Aula 17

Integrais de Dirichlet

$$\int_1^{+\infty} \frac{1}{x^\alpha}\,dx = \begin{cases} \frac{1}{\alpha-1}, & \text{se } \alpha > 1 \\ \text{Divergente}, & \text{se } \alpha \leq 1 \end{cases}$$

Integrais de Exponencial

$$\int_1^{+\infty} e^{\beta x}\,dx = \begin{cases} \frac{1}{\beta}, & \text{se } \beta < 0 \\ \text{Divergente}, & \text{se } \beta \geq 0 \end{cases}$$

Exercício (A)

$$\int_1^{+\infty} \frac{\sin^2 x}{x^{5/2}}\,dx \qquad \sin^2 x \in [0,1]$$

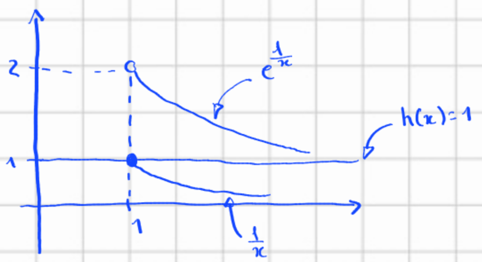
$$0 \leq \frac{\sin^2 x}{x^{5/2}} \leq \frac{1}{x^{5/2}}, \forall x \in [1,+\infty[$$

Como $\underbrace{\int_1^{+\infty} \frac{1}{x^{5/2}}\,dx}_{\frac{1}{\frac{5}{2}-1} = \frac{2}{3}}$ converge (Dirichlet) então $\int \frac{\sin^2 x}{x^{5/2}}\,dx$ converge

$\alpha = \frac{5}{2} > 1$

(A2)

$$\int_1^{+\infty} e^{1/x}\,dx$$



Na verdade, $0 \leq \frac{1}{x} \leq e^{1/x}, \forall x \in [1,+\infty[$

$$\frac{1}{x} \leq 1$$

$$e^{1/x} \geq 1$$

$$0 \leq \int_1^{+\infty} \frac{1}{x}\,dx \leq \int_1^{+\infty} e^{1/x}$$

$\int_1^{+\infty} \frac{1}{x}\,dx$ diverge (Dirichlet, $\alpha = 1 \leq 1$) então $\int_1^{+\infty} e^{1/x}\,dx$ diverge.

Outra solução

$$0 \leq \underbrace{10^{-10}}_{+\infty} \leq \underbrace{e^{1/x}}_{+\infty}$$

Logo $\int_1^{+\infty} e^{1/x}\,dx$ diverge

(A3)

$$\int_3^{+\infty} \frac{2x^2 - x}{3x^4 + x^3 + x + 1}\,dx$$

$$0 \leq \frac{2x^2 - x}{3x^4+x^3+x+1} \leq \frac{2x^2}{3x^4+x^3+x+1} \leq \frac{2x^2}{3x^4} \leq \frac{2}{3} \times \frac{1}{x^2}, \forall x \in [3,+\infty[$$

$\int_3^{+\infty} \frac{2x^2-x}{3x^4+x^3+x+1}\,dx$ Conv. porque $\int_3^{+\infty} \frac{2}{3} \times \frac{1}{x^2}\,dx = \underbrace{\overset{\neq 0}{\frac{2}{3}}}\, \underbrace{\int_3^{+\infty} \frac{1}{x^3}\,dx}_{\text{Converge}}$

Dirichlet, $x = 2 > 1$

Exercício (B)

Se $x \geq 5$, p.e. $x = 10^{10}$

$$\frac{x^2-2}{3x^3-x^2-70} \sim \frac{x^2}{3x^2} = \frac{1}{x}$$

$$\int_5^{+\infty} \frac{x^2-2}{3x^2-x^2-70}\, dx$$

$$L = \lim_{x\to+\infty} \frac{\frac{x^2-2}{3x^3-x^2-70}}{\frac{1}{3x}} = \lim_{x\to+\infty} \frac{3x^3-6x}{3x^3-x^2-70} = 1 \in \mathbb{R}^+,$$

logo $\int_5^{+\infty} \frac{x^2-2}{3x^3-x^2-70}\, dx$ tem a mesma natureza de $\int_5^{+\infty} \frac{1}{3x}\, dx = \frac{1}{3} \underbrace{\int_5^{+\infty} \frac{1}{x}\, dx}_{\text{Divergente}}$,

logo é Div. porque $\int_5^{+\infty} \frac{1}{x}\, dx$ é Div.

(B2)

Se $x \geq 0$

$$\frac{2}{e^{2x}-1} \sim \frac{2}{e^{2x}}$$

$$\int_1^{+\infty} \frac{2}{e^{2x}-1}\, dx$$

$$L = \lim_{x\to+\infty} \frac{\frac{2}{e^{2x}-1}}{\frac{2}{e^{2x}}} = \lim_{x\to+\infty} \frac{e^{2x}}{e^{2x}-1} = \lim_{x\to+\infty} \frac{\cancel{e^{2x}}}{\cancel{e^{2x}}(1-e^{-2x})} = \lim_{x\to+\infty} \frac{1}{1-e^{-2x}} = 1 \in \mathbb{R}^+$$

Como $\int_1^{+\infty} \frac{2}{e^{2x}}\, dx = 2\int_1^{+\infty} e^{-2x}\, dx$, converge $(\beta = -2 < 0)$

(B3)

Se $x \gg 1$

$$\frac{5x^2-3}{x^8+x-1} \sim \frac{5x^2}{x^8} = \frac{5}{x^6}$$

$$\int_1^{+\infty} \frac{5x^2-3}{x^8+x-1}$$

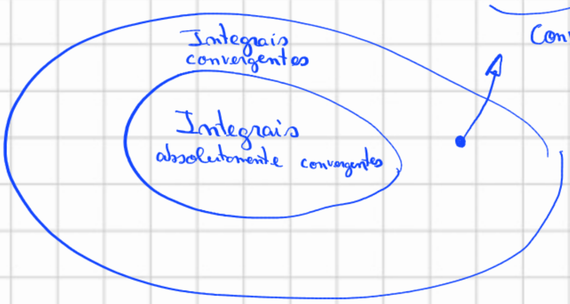
$$L = \lim_{x\to+\infty} \frac{\frac{5x^2-3}{x^8+x-1}}{\frac{5}{x^6}} = \lim_{x\to+\infty} \frac{\frac{1}{5}(5x^2-3)\times x^6}{x^8+x-1} = \lim_{x\to+\infty} \frac{x^8-\frac{3}{5}x^6}{x^8+x-1} = 1 \in \mathbb{R}^+$$

os integrais dado tem a mesma natureza de $5\int_1^{+\infty} \frac{1}{x^6}\, dx$

(conv. Dirichlet, $\alpha = 6 > 1$)

Exemplo de um integral convergente mas não absolutamente convergente.

$$\int_1^{+\infty} \left| \frac{\sin x}{x} \right| dx \quad \text{DIV} \quad \text{mas} \quad \underbrace{\int_1^{+\infty} \frac{\sin x}{x}\, dx}_{\text{Conv.}}$$



Integrais impróprios de 2ª Espécie

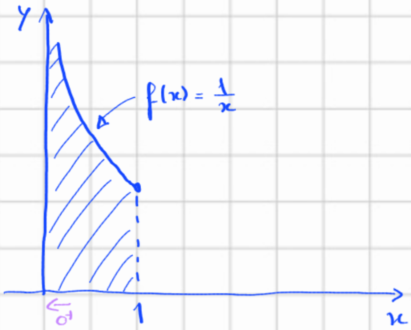
Integrais de Dirichlet (2ª Espécie)
$\int_0^1 \frac{1}{x^\alpha} dx = \begin{cases} \text{DIV. se } \alpha > 1 \\ \frac{1}{1-\alpha}, \text{ se } \alpha \in]0,1[\end{cases}$

Exemplo:

$$\int_0^1 \frac{1}{x} dx \;,\; \int_0^1 \frac{1}{(x-2)^2} dx$$

Não está def. para $x=0$ e até é ilimitada

Não está def. em $x=2$ e é ilimitada



$$\int_0^1 \frac{1}{x} dx := \lim_{t \to 0^+} \int_t^1 \frac{1}{x} dx = \lim_{t \to 0^+} \left([\ln x]_t^1 \right)$$

$$= \lim_{t \to 0^+} \left(\ln(1) - \ln(t) \right)$$

$$= \lim_{t \to 0^+} (-\ln t) = +\infty \quad \text{DIV}$$

Então e:

$$\int_0^1 \frac{1}{\sqrt{x}} dx := \lim_{t \to 0^+} \left(\int_t^1 x^{-1/2} dx \right)$$

$$= \lim_{t \to 0^+} \left(\left[\frac{x^{1/2}}{\frac{1}{2}} \right]_t^1 \right)$$

$$= \lim_{t \to 0^+} \left(2\sqrt{1} - 2\sqrt{t} \right) = 2$$

$$\frac{1}{1-\alpha} = \frac{1}{1-\frac{1}{2}} = 2$$

Exemplo C

$$\int_0^1 \frac{\ln(1-x)}{1-x} = \lim_{t\to 1^-} \int_0^t \frac{\ln(1-x)}{1-x}\,dx = \lim_{t\to 1^-}\left(\left[\frac{\ln^2(1-x)}{2}\right]_0^t\right)$$

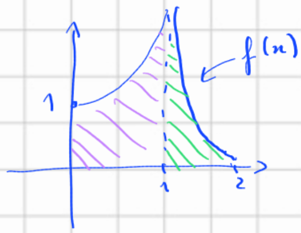
$$= \lim_{t\to 1^-}\left(\frac{\ln^2(1-t)}{2} - \frac{\cancel{\ln(1-0)}^{\,0}}{2}\right)$$

$$= \lim_{t\to 1^-}\left(\frac{\ln^2(1-t)}{2}\right) = +\infty \,, \quad \text{DIV}$$

C2

$$\int_0^2 \frac{1}{(x-1)^2}\,dx$$

$f(x)$ não está def. em $x=1$



Integral é da 2.ª espécie porque a função integrada não está definida em $x=1 \in]0,2[$

Este integral converge se convergirem os dois seguintes integrais:

$$\int_0^1 \frac{1}{(x-1)^2}\,dx \quad e \quad \int_1^2 \frac{1}{(x-1)^2}\,dx$$

2.ª espécie

$$\int_0^1 \frac{1}{(x-1)^2}\,dx = \lim_{t\to 1^-}\int_0^t \frac{1}{(x-1)^2}\,dx = \lim_{t\to 1^-}\left(\left[\frac{(x-1)^{-2+1}}{-2+1}\right]_0^t\right)$$

$$= \lim_{t\to 1^-}\left(\frac{1}{\cancel{1-t}}^{\,+\infty} - \frac{1}{1-0}\right) = +\infty \quad \text{DIV}$$

Logo, $\int_0^2 \frac{1}{(x-1)^2}\,dx$ diverge!