Walter B. Rudin: “Set Theory: An Offspring of Analysis” (1990)

Newton, Leibniz, late 17th century. Hard problems of calculus of variations, mathematical physics. Problem of the vibrating string.

Deform a string and let it go. How does it move?

Euler, Bernoulli, D’Alambert. Lagrange. 1750 controversy.

Multiple deformations in the string. Differential equation (wave equation) describes the height y_x of a point in the string at position x at time t after it’s released.

$$\frac{\partial^2 \mathrm{y}}{D\,\mathrm{x}^2} = \alpha^2 \frac{\partial^2 \mathrm{y}}{D\,\mathrm{t}^2} \tag{1}$$

α is a number which depends on the elastic property of the string.

Suppose you know that

$$y = f(x) \tag{2}$$

at time $t = 0$. What the shape of the string is at some later time? If the differential equation (1) can be solved for the initial condition (1) there is an answer. There are some functions $f(x)$ (the initial shape of the string) for which the solution to (1) can be written quite explicitly. Such simple functions are:

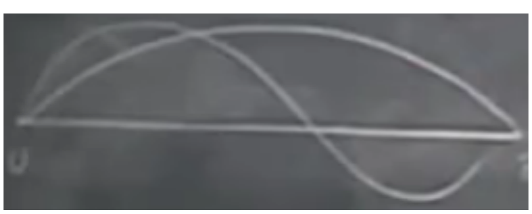
- The trigonometric functions

Graph of $\sin(x)$. If the initial shape of the string is this, an explicit solution can be written.

Graph of $\sin(n\,x)$ (n positive). Also.

If there are taken sums or (linear combinations?) $n\,f_1(x) + n\,f_2(x) + \ldots + n\,f_m(x)$ of those functions, also:

$$\sum_{n}^{c} \sin(n\,x). \tag{3}$$



Not all functions can be written as such finite sum.

Suppose an infinite sum $\sum_1^\infty c_n \sin(n\,x)$ of functions. We can still write down the solution.

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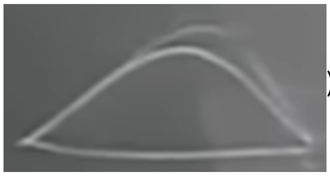
The controversy: But what is meant by this infinite sum (convergent series of signs)? Which functions can be written like this?

Bernoulli: any function, as long as its first and last points are 0.

Objection: these functions are differentiable. Not all functions are differentiable, for example, a “triangle”. (D’Alembert could not see that it could be approximated successively by ).

But the concepts were not well-defined. There was no consensus as to what “continuous” meant.

100 years later, a new objection. It was not conceivable that the same expression or formula could give a result on one interval and another result on another interval. To **Euler**, a function meant something that could be written as a formula. His objection was that first, for something to qualify as (1), it has to be an odd (?) function, or else the sign changes. And second, it has to be a *periodic* function. Period 2π. But take a parabola from a p_0 to a p_1 which coincides, in this intervals, with this periodic function’s points.

It is not a periodic function.

Controversy described by R. Langer book in 1947 by the Monthly.[1]

Soon after the time of the controversy, Fourier came to the same problem on heat conduction (instead of the string problem). Initial temperature distribution on a material, how does the temperature change.

$$\sum_{1}^{\infty} c_n \cos n x + d_n \sin n x \tag{4}$$

For Fourier, any function, he'll find the coefficients (c_n, d_n?), and the <u>series will converge to the function</u>. Euler already knew Fourier’s formulas for doing such. For Euler, the formulas only apply when the function is a sum of a trigonometric series. Fourier never proved, just ‘knew’ it worked, by examples.

Dirichlet (1829) first theorem. If f is a function with only finitely many local maxima and minima, it works.

Take the function, compute the Fourier coefficient, obtain a series; if the series converges, “the function at every point” (?).

In a function with a jump, the Fourier series will converge to the midpoint of the jump.

If the series is to converge **at every point** to the given function, it's "asking too much".

Also, requiring continuity of the function is not enough to assure convergence at every point in the function.

Dirichlet: there is a class of functions, not all functions. But the class is large. Gave the definition of **function** that's used now. Two sets: associate members of sets (in modern language). Correspondence.

Process up to now function ⟶ Fourier coefficient ⟶ Fourier series ⟶ convergence of the resulting series.

Riemann at the start of his paper (which?) defines **integral**. Had been around for 200 years, but Riemann defined a class of integral functions. The Riemann integral, now taught to undergrates in calculus. Questions: what functions can be sums of trigonometric series? Which functions can be written like (4) and can there be other reprentations? Maybe obtain the coefficients in another way? Never answered the question, but introduced a method.

Start with a series, instead of a function.

$$A_n(x) = a_n \cos n x + b_n \sin n x \tag{5}$$

Starting with a series $\sum A_n(x)$. Determine whether it **is** the Fourier series of the given function.

If the series converges, we know for every x the $\lim_{n\to\infty} A_n(x) = 0$. Riemann noticed this doesn't imply the a_ns in the coefficients tend to 0. Or, it does imply, but he realized it wasn't obvious.

If $a_n \to 0$ and $b_n \to 0$.

Things which are obvious now were difficult then.

$F(x) = \sum \frac{A_n(x)}{n^2}$.

$\lim_{h\to 0} \frac{F(x+h)+F(x-h)-2F(x)}{h^2} = \sum A_n(x)$.

Uniqueness theorem. If the series converges to f, it is a Fourier series.

Cantor (1870). If $\lim_{n\to\infty} A_n(x) = 0$ for all x in some interval, then $a_n \to 0$ and $b_n \to 0$ as $n \to \infty$.

Several people claimed to have proved this, but they always assumed that the limit was uniform. The proof is not hard (left as an exercise), but it was a difficult theorem at the time.

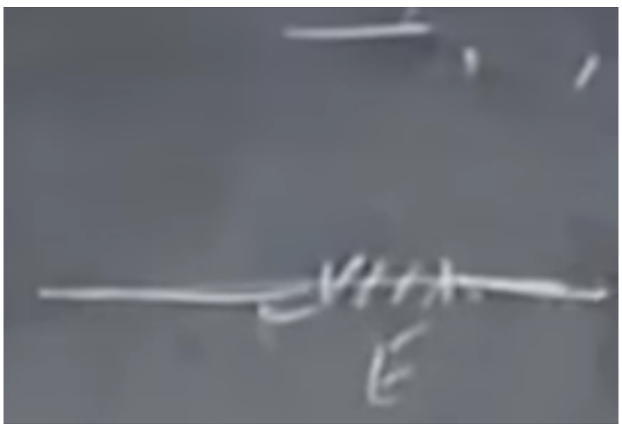
As a corollary, if $\sum_1^\infty A_n(x) = 0$ for every x, then $a_n = 0$ and $b_n = 0$ for all n.

In other words, the only trigonometric series which converges to 0 in any point is the series whose all coefficients are 0.

Suppose there are two series, $\sum A(x)$ and $\sum A'(x)$. If both converge to the same sum, then they're **term-to-term equal**. The **weakness** (?) theorem.

Taking a set E in the line in the interval of length 2π. If $\sum_1^\infty A(x) = 0$ *outside* E (for all x not in E), then $a_n = 0$ and $b_n = 0$. Trying to prove that the empty set is a U-set. (U for "uniqueness".)

Suppose a real interval (E). This is *not* a U-set. Take the function with a line $y > 0$ in the interval and 0 outside. By Dirichlet's theorem, the Fourier (?) series converges to zero in the 0 parts (outside the interval), but the series is not identical to 0.



So to be an U-set means to be a "small" set.

Quote from proof: "This cannot be proved, as commonly assumed, by term-to-term integration". Because term-to-term integration is not under your control.

One year later, proved that every finite set is a U-set.

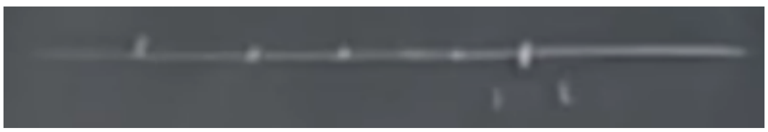
Taking finite sets of points:

It's known that the trigonometric series progresses to zero except inside the three central points. It's not know whether it converges, or whether it converges to zero. If there is convergence to zero, except at those (center) points, then it must be the zero series.

Used the Riemann function. The function $F(x)$ can't **take any corners**.

1822. Where it began to happen. ("Algebraic numbers" paper.) Suppose a set with one limit point.

In the paper, defined the notion of limit point. A limit point is a point such that every interval centered at this point contains infinitely many points of the set.



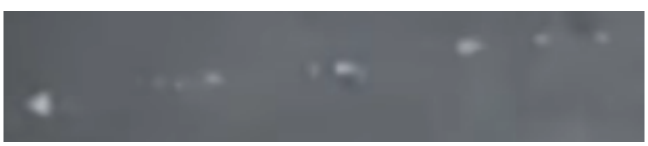
The definition is still used.

He also defined the concept of real number. A definition by Cauchy sequences of rational numbers; a real number can be thought of as an equivalence class of Cauchy sequences of rational numbers.

Dedekind also defined real numbers in a satisfactory way almost at the same time, by **cuts** in the rational numbers.

New notation. If E is a set, $E(1)$ is the set of all limit points. Called the derived set; still sometimes used.

Once with $E(1)$, can form $E(2)$. Each point in the sequence can itself be a **limit point of sequences**. Take the whole set, remove the (?) points, left with (?) points, and so on.



Take a set ⟶ take limit points ⟶ take limit points of that: $E \longrightarrow E(1) \longrightarrow E(2)$.

Until $E(n)$. Suppose $E(n)$ (or this process) is finite: **the next one $E(n+1)$ would be the empty set**. (A set of "type n".) Suppose all of these are U-sets.

1884. Even further in the same direction. Take every set E with countably many limit points in a U-set. Led to the countable ordinals.

Take a sequence of points converging to the last one.

Take a set of type 1 in the beginning (of the line) (in the first segment). Take a set of type 2 in the second segment. Take a set of type n in segment n.

When the (?) points are removed, one by one, “things” are lost in the middles, but you never “wipe out everything”. Call this remaining set of type ω (the first infinite ordinal). (Ordinals are the numbers that we can count — ω is the first infinite one.) Extend the process. Think of $2\,\omega$, $n\,\omega$. Gets to all countable ordinals.

A little about countable. 1874. Cantor proved that the set of all algebraic numbers is countable. The proof is not difficult. The interesting thing was that Cantor derived from that statement (about countability of all algebraic numbers) the existence of numbers which are not algebraic. Called transcendental numbers. Known earlier, but gave a different proof of it.

The first time a set was exhibited which was *not* countable. What really proved was that, looking at any interval, this set of numbers is uncountable. It isn’t possible to establish a one-to-one correspondence between the points in such an interval and the naturals.

He **didn’t** use the Cantor diagonal process. Used another method, thinking “quite” geometrically.

Suppose there is a sequence of numbers, between 0 and 1. To show that this sequence cannot cover the whole interval, there are points not in that sequence.

In the sequence, pick consecutive sequences of numbers inside ever-closer pairs of points.

There is a point (pointed at) which is *not* in the sequence (0 to 1) because it would be in one of the intervals. (Instead of choosing points for the intervals, the first that can be taken are always taken in the procedure.) ***And if that point were in the set, you would’ve picked it***.

Two types of infinity, countable and uncountable. Then, wrote papers on topology: infinite subsets of the line.

About the U-set problem. Which sets are of uniqueness? Still an open problem. Cantor sets let to Borel sets, Lebesgue integration.

Maybe every set of *measure* 0 is a U-set? Stream of papers mostly from Soviet Union and Poland showing more and more sets which were or were not sets of uniqueness.

Salem, about 1950.

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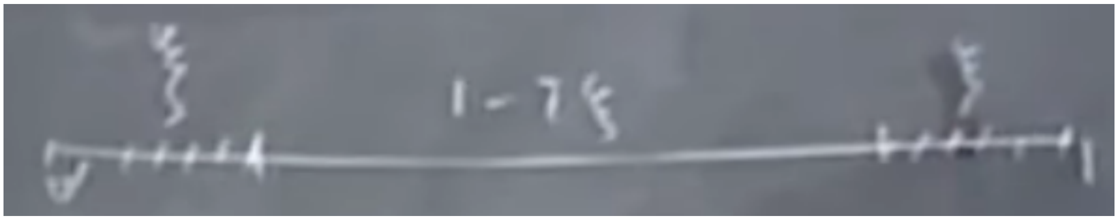

Subdivide the first segment in the same way, keeping the same ratios. Take an intersection. Gets a set E which depends on the number ξ. The theorem is E is a U-set, iff, $\frac{1}{\xi}$ is an algebraic integer all of which conjugates have absolute value < 1.

An algebraic number is a root of a polynomial whose leading coefficient is 1. The other roots are the conjugates.

These are Pisot numbers. Very strange properties.

In the “algebraic numbers” paper (where defines the real numbers), also raises the question now know as **“continuum hypothesis”**. The continuum hypothesis was proved to be consistent with the other axioms of set theory by Godël (1930s) and Paul Cohen (1963?) proved the denial. This started a new period in set theory. His method (Paul’s) revolutionized the field completely.

1 Rudolph E. Langer. The American Mathematical Monthly,Vol. 54, No. 7, Part 2: Fourier's Series: The Genesisand Evolution of a Theory (Aug. - Sep., 1947), pp. 82-86