) Describe the language given by the regular expression:

   - $(1+\varepsilon)(00^*1)^*0^*$

Algebraic Rules for Regular Expressions (REs)

Algebraic Rules for REs

- Two REs are equivalent if they define the same language
  - Two REs with variables are equivalent if, whatever the languages substituting the variables, both REs define the same language
- Main interest: simplify REs
- Commutativity
  - Union: L + M = M + L
  - Concatenation: does not exist!
- Associativity
  - Union: (L + M) + N = L + (N + M)
  - Concatenation: (LM)N = L(MN)

Algebraic Laws for REs (cont.)

- Identity
  - Union: $\emptyset + L = L + \emptyset = L$
  - Concatenation: $\varepsilon L = L\varepsilon = L$
- Absorption
  - Concatenation: $L\emptyset = \emptyset L = \emptyset$
  - Union: does not exist
- Distributive
  - Of the concatenation over the union
  - left: $L(M + N) = LM + LN$
  - right: $(M + N)L = ML + NL$

Algebraic Laws for REs (cont.)

- Idempotent
  - Union: $L + L = L$
  - Concatenation: don’t exist
- Example: simplify $0 + 01^*$
  - $0 + 01^* =$
  - $0\varepsilon + 01^* =$ identity of the concatenation
  - $0(\varepsilon + 1^*) =$ distributive of the concatenation over the union
  - 01^* because ε belongs to the language 1^*

Exercise 2

- Let's use the previous example (quiz):
  - RE = (0+1)*1(0+1)(0+1) + (0+1)*1(0+1) = (0+1)*1(0+1)(ε+0+1)
- How to simplify using RE rules?
  - (0+1)*1(0+1)(0+1) + (0+1)*1(0+1)
  - (0+1)*(1(0+1)(0+1) + 1(0+1))   [rule?]
  - (0+1)*1((0+1)(0+1) + (0+1))   [rule?]
  - (0+1)*1(0+1)((0+1) + ε)   [rule?]
  - (0+1)*1(0+1)(ε+0+1)   [rule?]

Algebraic Laws Involving Closure

- $(L^*)^* = L^*$
- $\varnothing^* = \varepsilon$ Why?
- $\varepsilon^* = \varepsilon$
- $L^+ = LL^* = L^*L$
- $L^* = L^+ + \varepsilon$
- $L? = \varepsilon + L$

- Exercise: in which conditions we obtain $L^* = L^+$?

Discovering New Laws

- Example: $(L + M)^* = (L^*M^*)^*$ is a law?
- Proof: →
  - Supposing $w \in (L+M)^*$
  - $w = w_1 w_2 \ldots w_k$, where $w_i \in L$ or $w_i \in M$
  - Then w_i is also in L^*M^*, because if it is in L then it is also in L^* and if we take $M^* = \varepsilon$ …
  - Needed to prove ← **(homework)**
- Alternative: transform the expression in a concrete RE and analyze the languages
  - $(L + M)^*$ can be transformed in the concrete $(a+b)^*$ and $(L^*M^*)^*$ to $(a^*b^*)^*$
  - In both cases we conclude that $L(E) = \Sigma^*$

Test for Algebraic Laws

- To test if E = F, where E and F are REs with the same set of variables
  - Convert E and F in the concrete REs C and D, substituting each variable by a symbol
  - Test if L(C) = L(D); if true then E=F is a law, else if not a law.
- Examples:
  - Is L* = L*L* ?
    - Converting: C=a* and D=a*a*; both are the set of all strings over {a}
    - Then L(C) = L(D) and "the concatenation of a closure language with itself produces the same language" is a law
  - Is L + ML = (L+M)L ?
    - C= a+ba, D= (a+b)a = aa + ba then L(C)≠L(D) and the statement is not a law

Limits of the Test

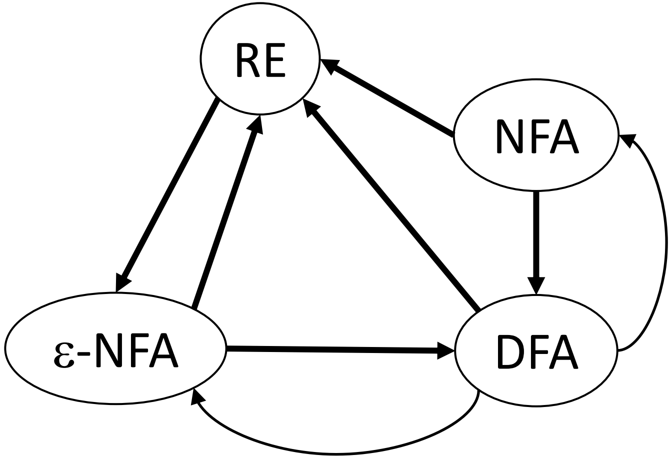
- The test becomes invalid if we consider other operators than the ones of the REs
- Example: add the interception operator to the algebra of the REs
  - Note: the $\cap$ operator does not empower the language (the languages we can define are the same)
  - Is $L \cap M \cap N = L \cap M$? The interception of 3 is the same as 2? Obviously false, but:
    - Substituting L=a, M=b, N=c we get $\{a\} \cap \{b\} \cap \{c\} = \{a\} \cap \{b\} = \varnothing$ and the test would give true.

Exercise 3

- Proof or give a counter-example for the following equalities:
  - $(R+S)^* = R^* + S^*$
  - $(RS+R)^*R = R(SR+R)^*$ (homework)
  - $(RS+R)^*RS = (RR^*S)^*$

Finite Automata (FAs) – Regular Expressions (REs) Equivalence

FA – RE Equivalence



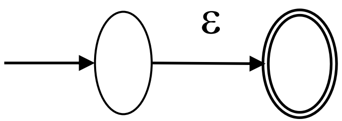
- Show that all the languages defined by FAs can be also defined by regular expressions (FA → RE)
- Show that all the languages defined by REs can be also defined by FAs (RE → ε-NFA)

From Regular Expressions (REs) to Finite Automata (FAs)

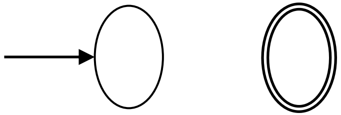
From REs to FAs

- Theorem: every language defined by a regular expression is also defined by an FA.
- Proof: structural induction over the definition of the regular expression
  - Basis step: ε, $\emptyset$ and a
  - Induction step: union, concatenation and closure
  - L = L(R) = L(E), E is a ε-NFA with
    - Exactly one accept state
    - Without input transitions to the start state
    - Without output transitions from the final state

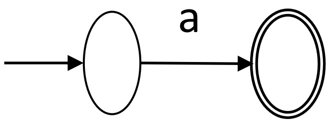
Basis Step



Empty String

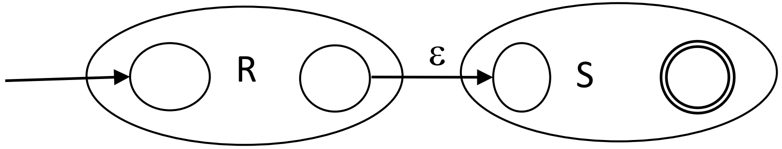


∅

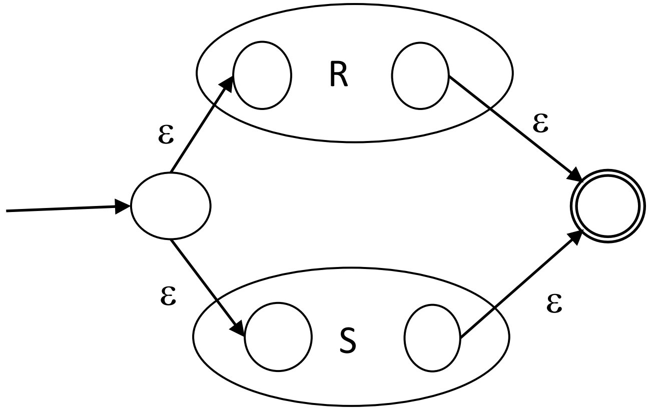


symbol

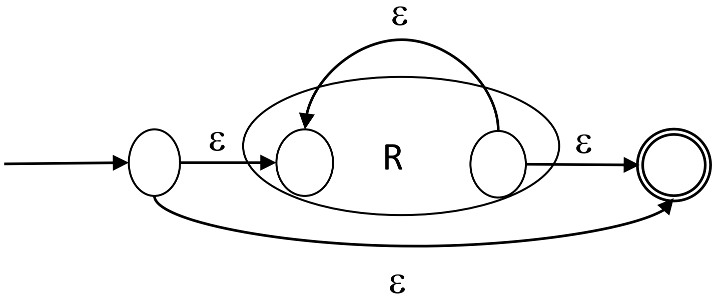
Inductive Step



Concatenation: RS



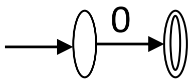
Union: R+S

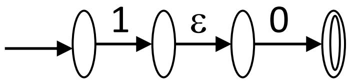


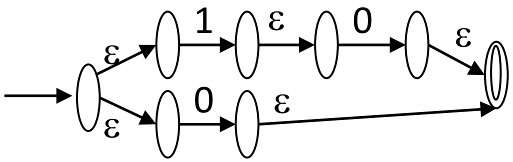
- Closure: R*

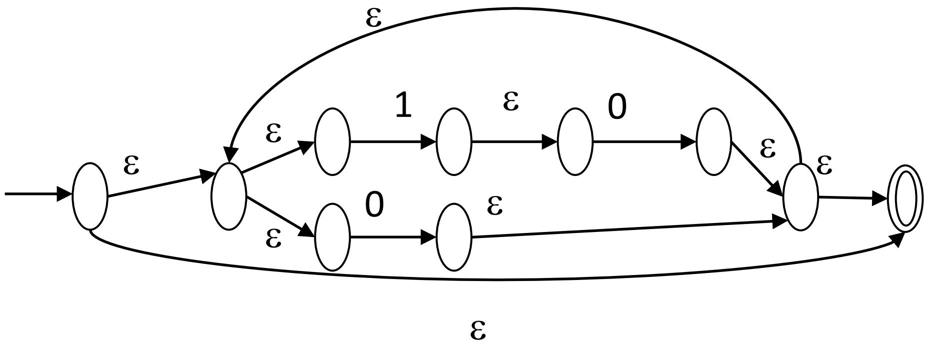
Example

- Draw an ε-NFA for the RE (0+10)*11(0+1)*
  - Try to simplify the FA

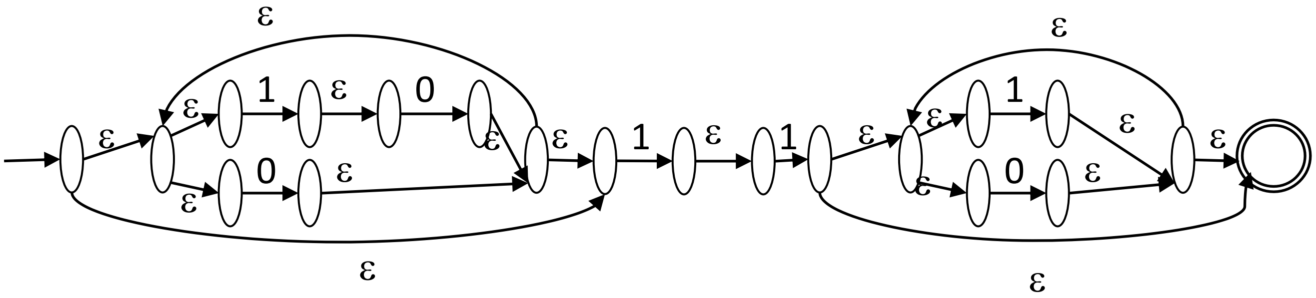


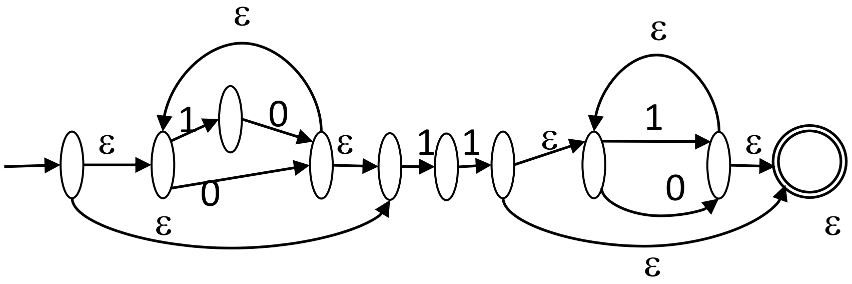




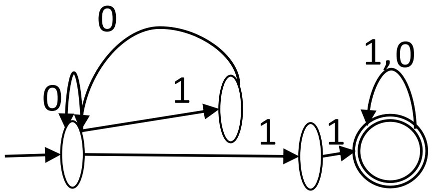


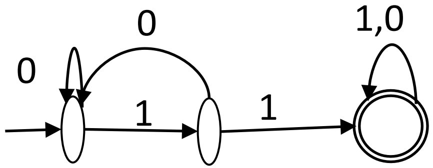
Example (cont.)





Simplification:





From FAs to REs

From FAs to REs

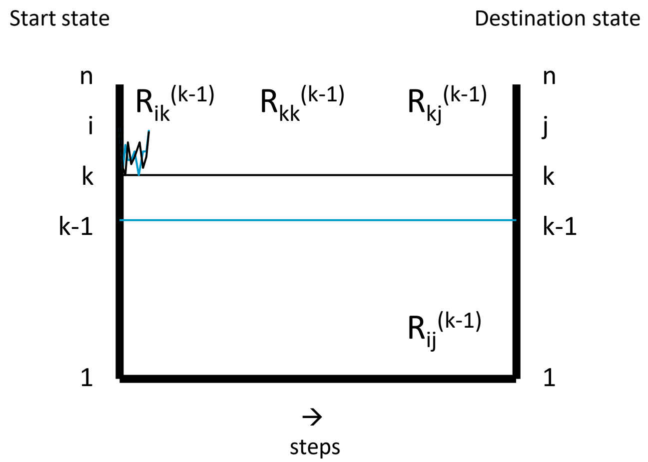
- **Theorem**: If L=L(A) for an FA (DFA, NFA, or ε-NFA) A then it exists a regular expression R such that L=L(R)
- Two methods:
  - **Construction of Paths:** Enumerate the states from 1 to n; build the REs that successively describe **paths** more complex in the FA, until they describe all the paths from the start state to each final state
  - **State Elimination:** Consider the transitions labeled by REs; **eliminate** the internal states substituting their “behavior” by REs

From FAs to REs

Construction of Paths

Construction of Paths

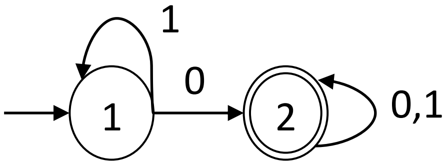
- Numerate the nodes (states) from 1 to n
- $R_{ij}^{(k)}$
  - Regular expression defining the language consisting of the set of strings such that w is the label of a path between nodes i and j, without passing in any intermediate node higher than k
- Induction in the number of nodes (k)



Construction of Paths

- Basis
  - k=0 means without intermediate nodes (the lowest node is the node labeled 1)
    - $R_{ij}^{(0)}$, edge from i to j
    - $i \neq j$: RE is the respective symbol; or $\emptyset$, if does not exist; or $a_1+a_2+\ldots+a_m$, if there are m edges
    - $i=j$: RE is ε or $\varepsilon+a_1+a_2+\ldots+a_m$, if there are m edges
- Induction
  - Hypothesis: the paths that use nodes until k-1 are already converted
  - Exist a path from i to j without passing in node k
    - $R_{ij}^{(k-1)}$
  - The path passes one or more times in k:
    - $R_{ij}^{(k)} = R_{ij}^{(k-1)} + R_{ik}^{(k-1)} (R_{kk}^{(k-1)})^* R_{kj}^{(k-1)}$
  - End: $R_{ij}^{(n)}$ paths between i and j considering all the nodes
- The RE of the language of the FA is the union of the regular expressions $R_{1j}^{(n)}$ such that j is a final state.

Example DFA $\Rightarrow$ RE



- FA that recognizes strings with at least one 0
- $R_{ij}^{(k)} = R_{ij}^{(k-1)} + R_{ik}^{(k-1)} (R_{kk}^{(k-1)})^* R_{kj}^{(k-1)}$
- $R_{ij}^{(1)} = R_{ij}^{(0)} + R_{i1}^{(0)} (R_{11}^{(0)})^* R_{1j}^{(0)}$

$R_{11}^{(0)}$	$\varepsilon+1$
$R_{12}^{(0)}$	0
$R_{21}^{(0)}$	$\varnothing$
$R_{22}^{(0)}$	$\varepsilon+0+1$

$R_{11}^{(1)}$	$\varepsilon+1+(\varepsilon+1)(\varepsilon+1)^*(\varepsilon+1)$	1^*
$R_{12}^{(1)}$	$0+(\varepsilon+1)(\varepsilon+1)^*0$	1^*0
$R_{21}^{(1)}$	$\varnothing+\varnothing(\varepsilon+1)^*(\varepsilon+1)$	$\varnothing$
$R_{22}^{(1)}$	$\varepsilon+0+1+\varnothing(\varepsilon+1)^*0$	$\varepsilon+0+1$

Simplification:

$(\varepsilon+1)^* = 1^*$

$\varnothing R = R\varnothing = \varnothing$

$\varnothing+R = R+\varnothing = R$

$R_{11}^{(2)}$	$1^* + 1^*0(\varepsilon+0+1)^*\varnothing$	1^*
$R_{12}^{(2)}$	$1^*0 + 1^*0(\varepsilon+0+1)^*(\varepsilon+0+1)$	$1^*0(0+1)^*$
$R_{21}^{(2)}$	$\varnothing + (\varepsilon+0+1)(\varepsilon+0+1)^*\varnothing$	$\varnothing$
$R_{22}^{(2)}$	$\varepsilon+0+1+(\varepsilon+0+1)(\varepsilon+0+1)^*(\varepsilon+0+1)$	$(0+1)^*$

$R = 1^*0(0+1)^*$

From FAs to REs

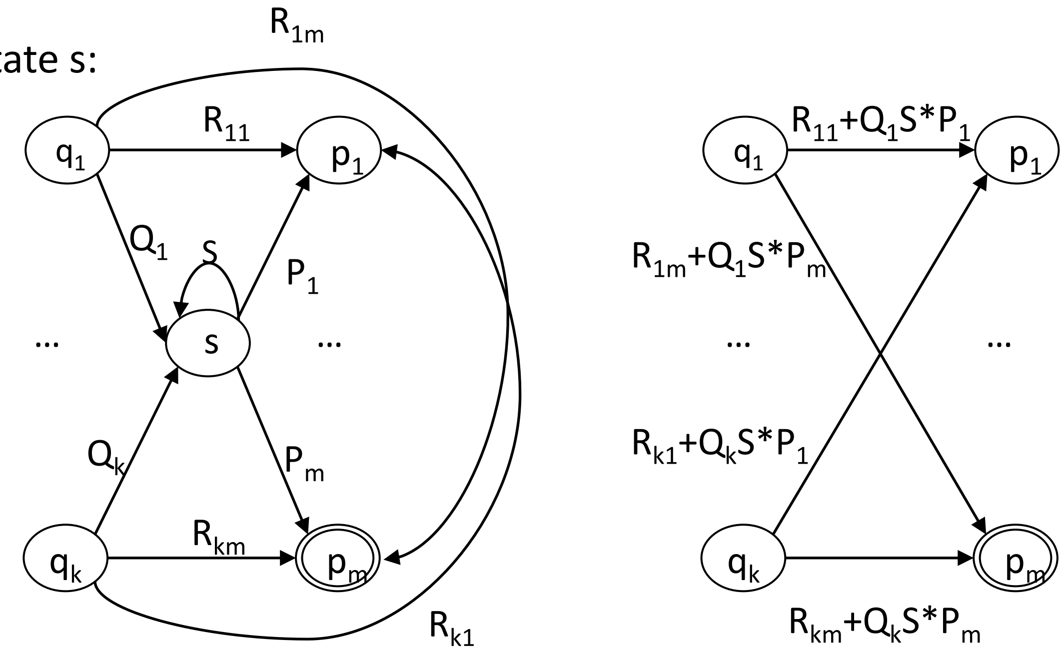
State Elimination

State Elimination

- The construction of paths has many repetitions, is onerous, and may provide long/complex REs if we don't simplify them
- State elimination technique
  - Build REs representing all the implicit strings in the part of the diagram we are substituting
  - Simplify the diagram making more complex the labels of the edges that remain
- State to eliminate: *s*
  - States q_i include all the source states of *s*
  - States p_j include all the sink states of *s* (they can intersect the states q_i)
  - Remove *s* and all the edges that connect s, adding in all the edges from q_i to p_j a part of the eventual path from q_i to p_j, over *s*, including the cycle in *s*: $Q_i S^* P_j$

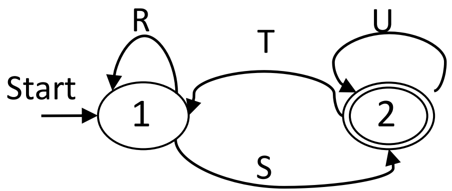
State Elimination

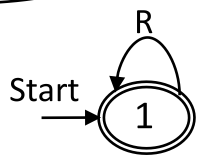
- Eliminating state s:



Strategy

- Consider one FA per final state
  - Eliminate the intermediate states, maintaining the start and the final state, until you have a single edge with the respective RE from the start to the final state
  - If q≠q0, we obtain an FA with 2 states
    - (R+SU*T)*SU*
  - If not, we obtain an FA with a single state
    - R*
- Final RE = union of the REs (1 per FA)

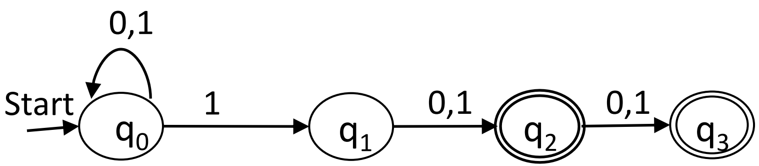


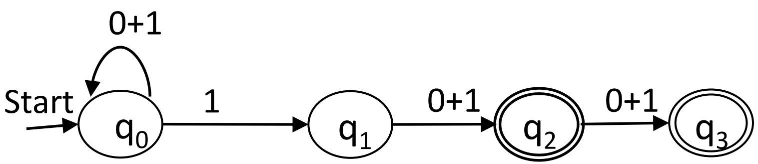


Example

- ❶Start by substituting the labels in the transitions to REs
- ❷Successively eliminate the nodes that are neither start nor accept and substitute each node eliminated by the respective RE
- Note: consider one FA for each accept state and the final RE is obtained by the union of the individual REs (one per FA)

❶



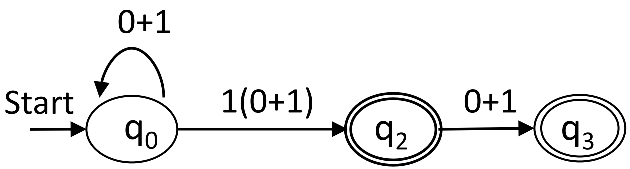


- $Q_1 = 1$, $P_1 = 0+1$, $R_{02} = \varnothing$, $S = \varnothing$
- New edge q_0-q_2: $\varnothing + 1\varnothing^*(0+1) = 1(0+1)$
  - Since: $L(\varnothing^*)=\{\varepsilon\} \cup L(\varnothing) \cup L(\varnothing)\, L(\varnothing)\ldots$

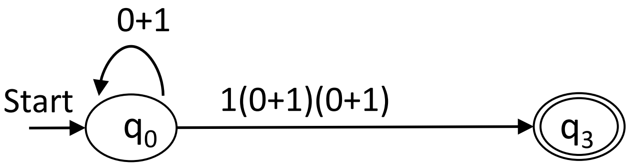
Example (cont.)

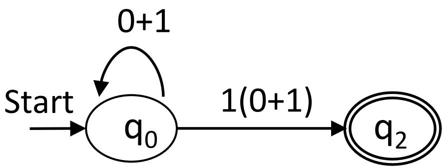
❶Start by substituting the labels in the transitions to REs
❷Successively eliminate the nodes that are neither start nor accept and substitute each node eliminated by the respective RE

❷



Consider one FA for each accept state:





The final RE is obtained by the union of the individual REs (one per FA):

- RE = $(0+1)^*1(0+1)(0+1) + (0+1)^*1(0+1) = (0+1)^*1(0+1)(\varepsilon+0+1)$

Exercise 4

- Consider a DFA with the transition table given below
  1. Construction of paths technique:
     a) Calculate all the expressions $R_{ij}^{(0)}$ (i is the number of state q_i)
     b) Calculate all the expressions $R_{ij}^{(1)}$ and simplify them
     c) Obtain a regular expression for the language of the DFA
  2. Draw the transition (state) diagram of the DFA and obtain a regular expression for its language using the state elimination technique

	0	1
$\rightarrow q_1$	q_2	q_1
q_2	q_3	q_1
$*q_3$	q_3	q_2

Conclusion

- Regular Expressions (REs) provide a way to specify languages (named as regular languages)
- REs can be converted in ε-NFAs
- FAs can be converted into REs