Aula 06

Determinantes (a mesma definição pode ser dada em termos das linhas A) !

Não é necessário saber mas...

Existe uma função det tal que a cada matriz $m\times m$ de colunas c_1, c_2, c_m

$A = [c_1 | c_2 | \dots | c_m]$ faz corresponder um nº real que verifica

$$\det : \text{cont. de matriz } m\times m \longrightarrow \mathbb{R}$$

$$A \longrightarrow \det(A)$$

Propriedades

poderia ser colunas

$$\det\left(\begin{bmatrix}1&2&3\\1&2&3\\4&5&6\end{bmatrix}\right) = 0$$

1. $\det(I_m) = 1$
2. $\det[c_1|\dots|c_i|\dots|c_j|\dots|c_m] = 0$ se $c_i = c_j$
3. Para cada $\alpha = \mathbb{R}$ e cada $i \in \{1,\dots,m\}$:

$$\det([c_1|\dots|\alpha c_i|\dots|c_m]) = \alpha \det([c_1|\dots|c_i|\dots|c_m])$$

$$\det\left(\begin{bmatrix}2&1\\4&9\end{bmatrix}\right) \to 2\det\left(\begin{bmatrix}1&1\\2&9\end{bmatrix}\right)$$

4. Para cada $i \in \{1,\dots,m\}$:

$$\det[c_1|\dots|c_i + c_i'|\dots c_m]$$

$$= \det[c_1|\dots|c_i|\dots|c_m] + \det[c_1|\dots|c_i'|\dots|c_m]$$

$$\det\begin{bmatrix}2&2+4&3\\1&3+1&0\\0&2+2&1\end{bmatrix} = \det\begin{bmatrix}2&2&3\\1&3&0\\0&2&1\end{bmatrix} + \det\begin{bmatrix}2&4&3\\1&1&0\\0&2&1\end{bmatrix}$$

Determinantes de matrizes 2×2

$$\begin{vmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

det

$$\begin{vmatrix}3&4\\5&7\end{vmatrix} = 3\times 7 - 4\times 5 = 1$$

$$\begin{vmatrix}0&3\\1&2\end{vmatrix} = -3 \qquad \begin{vmatrix}\sin\alpha & -\cos\alpha\\ \cos\alpha & \sin\alpha\end{vmatrix} = \sin^2\alpha \cos^2\alpha = 1$$

Aplicação geométrica (cálculo da área do paralelogramo)

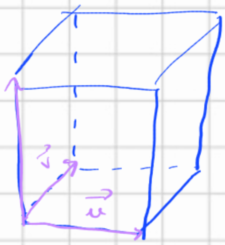
$A(\vec{u},\vec{v}) = |\det(A)|$ onde $A = [u, v]$ é uma matriz 2×2

$$u = \begin{bmatrix}-1\\1\end{bmatrix}, \quad v = \begin{bmatrix}0\\2\end{bmatrix}$$

$$A = \begin{bmatrix}-1&0\\1&2\end{bmatrix}$$

Cálculo do volume de um paralelipípedo

$$vol(u, v, w) = |\det(A)| \quad \text{onde } \underset{3\times 3}{A} = [u, v, w]$$



Menor e Co-fator

$A(n \times n)$, $A = [a_{ij}]$

$M_{ij} \to$ Matriz que se obtém de A eliminando a linha i e coluna j de A

Chama-se menor de a_{ij} a $\det(M_{ij})$

Chama-se co-fator ou complemento algébrico de a_{ij}, a

$$A_{ij} = (-1)^{i+j} \det(M_{ij})$$

__Exemplo:__

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 7 & 0 & 8 \\ -1 & 0 & -1 \end{bmatrix}$$

, determine o menor de $(2,3)$

$$\begin{bmatrix} 1 & 2 & 3 \\ 7 & 0 & 8 \\ -1 & 0 & -1 \end{bmatrix}$$

$$\det\left(\begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}\right) = \underline{\underline{2}}$$

, complemento algébrico de $(2,3)$

$$A_{23} = (-1)^{2+3} \times \underline{\underline{2}} = -1 \times 2 = -2$$

Teorema de Laplace

$$i\begin{bmatrix} \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \end{bmatrix} \quad \det(A) = a_{i1}A_{i1} + a_{i2}A_{i2} + \dots + a_{in}A_{in}$$

$\to$ desenvolvimento de Laplace do $\det(A)$ a partir da $\boxed{\text{linha}}$ i para $i \in \{1, \dots, n\}$

ou

$$i\begin{bmatrix} \vdots & a_{1j} & \vdots \\ \cdots & a_{ij} & \cdots \\ \vdots & a_{nj} & \vdots \end{bmatrix} \quad \det(A) = a_{j1}A_{j1} + a_{j2}A_{j2} + \dots + a_{jn}A_{jn}$$

$\to$ desenvolvimento de Laplace do $\det(A)$ a partir da $\boxed{\text{coluna}}$ j para $j \in \{1, \dots, n\}$

__Exemplo:__

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 3 & 5 & 2 & 2 \\ 0 & 1 & 0 & 1 \\ -1 & 2 & 1 & 1 \end{bmatrix}$$

, calcule o menor do elemento $(2,1)$

$$M_{21} = \begin{vmatrix} 2 & 0 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{vmatrix}$$

$\to$ (pelo teorema de laplace, 2ª coluna)

$$|M_{21}| \underset{\substack{T.L. \\ 2^a\text{ coluna}}}{=} 0 \times (-1)^{1+2} \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} + 0 \times (-1)^{2+2} + 1 \times (-1)^{3+2} \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = -(2-1) = -1$$

ou

$$|M_{21}| \underset{\substack{T.L.\\ 1^a \text{ linha}}}{=} 2 \times (-1)^{1+1} \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} + 1 \times (-1)^{1+3} \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = -2+1=-1$$

Determinantes de matrizes 3 x 3

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$|A| \underset{\substack{T.L.\\ 1^a \text{ linha}}}{=} \underbrace{a_{11}(-1)^{1+1}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}}_{a_{22}a_{33}-a_{23}a_{32}} + \underbrace{a_{12}(-1)^{1+2}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}}_{a_{21}a_{33}-a_{31}a_{23}} + \underbrace{a_{13}(-1)^{1+3}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}}_{a_{21}a_{32}-a_{31}a_{22}}$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{31}a_{23} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{31}a_{22}$$

Regra de Sarrus

1ª regra:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$
$$\begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{matrix}$$

$$= a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} - a_{13}a_{22}a_{31} - a_{23}a_{32}a_{11} - a_{33}a_{12}a_{21}$$

Melhor!

2ª regra:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

Igual!

Exemplo:

$$\begin{vmatrix} 2 & -1 & 4 \\ -1 & 0 & -1 \\ 3 & 2 & -1 \end{vmatrix} \begin{matrix} 2 & -1 \\ -1 & 0 \\ 3 & 2 \end{matrix}$$

$$= \cancel{(2\times0\times1)} + ((-1)\times(-1)\times3) + (4\times(-1)\times2)$$
$$- \cancel{(4\times0\times3)} - (2\times(-1)\times(2)) - ((-1)\times(-1)\times(-1))$$
$$= 3 - 8 + 4 + 1 = 0$$

Propriedades dos determinantes

- $\det(A) = \det(A^T)$

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix}$$

- Se A tem duas linhas (ou colunas) iguais

$\det(A) = 0$

$$\begin{vmatrix} 1 & 1 & 3 \\ 1 & 1 & 5 \\ 1 & 1 & 7 \end{vmatrix} = 0$$

- Se B resulta de A por uma troca de linhas (colunas) então

$$\det(B) = - \det(A) \qquad \begin{matrix}(L_i \leftrightarrow L_j)\\(c_i \leftrightarrow c_j)\end{matrix} \quad \begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} \underset{L_1 \leftrightarrow L_2}{=} - \begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix} \underset{c_1 \leftrightarrow c_2}{=} (-1)\times(-1) \begin{vmatrix} 2 & 3 \\ -1 & 2 \end{vmatrix}$$

- Se B resulta de A por multiplicação de uma linha (ou coluna) de A por um escalar α, então:

$$\det(B) = \alpha \det(A) \qquad \begin{matrix}(L_i := \alpha L_i)\\(c_i := \alpha c_i)\end{matrix}$$

$$\begin{vmatrix} 2 & 6 \\ 1 & 12 \end{vmatrix} = 2 \begin{vmatrix} 1 & 3 \\ 1 & 12 \end{vmatrix} = 2 \times 3 \begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} = 6 \times (4-1) = 18$$

- Se B resulta de A substituindo a linha (ou coluna) i pela sua soma com o múltiplo da linha (ou coluna) j:

$$L'_i := L_i + \alpha L_j \quad (c'_i := c_i + \alpha c_j)$$

, então $\det(A) = \det(B)$

$$\begin{vmatrix} 1 & 2 & 3 \\ 3 & 6 & 5 \\ 2 & 1 & 4 \end{vmatrix} \underset{\substack{L'_2 := L_2 - 3L_1 \\ L'_3 := L_3 - 2L_1}}{=} \begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & -4 \\ 0 & -3 & -2 \end{vmatrix} \underset{\substack{T.L. \\ 1^{a}\text{ coluna}}}{=} 1 \times (-1)^{1+1} \begin{vmatrix} 0 & -4 \\ -3 & -2 \end{vmatrix} = -12$$

- Se A tem uma linha (ou coluna) de zeros o $\det(A) = 0$
- Se $A = [a_{ij}]$ é triangular, então $\det(A) = a_{11}, a_{22}, \dots, a_{nn}$

$$\begin{vmatrix} 2 & 9 & 3 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2 \times 4 \times 1 = 8$$

- $\det(AB) = \det(A)\det(B)$

- Se A é invertível então $\det(A) \neq 0$ e $\det(A^{-1}) = \dfrac{1}{\det(A)}$

$\boxed{\text{Será que é verdade que } \det(A+B) = \det(A) + \det(B) \text{ ?}}$

Falso!

$\exists A, B,\ \exists n \in \mathbb{N}:$

$$\det(A+B) \neq \det(A) + \det(B)$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad [A+B] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \qquad \det(B) = \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} = 1$$

$$\det(A+B) = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = 0$$

, logo $\det(A+B) \neq \det(A) + \det(B)$

T.P.C. → Sejam A e B matrizes reais quadrados de ordem 3 tais que $\det(A) = -2$ e $\det(B) = 1/4$

(1) Calcule $\det(3A)$
(2) Calcule $\det(A\,B^{-1}\,A^{T})$
(3) $\det(-B)$
(4) $\det(B^{-1}\,A^{4}\,B)$
(5) $\det\left(-\frac{1}{2}\,(B^{T})^{-1}\right)$

(1) $\det(3A) = 3^3 \det(A) = 27 \times (-2) = -54$

(2) $\det(A\,B^{-1}\,A^{T}) = \det(A)\det(B^{-1})\det(A^{T})$

$$= -2 \times \frac{1}{\det(B)} \det(A)$$
$$= (-2) \times 4 \times (-2)$$
$$= 16$$

(3) $\det(-B) = -1^3 \times \det(B) = -\frac{1}{4}$

(4)

$$\det(B^{-1}\,A^{4}\,B) = \det(B^{-1})\det(A^{4})\det(B)$$

$$= \frac{1}{\det(B)}\left[\det(A)\right]^4 \times \frac{1}{4}$$

$$= \cancel{4} \times \cancel{\frac{1}{4}} \times (-2)^4$$

$$= 16$$

(5)

$$\det\left(-\frac{1}{2}\,(B^{T})^{-1}\right) = \left(-\frac{1}{2}\right)^3 \times \frac{1}{\det(B^{T})}$$

$$= -\frac{1}{8} \times 4$$

$$= -\frac{1}{2}$$