Nota: Referências a (T'_s) são da folha (Transformadas de Laplace)

Aula 23

Transformada de Laplace

$$f : \mathbb{R}_0^+ \longrightarrow \mathbb{R}$$
$$t \longmapsto f(t)$$

[Domínio do tempo] | [Domínio da Frequência]

relação de 1 para 1

$$f(t) \xrightarrow{\mathcal{L}} F(s) = \boxed{\int_0^{+\infty} e^{-st} f(t)\, dt} = \mathcal{L}\{f(t)\}$$

Transformada de Laplace de $f(t)$

$$\mathcal{L}^{-1}\{f(t)\} = f(t) \xleftarrow{\mathcal{L}^{-1}} F(s)$$

Operadores Funcionais:

$$\frac{d}{dt} f(t) = f'(t)$$

$$\int f(t)\, dt = F(t) + c_1, \ c \in \mathbb{R} \text{ tal que } f(0) = 0$$

$$2 \times f(t) = g(t)$$

ex: (Slide #4)

① Calcule a T.L. de:

a) $f(t) = \underbrace{1}_{= t^0}$

$$\mathcal{L}\{f(t)\} = \int_0^{+\infty} e^{-st}(1)\, dt = \lim_{b \to +\infty} \int_0^b e^{-st}(1)\, dt = \lim_{b \to +\infty} \left[\frac{e^{-st}}{-s}\right]_0^b$$

$$= \lim_{b \to +\infty} \left[\frac{e^{-sb}}{-s} - \frac{1}{-s}\right] = \frac{1}{s}, \ s > 0 \longrightarrow s \in \mathbb{R}^+$$

$e^{-sb} \to 0$ se $s > 0$ ou $+\infty$ se $s < 0$

Provei (T1) para $n = 0$. $\mathcal{L}\{\underbrace{t^0}_{t^0 = 1}\} = \frac{1}{s}, \ s > 0$

b) $f(t) = \underbrace{t}_{= t^1}$

- Para $s = 0$: $F(0) = \int_0^{+\infty} t\, dt = \lim_{b \to +\infty} \left[\frac{t^2}{2}\right]_0^b = +\infty$

- Para $s \neq 0$:

$$F(s) = \int_0^{+\infty} e^{-st}\, t\, dt = \lim_{b \to +\infty} \left[\int_0^b \underbrace{e^{-st}}_{v'} \underbrace{t}_{u}\, dt\right] = \lim_{b \to +\infty} \left[\left[\frac{e^{-st}}{-s}\, t\right]_0^b - \int_0^b \frac{e^{-st}}{-s}\, dt\right]$$

$v = \frac{e^{-st}}{-s}$ $\quad u' = 1$ $\quad$ Primitiva por partes

$$= \lim_{b \to +\infty} \left[-\frac{e^{-sb}\, b}{s} - \frac{1}{s^2}\left[e^{-st}\right]_0^b\right] = \lim_{b \to +\infty} \left[-\frac{e^{-sb}\, b}{s} - \frac{e^{-sb}}{s^2} + \frac{1}{s^2}\right]$$

$$= \lim_{b \to +\infty} \left[-\left(\frac{b}{s} + \frac{1}{s^2} \right) e^{-sb} + \frac{1}{s^2} \right]$$

0 se $s>0$ ou $+\infty$ se $s \leq 0$

$$= \begin{cases} \frac{1}{s^2}, & \text{se } s > 0 \\ -\infty, & \text{se } s < 0 \end{cases}$$

Logo: $\mathcal{L}\{t\} = \frac{1}{s^2}, \; s > 0$. T1 com $n = 1$.

Nota: com $n \in \mathbb{N}$, usa-se indução matemática (ver pags. 12-13, Caderno A)

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \; s > 0$$

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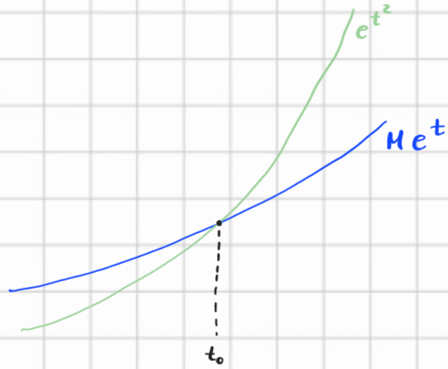
Curiosidade:

- Se $M \in \mathbb{R}^+$ e $a \in \mathbb{R}^+$:

$$e^{t^2} > M e^{at}, \text{ se } t \geq t_0, \text{ para algum } t_0 \in \mathbb{R}^+$$



$\Longrightarrow$ Portanto, $\mathcal{L}\{e^{t^2}\}$ não existe para nenhum $s \in \mathbb{R}$

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ex: (Slide #7)

(2)

a) $\mathcal{L}\{t+1\} = \mathcal{L}\{t^0\} + \mathcal{L}\{t\}$ (P1 Linearidade; T1 $n=0$; T2 $n=1$)

$$= \underset{(s>0)}{\frac{1}{s}} + \underset{(s>0)}{\frac{1}{s^2}}$$

$$= \frac{s+1}{s^2}, \; s > 0$$

b) $\mathcal{L}\{5+3t\} = 5\mathcal{L}\{t^0\} + 3\mathcal{L}\{t\}$ (P2)

$$= \underset{(s>0)}{\frac{5}{s}} + \underset{(s>0)}{\frac{3}{s^2}}$$

$$= \frac{5s+3}{s^2}, \; s > 0$$

ex: (Slide #8)

(3) T2

a) $\mathcal{L}\{e^{at}\}$, $a \in \mathbb{R}$

- Estude-se em função de $s \in \mathbb{R}$ o integral impróprio:

$$\int_0^{+\infty} e^{at} e^{-st}\, dt = \int_0^{+\infty} e^{(a-s)t}\, dt$$

- Para $s = a$, o integral diverge

- Para $s \neq a$, temos para todo o $b>0$:

0 se $s>a$ ou $+\infty$ se $s<a$

$$\int_0^b e^{(a-s)} dt = \left[\frac{1}{a-s} e^{(a-s)t} \right]_0^b = \frac{1}{a-s} e^{(a-s)b} - \frac{1}{a-s}$$

Logo:

$$\int_0^{+\infty} e^{(a-s)t} dt = \lim_{b \to +\infty} \left[\frac{1}{a-s} e^{(a-s)b} + \frac{1}{s-a} \right]$$

$$= \begin{cases} \frac{1}{s-a}, & \text{se } s>a \\ +\infty, & \text{se } s<a \end{cases}$$

$$\boxed{\mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \quad s>a}$$ — T2

b) $\mathcal{L}\{\cos(at)\}$ T.P.C. → Faz-se de modo parecido à alínea seguinte

c) $\mathcal{L}\{\operatorname{sen}(at)\}$

- Estuda-se em função de $s \in \mathbb{R}$, o integral:

$$\int_0^{+\infty} e^{-st} \sin(at)\, dt = \lim_{b \to +\infty} \int_0^b e^{-st} \sin(at)\, dt$$

$$= \underset{\text{Primitiva por partes}}{(\cdots)} = \frac{a}{s^2+a^2}, \quad s>0$$

d) $$\mathcal{L}\{\cosh(at)\} = \mathcal{L}\left\{\frac{e^{at}+e^{-at}}{2}\right\}$$

$$= \frac{1}{2}\mathcal{L}\{e^{at}\} + \frac{1}{2}\mathcal{L}\{e^{-at}\}$$

$$= \frac{1}{2} \underset{(s>a)}{\frac{1}{s-a}} + \frac{1}{2} \underset{(s>-a)}{\frac{1}{s+a}}$$

$$= \frac{1}{2}\frac{1}{s-a} + \frac{1}{2}\frac{1}{s+a}, \quad s > |-a|$$

$$\cosh(at) = \frac{e^{at}+e^{-at}}{2}$$

$$\sinh(at) = \frac{e^{at}-e^{-at}}{2}$$

T2 → $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \; s>a$

$\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}, \; s>-a$

$$= \frac{(s+\cancel{a}) + (s-\cancel{a})}{2(s+a)(s-a)} = \frac{\cancel{2}s}{\cancel{2}(s+a)(s-a)}$$

T6

$$\hookrightarrow = \frac{s}{(s+a)(s-a)}, \quad s > |a|$$

ex: (Slide #9)

4

a) $f(t) = t^2 + \cos(3t) + \pi$

Linearidade:

$$\mathcal{L}\{f(t)\} = \underbrace{\mathcal{L}\{t^2\}}_{T1,\ n=2} + \underbrace{\mathcal{L}\{\cos(3t)\}}_{T2,\ n=3} + \pi\underbrace{\mathcal{L}\{t^0\}}_{T1,\ n=0}$$

$$= \underset{(s>0)}{\frac{2}{s^3}} + \underset{(s>0)}{\frac{s}{s^2+3^2}} + \underset{(s>0)}{\frac{\pi}{s}}, \quad s>0$$

$= (\dots) \rightarrow$ Simplificava-se ...

b) $g(t) = 3e^{-2t} + \sin\left(\frac{t}{6}\right) + \cosh(4t)$

T2, T3, T6

$$\mathcal{L}\{g(t)\} = \underset{(s>-2)}{\frac{3}{s-(-2)}} + \underset{(s>0)}{\frac{1/6}{s^2+1/6^2}} + \underset{(s>4)}{\frac{s}{s^2+4^2}}$$

$$= \frac{3}{s+2} + \frac{1}{6\left(s^2+\frac{1}{36}\right)} + \frac{s}{s^2+16}, \quad s>4$$

c) $h(t) = t^{10} + \frac{e^t}{3} + \boxed{\cos^2 t}$ (?)

$$\mathcal{L}\{h(t)\} = \mathcal{L}\{t^{10}\} + \frac{1}{3}\mathcal{L}\{e^t\} + \frac{1}{2}\mathcal{L}\{t^0\} + \frac{1}{2}\mathcal{L}\{\cos(2t)\}$$

$$= \underset{(s>0)}{\frac{10!}{s^{11}}} + \frac{1}{3}\underset{(s>1)}{\frac{1}{s-1}} + \underset{(s>0)}{\frac{1/2}{s}} + \underset{(s>0)}{\frac{s/2}{s^2+4}}, \quad \boxed{s>1}$$

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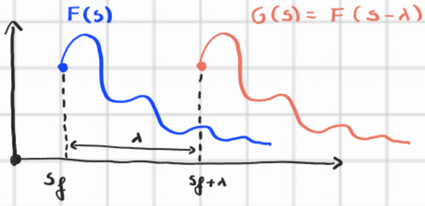
Deslocamento da Transformada



P3

$$\hookrightarrow \mathcal{L}\{e^{\lambda t} f(t)\} = F(s-\lambda), \quad s > s_f + \lambda$$

ex: 5

a) $\varphi(t) = \underset{\lambda=2}{e^{2t}} \cdot \overset{f(t)}{t^2}$

Defino $f(t) = t^2$, logo $\overbrace{\mathcal{L}\{f(t)\} = \frac{2}{s^3}}^{F(s)}, \quad s>0$

$$\mathcal{L}\{\varphi(t)\} = \mathcal{L}\{e^{-2t}f(t)\} = F(s-2) = \frac{2}{(s-2)^3}, \quad s > 0+2$$

b) $f(t) = e^{-3t}\sin(2t)$

$\lambda = -3$

$f(t) = \sin(2t)$

$F(s) = \dfrac{2}{s^2+4}, \quad s>0$

$$\mathcal{L}\{e^{-3t}\sin(2t)\} = F(s-(-3))$$
$$= F(s+3)$$
$$= \frac{2}{(s+3)^2+4}, \quad s > 0-3$$

Transformada do Deslocamento

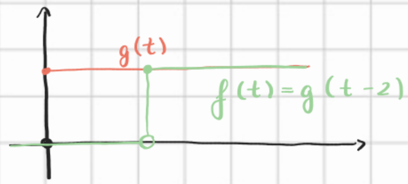
P5 $\longrightarrow$ $\mathcal{L}\{f(t-a)\} = e^{-at}F(s), \quad s > s_f$

$F(s) = \mathcal{L}\{f(t)\}$

↳ $g(t) = 0, \ \forall t \in \mathbb{R}^-$

ex: 6

a) $f(t) = \begin{cases} 0 & \text{se } t < 2 \\ 1 & \text{se } t \geq 2 \end{cases}$



• Defino a função auxiliar $\boxed{g(t) = \begin{cases} 0 & \text{se } t<0 \\ 1 & \text{se } t \geq 0 \end{cases}}$

Degrau de Heaviside ou step

Note: $f(t) = g(t-2)$

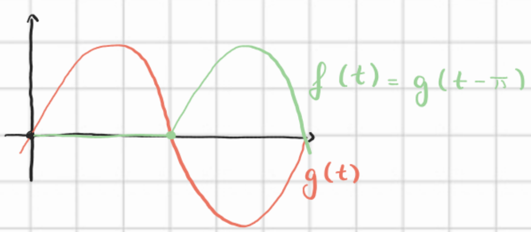
$$\mathcal{L}\{g(t)\} = G(s) = \frac{1}{s}, \quad s>0$$

então pela P5 : $\mathcal{L}\{f(t)\} = \mathcal{L}\{g(t-2)\} = e^{-2s}G(s) = \dfrac{e^{-2s}}{s}, \quad s>0$

b)

$f(t) = \begin{cases} 0 & , \text{ se } t < \pi \\ \text{sen}(t-\pi) & , \text{ se } t \geq \pi \end{cases}$

$g(t) = \begin{cases} 0 & , \text{ se } t<0 \\ \sin t & , \text{ se } t \geq 0 \end{cases}$



Note: $f(t) = g(t-\pi)$

Como: $\mathcal{L}\{g(t)\} = G(s) = \dfrac{1}{s^2+1}, \quad s>0$

P5

↳ $\mathcal{L}\{f(t)\} = \mathcal{L}\{g(t-\pi)\} = \dfrac{e^{-\pi s}}{s^2+1}, \quad s>0$

Mudança de escala (contração/expansão)

$$\mathcal{L}\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right), \quad s > a \cdot s_f,$$

onde $F(s) = \mathcal{L}\{f(t)\}$, $s > s_f$

ex: (slide #12)

a) $g(t) = \cos(4t)$

Defino $f(t) = \cos t$

$$F(t) = \frac{s}{s^2+1}, \quad s > 0 = s_f$$

Pela P6

$$\mathcal{L}\{g(t)\} = \mathcal{L}\{f(4t)\} = \frac{1}{4} F\left(\frac{s}{4}\right), \quad s > 4 \cdot (0) = 0$$

Universidade de Aveiro
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CÁLCULO II - Agrupamento 2 - 2017/18

Formulário (Transformada de Laplace)

#	função	transformada	OBS:
T1	$t^n \ (n \in \mathbb{N}_0)$	$\frac{n!}{s^{n+1}}, \ s > 0$	T.L. DA POTÊNCIA
T2	$e^{at} \ (a \in \mathbb{R})$	$\frac{1}{s-a}, \ s > a$	T.L. DA EXPONENCIAL
T3	$\text{sen}(at) \ (a \in \mathbb{R})$	$\frac{a}{s^2+a^2}, \ s > 0$	T.L. DO SENO
T4	$\cos(at) \ (a \in \mathbb{R})$	$\frac{s}{s^2+a^2}, \ s > 0$	T.L. DO COSENO
T5	$\text{senh}(at) \ (a \in \mathbb{R})$	$\frac{a}{s^2-a^2}, \ s > \lvert a \rvert$	T.L. DO SENO HIPERBÓLI
T6	$\cosh(at) \ (a \in \mathbb{R})$	$\frac{s}{s^2-a^2}, \ s > \lvert a \rvert$	T.L. DO COSENO HIPERBÓLIC
P1	$f(t) + g(t)$	$F(s) + G(s), \ s > \max\{s_f, s_g\}$	LINEARIDADE
P2	$\alpha f(t) \ (\alpha \in \mathbb{R})$	$\alpha F(s), \ s > s_f$	DA T.L.
P3	$e^{\lambda t} f(t) \ (\lambda \in \mathbb{R})$	$F(s-\lambda), \ s > s_f + \lambda$	DESLOCAMENTO DA T.L
P4	$t^n f(t) \ (n \in \mathbb{N})$	$(-1)^n F^{(n)}(s), \ s > s_f$	DERIVADA DA T.L.
P5	$f(t-a) \ (a > 0)$	$e^{-as} F(s), \ s > s_f$	T.L. DO DESLOCAMENTO
P6	$f(at) \ (a > 0)$	$\frac{1}{a} F\left(\frac{s}{a}\right), \ s > a\, s_f$	T.L. COM MUDANÇA DE ESCAL
P7	$f^{(n)}(t) \ (n \in \mathbb{N})$	$s^n F(s) - \sum_{k=1}^{n} s^{n-k} f^{(k-1)}(0)$, onde $f^{(0)} \equiv f$, $s > \max\{s_f, s_{f'}, s_{f''} \dots, s_{f^{(n-1)}}\}$	T.L. DA DERIVADA

Notas:

1. F denota a transformada de Laplace da função f, $F(s) = \mathcal{L}\{f(t)\}(s)$;
2. O facto de se indicarem restrições numa dada linha do quadro acima não significa que não haja restrições adicionais a considerar para que a fórmula indicada nessa linha seja válida.

DEFINIÇÃO DE T.L.

Seja $f: \mathbb{R}_0^+ \longrightarrow \mathbb{R}$, $t \mapsto f(t)$, integrável em $[0,b]$, $\forall b \in \mathbb{R}^+$.

$$\mathcal{L}\{f(t)\}(s) = \int_0^{+\infty} e^{-st} f(t)\, dt = F(s), \ s \in S \subset \mathbb{R}$$

$F(s)$ é a T.L. da função f. S é o conjunto de valores de s para os quais $F(s)$ existe.