Fractional space

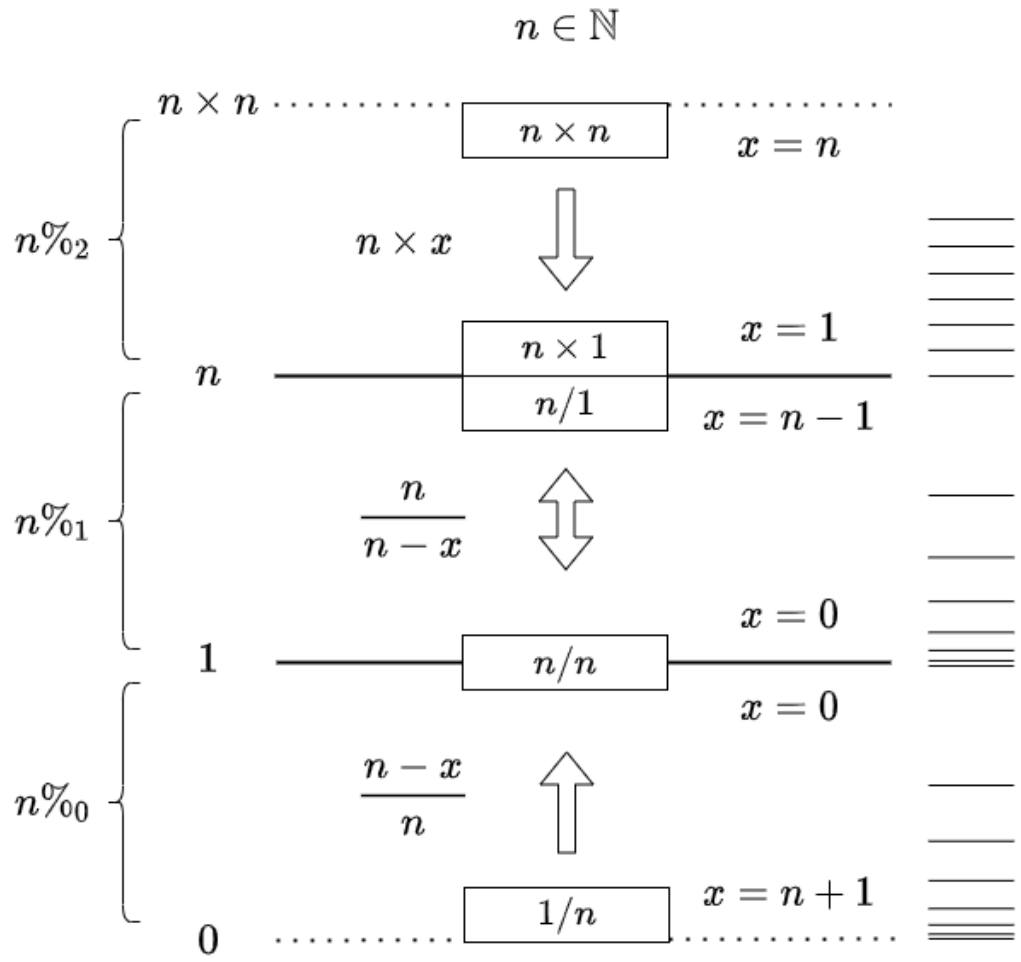
Let a fractional space be the union of the finite sequences

$\left\{\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n}\right\}$, or $n\,\%_0$, where $0 < n\,\%_0 \leq 1$

and

$\left\{\frac{n}{n}, \frac{n}{n-1}, \ldots, \frac{n}{1}\right\}$, or $n\,\%_1$, where $1 \leq n\,\%_1 \leq n$

for a specific $n \in \mathbb{N}$.



Let's describe each sequence.

$n\,\%_0$

In[1]:=
```
n0Display=.;n0Display=Function[{x},Table[DisplayForm@FractionBox[n,x],{n,1,x}]];
```

In[•]:=
```
n0Display@10
```

Out[•]=
$$\left\{\frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \frac{4}{10}, \frac{5}{10}, \frac{6}{10}, \frac{7}{10}, \frac{8}{10}, \frac{9}{10}, \frac{10}{10}\right\}$$

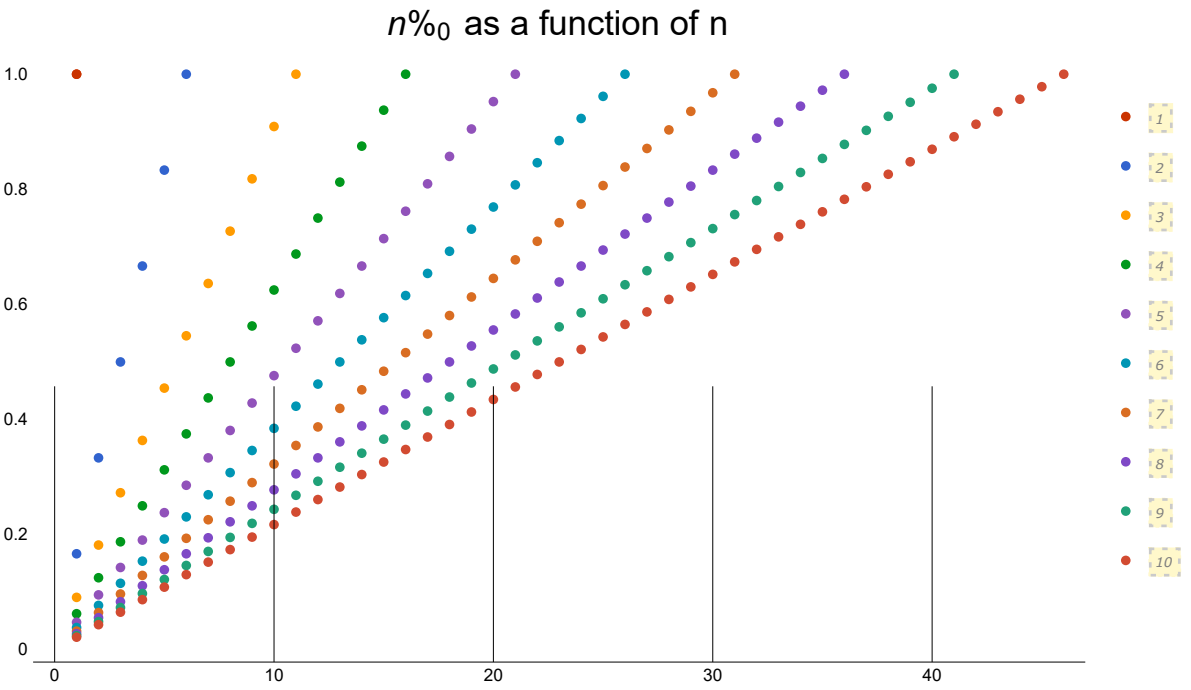
In[•]:=
```
n0Display@20
```

Out[•]=
$$\left\{\frac{1}{20}, \frac{2}{20}, \frac{3}{20}, \frac{4}{20}, \frac{5}{20}, \frac{6}{20}, \frac{7}{20}, \frac{8}{20}, \frac{9}{20}, \frac{10}{20}, \frac{11}{20}, \frac{12}{20}, \frac{13}{20}, \frac{14}{20}, \frac{15}{20}, \frac{16}{20}, \frac{17}{20}, \frac{18}{20}, \frac{19}{20}, \frac{20}{20}\right\}$$

In[2]:=
```
n0=.;n0=Function[{x},Table[n/x,{n,1,x}]];
```

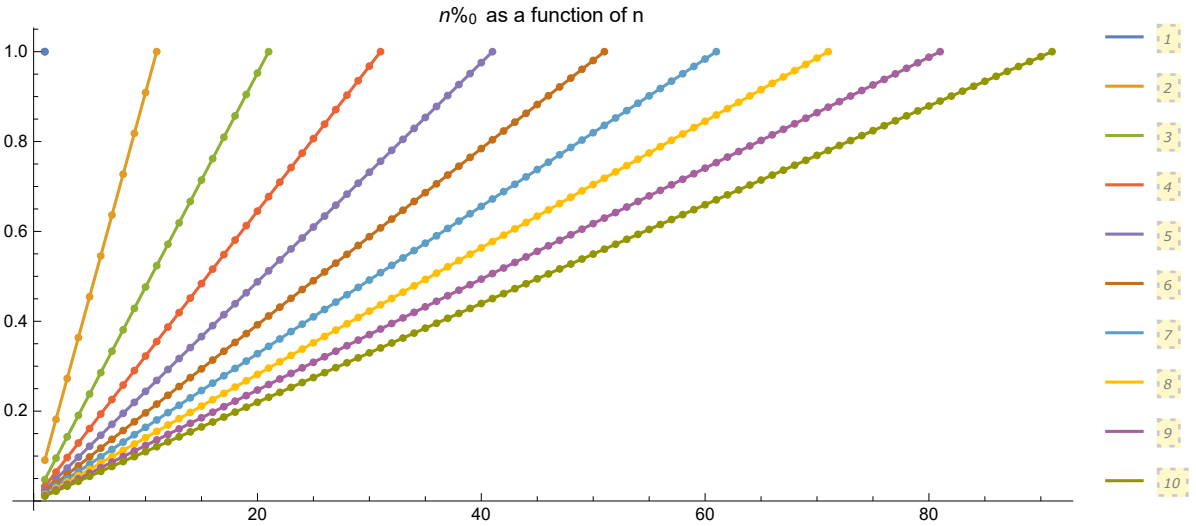
In[•]:=
```
ListPlot[Table[n0@x, {x, 1, 50, 5}], ImageSize → Large, PlotLegends → Automatic,
 PlotLabel → Style[#, 18] & /@ {"n%₀ as a function of n"}, PlotTheme → "Web"]
```

Out[•]=

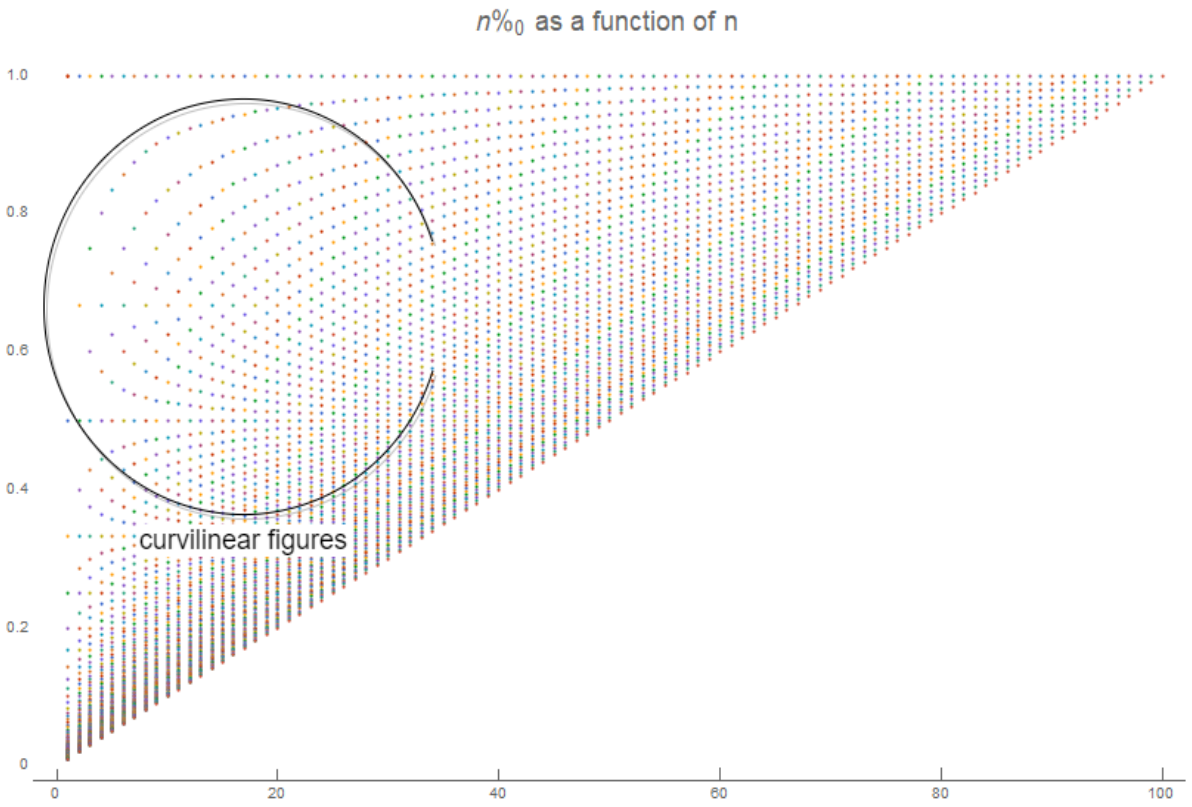


In[•]:=
```
ListPlot[Table[n0@x, {x, 1, 100, 10}], ImageSize → 800, Joined → True,
 Mesh → All, PlotLegends → Automatic, PlotLabel → "n%₀ as a function of n"]
```

Out[•]=



In[•]:=
```
ListPlot[Table[n0@x, {x, 1, 100, 1}], ImageSize → 800,
 PlotTheme → "Web", PlotLabel → Style[#, 18] & /@ {"n%₀ as a function of n"}]
```



In[•]:=
```
{(n0@10)[[1]], (n0@100)[[10]], (n0@1000)[[100]], (n0@500)[[50]]}
```

Out[•]=
$$\left\{\frac{1}{10}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10}\right\}$$

In[•]:=
```
{(n0@11)[[3]], (n0@110)[[30]], (n0@21)[[7]], (n0@1461)[[487]]}
```

Out[•]=
$$\left\{\frac{3}{11}, \frac{3}{11}, \frac{1}{3}, \frac{1}{3}\right\}$$

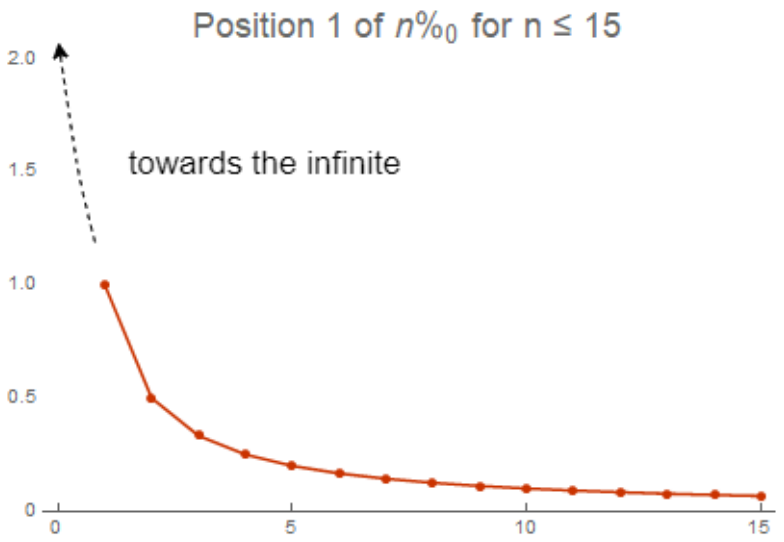
In[3]:=
```
n0posDisplay=.;n0posDisplay=Function[{n,pos},DisplayForm@FractionBox[1+pos,n]];
```

In[•]:=
```
Table[n0posDisplay[n, 0], {n, 0, 25}]
```

Out[•]=
$$\left\{\frac{1}{0}, \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}, \frac{1}{11}, \frac{1}{12}, \frac{1}{13}, \frac{1}{14}, \frac{1}{15}, \frac{1}{16}, \frac{1}{17}, \frac{1}{18}, \frac{1}{19}, \frac{1}{20}, \frac{1}{21}, \frac{1}{22}, \frac{1}{23}, \frac{1}{24}, \frac{1}{25}\right\}$$

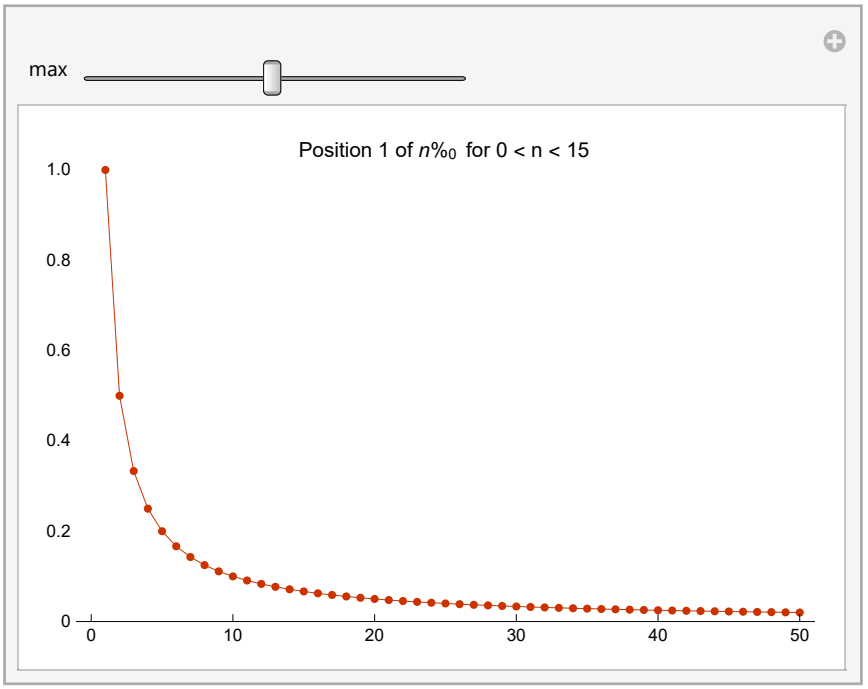
In[4]:=
```
n0pos=.;n0pos=Function[{n,pos},(1+pos)/n];
```

In[•]:=
```
ListPlot[Table[n0pos[n, 0], {n, 1, 15}],
 PlotLabel → Style[#, 18] & /@ {"Position 1 of n%0 for n ≤ 15"}, PlotRange → {0, 2},
 Joined → True, Mesh → All, PlotTheme → "Web", PlotStyle → {Thickness@0.004}, ImageSize → 400]
```



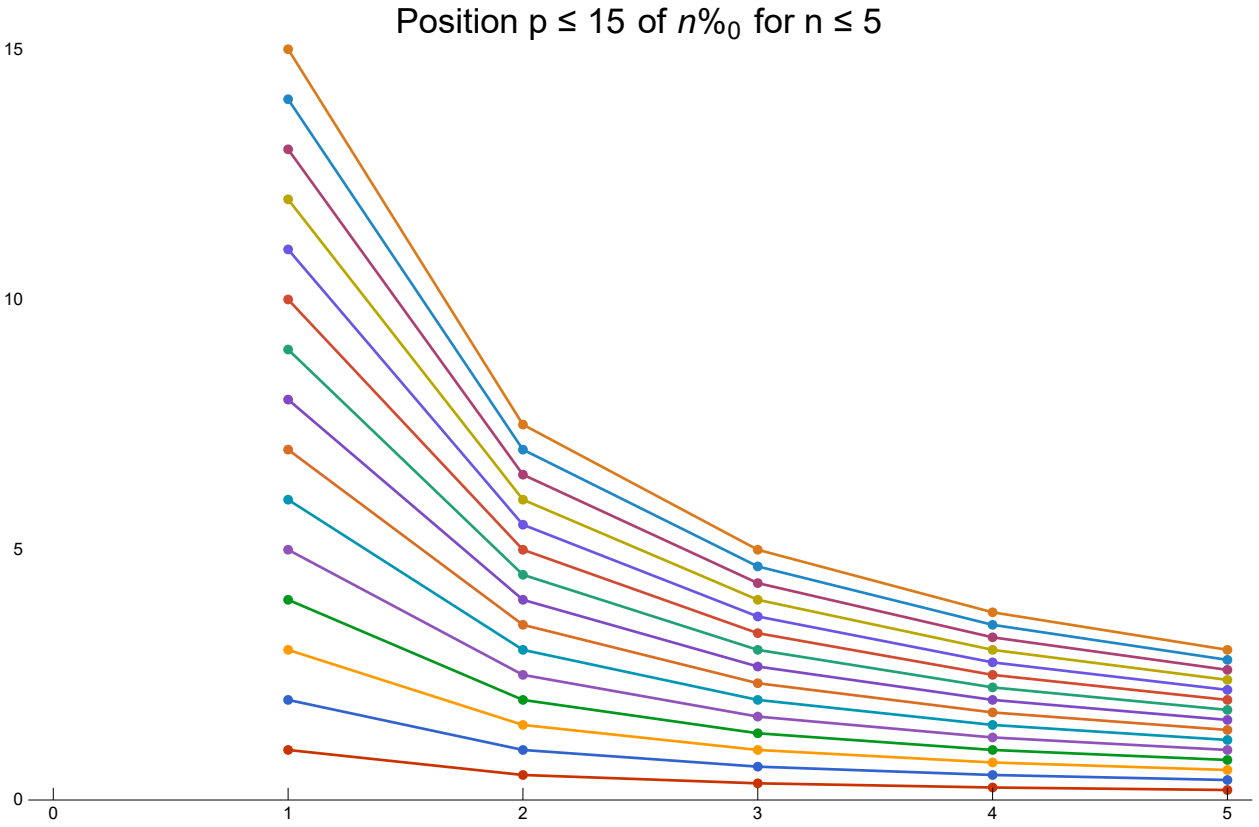
In[9]:=
```
Manipulate[
 ListPlot[Table[n0pos[n, 0], {n, 1, max}], PlotLabel → "Position 1 of n%₀ for 0 < n < 15",
  PlotRange → {0, 1.01}, Joined → True, Mesh → All, ImageSize → 400,
  PlotTheme → "Web", PlotStyle → {Thickness@0.001}], {{max, 50}, 1, 100, 1, Slider}]
```

Out[9]=



In[ ]:=
```
ListPlot[Table[Table[n0pos[n, pos], {n, 1, 5}], {pos, 0, 14, 1}],
 PlotLabel → Style[#, 18] & /@ {"Position p ≤ 15 of n%₀ for n ≤ 5"},
 PlotRange → {0, 15.1}, Joined → True, Mesh → All, ImageSize → 650, PlotTheme → "Web",
 PlotStyle → {Thickness@0.0022}, MeshStyle → Directive[PointSize@0.008]]
```

Out[ ]=



In[ ]:=
```
With[{maxpos = 10},
 Grid[Table[Table[n0posDisplay[n, pos], {n, 1, maxpos}], {pos, 0, maxpos - 1, 1}],
  ItemStyle → Directive[FontSize → 20], Spacings → {1, 1}]]
```

Out[ ]=

1/1	1/2	1/3	1/4	1/5	1/6	1/7	1/8	1/9	1/10
2/1	2/2	2/3	2/4	2/5	2/6	2/7	2/8	2/9	2/10
3/1	3/2	3/3	3/4	3/5	3/6	3/7	3/8	3/9	3/10
4/1	4/2	4/3	4/4	4/5	4/6	4/7	4/8	4/9	4/10
5/1	5/2	5/3	5/4	5/5	5/6	5/7	5/8	5/9	5/10
6/1	6/2	6/3	6/4	6/5	6/6	6/7	6/8	6/9	6/10
7/1	7/2	7/3	7/4	7/5	7/6	7/7	7/8	7/9	7/10
8/1	8/2	8/3	8/4	8/5	8/6	8/7	8/8	8/9	8/10
9/1	9/2	9/3	9/4	9/5	9/6	9/7	9/8	9/9	9/10
10/1	10/2	10/3	10/4	10/5	10/6	10/7	10/8	10/9	10/10

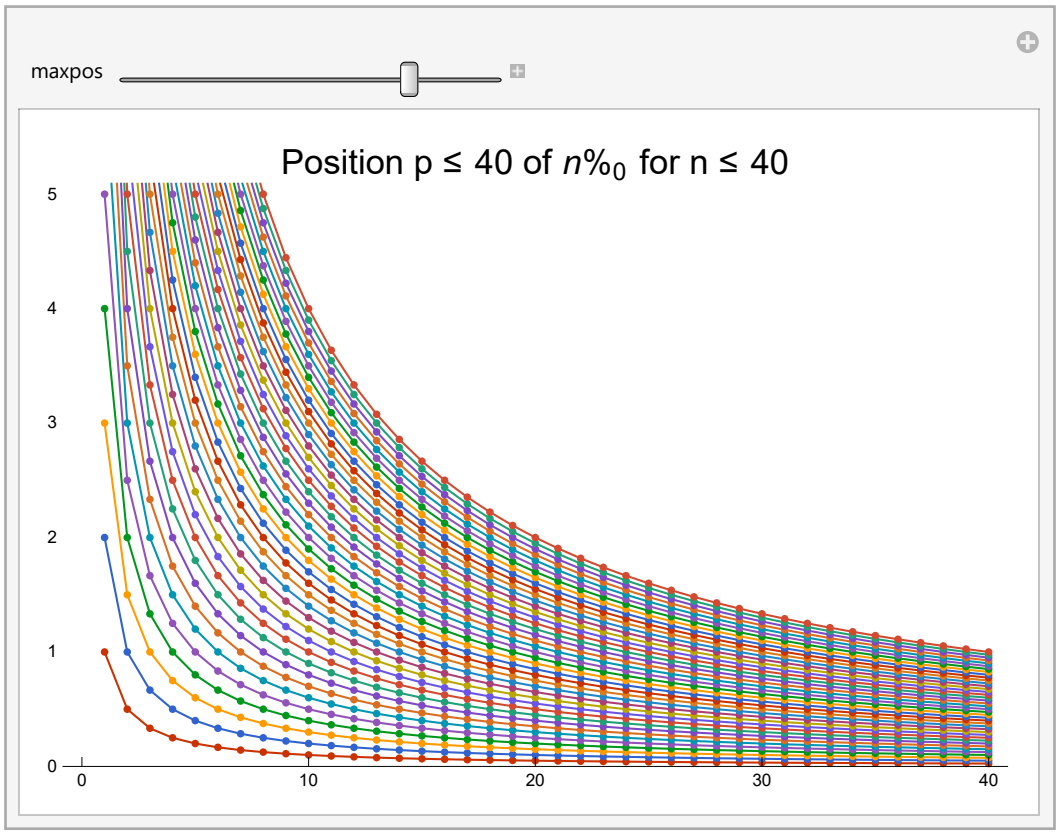
In[ ]:=
```
With[{maxpos = 10}, Grid[Table[Table[n0pos[n, pos], {n, 1, maxpos}], {pos, 0, maxpos - 1, 1}],
  ItemStyle → Directive[FontSize → 20], Spacings → {1, 1}]]
```

Out[ ]=

1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{8}$	$\frac{1}{9}$	$\frac{1}{10}$
2	1	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{2}{5}$	$\frac{1}{3}$	$\frac{2}{7}$	$\frac{1}{4}$	$\frac{2}{9}$	$\frac{1}{5}$
3	$\frac{3}{2}$	1	$\frac{3}{4}$	$\frac{3}{5}$	$\frac{1}{2}$	$\frac{3}{7}$	$\frac{3}{8}$	$\frac{1}{3}$	$\frac{3}{10}$
4	2	$\frac{4}{3}$	1	$\frac{4}{5}$	$\frac{2}{3}$	$\frac{4}{7}$	$\frac{1}{2}$	$\frac{4}{9}$	$\frac{2}{5}$
5	$\frac{5}{2}$	$\frac{5}{3}$	$\frac{5}{4}$	1	$\frac{5}{6}$	$\frac{5}{7}$	$\frac{5}{8}$	$\frac{5}{9}$	$\frac{1}{2}$
6	3	2	$\frac{3}{2}$	$\frac{6}{5}$	1	$\frac{6}{7}$	$\frac{3}{4}$	$\frac{2}{3}$	$\frac{3}{5}$
7	$\frac{7}{2}$	$\frac{7}{3}$	$\frac{7}{4}$	$\frac{7}{5}$	$\frac{7}{6}$	1	$\frac{7}{8}$	$\frac{7}{9}$	$\frac{7}{10}$
8	4	$\frac{8}{3}$	2	$\frac{8}{5}$	$\frac{4}{3}$	$\frac{8}{7}$	1	$\frac{8}{9}$	$\frac{4}{5}$
9	$\frac{9}{2}$	3	$\frac{9}{4}$	$\frac{9}{5}$	$\frac{3}{2}$	$\frac{9}{7}$	$\frac{9}{8}$	1	$\frac{9}{10}$
10	5	$\frac{10}{3}$	$\frac{5}{2}$	2	$\frac{5}{3}$	$\frac{10}{7}$	$\frac{5}{4}$	$\frac{10}{9}$	1

In[10]:=
```
Manipulate[ListPlot[Table[Table[n0pos[n, pos], {n, 1, maxpos}], {pos, 0, maxpos - 1, 1}],
  PlotLabel → Style[#, 18] & /@
   {"Position p ≤ " <> ToString@maxpos <> " of n%₀ for n ≤ " <> ToString@maxpos},
  PlotRange → {0, 5.1}, Joined → True, Mesh → All, ImageSize → 500,
  PlotTheme → "Web", PlotStyle → {Thickness@0.0022},
  MeshStyle → Directive[PointSize@0.008]], {{maxpos, 40}, 3, 50, 1}]
```

Out[10]=



$n\%_1$

In[5]:=
```
n1Display=.;n1Display=Function[{x},Table[DisplayForm@FractionBox[x,x-n],{n,0,x-0}]];
n1=.;n1=Function[{x},Table[x/(x-n),{n,0,x-1}]];
n1pos=.;n1pos=Function[{n,pos},If[pos≠n,n/(n-pos),Style[∞,Directive[Bold,Blue]]]];
n1posDisplay=.;n1posDisplay=Function[{n,pos},If[pos≠n,DisplayForm@FractionBox[n,n-pos],Style[∞,Direc
```

In[•]:=
```
n1Display@10
```

Out[•]=
$$\left\{\frac{10}{10}, \frac{10}{9}, \frac{10}{8}, \frac{10}{7}, \frac{10}{6}, \frac{10}{5}, \frac{10}{4}, \frac{10}{3}, \frac{10}{2}, \frac{10}{1}, \frac{10}{0}\right\}$$

In[•]:=
```
n1@20
```

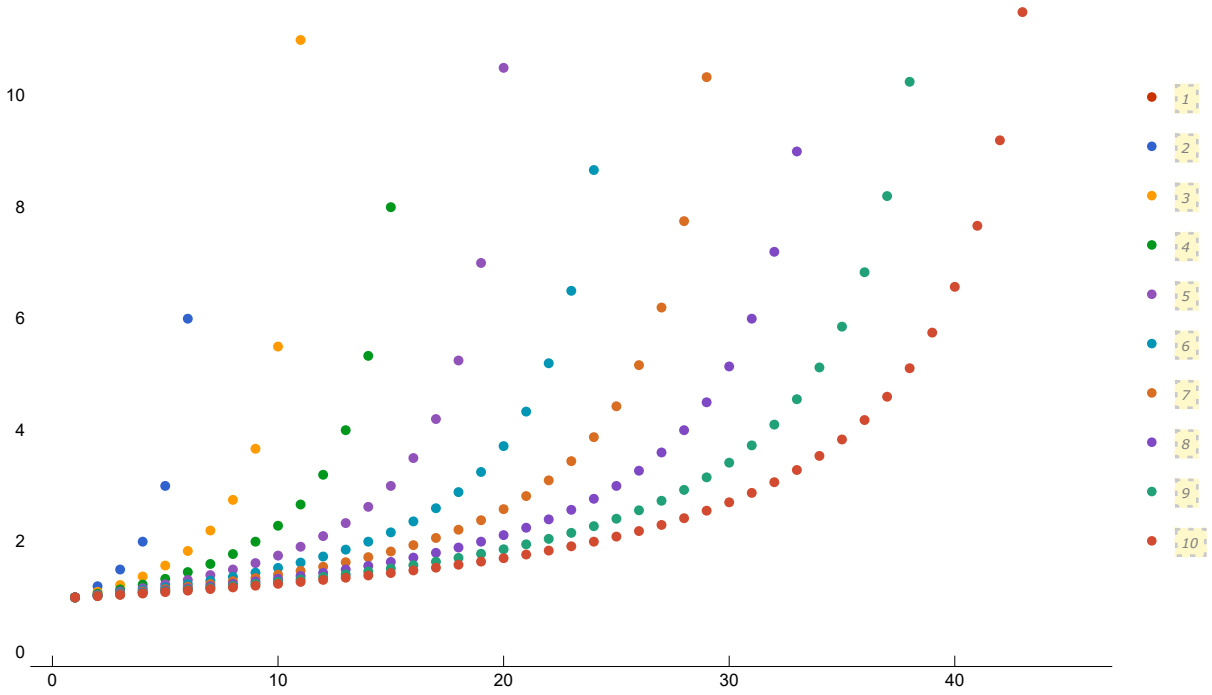
Out[•]=
$$\left\{1, \frac{20}{19}, \frac{10}{9}, \frac{20}{17}, \frac{5}{4}, \frac{4}{3}, \frac{10}{7}, \frac{20}{13}, \frac{5}{3}, \frac{20}{11}, 2, \frac{20}{9}, \frac{5}{2}, \frac{20}{7}, \frac{10}{3}, 4, 5, \frac{20}{3}, 10, 20\right\}$$

In[•]:=
```
{n1pos[20, 0], n1pos[20, 20]}
```

Out[•]=
$$\{1, \infty\}$$

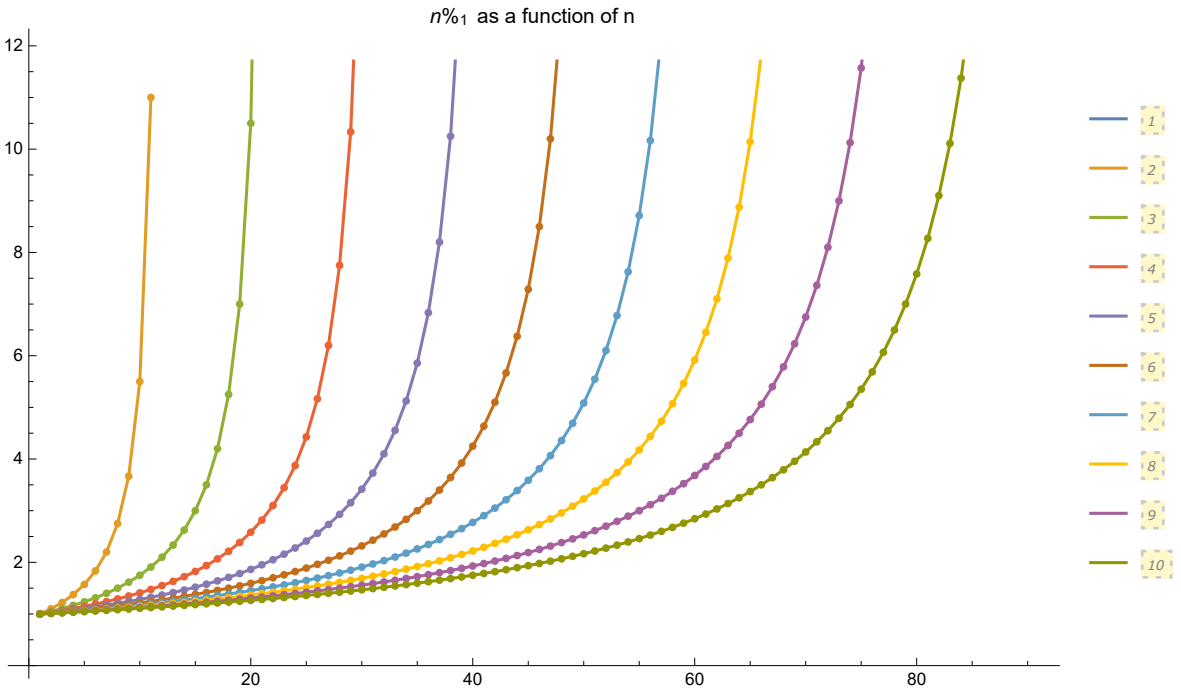
In[•]:=
```
ListPlot[Table[n1@x, {x, 1, 50, 5}], ImageSize → Large, PlotLegends → Automatic,
 PlotLabel → Style[#, 18] & /@ {"n%₁ as a function of n"}, PlotTheme → "Web"]
```

Out[•]=



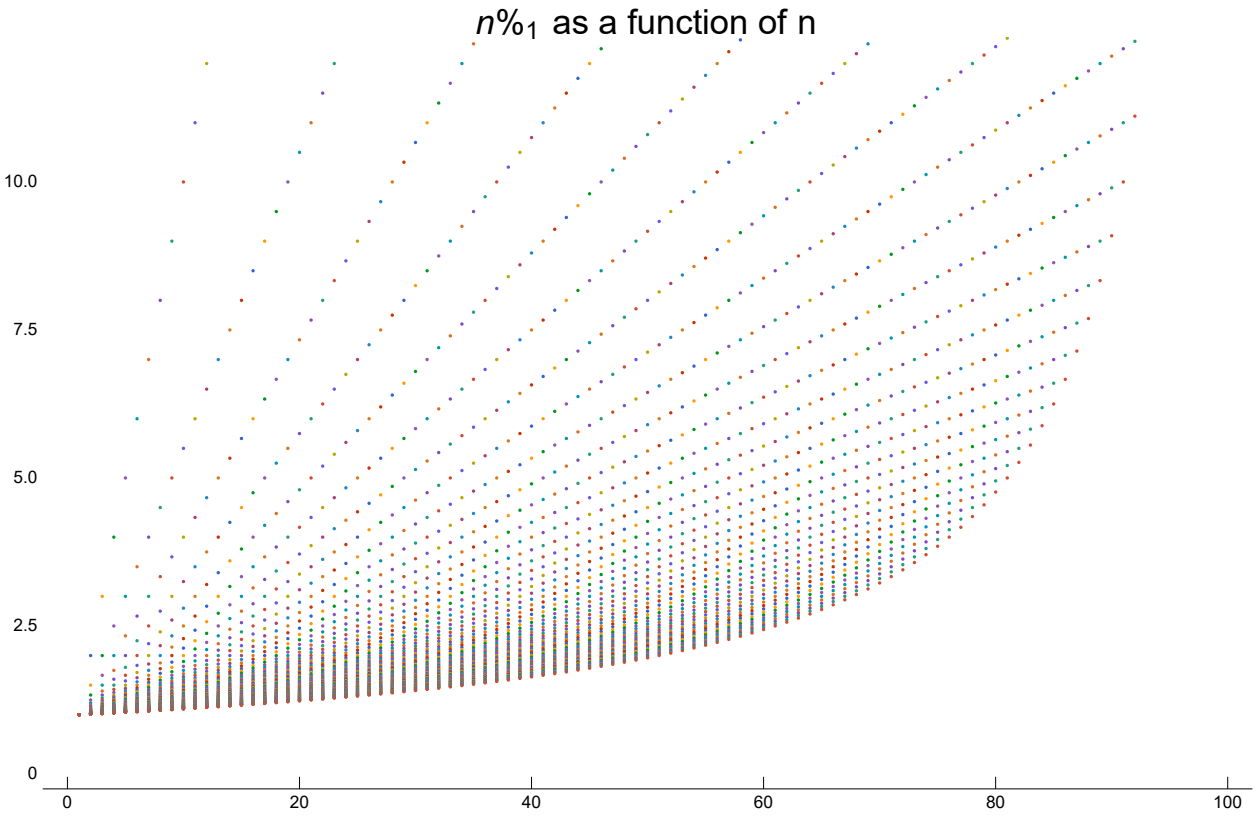
In[12]:=
```
ListPlot[Table[n1@x, {x, 1, 100, 10}], ImageSize → 550, Joined → True,
 Mesh → All, PlotLegends → Automatic, PlotLabel → "n%₁ as a function of n"]
```

Out[12]=



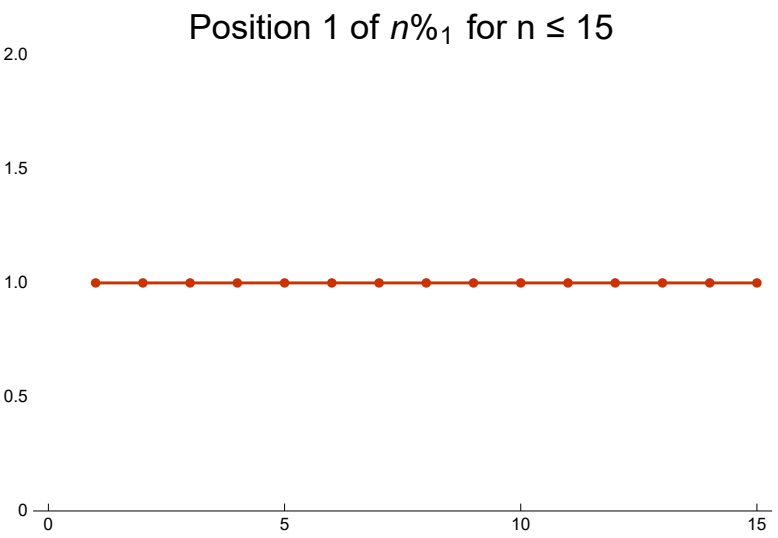
In[14]:=
```
ListPlot[Table[n1@x, {x, 1, 100, 1}], ImageSize → 650,
 PlotTheme → "Web", PlotLabel → Style[#, 18] & /@ {"n%₁ as a function of n"}]
```

Out[14]=



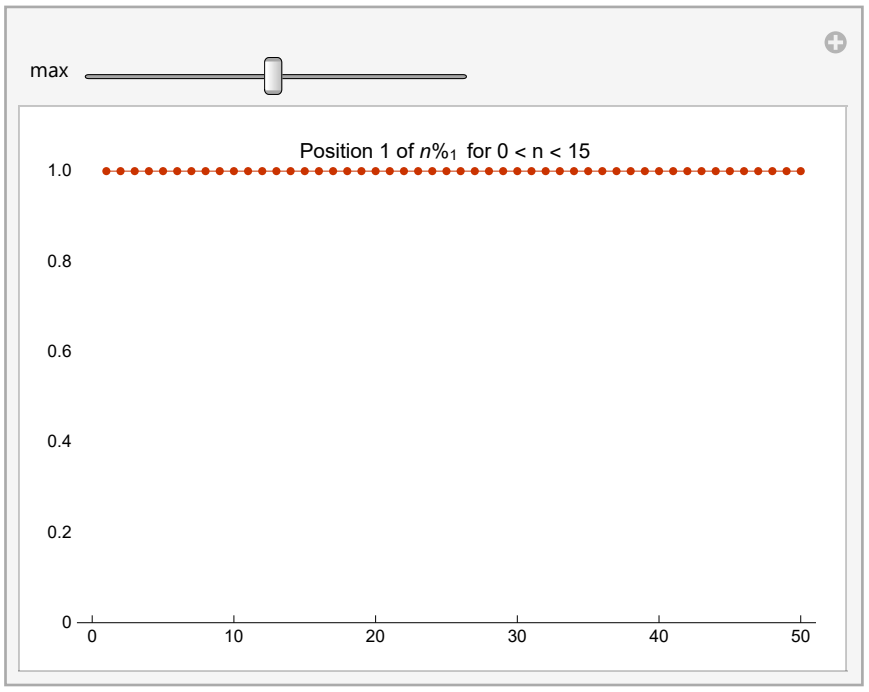
In[ ]:=
```
ListPlot[Table[n1pos[n, 0], {n, 1, 15}],
 PlotLabel → Style[#, 18] & /@ {"Position 1 of n%₁ for n ≤ 15"}, PlotRange → {0, 2},
 Joined → True, Mesh → All, PlotTheme → "Web", PlotStyle → {Thickness@0.004}, ImageSize → 400]
```

Out[ ]=



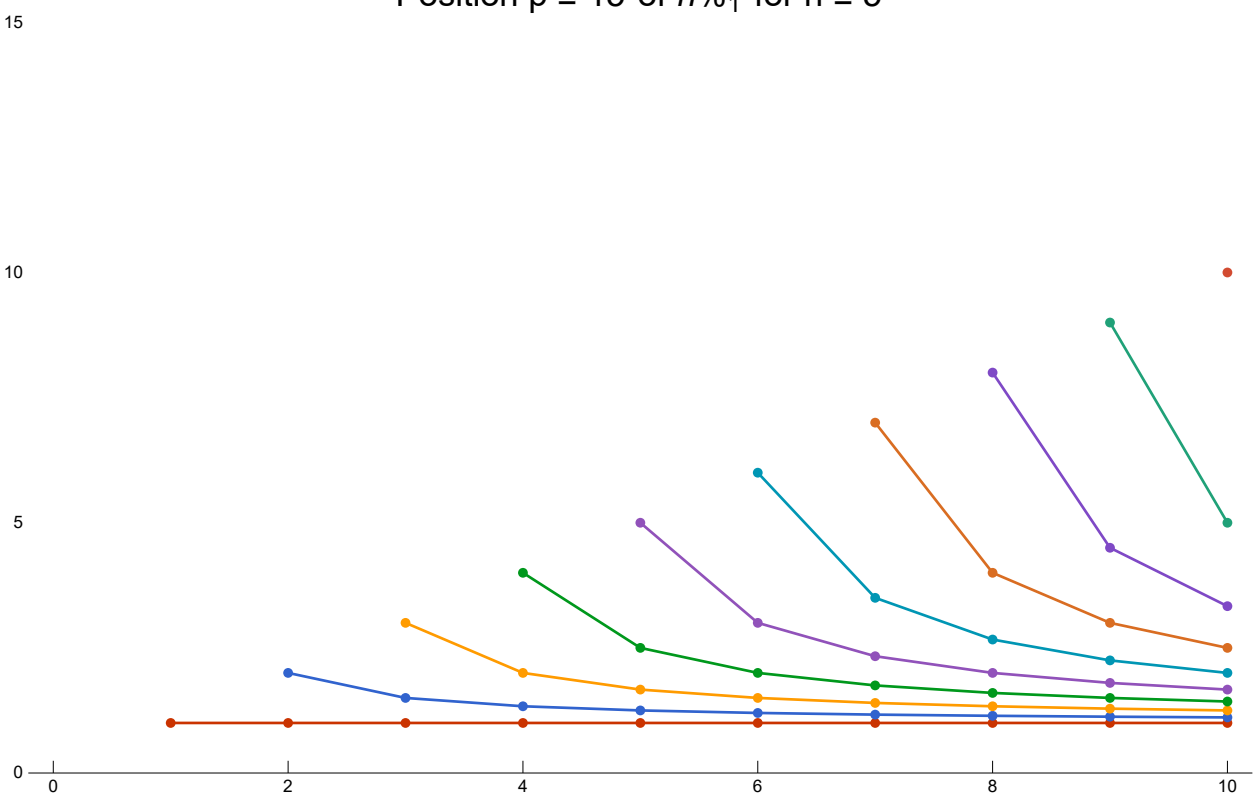
In[15]:=
```
Manipulate[
 ListPlot[Table[n1pos[n, 0], {n, 1, max}], PlotLabel → "Position 1 of n%₁ for 0 < n < 15",
  PlotRange → {0, 1.01}, Joined → True, Mesh → All, ImageSize → 400,
  PlotTheme → "Web", PlotStyle → {Thickness@0.001}], {{max, 50}, 1, 100, 1, Slider}]
```

Out[15]=



In[ ]:=
```
ListPlot[Table[Table[n1pos[n, pos], {n, 1, 10}], {pos, 0, 14, 1}],
 PlotLabel → Style[#, 18] & /@ {"Position p ≤ 15 of n%₁ for n ≤ 5"},
 PlotRange → {0, 15.1}, Joined → True, Mesh → All, ImageSize → 650, PlotTheme → "Web",
 PlotStyle → {Thickness@0.0022}, MeshStyle → Directive[PointSize@0.008]]
```

Out[ ]=



In[ ]:=
```
With[{maxpos = 10},
 Grid[Table[Table[n1posDisplay[n, pos], {n, 1, maxpos}], {pos, 0, maxpos - 1, 1}],
  ItemStyle → Directive[FontSize → 20], Spacings → {1, 1}]]
```

Out[ ]=

1/1	2/2	3/3	4/4	5/5	6/6	7/7	8/8	9/9	10/10
∞	2/1	3/2	4/3	5/4	6/5	7/6	8/7	9/8	10/9
1/−1	∞	3/1	4/2	5/3	6/4	7/5	8/6	9/7	10/8
1/−2	2/−1	∞	4/1	5/2	6/3	7/4	8/5	9/6	10/7
1/−3	2/−2	3/−1	∞	5/1	6/2	7/3	8/4	9/5	10/6
1/−4	2/−3	3/−2	4/−1	∞	6/1	7/2	8/3	9/4	10/5
1/−5	2/−4	3/−3	4/−2	5/−1	∞	7/1	8/2	9/3	10/4
1/−6	2/−5	3/−4	4/−3	5/−2	6/−1	∞	8/1	9/2	10/3
1/−7	2/−6	3/−5	4/−4	5/−3	6/−2	7/−1	∞	9/1	10/2
1/−8	2/−7	3/−6	4/−5	5/−4	6/−3	7/−2	8/−1	∞	10/1

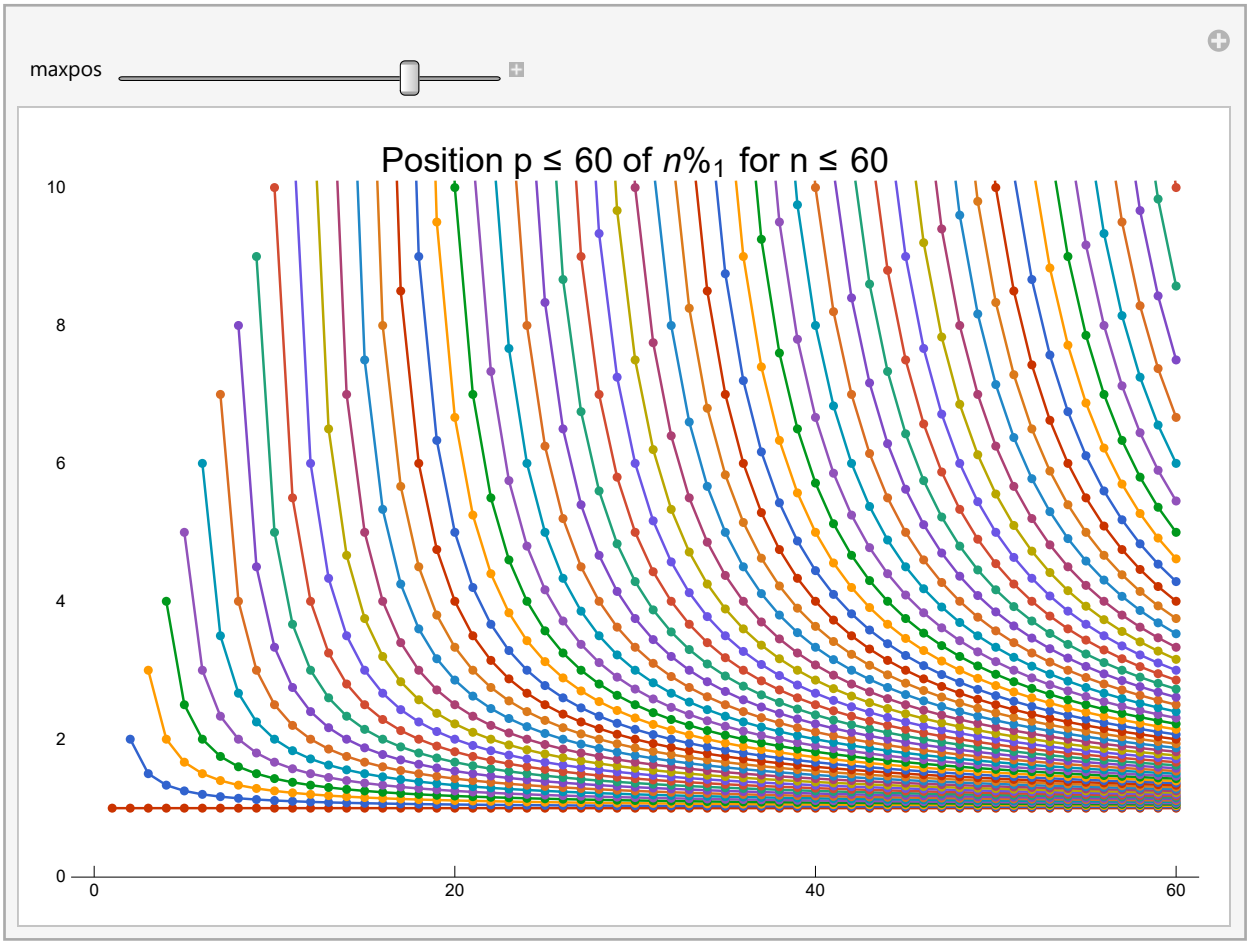
In[ ]:=
```
With[{maxpos = 10}, Grid[Table[Table[n1pos[n, pos], {n, 1, maxpos}], {pos, 0, maxpos - 1, 1}],
  ItemStyle → Directive[FontSize → 20], Spacings → {1, 1}]]
```

Out[ ]=

1	1	1	1	1	1	1	1	1	1
∞	2	$\frac{3}{2}$	$\frac{4}{3}$	$\frac{5}{4}$	$\frac{6}{5}$	$\frac{7}{6}$	$\frac{8}{7}$	$\frac{9}{8}$	$\frac{10}{9}$
−1	∞	3	2	$\frac{5}{3}$	$\frac{3}{2}$	$\frac{7}{5}$	$\frac{4}{3}$	$\frac{9}{7}$	$\frac{5}{4}$
$-\frac{1}{2}$	−2	∞	4	$\frac{5}{2}$	2	$\frac{7}{4}$	$\frac{8}{5}$	$\frac{3}{2}$	$\frac{10}{7}$
$-\frac{1}{3}$	−1	−3	∞	5	3	$\frac{7}{3}$	2	$\frac{9}{5}$	$\frac{5}{3}$
$-\frac{1}{4}$	$-\frac{2}{3}$	$-\frac{3}{2}$	−4	∞	6	$\frac{7}{2}$	$\frac{8}{3}$	$\frac{9}{4}$	2
$-\frac{1}{5}$	$-\frac{1}{2}$	−1	−2	−5	∞	7	4	3	$\frac{5}{2}$
$-\frac{1}{6}$	$-\frac{2}{5}$	$-\frac{3}{4}$	$-\frac{4}{3}$	$-\frac{5}{2}$	−6	∞	8	$\frac{9}{2}$	$\frac{10}{3}$
$-\frac{1}{7}$	$-\frac{1}{3}$	$-\frac{3}{5}$	−1	$-\frac{5}{3}$	−3	−7	∞	9	5
$-\frac{1}{8}$	$-\frac{2}{7}$	$-\frac{1}{2}$	$-\frac{4}{5}$	$-\frac{5}{4}$	−2	$-\frac{7}{2}$	−8	∞	10

In[16]:=
```
Manipulate[ListPlot[Table[Table[n1pos[n, pos], {n, 1, maxpos}], {pos, 0, maxpos - 1, 1}],
  PlotLabel → Style[#, 18] & /@
   {"Position p ≤ " <> ToString@maxpos <> " of n%₁ for n ≤ " <> ToString@maxpos},
  PlotRange → {0, 10.1}, Joined → True, Mesh → All, ImageSize → 600,
  PlotTheme → "Web", PlotStyle → {Thickness@0.0022},
  MeshStyle → Directive[PointSize@0.008]], {{maxpos, 60}, 3, 75, 1}]
```

Out[16]=



$n\%_0 \cup n\%_1$

This is the union of $n\%_0$ and $n\%_1$.

Let' s catenate both sequences.

```
In[ ]:= Grid[Table[Catenate@{n0Display@x, n1Display@x}, {x, 1, 10, 1}],
         ItemStyle → Directive[FontSize → 20], Spacings → {1, 1}]
```

Out[ ]=

1/1	1/1	1/0														
1/2	2/2	2/2	2/1	2/0												
1/3	2/3	3/3	3/3	3/2	3/1	3/0										
1/4	2/4	3/4	4/4	4/4	4/3	4/2	4/1	4/0								
1/5	2/5	3/5	4/5	5/5	5/5	5/4	5/3	5/2	5/1	5/0						
1/6	2/6	3/6	4/6	5/6	6/6	6/6	6/5	6/4	6/3	6/2	6/1	6/0				
1/7	2/7	3/7	4/7	5/7	6/7	7/7	7/7	7/6	7/5	7/4	7/3	7/2	7/1	7/0		
1/8	2/8	3/8	4/8	5/8	6/8	7/8	8/8	8/8	8/7	8/6	8/5	8/4	8/3	8/2	8/1	8/0
1/9	2/9	3/9	4/9	5/9	6/9	7/9	8/9	9/9	9/9	9/8	9/7	9/6	9/5	9/4	9/3	9/2
1/10	2/10	3/10	4/10	5/10	6/10	7/10	8/10	9/10	10/10	10/10	10/9	10/8	10/7	10/6	10/5	10/4

In[ ]:=
```
Grid[Catenate@{Table[n0Display@x, {x, 1, 10, 1}], Table[n1Display@x, {x, 1, 10, 1}]},
 ItemStyle → Directive[FontSize → 20], Spacings → {1, 1}]
```

Out[ ]=

1/1									
1/2	2/2								
1/3	2/3	3/3							
1/4	2/4	3/4	4/4						
1/5	2/5	3/5	4/5	5/5					
1/6	2/6	3/6	4/6	5/6	6/6				
1/7	2/7	3/7	4/7	5/7	6/7	7/7			
1/8	2/8	3/8	4/8	5/8	6/8	7/8	8/8		
1/9	2/9	3/9	4/9	5/9	6/9	7/9	8/9	9/9	
1/10	2/10	3/10	4/10	5/10	6/10	7/10	8/10	9/10	10/10

1/1	1/0									
2/2	2/1	2/0								
3/3	3/2	3/1	3/0							
4/4	4/3	4/2	4/1	4/0						
5/5	5/4	5/3	5/2	5/1	5/0					
6/6	6/5	6/4	6/3	6/2	6/1	6/0				
7/7	7/6	7/5	7/4	7/3	7/2	7/1	7/0			
8/8	8/7	8/6	8/5	8/4	8/3	8/2	8/1	8/0		
9/9	9/8	9/7	9/6	9/5	9/4	9/3	9/2	9/1	9/0	
10/10	10/9	10/8	10/7	10/6	10/5	10/4	10/3	10/2	10/1	10/0

In[ ]:=
```
Grid[Table[Catenate@{n0@x, n1@x}, {x, 1, 10, 1}],
 ItemStyle → Directive[FontSize → 20], Spacings → {1, 1}]
```

Out[ ]=

1	1																	
$\frac{1}{2}$	1	1	2															
$\frac{1}{3}$	$\frac{2}{3}$	1	1	$\frac{3}{2}$	3													
$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	1	$\frac{4}{3}$	2	4											
$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	1	1	$\frac{5}{4}$	$\frac{5}{3}$	$\frac{5}{2}$	5									
$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	1	1	$\frac{6}{5}$	$\frac{3}{2}$	2	3	6							
$\frac{1}{7}$	$\frac{2}{7}$	$\frac{3}{7}$	$\frac{4}{7}$	$\frac{5}{7}$	$\frac{6}{7}$	1	1	$\frac{7}{6}$	$\frac{7}{5}$	$\frac{7}{4}$	$\frac{7}{3}$	$\frac{7}{2}$	7					
$\frac{1}{8}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{3}{4}$	$\frac{7}{8}$	1	1	$\frac{8}{7}$	$\frac{4}{3}$	$\frac{8}{5}$	2	$\frac{8}{3}$	4	8			
$\frac{1}{9}$	$\frac{2}{9}$	$\frac{1}{3}$	$\frac{4}{9}$	$\frac{5}{9}$	$\frac{2}{3}$	$\frac{7}{9}$	$\frac{8}{9}$	1	1	$\frac{9}{8}$	$\frac{9}{7}$	$\frac{3}{2}$	$\frac{9}{5}$	$\frac{9}{4}$	3	$\frac{9}{2}$	9	
$\frac{1}{10}$	$\frac{1}{5}$	$\frac{3}{10}$	$\frac{2}{5}$	$\frac{1}{2}$	$\frac{3}{5}$	$\frac{7}{10}$	$\frac{4}{5}$	$\frac{9}{10}$	1	1	$\frac{10}{9}$	$\frac{5}{4}$	$\frac{10}{7}$	$\frac{5}{3}$	2	$\frac{5}{2}$	$\frac{10}{3}$	5	10

```
In[•]:= Grid[Catenate@{Table[n0@x, {x, 1, 10, 1}], Table[n1@x, {x, 1, 10, 1}]},
         ItemStyle → Directive[FontSize → 20], Spacings → {1, 1}]
```

Out[•]=

1									
$\frac{1}{2}$	1								
$\frac{1}{3}$	$\frac{2}{3}$	1							
$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1						
$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	1					
$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	1				
$\frac{1}{7}$	$\frac{2}{7}$	$\frac{3}{7}$	$\frac{4}{7}$	$\frac{5}{7}$	$\frac{6}{7}$	1			
$\frac{1}{8}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{3}{4}$	$\frac{7}{8}$	1		
$\frac{1}{9}$	$\frac{2}{9}$	$\frac{1}{3}$	$\frac{4}{9}$	$\frac{5}{9}$	$\frac{2}{3}$	$\frac{7}{9}$	$\frac{8}{9}$	1	
$\frac{1}{10}$	$\frac{1}{5}$	$\frac{3}{10}$	$\frac{2}{5}$	$\frac{1}{2}$	$\frac{3}{5}$	$\frac{7}{10}$	$\frac{4}{5}$	$\frac{9}{10}$	1

```
In[•]:= First@
         RepeatedTiming@Sort@DeleteDuplicates@Flatten@Table[Catenate@{n0@x, n1@x}, {x, 1, 1000, 1}]
Out[•]= 2.80

In[•]:= First@RepeatedTiming@Sort@DeleteDuplicates@
            Flatten@Catenate@{Table[n0@x, {x, 1, 1000, 1}], Table[n1@x, {x, 1, 1000, 1}]}
Out[•]= 2.75

In[•]:= DeleteDuplicates@Flatten@Table[Catenate@{n0@x, n1@x}, {x, 1, 10, 1}]
```

Out[•]=

$$\left\{1, \frac{1}{2}, 2, \frac{1}{3}, \frac{2}{3}, \frac{3}{2}, 3, \frac{1}{4}, \frac{3}{4}, \frac{4}{3}, 4, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{5}{4}, \frac{5}{3}, \frac{5}{2}, 5, \frac{1}{6},\right.$$
$$\frac{5}{6}, \frac{6}{5}, 6, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}, \frac{7}{6}, \frac{7}{5}, \frac{7}{4}, \frac{7}{3}, \frac{7}{2}, 7, \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}, \frac{8}{7}, \frac{8}{5}, \frac{8}{3}, 8,$$
$$\left.\frac{1}{9}, \frac{2}{9}, \frac{4}{9}, \frac{5}{9}, \frac{7}{9}, \frac{8}{9}, \frac{9}{8}, \frac{9}{7}, \frac{9}{5}, \frac{9}{4}, \frac{9}{2}, 9, \frac{1}{10}, \frac{3}{10}, \frac{7}{10}, \frac{9}{10}, \frac{10}{9}, \frac{10}{7}, \frac{10}{3}, 10\right\}$$

In[•]:= **Length@%61**

Out[•]= 63

In[•]:= **Sort@%61**

Out[•]= $\left\{\frac{1}{10}, \frac{1}{9}, \frac{1}{8}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{2}{9}, \frac{1}{4}, \frac{2}{7}, \frac{3}{10}, \frac{1}{3}, \frac{3}{8}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \frac{1}{2}, \frac{5}{9}, \frac{4}{7}, \frac{3}{5}, \frac{5}{8}, \frac{2}{3}, \frac{7}{10}, \frac{5}{7}, \frac{3}{4}, \frac{7}{9}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}, \frac{8}{9}, \frac{9}{10}, 1, \frac{10}{9}, \frac{9}{8}, \frac{8}{7}, \frac{7}{6}, \frac{6}{5}, \frac{5}{4}, \frac{9}{7}, \frac{4}{3}, \frac{7}{5}, \frac{10}{7}, \frac{3}{2}, \frac{8}{5}, \frac{5}{3}, \frac{7}{4}, \frac{9}{5}, 2, \frac{9}{4}, \frac{7}{3}, \frac{5}{2}, \frac{8}{3}, 3, \frac{10}{3}, \frac{7}{2}, 4, \frac{9}{2}, 5, 6, 7, 8, 9, 10\right\}$

In[•]:= **DeleteDuplicates@Flatten@Catenate@{Table[n0@x, {x, 1, 10, 1}], Table[n1@x, {x, 1, 10, 1}]}**

Out[•]= $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{5}{6}, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}, \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}, \frac{1}{9}, \frac{2}{9}, \frac{4}{9}, \frac{5}{9}, \frac{7}{9}, \frac{8}{9}, \frac{1}{10}, \frac{3}{10}, \frac{7}{10}, \frac{9}{10}, 2, \frac{3}{2}, 3, \frac{4}{3}, 4, \frac{5}{4}, \frac{5}{3}, \frac{5}{2}, 5, \frac{6}{5}, 6, \frac{7}{6}, \frac{7}{5}, \frac{7}{4}, \frac{7}{3}, \frac{7}{2}, 7, \frac{8}{7}, \frac{8}{5}, \frac{8}{3}, 8, \frac{9}{8}, \frac{9}{7}, \frac{9}{5}, \frac{9}{4}, \frac{9}{2}, 9, \frac{10}{9}, \frac{10}{7}, \frac{10}{3}, 10\right\}$

In[•]:= **Length@%64**

Out[•]= 63

In[•]:= **Sort@%64**

Out[•]= $\left\{\frac{1}{10}, \frac{1}{9}, \frac{1}{8}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{2}{9}, \frac{1}{4}, \frac{2}{7}, \frac{3}{10}, \frac{1}{3}, \frac{3}{8}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \frac{1}{2}, \frac{5}{9}, \frac{4}{7}, \frac{3}{5}, \frac{5}{8}, \frac{2}{3}, \frac{7}{10}, \frac{5}{7}, \frac{3}{4}, \frac{7}{9}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}, \frac{8}{9}, \frac{9}{10}, 1, \frac{10}{9}, \frac{9}{8}, \frac{8}{7}, \frac{7}{6}, \frac{6}{5}, \frac{5}{4}, \frac{9}{7}, \frac{4}{3}, \frac{7}{5}, \frac{10}{7}, \frac{3}{2}, \frac{8}{5}, \frac{5}{3}, \frac{7}{4}, \frac{9}{5}, 2, \frac{9}{4}, \frac{7}{3}, \frac{5}{2}, \frac{8}{3}, 3, \frac{10}{3}, \frac{7}{2}, 4, \frac{9}{2}, 5, 6, 7, 8, 9, 10\right\}$

In[•]:= **{{1, 2, 3} == {1, 2, 3}, {1, 2, 3} == {1, 2}, {1, 2, 3} == {2, 3, 1}}**

Out[•]= {True, False, False}

In[•]:= **%63 == %66**

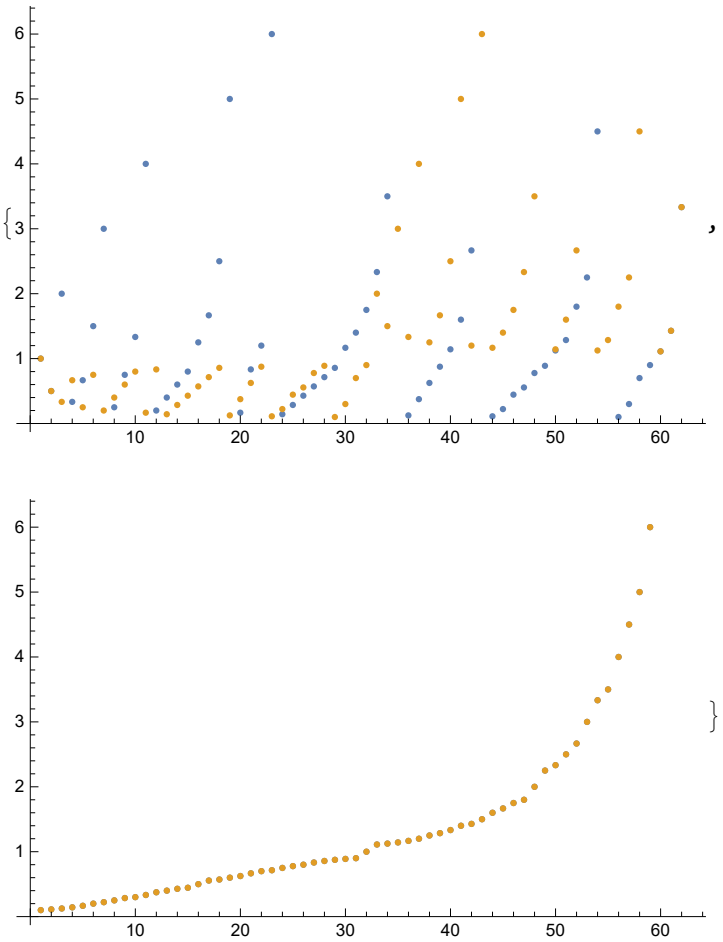
Out[•]= True

In[•]:= **{Min@%61, Max@%61, Min@%64, Max@%64}**

Out[•]= $\left\{\frac{1}{10}, 10, \frac{1}{10}, 10\right\}$

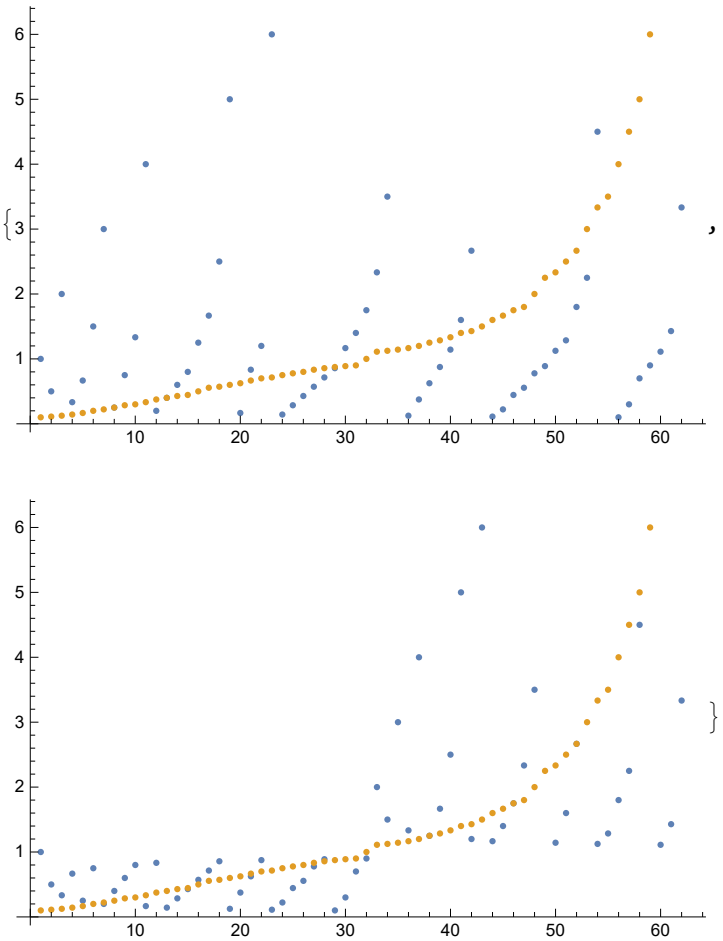
In[ ]:=
```
{ListPlot[{%61, %64}, ImageSize → Medium], ListPlot[{%63, %66}, ImageSize → Medium]}
```

Out[ ]=



In[ ]:=
```
{ListPlot[{%61, %63}, ImageSize → Medium], ListPlot[{%64, %66}, ImageSize → Medium]}
```

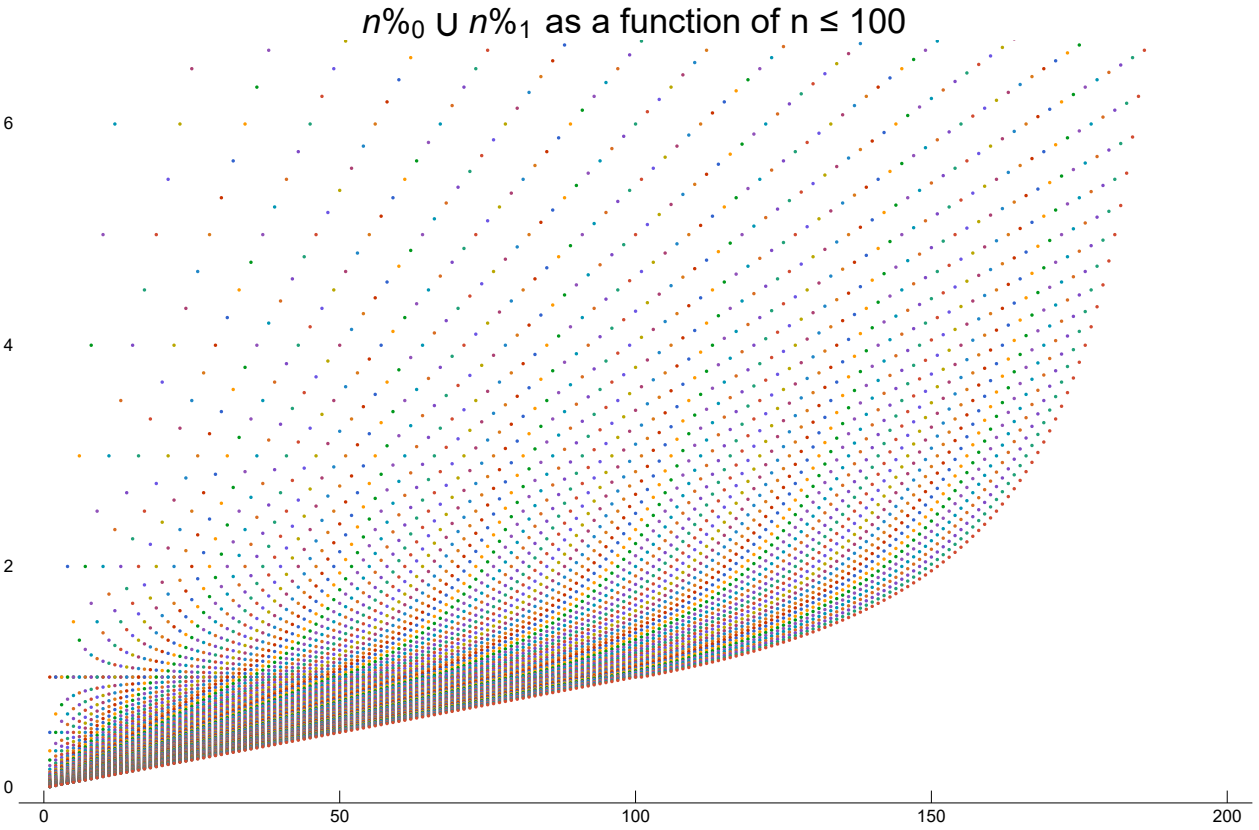
Out[ ]=



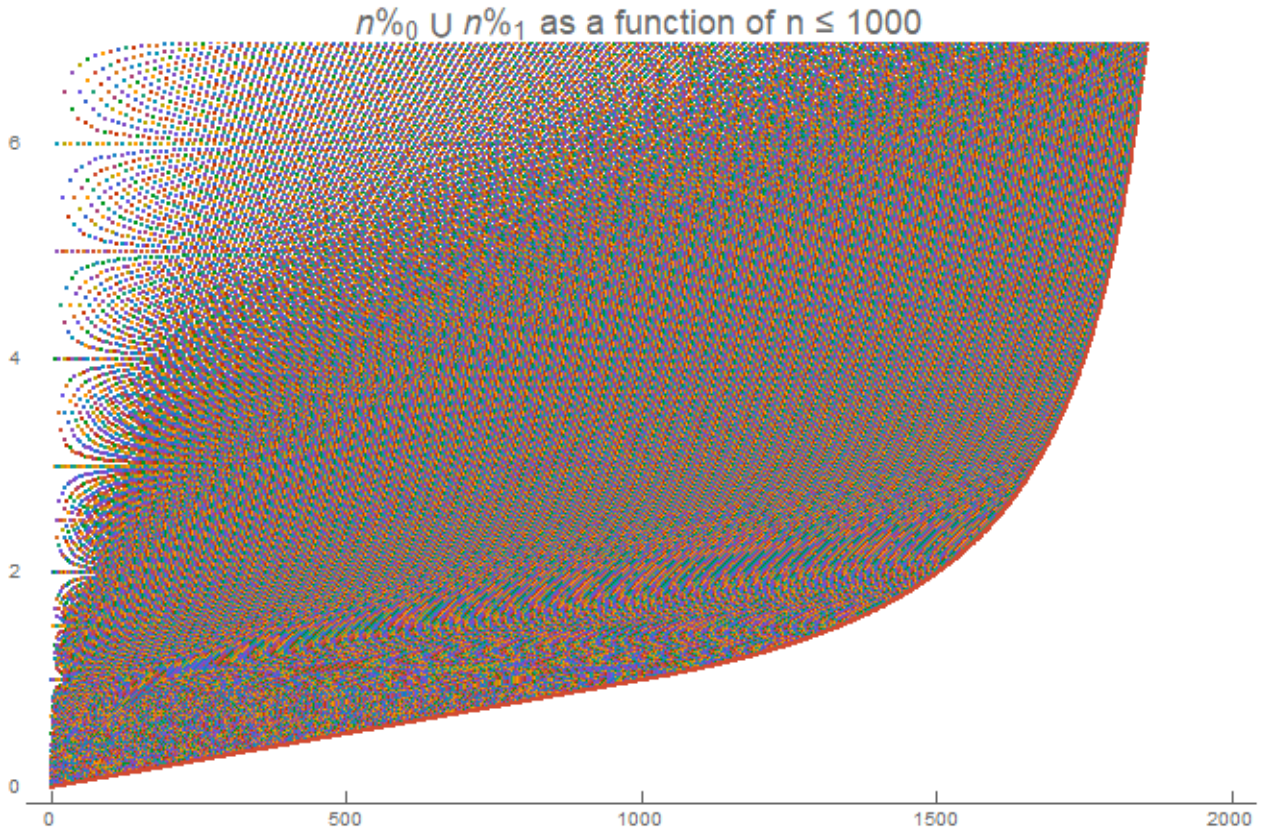
The unsorted forms doesn't show together accurately, but this is a problem with the plot function.

In[18]:=
```
ListPlot[Table[Catenate@{n0@x, n1@x}, {x, 1, 100, 1}], ImageSize → 650,
 PlotTheme → "Web", PlotLabel → Style[#, 18] & /@ {"n%₀ ⋃ n%₁ as a function of n ≤ 100"}]
```

Out[18]=



In[19]:=
```
ListPlot[Table[Catenate@{n0@x, n1@x}, {x, 1, 1000, 1}], ImageSize → 650,
 PlotTheme → "Web", PlotLabel → Style[#, 18] & /@ {"n%₀ ⋃ n%₁ as a function of n ≤ 1000"}]
```



$n\%_2$

To do...