Aula 24

Derivada da Transformada (Slide # 13)

$$\boxed{\mathcal{L}\{t^n f(t)\}(s) = (-1)^n F^{(n)}(s), \quad s > s_f} \text{ — P4}$$

a) $\varphi(t) = t\underbrace{e^{2t}}_{f(t)}$

Defino $f(t) = e^{2t}$, logo $F(s) = \mathcal{L}\{e^{2t}\} = \dfrac{1}{s-2}$, $s > 2$

Usando P4: $\mathcal{L}\{\varphi(t)\} = \mathcal{L}\{t^1 e^{2t}\} = (-1)^1 F'(s) = \dfrac{1}{(s-2)^2}$

b) $\varphi(t) = t^2 \cos t$

Defino $f(t) = \cos(t)$, logo $F(s) = \dfrac{s}{s^2+1}$, $s > 0$

Usando P4: $\mathcal{L}\{\varphi(t)\} = \mathcal{L}\{t^2 \cos(t)\} = (-1)^2 F''(s) = \left[\dfrac{s}{s^2+1}\right]'' = \left[\dfrac{1-s^2}{(s^2+1)^2}\right]'$

$$= \frac{2s^3 - 6s}{(s^2+1)^3}, \quad s > 0$$

c) $\varphi(t) = (t^2 - 3t + 2)\cdot \underbrace{\sin(3t)}_{f(t)}$

Defino $f(t) = \sin(3t)$, logo $F(s) = \dfrac{3}{s^2+9}$, $s > 0$

Usando a linearidade P1 e P2: $\mathcal{L}\{\varphi(t)\} = \underbrace{t^2 \cdot \sin(3t)}_{(-1)^2 F''(s)} - 3\underbrace{t \sin(3t)}_{(-1)^1 F'(s)} + 2\underbrace{\sin(3t)}_{(-1)^0 F(s)}$

$$\mathcal{L}\{\varphi(t)\} = (-1)^2 F''(t) + 3(-1) F'(s) + 2F(s)$$

$$= \frac{-4s}{(s^2+9)^3} - \frac{18s}{(s^2+9)^2} + \frac{6}{s^2+9}, \quad s > 0$$

$F'(s) = \dfrac{-6s}{(s^2+9)^2}$

$F''(s) = \dfrac{-4s}{(s^2+9)^3}$

Transformada da Derivada (Slide # 14)

P7:

$\boxed{n=1} \to \mathcal{L}\{f'(t)\}(s) = sF(s) - f(0), \quad s > s_f$

$\boxed{n=2} \to \mathcal{L}\{f''(t)\}(s) = s^2 F(s) - s f(0) - f'(0), \quad s > \max\{s_f, s_{f'}\}$

linea de existência de f — linea de existência de f'

$\boxed{n=3} \to \mathcal{L}\{f'''(t)\}(s) = s^3 F(s) - s^2 f(0) - s^1 f'(0) - f''(0), \quad s > \max\{s_f, s_{f'}, s_{f''}\}$

ex:

$\boxed{1}$

a) $\mathcal{L}\{f''(t)\}$ sabendo $\begin{cases} f(0) = 1 \\ f'(0) = 2 \end{cases}$

$$\mathcal{L}\{f''(t)\} = s^2 F(s) - s f(0) - f'(0)$$
$$= s^2 F(s) - s - 2$$

b) $\mathcal{L}\{f'''(t)\}$ sabendo $\begin{cases} f(0) = -2 \\ f'(0) = 0 \\ f''(0) = 1 \end{cases}$

$$\mathcal{L}\{f'''(t)\} = s^3 F(s) - s^2 f(0) - s f'(0) - f''(0)$$

$$= s^3 F(s) + 2s^2 - 1$$

c)

$\mathcal{L}\{y'' + 3y' - y\}$ sabendo $\begin{cases} y(0) = 3 \\ y'(0) = 0 \end{cases}$

Designo por $Y(s) = \mathcal{L}\{y(t)\}$

$$\mathcal{L}\{y'' + 3y' - y\} = \mathcal{L}\{y''\} + 3\mathcal{L}\{y'\} - \mathcal{L}\{y\}$$
$$= (\dots)$$

$\boxed{2}$

Determine $Y(s) = \mathcal{L}\{y(t)\}$ tal que $\begin{cases} y'' + y' = \cos t, \ t \geq 0 \\ y'(0) = 0 \\ y(0) = 2 \end{cases}$

$$y'' + y' = \cos t, \ t \in \mathbb{R}_0^+$$

$$\mathcal{L}\{y'' + y'\} = \mathcal{L}\{\cos t\}$$

$$\Leftrightarrow \mathcal{L}\{y''\} + \mathcal{L}\{y'\} = \frac{s}{s^2+1}, \quad s > 0$$

- Usando a propriedade da Transformada da derivada

$$\underbrace{\left[s^2 Y(s) - 2s\right]}_{s^2F(s) - sy(0) - y'(0)} + \underbrace{\left[s Y(s) - 2\right]}_{sF(s) - y(0)} = \frac{s}{s^2+1}, \quad s > 0$$

$$(s^2 + s)\, Y(s) = \frac{s}{s^2+1} + 2s + 2$$

$(\dots)$

$$Y(s) = \frac{1}{(s^2+1)(s+1)} + \frac{2}{s}, \quad s > 0$$

$$y = \mathcal{L}^{-1}\left\{ \frac{1}{(s^2+1)(s+1)} + \frac{2}{s} \right\}$$

Veremos asseguir!

Cálculo de TLI's usando decomposições e a Tabela

ex:

a) $\mathcal{L}^{-1}\left\{ \frac{5}{s^2+25} \right\}$

"Qual é a função $f: \mathbb{R}_0^+ \longrightarrow \mathbb{R}$ contínua tal que $\mathcal{L}\{f(t)\} = \frac{5}{s^2+25}$"

$$\mathcal{L}^{-1}\left\{ \frac{5}{s^2+25} \right\} \underset{\substack{T3 \\ a=5}}{=} \sin(5t),\ t \geq 0$$

b) $\mathcal{L}^{-1}\left\{ \frac{3}{s-4} \right\} = 3\,\mathcal{L}^{-1}\left\{ \frac{1}{s-4} \right\}$

$$y(t) \underset{\substack{T2 \\ a=4}}{=} 3e^{4t},\ t \geq 0$$

c) $\mathcal{L}\left\{ \frac{4}{s^7} \right\} = 4\,\mathcal{L}\left\{ \frac{1}{s^7} \right\} = 4 \times \frac{1}{6!} \times \mathcal{L}\left\{ \frac{6!}{s^7} \right\}$

T1 $n = 6$

$$= \frac{1}{180} t^6,\ t \geq 0$$

d) $\mathcal{L}^{-1}\left\{ \frac{s+2}{s^2+4s+40} \right\}$

Vamos ter de manipular a fração.

Raízes: $s^2+4s+40 = 0 \Leftrightarrow s = \frac{-4 \pm \sqrt{\underbrace{16-(4)(1)(40)}_{<0}}}{2} \notin \mathbb{R}$

Como há só um par de raízes complexos conjugados para raiz de $s^2+4s+40$

• Escrever $s^2+4s+40$ na forma $\boxed{pq}$!

$$s^2+4s+40 = \boxed{(s+p)^2+q}$$
$$= s^2+2ps+(p^2+q)$$

$$\Rightarrow \begin{cases} 2p = 4 \\ p^2+q = 40 \end{cases} \quad \begin{cases} p = 2 \\ q = 36 \end{cases}$$

• Logo: $s^2+4s+40 = (s+2)^2+36$

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{\overbrace{s+2}^{\lambda=-2}}{(s+2)^2+6^2} \right\}$$

$\hookrightarrow F(s-(-2))$

• Seja: $F(s) = \frac{s}{s^2+6^2} \Longrightarrow \mathcal{L}^{-1}\{F(s)\} = \cos(6t),\ t \geq 0$

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{(s+2)}{(s+2)^2+6^2} \right\} = \mathcal{L}^{-1}\{F(s-(-2))\}$$

• Assim, $\boxed{y(t) = e^{-2t}\cos(6t)\,,\ t \geq 0}$

e) $$y(t) = \mathcal{L}^{-1}\left\{ \frac{5}{s^2-6s-7} \right\}$$

$$s^2-6s-7 = (s+p)^2+q$$
$$= s^2+2ps+(p^2+q)$$

$$\Rightarrow \begin{cases} -6 = 2p \\ -7 = p^2+q \end{cases} \quad \begin{cases} p = -3 \\ q = -16 \end{cases} \Longrightarrow$$

$$s^2-6s-7 = (s-3)^2-16 = 0$$
$$(s-3)^2 = 16$$
$$s-3 = \pm 4$$
$$s = 3 \pm 4$$
$$\boxed{s = -1 \lor s = 7}$$

Calcular A e B:

$$\frac{5}{s^2-6s-7} = \frac{A}{s-7} + \frac{B}{s+1}$$

$$\frac{5}{s^2-6s-7} = \frac{A(s+1)+B(s-7)}{(s-7)(s+1)}$$

$$5 = (A+B)s + (A-7B)$$

$$\begin{cases} A+B=0 \\ A-7B=5 \end{cases} \quad \begin{cases} A = \frac{5}{8} \\ B = -\frac{5}{8} \end{cases}$$

$$\mathcal{L}^{-1}\left\{ \frac{5}{s^2-6s-7} \right\} = \mathcal{L}^{-1}\left\{ \frac{5/8}{s-7} - \frac{5/8}{s+1} \right\}$$

$$= \frac{5}{8}\mathcal{L}^{-1}\left\{ \frac{1}{s-7} \right\} - \frac{5}{8}\mathcal{L}^{-1}\left\{ \frac{1}{s+1} \right\}$$

$$= \frac{5}{8}e^{7t} - \frac{5}{8}e^{-t}\,,\ t \geq 0$$

$$= \frac{5}{4}e^{3t}\left(\frac{e^{4t}-e^{-4t}}{2} \right)$$

$$= \frac{5}{4}e^{3t}\sinh(4t)\,,\ t \geq 0$$

f) $$\mathcal{L}^{-1}\left\{ \frac{1}{s^2-3s} \right\} = \mathcal{L}^{-1}\left\{ \frac{1}{s(s-3)} \right\}$$

$$\frac{1}{s(s-3)} = \frac{A}{s} + \frac{B}{s-3}$$

$$\Leftrightarrow 1 = (A+B)s + (-3A)$$

$$\begin{cases} A+B=0 \\ -3A = 1 \end{cases} \quad \begin{cases} B = \frac{1}{3} \\ A = -\frac{1}{3} \end{cases}$$

Logo: $\mathcal{L}^{-1}\left\{\frac{-1/3}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{1/3}{s-3}\right\}$

$$= \frac{-1}{3} e^{0t} + \frac{1}{3} e^{3t}$$

$$= \boxed{\frac{1}{3} e^{3t} - \frac{1}{3}, \quad t \geq 0}$$

$\boxed{2}$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s^2+1)} + \frac{2}{s}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s^2+1)}\right\} + 2\,\mathcal{L}\left\{\frac{1}{s}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s^2+1)}\right\} + 2e^{0t}$$

$$\frac{1}{\underbrace{(s+1)}_{\text{raiz simples}}\underbrace{(s^2+1)}_{\text{um par de complexos simples}}} = \frac{A}{s+1} + \frac{Bs + C}{s^2+1}$$

$\Leftrightarrow 1 = A(s^2+1) + Bs(s+1) + C(s+1)$

$\Leftrightarrow 1 = (A+B)s^2 + (B+C)s + (A+C)$

$$\begin{cases} A + B = 0 \\ B + C = 0 \\ A + C = 1 \end{cases} \quad \begin{cases} A = 1/2 \\ B = -1/2 \\ C = 1/2 \end{cases}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1/2}{s+1} + \frac{(-1/2)s + 1/2}{s^2+1}\right\}$$

$$= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= \frac{1}{2} e^{-t} - \frac{1}{2}\cos t + \frac{1}{2}\sin t, \quad t \geq 0$$

falta a constante!