Theory of Computation

MIEIC, 2nd Year

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Outline

- Motivation
- Context
- The Church-Turing Thesis
- Turing Machine (TM)
- Languages of a TM
- Techniques to program a TM
- Extensions to TMs
- Summary

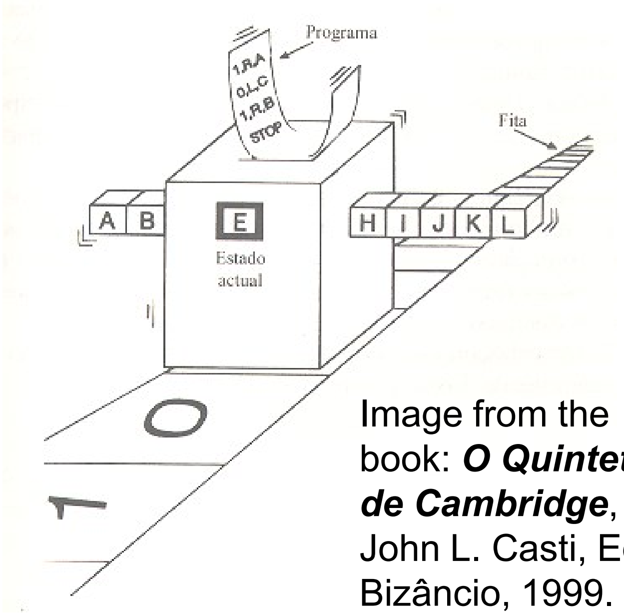


Image from the book: ***O Quinteto de Cambridge***, John L. Casti, Ed. Bizâncio, 1999.

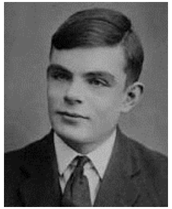
Motivation

- Undecidable Problems
  - There is no algorithm
- Intractable Problems
  - The known algorithms are very costly
  - Simplification and the use of heuristics
- Simple model to study the computability
  - Turing Machines
  - A model of a computer

Some “Theory of Computation” Pioneers

Context

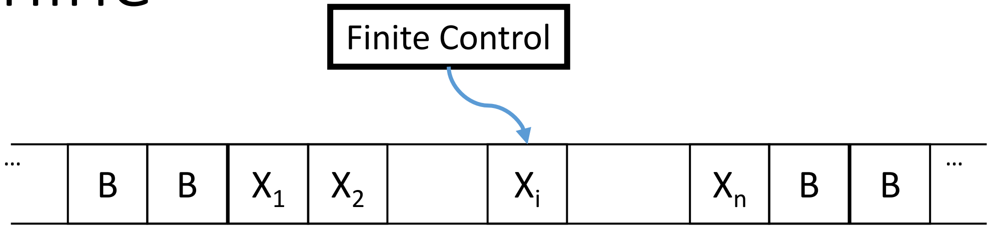
- David Hilbert (beginning of 20th century): https://en.wikipedia.org/wiki/David_Hilbert
  - Is there a way to determine whether any formula in the first-order predicate calculus, applied to integers, is true?
- Kurt Gödel (1931): https://en.wikipedia.org/wiki/Kurt_G%C3%B6del
  - Incompleteness theorem. He constructed a formula in the predicate calculus applied to integers, which asserted that the formula itself could be neither proved nor disproved within the predicate calculus
- Alan Turing (1936): https://en.wikipedia.org/wiki/Alan_Turing
  - proposed the Turing machine as a model of any possible computation
- A. Church (1930's): https://en.wikipedia.org/wiki/Alonzo_Church
  - Church-Turing hypothesis (unprovable)
  - "any general way to compute will allow us to compute only the partial-recursive functions" (or equivalently to what Turing machines or modern-day computers can compute)



The Church-Turing Thesis

- *"All reasonable models of (general-purpose) computers are equivalent. In particular, they are equivalent to a Turing machine."*
- First formulated by Alonzo Church in the 1930s and is usually referred to as Church's thesis, or the Church-Turing thesis
- A Turing Machine is, according to the Church-Turing thesis, a general model of computation, potentially able to execute any algorithm

Turing Machine



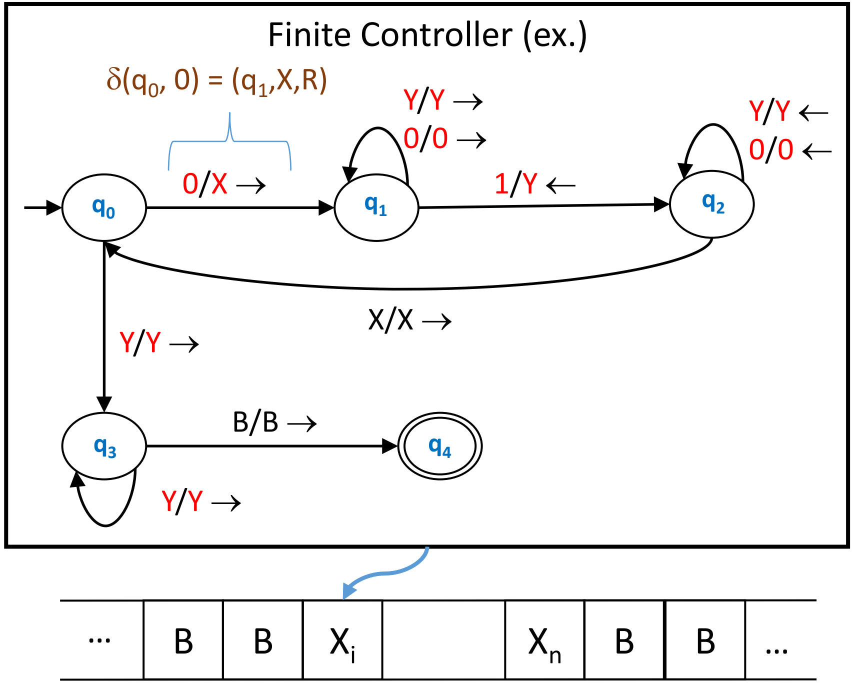
- Finite State Controller
  - Finite number of states
- Tape with infinite length and consisting of cells
  - Each cell can store a symbol
- Input
  - Finite string consisting of symbols of the input alphabet
  - Placed in the tape in the beginning - all other cells are marked with blank (B)
- Symbols in the tape
  - Input alphabet + blank + other symbols if needed

Turing Machine

- Head of the tape
  - Always positioned in a cell
  - In the beginning it is in the leftmost cell of the input string
- Movement or step of the machine
  - Function of the state of the control and of the symbol being read by the head
  - 1. State transition
    - It can be to the current state
  - 2. Write of a symbol in the cell where is the head
    - It can be the same symbol read
  - 3. Movement of the head by one cell left or right
    - A transition always implies a step (left or right) of the head in the tape

Formal Definition

- Turing Machine (TM) is described a 7-tuple, M= $(Q, \Sigma, \Gamma, \delta, q_0, B, F)$
  - Q: Finite set with the control states
  - Σ: Finite set of the input symbols
  - Γ: Finite set of symbols in the tape
  - δ: Transition function $\delta(q, X) = (p,Y,D)$
    - q is a state, X is a symbol in the tape
    - p is the next state (in Q);
    - Y is a symbol in Γ which substitutes X;
    - D is L or R, left or right ($\leftarrow$ or $\rightarrow$), direction in the movement of the head after the substitution of the symbol in the tape
  - q_0: initial state
  - B: blank, a symbol to represent the blank symbol and the tape is fully filled (excepting the cells with the input string) with this symbol
  - F: Set of final or accept states, $F \subseteq Q$



Computations

- Instantaneous descriptions
  - $X_1X_2\ldots X_{i-1}qX_iX_{i+1}\ldots X_n$
  - Cells from the first non-blank to the last non-blank (finite number)
    - With blank suffixes and prefixes depending where the head is
  - The state (q) and the cell (i) where the head is
- Step of the Turing Machine M ($\vdash_M$; $\vdash_M^*$: 0 or more steps)
  - Supposing δ(q,Xi) = (p,Y,L)
  - X1X2…Xi-1qXiXi+1…Xn $\vdash_M$ X1X2…Xi-2pXi-1YXi+1…Xn
    - Transition from q to p; Xi cell is changed to Y; head moves left
    - If i=1: qX1X2…Xn $\vdash$ pBYX2…Xn
    - If i=n and Y=B: X1X2…Xn-1qXn $\vdash$ X1X2…Xn-2pXn-1
    - Symmetric for δ(q,Xi) = (p,Y,R)

Example 0^n1^n

- TM to accept the language $\{0^n1^n \mid n \geq 1\}$
- Idea
  - In the beginning the tape contains the input string (0s and 1s)
  - Change the first 0 to X; move to right until the first 1 and change it to Y; move to left until the first X; move o right; repeat
  - If in a state there is a symbol in the tape not expected, the TM dies
    - If the input is not 0^n1^n
  - If in the iteration that marks the last 0 it also marks the last 1 then it accepts

Example 0^n1^n

State	0	1	X	Y	B
q_0	(q_1,X,R)			(q_3,Y,R)	
q_1	$(q_1,0,R)$	(q_2,Y,L)		(q_1,Y,R)	
q_2	$(q_2,0,L)$		(q_0,X,R)	(q_2,Y,L)	
q_3				(q_3,Y,R)	(q_4,B,R)
q_4					

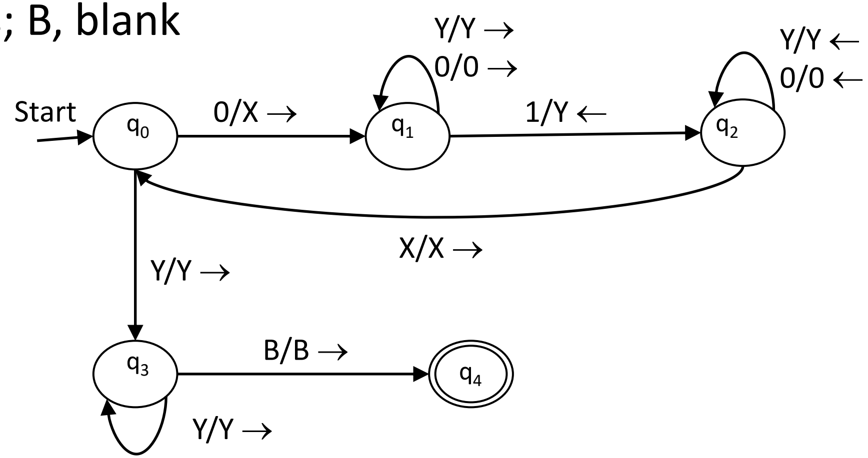
- $M = (\{q_0,q_1,q_2,q_3,q_4\}, \{0,1\}, \{0,1,X,Y,B\}, \delta, q_0, B, \{q_4\})$
  - q_0: changes 0 to X
  - q_1: moves to right until the first 1 and it is changed to Y
  - q_2: moves to the left until it finds an X and goes to q_0
  - If it has a 0 reinitiates the loop; if it has a Y goes to right; if it finds a blank goes to q_4 and accepts; otherwise dies without accepting

Computations for the Previous Example

- Input 0011
  - Initial instantaneous description (ID): $q_0 0011$
  - $q_0 0011 \vdash Xq_1 011 \vdash X0q_1 11 \vdash Xq_2 0Y1 \vdash q_2 X0Y1 \vdash$
  - $Xq_0 0Y1 \vdash XXq_1 Y1 \vdash XXYq_1 1 \vdash XXq_2 YY \vdash Xq_2 XYY \vdash$
  - $XXq_0 YY \vdash XXYq_3 Y \vdash XXYYq_3 B \vdash XXYYBq_4 B$
    - accepts
- Input 0010
  - $q_0 0010 \vdash Xq_1 010 \vdash X0q_1 10 \vdash Xq_2 0Y0 \vdash q_2 X0Y0 \vdash$
  - $Xq_0 0Y0 \vdash XXq_1 Y0 \vdash XXYq_1 0 \vdash XXY0q_1 B$
    - dies

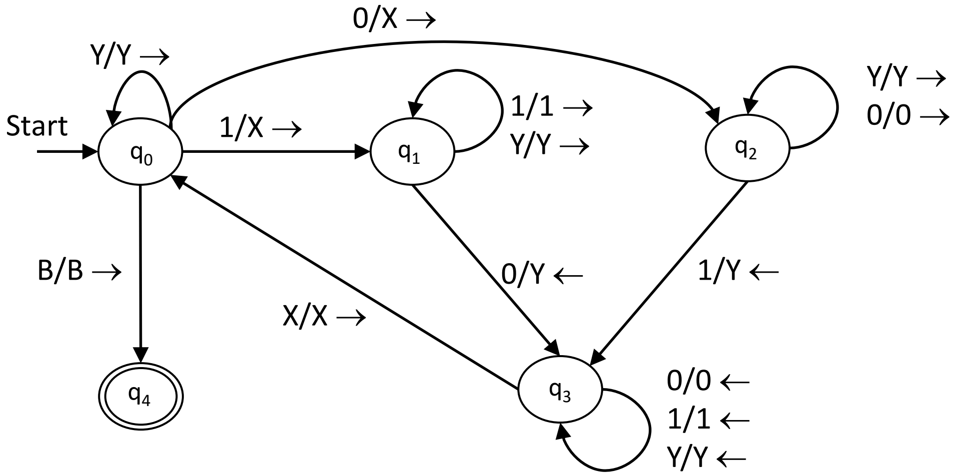
Transition Diagram

- Similar to PDA
  - Nodes are TM states
  - Edge from q state to p state with label X/YD
    - $\delta(q,X) = (p,Y,D)$, X and Y are symbols in the tape and D is L or R ($\leftarrow$ or $\rightarrow$)
  - "Start" arrow indicates the initial state; double circle represents final (accept) states; B, blank



Example

- Transition diagram for a TM which accepts the language of the strings with equal number of 0s and 1s



- $q_0 0110 \vdash Xq_2 110 \vdash q_3 XY10 \vdash Xq_0 Y10 \vdash XYq_0 10 \vdash XYXq_1 0 \vdash XYq_3 XY \vdash XYXq_0 Y \vdash XYXYq_0 B \vdash XYXYBq_4 B$
- $q_0 110 \vdash Xq_1 10 \vdash X1q_1 0 \vdash Xq_3 1Y \vdash q_3 X1Y \vdash Xq_0 1Y \vdash XXq_1 Y \vdash XXYq_1 B$

Example (cont.)

- Basic idea of the behavior of the MT
  - Identify in the tape 0-1 or 1-0 pairs, marking with X the first element and with Y the second element, until cannot find another pair (the case where the TM dies) or find blank (the case where the TM accepts)
- Meaning of the states
  - q_0: goes to right until it finds the first element of the next pair; if it finds a blank goes to accept state
  - q_1: found 1; goes to right until it finds a 0 or dies if it finds a blank;
  - q_2: found 0; goes to right until it finds a 1 or dies if it finds a blank;
  - q_3: found the second element of a pair; goes back to the left until it finds the rightmost X, case in which it transits to q_0;
  - q_4: accept.

Exercise 1

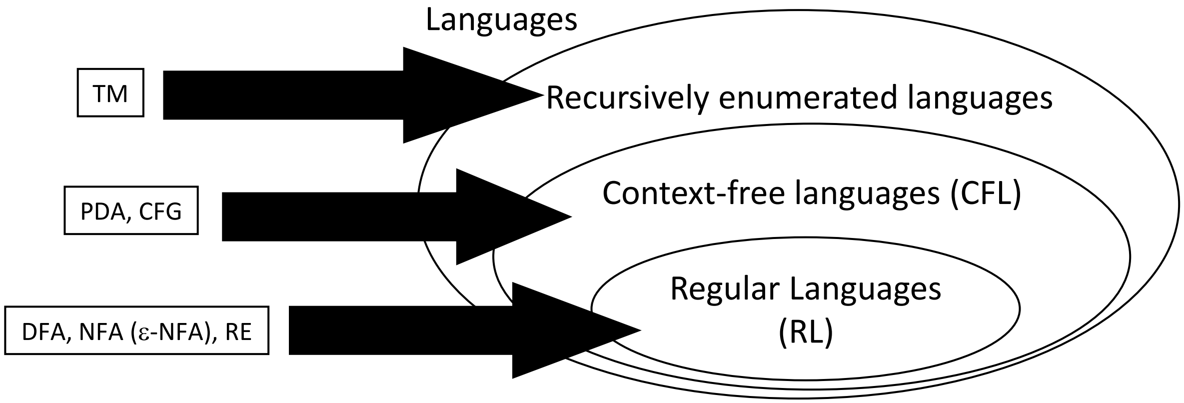
- Design a Turing Machine able to convert an input binary number to its two’s complement representation. In case of overflow, i.e., if the operation produces a number with more bits than the ones in the input number, that extra bit is ignored, maintaining the same number of bits.
  - Describe the meaning of each state.
  - Show the Computing trace when the input is 100.

Exercise 2

- Design a Turing Machine able to decrement by 1 an input binary number.
  - What are the strings accepted by the machine?
  - Where is the head of the machine in the end of the processing?
  - Show the sequence of instantaneous descriptions of the machine when it processes the input 100.

Language of a TM

- Input string placed in the tape
  - Head of the machine in the leftmost symbol of the input
- If the machine stops in an accept state, the input string is accepted
- Language of the TM M= (Q, Σ, Γ, δ, q_0, B, F)
  - Set of strings w in Σ^* such that $q_0 w \vdash^* \alpha p \beta$ and $p \in F$
  - Recursively Enumerated Languages (Turing-recognizable)



Stop Execution

- A TM stops if it enters in a state q, reads a symbol X and $\delta(q,X)$ is not defined
  - It allows to approach a TM as executing a computation with start and end
    - example: calculate the sum of two integers
  - TMs that always stop, accepting or not the input, constitute models of algorithms (recursive languages)
- We can assume that a TM always stops when accepts
- Unfortunately, it is not always possible to enforce that a TM stops when it does not accept (the halting problem)
  - Undecidability (recursively enumerated languages)
  - Possibility of a TM to refer to itself (it can be undecidable)

Techniques to program a TM

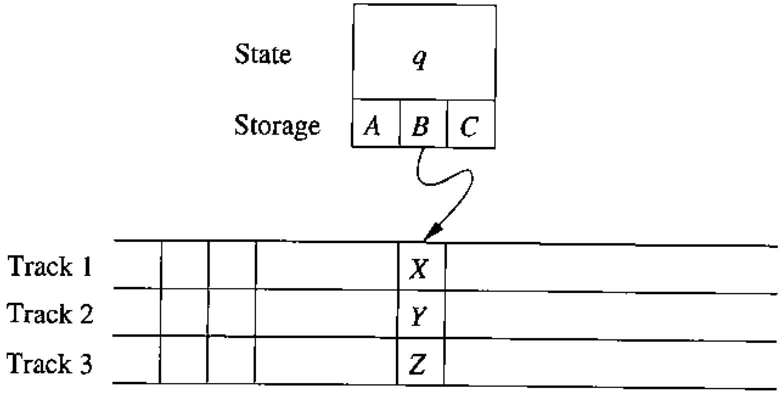
- A TM is as powerful as any actual computer (but too slow)
- Memory in the state
  - State = control + data memory
  - See the state as a tuple
- Additional symbols used in the tape can make easier the programming of the TM

Techniques to program a TM: subroutines

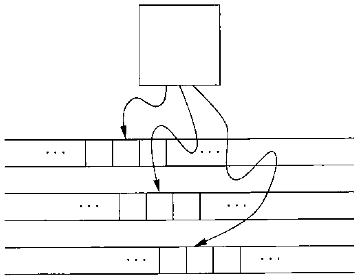
- A TM is a set of states which executes a process
  - It has an input and final states
- Seen as a subroutine of a major TM
  - Call goes to initial state
  - There is not notion of a return address
    - Return is done by using a state
  - If a “subroutine” is called from various distinct states, we copy it (like a macro) and return to the states where was called

Extensions to TMs

- Multiple tracks in a tape
  - A tape consists now by a finite number of tracks: a symbol in each track
  - Alphabet in tape consists of a tuple
- TM power is not changed



Extensions to TMs (cont.)



- TM with various tapes
  - Each one has a head
  - Input string only in the first tape
  - Movement: left, right, stationary
  - Equivalent to one tape
    - Quadratic temporal complexity
- Non-Deterministic TMs
  - Transition function gives a set of tuples (q,Y,D)
  - Equivalent to deterministic
    - Place in the tape a queue with the instantaneous descriptions to be processed
    - Simulate the non-determinism traversing them by breath first order
- They do not empower in terms of language recognition

Restrictions

- Machines with several stacks
  - A PDA with two stacks is equivalent to a Turing Machine
    - In the first stack we store what is in the left of the head of the TM
    - In the second stack we store what is on the right of the TM (the top contains the current symbol read by the head)
    - In the controller we simulate the control of the TM
    - Movement of the head of the TM is to do a pop in a stack and a push in the other

Restrictions (cont.)

- Machines with counters
  - The same structure of a machine with multiples stacks
  - Each stack is a counter
    - Contains a non-negative integer (number of X in stack)
    - We only distinguish if the counter is 0 or different of 0
  - Movement depends
    - State
    - Input symbol
    - Each one of the counters is 0
  - In the movement
    - Change state (or stay in the same)
    - Adds or subtracts 1 to each counter independently

Power of the machines with counters

- Theorem: the machines with 3 counters are equivalent to the TM, i.e., they accept the recursively enumerated languages
- A counter simulates a stack
  - A stack $X_1X_2 \ldots X_n$ in an alphabet with r-1 symbols can be seen as a number in base r $X_n r^{n-1}+X_{n-1}r^{n-2}+\ldots+X_2 r+X_1$
  - Pop is to divide by r and remove the remainder (X_1)
- Two counters for 2 stacks ($\equiv$ TM) and another one for the multiplication and division operations
- Theorem: the machines with 2 counters are equivalent to the TM
  - Code the 3 counters i, j, k in a counter $2^i 3^j 5^k$ (2,3,5 are primes)
  - Second counter for operations

TMs and computers

- A Computer is able to simulate a TM
  - The number of states and transitions is finite: states represented by strings and table of transitions
  - Number of symbols in the tape is finite and strings have fixed length
  - Tape infinite!
    - If the memory of the Computer is finite, it is a finite automaton and only accepts regular languages
    - Assuming the capability to change disks: it is reasonable to consider infinite memory
      - Stack of disks in the left and in the right of the tape: disk in use according to the position of the head
      - Recursively enumerable language

TMs and Computers (cont.)

- Simulate a Computer with a TM
  - memory: long sequence with words with an address
  - Program stored in memory
    - Simple instructions, assembly
    - Consider indirect addressing
  - Each instruction uses a limited number of words and changes one word at the maximum
  - Registers of the Computer considered as more memory
- TM with multiple tapes simulates a computer ($\equiv$ TM with 1 tape)
  - Memory (address-value pairs) + Instruction counter + memory address + Input file + area of temporaries
  - Simulate the instruction cycle; copy, sum, jump, …

The multi-tape TM simulates n steps of a Computer in $O(n^3)$ of its own steps; and a TM simulates n steps of a Computer in $O(n^6)$ TM steps [See Hopcroft book]

Universal Turing Machine

Universal Turing Machine

- A Turing Machine that is able to interpret a TM description and an input in their tape, i.e., a “programmable” Turing Machine

- Theorem:

The language:

$$A_{TM} = \{ < M, W > | \, M \, is \, a \, TM \, and \, M \, accepts \, W \}$$

is Turing *recognizable*

(but not Turing *decidable*)

Summary

Summary

- The TM is a valid representation of what a Computer can compute
- It is realistic to consider the TM as equivalent to a Computer
  - Relation between execution times is polynomial
  - The division between tractable and intractable problems is between the polynomial and greater than polynomial complexity
  - It allows to study the efficiency of the algorithms in the TM and not only the decidability

Selected videos

View the "Alan Turing at Bletchley Park" video via the IEEE Computer Society's YouTube channel at: http://www.youtube.com/watch?v=5nK_ft0Lf1s&list=PL54E3809040EE1338.

Turing machines explained visually https://www.youtube.com/watch?v=-ZS_zFg4w5k

Universal Turing Machine Theorem: https://www.youtube.com/watch?v=uM_Q-btwGms

Turing completeness

- What is the meaning of “Turing Complete”?
- What is the meaning of “Turing Equivalent”?