Theory of Computation

MIEIC, 2nd Year

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Outline

- Introduction to the topics of the course
- Concepts about Automata
- Proof method by induction

History (part of...)

- Automata theory: study of abstract computing devices [*machines*]
- Alan Turing (1930's)
  - Studied the limits of an abstract machine - equivalent to the current real ones!
  - Before the existence of computers!

- 1940's, 1950's
  - Study of finite automata to model the human brain
- Noam Chomsky (1950's)
  - Formal grammars – related to abstract automata and very useful in compilers

- Stephen Cook (1969)
  - Complexity theory – what is feasible or not to compute

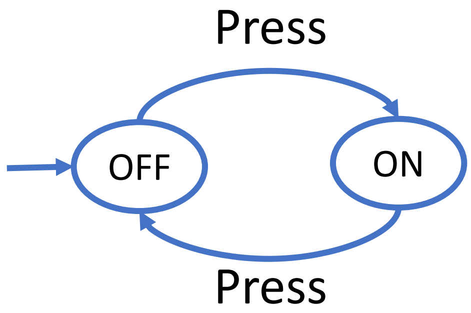
"Let us now return to the analogy of the theoretical computing machines ... It can be shown that a single special machine of that type can be made to do the work of all. It could in fact be made to work as a model of any other machine. ***The special machine may be called the universal machine*** *..."*

— Alan Turing 1947

Relevance of the Automata Theory

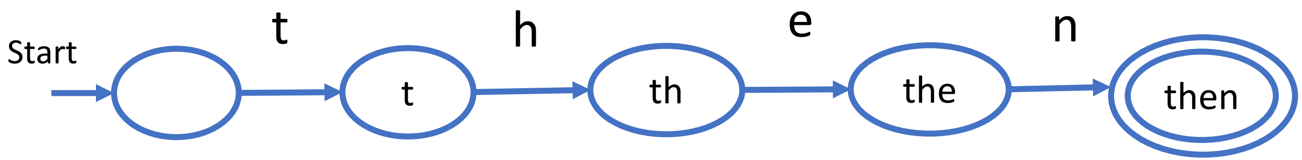
- Useful to model hardware and software
  - Design and test of digital circuits
  - Lexical analysis in compilers
  - Text processing, web search
  - State machines, communication protocols, security, cryptography, analytics, etc.
- Finite Automaton
  - System that in each instant is in one of a finite number of states
  - State memorizes the part relevant of the history of the system
  - Being finite, it needs to forget what is not relevant
  - It can be implemented with finite resources

Example of an Automaton: on/off switch



- Simple finite automaton – models switch
  - Two **states** [circles]: on and off
  - Only one **input** [labels in edges]: Press
    - Represents the external influence on the system [state transition]
    - Push button has an effect dependent of the state
  - **Initial state** represented by an arrow with label Start
  - There can exist one or more **final (acceptance) states**, represented by double circles

Example of an Automaton: recognizer



- If the input is the string *"then"* the automaton goes from the initial to the final state
  - Accumulate the history of the input
  - The goal is to recognize the string "then"

Structural Representations

- Regular Expressions
  - Describe the structure of strings
  - Example: [1-9][0-9][0-9][0-9][-][0-9][0-9][0-9][ ][A-Z][a-z]*
    - Describe “4200-465 Porto”, but not “5505-032 Vila Real”
    - Correction: [1-9][0-9][0-9][0-9][-][0-9][0-9][0-9][ ][A-Z][a-z]*([ ][A-Z][a-z]*)*
    - Correction does not describe, e.g., “Vila Nova de Gaia”!
- Grammars
  - Process data with recursive structure [expressions]
  - Example of a grammar rule: $E \rightarrow E + E$
    - One expression may consist of two expressions “connected” by “+”
  - Used in syntactic analyzers [parsers], e.g., in compilers

Proof Methods

- Formal proofs are important for informatics engineering
  - E.g., for demonstration of correctness of a given algorithm
- Statements
  - **if ... then**
    - if A then B ($A \to B$)
  - **if and only if – iff**
    - A iff B ($A \leftrightarrow B$, prove: $A \to B$ and $B \to A$)

Proof Methods

- There are several proof methods (e.g., by deduction)
- if H then C ($H \rightarrow C$)
  - By contradiction (reduction to absurdity): H and not C implies falsehood
  - By counter-example: show an example that proves the proposition is false
  - By counter-positive : if not C then not H (proving one is proving the other)
  - By induction (see the following slides)

Proof by Induction

- Proving a statement S(n) over an integer n (or a structure defined inductively, such as a tree or a graph)
  - Basis (base step): prove S(i) for some small i's, typically i=0 or i=1
  - Inductive step: assuming by **hypothesis** that S(k), k=n, is true, show that S(k+1) holds
  - Being k general, the property verifies for all k (and n!)
- Elements of an inductive proof
  - Structure over which we apply induction
    - Integers, trees, graphs, sets, strings, etc.
  - Statement S(n) which we intend to prove (*n* is de step)
  - Base case (basis)
  - Induction/inductive step

Induction proofs

- The principle of induction
  - If we prove S(i) and prove that for $n \geq i$, S(n) implies S(n + 1), then we can conclude that S(n) is true for any $n \geq i$

Example 1 (proof)

- Proof:

  Statement S(n): the sum of the first n natural numbers, i.e., 1+2+..+n, is n(n+1)/2

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Statement S(n): the sum of the first n natural numbers, i.e., 1+2+..+n, is n(n+1)/2

Basis: the first natural number is 1 (sum is equal to 1) and 1(1+1)/2 = 1 [basis is true]

Induction step: Let k be a natural number for which S(k) is true

S(k): the sum of the first k natural numbers is k(k+1)/2, by **hypothesis**

S(k+1): sum of the first k+1 natural numbers is (k+1)(k+2)/2

The sum of the first k+1 natural numbers is 1+2+…+k+(k+1)

as by hypothesis 1+2+…+k = k(k+1)/2, then 1+2+…+k+(k+1) = k(k+1)/2 + k+1

(k+1)(k/2 +1) = (k+1)(k+2)/2

which is exactly the expression of the sum of the first natural numbers given by the expression in the statement S(k+1) = (k+1)(k+2)/2

Q.E.D. (*quod erat demonstrandum*)

Widening the scope of the concept

- To prove statements of the form:
  - $\forall n\ [P(n) \rightarrow S(n)]$
- Induction necessary when P(n) is based on an inductive definition

Example 2

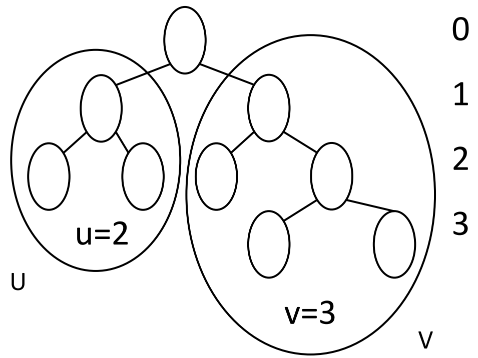
- Consider the following inductive definition of quasi-complete binary tree (qcBT)
- Inductive definition of a quasi-complete binary tree (qcBT):
  - An isolated node is an qcBT
  - If U and V are qcBT, then a node with U and V as children is an qcBT
- Prove that a quasi-complete binary tree with k leaves has 2k-1 nodes

Example 2 (proof)

- Prove that a quasi-complete binary tree (qcBT) with k leaves has 2k-1 nodes
- Proof based on the structure of the tree:
  - Structure: set of quasi complete binary trees
  - n step: quasi complete binary trees with height n
  - We could have selected the number of nodes, but we preferred to use the height

Example 2 (proof)

- Statement S(T): if T is a qcBT with k leaves then T has 2k-1 nodes
- Basis: an qcBT with height 0, only root, has 1 leaf and 2x1-1=1 node
- Induction step: Assume S(U) for the qcBT of height until n and in particular for the subtrees of T
  - T is an qcBT with height n+1 with root and two ABqc subtrees U and V (at least one of height n)
  - If U and V have u and v leaves, respectively, then T has t=u+v leaves
  - By hypothesis U and V have 2u-1 and 2v-1 nodes, respectively
  - By the definition of the tree, T has 1+(2u-1)+(2v-1) = 2(u+v)-1 = 2t-1 nodes
  - So, S(T) is true
- **Important**: we consider that the hypothesis is true for all the cases $\leq$ n



Exercise 1

- Prove that for any natural number n, the sum of the first n squares is $n(n+1)(2n+1)/6$

Exercise 2

- Prove that for any natural number x greater of equal than 4, $2^x \geq x^2$

Exercise 3

- Prove that the sum of the first n perfect cubes is a perfect square.
  - Examples:
    - $1^3+2^3+3^3=36=6^2$
    - $1^3+2^3+3^3+4^3+5^3=225=15^2$
- Solution:
  - Induction using integers
  - Statement: $\sum_{i=1}^{n} i^3 = a^2$
  - Basis: n=1
    - $a=1,\ 1^3 = a^2 = 1^2$
  - Induction step
    - $\sum_{i=1}^{n+1} i^3 = \sum_{i=1}^{n} i^3 + (n+1)^3 = a^2 + (n+1)^3 = b^2$ Which b?

Exercise 3: Inventor's paradox

- Solution: reformulate the statement to prove in order to make it stronger
  - Instead of “one” perfect square, say which is “the” square: the sum of the numbers
  - Prove that exists one and we identify it, “invent” an extra restriction which serves to proceed with the proof $\rightarrow$ Inventor's paradox
  - New statement: $\sum_{i=1}^{n} i^3 = (\sum_{i=1}^{n} i)^2$
  - Induction step (basis: the same as before)
    - $\sum_{i=1}^{n+1} i^3 = (\sum_{i=1}^{n+1} i)^2$ objective
    - $\sum_{i=1}^{n} i^3 + (n+1)^3 = (\sum_{i=1}^{n} i + (n+1))^2$ algebra
    - $\sum_{i=1}^{n} i^3 + (n+1)^3 = (\sum_{i=1}^{n} i)^2 + 2(\sum_{i=1}^{n} i)(n+1) + (n+1)^2$ algebra
    - $(n+1)^3 = 2(\sum_{i=1}^{n} i)(n+1) + (n+1)^2$ hypothesis
    - $(n+1)^3 = 2(n/2)(n+1)(n+1) + (n+1)^2$ sum of the arithmetic series
    - $(n+1)^3 = n^3+3n^2+3n+1$ Q.E.D.

Example 4 – Balanced parenthesis

- Two definitions of balanced parenthesis:
  - Grammatically (EG)
    - The empty string ε is balanced
    - If w is balanced then “(w)” is balanced
    - If w and x are balanced then wx is balanced
  - By scanning (EV)
    - w is balanced if and only if (*iff*)
      - Has an equal number of ‘(‘ and ‘)’
      - Each prefix of w has at least as many ‘(‘ as ‘)’
- Theorem: a string of parenthesis is EG ***iff*** is EV
  - Bidirectional proof

Example 4 – balanced parenthesis (proof)

- EG ← EV
  - Proof by induction base on the length of the string w (+ conditional proof)
  - Basis: $w = \varepsilon$, $|w| = 0$
    - $w = \varepsilon$ é EG, by the first rule
  - Induction step
    - For $|w|=n+1$ there are two cases
    - I) w does not have a non-empty prefix with the same number of (and)
      - Then w must begin with (and finish with), i.e., w = (x)
      - x must be EV→ |w| even
      - $|x| <= n$, so, by hypothesis x is EG
      - By the second rule, w = (x) is also EG
    - II) w has a non-empty prefix with the same number of (and)
      - Then w = xy, in which x is the shorter of those prefixes and $y \neq \varepsilon$
      - x and y are EV; by hypothesis, x and y are EG
      - By the third rule w is EG

Example 4 – balanced parenthesis (proof)

- EG $\rightarrow$ EV
  - Prove by induction based on the structure EG of the string w, i.e., in the number of applications of the rules of the EG definition (+ conditional proof)
  - Basis: w = ε, n = 1, first rule of EG
    - w = ε is EV (trivial)
  - Induction step
    - For n+1 applications of EG rules there are two cases
    - I) w is EG because of the second rule, i.e., w = (x) and x is EG
      - Then, by hypothesis, x is EV
      - As x has the same number of (and), (x) also has
      - As x does not have prefix with more) than (, (x) also does not
    - II) w is EG because of the third rule, i.e., w = xy and x and y are EG
      - By hypothesis, x and y are EV (rigorously, the hypothesis is EG $\rightarrow$ EV for a number of rules $\leq$ n)
      - As x and y have equal number of (and), w also has
      - If w had a prefix with more) than (, then or x would have such a prefix (in contradiction for being EV) or would have it x followed by a prefix of y (in contradiction to y being EV) (*proof by contradiction*)
    - Q.E.D.

Summary

- Introduction to the Theory of Computation
- Introduction to finite automata
- Proof methods with emphasis on the proofs by the induction method (revision)