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An Introduction to Optimization

Exercises

Part 1.

1.1 Consider the problem

$$\begin{aligned} \min_{\mathbf{x}} \quad & 3{x_1}^2 + 0.5{x_2}^2 + 2x_2 + 2 \\ \text{s.t.} \quad & x_1, x_2 \geq 0. \end{aligned}$$

Verify whether the FONC are satisfied at the following points : $[1\ \ 2]^T$, $[0\ \ 3]^T$, $[1\ \ 0]^T$ and $[0\ \ 0]^T$.

1.2 Find the saddle points, local minimizers and local maximizers (if they exist) of the following functions :

$$\begin{aligned} f_1(\mathbf{x}) &= 4.5 - x_1 + 2x_2 - 0.5{x_1}^2 - 2{x_2}^2 \\ f_2(\mathbf{x}) &= 2{x_1}^3 + x_1{x_2}^2 + 5{x_1}^2 + {x_2}^2 \end{aligned}$$

1.3 Investigate (using two different methods) whether $\mathbf{d} = [2\ \ -1]^T$ is a descent or an increase direction at $\mathbf{x}_0 = [1\ \ 1]^T$ with respect to the function

$$f(\mathbf{x}) = {x_1}^2 + x_1 x_2 - 4{x_2}^2 + 5$$

1.4 Solve the following unconstrained quadratic programming problem :

$$\begin{aligned} \min_{\mathbf{x}} \quad & f(\mathbf{x}) = (x_2 + x_1 - 3)^2 + 2(x_2 - x_1 + 1)^2 \\ \text{s.t.} \quad & \mathbf{x} \in \mathbb{R}^2 \end{aligned}$$

In order to verify your answer (graphically), plot using Matlab the level sets or the graph of the cost function in the domain :

$$\begin{aligned} -5 \ \leq & \ x_1 \leq \ 5 \\ -5 \ \leq & \ x_2 \leq \ 5 \end{aligned}$$

Part 2.

2.1 Write a Matlab routine to solve the problem :

$$\begin{aligned} \min_{x} \quad & f(x) = x^4 + 4x^3 + 9x^2 + 6x + 6 \\ \text{s.t.} \quad & x \in [-2, 2] \end{aligned}$$

using the Golden section with a tolerance $\epsilon = 10^{-2}$.

2.2 Write a Matlab routine to solve the problem :

$$\begin{aligned} \min_{x} \quad & f(x) = 2x^4 - 5x^3 + 100x^2 + 30x - 75 \\ \text{s.t.} \quad & x \in \mathbb{R} \end{aligned}$$

with a stopping criterion $|\frac{df}{dx}(x_k)| \leq 10^{-4}$ using :

1. Newton's method, with an initial condition $x_0 = 2$
2. Secant method, with initial conditions $x_0 = 2.1$ and $x_1 = 2$

Part 3.

3.1 Consider the following problem

$$\begin{aligned} \min_{\mathbf{x}} \quad & f(\mathbf{x}) = 1 + 2x_1 e^{-{x_1}^2 - {x_2}^2} \\ \text{s.t.} \quad & \mathbf{x} \in \mathbb{R}^2 \end{aligned}$$

Write a Matlab routine to solve the problem using the steepest descent method. Consider the stopping criterion $\|\nabla f(\mathbf{x}_k)\| < \epsilon = 10^{-3}$. For the initial condition $\mathbf{x}_0$ consider the cases $[-0.5 \;\; 0.5]^T$, $[0.5 \;\; -0.5]^T$ and $[1 \;\; 1]^T$.
Solve the line search problem analytically.

3.2 Consider the problem of finding the minimizer of Rosenbrock's function :

$$\begin{aligned} \min_{\mathbf{x}} \quad & f(\mathbf{x}) = 100(x_2 - {x_1}^2)^2 + (1 - x_1)^2 \\ \text{s.t.} \quad & \mathbf{x} \in \mathbb{R}^2 \end{aligned}$$

1. Show analytically that $[1 \;\; 1]^T$ is the unique global minimizer.
2. Implement the steepest descent method with a stopping criterion $\|\mathbf{x}_{k+1} - \mathbf{x}_k\| / \|\mathbf{x}_k\| < \epsilon_1 = 10^{-3}$, starting from $\mathbf{x}_0 = [-1 \;\; 2]^T$.
   At each iteration, use Newton's method for the line search, with initial value $\alpha_0 = 0.1$, and a stopping criterion $\frac{|\alpha_{k+1} - \alpha_k|}{\alpha k} < \epsilon_2 = 10^{-3}$.

3.3 Consider the problem

$$\begin{aligned} \min_{\mathbf{x}} \quad & f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T Q \mathbf{x} + \mathbf{q}^T \mathbf{x} \\ \text{s.t.} \quad & \mathbf{x} \in \mathbb{R}^2 \end{aligned}$$

for the cases :

$$Q = \lambda I, \ \lambda > 0, \qquad \forall \mathbf{q} \in \mathbb{R}^2 \qquad \forall \mathbf{x}_0 \in \mathbb{R}^2$$

$$Q = \begin{bmatrix} 10 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \mathbf{q} = \begin{bmatrix} -3 \\ -3 \end{bmatrix}, \qquad \mathbf{x}_0 = \begin{bmatrix} -2 \\ -7 \end{bmatrix}$$

$$Q = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \mathbf{q} = \begin{bmatrix} -3 \\ -3 \end{bmatrix}, \qquad \mathbf{x}_0 = \begin{bmatrix} -2 \\ -7 \end{bmatrix}$$

1. Rank the three cases in term of speed of convergence when applying the steepest descend method.
2. Write a Matlab routine to implement the steepest descend method and the conjugate gradient method, and compare their performance.

3.4 Consider the function

$$f(\mathbf{x}) = \frac{5}{2}{x_1}^2 + \frac{1}{2}{x_2}^2 + 2x_1x_2 - 3x_1 - x_2$$

1. Express the function in a standard quadratic form.
2. Write the steps of the conjugate gradient algorithm to find the minimizer of $f(\cdot)$ starting from $\mathbf{x}_0 = [0 \ \ 0]^T$.

3.5 Using the DFP algorithm, we want to find the solution to the following problem

$$\begin{aligned} \min_{\mathbf{x}} \quad & f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T \underbrace{\begin{bmatrix} 2 & 2 \\ 2 & 10 \end{bmatrix}}_{Q} \mathbf{x} + \underbrace{[2 \ \ 0]}_{\mathbf{q}^T} \mathbf{x} \\ \text{s.t.} \quad & \mathbf{x} \in \mathbb{R}^2 \end{aligned}$$

1. Find the formula for α_k in terms of Q, $\mathbf{g}_k = \nabla f(\mathbf{x}_k)$ and $\mathbf{d}_k$.
2. Implement the algorithm using Matlab starting from $\mathbf{x}_0 = [0 \ \ 0]^T$.