Aula 17

EDO's Homogéneos

São EDO's de 1.ª ordem que na forma normal se escrevem:

$$y' = f(x,y)$$

e, além disso:

$$f(\lambda x, \lambda y) = f(x,y), \quad \begin{array}{l} \forall \lambda \in \mathbb{R} \\ \forall (\lambda x, \lambda y) \in D_f \end{array}$$

Usando uma mudança de variável $\boxed{y \longrightarrow z}$

$$\boxed{y = z x \Longrightarrow z = \frac{y}{x}}$$

y é função de x
Logo, $z = \frac{y}{x}$ é função de x

$$y' = (z x)' = z' x + z \cancel{x'}^{\,1} = z' x + z \Longrightarrow \boxed{y' = z' x + z}$$

Obtemos uma EDO de variáveis separáveis nas variáveis x e z

ex:

(A) $x^2 + y^2 - 2x^2 y' = 0$

Forma normal: $y' = \underbrace{\frac{x^2 + y^2}{2x^2}}_{f(x,y)}, \quad x \neq 0$

Vamos ver se $f(x,y)$ tem a propriedade da homogeneidade:

$$f(\lambda x, \lambda y) = \frac{(\lambda x)^2 + (\lambda y)^2}{2(\lambda x)^2}$$

$$= \frac{\lambda^2 x^2 + \lambda^2 y^2}{2\lambda^2 x^2} = \frac{x^2 + y^2}{2x^2} = f(x,y), \quad \begin{array}{l} \forall x \in \mathbb{R} \\ \forall (\lambda x, \lambda y) \in D_f \end{array}$$

• Então a EDO é homogénea.

Usando a mudança de variável $y = zx$ temos $\boxed{y' = z + z'x}$

Obtemos:

EDO de variáveis separadas nas var. z e x

$$\underbrace{z + x z'}_{y'} = \frac{x^2 + (\overbrace{zx}^{y})^2}{2x^2} \iff x z' = \frac{\cancel{x^2} + z^2 \cancel{x^2} - 2z\cancel{x^2}}{2\cancel{x^2}} \iff z' = \frac{z^2 - 2z + 1}{2x}$$

$P(x)$, só depende de x

$$z' = \frac{\frac{1}{2x}}{\frac{1}{(z-1)^2}}$$

$Q(z)$ só depende de z

$$\frac{dz}{dx} = \frac{\frac{1}{2x}}{\frac{1}{(z-1)^2}} \quad , \quad z \neq 1$$

$$\Leftrightarrow \frac{1}{(z-1)^2}\, dz = \frac{1}{2x}\, dx$$

$$\Leftrightarrow \int \frac{1}{(z-1)^2}\, dx = \int \frac{1}{2x}\, dx$$

$$\Leftrightarrow -(z-1)^{-1} + c_1 = \frac{1}{2} \ln|x| + c_2$$

$$\Leftrightarrow -\frac{1}{z-1} = \frac{1}{2} \ln|x| + c_1 - c_2$$

" $c \in \mathbb{R}$

$$\Leftrightarrow \frac{-2}{z-1} = \ln|x| + c$$

$$\Leftrightarrow \boxed{z = \frac{-2}{\ln|x| + c} + 1}$$

Regresso à variável y, $y = zx \Leftrightarrow z = \frac{y}{x}$

$$\frac{y}{x} = \frac{-2}{\ln|x| + c} + 1$$

$$\boxed{y = \frac{-2x}{\ln|x| + c} + x}$$

Exercício de teste

ex:

(B) $x^2 e^{\frac{x}{y}} y' = y^2 + xy\, e^{\frac{x}{y}}$

Sugestão: use a mudança de var. $y = zx$

$$z = \frac{y}{x} \Rightarrow y = xz \Rightarrow y' = xz' + z$$

$x \neq 0$

$$x^2 e^{\frac{1}{z}} (z + xz') = (xz)^2 + x(xy)\, e^{\frac{1}{z}}$$

$$x^2 \cancel{z e^{\frac{1}{z}}} + x^3 z' e^{\frac{1}{z}} = x^2 z^2 + x^2 \cancel{y e^{\frac{1}{z}}}$$

$$x^{\cancel{3}} z' e^{\frac{1}{z}} = \cancel{x^2} z^2$$

$$z' = \frac{z^2}{x e^{\frac{1}{z}}} = \frac{\frac{1}{x}}{\frac{e^{\frac{1}{z}}}{z^2}}$$

$$\frac{dz}{dx} = \frac{\frac{1}{x}}{\frac{e^{1/z}}{z^2}}$$

$$\Leftrightarrow \int \frac{1}{x}\,dx = \int z^{-2} e^{1/z}\,dz$$

$$\Leftrightarrow \ln|x| + c_2 = -e^{1/z} + c_1 \ , \ c_1, c_2 \in \mathbb{R}$$

$$\Leftrightarrow e^{1/z} = -\ln|x| + c \ , \ c \in \mathbb{R}$$

Regresso à var. inicial: $\boxed{e^{\frac{x}{y}} = -\ln|x| + c}$, $c \in \mathbb{R}$

vex:

Ⓒ $(x^3 + y^3)\,dx = 3y^2x\,dy$

a)

$$x^3 + y^3 = 3y^2x\,\frac{dy}{dx}$$

$$\Leftrightarrow x^3 + y^3 = 3y^2x\,y'$$

$$\Leftrightarrow \boxed{y' = \frac{x^3 + y^3}{3y^2x}} \text{ — } ⊛$$

↳ $f(\lambda x, \lambda y)$

b)

$$f(\lambda x, \lambda y) = \frac{\cancel{\lambda^3}x^3 + \cancel{\lambda^3}y^3}{3\cancel{\lambda^2}y^2\cancel{\lambda}x} = \frac{x^3 + y^2}{3y^2x} = f(x,y) \ , \ \forall x \in \mathbb{R}, \ \forall (\lambda x, \lambda y) \in D_f$$

↳ Logo, a EDO é homogénea.

c)

Substituição: $y = zx$

$\Leftrightarrow z = \frac{y}{x}$, $x \neq 0$

Substituindo em $y' = xz' + z$

$$⊛ = z + z'x = \frac{\cancel{x^3} + z^3\cancel{x^3}}{3z^2\cancel{x^2}\cancel{x}}$$

$$\Leftrightarrow xz' = \frac{1 + z^3}{3z^2} - z$$

$$\Leftrightarrow z' = \frac{1 + z^3 - 3z^3}{3z^2x}$$

$$\Leftrightarrow z' = \frac{1 - 2z^3}{3z^2x}$$

$$\Leftrightarrow z' = \frac{\frac{1}{x}}{\frac{3z^2}{1-2z^3}}$$ → EDO de var. sep.

$$\Leftrightarrow \int \frac{3z^2}{1-2z^3}\,dz = \int \frac{1}{x}\,dx$$

$$\Leftrightarrow -\frac{1}{2}\int \frac{-6z^2}{1-2z^3}\,dz = \ln|x| + c_1$$

$$\Leftrightarrow -\frac{1}{2}\ln|1-2z^3| + c_2 = \ln|x| + c_1 \quad , c_1, c_2 \in \mathbb{R}$$

$$\Leftrightarrow \ln|x| = -\frac{1}{2}\ln|2z^3-1| + c_3, \quad c_3 \in \mathbb{R}$$

$$\Leftrightarrow \ln|2z^3-1| = -2\ln|x| - 2c_3$$

$$\Leftrightarrow \ln|2z^3-1| = \ln(x^{-2}) + c, \quad c \in \mathbb{R}$$

$$\Leftrightarrow \cancel{e}^{\ln|2z^3-1|} = e^{\ln(1/x^2)+c}$$

$$\Leftrightarrow |2z^3-1| = \cancel{e}^{\ln(1/x^2)} \times \underbrace{e^c}_{=K \in \mathbb{R}^+}$$

$$\Leftrightarrow |2z^3-1| = \frac{K}{x^2}, \quad K \in \mathbb{R}^+$$

É equivalente a:

$$2z^3-1 = \frac{K}{x^2} \quad \vee \quad 2z^3-1 = -\frac{K}{x^2}, \quad K \in \mathbb{R}^+$$

ou ainda:

$$2z^3-1 = \frac{K}{x^2}, \quad K \in \mathbb{R}\setminus\{0\}$$

se $K=0$ dá alguma coisa que se aproveite:

$$2z^3-1=0$$

$$\Leftrightarrow z = \sqrt[3]{\tfrac{1}{2}} \Rightarrow z' = 0$$

$$\Leftrightarrow z = \sqrt[3]{\frac{K}{2x^2} + \frac{1}{2}}, \quad K \in \mathbb{R}$$

Substituição inversa: $z = \frac{y}{x}$

$$\frac{y}{x} = \sqrt[3]{\frac{K}{2x^2} + \frac{1}{2}} \Rightarrow y = x\sqrt[3]{\frac{K}{2x^2} + \frac{1}{2}}, \quad K \in \mathbb{R}$$

EDO's Lineares de 1.ª ordem (Técnica do Fator Integrante)

$\boxed{1}$ Escreve-se a EDO na forma

$$y' + p(x)y = q(x)$$

$\boxed{2}$ Determina-se o fator integrante

$$\mu(x) = \exp\left(\int p(x)\,dx\right)$$

$$\mu(x) = e^{\int p(x)\,dx}$$

$\boxed{3}$ Multiplica-se a EDO por $\mu(x)$:

$$\underbrace{\mu(x)\left[y' + p(x)\,y\right]}_{\left[\mu(x)\cdot y\right]'} = \mu(x)\cdot q(x)$$

$$\left[\mu(x)\cdot y\right]' = \mu(x)\cdot y' + \mu(x)'\cdot y$$
$$= \mu(x)\cdot y' + \left[e^{\int p(x)dx}\right]'\cdot y$$
$$= \mu(x)\cdot y' + \underbrace{e^{\int p(x)dx}}_{=\mu(x)}\cdot p(x)\cdot y$$
$$= \mu(x)\cdot \left[\underbrace{y' + p(x)y}_{q(x)}\right]$$
$$= \mu(x)\cdot q(x)$$

c.q.m.

$\boxed{4}$ $\left[\mu(x)\cdot y\right]' = \mu(x)\cdot q(x)$

$$\mu(x)\cdot y = \int \mu(x)\cdot q(x)\,dx$$

$$y = \frac{1}{\mu(x)}\int \mu(x)\cdot q(x)\,dx$$

ex:

ⓒ $(1+x^2)\,y' + xy + x = 0$

$\boxed{1}$ $y' + \dfrac{xy}{(1+x^2)} + \dfrac{x}{(1+x^2)} = 0 \iff y' + \underbrace{\dfrac{x}{(1+x^2)}}_{=p(x)}\,y = \underbrace{-\dfrac{x}{(1+x^2)}}_{=q(x)}$

Trata-se de uma EDO linear de 1ª ordem

$\boxed{2}$ Determinar o fator integrante

$$\mu(x) = e^{\int \frac{x}{1+x^2}dx} = e^{\frac{1}{2}\int\frac{2x}{1+x^2}dx}$$

$$\mu(x) = e^{\frac{1}{2}\ln(1+x^2)} = \cancel{e}^{\cancel{\ln}(\sqrt{1+x^2})} = \sqrt{1+x^2}$$

$\boxed{3}$ Multiplico a EDO por $\mu(x)$:

$$\underbrace{\mu(x)\left[y' + \frac{x}{1+x^2}\,y\right]}_{\parallel} = \mu(x)\cdot\left[-\frac{x}{1+x^2}\right]$$

$$\iff \left[\mu(x)\cdot y\right]' = \sqrt{1+x^2}\left[-\frac{x}{1+x^2}\right]$$

$$\iff \left[\mu(x)\cdot y\right]' = -\frac{x\sqrt{1+x^2}}{1+x^2}$$

$$\frac{\sqrt{1+x^2}}{1+x^2} = \frac{\cancel{\sqrt{1+x^2}}}{\cancel{\sqrt{1+x^2}}\sqrt{1+x^2}} = \frac{1}{\sqrt{1+x^2}}$$

$$\iff \left[\mu(x)\cdot y\right]' = -\frac{x}{\sqrt{1+x^2}}$$

$$\iff \left[\sqrt{1+x^2}\cdot y\right] = -\int\frac{x}{\sqrt{1+x^2}}\,dx$$

(...) Fazer $\int \frac{x}{\sqrt{1+x^2}}dx$ e confirmar...

$$\iff \sqrt{1+x^2}\cdot y = -\underline{\sqrt{1+x^2}} + C$$

$$\iff y = \frac{C}{\sqrt{1+x^2}} - 1\,,\; C\in\mathbb{R}$$

1ex:

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Tenho foto : 2/5