Resolvidos - Slides AB 1

1.6

$f(x) = 3x - \pi$

$f'(x) = 3 > 0$, f é estritamente crescente em $\mathbb{R}$

Logo, f é injetiva em $\mathbb{R}$

$g(x) = x^2 + 4$

$g(1) = 1 + 4 = 5$
$g(-1) = 1 + 4 = 5$

Logo como $1 \neq -1 \wedge g(1) = g(-1)$, a função g não é injetiva

1.17

a) * $y = e^{1-2x} \Leftrightarrow \ln(y) = 1 - 2x$

$\Leftrightarrow \dfrac{1 - \ln(y)}{2} = x$

Inversa: $f^{-1}(x) = \dfrac{1 - \ln(x)}{2}$

$f^{-1}: \mathbb{R}^+ \longrightarrow \mathbb{R}$

$x \longmapsto y = \dfrac{1 - \ln(x)}{2}$

$CD_{f^{-1}} = \mathbb{R}$

* Devíamos verificar se a função é invertível
→ derivada
→ monotonia
→ é injetiva
→ logo é invertível

b) * $y = \dfrac{5\ln(x-3) - 1}{4} \Leftrightarrow \dfrac{4y+1}{5} = \ln(x-3)$

$\Leftrightarrow x - 3 = e^{\frac{4y+1}{5}}$

$\Leftrightarrow x = e^{\frac{4y+1}{5}} + 3$

Inversa: $f^{-1}(x) = e^{\frac{4x+1}{5}} + 3$

$f^{-1}: \mathbb{R} \longrightarrow \mathbb{R}$

$x \longmapsto y = e^{\frac{4x+1}{5}} + 3$

$CD_{f^{-1}} =]3, +\infty]$

1.21

→ Restrição principal da função seno $(D=[-\frac{\pi}{2},\frac{\pi}{2}])$

a) $y = \frac{1}{2}\sin\left(x+\frac{\pi}{2}\right) \Leftrightarrow 2y = \sin\left(x+\frac{\pi}{2}\right)$

$\Leftrightarrow \arcsin(2y) = x + \frac{\pi}{2}$

$\Leftrightarrow x = \arcsin(2y) - \frac{\pi}{2}$

"$\frac{\pi}{2}$", x é mentiroso

$-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

$-\pi \leq x+\frac{\pi}{2} \leq 0$

$D_f = [-\pi, 0]$

$-1 \leq \sin\left(x+\frac{\pi}{2}\right) \leq 1$

$-\frac{1}{2} \leq \frac{1}{2} \times \sin\left(x+\frac{\pi}{2}\right) \leq \frac{1}{2}$

$CD_f = \left[-\frac{1}{2}, \frac{1}{2}\right]$

Inversa: $f^{-1}(x) = \arcsin(2x) - \frac{\pi}{2}$

$f^{-1}: \left[-\frac{1}{2}, \frac{1}{2}\right] \longrightarrow \mathbb{R}$

$x \longrightarrow y = \arcsin(2x) - \frac{\pi}{2}$

$CD_{f^{-1}} = D_f = [-\pi, 0]$

b) $y = \frac{\pi}{2} - \frac{2\arcsen(1-x)}{3} \Leftrightarrow \frac{3\pi}{4} - \frac{3}{2}y = \arcsen(1-x)$

é um seno

$\Leftrightarrow \sen\left(\frac{3}{2}y - \frac{3\pi}{4}\right) = 1-x$

$\Leftrightarrow x = 1 - \sen\left(\frac{3}{2}y - \frac{3\pi}{4}\right)$

é um seno

$-1 \leq 1-x \leq 1$

$\Leftrightarrow -2 \leq -x \leq 0$

$\Leftrightarrow 0 \leq x \leq 2$

$D_f = [0, 2]$

$-\frac{\pi}{2} \leq \arcsen(1-x) \leq \frac{\pi}{2}$

$\Leftrightarrow -\frac{\pi}{3} \leq \frac{2}{3}\arcsin(1-x) \leq \frac{\pi}{3}$

$\Leftrightarrow -\frac{\pi}{3} \leq -\frac{2}{3}\arcsin(1-x) \leq \frac{\pi}{3}$

$\Leftrightarrow -\frac{\pi}{3} + \frac{\pi}{2} \leq f(x) \leq \frac{\pi}{3} + \frac{\pi}{2}$

$\Leftrightarrow \frac{\pi}{6} \leq f(x) \leq \frac{5\pi}{6}$

$CD_f = \left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$

$D_{f^{-1}} = CD_f = \left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$

Inversa: $f^{-1}(x) = 1 - \sen\left(\frac{3}{2}x - \frac{3\pi}{4}\right)$

$f^{-1}: \left[\frac{\pi}{6}, \frac{5\pi}{6}\right] \longrightarrow \mathbb{R}$

$x \longrightarrow y = 1 - \sen\left(\frac{3}{2}x - \frac{3\pi}{4}\right)$

$CD_{f^{-1}} = D_f = [0, 2]$

$\boxed{1.29}$

a) $$f(x) = tg\left(\frac{\pi}{2-x}\right), \quad x \neq 2$$

$-\frac{\pi}{2} < \frac{\pi}{2-x} < \frac{\pi}{2}$

$-\frac{1}{2} < \frac{1}{2-x}$

$-\frac{(2-x)}{2} > 1$

$-2+x > 2$

$x > 4$

Como $2-x<0$ invertemos o sinal

Para $x > 2$:
$\hookrightarrow 2-x<0$

$$\begin{cases} -\frac{\pi}{2} < \frac{\pi}{2-x} \\ \frac{\pi}{2-x} < 0 \end{cases} \quad \begin{cases} \frac{1}{-2} < \frac{1}{2-x} \\ 2-x<0 \end{cases} \quad \begin{cases} -(2-x) > 2 \\ -x < -2 \end{cases}$$

$$\begin{cases} -2+x > 2 \\ x > 2 \end{cases} \quad \begin{cases} x > 4 \\ x > 2 \end{cases}, \text{ logo } x \in]4, +\infty[$$

Para $x < 2$:
$\hookrightarrow 2-x>0$

$$\begin{cases} \frac{\pi}{2-x} < \frac{\pi}{2} \\ \frac{\pi}{2-x} > 0 \end{cases} \quad \begin{cases} 2 < 2-x \\ 2-x>0 \end{cases} \quad \begin{cases} -x > 0 \\ -x > -2 \end{cases} \quad \begin{cases} x < 0 \\ x < 2 \end{cases}, \text{ logo } x \in]-\infty, 0[$$

$$D_f =]-\infty, 0[\cup]4, +\infty[$$

Quando x percorre D_f $\frac{\pi}{2-x}$ percorre $]-\frac{\pi}{2}, 0[\cup]0, \frac{\pi}{2}[$, logo f percorre $\mathbb{R} \setminus \{0\}$

$CD_f = \mathbb{R} \setminus \{0\}$

$$y = tg\left(\frac{\pi}{2-x}\right) \Leftrightarrow arctg(y) = \frac{\pi}{2-x} \Leftrightarrow 2-x = \frac{\pi}{arctg(y)}$$

$$\Leftrightarrow x = 2 - \frac{\pi}{arctg(y)}$$

$$f^{-1}: \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$$
$$x \longrightarrow y = 2 - \frac{\pi}{arctg(y)}$$

Com $CD_{f^{-1}} = D_f =]-\infty, 0[\cup]4, +\infty[$

b) $f(x) = \frac{\pi}{2} - \operatorname{arctg}(1-x) \longrightarrow (1-x) \in \mathbb{R}$

$$-\frac{\pi}{2} < \operatorname{arctg}(1-x) < \frac{\pi}{2}$$

$$\frac{\pi}{2} > -\operatorname{arctg}(1-x) > -\frac{\pi}{2}$$

$$0 < f(x) < \pi$$

$D_f = \mathbb{R}$

$CD_f =]0, \pi[$

$$y = \frac{\pi}{2} - \operatorname{arctg}(1-x) \Leftrightarrow \frac{\pi}{2} - y = \operatorname{arctg}(1-x)$$

$$\Leftrightarrow \operatorname{tg}\left(\frac{\pi}{2} - y\right) = 1 - x$$

$$\Leftrightarrow x = 1 - \operatorname{tg}\left(\frac{\pi}{2} - y\right)$$

$$f^{-1}: \;]0, \pi[\longrightarrow \mathbb{R}$$

$$x \longrightarrow y = 1 - \operatorname{tg}\left(\frac{\pi}{2} - x\right)$$

Com $CD_{f^{-1}} = D_f = \mathbb{R}$

$\boxed{1.44}$

$g(x) = \operatorname{sen} x \qquad f(x) = x^3$

a) $(g \circ f)'(x) = f'(x) \cdot g'(f(x))$

$\boxed{CD_f = \mathbb{R} \subset D_g = \mathbb{R}}$

$$= 3x^2 \cdot \operatorname{sen}'(x^3)$$

$$= 3x^2 \cdot \cos(x^3)$$

b) $(f \circ g)'(x) = \left(f(g(x))\right)'$

$\boxed{CD_g = [-1, 1] \subset D_f = \mathbb{R}}$

$$= (\sin^3 x)'$$

$$= 3 \sin^2 x \times (\sin x)'$$

$$= 3 \sin^2 x \times \cos x$$

$\boxed{1.46}$

$\boxed{1}$

$f: [1, 4] \longrightarrow \mathbb{R}$

$f'(x) > 0, \; \forall x \in [1, 4]$

$f(2) = 7 \qquad f'(2) = \frac{2}{3}$

$$(f^{-1})'(7) = (f^{-1})'(f(2)) = \frac{1}{f'(2)} = \frac{1}{\frac{2}{3}} = \frac{3}{2}$$

$\boxed{2}$

$$f(x) = 4x^3 + x + 2$$

Seja $f(x_0) = 2$

$$(f^{-1})'(2) = (f^{-1})'(f(x_0)) = \frac{1}{f'(x_0)}$$

$$\cancel{2} = 4x_0^3 + x_0 + \cancel{2} \Leftrightarrow 0 = x_0(4x^2 + 1)$$

$$\Leftrightarrow x_0 = 0 \vee \underbrace{x^2 = -\frac{1}{4}}_{\text{C.I., } x^2 > 0,\ \forall x \in \mathbb{R}}$$

$$\Leftrightarrow x_0 = 0$$

$$f(0) = 2$$

$$(f^{-1})'(2) = \frac{1}{f'(0)}$$

$$f'(x) = 12x^2 + 1$$

$$f'(0) = 1$$

Logo, $(f^{-1})'(2) = 1$

$\boxed{3}$

$$f(x) = x^3 \qquad f'(x) = 3x^2$$

$$(f^{-1})'(x) = ? \qquad y_0 = x_0^3$$

$$(f^{-1})'(y_0) = (f^{-1})' \cdot f(x_0) = \frac{1}{f'(x_0)}$$

$$f'(x_0) = 3x_0^2$$

$$(f^{-1})'(y_0) = \frac{1}{3x_0^2}$$

Como $y_0 = x_0^3$

$$\Leftrightarrow x_0 = \sqrt[3]{y_0}$$

$$\frac{1}{3x_0^2} = \boxed{\frac{1}{3\sqrt[3]{y_0^2}}}$$

Confirmo definindo a inversa

$$y = x^3 \Leftrightarrow x = \sqrt[3]{y}$$

$$f^{-1}(x) = \sqrt[3]{x} = (x)^{\frac{1}{3}}$$

$$(f^{-1})'(x) = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3} \cdot \frac{1}{(x^2)^{\frac{1}{3}}} = \boxed{\frac{1}{3\sqrt[3]{x^2}}}$$

$\boxed{1.47}$ Utilizando o T. da derivação da função inversa

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}, \quad \forall x \in]-1,1[$$

$y_0 = \sin x_0$

$$(\arcsin y_0)' = \Big(\arcsin(\overbrace{\sin x_0}^{[-1,1]})\Big)' = \frac{1}{(\sin(x_0))'} = \frac{1}{\cos x_0}$$

- Como $y_0 = \sin x_0 \Leftrightarrow x_0 = \arcsin y_0$

$$\frac{1}{\cos x_0} = \frac{1}{\cos(\arcsin y_0)}$$

- $\sin^2 x + \cos^2 x = 1 \Leftrightarrow \cos \underset{[0,\pi] \Rightarrow \cos x > 0}{x} = +\sqrt{1-\sin^2 x}$

- $\cos(\arcsin y_0) = +\sqrt{1-(\sin(\arcsin y_0))^2}$

$\Leftrightarrow \cos(\arcsin y_0) = +\sqrt{1-y_0^2}$

$$\boxed{\text{Logo, } (\arcsin y_0)' = \frac{1}{\cos(\arcsin y_0)} = \frac{1}{\sqrt{1-y_0^2}}}$$

Exercícios extra:

a) $(x \arcsin \overset{\to[-1,1]}{x})' = \arcsin x + x(\arcsin x)'$

$$= \arcsin x + \frac{x}{\underset{\to[-1,1]}{\sqrt{1-x^2}}}$$

b) $\left[\dfrac{(1+x^2)\operatorname{arctg} \overset{\to \mathbb{R}}{x} - x}{2}\right]' = \dfrac{[(1+x^2)\operatorname{actg} x - x]' \times 2}{4}$

$$= \frac{(1+x^2)'\operatorname{actg} x + (1+x^2)(\operatorname{arctg} x)' - 1}{2}$$

$$= \cancel{2}x\,\underline{\operatorname{arctg} x} + \frac{1+\cancel{x^2}}{1\cancel{+x^2}} - \cancel{1}$$

a) $[x \arcsin \overset{\to[-1,1]}{x}]'$

b) $\left[\dfrac{(1+x^2)\operatorname{arctg} \overset{\to \mathbb{R}}{x} - x}{2}\right]'$

c) $[\sqrt{1+\underset{\to[-1,1]}{\arcsin}}]'$

d) $[\underset{\mathbb{R}_0^+}{\sqrt{\operatorname{arctg} x}} - \underset{[-1,1]}{(\arcsin x)^3}]'$ $\quad D = \mathbb{R}_0^+ \cap [-1,1] = [0,1]$

c) $\left[\sqrt{1+\arcsin x}\right]' = \left[(1+\arcsin x)^{\frac{1}{2}}\right]' = \frac{1}{2} \times (1+\arcsin x)^{-\frac{1}{2}} \times \frac{1}{\sqrt{1-x^2}}$

$= \frac{1}{2 \times \sqrt{1+\arcsin x} \times \sqrt{1-x^2}} = \frac{1}{2\sqrt{(1+\arcsin x)\times(1-x^2)}}$

d) $\left[\sqrt{\operatorname{arctg}(x)} - (\arcsin(x))^3\right]' = \frac{1}{2} \times (\operatorname{arctg} x)^{-\frac{1}{2}} \times \frac{1}{1+x^2} - 3\arcsin^2 x \times \frac{1}{\sqrt{1-x^2}}$

$\mathbb{R}_0^+$ $\quad [-1,1]$

$= \frac{1}{2\sqrt{\operatorname{arctg} x} \times (1+x^2)} - \frac{3\arcsin^2 x}{\sqrt{1-x^2}}$

$D_{f'} = \mathbb{R}_0^+ \cap [-1,1]$
$= [0,1]$

$\boxed{2.3}$ Admito que a tese é falsa e vou verificar que a hipótese é falsa

Forma de resolução

- Admita-se que f troca de sinal em $]a,b[$

→ Pelo T. Bolzano tem de existir outro zero de f num ponto $c \in]a,b[$. Ora isso contraria a hipótese (que diz que $x=a$ e $x=b$ são os únicos zeros de f)

$\boxed{u \in]a,b[: f(u) > 0 \qquad v \in]a,b[: f(v) < 0}$

⇓

Então, existiria um valor w entre u e v : $f(w) = 0$

$\boxed{2.13}$ I tenho foto

$\boxed{2}$ $f(x) = x^3 + ax + b$, onde $a > 0$, $b \in \mathbb{R}$

- Ver se f é diferenciável

$f'(x) = \underbrace{3x^2 + a}_{\substack{\geq 0 \ + \ > 0 \\ > 0}} > 0, \ \forall x \in \mathbb{R}$

Impossível (f' não tem zeros)

- Suponhamos que f tem dois zeros distintos u e v
- Como f é regular em qualquer intervalo compacto $[a,b] \subset \mathbb{R}$

T. de Rolle (Corolário I): se f tem dois zeros distintos u, v ($u < v$) então exis um zero da derivada $w \in]u,v[$

$\exists\, w \in]u,v[: f'(w) = 0$

Técnicas do 12.º ano:

→ f é estritamente crescente
→ $\lim_{x \to +\infty} f(x) = +\infty$ e $\lim_{x \to -\infty} f(x) = -\infty$

f troca de sinal

T. Bolzano, se troca de sinal tem um único zero