Aula 09

Exercício A

$$\int \frac{x^2}{(x-1)^3}\,dx$$

$$\frac{x^2}{(x-1)^3} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3}$$

$$\Leftrightarrow \frac{x^2}{\cancel{(x-1)^3}} = \frac{A(x-1)^2 + B(x-1) + C}{\cancel{(x-1)^3}}$$

$$\Leftrightarrow x^2 = A(x^2-2x+1) + B(x-1) + C$$

$$\Leftrightarrow x^2 = Ax^2 + (-2A+B)x + A - B + C$$

$$\begin{cases} A = 1 \\ -2A + B = 0 \\ A - B + C = 0 \end{cases} \quad \begin{cases} A = 1 \\ B = 2 \\ C = 1 \end{cases}$$

$$\frac{x^2}{(x-1)^3} = \frac{1}{(x-1)} + \frac{2}{(x-1)^2} + \frac{1}{(x-1)^3}$$

$$\rightarrow \int 1 \times (x-1)^{-2}\,dx = \frac{(x-1)^{-2+1}}{-2+1} = -\frac{1}{(x-1)}$$

$$\int \frac{x^2}{(x-1)^3}\,dx = \int \frac{1}{x-1}\,dx + 2\int \frac{1}{(x-1)^2}\,dx + \int \frac{1}{(x-1)^3}\,dx$$

$$= \ln|x-1| + 2 \times \left(-\frac{1}{x-1}\right) + \left(-\frac{1}{2(x-1)^2}\right) + C$$

$$= \ln|x-1| - \frac{2}{x-1} - \frac{1}{2 \times (x-1)^2} + C,\ C \in \mathbb{R} \text{ em intervalos}$$

Exercício B

$$\int \frac{x+1}{x^2+4x+5}\,dx$$

$x^2+4x+5 = (x+p)^2+q$
$= x^2+2px+(p^2+q)$

$x^2+4x+5=0$
$\Leftrightarrow x = \frac{-4\pm\sqrt{16-4\times1\times5}}{2}$

Não tem zeros reais

$\begin{cases} 1=1 \\ 2p=4 \\ p^2+q=5 \end{cases}$ $\begin{cases} 1=1 \\ p=2 \\ q=1 \end{cases}$

$$\int \frac{x+1}{x^2+4x+5}\,dx = \int \frac{x}{x^2+4x+5}\,dx + \int \frac{1}{(x+2)^2+1}\,dx = \frac{1}{2}\int \frac{2x+4-4}{x^2+4x+5} + \operatorname{arctg}(x+2)+c$$

$$= \frac{1}{2}\times\int\frac{2x+4}{x^2+4x+5}\,dx + \frac{1}{2}\times\int\frac{-4}{x^2+4x+5}\,dx + \operatorname{arctg}(x+2)+c$$

$$= \frac{1}{2}\times\ln(x^2+4x+5) + \frac{1}{2}\times(-4)\times\int\frac{1}{(x+2)^2+1}\,dx + \operatorname{arctg}(x+2)+c$$

$$= \frac{1}{2}\times\ln(x^2+4x+5) - 2\operatorname{arctg}(x+2) + \operatorname{arctg}(x+2)+c$$

$$= \frac{1}{2}\times\ln(x^2+4x+5) - \operatorname{arctg}(x+2)+c$$

Exercício C

$$\int \frac{1}{(1+x^2)^2}\,dx$$

Fração própria. Tem um par de raízes complexas com multiplicidade 2

$$\frac{1}{(1+x^2)^2} = \frac{A^{=0}}{1+x^2} + \frac{B^{=1}}{(1+x^2)^2},$$ resta calcular a primitiva de $\frac{1}{(1+x^2)^2}$

Sabendo que: $\int \frac{1}{(1+x^2)} \, dx = \operatorname{arctg}(x) + c, \; c \in \mathbb{R}$ em intervalos

Por outro lado,

$$\int \underbrace{1}_{u'} \times \underbrace{\frac{1}{1+x^2}}_{v} \, dx = x \times \frac{1}{(1+x^2)} - \int \frac{x \times -2x}{(1+x^2)^2} \, dx$$

$$\boxed{\begin{array}{ll} u = x & v = \frac{1}{(1+x^2)} \\ u' = 1 & \\ & v' = \frac{-2x}{(1+x^2)^2} \end{array}}$$

$$= \frac{x}{1+x^2} + 2 \int \frac{x^2+1-1}{(1+x^2)^2} \, dx$$

$$= \frac{x}{1+x^2} + 2 \int \frac{\cancel{1+x^2}}{(1+x^2)^{\cancel{2}}} \, dx - 2 \int \frac{1}{(1+x^2)^2} \, dx$$

$$= \frac{x}{1+x^2} + 2 \times \operatorname{arctg} x - 2 \boxed{\int \frac{1}{(1+x^2)^2} \, dx}$$

É o que nós queremos

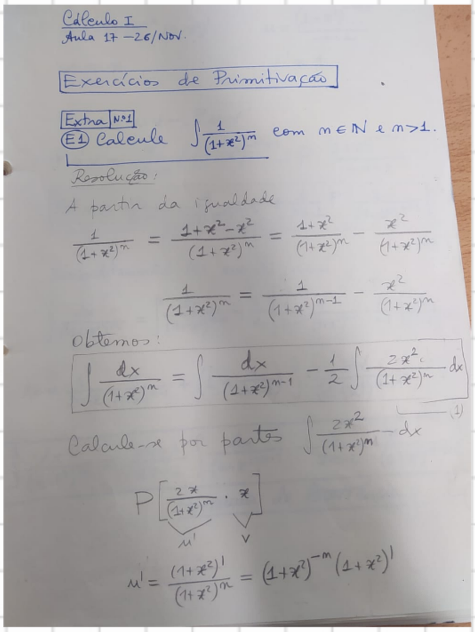
$$\int \frac{1}{(1+x^2)} \, dx = \frac{x}{1+x^2} + 2 \operatorname{arctg} x - 2 \int \frac{1}{(1+x^2)^2} \, dx$$

$= \operatorname{arctg} x$

$$(\Leftrightarrow) \int \frac{1}{(1+x^2)^2} \, dx = \frac{x}{2+2x^2} + \frac{2 \operatorname{arctg} x}{2} - \frac{\operatorname{arctg} x}{2} + c$$

$$(\Leftrightarrow) \int \frac{1}{(1+x^2)^2} \, dx = \frac{1}{2} \times \frac{x}{1+x^2} + \frac{1}{2} \times \operatorname{arctg} x + c, \; c \in \mathbb{R} \text{ em intervalos}$$

Exercício $\rightarrow$ $\int \frac{1}{(1+x^2)^m} \, dx$ $\longrightarrow$ Não sai no teste
"O professor contra da esta matéria!"


Cálculo I
Aula 17 – 26/Nov.

Exercícios de Primitivação

Extra | N.º1
(E1) Calcule $\int \frac{1}{(1+x^2)^m}$ com $m \in \mathbb{N}$ e $m > 1$.

Resolução:

A partir da igualdade

$$\frac{1}{(1+x^2)^m} = \frac{1+x^2-x^2}{(1+x^2)^m} = \frac{1+x^2}{(1+x^2)^m} - \frac{x^2}{(1+x^2)^m}$$

$$\frac{1}{(1+x^2)^m} = \frac{1}{(1+x^2)^{m-1}} - \frac{x^2}{(1+x^2)^m}$$

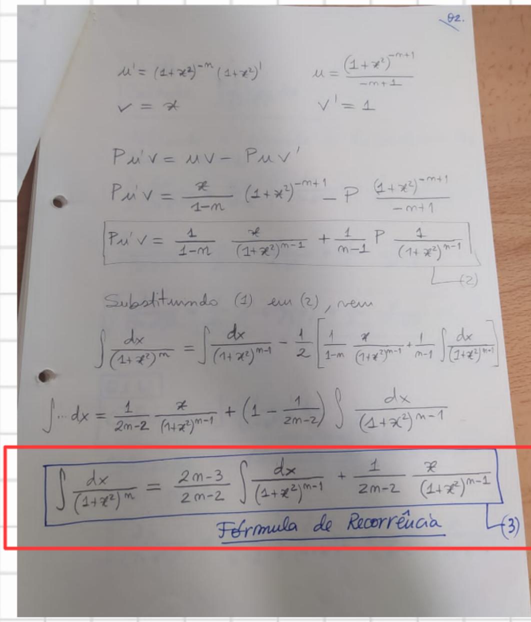
Obtemos:

$$\boxed{\int \frac{dx}{(1+x^2)^m} = \int \frac{dx}{(1+x^2)^{m-1}} - \frac{1}{2} \int \frac{2x^2}{(1+x^2)^m} \, dx} \quad (1)$$

Calcule-se por partes $\int \frac{2x^2}{(1+x^2)^m} \, dx$

$$P\Big[\underbrace{\frac{2x}{(1+x^2)^m}}_{u'} \cdot \underbrace{x}_{v}\Big]$$

$$u' = \frac{(1+x^2)'}{(1+x^2)^m} = (1+x^2)^{-m}(1+x^2)'$$


02.

$u' = (1+x^2)^{-m}(1+x^2)'$ $\qquad u = \frac{(1+x^2)^{-m+1}}{-m+1}$

$v = x$ $\qquad v' = 1$

$$Pu'v = uv - Puv'$$

$$Pu'v = \frac{x}{1-m}(1+x^2)^{-m+1} - P\frac{(1+x^2)^{-m+1}}{-m+1}$$

$$\boxed{Pu'v = \frac{1}{1-m} \frac{x}{(1+x^2)^{m-1}} + \frac{1}{m-1} P \frac{1}{(1+x^2)^{m-1}}} \quad (2)$$

Substituindo (1) em (2), vem

$$\int \frac{dx}{(1+x^2)^m} = \int \frac{dx}{(1+x^2)^{m-1}} - \frac{1}{2}\left[\frac{1}{1-m} \frac{x}{(1+x^2)^{m-1}} + \frac{1}{m-1} \int \frac{dx}{(1+x^2)^{m-1}}\right]$$

$$\int \dots dx = \frac{1}{2m-2} \frac{x}{(1+x^2)^{m-1}} + \left(1 - \frac{1}{2m-2}\right) \int \frac{dx}{(1+x^2)^{m-1}}$$

$$\boxed{\int \frac{dx}{(1+x^2)^m} = \frac{2m-3}{2m-2} \int \frac{dx}{(1+x^2)^{m-1}} + \frac{1}{2m-2} \frac{x}{(1+x^2)^{m-1}}} \quad (3)$$

Fórmula de Recorrência

Se for preciso dão a fórmula