Aula 16

Integrais impróprios

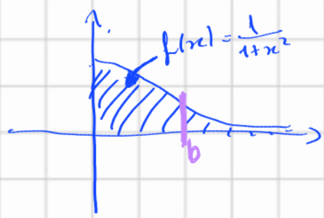
1.ª Espécie

$$\int_a^{+\infty} f(x)\,dx := \lim_{b\to+\infty} \int_a^b f(x)\,dx\,,\ f \text{ é integrável em } [a,b],\ \forall b\in\mathbb{R}^+$$

$\boxed{\text{Exemplo A}}$

$$\int_0^{+\infty} \frac{1}{1+x^2}\,dx = \lim_{b\to+\infty} \int_0^b \frac{1}{1+x^2}\,dx$$

$$= \lim_{b\to+\infty} \left[\operatorname{arctg} x\right]_0^b = \lim_{b\to+\infty} \left(\operatorname{arctg} b - \cancel{\operatorname{arctg} 0}^{\,0}\right)$$



$$= \frac{\pi}{2}$$

↳ converge

$\boxed{\text{Exemplo (B)}}$

a) $$\int_\pi^{+\infty} \cos x\,dx = \lim_{b\to+\infty} \int_\pi^b \cos x\,dx = \lim_{b\to+\infty} \left(\sin b - \cancel{\sin \pi}^{\,0}\right)$$

$$= \lim_{b\to+\infty} \sin b$$

↳ Diverge ⟹ Não existe!

b) $$\int_2^{+\infty} \frac{1}{(x+2)^2}\,dx = \lim_{b\to+\infty} \left[\frac{(x+2)^{-1}}{-1}\right]_2^b = \lim_{b\to+\infty} \left(\cancel{\frac{-1}{b+2}}^{\,0} + \frac{1}{4}\right)$$

$$= \frac{1}{4}$$

↳ Converge ⟹ Existe!

c) $$\int_1^{+\infty} \frac{(\ln x)^3}{x}\,dx = \lim_{b\to+\infty} \left[\frac{(\ln x)^4}{4}\right]_1^b = \lim_{b\to+\infty} \left(\frac{(\ln b)^4}{4} - \frac{(\ln 1)^4}{4}\right)$$

$$= \frac{(\ln +\infty)^4}{4} = +\infty$$

→ Diverge!

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$$\int_1^{+\infty} \frac{1}{x^\alpha}\,dx = \begin{cases} \frac{1}{\alpha-1}, & \alpha > 1 \to \text{Converge} \\ +\infty, & x \leq 1 \to \text{Diverge} \end{cases}$$

- $\boxed{\text{Se } \alpha = 1}$

$$\int_1^{+\infty} x\,dx = \lim_{b\to+\infty} [\ln x]_1^b = \lim_{b\to+\infty} (\ln b - \cancel{\ln 1}) = +\infty \to \text{Diverge}$$

- $\boxed{\text{Se } x \neq 1}$ ← Para ser mais fácil tínhamos feito → Se $x > 1$ e $x < 1$

$$\int_1^{+\infty} x^{-\alpha}\,dx = \lim_{b\to+\infty} \left[\frac{x^{-\alpha+1}}{-\alpha+1}\right]_1^b = \lim_{b\to+\infty} \left(\frac{b^{1-\alpha}}{1-\alpha} - \frac{1}{1-\alpha}\right) = \begin{cases} \frac{1}{\alpha-1}, & \text{se } x > 1 \\ +\infty & \text{se } x < 1 \end{cases}$$

$b^{1-\alpha}$: $\to +\infty$, se $\alpha < 1$; $\to 0$, se $x > 1$

Integrais de Dirichlet

$$\int_1^{+\infty} \frac{1}{x^\alpha}\,dx = \begin{cases} \frac{1}{\alpha-1}, & x > 1 \\ +\infty, & \text{se } x \leq 1 \end{cases} \qquad \int_0^{+\infty} e^{\beta x}\,dx = \begin{cases} -\frac{1}{\beta}, & \beta < 0 \\ +\infty, & \beta \geq 0 \end{cases}$$

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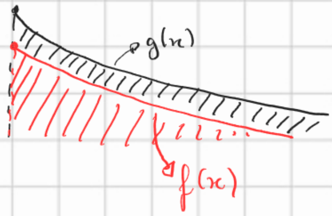
$$\int_0^{+\infty} e^{\beta x}\,dx = \begin{cases} -\frac{1}{\beta} & \text{se } \beta < 0 \\ +\infty & \text{se } \beta \geq 0 \end{cases}$$

$$\lim_{b\to+\infty} \int_0^b e^{\beta x}\,dx = \lim_{b\to+\infty} \left[\frac{e^{\beta x}}{\beta}\right]_0^b = \lim_{b\to+\infty} \left(\frac{e^{\beta b}}{\beta} - \frac{1}{\beta}\right)$$

$$\int_0^{+\infty} e^{\beta x}\,dx = \begin{cases} -\frac{1}{\beta}, & \beta < 0 \\ +\infty, & \beta \geq 0 \end{cases}$$

Critério de Comparação



$$0 \leq f(x) \leq g(x) \;, \; \forall x \in [a, +\infty[$$

$$\int_a^{+\infty} 0 \, dx \leq \int_a^{+\infty} f(x) \, dx \leq \int_a^{+\infty} g(x) \, dx$$

Finito $\Longleftarrow$ finito

$+\infty \Longrightarrow +\infty$