Aula 07

Calcular o determinante de A:

$$A = \begin{bmatrix} a & 0 & 0 & 0 & b \\ b & a & 0 & 0 & 0 \\ 0 & b & a & 0 & 0 \\ 0 & 0 & b & a & 0 \\ 0 & 0 & 0 & b & a \end{bmatrix}$$

$$\det(A) \underset{\substack{\text{T.L.} \\ 1^{\text{a}}\text{ linha}}}{=} a \times (-1)^{1+1} \times \begin{vmatrix} a & 0 & 0 & 0 \\ b & a & 0 & 0 \\ 0 & b & a & 0 \\ 0 & 0 & b & a \end{vmatrix} + b\,(-1)^{1+5} \begin{vmatrix} b & a & 0 & 0 \\ 0 & b & a & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & b \end{vmatrix}$$

$$= a \times a^4 + b \times b^4 = a^5 + b^5$$

$$\begin{vmatrix} 2 & 4 & 6 & 8 \\ 3 & 6 & 5 & 9 \\ 2 & 1 & 4 & 7 \\ 1 & 2 & 2 & 2 \end{vmatrix} = 2 \begin{vmatrix} 1 & 2 & 3 & 4 \\ 3 & 6 & 5 & 9 \\ 2 & 1 & 4 & 7 \\ 1 & 2 & 2 & 2 \end{vmatrix} \underset{\substack{L'_2 := L_2 - 3L_1 \\ L'_3 := L_3 - 2L_1 \\ L'_4 := L_4 - L_1}}{=} 2 \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & -4 & -3 \\ 0 & -3 & -2 & -1 \\ 0 & 0 & -1 & -2 \end{vmatrix}$$

$$\underset{\substack{\text{T.L.} \\ 1^{\text{a}}\text{ coluna}}}{=} 1 \times (-1)^{1+1} \times 2 \times \begin{vmatrix} 0 & -4 & -3 \\ -3 & -2 & -1 \\ 0 & -1 & -2 \end{vmatrix}$$

$$\underset{\substack{\text{T.L.} \\ 1^{\text{a}}\text{ coluna}}}{=} 2 \times \left[-3 \times (-1)^{2+1} \times \begin{vmatrix} -4 & -3 \\ -1 & -2 \end{vmatrix} \right] = 2 \times \left[3 \times (8-3) \right] = 2 \times 15 = 30$$

Adjunta de uma matriz $A\,(n \times n)$

$$\operatorname{adj}(A) = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}^T$$

Exemplo:

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\text{adj}(A) = \begin{bmatrix} (-1)^{1+1}\begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} & (-1)^{1+2}\begin{vmatrix} 3 & -1 \\ 0 & 0 \end{vmatrix} & (-1)^{1+3}\begin{vmatrix} 3 & 1 \\ 0 & 1 \end{vmatrix} \\ (-1)^{2+1}\begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} & (-1)^{2+2}\begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} & (-1)^{2+3}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \\ (-1)^{3+1}\begin{vmatrix} 0 & 2 \\ 1 & -1 \end{vmatrix} & (-1)^{3+2}\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} & (-1)^{3+3}\begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} \end{bmatrix}^T$$

↳ linha de zeros

$$= \begin{bmatrix} 1 & 0 & 3 \\ 2 & 0 & -1 \\ -2 & -7 & 1 \end{bmatrix}$$

Importante!

A é invertível ⟺ O sistema hom. tem apenas o solução trivial

negando

O sistema hom. tem uma solução não trivial ⟺ A não é invertível ⟹ $\det(A) = 0$

Teorema:

- A é invertível ⟺ $\det(A) \neq 0$

Corolário:

- Seja $A(n \times n)$. O sistema homogénio $AX = 0$ tem uma solução não trivial se $\det(A) = 0$

Teorema:

- Seja A $(n \times n)$ invertível. Então:

→ $\det(A^{-1}) = \dfrac{1}{\det(A)}$

→ $A^{-1} = \dfrac{1}{\det(A)} \text{adj}(A)$ → outra forma de calcular a inversa !

Pelo exemplo anterior:

$$\text{adj}(A) = \begin{bmatrix} 1 & 2 & -2 \\ 0 & 0 & 7 \\ 3 & -1 & 1 \end{bmatrix}, \text{ onde } A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} 1 & 0 & 2 \\ 3 & 1 & -1 \\ 0 & 1 & 0 \end{vmatrix} \underset{\substack{\text{T.L.} \\ 3^{\text{a}} \text{ linha}}}{=} (-1)^{3+2} \begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} = -1 \times (-7) = 7$$

$$A^{-1} = \frac{1}{\det(A)} \times \text{adj}(A) = \frac{1}{7} \times \begin{bmatrix} 1 & 2 & -2 \\ 0 & 0 & 7 \\ 3 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1/7 & 2/7 & -2/7 \\ 0 & 0 & 1 \\ 3/7 & -1/7 & 1/7 \end{bmatrix}$$

Regra de Cramer

- Seja $A(n\times n)$ tal que $\det(A)\neq 0$

- Então o sistema $AX=B$ é possível e determinado e a sua única solução é $x=\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}$, onde $x_j=\dfrac{\det(A_j)}{\det(A)}$, $j\in\{1,2,\cdots n\}$,
  onde A_j é a matriz que se obtém de A substituindo a coluna j pela coluna B

__Exemplo:__

$$\begin{cases}x_1+x_2+x_3=1\\2x_1-x_2+x_3=1\\x_1+2x_3=0\end{cases}$$, resolver pela regra de Cramer

$$A=\begin{bmatrix}1&1&1\\2&-1&1\\1&0&2\end{bmatrix}\qquad \det(A)=\begin{vmatrix}1&1&1\\2&-1&1\\1&0&2\end{vmatrix}\underset{\substack{L_2':=L_2-2L_1\\L_3':=L_3-L_1}}{=}\begin{vmatrix}1&1&1\\0&-3&-1\\0&-1&1\end{vmatrix}\underset{L_3':=L_3-\frac{1}{3}L_2}{=}\begin{vmatrix}1&1&1\\0&-3&-1\\0&0&4/3\end{vmatrix}$$

$$=1\times(-\cancel{3})\times\frac{4}{\cancel{3}}=-4$$

$$x_1=\frac{\begin{vmatrix}1&1&1\\1&-1&1\\0&0&2\end{vmatrix}}{-4}\underset{L_2':=L_2-L_1}{=}\frac{\begin{vmatrix}1&1&1\\0&-2&0\\0&0&2\end{vmatrix}}{-4}=\frac{-2\times2}{-4}=1$$

$$x_2=\frac{\begin{vmatrix}1&1&1\\2&1&1\\1&0&2\end{vmatrix}}{-4}\underset{\substack{L_2':=L_2-2L_1\\L_3':=L_3-L_1}}{=}\frac{\begin{vmatrix}1&1&1\\0&-1&-1\\0&-1&1\end{vmatrix}}{-4}\underset{L_3':=L_3-L_2}{=}\frac{\begin{vmatrix}1&1&1\\0&-1&-1\\0&0&2\end{vmatrix}}{-4}=\frac{-2}{-4}=\frac{1}{2}$$

$$x_3=\frac{\begin{vmatrix}1&1&1\\2&-1&1\\1&0&0\end{vmatrix}}{-4}\underset{\substack{L_3':=L_3-L_1\\L_2':=L_2-2L_1}}{=}\frac{\begin{vmatrix}1&1&1\\0&-3&-1\\0&-1&-1\end{vmatrix}}{-4}\underset{L_3':=L_3-\frac{1}{3}L_2}{=}\frac{\begin{vmatrix}1&1&1\\0&-3&-1\\0&0&-2/3\end{vmatrix}}{-4}=\frac{-\cancel{3}\times(-2/\cancel{3})}{-4}=-\frac{1}{2}$$

Seja $A=\begin{bmatrix}\beta&6&1\\0&\beta-1&1\\0&1&\beta+5\end{bmatrix}$, determine todos valores de β para os quais o sistema $AX=0$ admite apenas sol. trivial

- Vimos que o sistema $AX=0$ tem uma solução não trivial $\Leftrightarrow \det(A)=0$

- Logo se $\det(A)\neq 0$ então $AX=0$ tem apenas a solução trivial

$$|A| \neq 0 \qquad \begin{vmatrix} \beta & 6 & 1 \\ 0 & \beta-1 & 1 \\ 0 & 1 & \beta+5 \end{vmatrix} \underset{\substack{T.L. \\ 1^{a}\ coluna}}{=} \beta \times (-1)^{1+1} \times \begin{vmatrix} \beta-1 & 1 \\ 1 & \beta+5 \end{vmatrix} = \underbrace{\beta \times \left(\beta^2 + 4\beta - 6 \right) \neq 0}$$

$$\beta \neq 0 \ \wedge \ (\beta^2 - 4\beta - 6 \neq 0)$$

$$x^2 + 4x - 6 = 0$$
$$\Leftrightarrow x = \frac{-4 \pm \sqrt{16 - 4 \times 1 \times -6}}{2 \times 1}$$
$$\Leftrightarrow x = \frac{-4 \pm \sqrt{40}}{2}$$
$$\Leftrightarrow x = \frac{-4 - \sqrt{40}}{2} \vee x = \frac{-4 + \sqrt{40}}{2}$$
$$\Leftrightarrow x = -2 - \sqrt{10} \vee x = -2 + \sqrt{10}$$

Logo, como $|A| \neq 0 \Leftrightarrow \beta \neq 0 \wedge \beta^2 + 4\beta - 6 \neq 0$

$\Leftrightarrow \beta \neq 0 \wedge \beta \neq -2 - \sqrt{10} \wedge \beta \neq -2 + \sqrt{10}$

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 2 & -1 & 0 & 1 \\ -1 & 1 & 0 & 1 \end{bmatrix}$$

, calcule o elemento $(4,1)$ da inversa A, A^{-1}, sem determinar a inversa

! O sempre no teste

$$A^{-1} = \frac{1}{\det(A)} \times \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix}^T$$

$$= \frac{1}{\det(A)} \times \begin{bmatrix} A_{11} & A_{21} & A_{31} & A_{41} \\ A_{12} & A_{22} & A_{32} & A_{42} \\ A_{13} & A_{23} & A_{33} & A_{43} \\ A_{14} & A_{24} & A_{34} & A_{44} \end{bmatrix}$$

O elemento $(4,1)$ da inversa de A é:

$$A_{14} \times \frac{1}{\det(A)}$$

→ comp. algébrico do elemento $(1,4)$

$$A_{14} = (-1)^{1+4} \times \begin{vmatrix} 0 & 1 & 2 \\ 2 & -1 & 0 \\ -1 & 1 & 0 \end{vmatrix} \underset{\substack{T.L. \\ coluna\ 3}}{=} 2 \times (-1)^2 \times \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} = 2 \times (2-1) = 2$$

$$\det(A) = \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 2 & -1 & 0 & 1 \\ -1 & 1 & 0 & 1 \end{vmatrix} \underset{\substack{L_3' := L_3 - 2L_1 \\ L_4' := L_4 + L_1}}{=} \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & -1 & -2 & 1 \\ 0 & 1 & 1 & 1 \end{vmatrix} \underset{L_3' := L_3 + L_2}{=} \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 1 & 1 \end{vmatrix} \underset{L_4' := L_4 - L_2}{=} \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -1 & 0 \end{vmatrix}$$

$$= - \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 \end{vmatrix} = -(-1 \times 2) = 2$$

R: $\frac{1}{\cancel{2}} \times \cancel{2} = 1$