Aula 11

$$\int \frac{\sqrt{x^2-9}}{x}\,dx = \int \frac{\sqrt{9\sec^2 t-9}}{3\cancel{\sec t}} \times 3\operatorname{tg} t\,\cancel{\sec t}\,dt = \int \frac{\cancel{3}\sqrt{\sec^2 t-1}}{\cancel{3}} \times 3\operatorname{tg} t$$

$x \longrightarrow t$

$x = g(t) = 3\sec t,\ t \in]0, \frac{\pi}{2}[$

$g'(t) = 3\operatorname{tg} t \sec t,\ t \in]0, \frac{\pi}{2}[$

g é diferenciável e invertível em

$dx = 3\operatorname{tg} t \sec t\,dt$

$$= 3\int \operatorname{tg}^2 t\,dt$$

$$= 3\int \sec^2 t - 1\,dt$$

$$= 3 \times \operatorname{tg} t - 3t + c,\ c \in \mathbb{R} \text{ em intervalos}$$

$$3\sec t = x \Leftrightarrow \sec t = \frac{x}{3}$$

$$\Leftrightarrow \frac{1}{\cos t} = \frac{x}{3} \Leftrightarrow \arccos\left(\frac{3}{x}\right) = t$$

$$\operatorname{tg} t = \operatorname{tg}\left(\arccos\left(\frac{3}{x}\right)\right)$$

$$= \frac{\sin\left(\arccos\left(\frac{3}{x}\right)\right)}{\cos\left(\arccos\left(\frac{3}{x}\right)\right)} = \frac{\sqrt{1-\cos\left(\arccos\left(\frac{3}{x}\right)\right)^2}}{\frac{3}{x}} = \frac{x\sqrt{1-\left(\frac{3}{x}\right)^2}}{3} = \frac{\sqrt{x^2-9}}{3}$$

Voltando $t \to x$

$$\int \frac{x^2-9}{x}\,dx = \frac{\sqrt{x^2-9}}{3} - 3\arccos\left(\frac{3}{x}\right)$$

Casos mais complexos com substituição trigonométrica

E.g.: $\sqrt{a^2+b^2x^2} = \sqrt{b^2\left(\left(\frac{a^2}{b^2}\right)+x\right)} = b\sqrt{c^2+x^2},\ c = \frac{a}{b}$

$\sqrt{a^2-b^2x^2} = \sqrt{b^2\left(\left(\frac{a^2}{b^2}\right)-x^2\right)} = b\sqrt{c^2-x^2},\ c = \frac{a}{b}$

$\sqrt{-a^2+b^2x^2} = \sqrt{b^2\left(\left(-\frac{a^2}{b^2}\right)+x^2\right)} = b\sqrt{-c^2+x^2},\ c = \frac{a}{b}$

$\boxed{\sqrt{\underbrace{-a^2-b^2x^2}_{<0}}}$ é Impossível!

$$\int \sqrt{4+5x^2}\,dx = \int \sqrt{5\left(\frac{4}{5}+x^2\right)}\,dx = \sqrt{5} \times \int \sqrt{\frac{4}{5}+x^2}\,dx$$

$\frac{4}{5} = \left(\frac{2}{\sqrt{5}}\right)^2$

$x \longrightarrow t$

$x = g(t) = \frac{2}{\sqrt{5}}\, tg\, t$

$g'(t) = \frac{2}{\sqrt{5}} \sec^2 t, \quad t \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$

g é inv. e dif. em

$dx = \frac{2}{\sqrt{5}} \sec^2 t\, dt$

$$= \sqrt{5} \times \int \sqrt{\frac{4}{5}+\frac{4}{5} tg^2 t} \times \frac{2}{\sqrt{5}} \times \sec^2 t$$

$$= \sqrt{5} \times \sqrt{\frac{4}{5}} \times \frac{2}{\sqrt{5}} \times \int \sqrt{1+tg^2 t} \times \sec^2 t\, dt$$

$$= \frac{4}{\sqrt{5}} \times \boxed{\int \sec^3 t\, dt}$$

(ver atrás) faz-se pela primitivação por partes

$= \frac{1}{2} tg \sec t + \frac{1}{2} \ln|tg t + \sec t| + c$

$$= \frac{4}{\sqrt{5}} \times \left[\frac{1}{2} tg \sec t + \frac{1}{2} \times \ln|tg\, t + \sec t|\right] + c, \quad c \in \mathbb{R}$$

$$= \frac{2}{\sqrt{5}} tg\, t \sec t + \frac{2}{\sqrt{5}} \ln|tg\, t + \sec t| + c, \quad c \in \mathbb{R} \text{ em int.}$$

$x = \frac{2}{\sqrt{5}} tg\, t$

$\Leftrightarrow tg\, t = \frac{\sqrt{5}}{2} x$

$\sec t = ?$

$tg^2 t = \frac{5}{4} x^2$

$tg^2 t + 1 = \frac{5}{4} x^2 + 1$

$\sec^2 t = \frac{5}{4} x^2 + 1$

$\sec t = \sqrt{\frac{5}{4} x^2 + 1} = \frac{\sqrt{5x^2+4}}{2}$

$t \longrightarrow x$

$tg\, t = \frac{\sqrt{5}}{2} x$

$\sec t = \frac{\sqrt{5x^2+4}}{2}$

$$\int \sqrt{4+5x^2}\,dx = \frac{2}{5} \times \frac{\sqrt{5}}{2} x \times \frac{\sqrt{5x^2+4}}{2} + \frac{2}{\sqrt{5}} \ln\left|\frac{\sqrt{5}}{2} x + \frac{\sqrt{5x^2+4}}{2}\right| + c, \quad c \in \mathbb{R} \text{ em intervalos}$$

Caso mais geral!

- $\sqrt{ax^2+bx+c}$, $a \neq 0$, $b, c \in \mathbb{R}$
- Reduz-se aos racionais que anteriormente estudamos, escrevemos o polinómio:

$$ax^2+bx+c = a(x+p)^2+q$$ (e pondo em evidência constantes)

$$\int \frac{x}{\sqrt{8+2x-x^2}}\,dx = \int \frac{x}{\sqrt{-(x-1)^2+9}}\,dx = \int \frac{x}{\sqrt{9\left(1-\left(\frac{x-1}{3}\right)^2\right)}}\,dx$$

$$= \frac{1}{3}\int \frac{x}{\sqrt{1-\left(\frac{x-1}{3}\right)^2}}\,dx$$

$\longrightarrow \sin t \longrightarrow$ "o enunciado dizia"

$$8+2x-x^2 = -(x+p)^2+q = -x^2-2xp\,(-p^2+q)$$

$$\begin{cases} -1=-1 \\ 2=-2p \\ 8=-p^2+q \end{cases} \qquad \begin{cases} -1=-1 \\ p=-1 \\ 9=q \end{cases}$$

$$8+2x-x^2 = -(x-1)^2+9 = 9\left(1-\left(\frac{x-1}{3}\right)^2\right)$$

$$\boxed{\begin{array}{l} x \longrightarrow t \qquad \left(\frac{x-1}{3}\right)=\sin t \\ x=g(t)=3\sin t+1 \\ g'(t)=3\cos t,\ t\in\left]0,\frac{\pi}{2}\right[\\ g \text{ é inv. e dif em} \\ dx = 3\cos t\,dt \end{array}}$$

$$= \frac{1}{\cancel{3}}\int \frac{3\sin t+1}{\sqrt{1-(\sin t)^2}} \times \cancel{3}\cos t\,dt$$

$$= \int \frac{3\sin t+1}{\cancel{\sqrt{1-\sin^2 t}}}\,\cancel{\cos t}\,dt$$

$$= \int 3\sin t+1\,dt = t+c+3\int \sin t\,dt$$

$$= t-3\cos t+c,\ c\in\mathbb{R} \text{ em int.}$$

$$\frac{x-1}{3} = \sin t$$

$$\Leftrightarrow x = 3\sin t+1$$

$$\Leftrightarrow t = \arcsin\left(\frac{x-1}{3}\right)$$

$$\cos t = \cos\left(\arcsin\left(\frac{x-1}{3}\right)\right)$$

$$= \sqrt{1-\operatorname{sen}\left(\arcsin\left(\frac{x-1}{3}\right)\right)^2}$$

$$= \sqrt{1-\left(\frac{x-1}{3}\right)^2} = \sqrt{\frac{9}{9}-\frac{x^2-2x+1}{9}} = \frac{\sqrt{-x^2+2x+8}}{3}$$

$$\boxed{\begin{array}{l} t \longrightarrow x \\ t=\arcsin\left(\frac{x-1}{3}\right) \\ \cos t = \dfrac{\sqrt{-x^2+2x+8}}{3} \end{array}}$$

$$\int \frac{x}{\sqrt{8+2x-x^2}}\,dx = \arcsin\left(\frac{x-1}{3}\right) - \sqrt{-x^2+2x+8}+c,\ c\in\mathbb{R} \text{ em int.}$$

Regresso às Frações Racionais

Recorrendo a Fórmula de Recorrência:

$$\begin{cases} \displaystyle\int \frac{dx}{(1+x^2)^m} = \frac{2m-3}{2m-3}\int \frac{dx}{(1+x^2)^{m-1}} + \frac{1}{2m-2}\,\frac{x}{(1+x^2)^{m-1}},\ m=2,3,\dots \\ \displaystyle\int \frac{1}{1+x^2}\,dx = \operatorname{arctg} x + c,\ c\in\mathbb{R} \end{cases}$$

$$\int \frac{1}{(x^2+4x+9)^3}\,dx = \int \frac{1}{[(x+2)^2+5]^3}\,dx = \int \frac{1}{\left[5\left[\left(\frac{x+2}{\sqrt{5}}\right)^2+1\right]\right]^3}\,dx$$

$$x^2+4x+9 = (x+p)^2+q = x^2+2xp+(p^2+q)$$

$$= \frac{1}{125}\int \frac{1}{\left[\left(\frac{x+2}{\sqrt{5}}\right)^2+1\right]^3}\,dx$$

$$\begin{cases} 1=1 \\ 4=2p \\ 9=p^2+q \end{cases} \qquad \begin{cases} p=2 \\ q=5 \end{cases}$$

$$x^2+4x+9 = (x+2)^2+5$$

$$\boxed{\begin{array}{l} x \to t \\ t = \frac{x+2}{\sqrt{5}} \Leftrightarrow x = -2+\sqrt{5}t \\ dx = \sqrt{5}\,dt \end{array}}$$

$$= \frac{1}{125}\int \frac{1}{[t^2+1]^3}\sqrt{5}\,dt$$

$$= \frac{\sqrt{5}}{125}\boxed{\int \frac{1}{[1+t^2]^3}}\,dt$$

Agora usariamos a fórmula de recorrência!

Nota: No fim, fazia-se substituição inversa.
Onde estivesse t colocava-se $\frac{x+2}{\sqrt{5}}$

Primitivas potências: $\sin x$, $\cos x$, $\sec x$

- Usamos primitivação por partes e fórmulas trigonométricas:

$\to \sin^2 x + \cos^2 x + 1$ $\quad\to 1 + tg^2 x = \sec x$

$\to \cos(2x) = \cos^2 x - \sin^2 x$

$\to \sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x)$ $\quad\to \cos^2 x = \frac{1}{2} + \frac{1}{2}\cos(2x)$

Exemplo:

$$\int \cos^5 x\,dx = \int \underbrace{\cos x}_{u'}\ \underbrace{\cos^4 x}_{v}\,dx = \sin x \times \cos^4 x + \int \sin^2 x \times 4\cos^3 x\,dx$$

$u = \sin x$ $\quad v = \cos^4 x$

$u' = \cos x$ $\quad v' = -4\cos^3 x \times \sin x$

$$= \sin x \times \cos^4 x + 4\int (1-\cos^2 x)\cos^3 x\,dx$$

$$= \sin x\cos^4 x + 4\int \underbrace{\cos^3 x}_{(1-\sin^2 x)\cdot\cos x}\,dx - 4\int \cos^5 x\,dx$$

loop de cálculo
voltávamos ao inicial e acabávamos

Faltava acabar

$$\int \cos^3 x\,dx = \int (1-\sin^2 x)\cos x\,dx = \int \cos x\,dx - \int \sin^2 x \cos x\,dx$$

$$= \operatorname{sen} x - \frac{\sin^3 x}{3} + C,\ C \in \mathbb{R} \text{ em intervalos}$$

— // —

está no formulário

$$\int \cos^6 x\,dx = \int (\cos^2 x)^3\,dx = \int \left[\frac{1}{2} + \frac{1}{2}\cos(2x)\right]^3 dx = \frac{1}{8}\int [1+\cos(2x)]^3 dx$$

$$= \frac{1}{8} \times \int 1 + 3\cos(2x) + 3\cos^2(2x) + \cos^3(2x)\,dx$$

:: falta muito, mas não é nada de diferente

Potências ímpares de $\sin x$ ⟶ idêntico às pot. ímpares de $\cos x$

Potências pares de $\sin x$ → Usar $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x)$

é idêntica às pot. pares de $\cos x$

Primitivar $\boxed{\sin^m x \ \cos^n x}$ (um dos expoentes é um nº ímpar ≥ 3)

$$\int \sin^5 x \cos^2 x\,dx = \int \sin^4 x \cos^2 x \sin x\,dx$$

$$= \int (1-\cos^2 x)^2 \cos^2 x \sin x\,dx$$

$$= \int (1 - 2\cos^2 x + \cos^4 x)\cos^2 x \sin x$$

:: continua...

$$\int \sin^4 x \cos^6 x\,dx$$

$$= \int (\sin^2 x)^2 \cos^6 x\,dx = \int (1-\cos^2 x)^2 \cos^6 x\,dx$$

$$= \int \cos^6 x - 2\cos^8 x + \cos^{10} x\,dx$$

:: continua