Aula 07

Operações permitidos para obter representações na SP:

1.º) Multiplicar por constante ($\neq 0$)

2.º) Mudança de variável ($\bullet \leftarrow (\cdots)$)

3.º) Reunir numa só série duas outras séries (tal como fizemos com sinh)

4.º) Integrar termo a termo

5.º) Derivar termo a termo

Exercícios

$\boxed{1}$

Sabendo que $\frac{1}{1-x} = \sum_{n=0}^{+\infty} x^n$, $x \in]-1,1[$, obtenha representações em SP para:

a) $f(x) = \frac{1}{1+x^2}$

Como

$$\frac{1}{1-\bullet} = \sum_{n=0}^{+\infty} \bullet^n, \quad \bullet \in]-1,1[$$

, onde $\bullet$ é uma expressão aritmética na variável x

$$\bullet \leftarrow (-x^2)$$

$$\frac{1}{1-(-x^2)} = \sum_{n=0}^{+\infty} (-x^2)^n, \quad |-x^2| < 1 \Leftrightarrow |x| < 1$$

Logo, $\frac{1}{1+x^2} = \sum_{n=0}^{+\infty} (-1)^n x^{2n}, \quad |x| < 1$

b) $g(x) = \text{arctg}\, x$

Nota: Repare que $\frac{1}{1+x^2} = [\text{arctg}(x)]'$

Então:

$$\text{arctg}\, x = \int_0^x \frac{1}{1+t^2}\, dt = \int_0^x \left(\sum_{n=0}^{+\infty} (-1)^n t^{2n}\right) dt, \quad |x| < 1$$

centro da SP

Integrando termo a termo:

$$\text{arctg}\, x = \sum_{n=0}^{+\infty} \left(\int_0^x (-1)^n t^{2n}\, dt\right)$$

Primitivo e uso Regra de Barrow

$$= \sum_{n=0}^{+\infty} \left(\left[(-1)^n \frac{t^{2n+1}}{2n+1} \right]_0^x \right)$$

$$= \sum_{n=0}^{+\infty} \left(\left[(-1)^n \frac{x^{2n+1}}{2n+1} - \cancel{(-1)^n \frac{0^{2n+1}}{2n+1}} \right] \right)$$

$$= \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \quad , |x| < 1$$

c) $f(x) = \dfrac{1}{(1-x)^2}$ $\quad$ c.c. $\quad \left(\frac{1}{1-x}\right)' = \dfrac{1}{(1-x)^2}$

$$\frac{1}{(1-x)^2} = \left(\frac{1}{1-x}\right)' = \left[\sum_{n=0}^{+\infty} x^n\right]'$$

<u>Deriva termo a termo:</u>

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{+\infty} [x^n]'$$

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{+\infty} n\, x^{n-1} = \sum_{n=1}^{+\infty} n\, x^{n-1}$$

Cuidado que pode ingorar em escolha múltipla !

$k = n-1 \Leftrightarrow n = k+1$

Se $n = 1 \Rightarrow k = 0$

$$= \sum_{n=0}^{+\infty} (k+1)\, x^k = \sum_{n=0}^{+\infty} (n+1)\, x^n, \quad |x| < 1$$

d) $f(x) = \dfrac{1}{1+2x}$

$$\frac{1}{1-(-2x)} = \sum_{n=0}^{+\infty} (-2x)^n \quad , |-2x| < 1$$

$$= \sum_{n=0}^{+\infty} (-1)^n\, 2^n x^n, \quad |x| < \frac{1}{2}$$

e) $\ln(1+2x)$

Como $\ln(1+x) = \displaystyle\sum_{n=0}^{+\infty} (-1)^n \frac{x^{n+1}}{n+1}, \ |x| < 1$

$$\ln(1+2x) = \sum_{n=0}^{+\infty} (-1)^n \frac{(2x)^{n+1}}{n+1} \quad , |2x| < 1$$

$$= \sum_{n=0}^{+\infty} \frac{(-1)^n \, 2^{n+1}}{n+1} x^{n+1}, \quad |x| < \frac{1}{2}$$

e) $f(x) = \dfrac{2}{(1-x)^3}$ Como $\left[\dfrac{1}{(1-x)^2}\right]' = \dfrac{2(1-x)}{(1-x)^4} = \dfrac{2}{(1-x)^3}$

Do resultado da alínea c:

$$\frac{2}{(1-x)^3} = \left[\sum_{n=0}^{+\infty} (n+1)\, x^n\right]', \quad |x| < 1$$

Derivar termo a termo:

derivada em x!

$$\frac{2}{(1-x)^3} = \sum_{n=0}^{+\infty} \left((n+1)\, x^n\right)' = \sum_{n=0}^{+\infty} n(n+1)\, x^{n-1} = \sum_{n=1}^{+\infty} n(n+1)\, x^{n-1}, \quad |x| < 1$$

$\boxed{2}$ Obter $\sin\left(x - \frac{\pi}{4}\right)$ em SP e indicar I_c:

$$\sin x = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad x \in \mathbb{R}$$

$$x \leftarrow x - \frac{\pi}{4}$$

$$\sin\left(x - \frac{\pi}{4}\right) = \sum_{n=0}^{+\infty} (-1)^n \frac{\left(x - \frac{\pi}{4}\right)^{2n+1}}{(2n+1)!}, \quad \underbrace{\left|x - \tfrac{\pi}{4}\right| \in \mathbb{R}}_{x \in \mathbb{R}}$$

$\boxed{\text{Extra}}$

a) $f(x) = e^{-x^2}$

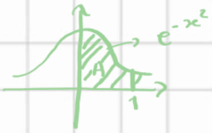
$$e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!}, \quad x \in \mathbb{R}$$

$$x \leftarrow (-x^2)$$

$$e^{-x^2} = \sum_{n=0}^{+\infty} \frac{(-x^2)^n}{n!}, \quad -x^2 \in \mathbb{R} \Leftrightarrow x \in \mathbb{R}$$

$$e^{-x^2} = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n}}{n!}, \quad x \in \mathbb{R}$$

b) $g(x) = \int e^{-x^2} dx$ (Não se faz com técnicas de Cálculo I)



$$g(x) = \int_0^x \left(\sum_{n=0}^{+\infty} (-1)^n \frac{t^{2n}}{n!}\right) dt$$

Primitivando termo a termo

$$= \sum_{m=0}^{+\infty} \left(\int_0^x (-1)^m \frac{t^{2m}}{m!} dt \right) = \sum_{m=0}^{+\infty} \left[(-1)^m \frac{t^{2m+1}}{m!(2m+1)} \right]_0^x$$

$$= \sum_{m=0}^{+\infty} (-1)^m \frac{x^{2m+1}}{m!(2m+1)}$$

$$\int e^{-x^2} dx = \sum_{m=0}^{+\infty} (-1)^m \frac{x^{2m+1}}{m!(2m+1)} + C \, , \, C \in \mathbb{R}$$

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$$A = \int_0^1 e^{-x^2} dx$$

$$= \sum_{m=0}^{+\infty} \left[(-1)^m \frac{x^{2m+1}}{m!(2m+1)} \right]_0^1 = \sum_{m=0}^{+\infty} \frac{(-1)^m}{m!(2m+1}$$

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Considere: $f : \mathbb{R} \longrightarrow \mathbb{R}$

$x \longmapsto f(x) = e^{x^2} + \cos(2x)$

a) Desenvolva a série de MacLaurin indicando para que valores o desenvolvimento é válido

MacLaurin para $e^x = \sum_{m=0}^{+\infty} \frac{x^m}{m!} \Longrightarrow e^{x^2} = \sum_{m=0}^{+\infty} \frac{x^{2m}}{m!} \, , \, x \in \mathbb{R}$

MacLaurin para $\cos x = \sum_{m=0}^{+\infty} (-1)^m \frac{x^{2m}}{(2m)!} \Longrightarrow \cos(2x) = \sum_{m=0}^{+\infty} (-1)^m \frac{2^{2m} x^{2m}}{(2m)!}$

$$f(x) = e^{x^2} + \cos(2x) = \sum_{m=0}^{+\infty} \frac{x^{2m}}{m!} + \sum_{m=0}^{+\infty} (-1)^m \frac{4^m x^{2m}}{(2m)!}$$

$$= \sum_{m=0}^{+\infty} \left(\left[\frac{1}{m!} + (-1)^m \frac{4^m}{(2m)!} \right] x^{2m} \right)$$

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