# P01: Exercises about Proofs by Induction **[SELECTED]**

Solutions for the selected exercises: 1, 2, 3, 4

---

1.

Hypothesis: $1 + 3 + 5 + \ldots + (2n+1) = (n+1)^2$

Base case: $n=0$: $1 = (0+1)^2=1$ ✓

Induction step: n+1

$1 + 3 + 5 + \ldots + (2n+1) + (2(n+1)+1) = ((n+1)+1)^2$

*By hypothesis:*

$(n+1)^2 + (2n+2+1) = (n+2)^2$

$n^2 + 2n + 1 + 2n + 3 = n^2 + 4n + 4$

$n^2 + 4n + 4 = n^2 + 4n + 4$ *qed*

---

2.

Definition:

1. Simple node is a tree.
2. T1, T2, …,Tk trees, structure with a new node N and T1, T2, …, Tk as the tree's children.
3. Nothing more is a tree.

**Hypothesis**: number of nodes is 1 unity higher than the number of edges.

$$\forall a\, (Tree(a) \rightarrow (N(a) = E(a) + 1))$$

**Induction proof**:

Case base: when T is a simple node, N=1 and E = 0, therefore it is verified that N=E+1.

Induction step:

Consider T a tree built by the induction step (2) see in the definition.

By hypothesis, S(Ti) is verified, by other words, Ti has Ni=Ei+1.

The T nodes are the N and the nodes of all the Ti's. Then, there are 1+N1+N2+…+Nk nodes in T.

The T edges are the k edges added in the inductive definition plus the edges of Ti's. Therefore, T has k+E1+E2+…+Ek edges .

Replacing Ni for Ei+1 we have that T has

$$1 + (E1+1) + (E2+1) + \ldots + (Ek+1)$$

nodes. Since there are k terms "+1" in the expression, we have:

$$1 + k + E1 + E2 + \ldots + Ek$$

which can be interpreted as 1 unity more than the number of edges in T.

Thus, the number of nodes in T is the number of edges plus 1.

3.

Definition:

1. Letter is pal
2. If $\alpha$ is pal, then the result of $l\alpha l$ is pal, with 1 letter.
3. Nothing more, except the results 1 and 2, are pal.

**Hypothesis**: all pal is a palindrome

$\forall x\ (Pal(x) \rightarrow Palindrome(x))$

**Induction proof**:

Case base: each letter of the alphabet is a pal, and also a palindrome: one letter can be read as the same from left to the right and the right to the left.

Induction step:

Consider a p a pal with length c. By hypothesis, p is a palindrome. To obtain a pal with length higher than c, it's necessary to use second rule: add the same letter to the beginning and the end of the p:

$p_{c+2} = lpl$

If p is a palindrome, $p_{c+2}$ is also a palindrome.

*A palindrome starts and ends with same character. Thus, if w is a pal (and, by hypothesis, a palindrome) and from it we create (by the 2), XwX (where X represents a letter), we also obtain a palindrome (keeping in mind that a letter is also a character).*

However, the opposite is not true:

- Palindromes with even number of characters are not pal; the definition would have to include ε (empty) as pal[1].
- Not all the characters are letters: the definition would have to open the application context to include characters, not only letters.

---

4.

ATM machine only has 20 and 50 bills.

Show that we can supply an amount multiple of 10, ≥40

**Structure**: the amount multiple of 10 and ≥40

1. 40€ is an amount
2. If q is an amount, q+10 is also.
3. Nothing more is an amount.

**Hypothesis**: all amount can be composed by 20€ and 50€ bills.

$\forall q\ (Amount\ (q) \rightarrow \exists x \geq 0,\ y \geq 0\ q = x*20+y*50)$

[1] Considering that the empty *string*, ε, is a palindrome. In case we don't consider ε a palindrome we will be able to add the following rule: two equal letters are a pal.

**<u>Proof by induction</u>**:

<u>Base case</u>: the <u>amount</u> is 40

$40 = 2*20 + 0*50$ ✓

Since there is not an amount less than 40:

- x and y cannot be 0 at the same time.
- if y is 0, x is $\geq 2$

[Here we could use the 'inventor': make the property stringer, including those 2 conditions.]

<u>Induction step</u>:

<u>Amount</u> q verifies a property:

$$q = k1*20 + k2*50$$

<u>Amount</u> q+10 is:

$$q+10 = k1*20 + k2*50 + 10$$

Cases:

- $k2 \neq 0$

  If k2 is at least less than 1 we can rewrite q+10:

  $$q+10 = k1*20 + (k2-1)*50 + 60 =$$
  $$= (k1+3)*20 + (k2-1)*50$$

  Extra properties are kept: if k1 was 0 and is no longer (a $\geq 3$); if k2 became 0 k1 is no longer 0.

- $k2 = 0$

  Then k1 is at least 2

  $$q+10 = (k1-2)*20 + 1*50$$

In both cases, q+10 is also composed by bills of 20 and 50.

Since all these amounts use the second rule, the property is proved.