$\boxed{1}$

a)

$$\mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

$$\phi(x,y) = (x+1, y, x+y)$$

$\forall (x,y,z), (x',y',z') \in \mathbb{R}^3$

$$\phi((x,y)+(x',y')) = \phi(x,y) + \phi(x',y') \,?$$

$$\phi((x,y)+(x',y')) = \phi((x+x', y+y'))$$

$$= ((x+x')+1, y+y', x+x'+y+y')$$

$$= ((x+1)+x', y+y', (x+y)+(x'+y'))$$

$$= \phi(x,y) + (x', y', x'+y')$$

Como $(x',y',x'+y') \neq (x'+1, y', x'+y')$, logo ϕ não é uma transformação linear ✗

b) Será linear a aplicação definida de $\mathbb{R}^3$ em $\mathbb{R}^3$ por

$$L(x,y,z) = (x+y, y, x-z) \,?$$

$\forall (x,y,z), (x',y',z') \in \mathbb{R}^3$

$$L((x,y,z)+(x',y',z')) = \underbrace{L(x,y,z) + L(x',y',z')}_{(x+y,\, y,\, x-z) \,+\, (x'+y',\, y',\, x'-z')}$$

$$L((x,y,z)+(x',y',z')) = L(x+x', y+y', z+z')$$

$$= ((x+x')+(y+y'), y+y', (x+x')-(z+z'))$$

$$= ((x+y)+(x'+y'), y+y', (x-z)+(x'-z'))$$

$$= (x+y, y, x-z) + (x'+y', y', x'-z)$$

$$= L(x,y,z) + L(x',y',z')$$

$\forall \alpha \in \mathbb{R}, \forall (x,y,z) \in \mathbb{R}^3$

$$L(\alpha(x,y,z)) = \alpha L(x,y,z) \,?$$

$$L(\alpha(x,y,z)) = L(\alpha x, \alpha y, \alpha z)$$
$$= (\alpha x + \alpha y, \alpha y, \alpha x - \alpha z)$$
$$= \alpha(x+y, y, x-z)$$
$$= \alpha L(x,y,z)$$ ✓

c)

$$\mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$\phi(x,y,z) = (x+y, 0, 2x-z)$$

$$\phi((x,y,z)+(x',y',z')) = \phi(x,y,z) + \phi(x',y',z') \ ?$$

$\forall (x,y,z),(x',y',z') \in \mathbb{R}^3$

$$\phi((x,y,z)+(x',y',z')) = \phi(x+x', y+y', z+z')$$

$$= (x+x'+y+y', 0, 2(x+x')-z-z')$$

$$= ((x+y)+(x'+y'), 0+0, (2x-z)+(2x'-z'))$$

$$= \phi(x,y,z) + \phi(x',y',z')$$

$\forall \alpha \in \mathbb{R}, (x,y,z) \in \mathbb{R}^3$

$$\phi(\alpha(x,y,z)) = \alpha\,\phi(x,y,z)$$

$$\phi(\alpha(x,y,z)) = \phi(\alpha x, \alpha y, \alpha z) = (\alpha x + \alpha y, 0, 2\alpha x - \alpha z)$$

$$= (\alpha(x+y), \alpha(0), \alpha(2x-z))$$

$$= \alpha\,\phi(x,y,z) \ ✓$$

d)

$$\phi(x,y,z) = (x-y, x^2, 2z)$$

$\forall (x,y,z),(x',y',z') \in \mathbb{R}^3$

$$\phi((x,y,z)+(x',y',z')) = \phi(x+x', y+y', z+z')$$

$$= (x+x'-y-y', (x+x')^2, 2z+2z')$$

$$= ((x-y)+(x'-y'), x^2+2xx'+(x')^2, 2z+2z')$$

$$= \phi(x,y,z) + (x'-y', 2xx'+(x')^2, 2z')$$

$$\neq$$

$$(x'-y', (x')^2, 2z')$$

✗

e)

$$\phi : \mathcal{P}_2 \longrightarrow \mathcal{P}_1$$

$$at^2 + bt + c \longrightarrow at + b + 1$$

$\forall\ at^2 + bt + c\ ,\ a't^2 + b't + c' \in \mathcal{P}_2$

$$\phi\left((at^2 + bt + c) + (a't^2 + b't + c')\right) = \phi(at^2 + bt + c) + \phi(a't^2 + b't + c')\ ?$$

$$= \phi\left((a + a')t^2 + (b + b')t + (c + c')\right) = (a + a')t + (b + b') + 1$$

$$= at + b + 1 + a't + b'$$

$$= \phi(at^2 + bt + c) + a't + b' + 0$$

Como $0 \neq 1 \Rightarrow \neq \phi(a't^2 + b't + c')$
$= a't + b't + 1$

f)

$$\phi : \mathcal{P}_2 \longrightarrow \mathcal{P}_2$$

$$at^2 + bt + c \longrightarrow a + (t+1)(bt + c)$$
$$= a + bt^2 + ct + bt + c$$
$$= bt^2 + (c + b)t + a + c$$

$\forall\ at^2 + bt + c\ ,\ a't^2 + b't + c' \in \mathcal{P}_2$

$$\phi\left((at^2 + bt + c) + (a't^2 + b't + c')\right) = \phi(at^2 + bt + c) + \phi(a't^2 + b't + c')\ ?$$

$$\phi(at^2 + a't^2 + bt + b't + c + c') = \phi((a + a')t^2 + (b + b')t + c + c')$$

$$= (b + b')t^2 + (c + c' + b + b')t + a + a' + c + c'$$

$$= \left(bt^2 + (c + b)t + a + c\right) + \left(b't + (c' + b')t + a' + c'\right)$$

$$= \phi(at^2 + bt + c) + \phi(a't^2 + b't + c')$$

$\forall \alpha \in \mathbb{R},\ \forall\ at^2 + bt + c \in \mathcal{P}_2$

$$\phi(\alpha(at^2 + bt + c)) = \alpha(\phi(at^2 + bt + c))\ ?$$

$$\phi((a\alpha)t^2 + (b\alpha)t + c\alpha) = (b\alpha)t^2 + (c\alpha - b\alpha)t + a\alpha + c\alpha$$
$$= \alpha\left(bt^2 + (c - b)t + a + c\right)$$
$$= \alpha\,\phi(at^2 + bt + c)$$

$\boxed{2}$

$$\phi : \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}^{n\times n}$$

$$A \longmapsto \phi(A) = \begin{cases} A^{-1} & \text{se } A \text{ não é singular} \\ 0 & \text{se } A \text{ é singular} \end{cases}$$

$\forall A, A' \in \mathbb{R}^{n\times n}$

$\phi(A+A') = \phi(A) + \phi(A')$?

$$\phi(A+A') = \begin{cases} (A+A')^{-1} & \text{se } (A+A') \text{ não é singular} \\ 0 & \text{se } (A+A') \text{ é singular} \end{cases}$$

Como $(A+A')^{-1} \neq A^{-1} + (A')^{-1}$

$\Rightarrow \phi(A+A') \neq \phi(A) + \phi(A')$

$\boxed{3}$

$$\phi : \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}^{n\times n}$$

$$B \longmapsto \phi(B) = AB - BA$$

$\forall B, B' \in \mathbb{R}^{n\times n}$

$\phi(B+B') = \phi(B) + \phi(B')$?

$$\begin{aligned} \phi(B+B') &= A(B+B') - (B+B')A \\ &= AB + AB' - BA - B'A \\ &= AB - BA + (AB' - B'A) \\ &= \phi(B) + \phi(B') \end{aligned}$$

$\forall \alpha \in \mathbb{R}, \forall B \in \mathbb{R}^{n\times n}$

$\phi(\alpha B) = \alpha \phi(B)$?

$$\begin{aligned} \phi(\alpha B) &= A(\alpha B) - (\alpha B)A \\ &= \alpha(AB - BA) \\ &= \alpha \phi(B) \end{aligned}$$

$\boxed{4}$

$$\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$\phi(1,1) = (2,-3)$$
$$\phi(0,1) = (1,2)$$

$B = \{(1,1),(0,1)\}$ e $\dim \mathbb{R}^2 = \dim B = 2$, logo B é uma base de $\mathbb{R}^2$ se os vetores forem l.i.

Seja $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, os vetores são l.i. se $\det(A) \neq 0$

$\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1 - 0 = 1 \neq 0$, logo B é uma base

$$(x,y) = \alpha(0,1) + \beta(1,1)$$

$$\Leftrightarrow \begin{cases} x = \beta \\ y = \alpha + \beta \end{cases} \quad \begin{cases} \beta = x \\ \alpha = y - x \end{cases}$$

$$(x,y) = \underbrace{(y-x)}_{\mathbb{R}}(0,1) + \underbrace{x}_{\mathbb{R}}(1,1), \quad x,y \in \mathbb{R}$$

$$\phi(x,y) = \phi((y-x)(0,1) + x(1,1))$$

como ϕ é ap. linear

$$= (y-x)\,\phi(0,1) + x\,\phi(1,1)$$
$$= (y-x)(1,2) + x(2,-3)$$
$$= (y - x + 2x,\ 2y - 2x - 3x)$$
$$= (y + x,\ 2y - 5x)$$

$$\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$(x,y) \longmapsto \phi(x,y) = (y + x,\ 2y - 5x)$$

a) $\phi(3,-2) = (3-2,\ 2\times(-2) - 5\times 3)$
$= (1,\ -4 - 15)$
$= (1,\ -19)$

b) $\phi(a,b) = (b + a,\ 2b - 5a)$

$\boxed{5}$

$\phi : \mathcal{P}_2 \longrightarrow \mathcal{P}_3$, onde $\phi(1) = 1$, $\phi(t) = t^2$ e $\phi(t^2) = t^3 + t$

$B = \{t^2, t, 1\}$ é uma base canónica de $\mathcal{P}_2$, onde $\dim \mathcal{P}_2 = \dim B$ e os vetores são l.i.

a)

$$\phi(2t^2 - 5t + 3) = 2\phi(t^2) - 5\phi(t) + 3\phi(1)$$
$$= 2t^3 + 2t - 5t^2 + 3$$

b) $$\phi(at^2 + bt + c) = a\phi(t^2) + b\phi(t) + c\phi(1)$$
$$= at^3 + at + bt^2 + c$$

$\boxed{6}$

$$\phi : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$(x,y,z) \hookrightarrow \phi(x,y,z) = (x+2y+z,\ 2x-y,\ 2y+z)$$

$$B = ((1,0,1),(0,1,1),(0,0,1))$$

a)

$\mathcal{C} = ((1,0,0),(0,1,0)(0,0,1))$

$$M(\phi;\mathcal{C},\mathcal{C}) = \begin{array}{cc} \begin{array}{ccc} \phi(1,0,0) & \phi(0,1,0) & \phi(0,0,1) \end{array} & \\ \begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 0 \\ 0 & 2 & 1 \end{bmatrix} & \begin{array}{c} (1,0,0) \\ (0,1,0) \\ (0,0,1) \end{array} \end{array}$$

$\phi(1,0,0) = (1,2,0)$

$\phi(0,1,0) = (2,-1,2)$

$\phi(0,0,1) = (1,0,1)$

$$[\phi(1,1,-2)]_{\mathcal{C}} = M(\phi;\mathcal{C},\mathcal{C}) \cdot [(1,1,-2)]_{\mathcal{C}}$$

$$= \begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\phi(1,1,-2) = \boxed{1} \times (1,0,0) + \boxed{1} \times (0,1,0) + (0) \times (0,0,1)$$

$$= (1,1,0)$$

b)

$$M(\phi;\mathcal{C},B) = \begin{array}{cc} \begin{array}{ccc} \phi(1,0,0) & \phi(0,1,0) & \phi(0,0,1) \end{array} & \\ \begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 0 \\ -3 & 1 & 0 \end{bmatrix} & \begin{array}{c} (1,0,1) \\ (0,1,1) \\ (0,0,1) \end{array} \end{array}$$

$$\phi(1,0,0) = (1,2,0) = c_1 \times (1,0,1) + c_2(0,1,1) + c_3(0,0,1)$$

$$\Rightarrow c_1 = 1 \wedge c_2 = 2 \wedge c_3 = -3$$

$$\phi(0,1,0) = (2,-1,2) = c'_1 \times (1,0,1) + c'_2(0,1,1) + c'_3(0,0,1)$$

$$\Rightarrow c'_1 = 2 \wedge c'_2 = -1 \wedge c'_3 = 1$$

$$\phi(0,0,1) = (1,0,1) = c''_1 \times (1,0,1) + c''_2(0,1,1) + c''_3(0,0,1)$$

$$\Rightarrow c''_1 = 1 \wedge c''_2 = 0 \wedge c''_3 = 0$$

$$[\phi(1,1,-2)]_B = M(\phi;\mathcal{C},B) \cdot [(1,1,-2)]_{\mathcal{C}}$$

$$= \begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 0 \\ -3 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

$$\phi(1,1,-2) = \boxed{1} \times (1,0,1) + \boxed{1} \times (0,1,1) + \boxed{(-2)} \times (0,0,1)$$

$$= (1,1,0)$$

c)

$$M(\phi; B, \mathcal{C}) = \begin{array}{c} \begin{array}{ccc} \phi(1,0,1) & \phi(0,1,1) & \phi(0,0,1) \end{array} \\ \begin{bmatrix} 2 & 3 & 1 \\ 2 & -1 & 0 \\ 1 & 3 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ (1,0,0) \\ (0,1,0) \\ (0,0,1) \end{array}$$

$\phi(1,0,1) = (2,2,1)$
$\phi(0,1,1) = (3,-1,3)$
$\phi(0,0,1) = (1,0,1)$

$[\phi(1,1,-2)]_{\mathcal{C}} = M(\phi; B, \mathcal{C}) \cdot [(1,1,-2)]_B$

$(1,1,-2) = c_1 \times (1,0,1) + c_2(0,1,1) + c_3 \times (0,0,1)$
$\Rightarrow c_1 = 1 \wedge c_2 = 1 \wedge c_3 = -4$

$$= \begin{bmatrix} 2 & 3 & 1 \\ 2 & -1 & 0 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = (1,1,0)$$

$\phi(1,1,-2) = \boxed{1} \times (1,0,0) + \boxed{1} \times (0,1,0) + \boxed{0} \times (0,0,1)$
$= (1,1,0)$ ✓

d)

$$M(\phi; B, B) = \begin{array}{c} \begin{array}{ccc} \phi(1,0,1) & \phi(0,1,1) & \phi(0,0,1) \end{array} \\ \begin{bmatrix} 2 & 3 & 1 \\ 2 & -1 & 0 \\ -3 & 1 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ (1,0,1) \\ (0,1,1) \\ (0,0,1) \end{array}$$

$\phi(1,0,1) = (2,2,1) = c_1(1,0,1) + c_2(0,1,1) + c_3(0,0,1)$
$\Rightarrow c_1 = 2 \wedge c_2 = 2 \wedge c_3 = -3$
$\phi(0,1,1) = (3,-1,3) = c'_1(1,0,1) + c'_2(0,1,1) + c'_3(0,0,1)$
$\Rightarrow c'_1 = 3 \wedge c'_2 = -1 \wedge c'_3 = 1$
$\phi(0,0,1) = (1,0,1) = c''_1(1,0,1) + c''_2(0,1,1) + c''_3(0,0,1)$
$\Rightarrow c''_1 = 1 \wedge c''_2 = 0 \wedge c''_3 = 0$

$[\phi(1,1,-2)]_B = M(\phi; B, B) \cdot [(1,1,-2)]_B$

$(1,1,-2) = c'''_1(1,0,1) + c'''_2(0,1,1) + c'''_3(0,0,1)$
$= c'''_1 = 1 \wedge c'''_2 = 1 \wedge c'''_3 = -4$

$$[\phi(1,1,-2)]_B = \begin{bmatrix} 2 & 3 & 1 \\ 2 & -1 & 0 \\ -3 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

$\phi(1,1,-2) = \boxed{1} \times (1,0,1) + \boxed{1} \times (0,1,1) + \boxed{-2}(0,0,1)$
$= (1,1,0)$ ✓

$\boxed{7}$

a)

$$\phi\left[\begin{pmatrix} x \\ y \end{pmatrix}\right] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathcal{B}_2 = ((1,0),(0,1))$$

$$\mathcal{B}_3 = ((1,0,0),(0,1,0),(0,0,1))$$

$$M(\phi;\mathcal{B}_2,\mathcal{B}_3) = \begin{array}{c} \begin{array}{cc} \phi(1,0) & \phi(0,1) \end{array} \\ \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 2 \end{bmatrix} \end{array} \begin{array}{l} (1,0,0) \\ (0,1,0) \\ (0,0,1) \end{array} = \left[[\phi(1,0)]_{\mathcal{B}_3} \; [\phi(0,1)]_{\mathcal{B}_3} \right]$$

$$\phi(x,y) = (x+y,\ x-y,\ x+2y)$$

$$\phi\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \boxed{1}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \boxed{1}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \boxed{1}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\phi\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \boxed{1}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \boxed{-1}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \boxed{2}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

b)

$$S = \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \; ; \; \mathcal{T} = \left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right)$$

$$M(\phi; S, \mathcal{T}) = \begin{array}{c} \begin{array}{cc} \phi(1,-1) & \phi(0,1) \end{array} \\ \begin{bmatrix} 1 & -1/3 \\ 0 & 2/3 \\ -1 & 4/3 \end{bmatrix} \end{array} \begin{array}{l} (1,1,0) \\ (0,1,1) \\ (1,-1,1) \end{array} = \left[[\phi(1,-1)]_{\mathcal{T}} \; [\phi(0,1)]_{\mathcal{T}} \right]$$

$$\phi\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\phi\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} = c_1'\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2'\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3'\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 0 & 1 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 3 & 4 \end{array}\right] \rightsquigarrow \begin{cases} c_1 = 1 - c_3 \\ c_2 = -2 + 2c_3 \\ c_3 = \frac{4}{3} \end{cases} \Leftrightarrow \boxed{\begin{cases} c_1 = -1/3 \\ c_2 = 2/3 \\ c_3 = 4/3 \end{cases}}$$

$$\begin{cases} c'_1 + c'_3 = 0 \\ c'_1 + c'_2 - c'_3 = 2 \\ c'_2 + c'_3 = -1 \end{cases} \Leftrightarrow \boxed{\begin{cases} c'_1 = 1 \\ c'_2 = 0 \\ c'_3 = -1 \end{cases}}$$

Outra resolução: $\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$$\begin{array}{ccccc} v_1 & v_2 & v_3 & \phi(x_1) & \phi(x_2) \end{array}$$
$$\left[\begin{array}{ccc|cc} 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & -1 & 2 & -1 \\ 0 & 1 & 1 & -1 & 2 \end{array}\right] \underset{(\dots)}{\sim} \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 1 & -1/3 \\ 0 & 1 & 0 & 0 & 2/3 \\ 0 & 0 & 1 & -1 & 4/3 \end{array}\right]$$
$$\underbrace{\qquad\qquad}_{I_3} \quad \underbrace{\qquad\qquad}_{M(\phi;S,T)}$$

c)

$$\phi(2,-3) = \,?$$

$$[\phi(2,-3)]_{\mathcal{B}_3} = M(\phi;\mathcal{B}_2,\mathcal{B}_3) \cdot [(2,-3)]_{\mathcal{B}_2}$$

$$[(2,-3)]_{\mathcal{C}_2} = (2,-3)$$

$$M(\phi;\mathcal{B}_2,\mathcal{B}_3) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 2 \end{bmatrix}$$

$$[\phi(2,-3)]_{\mathcal{B}_3} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \\ -4 \end{bmatrix} \Rightarrow \phi(2,-3) = -1(1,0,0) + 5(0,1,0) - 4(0,0,1)$$
$$= (-1,5,-4)$$

$$[\phi(2,-3)]_T = M(\phi;S,T)\ [(2,-3)]_S \qquad (2,-3) = c_1(1,-1) + c_2(0,1)$$

$$\begin{cases} c_1 = 2 \\ c_2 = -1 \end{cases}$$

$$[\phi(2,-3)]_T = \begin{bmatrix} 1 & -1/3 \\ 0 & 2/3 \\ -1 & 4/3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 7/3 \\ -2/3 \\ -10/3 \end{bmatrix}$$

$$\phi(2,-3) = \frac{7}{3}(1,1,0) + \left(-\frac{2}{3}\right)(0,1,1) + \left(-\frac{10}{3}\right)(1,-1,1)$$
$$= (-1,5,-4)$$

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a)

$$M(\phi; S, \mathcal{V}) = \begin{array}{c} \begin{array}{ccc} \phi(t^2) & \phi(t) & \phi(1) \end{array} \\ \left[\begin{array}{ccc} & & \\ & & \\ & & \end{array} \right] \end{array} \begin{array}{l} t^2-1 \\ t \\ t-1 \end{array}$$

$$= \left[[\phi(t^2)]_{\mathcal{V}} \; [\phi(t)]_{\mathcal{V}} \; [\phi(1)]_{\mathcal{V}} \right]$$

$at^2 + bt + c$

$1t^2 + 0t + 0$

$$\phi(t^2) = (1 \cdot 2 \cdot 0)t^2 + (0-0)t + (1-0) = t^2 + 1 = a_1(t^2-1) + a_2(t) + a_3(t-1)$$

$$\phi(t) = t = a_1'(t^2-1) + a_2'(t) + a_3'(t-1)$$

$$\phi(1) = 2t^2 - t - 1$$

$$t^2 + 1 = a_1 t^2 + (a_2 + a_3)t - a_1 - a_3 \Rightarrow \begin{cases} a_1 = 1 \\ a_2 + a_3 = 0 \\ -a_1 - a_3 = 1 \end{cases} \Leftrightarrow \begin{cases} a_1 = 1 \\ a_2 = 2 \\ a_3 = -2 \end{cases}$$

$$t = a_1'(t^2-1) + a_2' t + a_3'(t-1) \Rightarrow \begin{cases} a_1' = 0 \\ a_2' = 1 \\ a_3' = 0 \end{cases}$$

$$2t^2 - t - 1 = a_1''(t^2-1) + a_2'' t + a_3''(t-1) \overset{(\dots)}{\Rightarrow} \begin{cases} a_1'' = 2 \\ a_2'' = 0 \\ a_3'' = -1 \end{cases}$$

Logo,

$$M(\phi; S, \mathcal{V}) = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ -2 & 0 & -1 \end{bmatrix}$$

b)

$$\phi(2t^2 - 3t + 1)$$

$$[\phi(2t^2 - 3t + 1)]_{\mathcal{V}} = M(\phi; S, \mathcal{V}) \, [2t^2 - 3t + 1]_S$$

$$[\phi(x)]_{\mathcal{V}} = M(\phi; S, \mathcal{V}) \, [x]_S$$

$$[2t^2 - 3t + 1]_S = \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}$$

$$S = (t^2, t, 1)$$

$$2t^2 - 3t + 1 = \boxed{2} \times t^2 + \boxed{(-3)} \times t + \boxed{1} \times 1$$

Atenção à ordem dos vetores na base

Observação: Se $S' = (1, t, t^2) \Rightarrow 2t^2 - 3t + 1 = \boxed{1} \times 1 + \boxed{-3}\, t + \boxed{2}\, t^2$

$$[\phi(2t-3t+1)]_{\gamma} = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ -2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \overset{(\dots)}{=} \begin{bmatrix} 4 \\ 1 \\ -5 \end{bmatrix}$$

Logo:

$$\phi(2t^2+2t+1) = 4(t^2-1) + 1t + (-5)(t-1)$$
$$= 4t^2 - 4t + 1$$

$\boxed{10, 11 \text{ e } 12}$ → Faltam!

$\boxed{13}$

a) e b)

$$\text{Ker}\,\phi = \{(x,y,z,w) \in \mathbb{R}^4 : \underbrace{\phi(x,y,z,w)}_{(x+y,\ z+w,\ x+z)} = (0,0,0,0)\}$$

$$\begin{cases} x+y=0 \\ z+w=0 \\ x+z=0 \end{cases} \Leftrightarrow \begin{cases} y=-x \\ -x+w=0 \\ z=-x \end{cases} \Leftrightarrow \begin{cases} y=-x \\ w=x \\ z=-x \end{cases}, x \in \mathbb{R}$$

$$\text{Ker}\,\phi = \{(x,-x,-x,x), x \in \mathbb{R}\}$$
$$= \langle (1,-1,-1,1) \rangle$$

, $(1,-1,-1,1) \neq (0,0,0,0) \Rightarrow \{(1,1,-1,1)\}$ é l.i. e gera Ker ϕ

$B = \{(1,1,-1,1)\}$ é uma base do Ker ϕ

$$\text{im}\,\phi = \{\phi(x,y,z,w), (x,y,z,w) \in \mathbb{R}^4\}$$
$$= \{(x+y,\ z+w,\ x+z), x,y,z,w \in \mathbb{R}\}$$
$$= \langle (1,0,1), (1,0,0), \cancel{(0,1,1)}, (0,1,0) \rangle$$

!

Obs: Como $\dim \mathbb{R}^3 = 3$ então 3 é o máximo nº de vetores l.i. Isto significa que um dos 4 vetores é comb. linear dos restantes. Pode ser retirado!

$$\text{im}\,\phi = \langle (1,0,1), (1,0,0), (0,1,0) \rangle$$

$\text{im}\,\phi = \langle (1,0,1), (1,0,0), (0,1,0) \rangle$

Como $\alpha(1,0,1) + \beta(1,0,0) + \gamma(0,1,0) = (0,0,0)$

$\Downarrow$ (…) faltava provar

$\alpha = \beta = \gamma = 0$, logo são (l.i)

Logo, $B' = \{(1,0,1), (1,0,0), (0,1,0)\}$ é uma base de im ϕ

c)

$$\text{Ker}(\phi) = \langle (1,-1,-1,1) \rangle$$

Como $(1,-1,-1,1) \in \text{Ker}\,\phi$ então $\text{Ker}\,\phi \neq \{(0,0,0,0)\}$, logo ϕ não é injetiva

Como $\dim \underbrace{\text{im}\,\phi = 3}_{\text{base com 3 elementos}} = \dim \underbrace{\mathbb{R}^3}_{\text{Espaço de chegada}}$ → ϕ é sobrejetiva

d)

$$\underbrace{\dim(\text{im}\,\phi)}_{3} + \underbrace{\dim(\text{Ker}\,\phi)}_{1} = \underbrace{\dim(V)}_{4}$$

$\boxed{14}$

a) $\ker \mathcal{L} = \{ at^2+bt+c \in \mathcal{P}_2 : \mathcal{L}(at^2+bt+c) = 0t^2+0t+0$

$\mathcal{L}(at^2+bt+c) = (a+c)t^2 + (b+c)t$

$\mathcal{L}(\overset{1}{t^2} \overset{-1}{-t} \overset{-1}{-1}) = -2t \Rightarrow t^2-t-1 \notin \ker \mathcal{L}$

$\mathcal{L}(\overset{1}{t^2} \overset{1}{+t} \overset{-1}{-1}) = 0 \Rightarrow t^2+t-1 \in \ker \mathcal{L}$

b) $2t^2-t$ e $t^2-t+2 \in \operatorname{im} \mathcal{L}$?

Para algum

$2t^2-t \in \operatorname{im} \mathcal{L} \Leftrightarrow 2t^2-t = \mathcal{L}(at^2+bt+c)$, $at^2+bt+c \in \mathcal{P}_2$

$\Leftrightarrow 2t^2-t = (a+c)t^2+(b+c)t$

$\Leftrightarrow \begin{cases} a+c=2 \\ b+c=-1 \end{cases} \Leftrightarrow \begin{cases} a=2-c \\ b=-1-c \end{cases}, c \in \mathbb{R}$

$2t^2-t = \mathcal{L}((2-c)t^2+(-1-c)t+c)$, $c \in \mathbb{R}$

$\underline{c=0} \Rightarrow 2t^2-t = \mathcal{L}(2t^2-t)$

$\underline{c=1} \Rightarrow 2t^2-t = \mathcal{L}(t^2-2t+1)$

Logo, $2t^2-t \in \operatorname{im} \phi$

$t^2-t+2 = \mathcal{L}(at^2+bt+c)$, $a,b,c \in \mathbb{R}$

$= (a+c)t^2+(b+c)t$

$\Rightarrow \begin{cases} 0=2 \\ a+c=1 \\ b+c=-1 \end{cases} \Rightarrow t^2-t+2 \notin \operatorname{im} \phi$

c)

$\ker \phi = \{ at^2+bt+c \in \mathcal{P}_2 : \underbrace{\phi(at^2+bt+c)}_{(a+c)t^2+(b+c)t} = 0t^2+0t+0 \}$

$\begin{cases} a+c=0 \\ b+c=0 \end{cases} \begin{cases} a=-c \\ b=-c \end{cases}$

Atenção ! Não passar para $\mathbb{R}^3$

$\ker \phi = \{ -ct^2-ct+c, c \in \mathbb{R} \} = \{ c(-t^2-t+1), c \in \mathbb{R} \}$

$= \boxed{\langle -t^2-t+1 \rangle}$

Como $-t^2 - t + 1 \neq 0$, logo $B = \{-t^2 - t + 1\}$ é l.i. $\Rightarrow$ B é uma base de Ker ϕ

Note-se $\dim(\text{Ker } \phi) = 1$

Sabemos pelo teorema das dimensões

$$\dim(\text{im } \phi) + \underbrace{\dim(\text{Ker } \phi)}_{1} = \underbrace{\dim(\mathcal{P}_2)}_{3}$$

$\Rightarrow \dim(\text{im } \phi) = 2$

$$\text{im } \phi = \{\phi(at^2+bt+c), (at^2+bt+c) \in \mathcal{P}_2\}$$

$$= \{(a+c)t^2 + (b+c)t, \ a, b, c \in \mathbb{R}\}$$

$$= \{at^2 + ct^2 + bt + ct, \ a, b, c \in \mathbb{R}\}$$

$$= \{at^2 + bt + c(t^2+t), \ a, b, c \in \mathbb{R}\}$$

$$= \langle t^2, t, \cancel{t^2+t} \rangle$$

$\Rightarrow$ $\{t, t^2, t^2+t\}$ não será uma base

dim im ϕ = 2

Logo, $\text{im } \phi = \langle t^2, t \rangle$

e $B = \{t^2, t\}$ é l.i.

Logo B é base de im ϕ

d)

ϕ não é sobrejetiva:

$$\text{im } \phi = \langle t, t^2 \rangle \neq \mathcal{P}_2$$

ϕ não é injetiva pq. $\text{Ker } \phi = \langle -t^2 - t + 1 \rangle \neq \{0t^2 + 0t + 0\}$

$(\dim(\text{im } \phi) \neq \dim \mathcal{P}_2) \Rightarrow \phi$ não é sobrej.

$\boxed{15}$

a)

$$\phi : \mathbb{R}^{2\times2} \longrightarrow \mathbb{R}^{2\times2}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto \phi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a-b & b-c \\ a-d & b-d \end{bmatrix}$$

$$\text{Ker}\,\phi = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times2} : \phi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

$$\phi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Leftrightarrow \begin{cases} a-b=0 \\ b-c=0 \\ a-d=0 \\ b-d=0 \end{cases} \quad \begin{cases} a=b \\ c=b \\ d=a \\ d=b \end{cases} \quad \begin{cases} a=b \\ c=b \\ d=b \end{cases}$$

$$\text{Ker}\,\phi = \left\{ \begin{bmatrix} b & b \\ b & b \end{bmatrix}, b \in \mathbb{R} \right\}$$

$= \left\langle \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\rangle$, como $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow$ l.i. e gera $\text{Ker}\,\phi$, logo é uma base

$$\text{im}(\phi) = \left\{ \phi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right), \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times2} \right\}$$

$$= \left\{ \begin{bmatrix} a-b & b-c \\ a-d & b-d \end{bmatrix}, a,b,c,d \in \mathbb{R} \right\}$$

$$= \left\{ a\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + b\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} + c\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix}, a,b,c,d \in \mathbb{R} \right\}$$

$$= \left\langle \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \cancel{\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}}, \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix} \right\rangle$$

$$\begin{array}{ll} \dim(\text{Ker}\,\phi) = 1 & \Rightarrow \quad 1 + \dim(\text{im}\,\phi) = 4 \\ \dim(\mathbb{R}^{2\times2}) = 4 & \Leftrightarrow \dim(\text{im}\,\phi) = 3 \end{array}$$

Logo, o conjunto anterior de vetores que gera $\text{im}\,\phi$ não é l.i. pois apenas existem no máximo 3 vetores l.i., retiramos um dos vetores e verificamos...

$$\alpha_1\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + \alpha_2\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} + \alpha_3\begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Leftrightarrow \begin{cases} \alpha_1 = 0 \\ \alpha_2 = 0 \\ \alpha_1 - \alpha_3 = 0 \\ \alpha_3 = 0 \end{cases} \quad \begin{cases} \alpha_1 = 0 \\ \alpha_2 = 0 \\ 0 = 0 \\ \alpha_3 = 0 \end{cases}$$

$\Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0$, logo $\left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix} \right\}$ é uma base de $\text{im}\,\phi$

b)

$$\phi : \mathbb{R}^{2\times2} \longrightarrow \mathbb{R}^{2\times2}$$

$$A \longmapsto \phi(A) = A^T$$

$\ker \phi = \{ A \in \mathbb{R}^{2\times 2} : \phi(A) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \}$

Seja $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$:

$$\phi(A) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Leftrightarrow A^T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Leftrightarrow A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow a = 0 \land b = 0 \land c = 0 \land d = 0$$

$\ker \phi = \{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \} = \{ 0_{\mathbb{R}^{2\times 2}} \}$, logo como $\dim(\ker \phi) = 0 \Rightarrow$ base de $\ker(\phi)$ é o vazio $B = \{\}$

$$\operatorname{im} \phi = \{ \phi(A), A \in \mathbb{R}^{2\times 2} \}$$
$$= \{ A^T, A \in \mathbb{R}^{2\times 2} \}$$
$$= \{ \begin{bmatrix} a & b \\ c & d \end{bmatrix}^T, A \in \mathbb{R}^{2\times 2} \}$$
$$= \{ \begin{bmatrix} a & c \\ b & d \end{bmatrix}, A \in \mathbb{R}^{2\times 2} \}$$
$$= \langle \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \rangle \Rightarrow \text{base canónica de } \mathbb{R}^{2\times 2}$$

$\boxed{16}$

a)

$$\phi : \mathbb{R}^4 \to \mathbb{R}^2$$
$$X \mapsto \phi(X) = AX = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 3 & 1 & 0 & -1 \end{bmatrix} X$$

$C_2 = ((1,0),(0,1)) \to$ base canónica de $\mathbb{R}^2$

$C_4 = ((1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)) \to$ " " $\mathbb{R}^4$

$$M(\phi; C_4, C_2) = \left[[\phi(1,0,0,0)]_{C_2} \ [\phi(0,1,0,0)]_{C_2} \ [\phi(0,0,1,0)]_{C_2} \ [\phi(0,0,0,1)]_{C_2} \right]$$

$$\phi(1,0,0,0) = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 3 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix}_{C_2} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

A mesma coisa para os outros ...

$$M(\phi; C_4, C_2) = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 3 & 1 & 0 & -1 \end{bmatrix} = A$$

c.q.m.

b)

$\dim(\mathrm{Ker}\,\phi) = \dim(N(A)) = \mathrm{nul}(A) = n - \mathrm{car}(A)$

$n = 4$

Como A tem 2 linhas $\Rightarrow$ $\mathrm{car}(A) \leq 2$

$\Rightarrow \mathrm{nul}(A) \geq 4 - 2$

$\Leftrightarrow \mathrm{nul}(A) \geq 2$

$\Leftrightarrow \dim(N(A)) \geq 2$

$\Leftrightarrow \dim(\mathrm{Ker}\,\phi) \geq 2$

c)

$S = ((1,1,1,0), (1,1,1,1), (1,0,1,1), (0,1,1,1))$

$\mathcal{T} = ((1,1), (1,-1))$

(i)

$$M(\phi; S, C_2) = \left[[\phi(1,1,1,0)]_{C_2} \; [\phi(1,1,1,1)]_{C_2} \; [\phi(1,0,1,1)]_{C_2} \; [\phi(0,1,1,1)]_{C_2} \right]$$

$$\phi(1,1,1,0) = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 3 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 4 \end{bmatrix}_{C_2} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Continuando para todos...

$$M(\phi; S, C_2) = \begin{bmatrix} 3 & 4 & 4 & 3 \\ 4 & 3 & 2 & 0 \end{bmatrix}$$

(ii) era o mesmo...