Exercício 1.

a) $A + B = \begin{bmatrix} 2 & 0 \\ 4 & 4 \\ 7 & 9 \end{bmatrix}$

b) $D^T - 2A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 0 & 2 \end{bmatrix} - 2\begin{bmatrix} 1 & -2 \\ 1 & 0 \\ 2 & 3 \end{bmatrix}$

$= \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 2 & -4 \\ 2 & 0 \\ 4 & 6 \end{bmatrix}$

$= \begin{bmatrix} -2 & 5 \\ -3 & 0 \\ -4 & -4 \end{bmatrix}$

c)

$AD = \underset{(3\times 2)}{\begin{bmatrix} 1 & -2 \\ 1 & 0 \\ 2 & 3 \end{bmatrix}} \underset{(2\times 3)}{\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 2 \end{bmatrix}}$

$\overset{(3\times 3)}{=} \begin{bmatrix} -2 & -1 & -4 \\ 0 & -1 & 0 \\ 3 & -2 & 6 \end{bmatrix}$

d)

$DA = \underset{(2\times 3)}{\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 2 \end{bmatrix}} \underset{(3\times 2)}{\begin{bmatrix} 1 & -2 \\ 1 & 0 \\ 2 & 3 \end{bmatrix}}$

$\overset{(2\times 2)}{=} \begin{bmatrix} -1 & 0 \\ 5 & 4 \end{bmatrix}$

e) $ACD = \underset{(3\times 2)}{\begin{bmatrix} 1 & -2 \\ 1 & 0 \\ 2 & 3 \end{bmatrix}} \underset{(2\times 2)}{\begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix}} \underset{(2\times 3)}{\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 2 \end{bmatrix}}$

$= \underset{(3\times 2)}{\begin{bmatrix} -1 & -3 \\ -1 & 1 \\ -2 & 8 \end{bmatrix}} \underset{(2\times 3)}{\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 2 \end{bmatrix}}$

$(3\times 3) = \begin{bmatrix} -3 & 1 & -6 \\ 1 & 1 & 2 \\ 8 & 2 & 16 \end{bmatrix}$

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

f)

$$\frac{1}{5}\left(I_2 - (DA)^2\right)$$

$$= \frac{1}{5}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \left(\underset{(2\times3)}{\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 2 \end{bmatrix}} \underset{(3\times2)}{\begin{bmatrix} 1 & -2 \\ 1 & 0 \\ 2 & 3 \end{bmatrix}}\right)^2\right)$$

C.A.

$$\begin{bmatrix} -1 & 0 \\ 5 & 4 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 5 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 15 & 16 \end{bmatrix}$$

$$= \frac{1}{5}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 5 & 4 \end{bmatrix}^2\right)$$

$$= \frac{1}{5}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 15 & 16 \end{bmatrix}\right) = \frac{1}{5}\left(\begin{bmatrix} 0 & 0 \\ -15 & -15 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ -3 & -3 \end{bmatrix}$$

g)

$A\,(3\times2)$ $\quad C\,(2\times2)$ $\quad D\,(2\times3)$ $\quad E\,(2\times1)$

Sendo: $C \times D \times A \times E$

$$\underbrace{(2\times2)\times(2\times3)}_{(2\times3)}\times(3\times2)\times(2\times1)$$

$$\underbrace{(2\times3)\times(3\times2)}_{(2\times2)}\times(2\times1)$$

$$\underbrace{(2\times2)\times(2\times1)}_{(2\times1)}$$

$$(2\times1) \checkmark$$

$$CDAE = \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 1 & 0 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 2 \\ 2 & 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 1 & 0 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 4 \\ 10 & 8 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 10 \\ 14 \end{bmatrix}$$

$\boxed{2}$

a) $(A+B)^2 = (A+B)(A+B)$

$= A^2 + AB + BA + B^2$

Se $AB \neq BA$, $(A+B)^2 \neq A^2 + 2AB + B^2$, logo a afirmação é falsa!

b)

$(AB)^2 = (AB)(AB)$

Contra-exemplo: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$(AB)^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$B^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A^2B^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Logo, como $A^2B^2 \neq (AB)^2$, a afirmação é falsa!

c)

$A + B = B + C \Leftrightarrow$

$\Leftrightarrow A + \cancel{B - B} = \cancel{B - B} + C$

$\Leftrightarrow \boxed{A = C}$ Verdadeira!

d)

$AB = AC \Rightarrow A = 0 \vee B = C$

Falso! Contraexemplo: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ e $B = \begin{bmatrix} 7 & 4 \\ 2 & 5 \end{bmatrix}$ e $C = \begin{bmatrix} 9 & 0 \\ 2 & 5 \end{bmatrix}$

$\boxed{AB = AC,\ A \neq 0 \wedge B \neq C}$

e)

$$AA^T = 0 \Rightarrow A = 0$$

Verdadeiro

f) $\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}^2 = \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} = \begin{bmatrix} x^2 & 0 \\ 0 & x^2 \end{bmatrix}$, Verdadeira a afirmação!

$\boxed{3}$

a) Seja $A = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$ e $A_T \begin{bmatrix} x_{11} & x_{21} & x_{31} \\ x_{12} & x_{22} & x_{32} \\ x_{13} & x_{23} & x_{33} \end{bmatrix}$

$$A + A^T = \begin{bmatrix} 2x_{11} & x_{21}+x_{12} & x_{13}+x_{31} \\ x_{21}+x_{12} & 2x_{22} & x_{23}+x_{32} \\ x_{13}+x_{31} & x_{32}+x_{23} & 2x_{33} \end{bmatrix}$$

b) $A - A^T = \begin{bmatrix} 0 & x_{12}-x_{21} & x_{13}-x_{31} \\ x_{21}-x_{12} & 0 & x_{23}-x_{32} \\ x_{31}-x_{13} & x_{32}-x_{23} & 0 \end{bmatrix}$

$$\begin{matrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 4 & 1 \end{matrix}$$

$(A - A^T)$ é uma matri simétrica se A for uma matriz:
- → simétrica
- → nula
- → diagonal

$\boxed{4}$

a) Escalonada: B, D
Escalonada reduzida: D

b) $A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 3 \\ 0 & 0 & 1 \end{bmatrix} \underset{L_2' := L_2 - 3L_1}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

$\underset{L_2' := L_2 - 3L_3}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underset{L_2' := L_2 \times \frac{1}{3}}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ → forma escalonada reduzida

Forma escalonada

$B = \begin{bmatrix} 3 & 4 & 5 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \underset{L_1' := L_1 - \frac{5}{2}L_2}{\sim} \begin{bmatrix} 3 & 4 & 0 & -5/2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \underset{\substack{L_1' := L_1 + \frac{3}{10}L_3 \\ L_2' := L_2 - \frac{1}{5}L_3}}{\sim} \begin{bmatrix} 3 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$\sim_{\substack{L_1:=L_1\times\frac{1}{3}\\ L_2:=L_2\times\frac{1}{2}\\ L_3:=L_3\times\frac{1}{5}}} \begin{bmatrix} 1 & 4/3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ → escalonada reduzida

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix} \sim_{L_3 \leftrightarrow L_4} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim_{L_2:=L_2\times\frac{1}{5}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ → escalonada reduzida

$$D = \begin{bmatrix} 1 & 14 & 0 & 10 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ → forma escalonada reduzida

$\boxed{5}$

a) $\begin{cases} 3x_1 - x_2 = 4 \\ 2x_1 - \frac{1}{2}x_2 = 1 \end{cases}$

$$\left[\begin{array}{cc|c} 3 & -1 & 4 \\ 2 & -1/2 & 1 \end{array}\right] \sim_{L_1':=L_1-L_2} \left[\begin{array}{cc|c} 1 & -1/2 & 3 \\ 2 & -1/2 & 1 \end{array}\right] \sim_{L_2':=L_2-2L_1} \left[\begin{array}{cc|c} 1 & -1/2 & 3 \\ 0 & 1/2 & -5 \end{array}\right]$$

$$\begin{cases} x_1 - \frac{x_2}{2} = 3 \\ \frac{x_2}{2} = -5 \end{cases} \Leftrightarrow \begin{cases} x_1 - \frac{-10}{2} = 3 \\ x_2 = -10 \end{cases} \Leftrightarrow \begin{cases} x_1 = -2 \\ x_2 = -10 \end{cases}$$ Possível e determinado

b) $\begin{cases} 2x_1 - 3x_2 = 4 \\ x_1 - 3x_2 = 1 \\ x_1 + 3x_2 = 2 \end{cases}$

$$\left[\begin{array}{cc|c} 2 & -3 & 4 \\ 1 & -3 & 1 \\ 1 & 3 & 2 \end{array}\right] \sim_{L_1 \leftrightarrow L_2} \left[\begin{array}{cc|c} 1 & -3 & 1 \\ 2 & -3 & 4 \\ 1 & 3 & 2 \end{array}\right] \sim_{\substack{L_2':=L_2-2L_1 \\ L_3':=L_3-L_1}} \left[\begin{array}{cc|c} 1 & -3 & 1 \\ 0 & 3 & 2 \\ 0 & 6 & 1 \end{array}\right]$$

$$\sim_{L_3':=L_3-2L_2} \left[\begin{array}{cc|c} 1 & -3 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & -3 \end{array}\right]$$ Impossível! $car(A) < car([A|B])$

c)

$$\begin{cases} x_1 + 2x_3 = 0 \\ -x_1 + x_2 + 3x_3 = 2 \\ 2x_1 - x_2 + x_3 = 2 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ -1 & 1 & 3 & 2 \\ 2 & -1 & 1 & 2 \end{array}\right] \underset{\substack{L'_2 := L_2 + L_1 \\ L'_3 := L_3 - 2L_1}}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 5 & 2 \\ 0 & -1 & -3 & 2 \end{array}\right] \underset{L'_3 := L_3 + L_2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 5 & 2 \\ 0 & 0 & 2 & 4 \end{array}\right]$$

$$\begin{cases} x_1 + 2x_3 = 0 \\ x_2 + 5x_3 = 2 \\ 2x_3 = 4 \end{cases} \Leftrightarrow \begin{cases} x_1 + 2\times 2 = 0 \\ x_2 + 10 = 2 \\ x_3 = 2 \end{cases} \Leftrightarrow \begin{cases} x_1 = -4 \\ x_2 = -8 \\ x_3 = 2 \end{cases}$$

Possível e determinado

$S = \{(-4, -8, 2)\}$

d)

$$\begin{cases} x_1 - 2x_2 + 2x_3 = 4 \\ -2x_1 + x_2 + x_3 = 1 \\ x_1 - 5x_2 + 7x_3 = -1 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 4 \\ -2 & 1 & 1 & 1 \\ 1 & -5 & 7 & -1 \end{array}\right] \underset{\substack{L'_2 := L_2 + 2L_1 \\ L'_3 := L_3 - L_1}}{\sim} \left[\begin{array}{ccc|c} 1 & -2 & 2 & 4 \\ 0 & -3 & 5 & 9 \\ 0 & -3 & 5 & -5 \end{array}\right] \underset{L'_3 := L_3 - L_2}{\sim} \left[\begin{array}{ccc|c} 1 & -2 & 2 & 4 \\ 0 & -3 & 5 & 9 \\ 0 & 0 & 0 & -14 \end{array}\right]$$

Impossível!

$car(A) < car([A|B])$

e)

$$\begin{cases} 4x_1 + 3x_2 + 2x_3 = 1 \\ x_1 + 3x_2 + 5x_3 = 1 \\ 3x_1 + 6x_2 + 9x_3 = 2 \end{cases}$$

$$\left[\begin{array}{ccc|c} 4 & 3 & 2 & 1 \\ 1 & 3 & 5 & 1 \\ 3 & 6 & 9 & 2 \end{array}\right] \underset{L_2 \leftrightarrow L_1}{\sim} \left[\begin{array}{ccc|c} 1 & 3 & 5 & 1 \\ 4 & 3 & 2 & 1 \\ 3 & 6 & 9 & 2 \end{array}\right] \underset{\substack{L'_2 := L_2 - 4L_1 \\ L'_3 := L_3 - 3L_1}}{\sim} \left[\begin{array}{ccc|c} 1 & 3 & 5 & 1 \\ 0 & -9 & -18 & -3 \\ 0 & -3 & -6 & -1 \end{array}\right]$$

$$\underset{L'_3 := L_3 - \frac{1}{3}L_2}{\sim} \left[\begin{array}{ccc|c} 1 & 3 & 5 & 1 \\ 0 & -9 & -18 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} x_1 + 3x_2 + 5x_3 = 1 \\ -9x_2 - 18x_3 = -3 \end{cases} \Leftrightarrow \begin{cases} x_1 + \cancel{1} - 6x_3 + 5x_3 = \cancel{1} \\ x_2 = \frac{1}{3} - 2x_3 \end{cases} \Leftrightarrow \begin{cases} x_1 = x_3 \\ x_2 = \frac{1}{3} - 2x_1 \end{cases}$$

$$S = \left\{ \left(x_1, \frac{1}{3} - 2x_1, x_1 \right) \right\}$$

Possível Indeterminado com grau de indeterminação 1

f)

$$\begin{cases} 3x_1 + 4x_2 - 5x_3 + 7x_4 = 0 \\ 2x_1 - 3x_2 + 3x_3 - 2x_4 = 0 \\ 4x_1 + 11x_2 - 13x_3 + 16x_4 = 0 \\ 7x_1 - 2x_2 + x_3 + 3x_4 = 0 \end{cases}$$

$$\left[\begin{array}{cccc|c} 3 & 4 & -5 & 7 & 0 \\ 2 & -3 & 3 & -2 & 0 \\ 4 & 11 & -13 & 16 & 0 \\ 7 & -2 & 1 & 3 & 0 \end{array}\right] \underset{L'_1 := L_1 - L_2}{\sim} \left[\begin{array}{cccc|c} 1 & 7 & -8 & 9 & 0 \\ 2 & -3 & 3 & -2 & 0 \\ 4 & 11 & -13 & 16 & 0 \\ 7 & -2 & 1 & 3 & 0 \end{array}\right]$$

$-3 - 14 = -17$

$11 - 28 = -17$

$-2 - 49 = -51$

$$\underset{\substack{L'_2 := L_2 - 2L_1 \\ L'_3 := L_3 - 4L_1 \\ L'_4 := L_4 - 7L_1}}{\sim} \left[\begin{array}{cccc|c} 1 & 7 & -8 & 9 & 0 \\ 0 & -17 & 19 & -20 & 0 \\ 0 & -17 & 19 & -20 & 0 \\ 0 & -51 & 57 & -60 & 0 \end{array}\right] \underset{L'_3 := L_3 - L_2}{\sim} \left[\begin{array}{cccc|c} 1 & 7 & -8 & 9 & 0 \\ 0 & -17 & 19 & -20 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -51 & 57 & -60 & 0 \end{array}\right]$$

$3 + 16 = 19$

$-13 + 32 = 19$

$1 + 8 \times 7 = 1 + 56 = 57$

$-2 - 18 = -20$

$16 - 36 = -20$

$3 - 7 \times 9 = 3 - 63 = -60$

$$\underset{L_3 \leftrightarrow L_4}{\sim} \left[\begin{array}{cccc|c} 1 & 7 & -8 & 9 & 0 \\ 0 & -17 & 19 & -20 & 0 \\ 0 & -51 & 57 & -60 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \underset{L'_3 := L_3 - 3L_2}{\sim} \left[\begin{array}{cccc|c} 1 & 7 & -8 & 9 & 0 \\ 0 & -17 & 19 & -20 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{r} 19 \\ \times\ 7 \\ \hline 133 \end{array}$$

$$\begin{cases} x_1 + 7x_2 - 8t_1 + 9t_2 = 0 \\ -17x_2 + 19t_1 - 20t_2 = 0 \\ x_3 = t_1 \\ x_4 = t_2 \end{cases} \Leftrightarrow \begin{cases} x_1 = -\frac{133}{17}t_1 + \frac{140}{17}t_2 + 8t_1 - 9t_2 \\ x_2 = \frac{19}{17}t_1 - \frac{20}{17}t_2 \\ x_3 = t_1 \\ x_4 = t_2 \end{cases}$$

$$\Leftrightarrow \begin{cases} x_1 = \frac{3}{17}t_1 - \frac{13}{17}t_2 \\ x_2 = \frac{19}{17}t_1 - \frac{20}{17}t_2 \\ x_3 = t_1 \\ x_4 = t_2 \end{cases}, \quad t_1, t_2 \in \mathbb{R}$$

Possível e Indeterminado (grau de indeterminação 2)

$$S = \left\{ \left(\frac{3}{17}t_1 - \frac{13}{17}t_2, \frac{19}{17}t_1 - \frac{20}{17}t_2, t_1, t_2 \right) \right\}, \ t_1, t_2 \in \mathbb{R}$$

$\boxed{6}$

a)

$$\begin{cases} \alpha x + y = 1 \\ x + \alpha y = 1 \end{cases}$$

$$\left[\begin{array}{cc|c} \alpha & 1 & 1 \\ 1 & \alpha & 1 \end{array}\right] \underset{L_1 \leftrightarrow L_2}{\sim} \left[\begin{array}{cc|c} 1 & \alpha & 1 \\ \alpha & 1 & 1 \end{array}\right] \underset{L_2' := L_2 - \alpha L_1}{\sim} \left[\begin{array}{cc|c} 1 & \alpha & 1 \\ 0 & 1-\alpha^2 & 1-\alpha \end{array}\right]$$

$$\begin{cases} x + \alpha y = 1 \\ (1 - x^2) y = 1 - x \end{cases}$$

i)

$1 - x^2 = 0 \quad \wedge \quad 1 - \alpha \neq 0$

$\Leftrightarrow (\alpha = -1 \vee \alpha = 1) \wedge \alpha \neq 1$

$\Leftrightarrow \boxed{\alpha = -1}$

ii)

$1 - x^2 \neq 0$

$\Leftrightarrow \alpha \neq -1 \wedge \alpha \neq 1$

$\Leftrightarrow \boxed{\alpha \in \mathbb{R} \setminus \{-1, 1\}}$

iii)

$1 - x^2 = 0 \quad \wedge \quad 1 - \alpha = 0$

$\Leftrightarrow (\alpha = -1 \vee \alpha = 1) \wedge \alpha = 1$

$\Leftrightarrow \boxed{\alpha = 1}$

b)

$$\begin{cases} x + (\alpha - 1) y + \alpha z = \alpha - 2 \\ (\alpha - 1) y = 1 \\ \alpha z = \alpha - 3 \end{cases}$$

$$\begin{bmatrix} 1 & \alpha - 1 & \alpha & \alpha - 2 \\ 0 & \alpha - 1 & 0 & 1 \\ 0 & 0 & \alpha & \alpha - 3 \end{bmatrix}$$

i) car(A) < car([A|B]) sse

$\alpha - 1 = 0 \quad \vee \quad (\alpha = 0 \wedge \alpha - 3 \neq 0)$

$\Leftrightarrow \alpha = 1 \quad \vee \quad \alpha = 0$

$\boxed{\alpha \in \{0, 1\}}$

ii) $\text{car}(A) = \text{car}([A|B]) = 3$ sse

$$\alpha - 1 \neq 0 \ \wedge \ \alpha \neq 0$$
$$\Leftrightarrow \alpha \neq 1 \ \wedge \ \alpha \neq 0$$
$$\Leftrightarrow \boxed{\alpha \in \mathbb{R} \setminus \{0, 1\}}$$

Após estudar se é possível e det. apenas estudamos a sua negação

$$\sim (a \wedge b)$$
$$\sim a \vee \sim b$$
$$\boxed{\alpha = 1 \vee \alpha = 0}$$

• $\alpha = 1$:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$\text{car}(A) = 2 < \text{car}([A|B]) = 3 \Rightarrow$ Sistema Impossível

• $\alpha = 0$:

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & -2 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -3 \end{array}\right]$$

$\text{car}(A) = 2 < \text{car}([A|B]) = 3 \Rightarrow$ Sistema Impossível

iii) $\text{car}(A) = \text{car}([A|B]) < 3$ sse

$$\alpha - 1 \neq 0 \ \wedge \ (\alpha = 0 \ \wedge \ \alpha - 3 = 0)$$
$$\Leftrightarrow \alpha \neq 1 \ \wedge \ (\underbrace{\alpha = 0 \ \wedge \ \alpha = 3}_{C.I.})$$

$$\boxed{\text{Logo, não existem valores de } \alpha \text{ para os quais o sistema tem uma infinidade de soluções}}$$

c)

$$\begin{cases} x + \alpha y + \alpha z = 0 \\ \alpha x + y + z = 0 \\ x + y + \alpha z = \alpha^2 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & \alpha & \alpha & 0 \\ \alpha & 1 & 1 & 0 \\ 1 & 1 & \alpha & \alpha^2 \end{array}\right] \underset{\substack{L_2' := L_2 - \alpha L_1 \\ L_3' := L_3 - L_1}}{\sim} \left[\begin{array}{ccc|c} 1 & \alpha & \alpha & 0 \\ 0 & 1-\alpha^2 & 1-\alpha^2 & 0 \\ 0 & 1-\alpha & 0 & \alpha^2 \end{array}\right]$$

Porque não continuamos?

i)

$$(1 - \alpha) = 0 \ \wedge \ \alpha^2 \neq 0$$
$$\Leftrightarrow \alpha = 1 \ \wedge \ \alpha^2 \neq 0$$
$$\Leftrightarrow \boxed{\alpha = 1}$$

ii)

$$1 - \alpha \neq 0 \ \wedge \ 1 - \alpha^2 \neq 0$$
$$\Leftrightarrow \alpha \neq 1 \ \wedge \ \alpha \neq -1 \ \wedge \alpha \neq 1$$
$$\Leftrightarrow \boxed{\alpha \in \mathbb{R} \setminus \{-1, 1\}}$$

iii)

$$\left((1 - \alpha = 0 \ \wedge \ \alpha^2 = 0) \ \vee \ 1 - \alpha^2 = 0 \right) \ \wedge \ \alpha \neq 1$$

O sistema poderia ser imp.

$$\Leftrightarrow \left((\underbrace{\alpha = 1 \ \wedge \ \alpha = 0}_{C.I.}) \ \vee \ \alpha = -1 \ \vee \ \alpha = 1 \right) \ \wedge \ \alpha \neq 1$$

$$\Leftrightarrow \boxed{\alpha = -1}$$

$\boxed{7}$

a)
$$\begin{cases} x - y - z = a \\ x + y + z = a \\ x - by + z = -b \end{cases}, \quad a, b \in \mathbb{R}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & a \\ 1 & 1 & 1 & a \\ 1 & -b & 1 & -b \end{array}\right] \underset{\substack{L_2' := L_2 - L_1 \\ L_3' := L_3 - L_1}}{\sim} \left[\begin{array}{ccc|c} 1 & -1 & -1 & a \\ 0 & 2 & 2 & 0 \\ 0 & -b+1 & 2 & -b-a \end{array}\right] \underset{L_3' := L_3 + \frac{b-1}{2} \times L_2}{\sim} \left[\begin{array}{ccc|c} 1 & -1 & -1 & a \\ 0 & 2 & 2 & 0 \\ 0 & 0 & b+1 & -b-a \end{array}\right], \quad a, b \in \mathbb{R}$$

i) $car(A) = car([A|B]) = 3$ sse

$b + 1 \neq 0 \Leftrightarrow b \neq -1 \Leftrightarrow \boxed{b \in \mathbb{R} \setminus \{-1\}}$

ii) $car(A) = car([A|B]) < 3$ sse

$b + 1 = 0 \land -b - a = 0$

$\Leftrightarrow \begin{cases} b = -1 \\ 1 - a = 0 \end{cases} \Leftrightarrow \begin{cases} b = -1 \\ a = 1 \end{cases} \Leftrightarrow \boxed{b = -1 \land a = 1}$

iii) $car(A) < car([A|B])$ sse

$b + 1 = 0 \land -b - a \neq 0$

$\Leftrightarrow \begin{cases} b = -1 \\ 1 - a \neq 0 \end{cases} \Leftrightarrow \begin{cases} b = -1 \\ a \neq 1 \end{cases} \Leftrightarrow \boxed{b = -1 \land a \in \mathbb{R} \setminus \{1\}}$

b)

$(1, -1, 1)$

$$\begin{cases} 1 + 1 - 1 = a \\ 1 - 1 + 1 = a \\ 1 + b + 1 = -b \end{cases} \Leftrightarrow \begin{cases} a = 1 \\ a = 1 \\ b = -1 \end{cases} \Leftrightarrow \begin{cases} a = 1 \\ b = -1 \end{cases}$$

Substituindo na forma escalonada obtida na alínea (a)

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{cases} x - y - z = 1 \\ 2y + 2z = 0 \\ z = t \end{cases} \Leftrightarrow \begin{cases} x + \cancel{t} - \cancel{t} = 1 \\ y = -t \\ z = t \end{cases}$$

$$\Leftrightarrow \begin{cases} x = 1 \\ y = -t \\ z = t \end{cases}, t \in \mathbb{R} \qquad S = \{(1, -t, t), t \in \mathbb{R}\}$$

Possível e indeterminado
(grau de indeterminação 1)

Exercício extra

$$\begin{cases} x+y=2 \\ x+2y+\alpha z=3 \\ \alpha y+z=2\alpha+1 \end{cases}, \alpha \in \mathbb{R}$$

Ver quando é
→ Possível..., impossível

a) Discuta o sistema em função de α

b) Verifique que as soluções do sistema obtido (quando indeterminado) são os pontos de uma reta que passa por $(1,1,0)$

a)

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 1 & 2 & \alpha & 3 \\ 0 & \alpha & 1 & 2\alpha+1 \end{array}\right] \underset{L_2' := L_2 - L_1}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & \alpha & 1 \\ 0 & \alpha & 1 & 2\alpha+1 \end{array}\right] \underset{L_3' := L_3 - L_1 \times \alpha}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & \alpha & 1 \\ 0 & 0 & 1-\alpha^2 & \alpha+1 \end{array}\right]$$

Possível e determinado sse

$car(A) = car([A|B]) = 3$ sse

$1-\alpha^2 \neq 0 \Leftrightarrow \alpha \neq -1 \wedge \alpha \neq 1$, $\alpha \in \mathbb{R} \setminus \{-1,1\}$

$\sim(\alpha \neq -1 \wedge \alpha \neq 1)$
$\alpha = -1 \vee \alpha = 1$

Para $\alpha = -1$:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$car(A) = 2$
$\parallel$
$car([A|B]) = 2$, sistema possível e indeterminado com grau de indeterminação $n - car(A) = 3-2 = 1$

Para $\alpha = 1$:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

$car(A) = 2$
$<$
$car([A|B]) = 3$, sistema impossível

b)

$\alpha = -1$:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} x+y=2 \\ y-z=1 \end{cases} \Leftrightarrow \begin{cases} x+1+z=2 \\ y=1+z \end{cases} \Leftrightarrow \begin{cases} x=1-z \\ y=1+z \end{cases}, z \in \mathbb{R}$$

$S = \{(1-z, 1+z, z), z \in \mathbb{R}\}$

$= \{(1,1,0) + (-z, z, z), z \in \mathbb{R}\}$

$= \{(1,1,0) + z(-1,1,1), z \in \mathbb{R}\}$

R: $\underbrace{(x, y, z) = (1,1,0) + k(-1,1,1),\ k \in \mathbb{R}}$

Conjunto dos pontos de uma reta que passa por $(1,1,0)$ e tem o vetor diretor $(-1,1,1)$

$\boxed{8}$

$nul(A) = 5 - car(A) = 5 - 3 = 2$

$car([A|B]) = 3$

Como $car(A) = car([A|B]) < 5$, o sistema é possível indeterminado com grau de indeterminação 2.

nº de incógnitos

$\boxed{9}$

$A = m \times n$

$A =$ 

$AX = B$

$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$

$$\begin{cases} x_1 + 2x_2 + x_3 = 2 \\ 2x_1 + x_2 + 4x_3 = 1 \\ x_1 + 3x_2 + 2x_3 = 3 \end{cases}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 4 \\ 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

$L_2 := L_2 - 2L_1$
$L_3 := L_3 - L_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & -3 & 2 & -3 \\ 0 & 1 & 1 & 1 \end{array}\right] \underset{L_2 \leftrightarrow L_3}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & -3 & 2 & -3 \end{array}\right] \underset{L_3 := L_3 + 3L_2}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 5 & 0 \end{array}\right]$$

$$\begin{cases} x + 2y + z = 2 \\ y + z = 1 \\ 5z = 0 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = 1 \\ z = 0 \end{cases}$$

$\boxed{10}$

c)

$$R: \begin{cases} x_1 + 2x_3 = 0 \\ -x_1 + x_2 + 3x_3 = 2 \end{cases} \qquad P: 2x_1 - x_2 + x_3 = 2$$

$$\begin{cases} \overbrace{\boxed{\begin{matrix} x_1 + 2x_3 = 0 \\ -x_1 + x_2 + 3x_3 = 2 \end{matrix}}}^{R} \\ \underbrace{\boxed{2x_1 - x_2 + x_3 = 2}}_{P} \end{cases} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ -1 & 1 & 3 & 2 \\ 2 & -1 & 1 & 2 \end{array}\right] \underset{\substack{L_2' := L_2 + L_1 \\ L_3' := L_3 - 2L_1}}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 5 & 2 \\ 0 & -1 & -3 & 2 \end{array}\right] \underset{L_3' := L_3 + L_2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 5 & 2 \\ 0 & 0 & 2 & 4 \end{array}\right]$$

- Como car(A) = car$([A|B])$ = 3, a reta R e o plano P são concorrentes.

$$\begin{cases} x + 2z = 0 \\ y + 5z = 2 \\ 2z = 4 \end{cases} \quad \begin{cases} x = -4 \\ y = 2 - 10 \\ z = 2 \end{cases} \quad \begin{cases} x = -4 \\ y = -8 \\ z = 2 \end{cases}$$

- A interseção da reta R e o plano P é o ponto $(-4, -8, 2)$

d)

$$\begin{cases} \boxed{\begin{matrix} x_1 - 2x_2 + 2x_3 = 4 \\ -2x_1 + x_2 + x_3 = 1 \end{matrix}} \to \text{reta } R \\ \boxed{x_1 - 5x_2 + 7x_3 = -1} \to \text{plano } P \end{cases} \qquad A = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 1 \\ 1 & -5 & 7 \end{bmatrix} \qquad B = \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}$$

$$[A|B] = \left[\begin{array}{ccc|c} 1 & -2 & 2 & 4 \\ -2 & 1 & 1 & 1 \\ 1 & -5 & 7 & -1 \end{array}\right] \underset{\substack{L_2' := L_2 + 2L_1 \\ L_3' := L_3 - L_1}}{\sim} \left[\begin{array}{ccc|c} 1 & -2 & 2 & 4 \\ 0 & -3 & 5 & 9 \\ 0 & -3 & 5 & -5 \end{array}\right] \underset{L_3' := L_3 - L_2}{\sim} \left[\begin{array}{ccc|c} 1 & -2 & 2 & 4 \\ 0 & -3 & 5 & 9 \\ 0 & 0 & 0 & -14 \end{array}\right]$$

- Como car(A) < car$([A|B])$, o sistema é impossível.
- Logo a reta R é paralela ao plano P, a interseção é o conjunto vazio.

e)

$$\begin{cases} 4x_1 + 3x_2 + 2x_3 = 1 \\ x_1 + 3x_2 + 5x_3 = 1 \\ 3x_1 + 6x_2 + 9x_3 = 2 \end{cases}$$

$$A = \begin{bmatrix} 4 & 3 & 2 \\ 1 & 3 & 5 \\ 3 & 6 & 9 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \qquad [A|B] = \left[\begin{array}{ccc|c} 4 & 3 & 2 & 1 \\ 1 & 3 & 5 & 1 \\ 3 & 6 & 9 & 2 \end{array}\right] \underset{L_1 \leftrightarrow L_2}{\sim} \left[\begin{array}{ccc|c} 1 & 3 & 5 & 1 \\ 4 & 3 & 2 & 1 \\ 3 & 6 & 9 & 2 \end{array}\right]$$

$$\underset{\substack{L_2' := L_2 - 4L_1 \\ L_3' := L_3 - 3L_1}}{\sim} \left[\begin{array}{ccc|c} 1 & 3 & 5 & 1 \\ 0 & -9 & -18 & -3 \\ 0 & -3 & -6 & -1 \end{array}\right] \underset{L_3 := L_3 - \frac{1}{3}L_2}{\sim} \left[\begin{array}{ccc|c} 1 & 3 & 5 & 1 \\ 0 & -9 & -18 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

• Como $car(A) = car([A|B]) < 3$, o plano P contém a reta R.

$\to$ nº de incógnitas

Posição relativa!

$$\begin{cases} x + 3y + 5z = 1 \\ -9y - 18z = -3 \\ z = t \end{cases} \quad \begin{cases} x = \cancel{1} - \cancel{3}\times\frac{\cancel{1}}{\cancel{3}} + 6t - 5t \\ y = \frac{1}{3} - 2t \\ z = t \end{cases} \quad \begin{cases} x = t \\ y = \frac{1}{3} - 2t \\ z = t \end{cases}$$

$$S = \left\{\left(t, \frac{1}{3} - 2t, t\right)\right\} = \left\{\left(1, \frac{1}{3}, 1\right) + t(1, -2, 1)\right\}, t \in \mathbb{R}$$

Intersecção da reta com o plano: $(x_1, x_2, x_3) = \left(1, \frac{1}{3}, 1\right) + K(1, -2, 1), K \in \mathbb{R}$

$\hookrightarrow$ reta R

$\boxed{11}$

11. Considere os planos $\mathcal{P}$ e $\mathcal{P}_{a,b}$ de equações $x + y + 2z = 3$ e $ax + 2y + 4z = b$, respectivamente, com $a, b \in \mathbb{R}$. Discuta a posição relativa dos planos $\mathcal{P}$ e $\mathcal{P}_{a,b}$ em função dos parâmetros reais a e b.

$P: x + y + 2z = 3 \qquad P_{a,b} = ax + 2y + 4z = b \quad , a, b \in \mathbb{R}$

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ a & 2 & 4 & b \end{array}\right] \underset{L_2 := L_2 - aL_1}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 3 \\ 0 & 2-a & 4-2a & b-3a \end{array}\right]$$

• P é paralelo a $P_{a,b}$ sse

$car(A) < car([A|B])$ sse

$2 - a = 0 \wedge 4 - 2a = 0 \wedge b - 3a \neq 0$

$\Leftrightarrow a = 2 \wedge a = 2 \wedge b \neq 3a$

$\Leftrightarrow a = 2 \wedge b \neq 6$

$\to$ Logo, se $a = 2$ e $b \in \mathbb{R} \setminus \{6\}$ os planos P e $P_{a,b}$ são paralelos

• P intersecta $P_{a,b}$ em uma reta sse

$car(A) = car([A|B]) = 2 < 3$ sse

$\to$ nº de incógnitas

$2 - a \neq 0 \vee 4 - 2a \neq 0$

$\Leftrightarrow a \neq 2 \vee a \neq 2$

$\Leftrightarrow a \neq 2$

$\to$ Logo, se $a \in \mathbb{R} \setminus \{2\}$ e $b \in \mathbb{R}$ os planos P e $P_{a,b}$ intersetam-se numa reta.

- P e $P_{a,b}$ são coincidentes sse

$$car(A) = car([A|B]) = 1 < \overset{\to \text{ nº de incógnitas}}{3} \text{ sse}$$

$$2-a=0 \wedge 4-2a=0 \wedge b-3a=0$$
$$\Leftrightarrow a=2 \wedge b=6$$

→ Logo, se $a=2$ e $b=6$ os planos P e $P_{a,b}$ são coincidentes.

Resumindo:

→ Planos concorrentes numa reta se $a \neq 2$

→ Planos estritamente paralelos se $a=2 \wedge b \neq 6$

→ Planos coincidentes se $a=2 \wedge b=6$

$\boxed{12}$

a)
$$[A|B] = \left[\begin{array}{ccc|c} 1 & 0 & 2 & -5 \\ 0 & 15 & -5 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$car(A) = car([A|B]) = 3 = \overset{\to 3}{\text{nº de incógnitas}}$$
$$\Downarrow$$
r e s são concorrentes

b)
$$[A|B] = \left[\begin{array}{ccc|c} 1 & 0 & 2 & -1 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$car(A) = car([A|B]) = 3 < \text{nº de incógnitas}$$
$$\Downarrow$$
r e s são coincidentes

c)
$$[A|B] = \left[\begin{array}{ccc|c} 1 & 0 & 2 & -2 \\ 0 & 2 & -2 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

$$\overset{\to 3}{car(A)} < \overset{\to 4}{car([A|B])}$$
$$\Downarrow$$
r e s são enviesados

d)
$$[A|B] = \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$car(A) = 2 < car([A|B]) = 3$$
$$\Downarrow$$
r e s são estritamente paralelos

$\boxed{13}$

$r: x + 2y + z = 1$

$s_{a,b}: (x, y, z) = (a, 0, 1) + \alpha(0, 2, b)\ ,\ a, b \in \mathbb{R}$

$$\begin{cases} x = a \\ y = 2\alpha \\ z = 1 + \alpha b \end{cases} \Leftrightarrow \begin{cases} x = a \\ \alpha = \frac{y}{2} \\ \alpha = 1 + \frac{y}{2} \times b \end{cases} \Leftrightarrow \begin{cases} x = a \\ \alpha = y/2 \\ 2z - by = 2 \end{cases}$$

As equações cartesianas de $s_{a,b}$ são:

$$\begin{cases} x = a \\ by - 2z = -2 \end{cases} , a, b \in \mathbb{R}$$

$\boxed{r: \quad x = 2y + z = 1}$

$$\begin{cases} x = 2y + z \\ 2y + z = 1 \\ x = 1 \end{cases} \qquad \text{temos para } r: \begin{cases} 2y + z = 1 \\ x = 1 \end{cases}$$

$r \cap s_{a,b}$?

$$\begin{cases} x = 1 \\ 2y + z = 1 \\ x = a \\ by - 2z = 2 \end{cases} \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 2 & 1 & 1 \\ 1 & 0 & 0 & a \\ 0 & b & -2 & -2 \end{array}\right] \underset{L_3' := L_3 - L_1}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & a-1 \\ 0 & b & -2 & -2 \end{array}\right] \underset{L_4' := L_4 - \frac{b}{2}L_2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & a-1 \\ 0 & 0 & -2-\frac{b}{2} & -2-\frac{b}{2} \end{array}\right]$$

$$\underset{L_3 \leftrightarrow L_4}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & \frac{-4-b}{2} & \frac{-4-b}{2} \\ 0 & 0 & 0 & a-1 \end{array}\right] \underset{L_3' := L_3 \times 2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & -4-b & -4-b \\ 0 & 0 & 0 & a-1 \end{array}\right]$$

• Se $-4-b \neq 0 \Leftrightarrow b \neq 4$

(i) se $a-1=0 \Leftrightarrow a=1$

$car(A) = car([A|B]) = 3 = n$,

sistema possível e det.
(os retas são concorrentes)

(ii) se $a-1 \neq 0 \Leftrightarrow a \neq 1$

$car(A) = 3 < car([A|B]) = 4$,

sistema impossível (os retas são enviesados

• Se $-4-b=0$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a-1 \end{array}\right] \underset{L_2 \leftrightarrow L_3}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & a-1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(i) Se $a-1=0 \Leftrightarrow a=1$

$car(A) = car([A|B]) = 2 < 3$, o sistema é possível e ind.
com grau de indeterminação 1,
logo os retas são coincidentes

(ii) Se $a-1 \neq 0$

$car(A) = 2 < car([A|B]) = 3$, o sistema é impossível e os retas
são estritamente paralelos

Resumindo:

- Se $b \neq -4$ e $a=1 \Longrightarrow$ retas concorrentes
- Se $b \neq -4$ e $a \neq 1 \Longrightarrow$ retas enviesados
- Se $b=-4$ e $a=1 \Longrightarrow$ retas coincidentes
- Se $b=-4$ e $a \neq 1 \Longrightarrow$ retas estritamente paralelos

$\boxed{14}$

a)

$$\left[\begin{array}{cc|cc} 3 & 2 & 1 & 0 \\ -6 & -4 & 0 & 1 \end{array}\right] \underset{L_2' := L_2 + 2L_1}{\sim} \left[\begin{array}{cc|cc} 3 & 4 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{array}\right]$$, logo é uma matriz singular

b)

$$\left[\underbrace{\begin{matrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{matrix}}_{A} \middle| \overbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}^{I_3}\right] \underset{\substack{L_1' := L_1 - L_2 \\ L_2' := L_2 - L_3}}{\sim} \left[\overbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}^{I_3} \middle| \underbrace{\begin{matrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{matrix}}_{A^{-1}}\right]$$, logo é uma matriz não singular (invertível)

$$A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

c)

$$\left[\begin{array}{ccc|ccc} 0 & 2 & -1 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 1 & 0 \\ -2 & -5 & 4 & 0 & 0 & 1 \end{array}\right] \underset{L_1 \leftrightarrow L_2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 1 & 0 & 0 \\ -2 & -5 & 4 & 0 & 0 & 1 \end{array}\right] \underset{L_3' := L_3 + 2L_1}{\sim} \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 1 & 0 & 0 \\ 0 & -3 & 2 & 0 & 2 & 1 \end{array}\right]$$

$$\underset{L_2' := \frac{1}{2} \times L_2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1/2 & 1/2 & 0 & 0 \\ 0 & -3 & 2 & 0 & 2 & 1 \end{array}\right] \underset{L_3' := L_3 + 3L_2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 3/2 & 2 & 1 \end{array}\right] \underset{L_3' := L_3 \times 2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 4 & 2 \end{array}\right]$$

$$\underset{L_1' := L_1 - L_2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & -1/2 & -1/2 & 1 & 0 \\ 0 & 1 & -1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 4 & 2 \end{array}\right] \underset{L_1' := L_1 + \frac{1}{2} L_3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 3 & 1 \\ 0 & 1 & -1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 3 & 4 & 2 \end{array}\right] \underset{L_2' := L_2 + \frac{1}{2} L_3}{\sim} \left[\underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I_3} \middle| \underbrace{\begin{matrix} 1 & 3 & 1 \\ 2 & 2 & 1 \\ 3 & 4 & 2 \end{matrix}}_{A^{-1}}\right],$$

logo é não singular (invertível). $A^{-1} = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 2 & 1 \\ 3 & 4 & 2 \end{bmatrix}$

d)

$$\left[\begin{array}{cccc|cccc} 2 & 3 & 4 & 5 & 1 & 0 & 0 & 0 \\ 3 & 3 & 4 & 5 & 0 & 1 & 0 & 0 \\ 4 & 4 & 4 & 5 & 0 & 0 & 1 & 0 \\ 5 & 5 & 5 & 5 & 0 & 0 & 0 & 1 \end{array}\right] \underset{L_4' := \frac{1}{5} L_4}{\sim} \left[\begin{array}{cccc|cccc} 2 & 3 & 4 & 5 & 1 & 0 & 0 & 0 \\ 3 & 3 & 4 & 5 & 0 & 1 & 0 & 0 \\ 4 & 4 & 4 & 5 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1/5 \end{array}\right] \underset{L_1 \leftrightarrow L_4}{\sim} \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1/5 \\ 3 & 3 & 4 & 5 & 0 & 1 & 0 & 0 \\ 4 & 4 & 4 & 5 & 0 & 0 & 1 & 0 \\ 2 & 3 & 4 & 5 & 1 & 0 & 0 & 0 \end{array}\right]$$

$$\underset{\substack{L_2' := L_2 - 3L_1 \\ L_3' := L_3 - 4L_1 \\ L_4' := L_4 - 2L_1}}{\sim} \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1/5 \\ 0 & 0 & 1 & 2 & 0 & 1 & 0 & -3/5 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & -4/5 \\ 0 & 1 & 2 & 3 & 1 & 0 & 0 & -2/5 \end{array}\right] \underset{L_4 \leftrightarrow L_2}{\sim} \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1/5 \\ 0 & 1 & 2 & 3 & 1 & 0 & 0 & -2/5 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & -4/5 \\ 0 & 0 & 1 & 2 & 0 & 1 & 0 & -3/5 \end{array}\right] \underset{L_3 \leftrightarrow L_4}{\sim} \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1/5 \\ 0 & 1 & 2 & 3 & 1 & 0 & 0 & -1/5 \\ 0 & 0 & 1 & 2 & 0 & 1 & 0 & -3/5 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & -4/5 \end{array}\right]$$

$$\underset{L_1' := L_1 - L_2}{\sim} \left[\begin{array}{cccc|cccc} 1 & 0 & -1 & -2 & -1 & 0 & 0 & 3/5 \\ 0 & 1 & 2 & 3 & 1 & 0 & 0 & -2/5 \\ 0 & 0 & 1 & 2 & 0 & 1 & 0 & -3/5 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & -4/5 \end{array}\right] \underset{L_1' := L_1 + L_3}{\sim} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 1 & 0 & 0 & -2/5 \\ 0 & 0 & 1 & 2 & 0 & 1 & 0 & -3/5 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & -4/5 \end{array}\right]$$

$$\underset{L_2' := L_2 - 2L_3}{\sim} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & -2 & 0 & 4/5 \\ 0 & 0 & 1 & 2 & 0 & 1 & 0 & -3/5 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & -4/5 \end{array}\right] \underset{L_2' := L_2 + L_4}{\sim} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & 1 & 0 & -3/5 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & -4/5 \end{array}\right]$$

$$\underset{L_3' := L_3 - 2L_4}{\sim} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & -4/5 \end{array}\right]$$

, logo é não singular (invertível)

$$A^{-1} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -4/5 \end{bmatrix}$$

$\boxed{15}$

a)

$$ADB = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix} = \begin{bmatrix} 4 & 9 \\ 10 & 21 \end{bmatrix} \begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix} = \begin{bmatrix} 17 & -6 \\ 35 & -12 \end{bmatrix} = C$$

b)

$$[A|I_2] = \left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 5 & 7 & 0 & 1 \end{array}\right] \underset{L_1' := \frac{1}{2} \times L_1}{\sim} \left[\begin{array}{cc|cc} 1 & 3/2 & 1/2 & 0 \\ 5 & 7 & 0 & 1 \end{array}\right] \underset{L_2' := L_2 - 5L_1}{\sim} \left[\begin{array}{cc|cc} 1 & 3/2 & 1/2 & 0 \\ 0 & -1/2 & -5/2 & 1 \end{array}\right]$$

$$\underset{L_2' := -2 \times L_2}{\sim} \left[\begin{array}{cc|cc} 1 & 3/2 & 1/2 & 0 \\ 0 & 1 & 5 & -2 \end{array}\right] \underset{L_1' := L_1 - \frac{3}{2} L_2}{\sim} \left[\begin{array}{cc|cc} 1 & 0 & -7 & 3 \\ 0 & 1 & 5 & -2 \end{array}\right]$$

A^{-1}

$$A^{-1} = \begin{bmatrix} 7 & 3 \\ 5 & -2 \end{bmatrix} = B$$

c)

$$C^5 = (ADB)^5 = AD^5B$$

$$= AD\underbrace{BA}_{I}D\underbrace{BA}_{I}D\underbrace{BA}_{I}D\underbrace{BA}_{I}DB$$

$$= ADIDIDIDIDB$$

$$= ADDDDDB$$

$$= AD^5B$$

d)

$$A \times D = B \longrightarrow \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} 2a+3c & 2b+3d \\ 5a+7c & 5b+7d \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} 4a+6c & 6b+9d \\ 10a+14c & 15b+21d \end{bmatrix} = \begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}$$

$$\begin{array}{cccc} a & b & c & d \end{array}$$
$$\left[\begin{array}{cccc|c} 4 & 0 & 6 & 0 & -7 \\ 0 & 6 & 0 & 9 & 3 \\ 10 & 0 & 14 & 0 & 5 \\ 0 & 15 & 0 & 21 & -2 \end{array}\right] \underset{\substack{L_3' := L_3 - L_1 \times \frac{5}{2} \\ L_4' := L_4 - L_2 \times \frac{5}{2}}}{\sim} \left[\begin{array}{cccc|c} 4 & 0 & 6 & 0 & -7 \\ 0 & 6 & 0 & 9 & 3 \\ 0 & 0 & -1 & 0 & 45/2 \\ 0 & 0 & 0 & -3/2 & -19/2 \end{array}\right]$$

$$\underset{\substack{L_3' = -L_3 \\ L_4' = -2L_4 \\ L_2' := \frac{1}{3}L_2}}{\sim} \left[\begin{array}{cccc|c} 4 & 0 & 6 & 0 & -7 \\ 0 & 2 & 0 & 3 & 1 \\ 0 & 0 & 1 & 0 & -45/2 \\ 0 & 0 & 0 & 3 & 19 \end{array}\right]$$

$$\begin{cases} 4a + 6c = -7 \\ 2b + 3d = 1 \\ c = -\frac{45}{2} \\ 3d = 19 \end{cases} \quad \begin{cases} 4a = -7 - 6 \times \left(-\frac{45}{2}\right) \\ 2b = 1 - 19 \\ c = -\frac{45}{2} \\ d = \frac{19}{3} \end{cases} \quad \begin{cases} a = 32 \\ b = -9 \\ c = -\frac{45}{2} \\ d = \frac{19}{3} \end{cases}$$

$$X = \begin{bmatrix} 32 & -9 \\ -45/2 & 19/3 \end{bmatrix}$$

$\boxed{16}$

a) $$M = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ -1 & -4 & 2 \end{bmatrix} \qquad M^3 = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ -1 & -4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ -1 & -4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ -1 & -4 & 2 \end{bmatrix}$$

$$M^2 = \begin{bmatrix} 6 & 8 & -3 \\ 4 & 5 & -2 \\ -11 & -14 & 5 \end{bmatrix} \qquad = \begin{bmatrix} 6 & 8 & -3 \\ 4 & 5 & -2 \\ -11 & -14 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ -1 & -4 & 2 \end{bmatrix}$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad = \begin{bmatrix} 25 & 32 & -12 \\ 16 & 21 & -8 \\ -44 & -56 & 21 \end{bmatrix}$$

$$M^3 - 4M^2 - I_3 = 0$$

$$\Leftrightarrow \begin{bmatrix} 25 & 32 & -12 \\ 16 & 21 & -8 \\ -44 & -56 & 21 \end{bmatrix} - 4 \times \begin{bmatrix} 6 & 8 & -3 \\ 4 & 5 & -2 \\ -11 & -14 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0$$

$$\Leftrightarrow \begin{bmatrix} 25 & 32 & -12 \\ 16 & 21 & -8 \\ -44 & -56 & 21 \end{bmatrix} - \begin{bmatrix} 24 & 32 & -12 \\ 16 & 20 & -8 \\ -44 & -56 & 20 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0$$

$$\Leftrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0$$

$$\Leftrightarrow 0 = 0 \checkmark$$

b)

$$\boxed{M^{-2} = M - 4I_3}$$

$$M^3 - 4M^2 - I_3 = 0$$

$$\Leftrightarrow M^3 - 4 \times I_3 \times M^2 = I_3 \qquad \boxed{M^2 = M^2 \times I_3}$$

$$\Leftrightarrow M^2 \underbrace{(M - 4I_3)}_{} = I_3$$

Se $AB = I$

$A \rightarrow M^2$

$B \rightarrow$ inversa de A

$$\boxed{M^2 \times (M^2)^{-1} = I_3} \Longrightarrow (M^2)^{-1} = M - 4I_3$$

$$\Leftrightarrow \boxed{M^{-2} = M - 4I_3}$$, ou seja a inversa de $M^2 = M - 4I_3$

c.q.d.

c)

$$\boxed{M^{-1} \times M = I_3}$$

$$M^{-2} = M - 4I_3 \Leftrightarrow M^{-1} \times M^{-1} = M - 4I_3$$

$$\Leftrightarrow M^{-1} = M(M - 4I_3)$$

$$\Leftrightarrow M^{-1} = M \times \begin{bmatrix} -3 & 2 & -1 \\ 2 & -3 & 0 \\ -1 & -4 & -2 \end{bmatrix}$$

$$\Leftrightarrow M^{-1} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ -1 & -4 & 2 \end{bmatrix} \times \begin{bmatrix} -3 & 2 & -1 \\ 2 & -3 & 0 \\ -1 & -4 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ -4 & 1 & -2 \\ -7 & 2 & -3 \end{bmatrix}$$

Verificar se $M^{-1} \times M = I$:

$$\begin{bmatrix} 2 & 0 & 1 \\ -4 & 1 & -2 \\ -7 & 2 & -3 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ -1 & -4 & 2 \end{bmatrix} = I$$

$$\Leftrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I \checkmark$$

$\boxed{17}$

a) $A\ (\underbrace{n \times n}_{\text{matriz quadrada}})$

$A^K = 0$, $K \in \mathbb{N} \rightarrow \{0, 1, 2, 3 \ldots\}$

Para $I_n - A$ ser invertível:

$$(I_n - A) \times (I_n - A)^{-1} = I_n$$

$$\Leftrightarrow I_n - A = (I_n - A) \times I_n$$

$$\Leftrightarrow I_n - A = I_n^2 - I_n A$$

$$\Leftrightarrow I_n - A = I_n - A \checkmark \text{, logo } I_n - A \text{ é invertível!}$$

Se $(I_n - A)^{-1} = I_n + A + A^2 + \ldots + A^{K-1}$ então:

$$(I_n - A) \times (I_n + A + A^2 + \ldots + A^{K-1}) = I_n$$

$$\Leftrightarrow (I_n)^2 + I_n A + I_n A^2 + \ldots + I_n A^{K-1} - A I_n - A^2 - A^3 - \ldots - A^{K-1} - A^K = I_n$$

$$\Leftrightarrow I_n + \cancel{A} + \cancel{A^2} + \ldots \cancel{A^{K-1}} - \cancel{A} - A^2 - \ldots - \cancel{A^{K-1}} - A^K = I_n$$

$$\Leftrightarrow I_n \underbrace{- A^K}_{=0} = I_n$$

$$\Leftrightarrow I_n = I_n \checkmark$$

b)

$$M = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Seja $M = I_n - A$
$\Leftrightarrow A = I_n - M \Leftrightarrow A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Descobrir o $K \in \mathbb{N}$: $A^K = 0$

$A^0 = I_n \neq 0$

$A^1 = A \neq 0$

$A^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq 0$

$A^3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$, logo $K = 3$

$M^{-1} = (I_n - A)^{-1} = I_n + A + A^2$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = M^{-1}$

$\boxed{18}$!

a)

$$\left[\left(B^{-1}\right)^{T} X\right]^{-1} A^{-1} = I \Leftrightarrow \left[\left(B^{T}\right)^{-1} X\right]^{-1} A^{-1} = I$$

$$\Leftrightarrow \left[\left(B^{T}\right)^{-1}\right]^{-1} X^{-1} A^{-1} = I \Leftrightarrow B^{T} X^{-1} = A \Leftrightarrow B^{T} = A X$$

$$\Leftrightarrow X = B^{T} A^{-1}$$

$$B^{T} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 0 & -2 \end{bmatrix}$$

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \underset{\substack{L_2' := L_2 - L_1 \\ L_1' := L_1 - L_3}}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

A^{-1}

$$A^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$X = B^{T} A^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

b)

$$\left(C^{T} D^{T} X\right)^{T} = E \Leftrightarrow X^{T} D C = E \Leftrightarrow X^{T} = E \times (DC)^{-1} \Leftrightarrow X = \left[E \times (DC)^{-1}\right]^{T}$$

$$\Leftrightarrow X = \left[(DC)^{-1}\right]^{T} \times E^{T}$$

$$DC = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 3 & 2 \end{bmatrix}$$

$$[DC|I] = \left[\begin{array}{cc|cc} -4 & 0 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{array}\right] \underset{L_2 \leftrightarrow L_1}{\sim} \left[\begin{array}{cc|cc} 3 & 2 & 0 & 1 \\ -4 & 0 & 1 & 0 \end{array}\right] \underset{L_1 := \frac{1}{3}L_1}{\sim} \left[\begin{array}{cc|cc} 1 & 2/3 & 0 & 1/3 \\ -4 & 0 & 1 & 0 \end{array}\right] \underset{L_2' := L_2 + 4L_1}{\sim} \left[\begin{array}{cc|cc} 1 & 2/3 & 0 & 1/3 \\ 0 & 8/3 & 1 & 4/3 \end{array}\right]$$

$$\underset{L_2' := L_2 \times \frac{3}{8}}{\sim} \left[\begin{array}{cc|cc} 1 & 2/3 & 0 & 1/3 \\ 0 & 1 & 3/8 & 1/2 \end{array}\right] \underset{L_1' := L_1 - \frac{2}{3}L_2}{\sim} \left[\begin{array}{cc|cc} 1 & 0 & -1/4 & 0 \\ 0 & 1 & 3/8 & 1/2 \end{array}\right] \rightarrow (DC)^{-1}$$

$$\left[(DC)^{-1}\right]^T = \begin{bmatrix} -1/4 & 3/8 \\ 0 & 1/2 \end{bmatrix}$$

$$E^T = \begin{bmatrix} 4 & 0 \\ -4 & 8 \end{bmatrix}^T = \begin{bmatrix} 4 & -4 \\ 0 & 8 \end{bmatrix}$$

$$X = \left[(DC)^{-1}\right]^T \times E^T = \begin{bmatrix} -1/4 & 3/8 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 4 & -4 \\ 0 & 8 \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 0 & 4 \end{bmatrix}$$

$\boxed{19}$

a)
$$\begin{cases} y + 3z = 1 \\ x + y + 3z = 0 \\ 2x + y + 4z = 1 \end{cases}$$

Seja A a matriz dos coeficientes do sistema

$$A = \begin{bmatrix} 0 & 1 & 3 \\ 1 & 1 & 3 \\ 2 & 1 & 4 \end{bmatrix}$$

$$[A \mid I_n] = \left[\begin{array}{ccc|ccc} 0 & 1 & 3 & 1 & 0 & 0 \\ 1 & 1 & 3 & 0 & 1 & 0 \\ 2 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \underset{L_1 \leftrightarrow L_2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 0 & 1 & 0 \\ 0 & 1 & 3 & 1 & 0 & 0 \\ 2 & 1 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\underset{L_2' := L_2 - 2L_1}{\sim} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 0 & 1 & 0 \\ 0 & 1 & 3 & 1 & 0 & 0 \\ 0 & -1 & -2 & 0 & -2 & 1 \end{array}\right] \underset{L_3' := L_3 + L_2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 0 & 1 & 0 \\ 0 & 1 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right]$$

$$\underset{L_1' := L_1 - L_2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right] \underset{L_2' := L_2 - 3L_3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -2 & 6 & -3 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right],$$

logo A é invertível.

$$A^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ -2 & 6 & -3 \\ 1 & -2 & 1 \end{bmatrix}$$

b)

$$[A \mid B] = \left[\begin{array}{ccc|c} 0 & 1 & 3 & 1 \\ 1 & 1 & 3 & 0 \\ 2 & 1 & 4 & 1 \end{array}\right] \underset{L_1 \leftrightarrow L_2}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 1 & 3 & 1 \\ 2 & 1 & 4 & 1 \end{array}\right] \underset{L_3' := L_3 - 2L_1}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & -1 & -2 & 1 \end{array}\right]$$

$$\underset{L_3' := L_3 + L_2}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

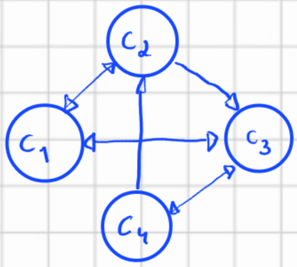
$$\left.\begin{array}{l} \text{car}(A) = 3 \\ \text{car}([A|B]) = 3 \end{array}\right\} = 3 = \text{nº de incógnitas},$$

logo o sistema é possível e determinado

$$\underset{L_1' := L_1 - L_2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right] \underset{L_2' := L_2 - 3L_3}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\begin{cases} x = -1 \\ y = -5 \\ z = 2 \end{cases} \qquad S = \{ (x = -1,\ y = -5,\ z = 2) \}$$

$\boxed{20}$



$C_1 \to C_2$, $C_1 \to C_3$

$C_2 \to C_1$, $C_2 \to C_3$

$C_3 \to C_1$, $C_3 \to C_4$

$C_4 \to C_2$, $C_4 \to C_3$

a)

$$A = [a_{ij}]_{4\times 4} = \begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{array} \begin{array}{c} \begin{array}{cccc} c_1 & c_2 & c_3 & c_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

b) $C_4 \longrightarrow C_1$

$$A^n = [a_{ij}^{(n)}]$$

i) $n = 1$

$A^1 = A$, e como $a_{41}^{(1)} = 0$, logo existem 0 itenerários que ligam C_4 a C_1 em apenas um voo

ii) $n = 2$

$$A^2 = A \times A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 1 & 1 \end{bmatrix}$$

$a_{41}^{(2)} = 2$, existem 2 itenerários para ir de C_4 a C_1 em dois voos.

1º: $C_4 \longrightarrow C_3 \longrightarrow C_1$

2º: $C_4 \longrightarrow C_2 \longrightarrow C_1$

iii) $n=3$

$$A^3 = A^2 \times A = \begin{bmatrix} 2 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 3 & 1 \\ 2 & 2 & 3 & 1 \\ 4 & 0 & 2 & 2 \\ 1 & 3 & 3 & 1 \end{bmatrix}$$

$a_{41}^{(3)} = 1$, apenas existe um itenerário para ir de C_4 a C_1 em três voos:

$$C_4 \longrightarrow C_2 \longrightarrow C_3 \longrightarrow C_1$$