$\boxed{1}$

a)

$$\begin{vmatrix} 3 & 4 \\ 5 & 7 \end{vmatrix} = 3\times 7 - 4\times 5 = 21-20=1$$

b) $$\begin{vmatrix} 0 & 3 \\ 1 & 2 \end{vmatrix} = 0-3=-3$$

c) $$\begin{vmatrix} \sin\alpha & -\cos\alpha \\ \cos\alpha & \sin\alpha \end{vmatrix} = \sin^2\alpha + \cos^2\alpha = 1$$

d) $$\left|\begin{matrix} 0 & 7 & 1 \\ 4 & 1 & 2 \\ 1 & 7 & 3 \end{matrix}\right|\begin{matrix} 0 & 7 \\ 4 & 1 \\ 1 & 7 \end{matrix} = 0+14+28-1-0-84 = 42-85=-43$$

e) $$\begin{vmatrix} 1 & 7 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{vmatrix} = 1\times 1\times 3=3$$

$\boxed{2}$

$$\boxed{\det(cA) = c^n \det(A)}$$

Seja $A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $(cA) = c \times \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} c\times a_{11} & c\times a_{12} \\ c\times a_{21} & c\times a_{22} \end{bmatrix}$

$\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$, logo $\det(cA) = \begin{vmatrix} c\times a_{11} & c\times a_{12} \\ c\times a_{21} & c\times a_{22} \end{vmatrix} = c\times c\times \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

$= c^2 \times \det(A)$

pois A é uma matriz 2×2

- Se A for uma matriz $(n\times n)$, $\det(cA) = c^n \times \det(A)$

$\boxed{3}$

a) $|A^T| = |A| = 3$

b) $|AB| = |A|\times|B| = 3\times(-5) = -15$

c) $|A^4| = |A\times A\times A\times A| = [\det(A)]^4 = 3^4 = 81$

d) $|B^{-1}| = \dfrac{1}{\det(B)} = -\dfrac{1}{5}$

e) $|2A| = 2|A| = 2^5 \times 3 = 96$

f) $|2A^{-1}| = 2^5 \times \frac{1}{de(A)} = \frac{32}{3}$

g) $|(2A)^{-1}| = |\frac{1}{2} \times A^{-1}| = \left(\frac{1}{2}\right)^5 \times \frac{1}{det(A)} = \frac{1}{32} \times \frac{1}{3} = \frac{1}{96}$

h) $|AB^{-1}A^T| = det(A) \times \frac{1}{det(B)} \times det(A^T)$

$= 3 \times \frac{1}{-5} \times det(A)$

$= -\frac{9}{5}$

$\boxed{4}$

$$\begin{cases} det(AB^{-1}) = 2 \\ det\left[(2A)^{-1} B (A^T)^2\right] = 8 \end{cases} \quad \begin{cases} \frac{det(A)}{det(B)} = 2 \\ det\left(\frac{1}{2} A^{-1} B (A^T)^2\right) = 8 \end{cases}$$

$$\begin{cases} \text{———} \\ \left(\frac{1}{2}\right)^2 \times \frac{1}{det(A)} \times det(B) \times det\left[(A^T)^2\right] = 8 \end{cases} \begin{cases} \frac{det(B)}{det(A)} = \frac{1}{2} \\ \frac{det(B)}{det(A)} \times [det(A)]^2 = 32 \end{cases}$$

$$\begin{cases} \text{———} \\ \frac{1}{2} \times [det(A)]^2 = 32 \end{cases}$$

$$\begin{cases} \text{———} \\ det(A) = \sqrt{64} \end{cases} \vee \begin{cases} \text{———} \\ det(A) = -\sqrt{64} \end{cases}$$

$$\begin{cases} det(B) = 4 \\ det(A) = 8 \end{cases} \vee \begin{cases} det(B) = -4 \\ det(A) = -8 \end{cases}$$

R: $(det(B) = 4 \wedge det(A) = 8) \vee (det(B) = -4 \wedge det(A) = -8)$

$\boxed{5}$

a) $f \times 0 \times c = 0$

b) como $1 = 1 = 1$, logo $\det = 0$

c)
$$\begin{vmatrix} a_1+b_1 & a_1-b_1 & c_1 \\ a_2+b_2 & a_2-b_2 & c_2 \\ a_3+b_3 & a_3-b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_1 & c_1 \\ a_2 & a_2 & c_2 \\ a_3 & a_3 & c_3 \end{vmatrix} + \begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix} - \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= - \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} - \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= -2 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$\boxed{6}$ (Podia ter feito com operações elementores)

a)
$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ b_1 & b_2 & b_3 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^T = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -3$$

b)
$$\begin{vmatrix} b_1 & b_3 & b_2 \\ c_1 & c_3 & c_2 \\ a_1 & a_3 & a_2 \end{vmatrix} = - \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = (-1) \times (-1) \times \begin{vmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (-1) \times (-1) \times (-1) \times \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= - \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -10$$

c)
$$\begin{vmatrix} a_1 & 2b_1 & 4c_1+a_1 \\ a_2 & 2b_2 & 4c_2+a_2 \\ a_3 & 2b_3 & 4c_3+a_3 \end{vmatrix} = 2 \left(\begin{vmatrix} a_1 & b_1 & 4c_1 \\ a_2 & b_2 & 4c_2 \\ a_3 & b_3 & 4c_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & a_1 \\ a_2 & b_2 & a_2 \\ a_3 & b_3 & a_3 \end{vmatrix} \right)$$

$$= 2 \times 4 \times \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 8 \times 2 = 16$$

d)

$$\begin{vmatrix} a_1+2b_1 & a_2+2b_2 & a_3+2b_3 \\ 3c_1+b_1 & 3c_2+b_2 & 3c_3+b_3 \\ -b_1 & -b_2 & -b_3 \end{vmatrix} = -\begin{vmatrix} a_1+2b_1 & a_2+2b_2 & a_3+2b_3 \\ 3c_1+b_1 & 3c_2+b_2 & 3c_3+b_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = -\begin{vmatrix} a_1 & a_2 & a_3 \\ 3c_1+b_1 & 3c_2+b_2 & 3c_3+b_3 \\ b_1 & b_2 & b_3 \end{vmatrix} - 2\begin{vmatrix} b_1 & b_2 & b_3 \\ 3c_1+b_1 & 3c_2+b_2 & 3c_3+b_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= -\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ b_1 & b_2 & b_3 \end{vmatrix} - 3\times\begin{vmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{vmatrix} - 2\times\begin{vmatrix} b_1 & b_2 & b_3 \\ b_1 & b_2 & b_3 \\ b_1 & b_2 & b_3 \end{vmatrix} - 6\times\begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= 3\times\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 3\times\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^T = 3\times\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 3\times 1 = 3$$

e)

$$\begin{vmatrix} a_1+a_2 & a_3 & 2a_3+5a_1 \\ b_1+b_2 & b_3 & 2b_3+5b_1 \\ c_1+c_2 & c_3 & 2c_3+5c_1 \end{vmatrix} = \begin{vmatrix} a_1 & a_3 & 2a_3+5a_1 \\ b_1 & b_3 & 2b_3+5b_1 \\ c_1 & c_3 & 5c_3+5c_1 \end{vmatrix} + \begin{vmatrix} a_2 & a_3 & 2a_3+5a_1 \\ b_2 & b_3 & 2b_3+5b_1 \\ c_2 & c_3 & 2c_3+5c_1 \end{vmatrix}$$

$$= 2\times\begin{vmatrix} a_1 & a_3 & a_3 \\ b_1 & b_3 & b_3 \\ c_1 & c_3 & c_3 \end{vmatrix} + 5\times\begin{vmatrix} a_1 & a_3 & a_1 \\ b_1 & b_3 & b_1 \\ c_1 & c_3 & c_1 \end{vmatrix} + 2\times\begin{vmatrix} a_2 & a_3 & a_3 \\ b_2 & b_3 & b_3 \\ c_2 & c_3 & c_3 \end{vmatrix} + 5\times\begin{vmatrix} a_2 & a_3 & a_1 \\ b_2 & b_3 & b_1 \\ c_2 & c_3 & c_1 \end{vmatrix}$$

$$= (-1)\times 5\times\begin{vmatrix} a_1 & a_3 & a_2 \\ b_1 & b_3 & b_2 \\ c_1 & c_3 & c_2 \end{vmatrix} = 5\times\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 5\times 1 = 1$$

$\boxed{7}$

$$\det(B) = \det(A)\times 2\times(-1) = (-2)\times 2\times(-1) = 4$$

$\boxed{8}$

a)

$$\det(A) = \begin{vmatrix} 2 & 3 & 0 \\ 4 & 5 & 1 \\ 5 & 10 & 4 \end{vmatrix} \underset{\substack{\text{T.L.} \\ 1^{a}\text{ linha}}}{=} 2\times(-1)^{1+1}\times\begin{vmatrix} 5 & 1 \\ 10 & 4 \end{vmatrix} + 3\times(-1)^{1+2}\times\begin{vmatrix} 4 & 1 \\ 5 & 4 \end{vmatrix}$$

$$= 2\times(20-10) - 3\times(16-5)$$

$$= 20 - 33$$

$$= -13$$

b)

$$\begin{vmatrix} 2 & -1 & 3 \\ 4 & 1 & -2 \\ 7 & 5 & -6 \end{vmatrix} \underset{L_2' = L_2 - 2L_1}{\sim} \begin{vmatrix} 2 & -1 & 3 \\ 0 & 3 & -8 \\ 7 & 5 & -6 \end{vmatrix} \underset{\substack{T.L. \\ 1^a \text{ coluna}}}{=} 2 \times (-1)^{1+1} \times \begin{vmatrix} 3 & -8 \\ 5 & -6 \end{vmatrix} + 7 \times (-1)^{3+1} \times \begin{vmatrix} -1 & 3 \\ 3 & -8 \end{vmatrix}$$

$$= 2 \times (-18 + 40) + 7 \times (8 - 9)$$

$$= 2 \times 22 - 7$$

$$= 44 - 7 = 37$$

c)

$$\begin{vmatrix} 3 & -2 & 7 & 0 \\ 1 & -2 & -3 & 8 \\ 6 & 0 & -1 & 8 \\ -1 & 2 & 5 & 2 \end{vmatrix} \underset{L_3' = L_3 - L_2}{\sim} \begin{vmatrix} 3 & -2 & 7 & 0 \\ 1 & -2 & -3 & 8 \\ 5 & 2 & 2 & 0 \\ -1 & 2 & 5 & 2 \end{vmatrix} \underset{\substack{T.L. \\ 4^a \text{ coluna}}}{=} 8 \times (-1)^{2+4} \times \begin{vmatrix} 3 & -2 & 7 \\ 5 & 2 & 2 \\ -1 & 2 & 5 \end{vmatrix} + 2 \times (-1)^{4+4} \times \begin{vmatrix} 3 & -2 & 7 \\ 1 & -2 & -3 \\ 5 & 2 & 2 \end{vmatrix}$$

$$\underset{\substack{T.L. \\ 2^a \text{ coluna}}}{=} 8 \times \left[-2 \times (-1)^{1+2} \times \begin{vmatrix} 5 & 2 \\ -1 & 5 \end{vmatrix} + 2 \times (-1)^{2+2} \times \begin{vmatrix} 3 & 7 \\ -1 & 5 \end{vmatrix} + 2 \times (-1)^{3+2} \times \begin{vmatrix} 3 & 7 \\ 5 & 2 \end{vmatrix} \right]$$

$$+ 2 \times \left[(-2) \times (-1)^{1+2} \times \begin{vmatrix} 1 & -3 \\ 5 & 2 \end{vmatrix} + (-2) \times (-1)^{2+2} \times \begin{vmatrix} 3 & 7 \\ 5 & 2 \end{vmatrix} + 2 \times (-1)^{2+3} \times \begin{vmatrix} 3 & 7 \\ 1 & -3 \end{vmatrix} \right]$$

$$= 8 \times \left[2 \times (25 + 2) + 2 \times (15 + 7) - 2 \times (6 - 35) \right]$$

$$+ 2 \times \left[2 \times (2 + 15) - 2 \times (6 - 35) - 2 \times (-9 - 7) \right]$$

$$= 8 \times (54 + 44 + 58) + 2 \times (34 + 58 + 32)$$

$$= 8 \times 156 + 2 \times 124$$

$$= 1248 + 248 = 1496$$

d)

$$\begin{vmatrix} 0 & 1 & 4 & 5 \\ -1 & -2 & -4 & 6 \\ 0 & 0 & -1 & 0 \\ 1 & 2 & 7 & 2 \end{vmatrix} \underset{\substack{T.L. \\ 3^a \text{ linha}}}{=} -1 \times (-1)^{3+3} \times \begin{vmatrix} 0 & 1 & 5 \\ -1 & -2 & 6 \\ 1 & 2 & 2 \end{vmatrix}$$

$$\underset{\substack{T.L. \\ 1^a \text{ coluna}}}{=} -\left(-1 \times (-1)^{1+2} \times \begin{vmatrix} 1 & 5 \\ 2 & 2 \end{vmatrix} + 1 \times (-1)^{1+3} \times \begin{vmatrix} 1 & 5 \\ -2 & 6 \end{vmatrix} \right) = -\left((2 - 10) + (6 + 10) \right)$$

$$= -(-8 + 16) = -8$$

e)

$$\begin{vmatrix} 1 & 2 & 1 & 1 \\ 1 & 3 & 2 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 \end{vmatrix} \underset{\substack{T.L. \\ 1^a \text{ coluna}}}{=} 1 \times (-1)^{1+1} \times \begin{vmatrix} 3 & 2 & 2 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{vmatrix} + 1 \times (-1)^{2+1} \times \begin{vmatrix} 2 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{vmatrix}$$

$$\underset{\substack{T.L \\ 2^a \text{ coluna}}}{=} 2 \times (-1)^{1+2} \times \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} + 1 \times (-1)^{3+2} \times \begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix}$$

$$- 1 \times (-1)^{1+2} \times \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} - 1 \times (-1)^{3+2} \times \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}$$

$$= -2 \times (1-2) - (6-2) + (1-2) + (4-1)$$

$$= 2 - 4 - 1 + 3 = 0$$

$\boxed{9}$

a)

$$\det(A) = \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ -2 & 1 & 1 \end{vmatrix} \underset{L_3' := L_3 + 2 \times L_1}{\sim} \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 1 & -1 \end{vmatrix} \underset{L_3' := L_3 - L_2}{\sim} \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \end{vmatrix} = 1 \times 1 \times (-3) = -3$$

b)

$$2\det(A) = 2 \times (-3) = -6, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -6 \end{bmatrix}, \text{ por exemplo.}$$

$$\det(B) = 2 \times \det(A)$$

c)

$$\operatorname{adj}(A) = \begin{bmatrix} (-1)^{1+1} \times \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} & (-1)^{1+2} \times \begin{vmatrix} 0 & 2 \\ -2 & 1 \end{vmatrix} & (-1)^{1+3} \times \begin{vmatrix} 0 & 1 \\ -2 & 1 \end{vmatrix} \\ (-1)^{2+1} \times \begin{vmatrix} 0 & -1 \\ 1 & 1 \end{vmatrix} & (-1)^{2+2} \times \begin{vmatrix} 1 & -1 \\ -2 & 1 \end{vmatrix} & (-1)^{2+3} \times \begin{vmatrix} 1 & 0 \\ -2 & 1 \end{vmatrix} \\ (-1)^{3+1} \times \begin{vmatrix} 0 & -1 \\ 1 & 2 \end{vmatrix} & (-1)^{3+2} \times \begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix} & (-1)^{3+3} \times \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} (1-2) & -4 & 2 \\ -1 & (1-2) & -1 \\ 1 & -2 & 1 \end{bmatrix}^T = \begin{bmatrix} -1 & -1 & 1 \\ -4 & -1 & -2 \\ 2 & -1 & 1 \end{bmatrix}$$

d) A é invertível se $\det(A) \neq 0$, como $\det(A) = -3 \Rightarrow A$ é invertível

$$A^{-1} = \frac{1}{\det(A)} \times \operatorname{adj}(A) = -\frac{1}{3} \times \operatorname{adj}(A) = \begin{bmatrix} 1/3 & 1/3 & -1/3 \\ 4/3 & 1/3 & 2/3 \\ -2/3 & 1/3 & -1/3 \end{bmatrix}$$

$\boxed{10}$

a)

$$A^{-1} = \frac{1}{\det(A)} \times \operatorname{adj}(A)$$

$$\det(A) = \begin{vmatrix} 3 & 4 & -1 \\ 0 & 5 & -4 \\ 0 & 0 & 4 \end{vmatrix} = 3 \times 5 \times 4 = 60 \neq 0,$$ logo A é invertível

$$\operatorname{adj}(A) = \begin{bmatrix} \begin{vmatrix} 5 & -4 \\ 0 & 4 \end{vmatrix} & -\begin{vmatrix} 0 & -4 \\ 0 & 4 \end{vmatrix} & \begin{vmatrix} 0 & 5 \\ 0 & 0 \end{vmatrix} \\ -\begin{vmatrix} 4 & -1 \\ 0 & 4 \end{vmatrix} & \begin{vmatrix} 3 & -1 \\ 0 & 4 \end{vmatrix} & -\begin{vmatrix} 3 & 4 \\ 0 & 0 \end{vmatrix} \\ \begin{vmatrix} 4 & -1 \\ 5 & -4 \end{vmatrix} & -\begin{vmatrix} 3 & -1 \\ 0 & -4 \end{vmatrix} & \begin{vmatrix} 3 & 4 \\ 0 & 5 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 20 & 0 & 0 \\ -16 & 12 & 0 \\ -11 & 12 & 15 \end{bmatrix}^T = \begin{bmatrix} 20 & -16 & -11 \\ 0 & 12 & 12 \\ 0 & 0 & 15 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1/3 & -4/15 & -11/60 \\ 0 & 1/5 & 1/5 \\ 0 & 0 & 1/4 \end{bmatrix}$$

b), c), d) Fazer quando tiver tempo! 😃

$\boxed{11}$

a)

$$\det(A) = \begin{vmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 1 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{vmatrix} \underset{\substack{L'_3 := L_3 - L_1 \\ L'_4 := L_4 - L_2}}{\sim} \begin{vmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -2 & 1 \\ 0 & 0 & 2 & 0 \end{vmatrix} \underset{L'_3 := L_3 - 2L_2}{\sim} \begin{vmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 2 & 0 \end{vmatrix}$$

$$\underset{L_3 \leftrightarrow L_4}{\sim} - \begin{vmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix} = \cancel{-}\left(1 \times 1 \times 2 \times \cancel{(-1)}\right) = 2$$

b)

$$\text{elemento } (2,3) \text{ da adjunta} = (-1)^{3+2} \times \overset{3,2}{\begin{vmatrix} 1 & 2 & -1 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{vmatrix}} = (-1) \times \left((-1)^{1+1} \times \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} \right)$$

$$= -(-1-1) = 2$$

elemento (2,3) da inversa $= \dfrac{1}{\det(A)} \times$ (elemento (2,3) da adjunta)

$$= \frac{1}{\cancel{2}} \times \cancel{2} = 1$$

$\boxed{12}$

a)

$$\det(A) = \begin{vmatrix} 2 & a-1 & 0 \\ -3 & 2 & 1 \\ 0 & a+1 & -2 \end{vmatrix} \underset{\substack{T.L.\\ 1^a \text{ coluna}}}{=} 2 \times (-1)^{1+1} \times \begin{vmatrix} 2 & 1 \\ a+1 & -2 \end{vmatrix} + (-3) \times (-1)^{2+1} \times \begin{vmatrix} a-1 & 0 \\ a+1 & -2 \end{vmatrix}$$

$$= 2 \times \left(-4 - (a+1)\right) + 3\,(-2a+2)$$

$$= 2 \times (-4 - a - 1) - 6a + 6$$

$$= -10 - 2a - 6a + 6$$

$$= -8a - 4$$

b)

Para A ser singular $\det(A) = 0 \Leftrightarrow -8a - 4 = 0 \Leftrightarrow a = -\frac{1}{2}$

c)

$a = -2 \Rightarrow \det(A) = 12 \Rightarrow$ A é invertível $\Rightarrow A = \begin{bmatrix} 2 & -3 & 0 \\ -3 & 2 & 1 \\ 0 & -1 & -2 \end{bmatrix}$

X = elemento (1,2) da inversa de A $= \dfrac{1}{\det(A)} \times$ (elemento (1,2) da adjacente de A)

elemento (1,2) da adjacente de A $= \underbrace{(-1)^{2+1} \times \begin{vmatrix} -3 & 0 \\ -1 & -2 \end{vmatrix}}_{\text{Complemento algébrico de } (2,1) \text{ de } A} = -(6) = -6$

!

$$X = \frac{1}{12} \times (-6) = -\frac{1}{2}$$

$\boxed{13}$

Para que o sistema homogénio $AX=0$ tenha apenas a solução trivial, A tem de ser invertível $\Longrightarrow \det(A)\neq 0$

$$\det(A) = \begin{vmatrix} \beta & 6 & 1 \\ 0 & \beta-1 & 1 \\ 0 & 1 & \beta+5 \end{vmatrix} = \beta \times (-1)^{1+1} \times \begin{vmatrix} \beta-1 & 1 \\ 1 & \beta+5 \end{vmatrix}$$

$$= \beta \times \left((\beta-1)(\beta+5)-1\right)$$

$$= \beta\left(\beta^2+5\beta-\beta-5-1\right)$$

$$= \beta^3+4\beta^2-6\beta$$

$$\det(A)=0 \Leftrightarrow \beta^3+4\beta^2-6\beta=0 \Leftrightarrow \beta\left(\beta^2+4\beta-6\right)=0$$

$$\Leftrightarrow \beta=0 \lor \beta^2+4\beta-6=0$$

$$\Leftrightarrow \beta=0 \lor \beta=\frac{-4\pm\sqrt{16-4\times 1\times(-6)}}{2}$$

$$\Leftrightarrow \beta=0 \lor \beta=\frac{-4\pm\sqrt{40}}{2}$$

$$\begin{array}{r|l} 40 & 2 \\ 20 & 2 \\ 10 & 2 \\ 5 & 5 \\ 1 & \end{array} \quad 2\sqrt{10}$$

$$\Leftrightarrow \beta=0 \lor \beta=-2-\sqrt{10} \lor \beta=-2+\sqrt{10}$$

Logo $\det(A)\neq 0$ sse $\beta\neq 0 \land \beta\neq -2-\sqrt{10} \land \beta\neq -2+\sqrt{10}$,

logo $\beta \in \mathbb{R}\setminus\{-2-\sqrt{10}, 0, -2+\sqrt{10}\}$

$\boxed{14}$! HARDCORE

$$\det(A_{n\times n})\neq 0$$

Pretende-se mostrar: $\boxed{A\,(\mathrm{adj}A) = \det(A)\times I_n \Longrightarrow \det(\mathrm{adj}A) = [\det(A)]^{n-1}}$

Sabemos: $A^{-1} = \dfrac{1}{\det(A)} \times \mathrm{adj}(A)$ $\boxed{E1}$

$A\times A^{-1} = I_n$, sendo A uma matriz do tipo $n\times n$ $\boxed{E2}$

$\boxed{E1}$

$$A^{-1} = \frac{\operatorname{adj}(A)}{\det(A)} \Leftrightarrow \operatorname{adj}(A) = A^{-1} \times \det(A)$$

<u>Substituindo</u>:

$$A \times (\operatorname{adj} A) = \underbrace{A \times A^{-1}}_{\boxed{E2}} \times \det(A)$$

$$= I_n \times \det(A)$$

$$= \det(A) \times I_n$$

c.q.d.

Se $\boxed{A \times \operatorname{adj}(A) = \det(A) \times I_n}$,

logo: $\det\left(A \operatorname{adj}(A)\right) = \det\left(\det(A) \times I_n\right)$

$$\Leftrightarrow \det(A) \times \det(\operatorname{adj} A) = [\det A]^n \times \underbrace{\cancel{\det(I_n)}}_{1}$$

$$\Leftrightarrow \det(\operatorname{adj} A) = \frac{[\det A]^n}{\det A}$$

$$\Leftrightarrow \det(\operatorname{adj} A) = (\det A)^{n-1}$$

c.q.d.

$\boxed{15}$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\operatorname{adj} A = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}^T = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

$$A \cdot \operatorname{adj} A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

Como $A \cdot (\operatorname{adj} A) = \det(A) \times I_n$, $\det(A) = -2$

$\boxed{16}$ A é uma matriz quadrada e $\det(A) \neq 0$

$\boxed{17}$

a)

$$\begin{cases} x - y - z = 0 \\ 4x + 2y - 4z = 6 \\ 3x + 2y - z = -1 \end{cases}$$

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 4 & 2 & -4 \\ 3 & 2 & -1 \end{bmatrix} \qquad \det(A) = \begin{vmatrix} 1 & -1 & -1 \\ 4 & 2 & -4 \\ 3 & 2 & -1 \end{vmatrix} = 2 \times \begin{vmatrix} 1 & -1 & -1 \\ 2 & 1 & -2 \\ 3 & 2 & -1 \end{vmatrix} \underset{\substack{L'_2 := L_2 - 2L_1 \\ L'_3 := L_3 - 3L_1}}{\sim} 2 \times \begin{vmatrix} 1 & -1 & -1 \\ 0 & 3 & 0 \\ 0 & 5 & 2 \end{vmatrix}$$

$$\underset{\substack{\text{T.L.} \\ 1^{a}\text{ coluna}}}{=} 2 \times 1 \times (-1)^{1+1} \times \begin{vmatrix} 3 & 0 \\ 5 & 2 \end{vmatrix} = 2 \times 6 = 12$$

Resolvendo pela <u>Regra de Cramer</u>:

$$x = \frac{\begin{vmatrix} 0 & -1 & -1 \\ 6 & 2 & -4 \\ -1 & 2 & -1 \end{vmatrix}}{12} \underset{L'_2 := L_2 + 6L_3}{\sim} \frac{\begin{vmatrix} 0 & -1 & -1 \\ 0 & 14 & -10 \\ -1 & 2 & -1 \end{vmatrix}}{12} \underset{\substack{\text{T.L.} \\ 1^{a}\text{ coluna}}}{=} \frac{(-1) \times \begin{vmatrix} -1 & -1 \\ 14 & -10 \end{vmatrix}}{12}$$

$$= -\frac{24}{12} = -2$$

$$y = \frac{\begin{vmatrix} 1 & 0 & -1 \\ 4 & 6 & -4 \\ 3 & -1 & -1 \end{vmatrix}}{12} \underset{C'_3 := C_3 + C_1}{\sim} \frac{\begin{vmatrix} 1 & 0 & 0 \\ 4 & 6 & 0 \\ 3 & -1 & 2 \end{vmatrix}}{12} \underset{\substack{\text{T.L.} \\ 1^{a}\text{ linha}}}{=} \frac{\begin{vmatrix} 6 & 0 \\ -1 & 2 \end{vmatrix}}{12} = \frac{12}{12} = 1$$

$$z = \frac{\begin{vmatrix} 1 & -1 & 0 \\ 4 & 2 & 6 \\ 3 & 2 & -1 \end{vmatrix}}{12} \underset{C'_2 := C_2 + C_1}{\sim} \frac{\begin{vmatrix} 1 & 0 & 0 \\ 4 & 6 & 6 \\ 3 & 5 & -1 \end{vmatrix}}{12} \underset{\substack{\text{T.L.} \\ 1^{a}\text{ linha}}}{=} \frac{\begin{vmatrix} 6 & 6 \\ 5 & -1 \end{vmatrix}}{12} = \frac{-6-30}{12} = \frac{-36}{12} = -3$$

$S = \{(-2, 1, -3)\}$

b)

$$\begin{cases} 4x - 3z = -2 \\ 2x - y = -2 \\ x - 3y + z = 4 \end{cases}$$

$$A = \begin{bmatrix} 4 & 0 & -3 \\ 2 & -1 & 0 \\ 1 & -3 & 1 \end{bmatrix} \qquad \det(A) = \begin{vmatrix} 4 & 0 & -3 \\ 2 & -1 & 0 \\ 1 & -3 & 1 \end{vmatrix} \underset{L'_1 := L_1 + 3L_3}{\sim} \begin{vmatrix} 7 & -9 & 0 \\ 2 & -1 & 0 \\ 1 & -3 & 1 \end{vmatrix} \underset{\substack{\text{T.L.} \\ 3^{a}\text{ coluna}}}{=} \begin{vmatrix} 7 & -9 \\ 2 & -1 \end{vmatrix} = -7 + 18 = 11$$

Resolvendo o sistema pela regra de Cramer:

$$x = \frac{\begin{vmatrix} -2 & 0 & -3 \\ -2 & -1 & 0 \\ 4 & -3 & 1 \end{vmatrix}}{11} \underset{L'_1 := L_1 + 3L_3}{\sim} \frac{\begin{vmatrix} 10 & -9 & 0 \\ -2 & -1 & 0 \\ 4 & -3 & 1 \end{vmatrix}}{11} \underset{\substack{\text{T.L.} \\ 3^{a}\text{ coluna}}}{=} \frac{\begin{vmatrix} 10 & -9 \\ -2 & -1 \end{vmatrix}}{11} = \frac{-10 - 18}{11} = -\frac{28}{11}$$

$$y = \frac{\begin{vmatrix} 4 & -2 & -3 \\ 2 & -2 & 0 \\ 1 & 4 & 1 \end{vmatrix}}{11} \underset{L'_1 := L_1 + 3L_3}{\sim} \frac{\begin{vmatrix} 7 & 10 & 0 \\ 2 & -2 & 0 \\ 1 & 4 & 1 \end{vmatrix}}{11} \underset{\substack{\text{T.L.} \\ 3^{a}\text{ coluna}}}{=} \frac{\begin{vmatrix} 7 & 10 \\ 2 & -2 \end{vmatrix}}{11} = \frac{-14 - 20}{11} = -\frac{34}{11}$$

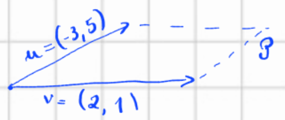
$$z = \frac{\begin{vmatrix} 4 & 0 & -2 \\ 2 & -1 & -2 \\ 1 & -3 & 4 \end{vmatrix}}{11} \underset{C'_1 := C_1 + 2C_3}{\sim} \frac{\begin{vmatrix} 0 & 0 & -2 \\ -2 & -1 & -2 \\ 9 & -3 & 4 \end{vmatrix}}{11} \underset{\substack{\text{T.L.} \\ 1^{a}\text{ linha}}}{=} \frac{(-2) \times \begin{vmatrix} -2 & -1 \\ 9 & -3 \end{vmatrix}}{11} = \frac{-2 \times (6 + 9)}{11}$$

$$= -\frac{30}{11}$$

c) A mesma coisa mas com muitos cálculos!

$\boxed{18}$

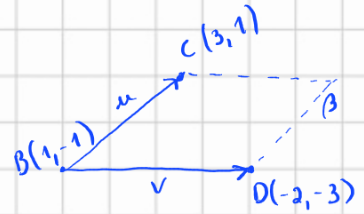
a)



Área de $\mathcal{P}$ é $|\det(A)|$, sendo $A = [u, v]$ $\quad u = \begin{bmatrix} -3 \\ 5 \end{bmatrix} \quad v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$A = \begin{bmatrix} -3 & 2 \\ 5 & 1 \end{bmatrix} \qquad |\det(A)| = \left| \begin{vmatrix} -3 & 2 \\ 5 & 1 \end{vmatrix} \right| = |-3 - 10| = |-13| = 13$$

b)



Seja u e v os vetores, $u = \overrightarrow{BC} = C - B = (3-1, 1+1) = (2,2)$

$v = \overrightarrow{BD} = D - B = (-2-1, -3+1) = (-3,-2)$

A área de $\mathcal{B}$ é $|\det(A)|$, sendo $A = [u, v]$ $u = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ $v = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$

$A = \begin{bmatrix} 2 & -3 \\ 2 & -2 \end{bmatrix}$ $\quad |\det(A)| = \begin{vmatrix} 2 & -3 \\ 2 & -2 \end{vmatrix} = -4 + 6 = 2$

$\boxed{19}$

$a \in \mathbb{R}$

O volume de $\mathcal{P}_a$ é $|\det(A)|$, sendo $A = [u, v, w]$ $u = \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}$, $v = \begin{bmatrix} a-1 \\ 2 \\ a+1 \end{bmatrix}$, $w = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$

$A = \begin{bmatrix} 2 & a-1 & 0 \\ -3 & 2 & 1 \\ 0 & a+1 & -2 \end{bmatrix}$ $\quad \det(A)$

$$V_{\mathcal{P}_a} = |\det(A)| = \left| \begin{vmatrix} 2 & a-1 & 0 \\ -3 & 2 & 1 \\ 0 & a+1 & -2 \end{vmatrix} \right| \underset{\substack{T.L. \\ 1^a \text{ coluna}}}{=} \left| 2 \times \begin{vmatrix} 2 & 1 \\ a+1 & -2 \end{vmatrix} + (-3) \times (-1) \times \begin{vmatrix} a-1 & 0 \\ a+1 & -2 \end{vmatrix} \right|$$

$$= |2 \times (-4 - a - 1) + 3 \times (-2a + 2)|$$

$$= |-10 - 2a - 6a + 6|$$

$$= |-8a - 4|$$

$\mathcal{P}_a = 4 \Leftrightarrow |-8a - 4| = 4 \Leftrightarrow -8a - 4 = -4 \lor -8a - 4 = 4$

$\Leftrightarrow -8a = 0 \lor -8a = 8$

$\Leftrightarrow a = 0 \lor a = -1$

$\Leftrightarrow a \in \{0, -1\}$

$\boxed{20}$

$A_{m \times m}$

$$A \times B = A \times C$$

Se $\det(A) \neq 0$, então A é invertível:

$$\Rightarrow A^{-1} \times (A \times B) = A^{-1}(A \times C)$$

$$\Leftrightarrow \underbrace{A^{-1} \times A}_{I_m} \times B = \underbrace{A^{-1} \times A}_{I_m} \times C$$

$$\Leftrightarrow I_m \times B = I_m \times C$$

$$\Leftrightarrow B = C$$

Se $\det(A) = 0$, por exemplo: $A = \begin{bmatrix} 0 & 7 \\ 0 & 2 \end{bmatrix}$

$B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$ $C = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}$

$$A \times B = A \times C$$

$$\Leftrightarrow \begin{bmatrix} 0 & 7 \\ 0 & 2 \end{bmatrix} \times \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} = \begin{bmatrix} 0 & 7 \\ 0 & 2 \end{bmatrix} \times \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} 7b_3 & 7b_4 \\ 2b_3 & 2b_4 \end{bmatrix} = \begin{bmatrix} 7c_3 & 7c_4 \\ 2c_3 & 2c_4 \end{bmatrix}$$

$$\Leftrightarrow \underbrace{b_3 = c_3 \wedge b_4 = c_4}$$

Mas não se sabe se $b_1 = c_1 \wedge b_2 = c_2$, logo B pode ser diferente de C

$\boxed{21}$

a) $\det(A \times A^T) = \det(A) \times \det(A^T) = \underbrace{[\det(A)]^2 \geq 0}_{C.V.}$

b)

$$AB = I_m \Leftrightarrow \det(A \times B) = \det(I_m)$$

$$\Leftrightarrow \det(A) \times \det(B) = 1,$$

logo $\det(A) \neq 0 \wedge \det(B) \neq 0$

c)

$$\det(A)=0$$

$$AB = X$$

$$\Leftrightarrow \det(AB) = \det(X)$$

$$\Leftrightarrow \underbrace{\det(A)}_{=0} \times \det(B) = \det(X)$$

$\Leftrightarrow \det(X)=0$, logo X é singular $\Rightarrow AB$ é uma matriz singular

d)

$$\det(A)\neq 0$$

$$A^2 = A$$

$$\Leftrightarrow \det(A^2) = \det(A)$$

$$\Leftrightarrow [\det(A)]^2 = \det(A)$$

Seja $\det(A) = x$:

$$\Rightarrow x^2 = x \Leftrightarrow x^2 - x = 0 \Leftrightarrow x(x-1)=0$$

$$\Leftrightarrow x = 0 \ \vee \ x = 1$$

Substituindo $x = \det(A)$

Como A é não singular $\det(A)\neq 0$

$$\Rightarrow \underline{\det(A)=0} \ \vee \ \det(A)=1$$

$$\Leftrightarrow \det(A)=1$$

e)

$$A = A^{-1}$$

$$\Leftrightarrow \det(A) = \det(A^{-1})$$

$$\Leftrightarrow \det(A) = \frac{1}{\det(A)}$$

$$\Leftrightarrow [\det(A)]^2 = 1$$

$$\Leftrightarrow \det(A) = \pm 1$$

22

a) $\boxed{\det(-A) = -\det(A)}$

$$\det(-A) = (-1)^n \times \det(A),$$

$n = 1$

$$\det(-A) = -\det(A)$$

$n = 2$

$$\det(-A) = \det(A)$$

, logo a afirmação é falsa

b)

$$A^T = A^{-1}$$

$$\Leftrightarrow \det(A^T) = \det(A^{-1})$$

$$\Leftrightarrow \det(A) = \frac{1}{\det(A)}$$

$$\Leftrightarrow [\det(A)]^2 = 1$$

$$\Leftrightarrow \det(A) = \pm 1$$

, logo a afirmação é falsa

c)

$\det(A) = 0$, por exemplo se $A = \begin{bmatrix} 0 & 5 \\ 0 & 2 \end{bmatrix}$ e $A \neq 0$, logo a afirmação é falsa

d)

Verdadeiro, pela regra de Cramer!

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Ex.:

$$x = \frac{\begin{vmatrix} 0 & b & c \\ 0 & e & f \\ 0 & h & i \end{vmatrix}}{\det(A)} = 0$$

$$y = \frac{\begin{vmatrix} a & 0 & c \\ b & 0 & f \\ c & 0 & i \end{vmatrix}}{\det(A)} = 0$$

$$z = \frac{\begin{vmatrix} a & b & 0 \\ d & e & 0 \\ g & h & 0 \end{vmatrix}}{\det(A)} = 0$$

logo, $S = \{(0,0,0)\}$

e)

$A \neq I_n$, não implica que $\det(A) \neq 1$, por exemplo $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $\det(A) = 1$ e $A \neq I_n$

$A^2 = A$

$\Leftrightarrow A = \boxed{A \times A^{-1}}$, só acontece se $\det(A) \neq 0$

$\Leftrightarrow \underbrace{A = I_n}_{\text{Impossível}}$, logo $\det(A) = 0$

Afirmação é Verdadeira

f)

$\det(AB) = 0$

$\Leftrightarrow \det(A) \times \det(B) = 0$

$\Leftrightarrow \det(A) = 0 \vee \det(B) = 0$

Afirmação é verdadeira

g) ! $\boxed{AB \neq BA \text{, não implica que } \det(AB) \neq \det(BA)}$

$AB \neq BA$

Verifica se: $\det(AB) \neq \det(BA)$

$\Leftrightarrow \det(A) \times \det(B) \neq \det(B) \times \det(A)$

$\Leftrightarrow \det(A) \times \det(B) \neq \det(A) \times \det(B)$

$\Leftrightarrow \underbrace{\det(AB) \neq \det(AB)}_{\text{C.I.}}$

Afirmação é falsa!