Aula 20

Matrizes simétricas (diagonalização)

Teorema: Toda a matriz simétrica $A(n\times n)$ é ortogonalmente diagonizável e admite como matriz diagnolizante ortogonal uma matriz cujas colunas são n vetores próprios o.n. de A

$$P^{-1}AP = D \qquad P^{T}AP = D \Rightarrow P^{T} = P^{-1}$$

Exemplo ①

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \text{ uma matriz simétrica} \Rightarrow A \text{ é ortogonalmente diagonalizável}$$

- Determine a matriz diagnolizante e ortogonal e a matriz D tal que: $P^{T}AP = D$

$$\det(\lambda I - A) = \begin{vmatrix} \lambda-1 & -2 \\ -2 & \lambda-1 \end{vmatrix} = 0 \Leftrightarrow (\lambda-1)^2 - 4 = 0 \Leftrightarrow (\lambda-1-2)(\lambda-1+2) = 0$$

$$\Leftrightarrow (\lambda-3)(\lambda+1) = 0$$

$$\Leftrightarrow \lambda = 3 \vee \lambda = -1$$

$$\lambda I - A = \begin{bmatrix} \lambda-1 & -2 \\ -2 & \lambda-1 \end{bmatrix}$$

$$U_{\lambda=3} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^{2\times 1} : \underbrace{\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}}_{3I-A} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

$$\left[\begin{array}{cc|c} 2 & -2 & 0 \\ -2 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$x - y = 0 \Leftrightarrow x = y$$

$$U_{\lambda=3} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : x = y \right\}$$

$$= \left\{ \begin{bmatrix} x \\ x \end{bmatrix}, x \in \mathbb{R} \right\} = \left\{ x\begin{bmatrix} 1 \\ 1 \end{bmatrix}, x \in \mathbb{R} \right\}$$

$$= \left\langle \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\rangle$$

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$U_{\lambda=-1} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : \begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

$$\left[\begin{array}{cc|c} -2 & -2 & 0 \\ -2 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad x + y = 0 \Leftrightarrow x = -y$$

$$U_{\lambda=-1} = \left\langle \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\rangle$$

$$v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

- Vetores próprios associados a valores próprios distintos são ortogonais $\Rightarrow v_1 \cdot v_2 = 0$

$$\text{Seja } \hat{v}_1 = \frac{1}{\|v_1\|} v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$

$$\hat{v}_2 = \frac{1}{\|v_2\|} v_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$$

$$\text{Seja } P = \begin{array}{c} \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \\ \begin{array}{cc} \lambda=3 & \lambda=-1 \end{array} \end{array}$$

Ou se fizermos ao contrário $\Longrightarrow P^T A P = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$

Assim,

$$P^T A P = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$$

Exemplo (2)

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Será A diagonalizável? Sim, A é simétrica. Logo, é ortogonalmente diagonalizável
Encontre a matriz diagonalizante ortogonal e a matriz diagonal D, tal que: $P^T A P = D$

$$\lambda I - A = \begin{bmatrix} \lambda & 0 & -1 \\ 0 & \lambda-1 & 0 \\ -1 & 0 & \lambda \end{bmatrix} \qquad \det(\lambda I - A) = \begin{vmatrix} \lambda & 0 & -1 \\ 0 & \lambda-1 & 0 \\ -1 & 0 & \lambda \end{vmatrix} = (\lambda-1) \begin{vmatrix} \lambda & -1 \\ -1 & \lambda \end{vmatrix} = (\lambda-1)^2(\lambda+1)^2 = 0$$

$$\Leftrightarrow \lambda = 1 \vee \lambda = -1$$

$$U_{\lambda=1} = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : \begin{bmatrix} -1 & 0 & -1 \\ 0 & -2 & 0 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$\left[\begin{array}{ccc|c} -1 & 0 & -1 & 0 \\ 0 & -2 & 0 & 0 \\ -1 & 0 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad x = -z \wedge y = 0$$

$$U_{\lambda=-1} = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : x = -z \wedge y = 0 \right\} = \left\{ \begin{bmatrix} -z \\ 0 \\ z \end{bmatrix}, z \in \mathbb{R} \right\}$$

$$= \left\langle \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\rangle$$

$$v_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$U_{\lambda=1} = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad x = z$$

$$U_{\lambda=1} = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : x = z \right\}$$

$$= \left\{ \begin{bmatrix} z \\ y \\ z \end{bmatrix}, z, y \in \mathbb{R} \right\} = \left\langle \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\rangle$$

$$v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad e \quad v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\left. \begin{array}{l} v_1 \cdot v_2 = 0 \\ v_1 \cdot v_3 = 0 \end{array} \right\}$$ Vetores próprios associados a valores próprios distintos são ortogonais

$$\hat{v}_1 = \frac{1}{\sqrt{2}} \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sqrt{2}/2 \\ 0 \\ \sqrt{2}/2 \end{bmatrix}$$

$$\hat{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$

$$\hat{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$P = \left[\hat{v}_1 \mid \hat{v}_2 \mid \hat{v}_3 \right]$$

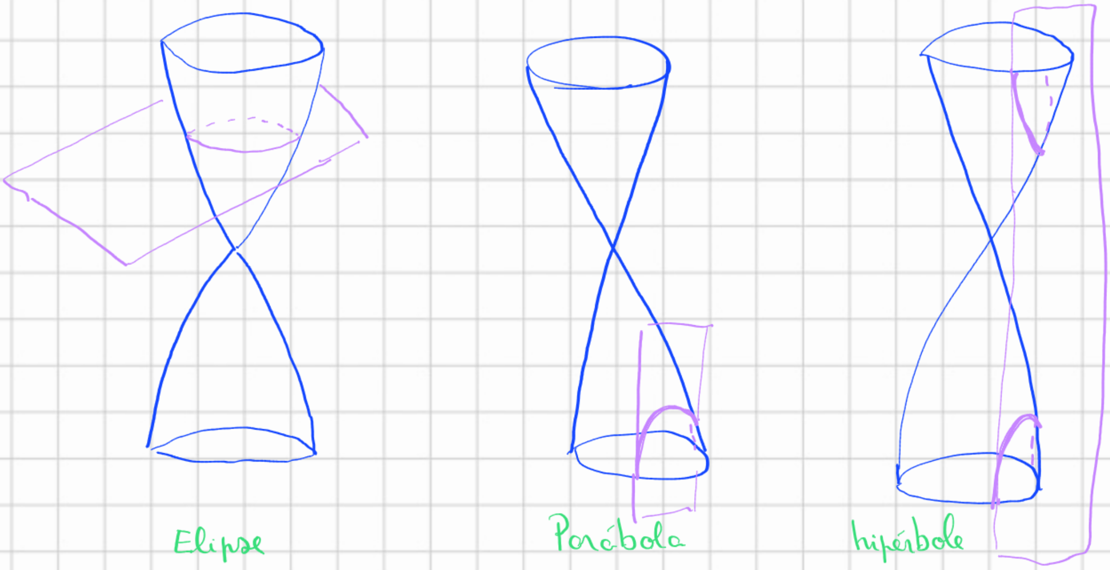
$$= \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{bmatrix}$$

$$\boxed{-1 \quad 1 \quad 1} \rightarrow \text{Valores próprios}$$

$$P^T A P = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Cónicas

cónica → curvas obtidas pela interseção de um cone com um plano



Equação geral de uma cónica:

$\alpha, \beta, \gamma \in \mathbb{R}$ não simultaneamente nulos e $\delta, \eta, \mu \in \mathbb{R}$

$$\alpha x^2 + \beta y^2 + 2\gamma \overbrace{xy}^{\text{termos cruzados}} + \delta x + \eta y + \mu = 0$$

$$X = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} \alpha & \gamma \\ \gamma & \beta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \delta & \eta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \mu = 0$$

$$X^T \underbrace{A}_{\substack{\text{matriz simétrica} \\ 2\times 2\text{, não nula}}} X + \underbrace{B}_{1\times 2} X + \mu = 0$$