From FAs to Regular Expressions - II

L.EIC, 2nd Year

João M. P. Cardoso
Email: jmpc@acm.org

U.PORTO
FEUP FACULDADE DE ENGENHARIA UNIVERSIDADE DO PORTO
DEI DEPARTAMENTO DE ENGENHARIA INFORMÁTICA

Conversion from FAs to Regular Expressions

- Given a Finite Automata (FA) how to generate an equivalent regular expression (RE)?
- Two techniques:
  - State Elimination
  - Construction of Paths
- Both algorithms work with Finite Automata (FA) as input, i.e., DFAs, NFAs, and ε-NFAs

Construction of Paths

- Hopcroft's formula (the *n* states of the FA are enumerated from 1 to *n*)
  - $R_{ij}^{(n)}$ is the regular expression of all paths from i to j (n is the number of states)
  - $R_{ij}^{(k)} = R_{ij}^{(k-1)}+R_{ik}^{(k-1)}.(R_{kk}^{(k-1)})^{*}.R_{kj}^{(k-1)}$
  - States are numbered 1 to n
  - $R_{ij}^{(k)}$ is regular expression of all paths from i to j passing through nodes less or equal than k
  - Computed for all i,j for k=0, then k=1,…,n
  - $R_{sf1}^{(n)}+\ldots+R_{sfk}^{(n)}$ is the regular expression of the DFA considering:
    - s is the start state, $f_1,\ldots,f_k$ are accepting states, n is the number of states.

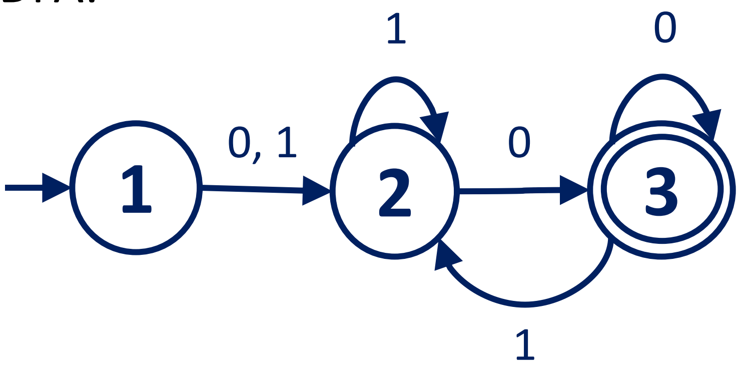
Algorithm based on Path Construction

- Numerate the nodes (states) from 1 to n
- $R_{ij}^{(k)}$
  - Regular expression defining the language consisting of the set of strings such that w is the label of a path between nodes i and j, without passing in any intermediate node higher than k
- Induction in the number of nodes (k)
  - $R_{ij}^{(k)} = R_{ij}^{(k-1)}+R_{ik}^{(k-1)}.(R_{kk}^{(k-1)})^{*}.R_{kj}^{(k-1)}$
  - Computed for all i,j for k=0, then k=1,…,n
- $R_{s,f1}(n)+\ldots+R_{s,fk}(n)$ is the regular expression of the DFA considering:
- s is the start state, f1,…,fk are accepting states, n is the number of states.

Algorithm based on Path Construction (example)

- DFA below
  - 3 states
  - 1 final state
  - DFA states already labeled from 1 to 3
- Regular expression representing the language of the DFA:
  - $RE = RE_{13}^{(3)}$

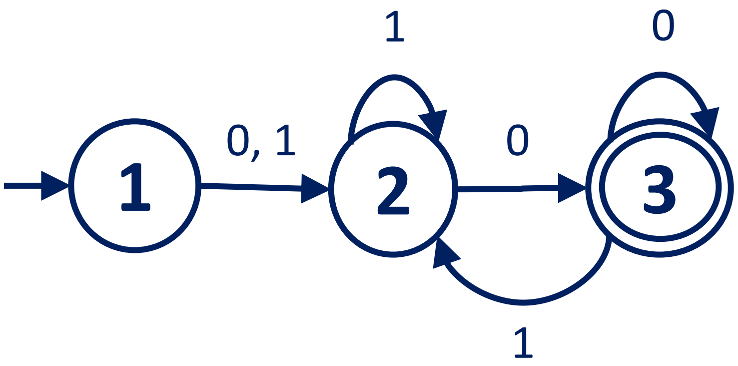
DFA:



Algorithm based on Path Construction

- $\boxed{R_{ij}^{(k)} = R_{ij}^{(k-1)} + R_{ik}^{(k-1)}.(R_{kk}^{(k-1)})^{*}.R_{kj}^{(k-1)}}$

$R_{11}^{(0)}$	ε
$R_{12}^{(0)}$	$0 + 1$
$R_{13}^{(0)}$	$\varnothing$
$R_{21}^{(0)}$	$\varnothing$
$R_{22}^{(0)}$	$\varepsilon + 1$
$R_{23}^{(0)}$	0
$R_{31}^{(0)}$	$\varnothing$
$R_{32}^{(0)}$	1
$R_{33}^{(0)}$	$\varepsilon + 0$



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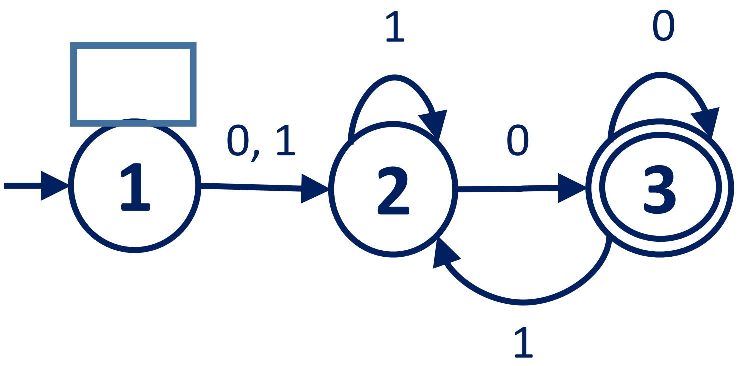

Algorithm based on Path Construction

- $R_{ij}^{(k)} = R_{ij}^{(k-1)} + R_{ik}^{(k-1)} . (R_{kk}^{(k-1)})^* . R_{kj}^{(k-1)}$

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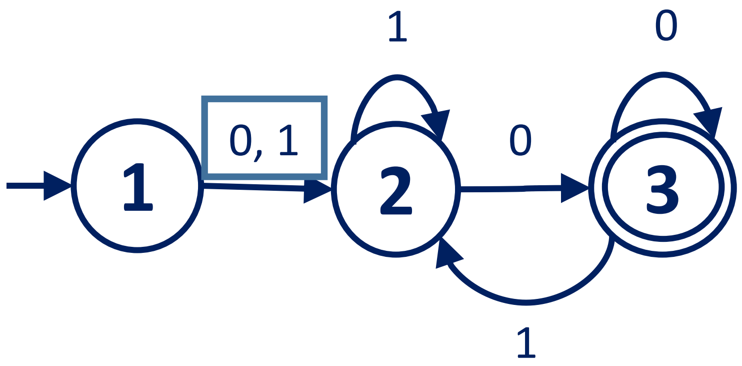

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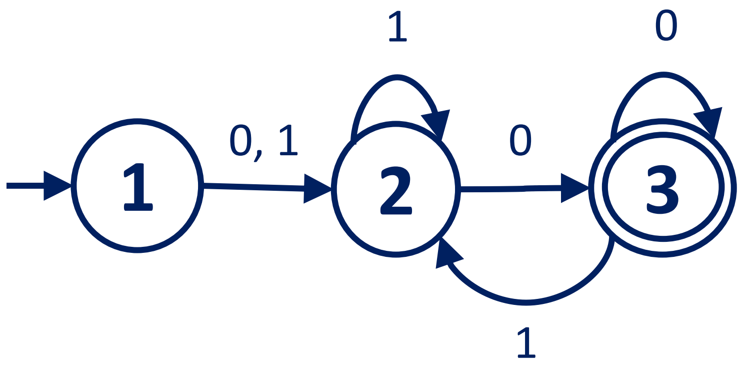

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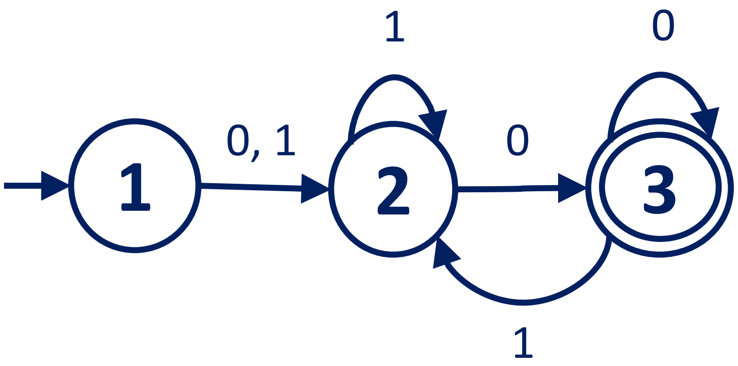

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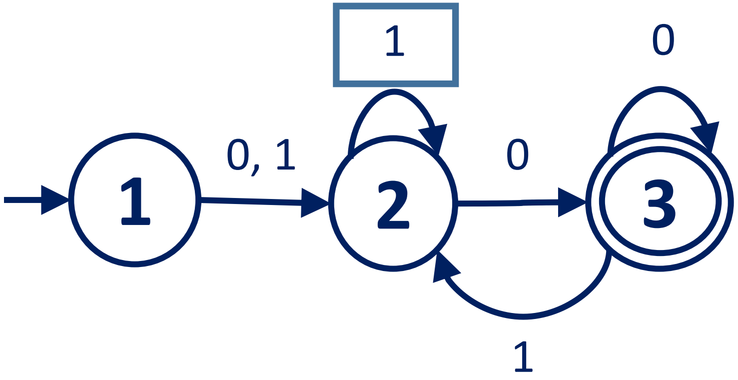

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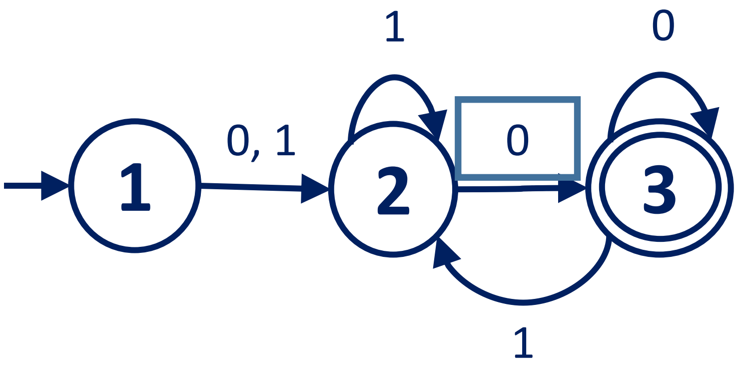

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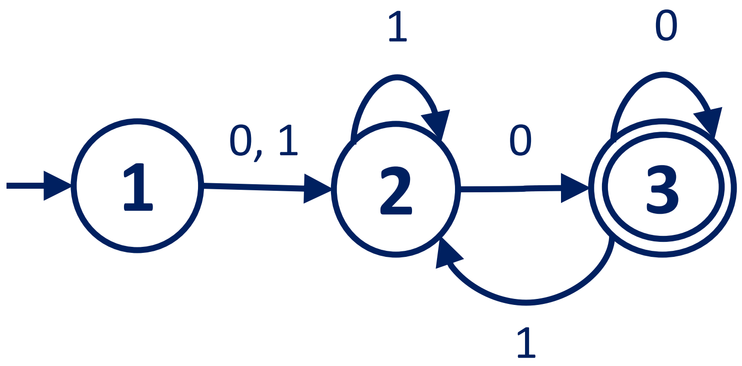

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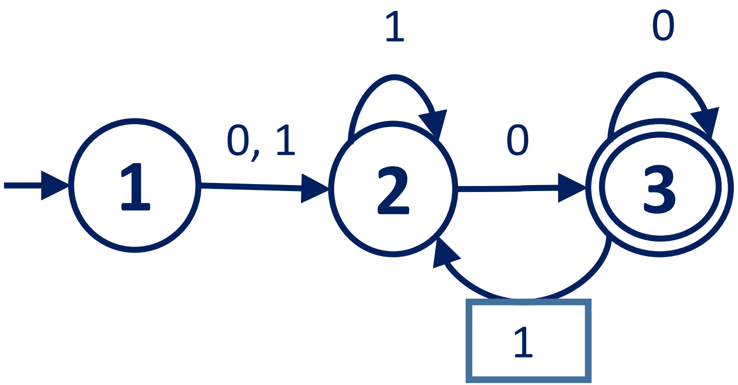

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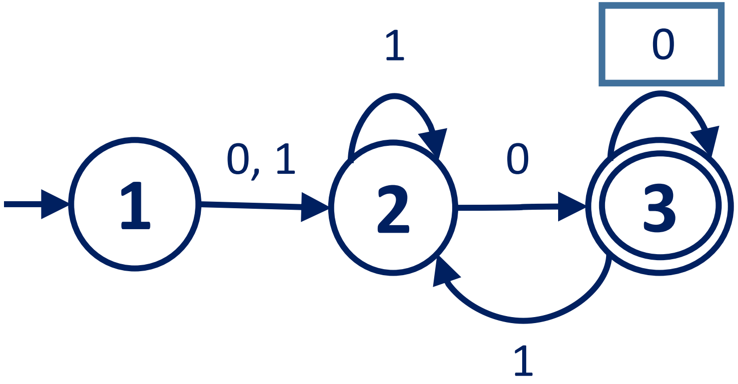

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$R_{11}^{(1)}$	$\varepsilon + \varepsilon . (\varepsilon)^{*} . \varepsilon$	ε
$R_{12}^{(1)}$		
$R_{13}^{(1)}$		
$R_{21}^{(1)}$		
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$R_{23}^{(1)}$		
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$R_{11}^{(1)}$	$\varepsilon + \varepsilon\ .\ (\varepsilon)^*\ .\ \varepsilon$	ε
$R_{12}^{(1)}$	$(0 + 1) + \varepsilon\ .\ (\varepsilon)^*\ .\ (0 + 1)$	$0 + 1$
$R_{13}^{(1)}$		
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$R_{12}^{(1)}$	$(0 + 1) + \varepsilon \,.\, (\varepsilon)^* \,.\, (0 + 1)$	$0 + 1$
$R_{13}^{(1)}$	$\varnothing + \varepsilon \,.\, (\varepsilon)^* \,.\, \varnothing$	$\varnothing$
$R_{21}^{(1)}$		
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$R_{12}^{(1)}$	$(0 + 1) + \varepsilon \,.\, (\varepsilon)^* \,.\, (0 + 1)$	$0 + 1$
$R_{13}^{(1)}$	$\varnothing + \varepsilon \,.\, (\varepsilon)^* \,.\, \varnothing$	$\varnothing$
$R_{21}^{(1)}$	$\varnothing + \varnothing \,.\, (\varepsilon)^* \,.\, \varepsilon$	$\varnothing$
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$R_{12}^{(1)}$	$(0 + 1) + \varepsilon\ .\ (\varepsilon)^*\ .\ (0 + 1)$	$0 + 1$
$R_{13}^{(1)}$	$\varnothing + \varepsilon\ .\ (\varepsilon)^*\ .\ \varnothing$	$\varnothing$
$R_{21}^{(1)}$	$\varnothing + \varnothing\ .\ (\varepsilon)^*\ .\ \varepsilon$	$\varnothing$
$R_{22}^{(1)}$	$(\varepsilon + 1) + \varnothing\ .\ (\varepsilon)^*\ .\ (0 + 1)$	$\varepsilon + 1$
$R_{23}^{(1)}$		
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$R_{12}^{(1)}$	$(0 + 1) + \varepsilon . (\varepsilon)^* . (0 + 1)$	$0 + 1$
$R_{13}^{(1)}$	$\varnothing + \varepsilon . (\varepsilon)^* . \varnothing$	$\varnothing$
$R_{21}^{(1)}$	$\varnothing + \varnothing . (\varepsilon)^* . \varepsilon$	$\varnothing$
$R_{22}^{(1)}$	$(\varepsilon + 1) + \varnothing . (\varepsilon)^* . (0 + 1)$	$\varepsilon + 1$
$R_{23}^{(1)}$	$0 + \varnothing . (\varepsilon)^* . \varnothing$	0
$R_{31}^{(1)}$		
$R_{32}^{(1)}$		
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$R_{11}^{(1)}$	$\varepsilon + \varepsilon . (\varepsilon)^* . \varepsilon$	ε
$R_{12}^{(1)}$	$(0 + 1) + \varepsilon . (\varepsilon)^* . (0 + 1)$	$0 + 1$
$R_{13}^{(1)}$	$\varnothing + \varepsilon . (\varepsilon)^* . \varnothing$	$\varnothing$
$R_{21}^{(1)}$	$\varnothing + \varnothing . (\varepsilon)^* . \varepsilon$	$\varnothing$
$R_{22}^{(1)}$	$(\varepsilon + 1) + \varnothing . (\varepsilon)^* . (0 + 1)$	$\varepsilon + 1$
$R_{23}^{(1)}$	$0 + \varnothing . (\varepsilon)^* . \varnothing$	0
$R_{31}^{(1)}$	$\varnothing + \varnothing . (\varepsilon)^* . \varepsilon$	$\varnothing$
$R_{32}^{(1)}$		
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$R_{11}^{(0)}$	ε
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$R_{11}^{(1)}$	$\varepsilon + \varepsilon . (\varepsilon)^* . \varepsilon$	ε
$R_{12}^{(1)}$	$(0 + 1) + \varepsilon . (\varepsilon)^* . (0 + 1)$	$0 + 1$
$R_{13}^{(1)}$	$\varnothing + \varepsilon . (\varepsilon)^* . \varnothing$	$\varnothing$
$R_{21}^{(1)}$	$\varnothing + \varnothing . (\varepsilon)^* . \varepsilon$	$\varnothing$
$R_{22}^{(1)}$	$(\varepsilon + 1) + \varnothing . (\varepsilon)^* . (0 + 1)$	$\varepsilon + 1$
$R_{23}^{(1)}$	$0 + \varnothing . (\varepsilon)^* . \varnothing$	0
$R_{31}^{(1)}$	$\varnothing + \varnothing . (\varepsilon)^* . \varepsilon$	$\varnothing$
$R_{32}^{(1)}$	$1 + \varnothing. (\varepsilon)^* . (0 + 1)$	1
$R_{33}^{(1)}$		

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$R_{12}^{(0)}$	$0 + 1$
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$R_{11}^{(1)}$	$\varepsilon + \varepsilon \,.\, (\varepsilon)^* \,.\, \varepsilon$	ε
$R_{12}^{(1)}$	$(0 + 1) + \varepsilon \,.\, (\varepsilon)^* \,.\, (0 + 1)$	$0 + 1$
$R_{13}^{(1)}$	$\varnothing + \varepsilon \,.\, (\varepsilon)^* \,.\, \varnothing$	$\varnothing$
$R_{21}^{(1)}$	$\varnothing + \varnothing \,.\, (\varepsilon)^* \,.\, \varepsilon$	$\varnothing$
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$R_{23}^{(1)}$	$0 + \varnothing \,.\, (\varepsilon)^* \,.\, \varnothing$	0
$R_{31}^{(1)}$	$\varnothing + \varnothing \,.\, (\varepsilon)^* \,.\, \varepsilon$	$\varnothing$
$R_{32}^{(1)}$	$1 + \varnothing. (\varepsilon)^* \,.\, (0 + 1)$	1
$R_{33}^{(1)}$	$(\varepsilon + 0) + \varnothing \,.\, (\varepsilon)^* \,.\, \varnothing$	$\varepsilon + 0$

$R_{11}^{(0)}$	ε
$R_{12}^{(0)}$	$0 + 1$
$R_{13}^{(0)}$	$\varnothing$
$R_{21}^{(0)}$	$\varnothing$
$R_{22}^{(0)}$	$\varepsilon + 1$
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$R_{11}^{(2)}$	$\varepsilon + (0 + 1) . (\varepsilon + 1)^{*}. \varnothing$	ε
$R_{12}^{(2)}$		
$R_{13}^{(2)}$		
$R_{21}^{(2)}$		
$R_{22}^{(2)}$		
$R_{23}^{(2)}$		
$R_{31}^{(2)}$		
$R_{32}^{(2)}$		
$R_{33}^{(2)}$		

$R_{11}^{(1)}$	ε
$R_{12}^{(1)}$	$0 + 1$
$R_{13}^{(1)}$	$\varnothing$
$R_{21}^{(1)}$	$\varnothing$
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$R_{11}^{(2)}$	$\varepsilon + (0 + 1) \, . \, (\varepsilon + 1)^* \, . \, \varnothing$	ε
$R_{12}^{(2)}$	$(0 + 1) + (0 + 1) \, . \, (\varepsilon + 1)^* . \, (\varepsilon + 1)$	$(0 + 1)1^*$
$R_{13}^{(2)}$		
$R_{21}^{(2)}$		
$R_{22}^{(2)}$		
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$R_{31}^{(2)}$		
$R_{32}^{(2)}$		
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$R_{11}^{(2)}$	$\varepsilon + (0 + 1) . (\varepsilon + 1)^* . \varnothing$	ε
$R_{12}^{(2)}$	$(0 + 1) + (0 + 1) . (\varepsilon + 1)^*. (\varepsilon +1)$	$(0 + 1)1^*$
$R_{13}^{(2)}$	$\varnothing + (0 + 1) . (\varepsilon + 1)^*.0$	$(0 + 1)1^*0$
$R_{21}^{(2)}$		
$R_{22}^{(2)}$		
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$R_{31}^{(2)}$		
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$R_{12}^{(2)}$	$(0 + 1) + (0 + 1) . (\varepsilon + 1)^*. (\varepsilon + 1)$	$(0 + 1)1^*$
$R_{13}^{(2)}$	$\varnothing + (0 + 1) . (\varepsilon + 1)^*.0$	$(0 + 1)1^*0$
$R_{21}^{(2)}$	$\varnothing + (\varepsilon + 1). (\varepsilon + 1)^*. \varnothing$	$\varnothing$
$R_{22}^{(2)}$		
$R_{23}^{(2)}$		
$R_{31}^{(2)}$		
$R_{32}^{(2)}$		
$R_{33}^{(2)}$		

$R_{11}^{(1)}$	ε
$R_{12}^{(1)}$	$0 + 1$
$R_{13}^{(1)}$	$\varnothing$
$R_{21}^{(1)}$	$\varnothing$
$R_{22}^{(1)}$	$\varepsilon + 1$
$R_{23}^{(1)}$	0
$R_{31}^{(1)}$	$\varnothing$
$R_{32}^{(1)}$	1
$R_{33}^{(1)}$	$\varepsilon + 0$

Algorithm based on Path Construction

- $R_{ij}^{(k)} = R_{ij}^{(k-1)} + R_{ik}^{(k-1)}.(R_{kk}^{(k-1)})^{*}.R_{kj}^{(k-1)}$

$R_{11}^{(2)}$	$\varepsilon + (0 + 1) . (\varepsilon + 1)^* . \varnothing$	ε
$R_{12}^{(2)}$	$(0 + 1) + (0 + 1) . (\varepsilon + 1)^*. (\varepsilon + 1)$	$(0 + 1)1^*$
$R_{13}^{(2)}$	$\varnothing + (0 + 1) . (\varepsilon + 1)^*.0$	$(0 + 1)1^*0$
$R_{21}^{(2)}$	$\varnothing + (\varepsilon + 1). (\varepsilon + 1)^*. \varnothing$	$\varnothing$
$R_{22}^{(2)}$	$(\varepsilon + 1) + (\varepsilon + 1). (\varepsilon + 1)^*. (\varepsilon + 1)$	1^*
$R_{23}^{(2)}$		
$R_{31}^{(2)}$		
$R_{32}^{(2)}$		
$R_{33}^{(2)}$		

$R_{11}^{(1)}$	ε
$R_{12}^{(1)}$	$0 + 1$
$R_{13}^{(1)}$	$\varnothing$
$R_{21}^{(1)}$	$\varnothing$
$R_{22}^{(1)}$	$\varepsilon + 1$
$R_{23}^{(1)}$	0
$R_{31}^{(1)}$	$\varnothing$
$R_{32}^{(1)}$	1
$R_{33}^{(1)}$	$\varepsilon + 0$

Algorithm based on Path Construction

- $R_{ij}^{(k)} = R_{ij}^{(k-1)} + R_{ik}^{(k-1)} . (R_{kk}^{(k-1)})^* . R_{kj}^{(k-1)}$

$R_{11}^{(2)}$	$\varepsilon + (0 + 1) . (\varepsilon + 1)^* . \varnothing$	ε
$R_{12}^{(2)}$	$(0 + 1) + (0 + 1) . (\varepsilon + 1)^* . (\varepsilon + 1)$	$(0 + 1)1^*$
$R_{13}^{(2)}$	$\varnothing + (0 + 1) . (\varepsilon + 1)^* . 0$	$(0 + 1)1^*0$
$R_{21}^{(2)}$	$\varnothing + (\varepsilon + 1) . (\varepsilon + 1)^* . \varnothing$	$\varnothing$
$R_{22}^{(2)}$	$(\varepsilon + 1) + (\varepsilon + 1) . (\varepsilon + 1)^* . (\varepsilon + 1)$	1^*
$R_{23}^{(2)}$	$0 + (\varepsilon + 1) . (\varepsilon + 1)^* . 0$	1^*0
$R_{31}^{(2)}$		
$R_{32}^{(2)}$		
$R_{33}^{(2)}$		

$R_{11}^{(1)}$	ε
$R_{12}^{(1)}$	$0 + 1$
$R_{13}^{(1)}$	$\varnothing$
$R_{21}^{(1)}$	$\varnothing$
$R_{22}^{(1)}$	$\varepsilon + 1$
$R_{23}^{(1)}$	0
$R_{31}^{(1)}$	$\varnothing$
$R_{32}^{(1)}$	1
$R_{33}^{(1)}$	$\varepsilon + 0$

Algorithm based on Path Construction

- $R_{ij}^{(k)} = R_{ij}^{(k-1)} + R_{ik}^{(k-1)}.(R_{kk}^{(k-1)})^{*}.R_{kj}^{(k-1)}$

$R_{11}^{(2)}$	$\varepsilon + (0 + 1) . (\varepsilon + 1)^* . \varnothing$	ε
$R_{12}^{(2)}$	$(0 + 1) + (0 + 1) . (\varepsilon + 1)^*. (\varepsilon + 1)$	$(0 + 1)1^*$
$R_{13}^{(2)}$	$\varnothing + (0 + 1) . (\varepsilon + 1)^*.0$	$(0 + 1)1^*0$
$R_{21}^{(2)}$	$\varnothing + (\varepsilon + 1). (\varepsilon + 1)^*. \varnothing$	$\varnothing$
$R_{22}^{(2)}$	$(\varepsilon + 1) + (\varepsilon + 1). (\varepsilon + 1)^*. (\varepsilon + 1)$	1^*
$R_{23}^{(2)}$	$0 + (\varepsilon + 1). (\varepsilon + 1)^*. 0$	1^*0
$R_{31}^{(2)}$	$\varnothing + 1. (\varepsilon + 1)^*. \varnothing$	$\varnothing$
$R_{32}^{(2)}$		
$R_{33}^{(2)}$		

$R_{11}^{(1)}$	ε
$R_{12}^{(1)}$	$0 + 1$
$R_{13}^{(1)}$	$\varnothing$
$R_{21}^{(1)}$	$\varnothing$
$R_{22}^{(1)}$	$\varepsilon + 1$
$R_{23}^{(1)}$	0
$R_{31}^{(1)}$	$\varnothing$
$R_{32}^{(1)}$	1
$R_{33}^{(1)}$	$\varepsilon + 0$

Algorithm based on Path Construction

- $R_{ij}^{(k)} = R_{ij}^{(k-1)} + R_{ik}^{(k-1)}.(R_{kk}^{(k-1)})^*.R_{kj}^{(k-1)}$

$R_{11}^{(2)}$	$\varepsilon + (0+1) . (\varepsilon+1)^* . \varnothing$	ε
$R_{12}^{(2)}$	$(0+1) + (0+1) . (\varepsilon+1)^*. (\varepsilon+1)$	$(0+1)1^*$
$R_{13}^{(2)}$	$\varnothing + (0+1) . (\varepsilon+1)^*.0$	$(0+1)1^*0$
$R_{21}^{(2)}$	$\varnothing + (\varepsilon+1). (\varepsilon+1)^*. \varnothing$	$\varnothing$
$R_{22}^{(2)}$	$(\varepsilon+1) + (\varepsilon+1). (\varepsilon+1)^*. (\varepsilon+1)$	1^*
$R_{23}^{(2)}$	$0 + (\varepsilon+1). (\varepsilon+1)^*. 0$	1^*0
$R_{31}^{(2)}$	$\varnothing + 1. (\varepsilon+1)^*. \varnothing$	$\varnothing$
$R_{32}^{(2)}$	$1 + 1 . (\varepsilon+1)^*. (\varepsilon+1)$	$11^*=1^+$
$R_{33}^{(2)}$		

$R_{11}^{(1)}$	ε
$R_{12}^{(1)}$	$0+1$
$R_{13}^{(1)}$	$\varnothing$
$R_{21}^{(1)}$	$\varnothing$
$R_{22}^{(1)}$	$\varepsilon+1$
$R_{23}^{(1)}$	0
$R_{31}^{(1)}$	$\varnothing$
$R_{32}^{(1)}$	1
$R_{33}^{(1)}$	$\varepsilon+0$

Algorithm based on Path Construction

- $R_{ij}^{(k)} = R_{ij}^{(k-1)} + R_{ik}^{(k-1)}.(R_{kk}^{(k-1)})^{*}.R_{kj}^{(k-1)}$

$R_{11}^{(2)}$	$\varepsilon + (0+1) . (\varepsilon+1)^* . \varnothing$	ε
$R_{12}^{(2)}$	$(0+1) + (0+1) . (\varepsilon+1)^*. (\varepsilon+1)$	$(0+1)1^*$
$R_{13}^{(2)}$	$\varnothing + (0+1) . (\varepsilon+1)^*.0$	$(0+1)1^*0$
$R_{21}^{(2)}$	$\varnothing + (\varepsilon+1). (\varepsilon+1)^*. \varnothing$	$\varnothing$
$R_{22}^{(2)}$	$(\varepsilon+1) + (\varepsilon+1). (\varepsilon+1)^*. (\varepsilon+1)$	1^*
$R_{23}^{(2)}$	$0 + (\varepsilon+1). (\varepsilon+1)^*. 0$	1^*0
$R_{31}^{(2)}$	$\varnothing + 1. (\varepsilon+1)^*. \varnothing$	$\varnothing$
$R_{32}^{(2)}$	$1 + 1 . (\varepsilon+1)^*. (\varepsilon+1)$	1^+
$R_{33}^{(2)}$	$(\varepsilon+0) + 1 . (\varepsilon+1)^*. 0$	$\varepsilon + 0 + 11^*0$

$R_{11}^{(1)}$	ε
$R_{12}^{(1)}$	$0+1$
$R_{13}^{(1)}$	$\varnothing$
$R_{21}^{(1)}$	$\varnothing$
$R_{22}^{(1)}$	$\varepsilon + 1$
$R_{23}^{(1)}$	0
$R_{31}^{(1)}$	$\varnothing$
$R_{32}^{(1)}$	1
$R_{33}^{(1)}$	$\varepsilon + 0$

Algorithm based on Path Construction

- $R_{ij}^{(k)} = R_{ij}^{(k-1)} + R_{ik}^{(k-1)}.(R_{kk}^{(k-1)})^{*}.R_{kj}^{(k-1)}$

$R_{11}^{(3)}$	$\varepsilon + ((0+1)1^*0).(\varepsilon + 0 + 11^*0)^*.\varnothing$	ε
$R_{12}^{(3)}$		
$R_{13}^{(3)}$		
$R_{21}^{(3)}$		
$R_{22}^{(3)}$		
$R_{23}^{(3)}$		
$R_{31}^{(3)}$		
$R_{32}^{(3)}$		
$R_{33}^{(3)}$		

$R_{11}^{(2)}$	ε
$R_{12}^{(2)}$	$(0+1)1^*$
$R_{13}^{(2)}$	$(0+1)1^*0$
$R_{21}^{(2)}$	$\varnothing$
$R_{22}^{(2)}$	1^*
$R_{23}^{(2)}$	1^*0
$R_{31}^{(2)}$	$\varnothing$
$R_{32}^{(2)}$	1^+
$R_{33}^{(2)}$	$\varepsilon + 0 + 11^*0$

Algorithm based on Path Construction

- $R_{ij}^{(k)} = R_{ij}^{(k-1)} + R_{ik}^{(k-1)}.(R_{kk}^{(k-1)})^*.R_{kj}^{(k-1)}$

$R_{11}^{(3)}$	$\varepsilon + ((0 + 1)1^*0).(\varepsilon + 0 + 11^*0)^*.\ \varnothing$	ε
$R_{12}^{(3)}$	$(0 + 1)1^* + ((0 + 1)1^*0).(\varepsilon + 0 + 11^*0)^*.\ 1+$	
$R_{13}^{(3)}$		
$R_{21}^{(3)}$		
$R_{22}^{(3)}$		
$R_{23}^{(3)}$		
$R_{31}^{(3)}$		
$R_{32}^{(3)}$		
$R_{33}^{(3)}$		

$R_{11}^{(2)}$	ε
$R_{12}^{(2)}$	$(0 + 1)1^*$
$R_{13}^{(2)}$	$(0 + 1)1^*0$
$R_{21}^{(2)}$	$\varnothing$
$R_{22}^{(2)}$	1^*
$R_{23}^{(2)}$	1^*0
$R_{31}^{(2)}$	$\varnothing$
$R_{32}^{(2)}$	1^+
$R_{33}^{(2)}$	$\varepsilon + 0 + 11^*0$

Algorithm based on Path Construction

- $R_{ij}^{(k)} = R_{ij}^{(k-1)} + R_{ik}^{(k-1)}.(R_{kk}^{(k-1)})^*.R_{kj}^{(k-1)}$

$R_{11}^{(3)}$	$\varepsilon + ((0 + 1)1^*0).(\varepsilon + 0 + 11^*0)^*. \varnothing$	ε
$R_{12}^{(3)}$	$(0 + 1)1^* + ((0 + 1)1^*0).(\varepsilon + 0 + 11^*0)^*. 1+$	
$R_{13}^{(3)}$	$(0 + 1)1^*0 + ((0 + 1)1^*0).(\varepsilon + 0 + 11^*0)^*. (\varepsilon + 0 + 11^*0)$	

$R_{11}^{(2)}$	ε
$R_{12}^{(2)}$	$(0 + 1)1^*$
$R_{13}^{(2)}$	$(0 + 1)1^*0$
$R_{21}^{(2)}$	$\varnothing$
$R_{22}^{(2)}$	1^*
$R_{23}^{(2)}$	1^*0
$R_{31}^{(2)}$	$\varnothing$
$R_{32}^{(2)}$	1^+
$R_{33}^{(2)}$	$\varepsilon + 0 + 11^*0$

$R_{s,f1}^{(n)}+\ldots+R_{s,fk}^{(n)}$ is the regular expression of the DFA

RE: $R_{13}^{(3)} = (0 + 1)1^*0 + ((0 + 1)1^*0).(\varepsilon + 0 + 11^*0)^*. (\varepsilon + 0 + 11^*0)$

Simplified: $(0+1)1^*0(0+11^*0)^*$