An Introduction to Optimization

Exercises

Part 4.

4.1 Let $A \in \mathbb{R}^{m\times n}$, $\mathbf{b} \in \mathbb{R}^m$ with $m \leq n$, rank $A = m$ and $\mathbf{x}_0 \in \mathbb{R}^n$. Consider the problem

$$\begin{aligned} \arg\min_{\mathbf{x}} \quad & \|\mathbf{x} - \mathbf{x}_0\|^2 \\ \text{s.t.} \quad & A\mathbf{x} = \mathbf{b} \end{aligned}$$

Show that the problem has the following unique solution

$$\mathbf{x}^\star = A^T(AA^T)^{-1}\mathbf{b} + (\mathbf{I} - A^T(AA^T)^{-1}A)\mathbf{x}_0$$

4.2 Plot the curves corresponding to the following equations

$$\begin{aligned} 4{x_2}^2 &= 20 - {x_1}^2 \\ x_2 &= {x_1}^4 - 10 \end{aligned}$$

1. Formalize a least square optimization problem which permits to find the intersections of the previous two curves.
2. Implement a Gauss-Newton method to solve the problem and run it for different initial conditions.

4.3 We want to find the parameters of a process that has an output which is linear in time. We conduct $m \geq 2$ measurements $\{y_1, \dots, y_m\}$ at different instants $\{t_1, \dots, t_m\}$, and we want to find the line $y = a^\star t + b^\star$ which has the least squared error with respect to the measurements. That is, we want to find $a^\star$ and $b^\star$ such that

$$\begin{aligned} [a^\star \ \ b^\star]^T \quad &= \arg\min_{\mathbf{x}} \quad F(\mathbf{x}) = \sum_{i=1}^{m}(y_i - at_i - b)^2 \\ \text{s.t.} \quad & \quad \mathbf{x} \in \mathbb{R}^2 \end{aligned}$$

Find analytically the solution $[a^\star \ \ b^\star]^T$. Write a Matlab routine to implement the solution for the data :

t	y
1.0779	6.9959
1.4268	9.4782
2.1801	12.0585
2.3138	14.5837
2.9755	17.1019
3.4526	19.4104
4.1062	22.0719
4.6181	24.5620
4.8747	26.8650

(1)

4.4 Consider the set of m perturbed measurements of a sinusoidal signal $\{y_1, \dots, y_m\}$ at different instants $\{t_1, \dots, t_m\}$ in (2). We want to fit a sinusoidal signal to the measured data :

$$y = a \sin(\omega t + \phi)$$

by solving a nonlinear least-squares problem

$$\begin{aligned} &\arg\min_{\mathbf{x}} \quad F(\mathbf{x}) = \sum_{i=1}^{m} f_i(\mathbf{x})^2 \\ &\text{s.t.} \qquad \mathbf{x} = [a \ \ \omega \ \ \phi]^T \in \mathbb{R}^3 \end{aligned}$$

1. Find the expression of $f_i(\mathbf{x})$ in the objective function.
2. Using Matlab, find a solution to the fitting problem using Gauss-Newton method. Consider $\mathbf{x}_0 = [0.5 \ 1.25 \ 0.1]^T$.

t	y
0	0.1128
0.2300	0.0876
0.4800	0.3971
0.7300	0.5766
0.9800	0.8538
1.2300	0.7856
1.4800	0.9596
1.7300	0.8259
1.9800	0.7542
2.2300	0.8918
2.4800	0.5151
2.7300	0.4231
2.9800	0.3269
3.2300	-0.1352
3.4800	-0.2729
3.7300	-0.4683
3.9800	-0.7837
4.2300	-0.8328
4.4800	-1.1336
4.7300	-0.8258
4.9800	-1.0839
5.2300	-0.9525
5.4800	-0.5864
5.7300	-0.5132
5.9800	-0.1718
6.2300	-0.0748

(2)

Part 5.

5.1 We want to find the minimum of the function

$$f(\mathbf{x}) = 2x_1$$

under the constraints

$$\begin{array}{rcll} x_1 - x_3 & = & 3 & \\ x_1 - x_2 - 2x_4 & = & 1 & \\ 2x_1 + x_4 & \leq & 7 & \\ x_i & \geq & 0 & i = 1, 2, 3, 4 \end{array}$$

1. Solve the problem using the two-phase simplex method. Give the minimizer $\mathbf{x}^\star$, and the minimum value $f(\mathbf{x}^\star)$.
2. Give all the possible optimal basic feasible solutions to the problem.

5.2 Consider the following problem : A farmer has an area of 120 hectares. He cultivates beets, corn and wheat. Beets cultivation yields 474 Euros per hectare and requires 10 hours of use of a tractor per hectare. Corn cultivation yields 774 Euros per hectare, but requires 30 hours of tractor per hectare. Finally, wheat cultivation yields 645 Euros per hectare and requires 30 hours of tractor per hectare.

The tractor is available for a maximum of 2500 hours a year and for storage reasons, the corn surface may not exceed 1/4 of the total surface, we seek the optimal distribution of the surfaces dedicated to the three kinds of crops that maximizes the farmer's profit.

1. Formulate the problem as a linear programming problem.
2. Solve the problem using Matlab's linprog function.